THEORIES WITH JUDGEMENT

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A *Theory* is given by a language $\mathcal{L}$, an inductively defined set of judgements $\mathcal{J}$, and an explanation of their meaning $[\![-]\!] : \mathbf{Set}^{\mathcal{J}}$. One such theory is the theory **Nat** of the natural numbers, whose terms are the numerals:

$$\begin{array}{rcl} \mathcal{L}_{\mathbf{Nat}} & ::= & \mathsf{zero} \\ & \mid & \mathsf{succ}\ \langle \mathcal{L}_{\mathbf{Nat}} \rangle \end{array}$$

The theory **Nat** has only a single form of judgement, which asserts that a term is a natural number.

$$\mathcal{J}_{\mathbf{Nat}} = \{n \text{ nat} \mid n : \mathcal{L}_{\mathbf{Nat}}\}$$

The judgement is then interpreted over the syntax recursively:

$$[\![\mathsf{zero}\text{ nat}]\!]_{\mathbf{Nat}} = \top$$
$$[\![\mathsf{succ}\ n\text{ nat}]\!]_{\mathbf{Nat}} = [\![n\text{ nat}]\!]_{\mathbf{Nat}}$$

Going forward, we'll equivalently present the judgements and their interpretations in terms of "canonical constructors" in the ambient metalanguage, as follows:

$$(-\text{ nat}) : \mathcal{J}_{\mathbf{Nat}}^{\mathcal{L}_{\mathbf{Nat}}}$$

$$\frac{}{\mathsf{zero}\text{ nat}} \qquad \frac{n\text{ nat}}{\mathsf{succ}\ n\text{ nat}}$$

Now, this is not a particularly interesting theory, since its single form of judgement is true at all instantiations, but it served to illustrate the construction of a theory with judgement.

A more interesting theory is that of names, **Nm**; we take $\mathcal{L}_{\mathbf{Nm}}$ to be the countably infinite set of strings of letters *abc*... Then, the judgements are given as follows:

(Apartness) $$(- \# -) : \mathcal{J}_{\mathbf{Nm}}^{\mathcal{L}_{\mathbf{Nm}} \times \mathcal{L}_{\mathbf{Nm}}}$$

We can take the interpretation of the apartness judgement as primitive.

Now we can define the theory of contexts of assumptions over some other theory **T**, which we will call **Ctx**[**T**]; we introduce the following syntax and judgements:

$$\begin{array}{rcl} \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} & ::= & \cdot \\ & \mid & \left\langle \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} \right\rangle, \langle \mathcal{L}_{\mathbf{Nm}} \rangle : \langle \mathcal{J}_{\mathbf{T}} \rangle \end{array}$$

$$(-\text{ ctx}) : \mathcal{J}_{\mathbf{Ctx}[\mathbf{T}]}^{\mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]}}$$
$$(- \notin -) : \mathcal{J}_{\mathbf{Ctx}[\mathbf{T}]}^{\mathcal{L}_{\mathbf{Nm}} \times \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]}}$$
$$(- \ni - : -) : \mathcal{J}_{\mathbf{Ctx}[\mathbf{T}]}^{\mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} \times \mathcal{L}_{\mathbf{Nm}} \times \mathcal{J}_{\mathbf{T}}}$$
$$(- \leq -) : \mathcal{J}_{\mathbf{Ctx}[\mathbf{T}]}^{\mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]} \times \mathcal{L}_{\mathbf{Ctx}[\mathbf{T}]}}$$

And the meanings of these judgements are given inductive-recursively in terms of the following rules:

$$\frac{}{\cdot\ \mathrm{ctx}} \qquad \frac{\Gamma\ \mathrm{ctx} \quad x \notin \Gamma}{\Gamma, x : J\ \mathrm{ctx}}$$

$$\frac{}{x \notin \cdot} \qquad \frac{x \notin \Gamma \quad x \# y}{x \notin (\Gamma, y : J)}$$

$$\frac{}{\Gamma, x : J \ni x : J} \qquad \frac{\Gamma \ni y : J'}{\Gamma, x : J \ni y : J'}$$

$$\frac{}{\cdot \leq \Gamma} \qquad \frac{\Delta \leq \Gamma \quad \Gamma \ni x : J}{\Delta, x : J \leq \Gamma}$$

Finally, we can iterate the process in order to get a theory **IPC** of intuitionistic propositional logic, which introduces a notion of hypothetical judgement: but note that we did not need to include that as part of the general framework, since we have defined it internally in the theory of contexts above.

$$\begin{array}{rcl} \mathcal{L}_{\mathbf{IPC}} & ::= & \bot \\ & | & \top \\ & | & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \land \langle \mathcal{L}_{\mathbf{IPC}} \rangle \\ & | & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \lor \langle \mathcal{L}_{\mathbf{IPC}} \rangle \\ & | & \langle \mathcal{L}_{\mathbf{IPC}} \rangle \supset \langle \mathcal{L}_{\mathbf{IPC}} \rangle \end{array}$$

The theory has a single form of judgement,

$$- \ \mathrm{true}\ [-] : \mathcal{J}_{\mathbf{IPC}}^{\mathcal{L}_{\mathbf{IPC}} \times \mathcal{L}_{\mathbf{Ctx[IPC]}}}$$

The definition of the theory **IPC** is circular, but not viciously so; it is inductive-recursive. Then, the meaning of this judgement is given by the following rules:

$$\frac{}{\top\ \mathrm{true}\ [\Gamma]} \qquad \frac{\bot\ \mathrm{true}\ [\Gamma]}{P\ \mathrm{true}\ [\Gamma]}$$

$$\frac{P\ \mathrm{true}\ [\Gamma] \quad Q\ \mathrm{true}\ [\Gamma]}{P \land Q\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \land Q\ \mathrm{true}\ [\Gamma] \quad R\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]; y : Q\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]]]}{R\ \mathrm{true}\ [\Gamma]}$$

$$\frac{P \lor Q\ \mathrm{true}\ [\Gamma]}{P\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \lor Q\ \mathrm{true}\ [\Gamma]}{Q\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \lor Q\ \mathrm{true}\ [\Gamma] \quad R\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]] \quad R\ \mathrm{true}\ [\Gamma, x : Q\ \mathrm{true}\ [\Gamma]]}{R\ \mathrm{true}\ [\Gamma]}$$

$$\frac{Q\ \mathrm{true}\ [\Gamma, x : P\ \mathrm{true}\ [\Gamma]]}{P \supset Q\ \mathrm{true}\ [\Gamma]} \qquad \frac{P \supset Q\ \mathrm{true}\ [\Gamma] \quad P\ \mathrm{true}\ [\Gamma]}{Q\ \mathrm{true}\ [\Gamma]}$$

$$\frac{\Gamma \ni x : P\ \mathrm{true}\ [\Delta] \quad \Delta \leq \Gamma}{P\ \mathrm{true}\ [\Gamma]}$$

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