THE THEORY OF LOGOI (M. ANEL)

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Abstract

These notes were prepared from a course given by Mathieu Anel at HoTT 2019 Summer School [Ane19b]; all mistakes are my own. See Anel and Joyal [AJ19] and Anel [Ane19a] for more detailed material on this topic.

(0.1) In these notes, everything is implicitly ∞-categorical; 1-categories are explicitly marked.

1. Course 2: the theory of logoi

(1.1) A topos is a space defined by a *logos of sheaves*; a logos is an algebraic structure on the category of sheaves on a space.

(1.2) The dichotomy between sets and classes is not enough to handle category theory. An inaccessible cardinal is a cardinal α such that $\mathbf{Set}_{<\alpha}$ is stable under dependent product and sum. We will use the terms atomic (subsingleton), finite, small, normal, large for several strictly increasing scales of inaccessible cardinal. The category of small categories is a normal category.

We will write $\mathcal{S}$ for the normal ∞-category of small spaces, and $\overline{\mathcal{S}}$ for the large ∞-category of normal spaces.

1.1. Free cocompletion

(1.1.1) Let a category C be called *cocomplete* if for any small category K, the functor of constant diagrams $C \longrightarrow C^K$ has a left adjoint. A functor $C \longrightarrow D$ between cocomplete functors should be *cocontinuous*; this gives a category of cocomplete categories. The forgetful functor from cocomplete categories to complete categories has a left adjoint (Lurie). This is the *free cocompletion*.

(1.1.2) If C is small, the $\mathbf{P}(C) = [C^{\mathrm{op}}, \mathcal{S}]$ is a category. If C is normal, then we take presheaves $\overline{\mathbf{P}}(C)$ into $\overline{\mathcal{S}}$ and take the smallest subcategory of this large presheaf category which is closed under small colimits. This can be obtained from a transfinite iteration.

An object $X : C$ is *small* if for all filtered $K \xrightarrow{Y} C$, we have $[X, \mathrm{colim}_k Y_k] = \mathrm{colim}_k [X, Y_k]$

(1.1.3) C is *presentable* if there exists a small diagram $K \longrightarrow C$ and a small diagram $W \to \mathbf{P}(K)^{\to}$, such that $\mathbf{L}(\mathbf{P}(K), W) \simeq C$; in other words, C is the localization of a presheaf category at some class of maps. A cocomplete category is small iff it is presentable; this statement must be verified within a specific *model* of ∞-categories, such as quasicategories.

1.2. Logos and descent

Assume that C is a category with finite limits (a *lex category*).

(1.2.1) A *family* in C parameterized by $B : C$ is a map $E \xrightarrow{p} B$. A morphism of families over the same base is a commuting triangle. The category of such things is the slice C/B. The reindexing of a family $E \xrightarrow{p} B$ along an arbitrary map $B' \longrightarrow B$ is simply the fiber product. From this we construct the functor $C^{\mathrm{op}} \xrightarrow{\mathbb{U}} \mathbf{CAT}$ which takes an object to its slice, and a morphism to the fiber product. We call $\mathbb{U}$ the *universe* of $\mathbb{C}$; this functor gives the fibers of the codomain fibration. In 1-category theory, $\mathbb{U}$ would be a pseudofunctor into a bicategory.

(1.2.2) We define U as the groupoid core of $\mathbb{U}$: it has the same objects as $\mathbb{U}$ but only the invertible arrows. This universe in the sense of type theory; but neither is actually an object of C; they are structures *over* C. Suppose we have a small diagram $I \xrightarrow{X} C$ and the colimit $X_\infty = \mathrm{colim}_i X_i$; what if we want to describe a family over the colimit, i.e. $E \longrightarrow X_\infty$? This is the topic of *descent*.

(1.2.3) We obtain a "Yoneda" embedding $C \xrightarrow{\text{よ}} [C^{\mathrm{op}}, \mathbf{CAT}]$, factoring through actual Yoneda embedding and the inclusion of groupoids into categories. The Yoneda lemma says that morphisms $\text{よ}B \xrightarrow{\phi} \mathbb{U}$ are the "same" as families $\overline{\phi} \in \mathbb{U}(B) = C/B$. $\mathbb{U}$ classifies families and all morphisms (not only isomorphisms as U, or only monos as Ω).

(1.2.4) Coming back to descent, we can say that families $E \longrightarrow X_\infty$ are the same as maps $\text{よ}X_\infty \longrightarrow \mathbb{U}$; by reindexing, we obtain maps $X_k \longrightarrow \mathbb{U}$ for each $k : K$. Going the other direction, can we start from such component families and get a family over the colimit? At first glance looks like we might be able to, but note that we are constructing a map $\text{よ}X_\infty \longrightarrow \mathbb{U}$ and the representable object on the left only acts as a colimit with respect to representable objects on the right: but $\mathbb{U}$ is not representable.

We want a unique lifting condition: if we define a family out of the $\mathbf{P}(C)$-colimit, can we get a unique map out of the Yoneda embedding of the C-colimit? In other words, can $\mathbb{U}$ distinguish between the colimit and the Yoneda embedding of the colimit? This is descent. Phrasing this more precisely, we want $[\text{よ}\,\mathrm{colim}\, X, \mathbb{U}] \longrightarrow [\mathrm{colim}\,\text{よ}X, \mathbb{U}]$ to be an equivalence of categories.

We can pull the colimit out of the hom as a limit, and then this equivalence is calculated to the more familiar descent condition $\lim_i [\text{よ}X_i, \mathbb{U}] \simeq [\text{よ}\,\mathrm{colim}_i X_i, \mathbb{U}]$, i.e. $\lim_i (C/X_i) \simeq C/\mathrm{colim}_i X_i$. There is an obvious adjunction here, given by reindexing and colimit. *Descent says that this adjunction is an equivalence.*

Two conditions for descent

(1.2.5) Start with $E \longrightarrow X_\infty$ and then pull it back to get the fiber product $E \times_{X_\infty} X_j$ and take the colimit; that $E = \mathrm{colim}_j (E \times_{X_\infty} X_j)$ is called the *universality of colimits*. This is satisfied by any cocomplete, locally Cartesian closed category.

(1.2.6) The hard direction is when you start from a family of diagrams $E_i \longrightarrow X_i$, take the colimit, and then pull it back. We want $(\mathrm{colim}_i E_i) \times_{X_\infty} E_j = E_j$

for each j; Mathieu calls this condition *effectivity*, by analogy with the effectivity of equivalence relations in classical topos theory.

(1.2.7) non-trivial n-categories do not have descent; ∞-categories can have descent.

2. Course 3: features of logoi

We begin by reviewing some counterexamples to descent.

(2.1) In **Set** the colimits are universal, but effectivity of colimits fails. He gives the counterexample with $S^0 = 2$:

$$\begin{tikzcd}
S^0 \arrow[d] & S^0 \times S^0 \arrow[l] \arrow[r] \arrow[d, "m"] & S^0 \arrow[d] \\
1 & S^0 \arrow[l] \arrow[r] & 1
\end{tikzcd}$$

Note that the colimit of the spans upstairs and downstairs is just 1, but if you take the pullbacks described in **(1.2.6)**, one does not obtain the desired fibers. For instance:

$$\begin{tikzcd}
1 \neq S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_\infty \arrow[d] \\
1 \arrow[r] & X_\infty
\end{tikzcd}$$

(2.2) In 1-**Gpd**, you get an analogous counterexample from S^1.

(2.3) A cocomplete lex category is called a *logos*; these are arranged into a category whose morphisms are cocontinuous lex functors. We will write **LOGOS** for the category of logoi. There are forgetful functors to cocomplete cateogires, lex categories, and categories.

(2.4) The *presentable logoi* are the logoi whose underlying cocomplete categories are presentable in the sense of **(1.1.3)**; the category of such presentable logoi is written **Logos**.

(2.5) An object of **Logos** is an ∞-topos in the sense of Lurie [Lur09].

(2.6) A general logos need not be locally Cartesian closed, and morphisms f^* between logoi need not have right adjoints. This is the price to pay for the category **LOGOS** to be nice.

(2.7) (Anel-Lejay) The following are equivalent:

1. C is a logos
2. The canonical cocontinuous functor $\mathbf{P}(C) \longrightarrow C$ is left exact. (Analogous to pre-topoi.)
3. The left Kan extension of every lex diagram $D \longrightarrow C$ along $D \longrightarrow \mathbf{P}(D)$ is lex.
4. Giraud axioms: like for ∞-topoi, removing presentability.

(2.8) There is an analogy with commutative algebra; presentable logoi are like finitely generated commutative rings. Grothendieck made general commutative rings the main object of study rather than the finitely generated ones, and Mathieu advocates that we do the same in topos theory; the category of general logoi is nicer than the category of presentable logoi.

2.1. Other examples of logoi

(2.1.1) The free logos on a category: one takes C to $\mathcal{S}[C] = \mathbf{P}(C^{lex})$. In particular, the free logos on one generator $S[X]$ is $[\mathbf{Fin}, \mathcal{S}]$.

(2.1.2) If T is a Lawvere theory, then it is a category with finite products. Let the *enveloping logos* $\mathcal{S}\langle T\rangle = \mathbf{P}(T^{lex/\times})$. This logos classifies models of the theory T.

(2.1.3) ∞-sheaves on a locale X; we can construct a 1-logos $\mathbf{Sh}(X)$ out of functors $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ which satisfy the sheaf condition for the canonical topology. This generalizes to ∞-logoi in a straightforward way, where the sheaf condition is strengthened to consider not just a couple levels of overlap, but infinitely many.

References

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