Tensorial structure of the lifting doctrine in constructive domain theory

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Abstract

We present a survey of the two-dimensional and tensorial structure of the *lifting doctrine* in constructive domain theory. We establish the universal property of lifting of directed-complete partial orders (dcpos) as the Sierpiński cone, from which we deduce (1) that lifting forms a Kock–Zöberlein doctrine, (2) that lifting algebras, pointed dcpos, and inductive partial orders form canonically equivalent locally posetal 2-categories, and (3) that the category of lifting algebras is cocomplete, with connected colimits created by the forgetful functor to dcpos. Finally we deduce the symmetric monoidal closure of the Eilenberg–Moore resolution of the lifting 2-monad by means of smash products; these are shown to classify both bilinear maps and strict maps, which we prove to coincide in the constructive setting. We provide several concrete computations of the smash product as dcpo coequalisers and lifting algebra coequalisers, and compare these with the more abstract results of Seal. Although all these results are well-known classically, the existing proofs do not apply in a constructive setting; indeed, the classical analysis of the Eilenberg–Moore category of the lifting monad relies on the fact that all lifting algebras are free, a condition that is not known to hold constructively.

1 Introduction

Axiomatic approaches to domain theory take place in a *monoidal adjunction* between a category of "predomains" and a category of "domains". The simplest notion of predomain is given by *directed complete partial orders* (dcpos) and Scott–continuous functions between them; a corresponding notion of domain arises by considering algebras for an appropriate commutative monad on the preorder-enriched category of predomains. Most commonly, domains are considered to be algebras for *lifting monad* $\mathbb{L}$ on the category of predomains that introduces partiality.

For the Eilenberg–Moore resolution $L \dashv U : \mathsf{dcpo}^{\mathbb{L}} \to \mathsf{dcpo}$ of the lifting monad to be monoidal, we of course presuppose that $\mathsf{dcpo}^{\mathbb{L}}$ has a monoidal product $\otimes$; then the left adjoint being strong monoidal means that we have coherent isomorphisms $L(A \times B) \cong LA \otimes LB$, *etc.* This property can be seen as a definition of the tensor on

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free domains (take the free domain on the cartesian product of the generators), but it does not immediately follow from this that we may extend the tensor to operate on non-free domains. In classical mathematics, this difficulty is side-stepped by virtue of the fact that *there are no non-free domains!*

Indeed, classically, every $\mathbb{L}$-algebra is a *free* $\mathbb{L}$-algebra — if X has a bottom element $\perp$, it can be seen that X is the lift of the dcpo $X \setminus \{\perp\}$ using the law of the excluded middle. Unfortunately, this simple description of $\mathbb{L}$-algebras does not carry over to the constructive mathematics of an elementary topos, as Kock [1995] has discussed at length. We can illustrate the problem by means of the following Brouwerian counterexample (Theorem 2) which follows by way of Lemma 1 below.

Lemma 1. *The lifting functor* $L \colon \mathsf{dcpo} \to \mathsf{dcpo}^{\mathbb{L}}$ *is conservative.*

Proof. Let $f \colon A \to B$ be a morphism of dcpos such that $Lf \colon LA \to LB$ is an isomorphism of $\mathbb{L}$-algebras. For arbitrary such $f \colon A \to B$ we have the following pullback square in the category of dcpos:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "\eta_A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, hook, "\eta_B"] \\ LA \arrow[r, "Lf"'] & LB \end{tikzcd} \tag{1}$$

Any pullback of an isomorphism is an isomorphism, so we may conclude that $f \colon A \to B$ is an isomorphism. ◻

Theorem 2. *The law of excluded middle holds if and only if every free* $\mathbb{L}$*-algebra is free on its non-bottom elements.*

Proof. If the law of excluded middle holds, then obviously every $\mathbb{L}$-algebra is free on its non-bottom elements. In the converse direction, we consider whether the $\mathbb{L}$-algebra Ω given by the collection of all propositions with their implication order, where suprema are computed by existential quantification, is free on its non-bottom elements; it is easy to see that Ω is the free $\mathbb{L}$-algebra on the terminal dcpo. Therefore the map $L!_{\Omega\setminus\{\perp\}} \colon L(\Omega \setminus \{\perp\}) \to L\mathbf{1}$ is an isomorphism; by assumption, we may conclude from Lemma 1 that $\Omega \setminus \{\perp\}$ is a singleton — or, equivalently, that a proposition ϕ is true if and only if $\phi \neq \perp$. Now let ψ be any proposition; to show that $\psi \vee \neg\psi$, by the above we may assume $\neg(\psi \vee \neg\psi)$ to prove a contradiction; our assumption is equivalent to $\neg\psi \wedge \neg\neg\psi$, which is clearly contradictory. ◻

Remark 3. Note that lifting of dcpos in constructive mathematics is *not* defined on points by taking the coproduct with $\mathbf{1}$, as Kock [1995] has pointed out; it is defined using the partial map classifier of the ambient topos — which degenerates to $(\mathbf{1} + -)$ in a boolean topos. The counterexample in Theorem 2 obtains *in spite* of the constructive definition of lifting.

Although Theorem 2 shows that it need not be the case that all $\mathbb{L}$-algebras are free on their non-bottom elements, with one might conjecture that every $\mathbb{L}$-algebra is nonetheless free on a *different* subdcpo. The most natural candidate for a subdcpo $X^+ \subseteq UX$ such that $LX^+ \cong X$ would be the one spanned by *positive* elements in the sense of de Jong and Escardó [2021] as adapted from Johnstone [1984]: an element x of an $\mathbb{L}$-algebra X is called ***positive*** when any semidirected subset of X whose suprema lies above x is directed. Noting that the subposet of an $\mathbb{L}$-algebra X spanned by positive elements is always a dcpo, we are naturally led to the following open question:

Open Question 1. *Does there exist an elementary topos containing an $\mathbb{L}$-algebra that is not free on its subdcpo of positive elements?*

Indeed, Kock [1995] has shown that an $\mathbb{L}$-algebra is free if and only if it is free on its positive elements; combining this with Lemma 1, we see that the *only* possible generators for a free $\mathbb{L}$-algebra dcpo are its positive elements (which coincide with the non-bottom elements in the classical setting). Therefore, an answer to Open Question 1 would determine altogether whether and how all $\mathbb{L}$-algebras can be free in constructive mathematics; I conjecture that the answer to Open Question 1 is "Yes", and so there may exist examples of non-free domains. Until and unless this expectation is contravened by mathematical evidence, the constructive version of the smash product must be defined on (potentially) non-free domains.

Lifting closed structure à la Kock and Seal It is a well-known result of category theory due to Kock [1971] that the category of algebras $\mathscr{V}^{\mathbb{T}}$ for a commutative monad $\mathbb{T} \equiv (T, \eta, \mu)$ on a symmetric monoidal closed category $\mathscr{V}$ with equalizers inherits *closed* structure from $\mathscr{V}$, and (moreover) that the Eilenberg–Moore resolution of $\mathbb{T}$ consists of closed functors, *i.e.* the left and right adjoints laxly preserve the internal hom. What is missing is the *monoidal* structure on $\mathbb{L}$-algebras that should extend the Eilenberg–Moore resolution $L \dashv U : \mathsf{dcpo}^{\mathbb{L}} \to \mathsf{dcpo}$ to a (symmetric) *monoidal* closed adjunction. Luckily, a further result of Seal [2013] provides sufficient conditions for a category of algebras to admit a tensor product by means of a construction dual to that of the internal hom and, moreover, for this tensor product to represent bilinear maps. That these conditions in fact hold constructively for dcpos and their lifting monad has not been verified until now, although they are not especially difficult.

Summary of contributions The contribution of the present paper is to provide a constructive analysis of the lifting doctrine for dcpos, embodied in the following results:

1. **Universal properties of Ω:** the top truth value $\top : \mathbf{1} \hookrightarrow \Omega$ is the universal Scott–open immersion (Theorem 24), and the inequality $\bot \sqsubseteq \top : \mathbf{1} \hookrightarrow \Omega$ satisfies the 2-categorical universal property of the Sierpiński space (Theorem 25).[1]

2. **Universal properties for lifting:** lifting enjoys both left- and right-handed universal properties in the 2-category of dcpos as a Sierpiński cone (Theorem 37) and as a partial product (Theorem 35) respectively. The former implies our most

[1]Although these results are known, they play a important role in what follows.

important technical lemma, that $\bot : \mathbf{1} \hookrightarrow LA$ and $\eta_A : A \hookrightarrow LA$ are jointly (lax) epimorphic (Corollary 39), enabling a restricted form of classical reasoning when establishing inequalities of the form $f \sqsubseteq g : LA \to B$.

3. **Lifting is a Kock–Zöberlein doctrine:** for any lifting algebra X, the structure map $\alpha_X : LX \to X$ is left adjoint to the unit $\eta_X : X \hookrightarrow LX$, and so lifting algebra structures are unique (Lemma 43).

4. **Monadicity of pointed dcpos and ipos:** lifting algebras, pointed dcpos, and inductive partial orders are all canonically equivalent locally posetal 2-categories (Corollary 50), and so pointed dcpos and ipos are monadic over dcpos (Corollary 51).

5. **Cocompleteness of lifting algebras:** the category of lifting algebras is closed under all colimits, with connected colimits created by the forgetful functor $U : \mathsf{dcpo}^{\mathbb{L}} \to \mathsf{dcpo}$ (Corollaries 53 and 55).

6. **Tensorial structure of lifting:** bilinear and bistrict maps coincide (Lemma 70) and are representable by the *smash product* (Theorem 66) for which we provide several computations as coequalisers in both dcpo and $\mathsf{dcpo}^{\mathbb{L}}$ (Corollary 67). Smash products extend to a full symmetric monoidal structure on $\mathsf{dcpo}^{\mathbb{L}}$, so that the adjunction $L \dashv U : \mathsf{dcpo}^{\mathbb{L}} \to \mathsf{dcpo}$ is symmetric monoidal (Corollary 77). Moreover, smash products are left adjoint to strict function spaces (Lemma 81) which make $L \dashv U$ into a *closed* adjunction.

Why does constructive domain theory matter? The generality of our results is important, as modern approaches to programming semantics routinely involve computing recursive functions in non-boolean topoi. Our interest in constructive domains is not rooted in the philosophy of intuitionism, but instead in the practical necessity to study computation in *variable and continuous sets* [Lawvere, 1975] as well as *effective sets* [Hyland, 1982, Bauer, 2006], whose dynamics generalize those of constant sets.

In fact, it happens that the constructive theory of dcpos has not received much attention in the literature outside the groundbreaking work of Kock [1995], Townsend [1996], de Jong and Escardó [2021], de Jong [2021, 2023]. Therefore many results that appear to be "obvious" have not in fact been established, and the constructive domains behave differently enough from the classical ones that it would not be safe to take these results for granted. This paper is one further step in the direction of a thorough and base-independent account of dcpos that is applicable in an arbitrary topos.

2 Preliminaries in constructive category theory

2.1 Creation of colimits

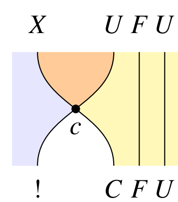
Due to the proliferation of numerous incompatible (and infelicitous, if not erroneous) descriptions of created (co)limits, we fix appropriate definitions and prove a few basic results that we will need. The reader familiar with standard results about monads and created colimits can safely skip this section.

Definition 4. Let $U: \mathscr{D} \to \mathscr{C}$ be a functor and let $\mathscr{I}$ be a category. The functor U is said to ***create colimits of $\mathscr{I}$-figures*** when for any diagram $D: \mathscr{I} \to \mathscr{D}$, if $UD: \mathscr{I} \to \mathscr{C}$ has a colimit then $D: \mathscr{I} \to \mathscr{D}$ has a colimit that is both preserved and reflected by U.

Lemma 5. *Let $\mathscr{C}$ be a category and let $\mathbb{T} \equiv (T, \eta, \mu)$ be a monad on $\mathscr{C}$. Suppose that the endofunctor T preserves colimits of $\mathscr{I}$-figures for a given category $\mathscr{I}$. Let $X: \mathscr{I} \to \mathscr{C}^{\mathbb{T}}$ be a diagram of $\mathbb{T}$-algebras such that $UX: \mathscr{I} \to \mathscr{C}$ has a universal cocone $c: UX \to \{C\}$. We may extend C to an essentially unique $\mathbb{T}$-algebra structure $\bar{C}$ over C in a canonical way such that $c: UX \to \{C\}$ lifts to a cocone of algebras $\bar{c}: X \to \{\bar{C}\}$ over c.*

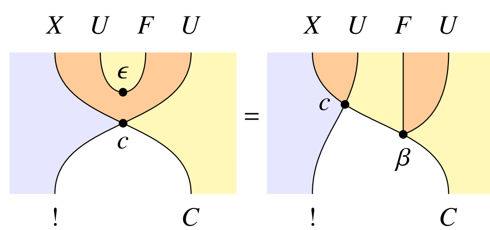
We will argue using the *string diagrammatic* language of the 2-category of categories, the advantage being that it clarifies reasoning that involves naturality. We refer to Hinze and Marsden [2023] for a thorough introduction to string diagrams in a 2-category; note, however, that we differ from *op. cit*. by having diagrams flow from the downward and to the right in keeping with the usual diagrammatic order of composition. In what follows, we let $F \dashv U$ be the Eilenberg–Moore resolution of $\mathbb{T}$.

Proof. By assumption, the following diagram is a universal cocone.



(2)

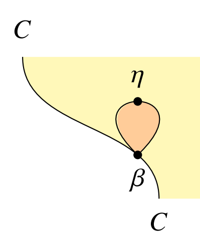
We define a further cocone on the left below, which by the universal property of Diagram 2 factors through a unique map $\beta: UFC \to C$ as depicted on the right:



(3)

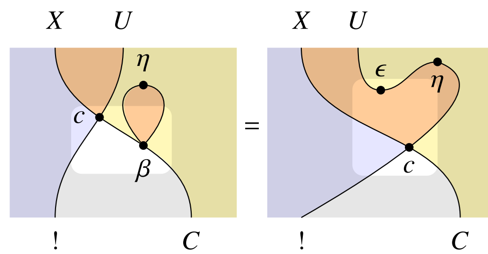
We will show that the map $\beta: UFC \to C$ satisfies the axioms of a $\mathbb{T}$-algebra.

1. The unit law asserts that Diagram 4 depicts the identity cell on C:



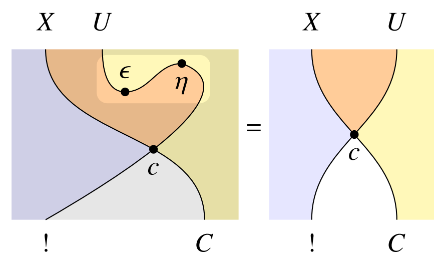
(4)

By the universal property of Diagram 2, it suffices to check that composition of Diagram 2 with Diagram 4 is equal to Diagram 2. Forming the composite, we first recall the defining property of $\beta\colon UFC \to C$ and rewrite accordingly:



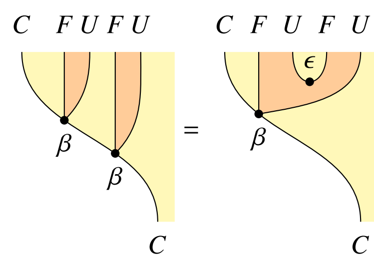
(5)

Finally, we rewrite using the snake identity of $F \dashv U$:



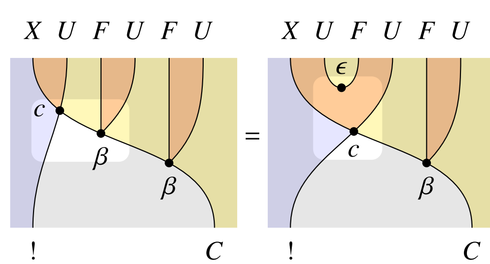
(6)

2. The multiplication law asserts that the following two diagrams are equal:



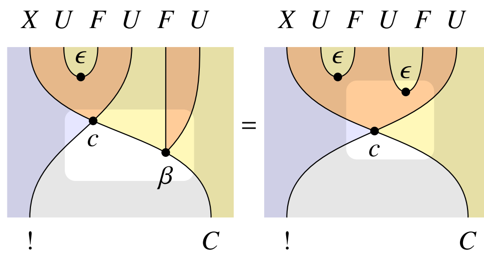
(7)

It suffices to consider their restriction along the cocone $TTc\colon TTUX \to \{TTC\}$, which is universal as T is assumed to preserve this colimit. We first use the defining property of β:



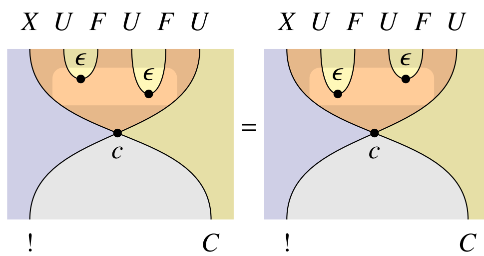
(8)

We use the defining property of β once more:



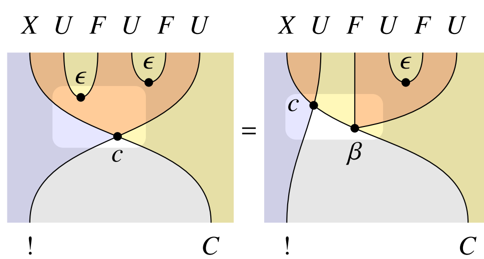
(9)

Naturality allows us to swap the order in which the counits are composed, corresponding to the depth of the depicted "sag".



(10)

Then the defining equation of β implies the result.

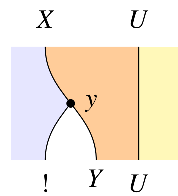


(11)

Hence we may define a $\mathbb{T}$-algebra structure $\bar{C}$ with $U\bar{C} = C$, setting $\alpha_{\bar{C}} : TC \to C$ to be β. That c lifts to a cocone of algebras is *exactly* the defining condition of $\alpha_{\bar{C}} = \beta$ via the universal property of $Tc : TUX \to \{TC\}$; uniqueness of the algebra structure follows from the same universal property. $\square$

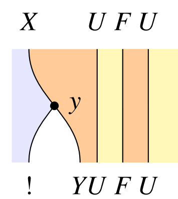
Lemma 6. *Let $\mathbb{T} \equiv (T, \eta, \mu)$ be a monad on a category $\mathscr{C}$. If T preserves colimits of $\mathscr{I}$-figures for a given category $\mathscr{I}$, then $U : \mathscr{C}^{\mathbb{T}} \to \mathscr{C}$ reflects colimits of $\mathscr{I}$-figures.*

Proof. Let $X : \mathscr{I} \to \mathscr{C}^{\mathbb{T}}$ be a diagram equipped with a cocone $y : X \to \{Y\}$ whose image $Uy\colon UX \to \{UY\}$ in $\mathscr{C}$ is universal.



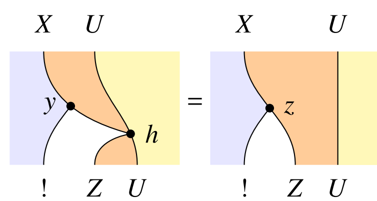
(12)

If therefore follows, by assumption, that $TUy\colon TUX \to \{TUY\}$ is universal:



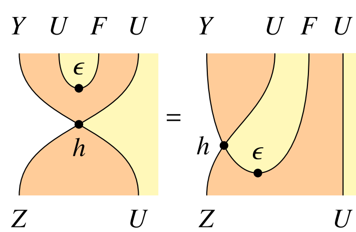
(13)

We aim to show that $y\colon X \to \{Y\}$ is universal in $\mathscr{C}^{\mathbb{T}}$. To check this universal property, we fix a further cocone $z\colon X \to \{Z\}$ in $\mathscr{C}^{\mathbb{T}}$; of course, we may factor $Uz\colon UX \to \{UZ\}$ through the universal cocone $Uy\colon UX \to \{UY\}$ through some unique $h\colon UY \to UZ$ as depicted below:



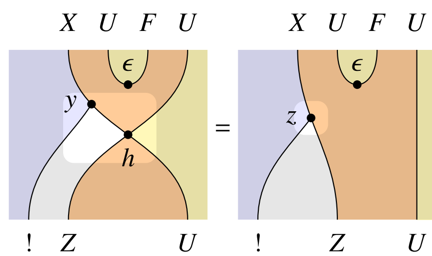
(14)

We will show that $h\colon UY \to UZ$ lies in the image of some $\bar{h}\colon Y \to Z$ in $\mathscr{C}^{\mathbb{T}}$; as the forgetful functor $U\colon \mathscr{C}^{\mathbb{T}} \to \mathscr{C}$ is necessarily faithful, this will establish that $y\colon X \to \{Y\}$ is a universal cocone. To exhibit $\bar{h}\colon Y \to Z$ over h is, by definition, the same as to check that the latter is a homomorphism of algebras in the sense depicted below:



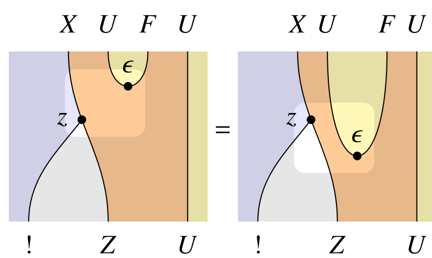
(15)

Because Diagram 13 is a universal cocone, we can check Eq. (15) by restricting both sides along Diagram 13. After doing so, we first use rewrite along Eq. (14):



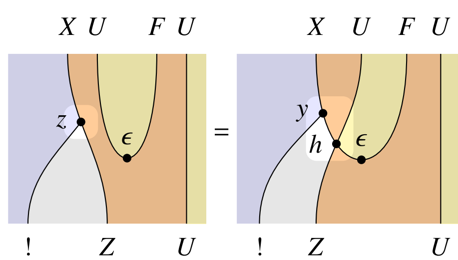
(16)

We can then swap the order in which z is composed with the counit, by naturality:



(17)

We finally use Eq. (14) one last time.



(18)

We have shown that $h\colon UY \to UZ$ satisfies the homomorphism property, and therefore lies in the image of some (unique) $\bar{h}\colon Y \to Z$, so we are done. □

Lemma 7. *Let $\mathbb{T} \equiv (T, \eta, \mu)$ be a monad on a category $\mathscr{C}$. If T preserves colimits of $\mathscr{I}$-figures for a given category $\mathscr{I}$, then $U\colon \mathscr{C}^{\mathbb{T}} \to \mathscr{C}$ creates colimits of $\mathscr{I}$-figures.*

Proof. Let $X\colon \mathscr{I} \to \mathscr{C}^{\mathbb{T}}$ be a diagram such that $UX\colon \mathscr{I} \to \mathscr{C}$ has a universal cocone $c\colon UX \to \{C\}$ in $\mathscr{C}$. We let $\bar{C} \in \mathscr{C}^{\mathbb{T}}$ with $U\bar{C} = C$ be the algebra structure on C given by Lemma 5, so that $c\colon UX \to \{UC\}$ lifts to a cocone of algebras $\bar{c}\colon X \to \{C\}$. As $U\colon \mathscr{C}^{\mathbb{T}} \to \mathscr{C}$ reflects colimits of $\mathscr{I}$-figures (Lemma 6), we conclude that the cocone $\bar{c}\colon X \to \{C\}$ is indeed universal in $\mathscr{C}^{\mathbb{T}}$. □

2.2 Geometry in a 2-category

In this section, we elucidate the 2-categorical universal properties that will play a role in the constructive study of the lifting doctrine on dcpos. Although we of course have need only for *poset-enriched* versions of what follows, we first work in as much generality as possible in order to lay the foundations for future investigations of higher-dimensional domain theory outside the locally posetal setting.

Definition 8. Let $\mathcal{K}$ be any 2-category with a terminal object; a ***Sierpiński space*** is then defined to be a cocomma object of the following form:

$$\begin{tikzcd} \mathbf{1} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow"] & \mathbf{1} \arrow[d, "\top"] \\ \mathbf{1} \arrow[r, "\bot"'] & \Sigma \end{tikzcd} \tag{19}$$

Equivalently, the Sierpiński space is the *tensor* $\Delta^1 \cdot \mathbf{1}$ where Δ^1 is the directed interval category $\{0 \to 1\}$.

Reading Definition 8 in the 2-category of dcpos, the Sierpiński space Σ is, if it exists, the smallest dcpo that contains two points $\bot, \top : \Sigma$ and an inequality $\bot \sqsubseteq \top$.[2] The Sierpiński space is a special case of a more general gluing construction called the Sierpiński cone:

Definition 9. The ***Sierpiński cone*** of an object $A : \mathcal{K}$ in a 2-category $\mathcal{K}$ with a terminal object is defined to be the following cocomma object:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "!_A"'] \arrow[dr, phantom, "\Nearrow"] & A \arrow[d, "\top"] \\ \mathbf{1} \arrow[r, "\bot"'] & \Sigma A \end{tikzcd} \tag{20}$$

The geometry of Definition 9 is that ΣA adjoins an additional point "to the left" of A, which forms the apex of a cone in A whose endpoints lie in A. Of course, we have $\Sigma = \Sigma\mathbf{1}$ and further generic "finite chain" figures can be obtained by iteration; for instance, $\Sigma_n :\equiv \Sigma^n\mathbf{1}$ would be the generic chain with n segments.

Observation 10. *Product 2-functors $A \times -$ in a cartesian closed 2-category preserve cocomma squares.*

Lemma 11 (Generalised Phoa's principle)**.** *Let $\mathcal{K}$ be a cartesian closed 2-category with a terminal object and Sierpiński space Σ; then for any $Y \in \mathcal{K}$, the following lax square*

[2] As we will see, this description does *not* imply that the Sierpiński dcpo has exactly two points!

induced by evaluation at the generic 2-cell $\bot \sqsubseteq \top$ is a comma square in $\mathcal{K}$:

$$\begin{tikzcd} Y^{\Sigma} \arrow[r, "-\top"] \arrow[d, "-\bot"'] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd} \tag{21}$$

Proof. Equivalently, we must check that Y^{Σ} is the power $\Delta^1 \pitchfork Y$. The proof is (2-)adjoint calisthenics, using the characterisation of Σ as the power $\Delta^1 \cdot \mathbf{1}$.

$$\begin{aligned} \mathcal{K}(X, Y^{\Sigma}) &\cong \mathcal{K}(X \times \Sigma, Y) \\ &\cong \mathcal{K}(X \times (\Delta^1 \cdot \mathbf{1}), Y) \\ &\cong \mathcal{K}(\Delta^1 \cdot X, Y) \\ &\cong \mathbf{Cat}(\Delta^1, \mathcal{K}(X, Y)) \\ &\cong \mathcal{K}(X, \Delta^1 \pitchfork Y) \end{aligned}$$

□

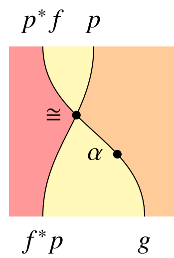
2.3 Partial products in a 2-category

Finally, we recall the notion of *cocartesian fibration* and *partial product* in a 2-category [Johnstone, 2002]. In this section, let $\mathcal{K}$ be a finitely complete 2-category. We will prefer the "Chevalley criterion" for cocartesian fibrations described below.

Definition 12 (Loregian and Riehl [2020])**.** A 1-cell $p\colon E \to B$ in $\mathcal{K}$ is called ***cocartesian fibration*** when the canonical arrow $\Delta^1 \pitchfork E \to p \downarrow B$ corresponding to the lax square below has a left adjoint right inverse:

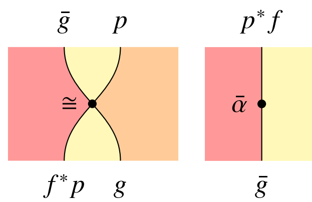
$$\begin{tikzcd} \Delta^1 \pitchfork E \arrow[r, "p \circ \partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\Nearrow"] & B \arrow[d, equal] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{22}$$

Construction 13 (Lifting 2-cells to generalised fibers)**.** As Hazratpour [2019] points out, a cocartesian fibration in the sense of Definition 12 can be equipped with operations corresponding to the more nuts-and-bolt description of internal cocartesian fibrations given by Johnstone [2002]. In particular, for a given 2-cell $\alpha\colon f \to g$ in $\mathcal{K}(C, B)$ we may define a 1-cell $\alpha^*E\colon f^*E \to g^*E$ between pullbacks. In particular, the 2-cell determines a 1-cell $f^*E \to p \downarrow B$, where $p \circ p^*f \cong f \circ f^*p$ is the canonical isomorphism of the pullback square:

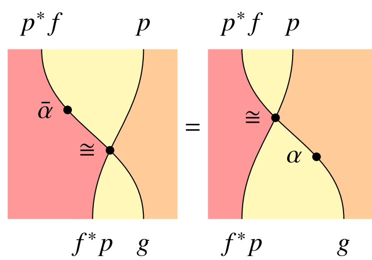


(23)

Postcomposing with the left adjoint right inverse to $\Delta^1 \pitchfork E \longrightarrow p \downarrow B$, we obtain the following cells and equations:



(24)



(25)

The isomorphism $p \circ \bar{g} \cong g \circ f^*p$ depicted in Diagram 24 is precisely the data of a suitable map $\alpha^*E\colon\ f^*E \longrightarrow g^*E$, considering the universal property of g^*E.

The following notion is described by Johnstone [2002] as a *partial product cone*.

Definition 14 (Johnstone [2002])**.** Let $p\colon E \longrightarrow B$ in be a cocartesian fibration in $\mathcal{K}$, and let A be a 0-cell in $\mathcal{K}$. A ***nondeterministic map*** from C to A with coefficients in $p\colon E \longrightarrow B$ is defined to consist of a 1-cell $u\colon C \longrightarrow B$ equipped with a further 1-cell $e\colon u^*E \longrightarrow A$ as depicted below:

$$\begin{tikzcd} A & u^*E \arrow[l, "e"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ & C \arrow[r, "u"'] & B \end{tikzcd} \tag{26}$$

A morphism of such nondeterministic maps from (u, e) to (u', e') is given by a 2-cell $\alpha\colon u \to u'$ together with a further 2-cell $\beta\colon e \to \alpha^* E * e'$ where $\alpha^* E\colon u^* E \to u'^* E$ is as described in Construction 13.

We shall write $\mathcal{K}^p(-, A)\colon \mathcal{K}^{\mathsf{op}} \to \mathbf{Cat}$ for the pseudofunctor sending 0-cells $C \in \mathcal{K}$ to the category of nondeterministic maps from C to A with coefficients in p.

Definition 15. The ***partial product*** of a cocartesian fibration $p\colon E \to B$ in $\mathcal{K}$ with a 0-cell A is a 0-cell $\mathcal{P}_\bullet(p, A)$ representing the pseudofunctor $\mathcal{K}^p(-, A)$ in the sense that we have a pseudonatural equivalence $\mathcal{K}(-, \mathcal{P}_\bullet(p, A)) \simeq \mathcal{K}^p(-, A)$.

When $\mathcal{P}_\bullet(p, A)$ is the partial product of $p\colon E \to B$ with $A \in \mathcal{K}$, we have in the generic case a nondeterministic map from $\mathcal{P}_\bullet(p, A)$ to itself, as depicted below:

$$\begin{tikzcd} A & u^* E \arrow[l, "e"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ & \mathcal{P}_\bullet(p, A) \arrow[r, "u"'] & B \end{tikzcd} \tag{27}$$

In this case, we shall refer to the above as the ***universal nondeterministic map out of*** A ***with coefficients in*** $p\colon E \to B$.

3 Basic notions in constructive domain theory

We recall the basics of the (constructive) theory of dcpos and their lifting monad, following the exposition of de Jong [2021], de Jong and Escardó [2021]. The main difference in relation to *op. cit*. is that we assume propositional resizing, as we are not concerned here with predicativity.

Definition 16. A subset $U \subseteq A$ of a partial order A is called ***semidirected*** when for any two $x, y \in U$ there exists an upper bound for x and y in U. A subset is called ***directed*** when it is both semidirected and inhabited.

Definition 17. A partial order A is called a ***directed-complete*** when any directed subset $U \subseteq A$ has a supremum in A. A morphism of directed-complete partial orders is a ***Scott–continuous function***, *i.e*. a function that preserves directed suprema.

We shall refer to directed-complete partial orders a *dcpo*s, writing writing dcpo for the category of dcpos and Scott continuous maps. Note that Scott–continuous functions are automatically monotone.

Definition 18. A partial order A is called ***inductive*** when any semidirected subset $U \subseteq A$ has a supremum in A. A morphism of inductive partial orders is an ***inductive function***, *i.e*. one that preserves semidirected suprema.

We shall abbreviate inductive partial orders as *ipos*, writing ipo for the category of ipos and morphisms of ipos.

Definition 19. A dpco A is called ***pointed*** when it has a bottom element $\bot$, *i.e.* such that $\bot \sqsubseteq a$ for all $a : A$.

Definition 20. A Scott–continuous map between pointed dcpos is called ***strict*** when it preserves the bottom element.

We shall abbreviate pointed dcpos as *dcppos* and write dcppo for the category of pointed dcpos and strict maps. Once we have introduced the lifting monad $\mathbb{L}$ on dcpo, we will ultimately show in Section 4.7 that $\mathsf{dcpo}^{\mathbb{L}} = \mathsf{ipo} = \mathsf{dcppo}$.

3.1 Open subspaces and their classifier

We recall the notion of Scott–open subset of a dcpo in the constructive setting, *e.g.* from de Jong [2021].

Definition 21. A subset $U \subseteq A$ of a dcpo A is called ***Scott–open*** when it is inaccessible by directed suprema in the sense that for any directed subset $S \subseteq A$ with $\bigsqcup S \in U$, there exists an element $s \in S$ such that $s \in U$.

(A Scott–open subset is automatically an upper set.)

Remark 22. Note that the appropriate notion of Scott–closed subset is *not* obtained by taking complements of Scott–open subsets, except in the case of continuous dcpos [de Jong, 2021]. We will not deal with closed subsets in this paper.

We shall refer to the subdcpo spanned by a given Scott–open subset as a ***Scott–open subspace***. A morphism of dcpos $i : A \to B$ factoring through an isomorphism onto an open subspace of B is called a ***Scott–open immersion***. We will observe that *universal monomorphism* $\top : \mathbf{1} \rightarrowtail \Omega$ in the category of sets[3] extends to a *universal Scott–open immersion* in the world of dcpos.

Lemma 23. *The universe Ω of all propositions is a dcpo with its implication order.*

Proof. Implication clearly gives rise to a partial order on Ω; the existential quantifier ensures that Ω is in fact a sup-lattice, and thus a dcpo. □

Theorem 24. *The morphism $\top : \mathbf{1} \hookrightarrow \Omega$ is the **universal Scott–open immersion** in* dcpo*, in the sense that $\top : \mathbf{1} \hookrightarrow \Omega$ is a Scott–open immersion and that for any other Scott–open immersion $i : U \hookrightarrow A$, there exists a unique cartesian square from i to $\top$ in* dcpo *as depicted below:*

$$\begin{tikzcd} U \arrow[r, dashed, "!_U"] \arrow[d, hook, "i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{1} \arrow[d, hook, "\top"] \\ A \arrow[r, dashed, "\exists!\,[i]"'] & \Omega \end{tikzcd} \tag{28}$$

[3]To be more precise, we mean the ambient topos when we speak of "sets".

Proof. Without loss of generality, we may consider the open immersion induced by a Scott–open subset U of A. As the forgetful functor from dcpos to their underlying sets is faithful, we can deduce our result from the universal property of $\top\colon \mathbf{1} \rightarrowtail \Omega$ as the universal monomorphism in the category of sets; in particular, it is enough to observe that the characteristic function of a subset of a dcpo is Scott–continuous if and only if the subset is Scott–open, recalling that joins in Ω are given by existential quantification. ◻

3.2 Geometry of the Scott–open subspace classifier

Both dcpo and ipo are easily seen to be enriched in posets; given $f, g\colon A \to B$ we define $f \sqsubseteq g$ if and only if $fx \sqsubseteq gx$ for all $x : A$. This enrichment turns dcpo and ipo into (locally posetal) 2-categories, and so we may consider 2-categorical limits and colimits.

We have seen a "right-handed" or limit-style universal property for Ω as the base of the universal Scott–open immersion (Theorem 24). In this section, we will see that Ω has an alternative left-handed universal property as the *Sierpiński space* (Definition 8) in the 2-category of dcpos. These two universal properties reflect the role of Ω as a dualising object in the algebro-geometric context of domain theory.

Theorem 25. *The following is a cocomma square in the 2-category* dcpo*, and so* Ω *is the Sierpiński space in the sense of Definition 8:*

$$\begin{tikzcd} \mathbf{1} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow"] & \mathbf{1} \arrow[d, "\top"] \\ \mathbf{1} \arrow[r, "\perp"'] & \Omega \end{tikzcd} \tag{29}$$

Proof. Consider an arbitrary lax square in the following configuration:

$$\begin{tikzcd} \mathbf{1} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow"] & \mathbf{1} \arrow[d, "c_1"] \\ \mathbf{1} \arrow[r, "c_0"'] & C \end{tikzcd} \tag{30}$$

The universal map $h\colon \Omega \to C$ factoring c_0 through $\perp$ and c_1 through $\top$ is defined so as to send $\phi : \Omega$ to $\bigvee_{\mathbf{1}+\phi}[c_0 \mid c_1]$, *i.e.* supremum of the union of $\{c_1 \mid \phi = \top\}$ and $\{c_0\}$. It is also observed easily that this assignment preserves directed suprema in Ω. That $h\colon \Omega \to C$ is unique with this factorization property follows from the uniqueness of suprema: any map factoring c_0 and c_1 in this sense is supremum of the same directed subset. ◻

By virtue of Theorem 25, we may define $\Sigma :\equiv \Omega$; therefore, unless the law of excluded middle holds, it need not be the case that Σ has exactly two points — although the law of non-contradiction ensures that no third point can be proved unequal to both $\bot$ and $\top$.[4]

Remark 26. It is perhaps surprising at first that the Sierpiński space in the 2-category of *posets* nonetheless has only two elements in constructive mathematics, in spite of Theorem 25. This is not so strange, however: the ideal completion 2-functor from posets to dcpos is *left adjoint* to the forgetful functor, and so it necessarily preserves Sierpiński objects. But in constructive mathematics, the set of ideals in $\mathbf{2} = \{0 \leq 1\}$ necessarily contains *all* directed downsets of $\mathbf{2}$ and not just the decidable ones; thus we see, by means of a more conceptual argument than that of Lemma 23, that the Sierpiński dcpo must be given by Ω.

Lemma 27. *The universal open immersion $\top\colon \mathbf{1} \hookrightarrow \Sigma$ is a cocartesian fibration of dcpos in the sense of Definition 12.*

Proof. Letting A be an arbitrary dcpo; we must check that the canonical morphism $\Delta^1 \pitchfork \mathbf{1} \to \top \downarrow \Sigma$ has a left adjoint right inverse. In fact, $\Delta^1 \pitchfork \mathbf{1} \cong \mathbf{1} \cong \top \downarrow \Sigma$, so we are done. □

Definition 28 (Paths between dpco morphisms)**.** Let $f, g\colon A \to B$ be a morphism of dcpos; a *path* from f to g is defined to be a morphism $\alpha\colon \Sigma \times A \to B$ such that $\alpha \circ (\bot, -) = f$ and $\alpha \circ (\top, -) = g$.

Corollary 29 (Path enrichment)**.** *The following properties of paths hold:*

1. *There is at most one path between any two morphisms $f, g\colon A \to B$ of dcpos.*
2. *For $f, g\colon A \to B$, there exists a path from f to g if and only if $f \sqsubseteq g$.*

Proof. These are immediate consequences of Theorem 25. □

3.3 Enriched cocompleteness of the category of dcpos

Our study of the Sierpiński space and the path-enrichment of dcpo (Corollary 29) implies the important property that any 1-categorical colimits of dcpos that we may construct are, in fact, 2-categorical colimits.

Corollary 30 (Enrichment of colimits)**.** *Colimits in* dcpo *are poset-enriched.*

Proof. This follows immediately from Corollary 29 and the fact that the product functor $\Sigma \times -$ has a right adjoint and is therefore cocontinuous. □

We have not seen, however, how to actually construct any given colimit of dcpos. Although it is not hard to see that dcpo is cocomplete in a classical metatheory using the adjoint functor theorem [Abramsky and Jung, 1995], it is unclear how to satisfy the solution set condition in constructive mathematics, although it may nonetheless be possible. Luckily, it happens that the constructive cocompleteness of dcpo is an immediate consequence of the (fully constructive) generalized coverage theorem of Townsend [1996].

[4] From the external point of view, there will generally be many distinct *global* points of the internal dcpo Σ. But even if p, q, r are distinct global points, the topos logic will not deduce $p \neq q \neq r$ unless the topos is the *empty topos* (*i.e.* the trivial category).

Lemma 31 (Townsend [1996, p. 72])**.** *The category of dcpos is closed under coequalisers, and is therefore cocomplete.*

Proof. Townsend [1996] has shown that the coequaliser of dcpos can be computed in their enveloping sup-lattices and then extracted by means of an image factorization that isolates the smallest subdcpo of the coequaliser sup-lattice containing the original dcpo that we wished to quotient. □

The argument of *op. cit*. is a more conceptual version of the explicit construction of dcpo quotients in terms of dcpo presentations [Jung et al., 2008], or the even more explicit constructions of Fiech [1996], Goubault-Larrecq [2019].

4 The lifting monad and its algebras

4.1 The partial map classifier

In this section, we shall study the structure of *partial maps* of dcpos in terms of 2-category–theoretic universal properties.

Definition 32. A ***partial map*** from A to B is given by a span $B \leftarrow U \hookrightarrow A$ in which $U \hookrightarrow A$ is a Scott–open immersion. An *inequality* from $B \leftarrow U \hookrightarrow A$ to $B \leftarrow U' \hookrightarrow A$ is given by an embedding $U \hookrightarrow U'$ over B such the composite $U \hookrightarrow U' \hookrightarrow B \to A$ is equal to $U \to A$.

Observation 33. *The partial order of partial maps from A to B is* precisely *the (posetal) category* $\mathsf{dcpo}^{\top}(A, B)$ *of nondeterministic maps from A to B with coefficients in the universal Scott–open immersion* $\top \colon \mathbf{1} \hookrightarrow \Sigma$.

Proof. This follows immediately from the universal property of $\top \colon \mathbf{1} \hookrightarrow \Sigma$ as the universal Scott–open immersion (Lemma 27). □

Thus the appropriate enriched / 2-categorical universal property for *classifying* partial maps is given by partial products (Definition 15). We will now give an explicit description of the classification of partial maps into B by the partial product $\mathcal{P}_{\bullet}(\top, B)$.

Construction 34 (The lifting operation on dcpos)**.** Given a dcpo A, the ***lifted dcpo*** LA is defined to have the (base of) the *partial map classifier* $LA :\equiv \sum_{\phi:\Omega} A^{\phi}$ of A as its underlying set, with the following partial order:

$$\begin{aligned}(\phi, u) \sqsubseteq_{LA} (\psi, v) &\iff \forall x : \phi.\, \exists y : \psi.\, ux \sqsubseteq vy \\ &\iff \forall a : A.\, a \in \eta^{-1}u \to \exists b : B.\, b \in \eta^{-1}v \\ &\iff (\phi \sqsubseteq_{\Sigma} \psi) \wedge \forall x : \phi, y : \psi.\, ux \sqsubseteq vy\end{aligned}$$

It is not difficult to show that if A is directed-complete, then so is LA; suprema are computed so that the (clearly monotone) projection $\pi_1 \colon LA \to \Sigma$ is Scott-continuous and so a morphism of dcpos.

Theorem 35. *Each lifted dcpo LB is the partial product of* $\top \colon \mathbf{1} \hookrightarrow \Sigma$ *with B.*

Proof. We must construct an isomorphism of posets from $\mathsf{dcpo}(A, LB)$ to the poset $\mathsf{dcpo}^{\top}(A, B)$ of partial maps from A to B. Given $f : A \to LB$, we choose the following partial map from A to B:

$$\begin{tikzcd}
B & \{x : A \mid \pi(fx) = \top\} \arrow[l, "\pi_2 \circ f"'] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, hook, "\top"] \\
& A \arrow[r, "\pi \circ f"'] & \Sigma
\end{tikzcd} \tag{31}$$

Monotonicity is immediate. Conversely, we consider an arbitrary partial map:

$$\begin{tikzcd}
B & U \arrow[l, "e"'] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, hook, "\top"] \\
& A \arrow[r, "p"'] & \Sigma
\end{tikzcd} \tag{32}$$

The above corresponds to the map $A \to LB$ sending $x : A$ to $(px, \lambda z.\, (x, z))$. ◻

Corollary 36. *Let A be a dcpo; then the evaluation map $e : U \to A$ in the universal nondeterministic map with coefficients in $\top : \mathbf{1} \hookrightarrow \Sigma$ is an isomorphism.*

$$\begin{tikzcd}
A & U \arrow[l, "e"'] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, hook, "\top"] \\
& \mathcal{P}_{\bullet}(\top, A) \arrow[r, "\pi"'] & \Sigma
\end{tikzcd} \tag{33}$$

4.2 Geometry of the partial map classifier

In Sections 3.1 and 3.2 we have seen that the classifier of Scott–open subsets has an additional left-handed universal property as a 2-categorical colimit: the Sierpiński space. In this section, we will upgrade this result to see that the partial map classifier of a given dcpo A has an additional left-handed universal property as the Sierpiński *cone* of A. From this, we will obtain the most important reasoning principle for the lifting doctrine in constructive domain theory, namely our Corollaries 38 and 39.

Theorem 37 (Lifting = Sierpiński cone)**.** *For any dcpo A, the following lax square involving the lifting operation is a co-comma square:*

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "!_A"'] \arrow[dr, phantom, "\Nearrow"] & A \arrow[d, hook, "\eta_A"] \\
\mathbf{1} \arrow[r, hook, "\perp"'] & LA
\end{tikzcd}
\tag{34}
$$

In other words, the lifted dcpo LA is in fact the Sierpiński cone *of A in* dcpo.

Proof. Consider an arbitrary lax square in the following configuration:

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & A \arrow[d, "c_1"] \\
\mathbf{1} \arrow[r, "c_0"'] & C
\end{tikzcd}
\tag{35}
$$

The universal map $h\colon LA \to C$ factoring c_0 through $\perp$ and c_1 through η_A is defined so as to send $u : LA$ to the supremum of the union of $\{c_0\}$ with $\{c_1 x \mid u = \eta_A x\}$. This set is evidently directed, and so each hu is well-defined; to see that the assignment $u \mapsto hu$ is continuous, we fix a directed subset $V \subseteq LA$:

$$
\begin{aligned}
h \bigsqcup V &= \bigsqcup\left(\{c_0\} \cup \left\{c_1 x \;\middle|\; \bigsqcup V = \eta_A x\right\}\right) \\
&= \bigsqcup(\{c_0\} \cup \{c_1 x \mid \eta_A x \in V\}) \\
&= \bigsqcup_{u \in V}(\{c_0\} \cup \{c_1 x \mid u = \eta_A x\}) \\
&= \bigsqcup_{u \in V} hu
\end{aligned}
$$

Lastly, we must check that $h\colon LA \to C$ is unique with this property. We will show that any two $h, h'\colon LA \to C$ factoring our lax square in the appropriate sense are equal, fixing $u : LA$.

$$
\begin{aligned}
hu &= h \bigsqcup(\{\perp\} \cup \{\eta_A x \mid u = \eta_A x\}) \\
&= \bigsqcup(\{h\perp\} \cup \{h(\eta_A x) \mid u = \eta_A x\}) \\
&= \bigsqcup(\{h'\perp\} \cup \{h'(\eta_A x) \mid u = \eta_A x\}) \\
&= h' \bigsqcup(\{\perp\} \cup \{\eta_A x \mid u = \eta_A x\}) \\
&= h'u
\end{aligned}
$$

□

From the universal property of LA as the Sierpiński cone of A, we can deduce the following important reasoning principle.

Corollary 38. *For any dcpo A, the two embeddings $\perp\colon \mathbf{1} \hookrightarrow LA$ and $\eta_A\colon A \hookrightarrow LA$ are jointly epimorphic; as such, we have an epimorphic embedding $[\perp \mid \eta_A]\colon \mathbf{1} + A \hookrightarrow\!\!\!\!\!\rightarrow LA$.*

Proof. This is an immediate consequence of Theorem 37: because LA is the Sierpiński cone of A, equality of maps $LA \to C$ can be checked by restriction along the embeddings $\perp\colon \mathbf{1} \hookrightarrow LA$ and $\eta_A\colon A \hookrightarrow LA$. □

It was Fiore [1995] who first argued for the importance of Corollary 38 for the general axiomatics of lifting monads as Kock–Zöberlein doctrines, *i.e. lax idempotent 2-monads*. In this paper, we consider a stronger *enriched* version of this statement.

Corollary 39. *For any dcpo A, the embedding* $[\perp \mid \eta_A]\colon \mathbf{1} + A \hookrightarrow\!\!\!\!\rightarrow LA$ *is* lax epimorphic *in the 2-category of dcpos, so that for any dcpo C the induced restriction map* $\mathsf{dcpo}([\perp \mid \eta_A], C)\colon \mathsf{dcpo}(LA, C) \to \mathsf{dcpo}(\mathbf{1} + A, C)$ *is an* order-embedding.

Proof. This is a consequence of Corollaries 30 and 38. □

4.3 Lifting as a 2-monad

It is not difficult to see that the lifting operation on dcpos is functorial and, indeed, a monad; on point-sets, these operations are the same as those of the (discrete) partial map classifier on sets — as the functorial action sends continuous maps to continuous maps, and both the unit and multiplication can be seen to be continuous. Moreover, the functorial action is in fact *monotone* in hom posets. Therefore:

Lemma 40 (Enrichment)**.** *Lifting gives rise to a 2-monad* $\mathbb{L} = (L, \eta, \mu)$ *on* dcpo.

Proof. This amounts to the fact that each functorial map taking $f\colon A \to B$ to $Lf\colon LA \to LB$ is *monotone* as a function on hom posets. That the unit and multiplication are 2-natural is automatic in the locally posetal setting. □

Essentially by definition, the Kleisli 2-category for $\mathbb{L}$ is given by dcpos with *partial* maps between them. The rest of this section is devoted to understanding the broader Eilenberg–Moore resolution of $\mathbb{L}$, which extends beyond the free lifting algebras to arbitrary lifting algebras. We will show in Section 4.7 that lifting algebras, pointed dcpos, and inductive partial orders give equivalent 2-categories; in Section 4.8, we will show that the category of lifting algebras is cocomplete.

Definition 41. We shall emphasise the property of dcpo morphism $f\colon UX \to YU$ tracking a morphism of $\mathbb{L}$-algebras by calling it ***linear***.

The following can be seen by unfolding definitions.

Observation 42. *Each unit map* $\eta_A\colon A \to LA$ *is an order-embedding.*

4.4 Lifting as a Kock–Zöberlein doctrine

The lifting 2-monad is *lax idempotent* and so gives rise to a Kock–Zöberlein doctrine on dcpos. We will see this doctrine takes the form of cocompletion under bottom elements, constructivising the classical viewpoint of dcpo lifting algebras as *pointed* dcpos.

Lemma 43. *The lifting 2-monad is lax idempotent: for any algebra* $X \in \mathsf{dcpo}^{\mathbb{L}}$*, the structure map* $\alpha_X\colon LUX \to UX$ *is left adjoint to the unit* $\eta_{UX}\colon UX \to LUX$ *in* dcpo.

Proof. The counit $\alpha_X \circ \eta_{UX} \sqsubseteq 1_{UX}$ is automatic (and invertible) by the unit law for monad algebras. To exhibit the unit $1_{LUX} \sqsubseteq \eta_{UX} \circ \alpha_X$, it suffices by Corollary 39 to check both $\bot \sqsubseteq \eta_{UX}\alpha_X\bot$ and $\eta_{UX} \sqsubseteq \eta_{UX}\alpha_X\eta_{UX}$. The former is immediate and the latter holds by the unit law for monad algebras. □

Corollary 44. *There is at most one lifting algebra structure on a dcpo.*

Proof. Left adjoints are unique! □

4.5 Pointed dcpos *vs*. ipos

Lemma 45. *A dcpo A is pointed if and only if it is inductive, i.e. semidirected-complete.*

Proof. Suppose that A is closed under suprema of semidirected subsets. Then the supremum of the *empty* subset (which is trivially semidirected) is can be seen to be the bottom element using the universal property of suprema.

Conversely, suppose that A is pointed and let $I \subseteq A$ be semidirected. Then we may replace $I \subseteq A$ by the *directed* subset $I' = I \cup \{\bot\}$; the inclusion $I \subseteq I'$ is clearly cofinal as $\bot$ lies below everything, so the supremum of I' is also the supremum of I. □

Lemma 46. *A Scott–continuous morphism between pointed dcpos is strict if and only if it is inductive, i.e. preserves suprema of semidirected subsets.*

Proof. An inductive morphism obviously preserves the bottom element. Conversely, let $f\colon A \to B$ preserve directed suprema and the bottom element and let $I \subseteq A$ be a semidirected subset of A. To show that $f \bigsqcup I = \bigsqcup_{i:I} fi$, we note that $I \subseteq I \cup \{\bot\}$ is a cofinal inclusion onto a *directed* subset, and so $f \bigsqcup I = f \bigsqcup(\{\bot\} \cup I) = \bigsqcup_{\mathbf{1}+I}[f\bot \mid f] = \bigsqcup_{\mathbf{1}+I}[\bot \mid f] = \bigsqcup_{i:I} fi$. □

4.6 Pointed dcpos *vs*. lifting algebras

Lemma 47 (Pointed dcpos are lifting algebras)**.** *Any pointed dcpo carries a lifting algeba structure.*

Of course, by Corollary 44 any lifting algebra structure we impose on a dcpo, pointed or not, is unique.

Proof. Let A be a pointed dcpo; we define the structure map $\alpha_A\colon LA \to A$ to take $u : LA$ to the supremum of the semidirected subset $\{x : A \mid u = \eta_A x\}$, computed via Lemma 45. The unit law is trivial, and the multiplication law follows from the fact that a supremum of suprema can be computed as the supremum of a single subset. □

Lemma 48 (Lifting algebras are pointed)**.** *For any lifting algebra $X \in \mathsf{dcpo}^{\mathbb{L}}$, the underlying dcpo UX is pointed.*

Proof. The bottom element of UX is obtained by applying the structure map to the bottom element of LUX, so we have $\bot :\equiv \alpha_X(\bot, *)$. That this does in fact compute the bottom element can be seen as follows: fixing $u : UX$, we note that $\alpha_X(\bot, *) \sqsubseteq_{UX} u$ is equivalent to $\bot \sqsubseteq_{LUX} \eta_{UX}u$ because $\alpha_X \dashv \eta_{UX}$ by Lemma 43 (lax idempotence). □

Lemma 49 (Strict maps *vs*. algebra homomorphisms)**.** *A Scott–continuous map between pointed dcpos is strict if and only if it tracks a lifting algebra homomorphism.*

Proof. It is clear from the proof of Lemma 48 that a homomorphism of algebras must preserve the bottom element. On the other hand, we suppose that $f : A \to B$ is strict to check that the following diagram commutes:

$$\begin{tikzcd} LA \arrow[r, "Lf"] \arrow[d, "\alpha_A"'] & LB \arrow[d, "\alpha_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{36}$$

By Corollary 38 and the fact that all maps in sight are strict, it is enough to consider the restriction of Diagram 36 along $\eta_A : A \hookrightarrow LA$; then we have $\alpha_B \circ Lf \circ \eta_A = \alpha_B \circ \eta_A \circ f = f = f \circ \alpha_A \circ \eta_A$ by the unit law for algebras. □

4.7 Characterisation of the category of lifting algebras

Corollary 50. *The 2-categories of lifting algebras, pointed dcpos, and inductive partial orders are all canonically equivalent.*

Proof. Having and preserving bottom elements, semidirected suprema, and lifting algebra structures are all *properties* (we have seen the latter in Corollary 44). Therefore, we may argue that these categories all arise as the same (non-full) subcategory of dcpo via Lemmas 45 to 49. □

Corollary 51 (Monadicity)**.** *The forgetful functors* $\mathsf{dcppo} \to \mathsf{dcpo}$ *and* $\mathsf{ipo} \to \mathsf{dcpo}$ *are both monadic.*

4.8 Cocompleteness of lift-algebras

Lemma 52. *The lifting endofunctor* $L : \mathsf{dcpo} \to \mathsf{dcpo}$ *preserves connected colimits.*

Proof. Let $A_\bullet : \mathscr{I} \to \mathsf{dcpo}$ be a connected diagram, *i.e.* such that $\mathscr{I}$ is inhabited and has a finite zigzag between any two objects; further suppose that there exists a universal cocone $a_\bullet : A_\bullet \to \{A_\infty\}$, to check that the lifted cocone $La_\bullet : LA_\bullet \to \{LA_\infty\}$ is also universal. We fix a cocone $b_\bullet : LA_\bullet \to \{B\}$ and must check that there exists a unique map $b_\infty : LA_\infty \to B$ factoring $b_\bullet$ through $La_\bullet$. We have shown in Theorem 37 that LA_∞ is the Sierpiński cone of A_∞, so a map $b_\infty : LA_\infty \to B$ is uniquely determined by an element $b_\infty^\perp : \mathbf{1} \to B$ and a map $b_\infty^\top : A_\infty \to B$ such that $b_\infty^\perp \circ !_{A_\infty} \sqsubseteq b_\infty^\top$.

We first define $b_\infty^\perp$ to be the unique element of B that is equal to $b_k \perp$ for all $k \in \mathscr{I}$; that this element is exists and is unique follows from connectedness of $\mathscr{I}$. Next, we define $b_\infty^\top\colon A_\infty \to B$ using the universal property of $a_\bullet\colon A_\bullet \to \{A_\infty\}$:

$$\begin{tikzcd} A_\bullet \arrow[r, "a_\bullet"] \arrow[d, hook, "\eta_{A_\bullet}"'] & \{A_\infty\} \arrow[d, dashed, "\{b_\infty^\top\}"] \\ LA_\bullet \arrow[r, "b_\bullet"'] & \{B\} \end{tikzcd} \tag{37}$$

Finally we check that $b_\infty^\perp \circ !_{A_\infty} \sqsubseteq b_\infty^\top$; by Corollary 30, it suffices to check that $b_\infty^\perp \circ !_{A_i} \sqsubseteq b_\infty^\top \circ a_i$ for each $i \in \mathcal{I}$; fixing $x : A_i$, we do indeed have $b_\infty^\perp = b_i\perp \sqsubseteq b_i(\eta_{A_i}x) = b_\infty^\top(a_i x)$ by monotonicity of $b_i\colon LA_i \to B$ on $\perp \sqsubseteq \eta_{A_i}x$.

Thus we have defined a map $b_\infty\colon LA_\infty \to B$ such that $b_\infty\perp = b_k\perp$ for all $k \in \mathcal{I}$ and $b_\infty(\eta_{A_\infty}x) = b_\infty^\top x$ for all $x : A_\infty$. We need to check that $b_\infty\colon LA_\infty \to B$ uniquely factors $b_\bullet\colon LA_\bullet \to \{B\}$ through $LA_\bullet\colon LA_\bullet \to \{LA_\infty\}$:

$$\begin{tikzcd} LA_\bullet \arrow[r, "La_\bullet"] \arrow[dr, "b_\bullet"'] & \{LA_\infty\} \arrow[d, "b_\infty"] \\ & \{B\} \end{tikzcd} \tag{38}$$

We check the factorization above using Corollary 38. In particular, it is enough to check that $b_\infty(La_i(\perp)) = b_i\perp$ and that $b_\infty(La_i(\eta_{LA_i}x)) = b_i(\eta_{A_i}x)$ for each $x : A_i$. The former holds as we have $b_\infty(La_i(\perp)) = b_\infty\perp = b_\infty^\perp = b_i\perp$, and the latter holds by $b_\infty(La_i(\eta_{LA_i}x)) = b_\infty(\eta_{LA_\infty}(a_i x)) = \beta_\infty^\top(a_i x) = b_i(\eta_{A_i}x)$. Finally, we check that any two factorizations $f, g\colon LA_\infty \to B$ of $b_\bullet$ through $LA_\bullet$ are equal. But this follows by construction via Corollary 38 and the universal property of the cocone $a_\bullet\colon A_\bullet \to \{A_\infty\}$. □

Corollary 53. *The category of lift-algebras is closed under connected colimits, and these are created by the forgetful functor $U\colon \mathsf{dcpo}^{\mathbb{L}} \to \mathsf{dcpo}$.*

Proof. By Lemmas 7, 31 and 52. □

Lemma 54 (Linton [1969])**.** *The category of lift-algebras is closed under coproducts.*

Proof. Coproducts in $\mathsf{dcpo}^{\mathbb{L}}$ are computed using a reflexive coequaliser involving the coproducts from dcpo. By Corollary 53, we know that $\mathsf{dcpo}^{\mathbb{L}}$ is closed under reflexive coequalisers and these are computed as in dcpo. □

Corollary 55. *The category of lift-algebras is cocomplete.*

Proof. By Corollary 53 and Lemma 54. □

5 Tensorial structure of the lifting adjunction

5.1 Enrichment and commutativity of the lifting monad

We shall view dcpo as a symmetric monoidal closed category via its cartesian product and exponential, canonically self-enriched. We first observe that $\mathsf{dcpo}^{\mathbb{L}} = \mathsf{dcppo} = \mathsf{ipo}$ inherits this dcpo-enrichment.

Lemma 56. *The category* dcppo *of pointed dcpos is* dcpo*-enriched in the sense that every hom poset* $\mathsf{dcppo}(A, B)$ *is closed under suprema of directed subsets.*

Proof. Given pointed dcpos A and B, we must check that the supremum of a directed set of strict maps from A to B, computed in the dcpo exponential B^A, is strict. This can be seen immediately from the fact that function application is continuous in its first argument:

$$\left(\bigsqcup_{i:I} f_i\right)\bot = \bigsqcup_{i:I} f_i\bot = \bigsqcup_{i:I} \bot = \bot$$

□

Lemma 57. *The category* dcppo *of pointed dcpos is closed under* dcpo-powers.

Proof. Let A be a dcpo and let B be a pointed dcpo. The *power* $A \pitchfork B$ of B by A has the dcpo exponential B^A as its underlying (pointed) dcpo. To check the universal property, we observe that a strict map from C to $A \pitchfork B$ is the same as a map from $C \times A$ to B that is strict in its first argument. Of course, this is the same as a Scott–continuous map from A to $\mathsf{dcppo}(C, B)$. Thus we have $\mathsf{dcppo}(C, A \pitchfork B) \cong \mathsf{dcpo}(A, \mathsf{dcppo}(C, B))$ and so we are done. □

Lemma 58. *The poset-enrichment of the lifting monad* $\mathbb{L}$ *on* dcpo *extends to a* dcpo*-enrichment.*

Proof. The functorial action and monadic operations can all be internalised as Scott–continuous operations □

Corollary 59. *The lifting monad* $\mathbb{L}$ *extends to a* strong *monad on* dcpo.

Proof. Strengths for a given monad on a cartesian closed category $\mathcal{V}$ correspond precisely to $\mathcal{V}$-enrichments of the monad [McDermott and Uustalu, 2022]. □

Lemma 60. *The* dcpo*-enriched lifting monad* $\mathbb{L}$ *is commutative.*

Proof. We use Kock's criterion for commutativity of a strong monad on a closed category. Fixing a pointed dcpo B and a dcpo A, we must check that the extension map $(-)^\dagger \colon A \pitchfork B \to LA \pitchfork B$ is *strict*. As the bottom element of any power $I \pitchfork B$ is pointwise, we are trying to check that $(\lambda x.\, \bot)^\dagger u = \bot$ for any $u : LA$. By Corollary 38, it suffices to observe that $(\lambda x.\, \bot)^\dagger \bot = \bot$ and $(\lambda x.\, \bot)^\dagger(\eta_A a) = (\lambda x.\, \bot)(a) = \bot$. □

Corollary 61. *The lifting monad* $\mathbb{L}$ *is symmetric monoidal.*

Proof. This is in fact equivalent to being commutative. □

Construction 62 (Commutator)**.** The commutator $\kappa_{A,B} \colon LA \times LB \to L(A \times B)$ is given by iterated (internal) Kleisli extension; the commutativity property ensures that it doesn't matter in which order these extensions are taken.

5.2 Smash products and the universal bistrict morphism

Lemma 63. *The following are equivalent for a morphism of dcpos $f \colon A \times B \to C$ where A and B are pointed:*

1. *Any of the following diagrams commute:*

$$A + B \underset{[(1_A, \bot) \mid (\bot, 1_B)]}{\overset{\bot \circ !_{A+B}}{\rightrightarrows}} A \times B \xrightarrow{f} C \tag{39}$$

$$L(A + B) \underset{[(1_A, \bot) \mid (\bot, 1_B)]^\dagger}{\overset{\bot \circ !_{L(A+B)}}{\rightrightarrows}} A \times B \xrightarrow{f} C \tag{40}$$

2. *For any $a : A$ and $b : B$ we have $f(\bot, b) = f(a, \bot)$.*

Proof. The last condition is immediately equivalent to Diagram 39 commuting. The equivalence between Diagrams 39 and 40 is deduced from Corollary 38, noting that the parallel maps in Diagram 40 are both strict. □

Lemma 64. *The following are equivalent for a (not necessarily strict) morphism $f \colon A \times B \to C$ of dcpos where A, B, and C are pointed:*

1. *Any of the equivalent conditions of Lemma 63.*

2. *Either of the following diagrams commute:*

$$A + B \underset{\eta_{A \times B} \circ [(1_A, \bot) \mid (\bot, 1_B)]}{\overset{\bot \circ !_{A+B}}{\rightrightarrows}} L(A \times B) \xrightarrow{f^\dagger} C \tag{41}$$

$$L(A + B) \underset{L[(1_A, \bot) \mid (\bot, 1_B)]}{\overset{\bot \circ !_{L(A+B)}}{\rightrightarrows}} L(A \times B) \xrightarrow{f^\dagger} C \tag{42}$$

Proof. Diagram 39 commutes if and only if Diagram 41 commutes, by the unit law for C as a lifting algebra; for the same reason, Diagram 40 commutes if and only if Diagram 42 commutes. □

Definition 65 (Bistrict morphism)**.** Let A, B, and C be pointed dcpos. A Scott–continuous morphism $f \colon A \times B \to C$ is called ***bistrict*** when any of the following equivalent conditions hold:

1. The morphism $f \colon A \times B \to C$ is strict and satisfies any of the equivalent conditions of Lemmas 63 and 64.

2. For any $a : A$ and $b : B$ we have $f(\bot, b) = f(a, \bot) = \bot$.

Theorem 66 (The universal bistrict map)**.** *For any pointed dcpos A and B, we may define a pointed $A \otimes B$ equipped with a* universal *bistrict map $\otimes_{A,B} \colon A \times B \to A \otimes B$, in the sense that any bistrict $f \colon A \times B \to C$ factors uniquely through it by a unique strict map $\bar{f} \colon A \otimes B \to C$ as depicted below:*

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[d, "\otimes_{A,B}"'] & C \\ A \otimes B \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd} \tag{43}$$

Moreover, the following diagram is a coequaliser in dcppo*:*

$$\begin{tikzcd} L(A+B) \arrow[r, shift left, "\perp \circ !_{L(A+B)}"] \arrow[r, shift right, "{[(1_A, \perp) \mid (\perp, 1_B)]^\dagger}"'] & A \times B \arrow[r, two heads, "\otimes_{A,B}"] & A \otimes B \end{tikzcd} \tag{44}$$

Proof. We may compute desired coequaliser, as we have already shown in Corollary 55 that $\mathsf{dcpo}^{\mathbb{L}} = \mathsf{dcppo}$ is cocomplete. The coequaliser map $\otimes_{A,B} \colon A \times B \twoheadrightarrow A \otimes B$ is bistrict by definition, as Diagram 44 is an instance of Diagram 40 from Lemma 63. The unique factorisation condition of Diagram 43 is, then, precisely the universal property of Diagram 44 as a coequaliser in dcppo. □

Corollary 67. *The following are coequaliser diagrams in both* dcppo *and* dcpo*:*

$$\begin{tikzcd} L(A+B) \arrow[r, shift left, "\perp \circ !_{L(A+B)}"] \arrow[r, shift right, "{[(1_A, \perp) \mid (\perp, 1_B)]^\dagger}"'] & A \times B \arrow[r, two heads, "\otimes_{A,B}"] & A \otimes B \end{tikzcd} \tag{44}$$

$$\begin{tikzcd} L(A+B) \arrow[r, shift left, "\perp \circ !_{L(A+B)}"] \arrow[r, shift right, "{L[(1_A, \perp) \mid (\perp, 1_B)]}"'] & L(A \times B) \arrow[r, two heads, "\otimes^\dagger_{A,B}"] & A \otimes B \end{tikzcd} \tag{45}$$

The following are coequaliser diagrams in dcpo*:*

$$\begin{tikzcd} A+B \arrow[r, shift left, "\perp \circ !_{A+B}"] \arrow[r, shift right, "{[(1_A, \perp) \mid (\perp, 1_B)]}"'] & A \times B \arrow[r, two heads, "\otimes_{A,B}"] & A \otimes B \end{tikzcd} \tag{46}$$

$$\begin{tikzcd} A+B \arrow[r, shift left, "\perp \circ !_{A+B}"] \arrow[r, shift right, "{[(1_A, \perp) \mid (\perp, 1_B)]}"'] & L(A \times B) \arrow[r, two heads, "\otimes^\dagger_{A,B}"] & A \otimes B \end{tikzcd} \tag{47}$$

Proof. We have seen in Corollary 53 that the forgetful functor $U \colon \mathsf{dcppo} \to \mathsf{dcpo}$ creates connected colimits; therefore, a coequaliser diagram dcppo is equally well a coequaliser diagram in dcpo. Diagram 44 is therefore a coequaliser in both categories by Theorem 66. That Diagrams 45 to 47 are all coequalisers follows from Lemma 64. □

Lemma 68. *Up to isomorphism, the lifting monad sends any cartesian product $A \times B$ to the smash product $LA \times LB$. In particular, the commutator $\kappa_{A,B} \colon LA \times LB \to L(A \times B)$ is the universal bistrict map in the sense of Theorem 66.*

Proof. It suffices to show that any bistrict map $f \colon LA \times LB \to C$ extends uniquely along $\kappa_{A,B} \colon LA \times LB \to L(A, B)$. We let $\bar{f} \colon L(A \times B) \to C$ be the extension of $f \circ \eta_A \times \eta_B \colon A \times B \to C$, which is automatically strict. Uniqueness of the extension is deduced using Corollary 38. ◻

5.3 Bilinear morphisms and Seal's general theory

Although we have developed smash products and their universal property with respect to bistrict morphisms in the concrete, Seal [2013] has provided a general theory for deducing tensorial structure from commutative monads. In this section, we show that *op. cit*.'s notion of *bilinear map* coincides with our bistrict maps and, moreover, that the tensor products of *op. cit*. satisfy the same universal property as our smash product.

Definition 69 (Bilinear morphism, Seal [2013])**.** Let A, B, and C be pointed dcpos. A Scott–continuous morphism $f \colon A \times B \to C$ is called ***bilinear*** when the following diagram commutes:

$$\begin{tikzcd} LA \times LB \arrow[r, "\kappa_{A,B}"] \arrow[d, "\alpha_A \times \alpha_B"'] & L(A \times B) \arrow[d, "f^\dagger"] \\ A \times B \arrow[r, "f"'] & C \end{tikzcd} \tag{48}$$

Lemma 70. *A morphism $f \colon A \times B \to C$ is bistrict if and only if it is bilinear.*

Proof. A bilinear map is clearly bistrict. Conversely, assume that $f \colon A \times B \to C$ is bistrict. By Corollary 38, both of the embeddings $[\bot \mid \eta_A] \colon \mathbf{1} + A \hookrightarrow\!\!\!\!\rightarrow LA$ and $[\bot \mid \eta_B] \colon \mathbf{1} + B \hookrightarrow\!\!\!\!\rightarrow LB$ are epimorphic, and therefore so is their cartesian product. Therefore, it suffices to consider the restriction of Diagram 48 from Definition 69 along $[\bot \mid \eta_A] \times [\bot \mid \eta_B] \colon (\mathbf{1} + A) \times (\mathbf{1} + B) \hookrightarrow\!\!\!\!\rightarrow LA \times LB$, or, equivalently, along each of the following four embeddings:

$$(\bot, \bot) \colon \mathbf{1} \hookrightarrow LA \times LB \tag{49}$$
$$(\eta_A, \bot) \colon A \hookrightarrow LA \times LB \tag{50}$$
$$(\bot, \eta_B) \colon B \hookrightarrow LA \times LB \tag{51}$$
$$(\eta_A, \eta_B) \colon A \times B \hookrightarrow LA \times LB \tag{52}$$

From this reduction, it is easily seen that bistrictness implies bilinearity. ◻

Now we recall Seal's construction of the tensor product.

Definition 71 (Seal [2013, §2.2]). The ***tensor product*** $A \boxtimes B$ of two pointed dcpos A and B is given by the following coequalier in dcppo, which exists by virtue of Corollaries 50 and 55:

$$L(LA \times LB) \underset{L(\alpha_A \times \alpha_B)}{\overset{\kappa^\dagger_{A,B}}{\rightrightarrows}} L(A \times B) \overset{q_{A,B}}{\dashrightarrow\!\!\!\!\rightarrow} A \boxtimes B \tag{53}$$

Seal [2013] proves a universal property for the tensor product with respect to bilinear morphisms.

Theorem 72 (Seal [2013]). *The tensor product $A \boxtimes B$ represents bilinear maps in the sense that for any bilinear morphism $f\colon A \times B \to C$ there exists a unique linear morphism $\bar{f}\colon A \boxtimes B \to C$ making the following triangle:*

$$\begin{tikzcd} L(A \times B) \arrow[r, "f^\dagger"] \arrow[d, "q_{A,B}"'] & C \\ A \boxtimes B \arrow[ur, dashed, "\exists!\bar{f}"'] & \end{tikzcd} \tag{54}$$

Moreover, let $\boxtimes_{A,B}\colon A \times B \to A \boxtimes B$ be the composite $A \times B \xrightarrow{\eta_{A\times B}} L(A \times B) \xrightarrow{q_{A,B}} A \boxtimes B$. Then for any linear morphism $h\colon A \boxtimes B \to C$, the restriction $h \circ \boxtimes_{A,B}\colon A \times B \to C$ is bilinear and induces h in the sense that $\overline{h \circ \boxtimes_{A,B}} = h$.

Proof. This follows from Seal [2013, Lemma 2.3.3] via Corollary 59. □

In order to show that Seal's tensor product satisfies the same universal property as our smash product, we must deduce a slight reformulation of Theorem 72.

Lemma 73 (Universal bilinear map). *The composite $\boxtimes_{A,B} = q_{A,B} \circ \eta_{A\times B}\colon A \times B \to A \otimes B$ is the **universal bilinear map** in the sense that any bilinear map $f\colon A \times B \to C$ factors uniquely through it in* dcppo *as depicted below:*

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[d, "\boxtimes_{A,B}"'] & C \\ A \boxtimes B \arrow[ur, dashed, "\exists!\bar{f}"'] & \end{tikzcd} \tag{55}$$

Proof. First of all, $\boxtimes_{A,B}\colon A \times B \to A \otimes B$ is indeed bilinear by the second part of Theorem 72. That any bilinear map $f\colon A \times B \to C$ factors uniquely through it follows from the first part of Theorem 72 via Corollary 38. Indeed, we first let $\bar{f}\colon A \boxtimes B \to C$

be the map determined by Theorem 72 as follows:

$$
\begin{tikzcd}
L(A \times B) \arrow[r, "f^\dagger"] \arrow[d, "q_{A,B}"'] & C \\
A \boxtimes B \arrow[ur, dashed, "\bar{f}"']
\end{tikzcd}
\tag{56}
$$

By Corollary 38, the diagram above commutes if and only if its restrictions along $\bot \colon \mathbf{1} \to L(A \times B)$ and $\eta_{A \times B} \colon A \times B$ commute. The former is automatic because all maps in sight are strict; the latter is precisely the property of $\bar{f}$ extending f along $\boxtimes_{A,B}$. □

Corollary 74. *There exists a unique bilinear / bistrict isomorphism $A \boxtimes B \to A \otimes B$ from Seal's tensor product to our smash product factoring the universal bistrict map through the universal bilinear map, and vice versa:*

$$
\begin{tikzcd}
A \times B \arrow[r, "\otimes_{A,B}"] \arrow[d, "\boxtimes_{A,B}"'] & A \otimes B \\
A \boxtimes B \arrow[ur, dashed, "\overline{\otimes_{A,B}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "\boxtimes_{A,B}"] \arrow[d, "\otimes_{A,B}"'] & A \boxtimes B \\
A \otimes B \arrow[ur, dashed, "\overline{\boxtimes_{A,B}}"']
\end{tikzcd}
\tag{57}
$$

Proof. This is an immediate consequence of the fact that bilinear and bistrict map coincide (Lemma 70). □

5.4 Symmetric monoidal structure of the smash product

The smash product of pointed dcpos from Section 5.2 extends to a full symmetric monoidal structure on $\mathsf{dcpo}^{\mathbb{L}} = \mathsf{dcppo} = \mathsf{ipo}$ with identity $I = L\mathbf{1}$; this result can be taken off the shelf from Seal [2013, Theorem 2.5.5], in combination with our own result $A \otimes B = A \boxtimes B$ from Corollary 74.

5.5 Symmetric monoidal structure of the lifting adjunction

Seal [2013] shows that under assumptions that we have established in this paper for the lifting monad $\mathbb{L}$ and its category of algebras $\mathsf{dcpo}^{\mathbb{L}} = \mathsf{dcppo} = \mathsf{ipo}$, the Eilenberg–Moore adjunction $L \dashv U \colon \mathsf{dcppo} \to \mathsf{dcpo}$ is *monoidal*: the left adjoint is *strong monoidal* (*cf*. our own Lemma 68) and the right adjoint is *lax monoidal*.

In this section, we extend the result of *op. cit*. in our specific case to show that $L \dashv U \colon \mathsf{dcppo} \to \mathsf{dcpo}$ is *symmetric* monoidal. We first recall the braiding $\beta^{\otimes}_{A,B} \colon A \otimes B \to B \otimes A$ of the smash product in dcppo in terms of the braiding of the Cartesian product on dcpo:

$$\begin{tikzcd}
A \times B \arrow[r, "\beta^{\times}_{A,B}"] \arrow[d, "\otimes_{A,B}"'] & B \times A \arrow[r, "\otimes_{B,A}"] & B \otimes A \\
A \otimes B \arrow[urr, dashed, "\exists ! \beta^{\otimes}_{A,B}"'] & &
\end{tikzcd} \tag{58}$$

Lemma 75. *The functor $L\colon \mathsf{dcpo} \to \mathsf{dcppo}$ is symmetric monoidal in the sense that the following diagram commutes in dcppo for dcpos A, B, C:*

$$\begin{tikzcd}
LA \otimes LB \arrow[r, "\beta^{\otimes}_{A,B}"] \arrow[d, "\bar{\kappa}_{A,B}"'] & LB \otimes LA \arrow[d, "\bar{\kappa}_{B,A}"] \\
L(A \times B) \arrow[r, "L(\beta^{\times}_{A,B})"'] & L(B \times A)
\end{tikzcd} \tag{59}$$

Proof. To check that Diagram 59 commutes, it suffices to consider its restriction along the universal bistrict map $\otimes_{LA,LB}\colon LA \times LB \to LA \otimes LB$. Therefore, to check that the lower inner square commutes in Diagram 60 below, it suffices to check that the outer square commutes in the sense that $L(\beta^{\times}_{A,B}) \circ \kappa_{A,B} = \kappa_{B,A} \circ \beta^{\times}_{LA,LB}$:

$$\begin{tikzcd}
LA \times LB \arrow[r, "\beta^{\times}_{LA,LB}"] \arrow[d, "\otimes_{LA,LB}"] \arrow[dd, bend right, "\kappa_{A,B}"'] & LB \times LA \arrow[d, "\otimes_{LB,LA}"'] \arrow[dd, bend left, "\kappa_{B,A}"] \\
LA \otimes LB \arrow[r, "\beta^{\otimes}_{A,B}"] \arrow[d, "\bar{\kappa}_{A,B}"] \arrow[dr, phantom, "?"] & LB \otimes LA \arrow[d, "\bar{\kappa}_{B,A}"'] \\
L(A \times B) \arrow[r, "L(\beta^{\times}_{A,B})"'] & L(B \times A)
\end{tikzcd} \tag{60}$$

By Corollary 38 and the fact that all maps in sight are strict, it suffices to consider just three cases:

$$\begin{aligned}
L(\beta^{\times}_{A,B})(\kappa_{A,B}(\eta_A x, \perp)) &= L(\beta^{\times}_{A,B})\perp \\
&= \perp \\
&= \kappa_{B,A}(\perp, \eta_A x)
\end{aligned}$$

$$
\begin{aligned}
&= \kappa_{B,A}(\beta^{\times}_{LA,LB}(\eta_A x, \bot)) \\
L(\beta^{\times}_{A,B})(\kappa_{A,B}(\bot, \eta_B y)) &= L(\beta^{\times}_{A,B})\bot \\
&= \bot \\
&= \kappa_{B,A}(\eta_B y, \bot) \\
&= \kappa_{B,A}(\beta^{\times}_{LA,LB}(\bot, \eta_B y)) \\
L(\beta^{\times}_{A,B})(\kappa_{A,B}(\eta_A x, \eta_b y)) &= L(\beta^{\times}_{A,B})(\eta_{A\times B}(x, y)) \\
&= \eta_{B\times A}(\beta^{\times}_{A,B}(x, y)) \\
&= \eta_{B\times A}(y, x) \\
&= \kappa_{B,A}(\eta_B y, \eta_A x) \\
&= \kappa_{B,A}(\beta^{\times}_{LA,LB}(\eta_A x, \eta_B y)) \qquad \square
\end{aligned}
$$

Lemma 76. *The forgetful functor* $U \colon \mathsf{dcppo} \to \mathsf{dcpo}$ *is symmetric monoidal in the sense that the following diagram commutes in* dcpo *for pointed dcpos* A, B, C*:*

$$
\begin{tikzcd}
A \times B \arrow[r, "\beta^{\times}_{A,B}"] \arrow[d, "\otimes_{A,B}"'] & B \times A \arrow[d, "\otimes_{B,A}"] \\
A \otimes B \arrow[r, "\beta^{\otimes}_{A,B}"'] & B \otimes A
\end{tikzcd}
\tag{61}
$$

Proof. That Diagram 61 commutes is in fact the *defining* property of the braiding $\beta^{\otimes}_{A,B}$ as constructed in Diagram 58. $\square$

Corollary 77. *The adjunction* $L \dashv U \colon \mathsf{dcppo} \to \mathsf{dcpo}$ *is symmetric monoidal in the sense that* $L \colon \mathsf{dcpo} \to \mathsf{dcppo}$ *is* strong *symmetric monoidal and* $U \colon \mathsf{dcppo} \to \mathsf{dcpo}$ *is* lax *symmetric monoidal.*

Proof. By Lemmas 75 and 76 via Seal [2013, Remark 2.7.3]. $\square$

5.6 Closed structure of the lifting adjunction

Kock [1971] has provided a method to lift the closed structure of dcpo to $\mathsf{dcpo}^{\mathbb{L}}$ by means of an equaliser of dcpos. Of course, the forgetful functor $U \colon \mathsf{dcppo} \to \mathsf{dcpo}$ is monadic (Corollary 50) and so creates limits; therefore we can slightly reformulate the construction of *op. cit*. by computing an equaliser of pointed dcpos directly.

Definition 78. Let A and B be pointed dcpos. We define the ***linear function space*** $A \multimap B$ to be the following equaliser in dcppo, where $\sigma_{A,B} \colon B^A \to B^{LA}$ is the internal extension map induced by the strength of L and the algebra structure of B:

$$A \multimap B \rightarrowtail B^A \underset{\sigma_{A,B}}{\overset{B^{\alpha_A}}{\rightrightarrows}} B^{LA} \tag{62}$$

The results of Kock [1971] then imply that the adjunction $L \dashv U : \mathsf{dcppo} \to \mathsf{dcpo}$ is closed with respect to the linear function space.

Definition 79. Let A and B be pointed dcpos. We define the ***strict function space*** $A \Rightarrow_\perp B$ to be the following equaliser in dcppo:

$$A \Rightarrow_\perp B \rightarrowtail B^A \underset{\perp \circ !_{B^A}}{\overset{B^\perp}{\rightrightarrows}} B \tag{63}$$

Lemma 80. *The strict and linear function spaces coincide.*

Proof. We will show that for any strict map $f : C \to B^A$, we have $B^\perp \circ f = \perp \circ !_{B^A} \circ f$ if and only if $B^{\alpha_A} \circ f = \sigma_{A,B} \circ f$. Fixing $x : C$, we must check that $fx\perp = \perp$ if and only if $f\,x \circ \alpha_A = \sigma_{A,B} \circ fx$. These are equivalent by Corollary 38 and the unit laws for algebras. □

By virtue of Lemma 80, we will freely write $A \multimap B$ for both the linear and strict function spaces.

Lemma 81. *For any pointed dcpo A, we have an adjunction* $- \otimes A \dashv A \multimap -$ *on* dcppo.

Proof. Fix $\bar{f} : C \otimes A \to B$ for some bistrict $f : C \times A \to B$. By definition, the mate $f^\sharp : C \to B^A$ in $- \times A \dashv (-)^A$ is strict and moreover satisfies the defining property of Diagram 63, so we may factor $f^\sharp : C \to B^A$ through $A \multimap B \rightarrowtail B^A$ by some unique strict map $\bar{f}^\sharp : C \to A \multimap B$. It can be seen that this assignment is naturally bijective. □

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