Crossing Boundaries in Program Semantics

Jonathan Sterling • 9 February 2023 • Marie Skłodowska-Curie Postdoctoral Fellow @ Aarhus University

“Everyday human activities such as building a house on a hill by a stream, laying a network of telephone conduits, navigating the solar system, require **plans that can work**. Planning any such undertaking requires the development of *thinking about space*. Each development involves many steps of thought and many related geometrical constructions on spaces. Because of the necessary multistep nature of thinking about space, **uniquely mathematical measures** must be taken to make it reliable. Only explicit principles of thinking (logic) and explicit principles of space (geometry) can guarantee reliability.”

F. William Lawvere, may he rest in peace

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Different models of computation surface different *facets* of program execution that we wish to study.

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Each of these ***mathematical models*** represents a different *abstraction* of the behavior of programs on physical hardware.

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  - This research program has led to the solution of several open problems in dependent type theory.

prelude:

Phases, a new dimension of composition

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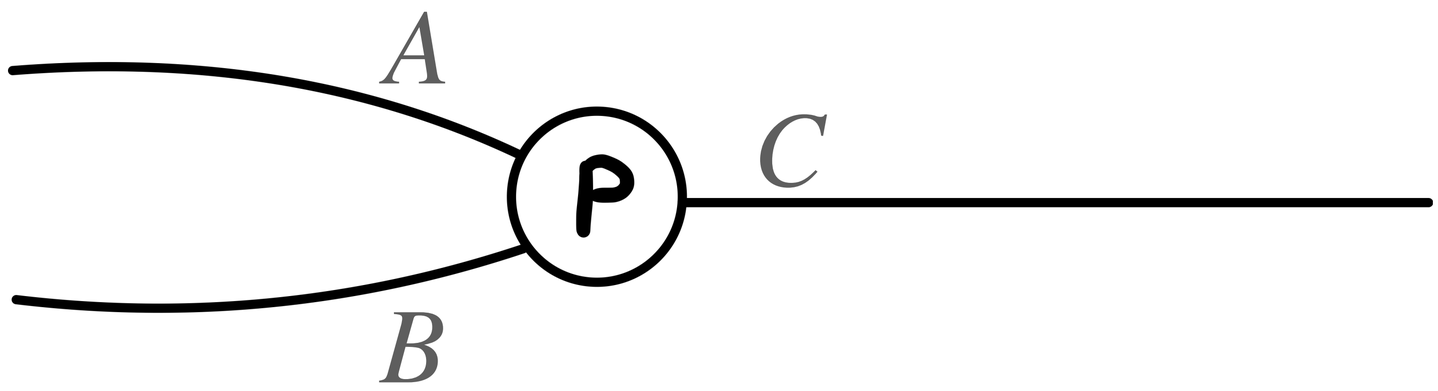

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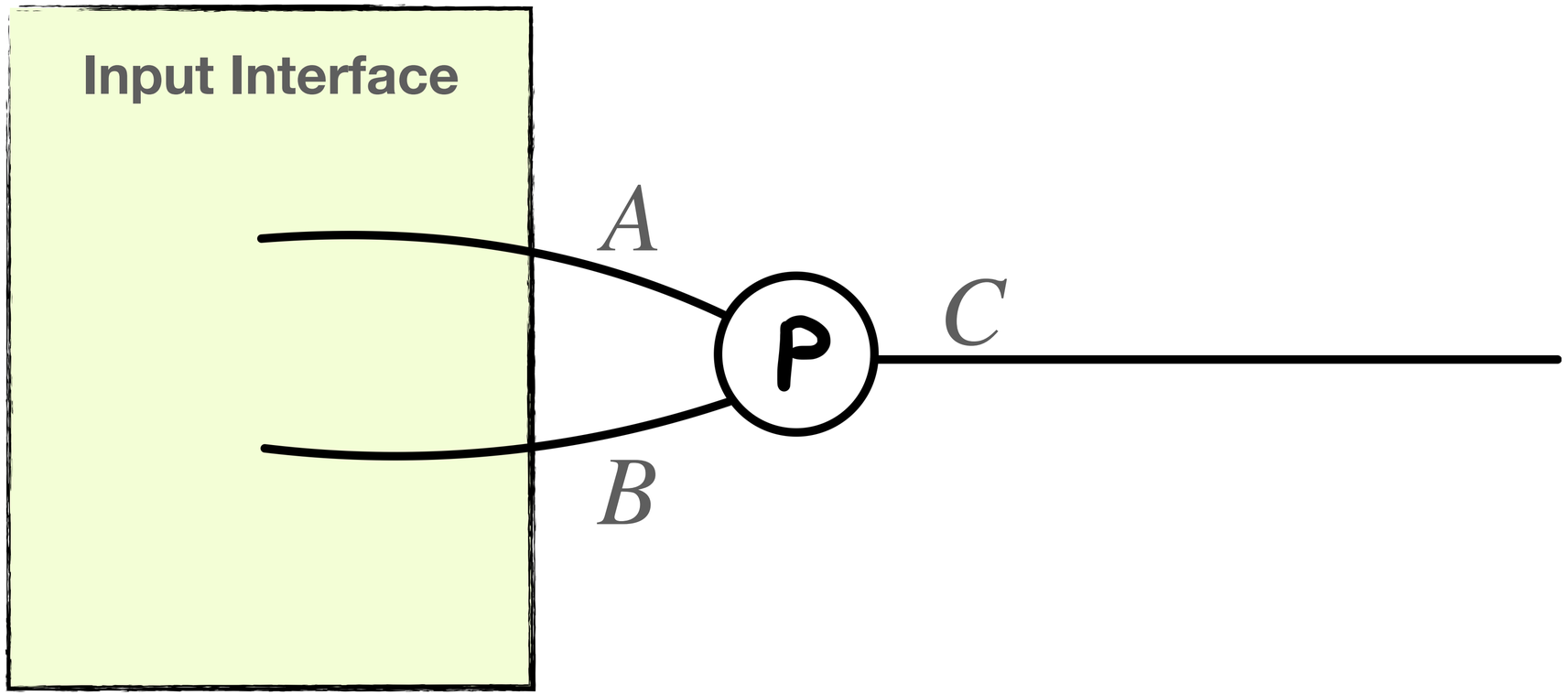

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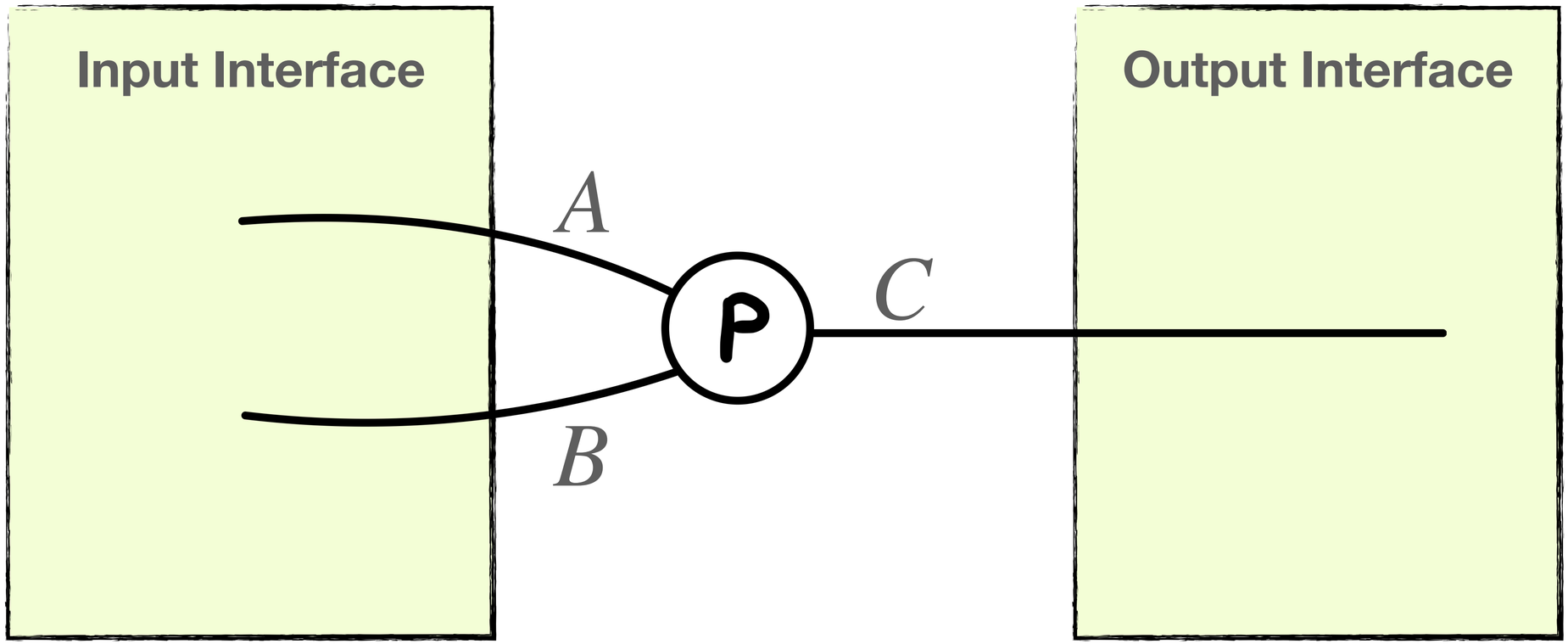

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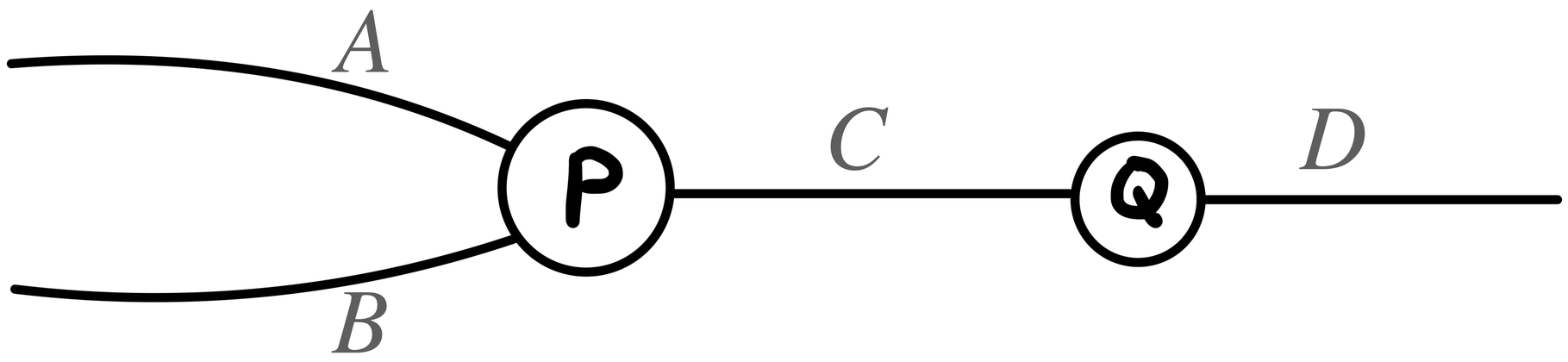

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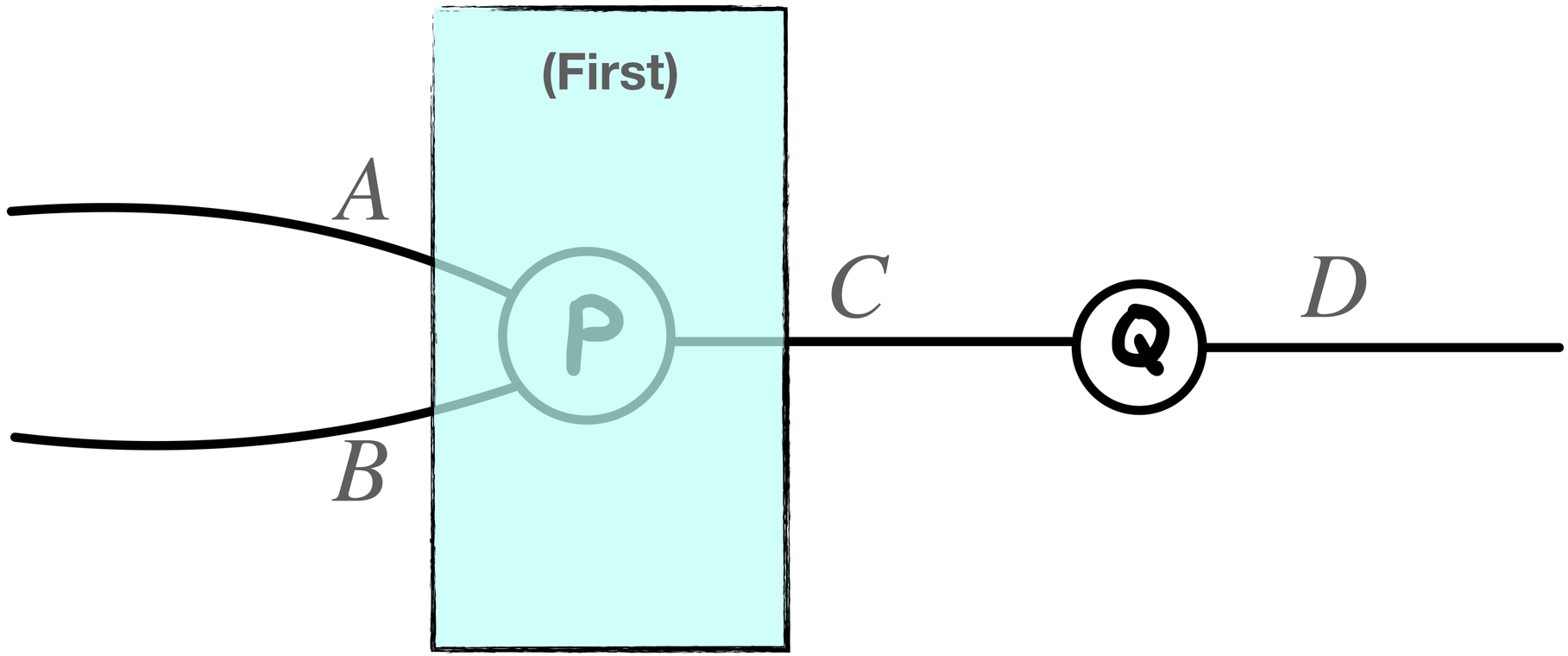

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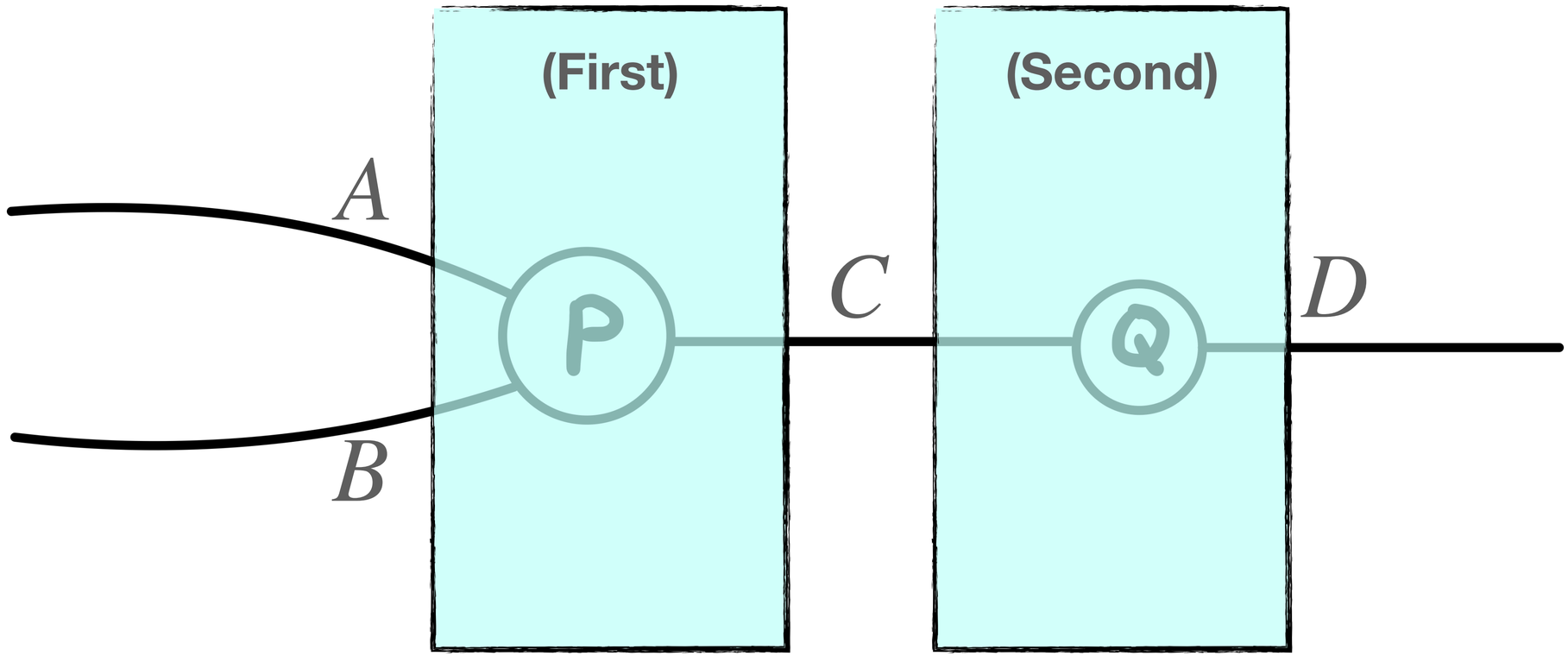

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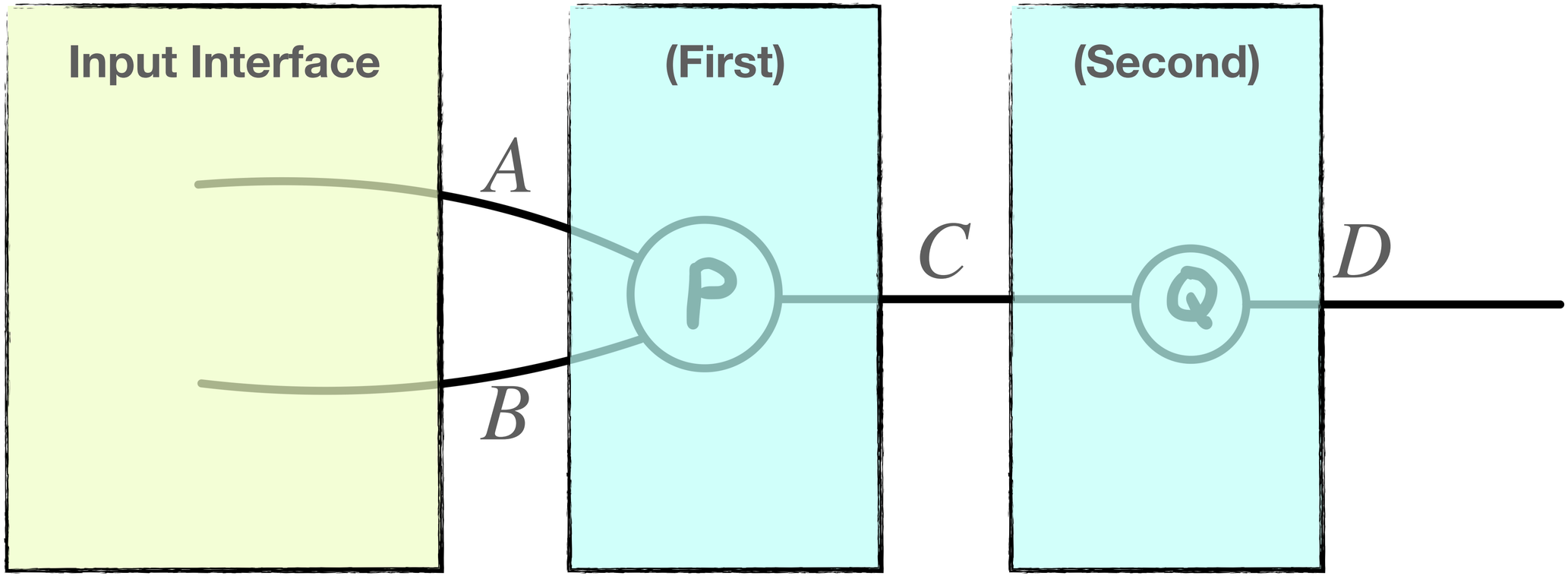

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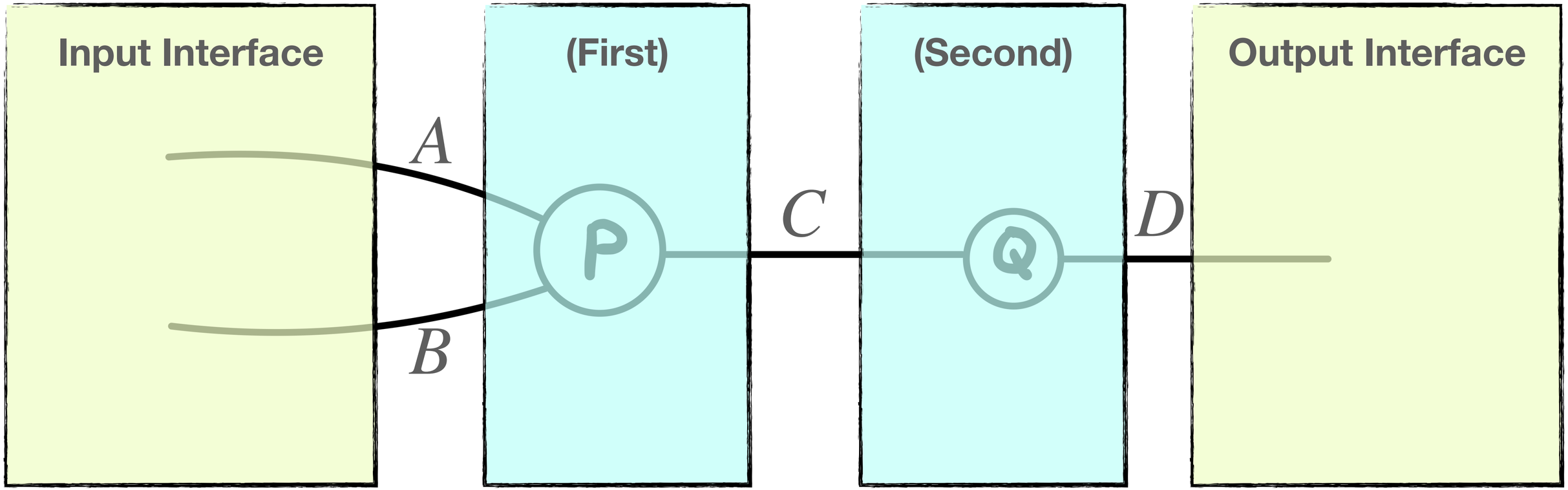

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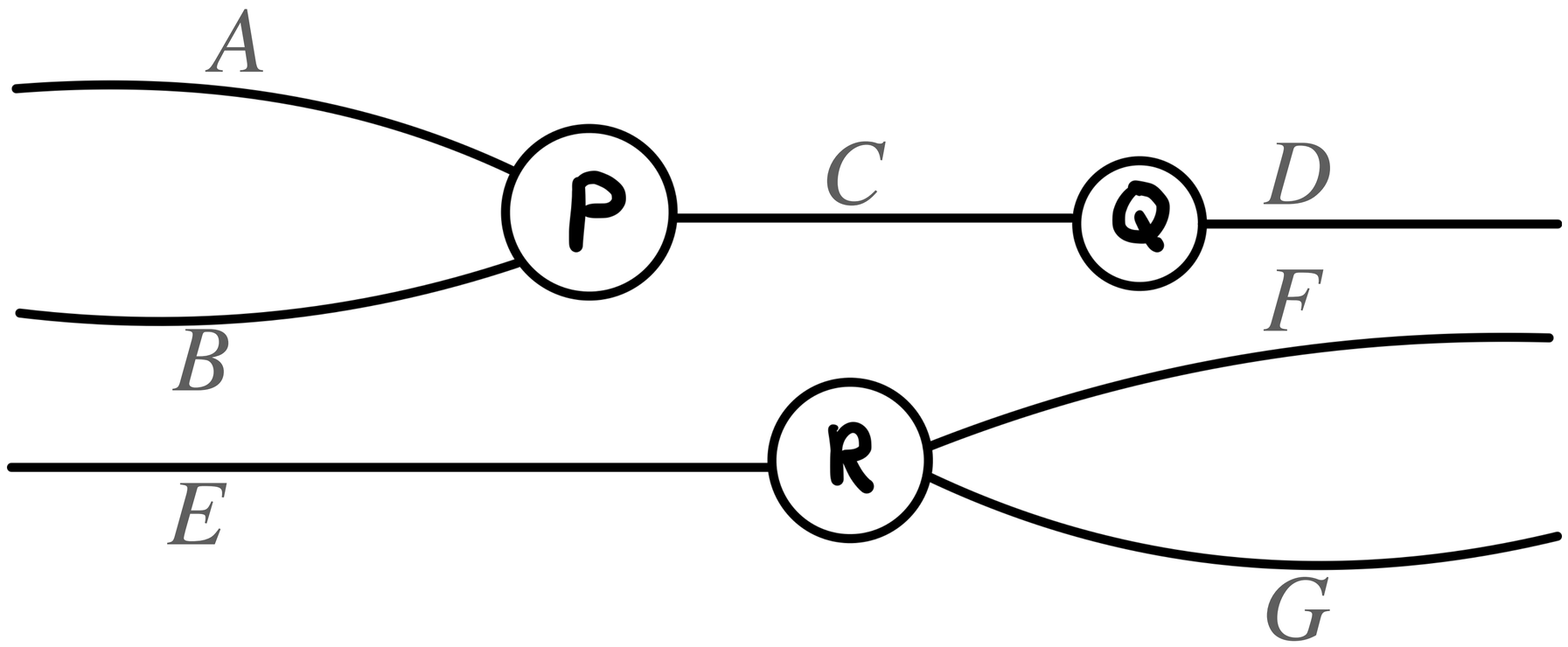

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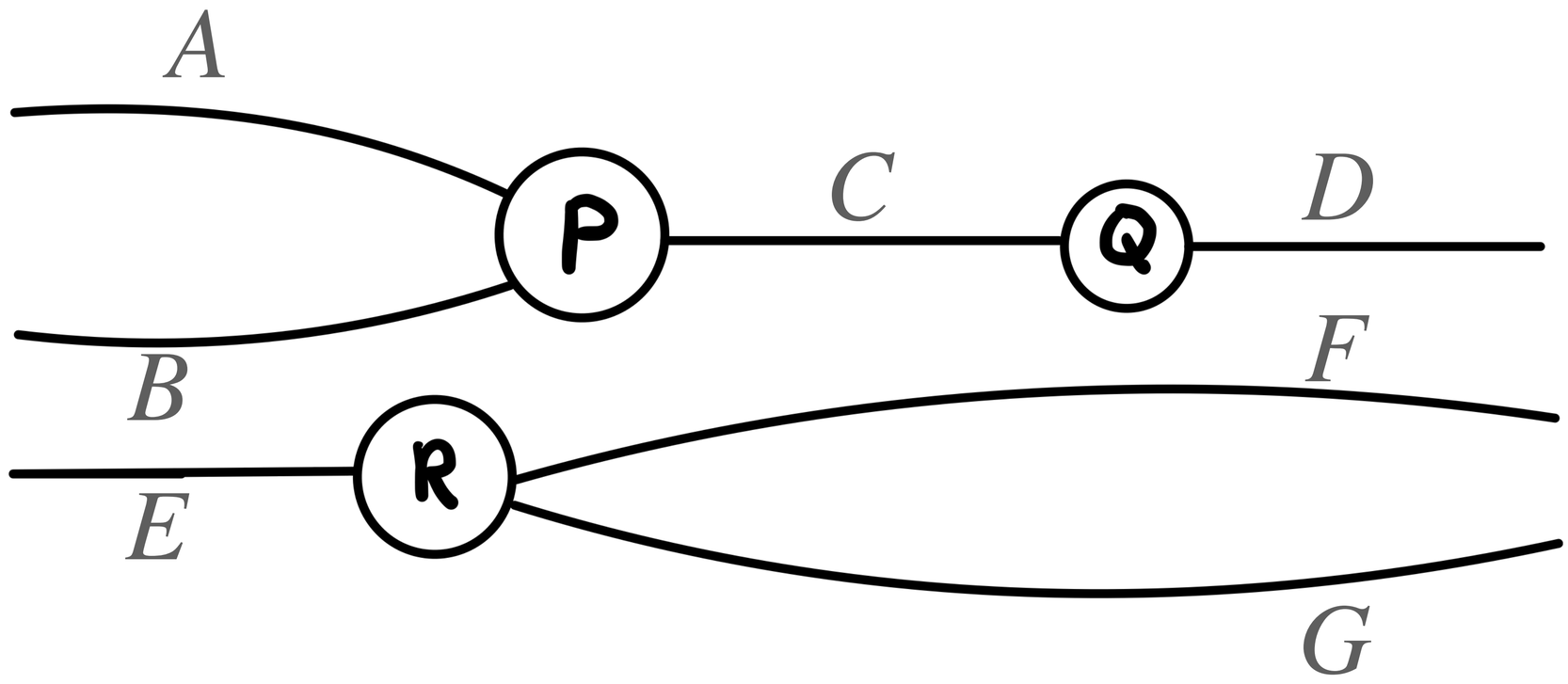

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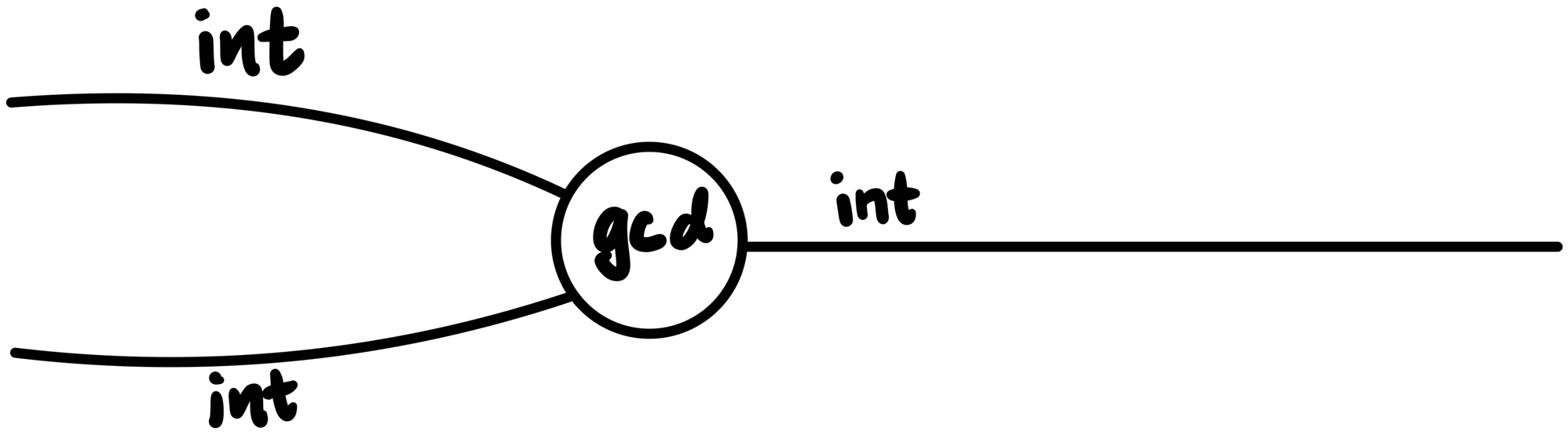

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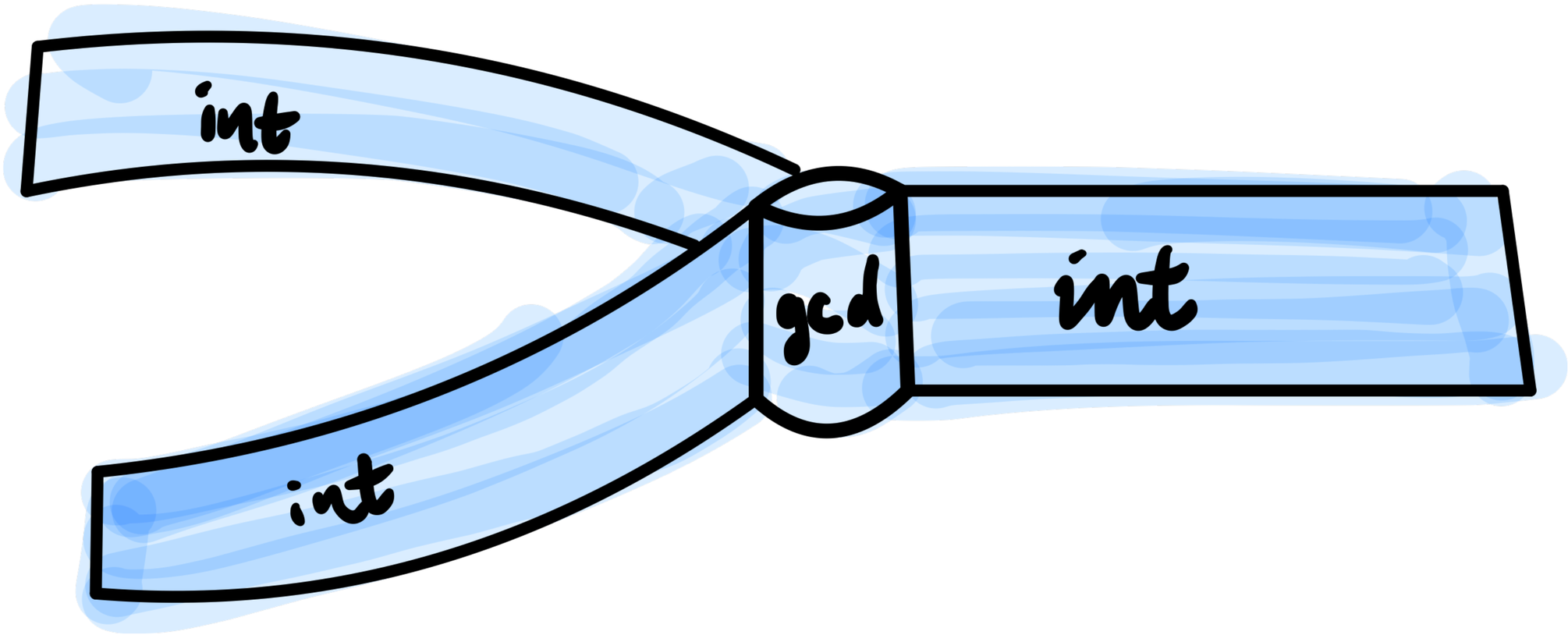

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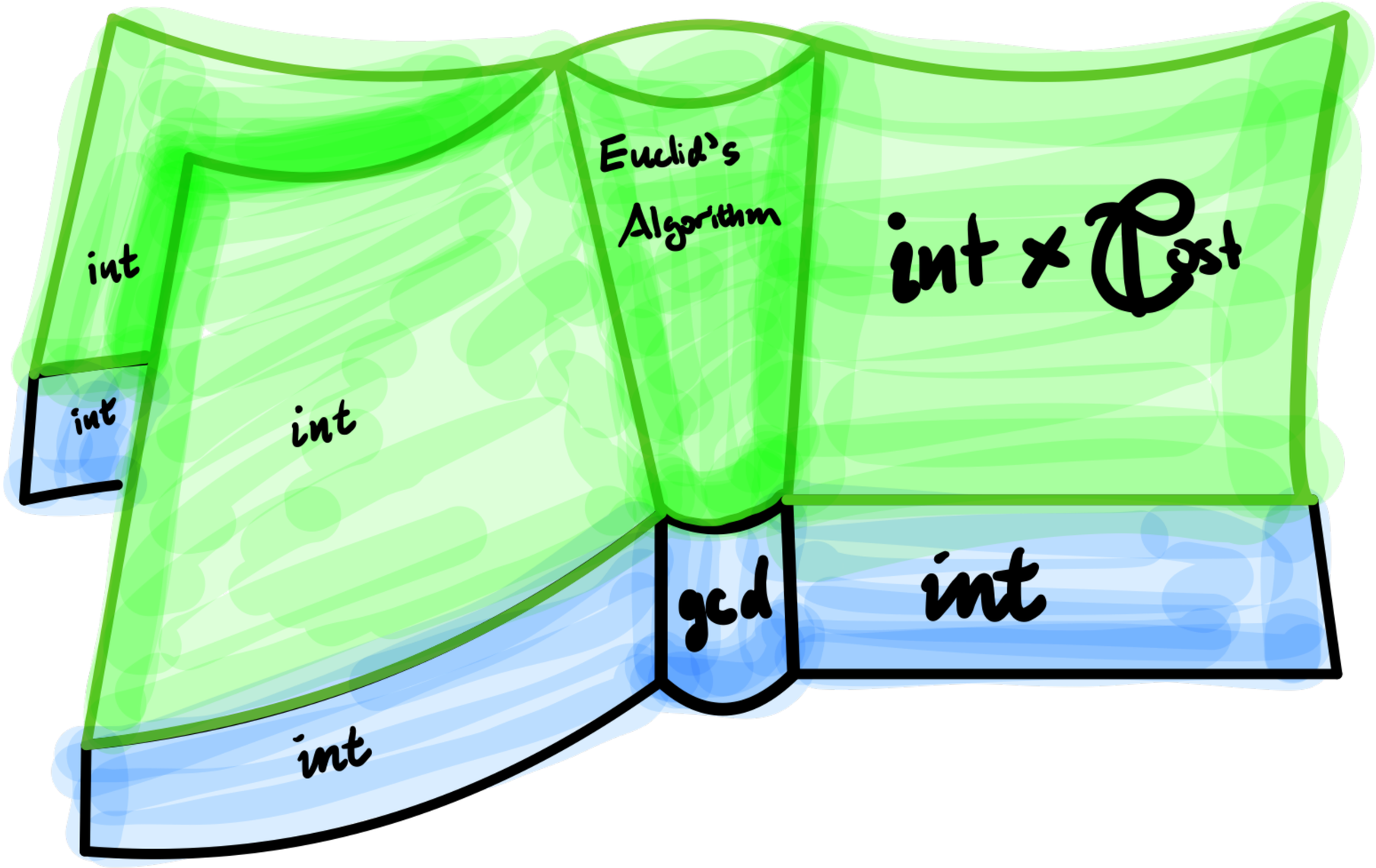

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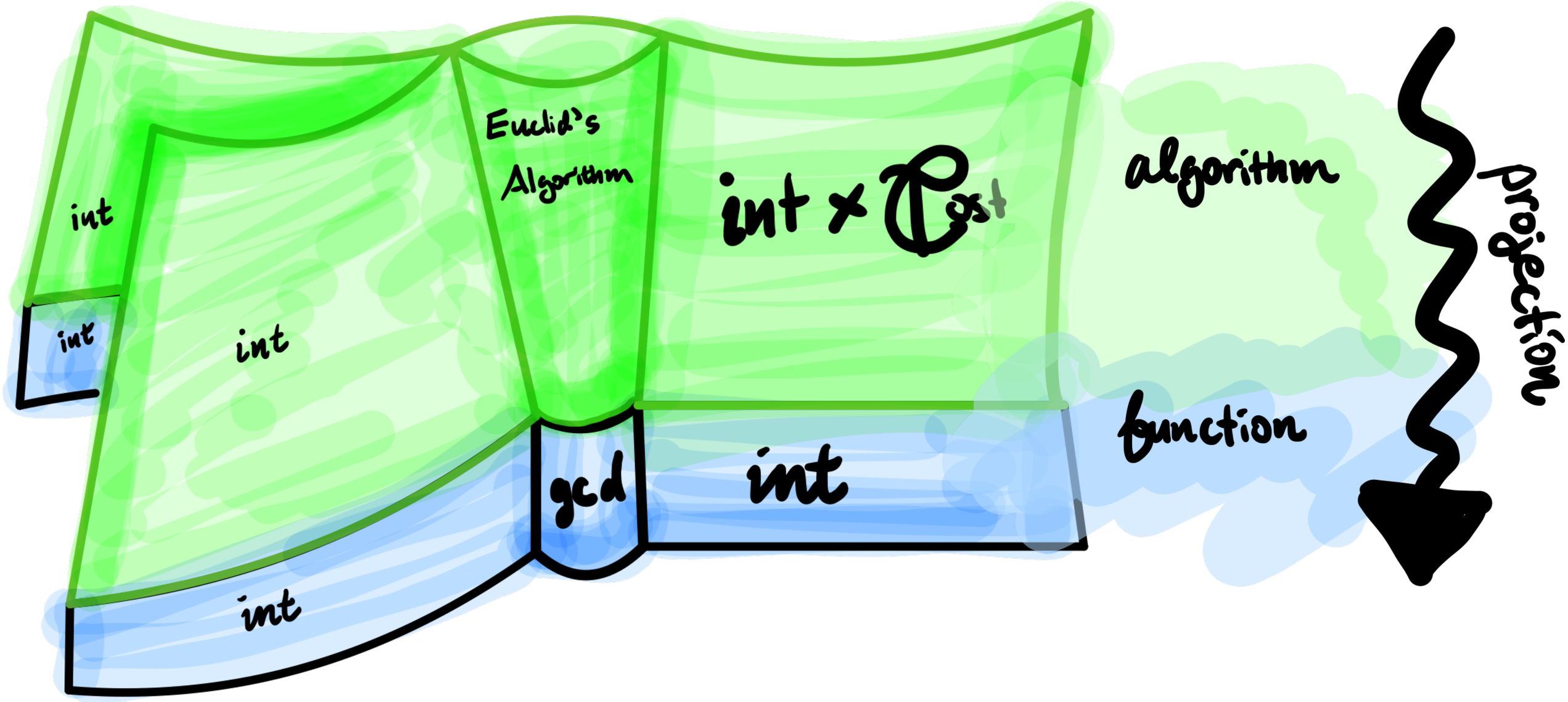

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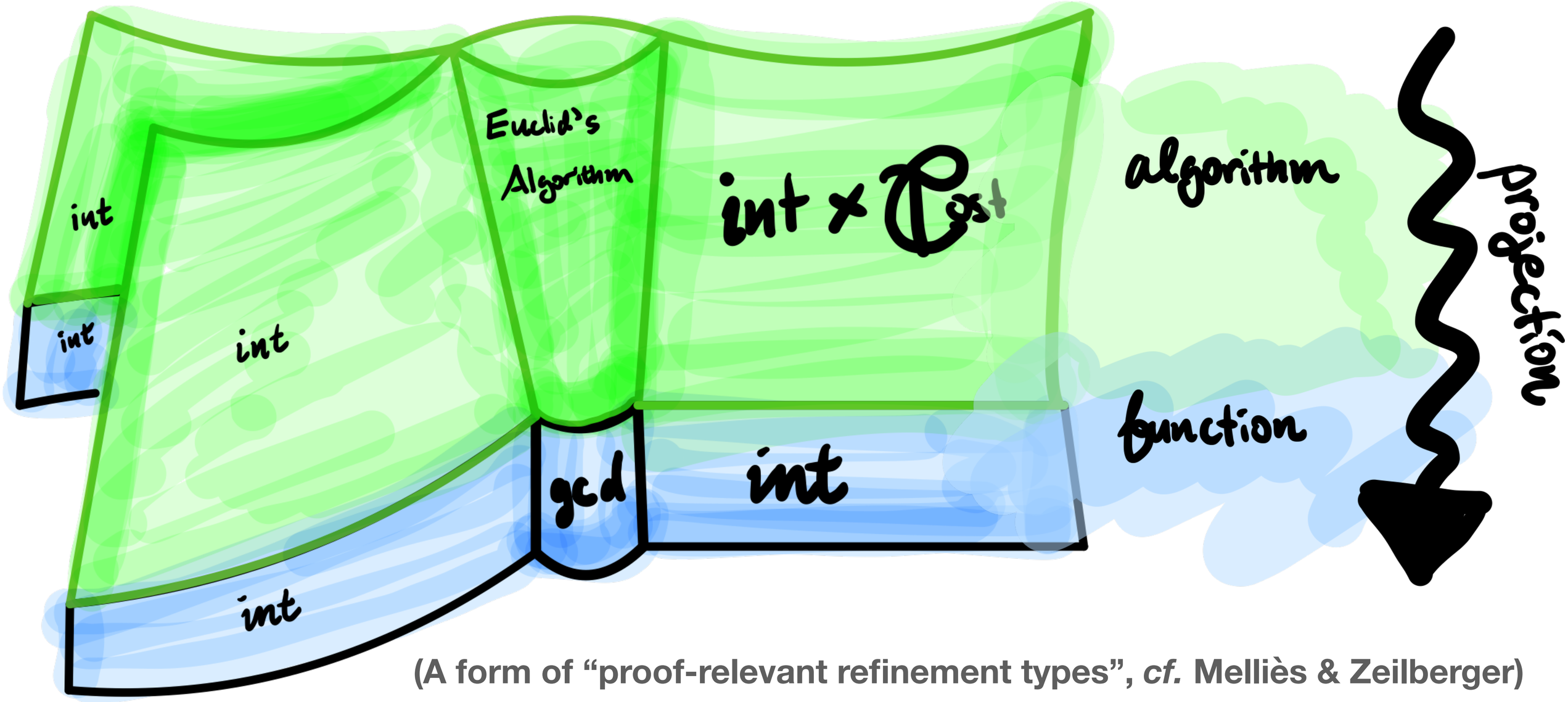

(A form of “proof-relevant refinement types”, *cf.* Melliès & Zeilberger)

New Spaces for Program Semantics:

gluing phases together in computational models

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- **Classical domain theory:** dcpos, ω-cpos, Scott domains, stable domains, coherent spaces, ultrametric spaces...

- **Synthetic domain theory:** "sets" in a topos. *Amenable to* ***gluing constructions*** *that correspond to* ***phase composition****.*

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These gluings provide **universal** ways to surface subtle facets of computation (see also the recent work of Matache, Moss, Staton).

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Niu, S., Grodin, Harper. *"A Cost-Aware Logical Framework."* **POPL '22.**

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S. & Harper. *"Sheaf semantics of termination-insensitive noninterference."* **FSCD '22.**

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- **Solution:** glue together three copies (left, right, and top) of your computational model. The left and right sides represent two implementations of a data structure, and the “top” carries a representation invariant between the two.

TOP: Representation Invariant

$$l, r_{front}, r_{back} : \text{List} \mid l = \text{append}(r_{front}, \text{reverse}(r_{back}))$$

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S., Gratzer, Birkedal. *“Denotational semantics of general store and polymorphism.”* Under review.

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Uemura. *"Normalization and coherence for ∞-type theories."* Unpublished manuscript.

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S. *"First Steps in Synthetic Tait Computability: The Objective Metatheory of Cubical Type Theory."*
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- ***Denotational semantics of higher-order reference types*** (with Gratzer, Birkedal) in synthetic guarded domain theory; ***higher-order separation logic over denotational semantics*** (with Aagaard, Birkedal). **Applying semantic methods to "realistic" programming languages rather than toy languages.**

Thanks!

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