NORMALIZATION FOR FREE CARTESIAN CLOSED CATEGORIES

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1. Free Cartesian closed categories

(1.1) The theory of categories can be expressed in the language of finite limits with classifying category $\mathbb{T}_{\mathbf{Cat}}$; likewise, the notion of Cartesian closure is expressed at the algebraic level by a lex functor $\mathbb{T}_{\mathbf{Cat}} \longrightarrow \mathbb{T}_{\mathbf{CCC}}$, exhibiting at the level of categories of algebras an adjunction $\mathsf{F} \dashv \mathsf{U} : \mathbf{CCC} \longrightarrow \mathbf{Cat}$. The monad of this adjunction $\mathbf{Cat} \xrightarrow{\mathsf{T}} \mathbf{Cat}$ freely adjoins products and exponentials to any category.

(1.2) Let $\mathbb{C}[\mathsf{x}] \equiv \mathsf{T}\{\mathsf{x}\}$ be the free Cartesian closed category generated by a single object. We intend to show that there is a normal form for morphisms in $\mathbb{C}[\mathsf{x}]$, and that the word problem for free Cartesian closed categories is consequently decidable.

2. Category of renamings

(2.1) The notion of normal form for an arrow in a Cartesian closed category does not *a priori* have a substitution action for arbitrary $\mathbb{C}[\mathsf{x}]$-arrows. Equipping normal forms with such an action will be a consequence of the present theorem.

However, any appropriate notion of normal form *will* possess an action for certain $\mathbb{C}[\mathsf{x}]$-arrows, namely the ones which arise from the structure of finite products only. In the language of syntax, these correspond to the substitutions which arise from structural rules (weakening, contraction, exchange).

(2.2) Let $\mathbb{R}[\mathsf{x}] \overset{j}{\hookrightarrow} \mathbb{C}[\mathsf{x}]$ be the *least wide subcategory* of $\mathbb{C}[\mathsf{x}]$ which has finite products; in other words $\mathbb{R}[\mathsf{x}]$ has the exact same objects as $\mathbb{C}[\mathsf{x}]$, but its morphisms are only the ones which arise from product cones and their universal maps.

3. A nerve from syntax to presheaves on renamings

(3.1) Let $\widehat{\mathbb{C}} \equiv [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ to denote the category of presheaves on any category $\mathbb{C}$. We will write $\mathbb{C} \overset{よ}{\hookrightarrow} \widehat{\mathbb{C}}$ for the Yoneda embedding, which takes $C : \mathbb{C}$ to the representable presheaf $\mathbb{C}[-, C]$. Given a functor $\mathbb{C} \xrightarrow{f} \mathbb{D}$, there is a corresponding *change of base* functor $\widehat{\mathbb{D}} \xrightarrow{f^*} \widehat{\mathbb{C}}$ which has both right and left adjoints, and is therefore left exact.

(3.2) In particular, we have a *nerve* functor $\mathbb{C}[\mathsf{x}] \xrightarrow{\mathsf{N}} \widehat{\mathbb{R}[\mathsf{x}]}$ defined by composing the Yoneda embedding with the change of base along $\mathbb{R}[\mathsf{x}] \xhookrightarrow{j} \mathbb{C}[\mathsf{x}]$ as in the following diagram:

$$\begin{tikzcd} \mathbb{C}[\mathsf{x}] \arrow[r, hook, "よ"] \arrow[dr, dotted, "\mathsf{N}"'] & \widehat{\mathbb{C}[\mathsf{x}]} \arrow[d, "j^*"] \\ & \widehat{\mathbb{R}[\mathsf{x}]} \end{tikzcd} \tag{1}$$

N is left exact, because it is a composite of left exact functors; it's worth noting that N is *not* a Cartesian closed functor, because the change of base j^* does not preserve the exponential.

4. Notions of neutral and normal forms

(4.1) Rather than immediately constructing an ad hoc characterization of normal and neutral forms, we will first specify exactly what is required of these notions for the main theorem, and then show that such a notion exists. In subsequent sections, we will then work abstractly with an arbitrary notion of normal form.

(4.2) Let $\mathbb{D}[\mathsf{x}] \xhookrightarrow{d} \mathbb{C}[\mathsf{x}]$ be the least wide subcategory of $\mathbb{C}[\mathsf{x}]$, i.e. the free category on the underlying set of objects of $\mathbb{C}[\mathsf{x}]$.

(4.3) A *notion of normal form* for Cartesian closed categories is defined to be a functor $? \xrightarrow{\mathbb{D}[\mathsf{x}]} ?$