THEORIES WITH JUDGEMENT

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A *Theory* is given by a language $\mathcal{L}$, an inductively defined set of judgements $\mathcal{J}$, and an explanation of their meaning $[\![-]\!] : \mathbf{Set}^{\mathcal{J}}$. One such theory is the theory **Nat** of the natural numbers, whose terms are the numerals:

$$\begin{array}{lcl} \mathcal{L}_{\mathbf{Nat}} & ::= & \mathsf{zero} \\ & | & \mathsf{succ}\ \langle \mathcal{L}_{\mathbf{Nat}} \rangle \end{array}$$

The theory **Nat** has only a single form of judgement, which asserts that a term is a natural number.

$$\mathcal{J}_{\mathbf{Nat}} = \{ n \text{ nat} \mid n : \mathcal{L}_{\mathbf{Nat}} \}$$

The judgement is then interpreted over the syntax recursively:

$$[\![\mathsf{zero}\ \text{nat}]\!]_{\mathbf{Nat}} = \top$$
$$[\![\mathsf{succ}\ n\ \text{nat}]\!]_{\mathbf{Nat}} = [\![n\ \text{nat}]\!]_{\mathbf{Nat}}$$

Going forward, we'll equivalently present the judgements and their interpretations in terms of "canonical constructors" in the ambient metalanguage, as follows:

$$(-\ \text{nat}) : \mathcal{J}_{\mathbf{Nat}}^{\mathcal{L}_{\mathbf{Nat}}}$$

$$\frac{}{\mathsf{zero}\ \text{nat}} \qquad \frac{n\ \text{nat}}{\mathsf{succ}\ n\ \text{nat}}$$

Now, this is not a particularly interesting theory, since its single form of judgement is true at all instantiations, but it served to illustrate the construction of a theory with judgement.

A more interesting theory is that of names, **Nm**; we take $\mathcal{L}_{\mathbf{Nm}}$ to be the countably infinite set of strings of letters *abc*... Then, the judgements are given as follows:

(Apartness) $$(- \# -) : \mathcal{J}_{\mathbf{Nm}}^{\mathcal{L}_{\mathbf{Nm}} \times \mathcal{L}_{\mathbf{Nm}}}$$

We can take the interpretation of the apartness judgement as primitive.

Now we can define the theory of contexts of assumptions over some other theory **T**, which we will call **Ctx**[**T**]; we introduce the following syntax and judgements:

$$\begin{array}{lcl} \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} & ::= & \cdot \\ & | & \left\langle \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} \right\rangle, \langle \mathcal{L}_{\mathbf{Nm}} \rangle : \langle \mathcal{J}_{\mathbf{T}} \rangle \end{array}$$

$$(-\ \text{ctx}) : \mathcal{J}_{\mathbf{Ctx}[\mathbf{T}]}^{\mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]}}$$
$$(- \notin -) : \mathcal{J}_{\mathbf{Ctx}[\mathbf{T}]}^{\mathcal{L}_{\mathbf{Nm}} \times \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]}}$$
$$(- \ni - : -) : \mathcal{J}_{\mathbf{Ctx}[\mathbf{T}]}^{\mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} \times \mathcal{L}_{\mathbf{Nm}} \times \mathcal{J}_{\mathbf{T}}}$$

And the meanings of these judgements are given inductive-recursively in terms of the following rules:

$$\frac{}{\cdot\ \mathrm{ctx}} \qquad \frac{\Gamma\ \mathrm{ctx} \quad x \notin \Gamma}{\Gamma, x : J\ \mathrm{ctx}}\ (x \in \mathcal{L}_{\mathbf{Nm}})$$

$$\frac{}{x \notin \cdot}\ (x \in \mathcal{L}_{\mathbf{Nm}}) \qquad \frac{x \notin \Gamma \quad x \mathrel{\#} y}{x \notin (\Gamma, y : J)}\ (x, y \in \mathcal{L}_{\mathbf{Nm}})$$

$$\frac{}{\Gamma, x : J \ni x : J}\ (x \in \mathcal{L}_{\mathbf{Nm}}) \qquad \frac{\Gamma \ni y : J'}{\Gamma, x : J \ni y : J'}\ (x, y \in \mathcal{L}_{\mathbf{Nm}})$$

Finally, we can iterate the process in order to get a theory **IPC** of intuitionistic propositional logic, which introduces a notion of hypothetical judgement: but note that we did not need to include that as part of the general framework, since we have defined it internally in the theory of contexts above.

$$\begin{array}{rcl} \mathcal{L}_{\mathbf{IPC}} & ::= & \bot \\ & \mid & \top \\ & \mid & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \wedge \langle \mathcal{L}_{\mathbf{IPC}} \rangle \\ & \mid & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \vee \langle \mathcal{L}_{\mathbf{IPC}} \rangle \\ & \mid & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \supset \langle \mathcal{L}_{\mathbf{IPC}} \rangle \end{array}$$

The theory has a single form of judgement,

$$-\ \mathrm{true}\ [-] : \mathcal{J}_{\mathbf{IPC}}^{\mathcal{L}_{\mathbf{IPC}} \times \mathcal{L}_{\mathbf{Ctx}[\mathbf{IPC}]}}$$

The definition of the theory **IPC** is circular, but not viciously so; it is inductive-recursive. Then, the meaning of this judgement is given by the following rules:

$$\frac{}{\top\ \mathrm{true}\ [\Gamma]} \qquad \frac{\bot\ \mathrm{true}\ [\Gamma]}{P\ \mathrm{true}\ [\Gamma]}$$

$$\frac{P\ \mathrm{true}\ [\Gamma] \quad Q\ \mathrm{true}\ [\Gamma]}{P \wedge Q\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \wedge Q\ \mathrm{true}\ [\Gamma] \quad R\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]; y : Q\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]]]}{R\ \mathrm{true}\ [\Gamma]}$$

$$\frac{P \vee Q\ \mathrm{true}\ [\Gamma]}{P\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \vee Q\ \mathrm{true}\ [\Gamma]}{Q\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \vee Q\ \mathrm{true}\ [\Gamma] \quad R\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]] \quad R\ \mathrm{true}\ [\Gamma, x : Q\ \mathrm{true}\ [\Gamma]]}{R\ \mathrm{true}\ [\Gamma]}$$

$$\frac{Q\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]]}{P \supset Q\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \supset Q\ \mathrm{true}\ [\Gamma] \quad P\ \mathrm{true}\ [\Gamma]}{Q\ \mathrm{true}\ [\Gamma]}$$

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