THEORIES WITH JUDGEMENT

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A *Theory* is given by a language $\mathcal{L}$, an inductively defined set of judgements $\mathcal{J}$, and an explanation of their meaning $[\![-]\!] : \mathbf{Set}^{\mathcal{J}}$. One such theory is the theory **Nat** of the natural numbers, whose terms are the numerals:

$$\begin{array}{lcl} \mathcal{L}_{\mathbf{Nat}} & ::= & \mathsf{zero} \\ & \mid & \mathsf{succ}\ \langle \mathcal{L}_{\mathbf{Nat}} \rangle \end{array}$$

The theory **Nat** has only a single form of judgement, which asserts that a term is a natural number.

$$\mathcal{J}_{\mathbf{Nat}} = \{ n \text{ nat} \mid n : \mathcal{L}_{\mathbf{Nat}} \}$$

The judgement is then interpreted over the syntax recursively:

$$\begin{aligned} [\![\mathsf{zero} \text{ nat}]\!]_{\mathbf{Nat}} &= \top \\ [\![\mathsf{succ}\ n \text{ nat}]\!]_{\mathbf{Nat}} &= [\![n \text{ nat}]\!]_{\mathbf{Nat}} \end{aligned}$$

Going forward, we'll equivalently present the judgements and their interpretations in terms of "canonical constructors" in the ambient metalanguage, as follows:

$$(- \text{ nat}) : \mathcal{J}_{\mathbf{Nat}}^{\mathcal{L}_{\mathbf{Nat}}}$$

$$\frac{}{\mathsf{zero} \text{ nat}} \qquad \frac{n \text{ nat}}{\mathsf{succ}\ n \text{ nat}}$$

Now, this is not a particularly interesting theory, since its single form of judgement is true at all instantiations, but it served to illustrate the construction of a theory with judgement.

A more interesting theory is that of names, **Nm**; we take $\mathcal{L}_{\mathbf{Nm}}$ to be the countably infinite set of strings of letters $\{\mathsf{a}, \mathsf{b}, \mathsf{c} \ldots\}$. Note that we refer to a canonical name in sans serif font, whereas we quantify schematically over variables $x : \mathcal{L}_{\mathbf{Nm}}$ in *italic* font. Then, the judgements are given as follows:

(Apartness) $$(- \# -) : \mathcal{J}_{\mathbf{Nm}}^{\mathcal{L}_{\mathbf{Nm}} \times \mathcal{L}_{\mathbf{Nm}}}$$

We can take the interpretation of the apartness judgement as primitive.

Now we can define the theory of contexts of assumptions over some other theory $\mathbf{T}$, which we will call $\mathbf{Ctx}[\mathbf{T}]$; we introduce the following syntax and judgements:

$$\begin{array}{lcl} \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} & ::= & \cdot \\ & \mid & \left\langle \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} \right\rangle, \langle \mathcal{L}_{\mathbf{Nm}} \rangle : \langle \mathcal{J}_{\mathbf{T}} \rangle \end{array}$$

$$
\begin{aligned}
(-\ \mathrm{ctx}) &: \mathcal{J}_{\mathbf{Ctx[T]}}^{\mathcal{L}_{\mathbf{Ctx[T]}}} \\
(- \notin -) &: \mathcal{J}_{\mathbf{Ctx[T]}}^{\mathcal{L}_{\mathbf{Nm}} \times \mathcal{L}_{\mathbf{Ctx[T]}}} \\
(- \ni - : -) &: \mathcal{J}_{\mathbf{Ctx[T]}}^{\mathcal{L}_{\mathbf{Ctx[T]}} \times \mathcal{L}_{\mathbf{Nm}} \times \mathcal{J}_{\mathbf{T}}} \\
(- \leq -) &: \mathcal{J}_{\mathbf{Ctx[T]}}^{\mathcal{L}_{\mathbf{Ctx[T]}} \times \mathcal{L}_{\mathbf{Ctx[T]}}}
\end{aligned}
$$

And the meanings of these judgements are given inductive-recursively in terms of the following rules:

$$
\frac{}{\cdot\ \mathrm{ctx}} \qquad \frac{\Gamma\ \mathrm{ctx} \quad x \notin \Gamma}{\Gamma, x : J\ \mathrm{ctx}}
$$

$$
\frac{}{x \notin \cdot} \qquad \frac{x \notin \Gamma \quad x \# y}{x \notin (\Gamma, y : J)}
$$

$$
\frac{}{\Gamma, x : J \ni x : J} \qquad \frac{\Gamma \ni y : J'}{\Gamma, x : J \ni y : J'}
$$

$$
\frac{}{\cdot \leq \Gamma} \qquad \frac{\Delta \leq \Gamma \quad \Gamma \ni x : J}{\Delta, x : J \leq \Gamma}
$$

Finally, we can iterate the process in order to get a theory **IPC** of intuitionistic propositional logic, which introduces a notion of hypothetical judgement: but note that we did not need to include that as part of the general framework, since we have defined it internally in the theory of contexts above.

$$
\begin{array}{lcl}
\mathcal{L}_{\mathbf{IPC}} & ::= & \bot \\
& | & \top \\
& | & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \wedge \langle \mathcal{L}_{\mathbf{IPC}} \rangle \\
& | & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \vee \langle \mathcal{L}_{\mathbf{IPC}} \rangle \\
& | & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \supset \langle \mathcal{L}_{\mathbf{IPC}} \rangle
\end{array}
$$

The theory has a single form of judgement,

$$
-\ \mathrm{true}\ [-] : \mathcal{J}_{\mathbf{IPC}}^{\mathcal{L}_{\mathbf{IPC}} \times \mathcal{L}_{\mathbf{Ctx[IPC]}}}
$$

The definition of the theory **IPC** is circular, but not viciously so; it is inductive-recursive. Then, the meaning of this judgement is given by the following rules:

$$\frac{}{\top \text{ true } [\Gamma]} \qquad \frac{\bot \text{ true } [\Gamma]}{P \text{ true } [\Gamma]}$$

$$\frac{P \text{ true } [\Gamma] \quad Q \text{ true } [\Gamma]}{P \wedge Q \text{ true } [\Gamma]} \qquad \frac{P \wedge Q \text{ true } [\Gamma] \quad R \text{ true } [\Gamma, x : P \text{ true } [\Gamma]; y : Q \text{ true } [\Gamma, x : P \text{ true } [\Gamma]]]}{R \text{ true } [\Gamma]}$$

$$\frac{P \vee Q \text{ true } [\Gamma]}{P \text{ true } [\Gamma]} \qquad \frac{P \vee Q \text{ true } [\Gamma]}{Q \text{ true } [\Gamma]} \qquad \frac{P \vee Q \text{ true } [\Gamma] \quad R \text{ true } [\Gamma, x : P \text{ true } [\Gamma]] \quad R \text{ true } [\Gamma, x : Q \text{ true } [\Gamma]]}{R \text{ true } [\Gamma]}$$

$$\frac{Q \text{ true } [\Gamma, x : P \text{ true } [\Gamma]]}{P \supset Q \text{ true } [\Gamma]} \qquad \frac{P \supset Q \text{ true } [\Gamma] \quad P \text{ true } [\Gamma]}{Q \text{ true } [\Gamma]}$$

$$\frac{\Gamma \ni x : P \text{ true } [\Delta] \quad \Delta \leq \Gamma}{P \text{ true } [\Gamma]}$$

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