NOTES ON THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

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Abstract. In these notes we present a short, mostly self-contained presentation of the independence of the continuum hypothesis. The proof is topos-theoretic: we shall be presenting a specific topos which can model ZFC for which the continuum hypothesis fails. This demonstrates the connection between the logical methods of forcing, Beth semantics and Grothendieck toposes. This development closely follows that given in MacLane and Moerdijk [14], a simplification of the original proof given in Cohen [2]. Each section (with the obvious exception of the background 1) contains an individual chunk of the proof and may be read largely independently.

1. Background

These notes are not meant to serve as a complete introduction to topos theory. Therefore, the background section of these notes, rather than being the first 5 chapters of MacLane and Moerdijk [14] will contain more or less an accumulation of definitions and lemmas that we will need. These will be more useful for ensuring that I have things to reference as well as ensuring a well established set of terminology for use in the proof.

Since this is a topos theoretic proof, it would behoove us to start with

Definition 1.1. *An elementary topos $\mathcal{E}$ is a category that*

- *has all finite limits*
- *is cartesian closed*
- *has a subobject classifier*

The topos that we are interested in will be ones that satisfy certain characteristics making them into a model of ZFC. Modeling the axiom of choice, the law of the excluded middle, and the existence of infinite sets in particular present challenges. The next set of definitions are the categorical analogs of these traits.

Definition 1.2. *A natural number object (NNO) is a category $N \in \mathbb{C}$ is an object with arrows $s : N \to N$ and $z : 1 \to N$ so that for any object $A \in \mathbb{C}$ with $f : 1 \to A$ and*

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$g : A \to A$, there exists a unique h so that

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "f"'] & N \arrow[r, "s"] \arrow[d, dashed, "h"] & N \arrow[d, dashed, "h"] \\
& A \arrow[r, "g"] & A
\end{tikzcd}$$

Example 1.3. *In the category of sets,* **Set***, the set of all natural numbers $\mathbb{N}$ forms a natural number object.*

Example 1.4. *In any presheaf $\widehat{\mathbb{C}}$, the constant presheaf $A \mapsto \mathbb{N}$ forms a natural number object.*

Example 1.5. *NNOs are reflected by geometric morphism. In particular, any reflective subtopos inherits an NNO from the full topos.*

Having categorified the definition of the set of natural numbers, we now turn to the logical aspects of a topos. The internal logic of a topos is, in general, intuitionistic and thus validates neither the law of the excluded middle nor the axiom of choice. We shall need both of these for the topos we're constructing. Therefore, we turn to defining what toposes *do* satisfy these principles.

Definition 1.6. *A topos is said to be boolean if the subobject classifier Ω forms an internal boolean algebra.*

Booleaness can be captured in several equivalent in useful ways.

Lemma 1.7. *The following conditions are equivalent*

(1) *$\mathcal{E}$ is boolean*

(2) $\neg\neg = 1 : \Omega \to \Omega$

(3) *The Heyting algebra* $\mathrm{Sub}\, A$ *for an $A \in \mathcal{E}$ is a boolean algebra*

(4) $\Omega \cong 1 + 1$ *with* $[\mathrm{true}, \neg \circ \mathrm{true}]$.

Proof. TODO fill out as an exercise. □

For our purposes of using a topos to model ZFC, booleaness will be essential. A boolean topos will validate the law of excluded middle[1] which is crucial for validating the rules of ZFC. There then arises the natural question of taking an existing topos and modifying it so that it is boolean. This is easily done using a Lawvere-Tierney topology

Definition 1.8. *A Lawvere-Tierney Topology is a map $j : \Omega \to \Omega$ so that*

(1) $j \circ \mathrm{true} = \mathrm{true}$

(2) $j \circ j = j$

(3) $\wedge \circ (j, j) = j \circ \wedge$

[1]Exercise, show that in a boolean topos $[\![\forall x.\ x \vee \neg x]\!]$ holds

Now in our case, we shall be interested in one particular topology, the double negation topology.

Example 1.9. *For any topos, $\neg\neg : \Omega \to \Omega$ forms a topology.*

The useful property of the double-negation topology for our purposes is that $\mathbf{Sh}_{\neg\neg}(\mathcal{E})$ will always form a boolean subtopos of $\mathcal{E}$. Later on, we will use this to construct a boolean topos out of a topos modeling the forcing construction we wish to implement.

Lemma 1.10. *$\mathbf{Sh}_{\neg\neg}(\mathcal{E})$ is boolean.*

Proof. TODO □

An interesting aside at this point is that the double-negation topology on a presheaf topos has a well known Grothendieck analog: the dense topology. That is, the topology given by

$$J(C) = \{S \mid \forall f : A \to C.\ \exists h.\ f \circ h \in S\}$$

I will not prove this but will use the phrase "dense topology" and "double negation topology" interchangeably in the notes[2]. We next turn to the topos-theoretic analog of the axiom of choice. Here we are presented with two possible ways. The first is a direct formulation of the principle from the category of sets. We wish to generalize *every surjection has a section.* We can generalize this to a topos by replacing surjection with epimorphism to get

Definition 1.11. *A topos satisfies the axiom of choice if every $e : A \twoheadrightarrow B$ has a section $s : B \to A$ so that $es = 1$*

However, it is often preferable to state a version termed the internal axiom of choice (TODO ELABORATE)

Definition 1.12. *A topos satisfies the internal axiom of choice if $(-)^A$ preserves epimorphisms.*

Remark 1.13. *Any topos that satisfies the axiom of choice satisfies the internal version. Any topos that satisfies the internal version is boolean. This is due to Diaconescu [6].*

This characterization of the axiom of choice is awkward to work with however in many toposes. A cleaner characterization can be given in terms of the what objects *generate* the topos.

Definition 1.14. *A collection of objects S is said to generate a category $\mathbb{C}$ if for any parallel arrows $f \neq g : A \to B$, there exists an arrow $h : C \to A$ for some $C \in S$ so that $fh \neq gh : C \to B$. In particular, if a topos is generated by 1 and is nontrivial then it is* well-pointed.

Well-pointedness captures a large number of logical principles that we will need for our topos. Most important among these is that for a Grothendieck topos, the axiom of choice follows from it. This is because in a Grothendieck topos the subobject preorders are all *complete* partial orders.

[2] It's because I'm a mean-spirited person

Lemma 1.15. *A well-pointed topos, $\mathcal{E}$ is boolean. Moreover, if* $\mathrm{Sub}(A)$ *is a complete poset for all A then $\mathcal{E}$ satisfies the axiom of choice.*

Proof. TODO □

With this we are in a position to start our proof because a topos which satisfies the AoC and has a NNO is powerful enough to provide a model of ZFC that we will use to validate the independence of the continuum hypothesis.

2. An Overview of the Proof

Before we dive into the details of this proof, I think it is helpful to step back and summarize it at a high level. The proof using a method called forcing[3]. The basic premise of the proof is that we want to construct a model of ZFC which forces a set much larger than 2^N, B, to have an injection into 2^N in the topos $\mathbf{Sh}_{\neg\neg}(P)$ for some poset P. It doesn't particularly matter what B is so long as it's strictly larger than 2^N. For simplicity, let us fix

$$B = \mathcal{P}\left(\mathcal{P}\left(N\right)\right)$$

for the remainder of the paper. Since this cannot be done ordinarily we use forcing to gradually introduce fragments of such an injection so that, internally to the topos, we may manipulate this arrow as if it were a complete injection. The "fragments" of the function will constitute the poset P that we're defining sheaves over. This is the essence of topos-theoretic forcing. This topos is developed in section 3.

Now in our model, we then have $\mathbf{a}(N) \rightarrowtail \mathbf{a}(B) \rightarrowtail \Omega_{\neg\neg}^N$. Having done this, we can actually force an object to appear strictly between N and $\Omega_{\neg\neg}^N$. This object however isn't B! Instead, we use $\mathbf{a}(\Omega^N)$, the powerset of the NNO from the presheaf topos included into $\mathbf{Sh}_{\neg\neg}(P)$. Since B was specifically chosen to be much larger than 2^N, we get an inclusion

$$\mathbf{a}(N) \rightarrowtail \mathbf{a}(\Omega^N) \rightarrowtail \mathbf{a}(B) \rightarrowtail \Omega_{\neg\neg}^N$$

All of these inequalities are developed in section 4. Moreover, those first two inclusions are strict in $\widehat{P}$. We then prove that posets like P satisfy what is called the Souslin property and that in this case $\mathbf{a}(-)$ preserves the strictness of inclusions. This is the most technical portion of the proof and is developed in 5.

Having done this, by transitivity we have then forced the existence of an object $\mathbf{a}(2^N)$ which lies between N and $\mathcal{P}\left(N\right)$ as required.

3. The Cohen Topos

3.1. The Partial Order P.

For the remainder of this development, we will work with the cohen topos. This is a subtopos of presheaves on a particular partial order. As mentioned above, we wish to design this partial order in order to help us construct a monomorphism from $B \to \Omega^N$. Now by transposition, such a morphism can always be represented as $B \times N \to \Omega$. This gives us an indication for

[3] Forcing was in fact developed to solve this problem

how to construct P, it can simply be subsets of $B \times N$. Now in order a function $B \to \Omega^N$ (transposed as $B \times N \to \Omega$) to be a monomorphism, it must be that for any p and any $b \neq b' \in B$ that there is some n so that $p(b, n) \neq p(b', n)$.

Therefore, our partial order P shall be disjoint collections of $P = \mathcal{P}_{fin}(B \times N \to \{0, 1\})$. For any $p \in P$, we shall treat p as a member of $B \times N \rightharpoonup 0, 1\}$ which is defined on only a finite number of inputs. Accordingly, we shall use $p(b, n) \downarrow$ and $p(b, n) \uparrow$ to indicate whether or not p is or isn't defined on a particular input respectively. P is called the collection of forcing conditions and each element is thus a condition. It should be thought of as a "constraint" on the map that we are trying to construct from $B \to \Omega^N$. If we are working at forcing condition p we are in effect stating that while we do not know the full contents of the map $B \to \Omega^N$, we know that it is at least a completion of p.

The finiteness of each set is crucial. It is used to imply that each completion *could* be part of a monomorphism. This is because for any $b, b' \in \mathrm{Dom}(p)$ for some $p \in P$, there exists an n so that $p(b, n) \uparrow$ and $p(b', n) \uparrow$. Therefore, it is always the case that a future condition may add some data to distinguish b and b'. It remains to define an order on P however. Let us say

$$q \leq p \triangleq \forall (b, n) \in B \times N.\ p(b, n) \downarrow \Longrightarrow p(b, n) = q(b, n)$$

This is the opposite of the traditional order that partial functions are endowed with and is clearly reflexive, antisymmetric, and transitive. This reversal is typical in forcing developments and may seem slightly confusing. The reason for it though is quite straight forward, each p is thought of representing our knowledge about some map from $B \to \Omega^N$. The more defined p is, the smaller it is according to our ordering, the fewer maps it corresponds to. The ordering is in effect the traditional subset ordering on the possibilities each condition allows. Hence, a larger condition has more information, it permits fewer possibilities and is therefore smaller.

3.2. **Presheaves on P.**

Let us now turn our attention to the category of presheaves on P, $\widehat{P}$. We want to develop the framework necessary for us to construct our cardinal inequalities that we hinted at in the overview 2.

3.3. **The Double Negation Topology on P.** Foo

4. The Cardinal Inequalities

5. The Preservation of Strictness

6. Forcing in a More General Context

Having proven the independence continuum hypothesis, I wanted to take a little time to discuss how this proof fits into a broader context. In general forcing is incredibly useful for establishing independence results in both set theory and type theory. This proof shows how syntactic forcing proofs can be smoothly translated into a proof about toposes. I am not capable of speaking of interesting results established in set theory using forcing but several developments in type theory have used a topos theoretic forcing technique. Specifically, Coquand and Jaber [4] represents a coherent introduction to some of the developments done in Coquand [3] and Coquand and Mannaa [5]. This work was developed in the Jaber [12]. While not directly using topos-theoretic forcing, Escardo [9] and Sterling [16] are both results about type theory done by a similar technique.

The alternative characterization of forcing in terms of boolean valued models was explored by Scott and others during the 60s and 70s. These are remarkably topos theoretic semantics in which the validity of a statement isn't a simple boolean but given in terms of an element of a complete boolean lattice. By quotienting this lattice by a particular ultrafilter one can translate forcing proofs into this framework. I am not well positioned to recommend literature on this but Jech [13] contains an approachable introduction and our own Clive Newstead has produced notes on this [15].

More familiar to logicians will be Kripke semantics and Beth semantics. In these we index the $\Vdash$ relation with a "world" at which we consider it. These worlds are assumed to form a preorder which is intended to represent time with $w_1 \leq w_2$ implying that w_2 is a possible future of w_1. Kripke semantics correspond closely to presheaf semantics on the poset of worlds. Beth semantics add the local character sheaves enjoy to the semantics. Accordingly then, Kripke semantics, step-indexed logical relation, and Beth semantics in general can be translated as a special case of the internal logic of some presheaf or sheaf topos. This idea for intuitionistic logic was discussed in Fourman [10] and Fourman [11]. Kripke and Beth semantics are given a lengthy consideration in Dummett [8] and Troelstra and van Dalen [17]. Recently, the more topos theoretic approach has been made explicit in the work of Birkedal et al. [1] and Dreyer et al. [7].

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