REMARK ON THE HYPOTHETICAL JUDGMENT

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Abstract. What is the proper explanation of intuitionistic hypothetical judgment, and thence propositional implication? The answer to this question is not clearly determinate from the writings of Brouwer, Heyting and Kolmogorov, since in their lifetimes they propounded multiple (sometimes conflicting) explanations of the hypothetical judgment. To my mind, the determination of an acceptable explanation must take into account its adequacy for the expression of the bar theorem and, more generally, the development of an open-ended framework for transcendental arguments in mathematics.

1. Judgments and Propositions

The distinction between the propositions and the judgments (assertions) is an old one, but prior to Martin-Löf, the significance of assertions was limited to the affirmation of the truth of propositions. Following Martin-Löf [8], forms of judgment other than P *true* are recognized, including P *prop*.

What is the difference between a judgment (assertion) on the one hand, and a proposition on the other hand? A judgment is an act or an experience, whereas a proposition is a mathematical object which may be experienced in different ways. For instance, the assertion of the truth of a proposition (i.e. P *true*) consists in the fulfillment of the intention expressed by the proposition, while the recognition of an object as a proposition (i.e. P *prop*) is the act of understanding this intention.

In addition to the categorical judgments above, higher-order forms of judgment are also explained, including the hypothetical judgment and the general judgment. Now, the primitive hypothetical judgment $\mathcal{J}_2\ (\mathcal{J}_1)$,[1] was explained by Martin-Löf in terms of hypothetical *proof* or *demonstration*, which he defined as follows:

> The notion of hypothetical proof [demonstration], in turn, which is a primitive notion, is explained by saying that it is

[1]Hypothetical judgment is to be distinguished from the sequent judgments $\Gamma \vdash \cdots$, which are not even higher-order judgments at all.

> a proof [demonstration] which, when supplemented by proofs [demonstrations] of the hypotheses, or antecedents, becomes a proof [demonstration] of the thesis, or consequent.[2] [8]

In 1956, Heyting propounded his version of what has come to be known as the *Brouwer-Heyting-Kolmogorov* interpretation of intuitionistic logic, by explaining the assertion conditions of the propositions. Note that where Heyting says "assert a proposition", in light of Martin-Löf's clarification, we must read "assert *the truth of* a proposition". Heyting's explanation of the assertion conditions for the truth of implication were as follows:

> The implication $\mathfrak{p} \supset \mathfrak{q}$ may be asserted if and only if we possess a construction $\mathfrak{r}$, which, joined to any construction proving $\mathfrak{p}$ (supposing the latter be effected), would automatically effect a construction of $\mathfrak{q}$. [5]

Now, Martin-Löf would probably consider the parenthetical "supposing the latter be effected" to be superfluous, since to assume a judgment is the same as to assume that you know it [8]. So, Heyting's definition might be rewritten today as:

> $P \supset Q$ *true* may be asserted if and only if we possess a construction $\mathfrak{r}$, which, joined to any demonstration of P *true*, would automatically effect a demonstration of Q *true*.

Martin-Löf explained the truth of an implication by appealing to the hypothetical judgment, so we should be able to factor Heyting's explanation through it in a similar way:

> $P \supset Q$ *true* may be asserted if and only if we may assert Q *true* (P *true*).

In fact, if we make this transformation, we shall have arrived at something very similar to Martin-Löf's definition of propositional implication. This inference is, at least, valid with respect to Martin-Löf's definition, but it is merely an extensional specification for the meaning of the judgment: it expresses the material equivalence of the assertions $P \supset Q$ *true* and Q *true* (P *true*), but it does not contain an actual *explanation* of $P \supset Q$ *true*, which would need to be in the form "To know $P \supset Q$ *true* is to know...".

It is tempting to rewrite the definition in the following way:

> (*) To know $P \supset Q$ *true* is to know Q *true* (P *true*).

[2] I prefer the term *demonstration* to the more ambiguous *proof*, since the former is clearly an act, whereas the latter may be read as either an act or as a mathematical object.

However, as a definition, this is impredicative. Following Dummett [2] and Martin-Löf, we must start from a distinction between the direct experience of truth (which we shall call *verification*) and the indirect experience (which we shall just call *truth*). Then, the intention of a proposition is its verification, and the truth of a proposition is the experience of a means of fulfillment for that intention:

To know $P \supset Q$ *ver* is to know Q *true* (P *true*).

Then, to know $P \supset Q$ *true* is to have a means of verifying $P \supset Q$, that is, to have a plan to experience $P \supset Q$ *ver*.

2. The Proof Interpretation

When we rewrote Heyting's explanation of the assertion conditions for the truth of an implication to appeal to hypothetical judgment, we implicitly assumed that the instantiation of Martin-Löf's hypothetical judgment would preserve the meaning of the original statement.

This, however, may be too much to ask, since in modern proof-theoretic accounts of meaning, the hypothetical judgment as explained by Martin-Löf must be understood in a very strong sense, where its proof shall be an object with a "hole" in it, which could be plugged with a proof for the antecedent to yield a proof for the consequent: that is, it is not enough that one should have a way of transforming the proof of the antecedent into a proof of the consequent, but one must have a uniform way to do so.

Anticipating the "proof interpretation" of intuitionistic logic, Brouwer also had come to a conclusion similar to this, if not quite equivalent. For Brouwer, a hypothetical assertion $\mathcal{J}_2$ ($\mathcal{J}_1$) was essentially an assertion of $\mathcal{J}_2$ which proceeds by embedding an actual construction of $\mathcal{J}_1$ into a matrix for a construction of $\mathcal{J}_2$. Now, this is not quite the same since it requires that $\mathcal{J}_1$ be proved, so it corresponds more closely with the notion of *cut* than it does with hypothetical judgment—but he does seem to agree as far as the uniformity of the partial proof of the consequent is concerned.

In case it is not entirely evident, let us reason through what would happen to a *proof theory* if proofs of hypothetical judgments could be non-uniform. For one, it would cease to be a proof theory, since two crucial properties would fail:

(1) Proofs are finitary objects.

(2) It is effectively determinable in finite time whether an object is a proof of a judgment.

If a proof of $\mathcal{J}_2$ $(\mathcal{J}_1)$ were construed as a means of converting proofs of $\mathcal{J}_1$ into proofs of $\mathcal{J}_2$ (as opposed to the proof-with-a-hole interpretation), proofs would certainly cease to be finitary objects: for instance, a proof of $|_n\ \mathcal{J}(n)$ $(n \in \mathbb{N})$ would be infinitely large, in that it would contain one branch for each natural number. Then, the failure of the second property (decidability of wellformedness) is immediate.

Proof-theoretic meaning, however, is hopeless anyway as an interpretation of intuitionistic logic if one has any intention to justify the creating subject [16], or even the bar theorem; indeed, the proof-theoretic reading of hypothetical judgment specifically rules out the kind of non-uniform evidence that is essential in the demonstration of the bar theorem.

3. The Proper Interpretation of Hypothetical Judgment

If we return to Heyting's original definition of the assertion $P \supset Q$ *true*, it is clear that he would have to accept any mathematical means of transforming the construction of P *true* into a construction of Q *true*, since the construction $\mathfrak{r}$ in his definition is not constrained by any uniformity condition: it is merely any effective operation which, when adjoined with a construction of the premise, effects a construction of the conclusion; and moreover, $\mathfrak{r}$ may proceed by appealing to any of the transcendental observations which are possible by virtue of the truth of P having been experienced by the subject.

And what of Martin-Löf? In his type theory, only the introduction rules for the types are given, and the elimination rules are "theorems" (or admissible rules) which are evident only in the non-uniform sense; that is, the justifications of the elimination rules proceed by introspection on the possible ways in which their premises could have been experienced. Therefore, if Martin-Löf's uniform hypothetical judgment is to be accepted, it is necessary that the statement of the elimination rules be effected using something that permits a non-uniform demonstration.

Sundholm and Van Atten [14], for instance, distinguish between a hypothetical proof of $\mathcal{J}_2$ $(\mathcal{J}_1)$ and an inference $\dfrac{\mathcal{J}_1}{\mathcal{J}_2}$ which expresses the closure of mathematics under a rule, and say that it is the latter which establishes the bar theorem, and not the former.

Martin-Löf on the other hand says specifically that an inference is to be read as a proof of a hypothetical judgment, and so no real progress is made:

> The difference between an inference and a logical consequence, or hypothetical judgement, is that an inference is a proof of a logical consequence. Thus an inference is the same as a hypothetical proof. [8]

In light of Martin-Löf's explanations of the elimination rules which follow this statement, we cannot accept his claim that inference is the same as hypothetical proof (as he has defined it), because under that definition even the following is not a valid inference:

$$\frac{P \wedge Q \ \mathit{true}}{P \ \mathit{true}}$$

Why not? This purports to be a proof of the judgment P *true* ($P \wedge Q$ *true*), which is not evident under the uniform explanation of hypothetical judgment. It could be made evident if the definition of a proposition consisted in the declaration of both its introduction rules and its elimination rules, but in fact, only the introduction rules are given, and the elimination rules are simply codifications of common patterns of reasoning from premise to conclusion.

Indeed, Martin-Löf justifies the elimination rules using the non-uniform (material, rather than logical) consequence. The above rule, for instance, is justified as follows:

Proof. If you know $P \wedge Q$ *true*, then you must have a means of verifying $P \wedge Q$, whence you must know both P *true* and Q *true*; the conclusion is now immediate. □

If we are to take the explanation of the elimination rules seriously, then, we must read Martin-Löf as having already at his disposal a kind of hypothetical judgment whose evidence consists in any effective means at all of transforming the demonstration of the premise into a demonstration of the conclusion. So, the use of the uniform hypothetical judgment elsewhere has not relieved us from the need to explain the inference from premise to conclusion in an elimination rule.

Contra Martin-Löf's explanation in the Siena lectures, the interpretation of hypothetical judgment as material consequence is crucial for the semantics of his type theory, as noted by Dybjer [3].

3.1. **What is the difference between an inference rule and a hypothetical judgment?** Is a rule of inference really the same as a consequence or hypothetical judgment, as Martin-Löf claimed? One way to elucidate the differences is to consider them in the context of a Beth or Kripke semantics.

The validity of an inference rule $\frac{\mathcal{J}_1}{\mathcal{J}_2}$ at a world lies in an effective transformation of experiences of $\mathcal{J}_1$ at that world to experiences of $\mathcal{J}_2$ at that world; on the other hand, to experience a hypothetical judgment $\mathcal{J}_2\ (\mathcal{J}_1)$ at a world is to have a means to transform experiences of $\mathcal{J}_1$ at any future world into experiences of $\mathcal{J}_2$.

Construed in this way, judgments can be explained by specifying when/where they are forced; it is a reasonable requirement that if we shall consider $\mathcal{J}$ to be a judgment, then for any worlds $u \preceq v$, from $u \Vdash \mathcal{J}$ we may conclude $v \Vdash \mathcal{J}$ (this is called *monotonicity*). The hypothetical judgment, at least as explained above, preserves the monotonicity of knowledge by definition, whereas rules of inference may not in general satisfy this property.

On the contrary, an "admissible rule" $\frac{\mathcal{J}_1}{\mathcal{J}_2}$ is sensitive to changes in the state of knowledge, and may cease to be valid if a previously unknown way to experience $\mathcal{J}_1$ is found. Such rules may only be construed as hypothetical judgments if the acts specified by the meanings of their premises are sufficiently circumscribed so as to satisfy the monotonicity requirement.

Martin-Löf's identification of the rules of logic with hypothetical judgments, then, only obtains because the concepts of *verification* of a proposition, and (secondarily) *truth* of a proposition are fixed in advance for all time by means of *canonical forms* and *computation* respectively.

4. Realizability and Type Theory

Now that we have settled upon an explanation for hypothetical judgment, let us return to the notions of judgment and proposition, and their respective concepts of "construction". Like "proof", the term "construction" is also ambiguous in that it may refer to an act of constructing, and it may also denote a concrete mathematical object.

A construction for a judgment is simply the act of coming to know it: this is what Martin-Löf calls a *demonstration*, and if it is to be thought of as an object, it is at least a tensed, ephemeral one. On the other hand, a construction for a proposition is a mathematical object, not an experience: it is the object that the subject constructs during the verification of a proposition.

This latter sort of construction is called a *witness*, or, following realizability, a *realizer*.

4.1. **Realizability Models as Unary Logical Relations.** In fact, we can replace the abstract/synthetic explanations of the propositions in terms of their verification acts with new explanations in terms of verification objects (i.e. their canonical witnesses); then, the verification act consists in constructing a verification object.

A proposition is verified just in case there exists a verification object, but it is important to understand that this is not to say that a (possibly unknown) verification object may exist outside the subject's experience (construction) of it. Rather, this is a trivial equivalence, since to say that an object exists is the same as to say that the subject has constructed it.[3]

A realizability model in this simple sense amounts to interpreting the propositions into unary logical relations. To define a proposition, then, is to define the unary relation $\mathcal{V}[\![P]\!]$ (which is the species of verification objects of the proposition P); then, a separate logical relation $\mathcal{E}[\![P]\!]$ is defined uniformly over all propositions P by appealing to the computation $M \Rightarrow N$ of witnesses to canonical form.

$$\frac{A \Rightarrow A' \quad \mathcal{V}[\![A']\!]\ \textit{defined}}{A\ \textit{set}} \qquad \frac{A \Rightarrow A' \quad \mathcal{E}[\![A']\!](M)}{M \in A}$$

$$\begin{aligned}
\mathcal{V}[\![\top]\!] &\equiv \{\star\} \\
\mathcal{V}[\![\bot]\!] &\equiv \{\} \\
\mathcal{V}[\![P \supset Q]\!] &\equiv \{(\lambda x)E \mid |_x\ E \in Q\ (x \in P)\} \\
\mathcal{V}[\![P \wedge Q]\!] &\equiv \{\langle M, N\rangle \mid M \in P, N \in Q\} \\
\mathcal{V}[\![P \vee Q]\!] &\equiv \{\mathsf{inl}(M) \mid M \in P\} \cup \{\mathsf{inr}(M) \mid M \in Q\} \\
\mathcal{E}[\![P]\!] &\equiv \{M \mid M \Rightarrow M', \mathcal{V}[\![P]\!](M')\}
\end{aligned}$$

The material interpretation of the hypothetical judgment is crucial in the realizability model; this is because only the verification objects (i.e. canonical witnesses) are given. All the non-canonical witnesses are explained

[3]The idea that verification or proof objects exist separately from our experience of them is part of the realist ontology which is now espoused by Martin-Löf [10], contrary to his position at the time of the Siena lectures; this view of course cannot be accepted by Brouwerians, who profess a thoroughly idealist ontology [14, 13].

via computation, and the use of logical consequence instead of material consequence in the explanation of $\mathcal{V}[\![P \supset Q]\!]$ would have been disastrous.

For instance, it should be the case that $(\lambda x)\langle\star,\star\rangle$ is a witness of $\bot \supset \top$. To see if this is the case in the model, let us translate this into a concrete statement:

$$(\lambda x)\langle\star,\star\rangle \in \bot \supset \top \tag{4.1}$$

$$(\lambda x)\langle\star,\star\rangle \Rightarrow (\lambda x)\langle\star,\star\rangle,\quad \mathcal{V}[\![\bot \supset \top]\!]((\lambda x)\langle\star,\star\rangle) \tag{4.2}$$

$$|_x\ \langle\star,\star\rangle \in \top\ (x \in \bot) \tag{4.3}$$

$$|_x\ \langle\star,\star\rangle \in \top\ (\mathcal{E}[\![\bot]\!](x)) \tag{4.4}$$

$$|_x\ \langle\star,\star\rangle \in \top\ (x \Rightarrow M, \mathcal{V}[\![\bot]\!](M)) \tag{4.5}$$

And at this time, we may discharge the entire hypothetical judgment, since we know that the unary relation $\mathcal{V}[\![\bot]\!]$ is empty, and so there can be no such M.

If we did not have the material consequence at our disposal, then this statement would not have been valid, unless we were to eschew the verificationist meaning explanation and also add "use" rules (i.e. direct eliminations) for each proposition in addition to the verification rules.

4.2. **Type Theories as Binary Logical Relations.** The unary logical relations express exactly the content of a realizability model, but they still do not yield a theory of sets which is sufficient for reasoning about mathematical objects, which have extensional identity. In order to consider the equality of sets and witnesses, the unary logical relations are replaced with binary ones (sc. partial equivalence relations), as follows:

$$\frac{A \Rightarrow A' \quad B \Rightarrow B' \quad \mathcal{V}[\![A']\!] \equiv \mathcal{V}[\![B']\!]}{A = B\ set} \qquad \frac{A = A\ set}{A\ set}$$

$$\frac{A \Rightarrow A' \quad M \Rightarrow M' \quad N \Rightarrow N' \quad \mathcal{V}[\![A']\!](M', N')}{M = N \in A} \qquad \frac{M = M \in A}{M \in A}$$

In order to make an important point about functionality, we will define intuitionistic existential and universal quantification rather than their special cases, conjunction and implication.

$$\begin{aligned}
\mathcal{V}[\![\top]\!] &\equiv \{(\star,\star)\}\\
\mathcal{V}[\![\bot]\!] &\equiv \{\}\\
\mathcal{V}[\![(\forall x \in A)B]\!] &\equiv \{((\lambda x)E, (\lambda x)E') \mid |_{y,z}\ [y/x]E = [z/x]E' \in [y/x]B\ (y = z \in A)\}\\
\mathcal{V}[\![(\exists x \in A)B]\!] &\equiv \{(\langle M, N\rangle, \langle M', N'\rangle) \mid M = M' \in A, N = N' \in [M/x]B\}\\
\mathcal{V}[\![A \vee B]\!] &\equiv \{(\mathsf{inl}(M), \mathsf{inl}(N)) \mid M = N \in A\} \cup \{(\mathsf{inr}(M), \mathsf{inr}(N)) \mid M = N \in B\}\\
\mathcal{E}[\![A]\!] &\equiv \{(M, N) \mid M \Rightarrow M', N \Rightarrow N', \mathcal{V}[\![A]\!](M', N')\}
\end{aligned}$$

Now, in contrast to the treatment of the quantifiers in formal intuitionistic logic (or in the Mitchell-Bénabou language of a topos [7]), in this setting it is part of their meaning that their verifications should respect the equality of the domain of discourse; this constraint is called functionality, and reflects the fact that universal quantification is reconstructed as a more general form of implication. Within the theory of sets, there is simply not a quantifier which expresses non-functional generality; in this way, contrary to the state of affairs in **BISH** [1], the theorem of choice is in fact verified by a choice *function*, not merely a choice *operation*.

With the extension to the binary logical relation, we now properly treat the equivalence of propositions (sets) and of witnesses, and we also have a definitive answer to the question, "What is the purpose of adding a language of types and witnesses to the existing system of judgments and their demonstrations?"

The judgments and demonstrations are the activity of the subject in performing and experiencing mathematics. On the other hand, the theory of sets that we have defined is, to my mind, the correct level at which to *do* mathematics, where objects are concrete and have an extensional identity. By guaranteeing pervasive functionality, we have embedded in the rich world of intuitionistic mathematics a haven invulnerable to the paradoxes that arise from failures of extensionality, such as Diaconescu's theorem [9, 12].

5. Related Discussion

In the logical framework which forms the basis of *Practical Foundations for Programming Languages* [4], Robert Harper treats both logical consequence $\mathcal{J}_1 \vdash \mathcal{J}_2$ and material consequence $\mathcal{J}_1 \vDash \mathcal{J}_2$, which express derivability and admissiblity respectively.

The material consequence (and its open-ended interpretation as a mapping from demonstrations of the antecedent to demonstrations of the consequent) formed the backbone of Zeilberger, Harper and Licata's work on higher-order focused calculi, which provide a convincing abstract notation for the traces of verification and use acts in a logic which mixes the verificationist and pragmatist meaning explanations [19, 6, 18].

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