Tutorial on the Toposes and Names

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This is intended to be an elementary tutorial on how atomic sites give rise to a form of *nominal indexing* which is useful for describing languages with *symbols* as in Harper's *Practical Foundations for Programming Languages* [4]. There are nearly always numerous different ways to define a mathematical object; in this tutorial, I prefer to optimize for minimal combinatorial complexity of definitions.

Texts in category theory are usually written for mathematicians who are already competent in areas of math which are far more complex than category theory; as a result, it is common for a virtue to be made of immediately unfolding an object into a purely analytic muddle.

Because this tutorial is meant to be useful for type theorists and logicians, I prefer a style based on abstract and synthetic definitions built up from well-understood and semantically rich components, as opposed to the more common "elementary" definitions which are more difficult to motivate and understand without having first consumed the common prerequisites (e.g. classical topology, algebraic topology, algebraic geometry, order theory, etc.).

To learn toposes and sheaves thoroughly, the reader is directed to study Mac Lane and Moerdijk's excellent and readable *Sheaves in Geometry and Logic* [6]; this tutorial serves only as a *supplement* to the existing material, intended to suggest intuitions to type theorists who, like the author, may not have a sufficiently broad mathematical background to build up the usual intuitions.

1 Toposes for concepts in motion

Do mathematical *concepts* vary over time and space? This question is the fulcrum on which the contradictions between the competing ideologies of mathematics rest. Let us review their answers:

Platonism No.

Constructivism Maybe.

Intuitionism Necessarily, but space is just an abstraction of time.

Vulgar constructivism No.[1]

[1] I mean, of course, the Markov school.

Brouwer's radical intuitionism was the first conceptualization of mathematical activity which took a *positive* position on this question; the incompatibility of intuitionism with classical mathematics amounts essentially to the fact that they take opposite positions as to the existence of mathematical objects varying over time.

Constructivism, as exemplified by Bishop [1] takes a more moderate position: we can neither confirm nor deny the variable character of mathematical concepts. In this way, mathematics in Bishop's sense is simultaneously the mathematics of *all* forms of variation, including the chaotic (classical) form.

This dispute has been partly trivialized under the unifying perspective of *toposes*,[2] which allow the scientific study of mathematical systems and their relationships, including Platonism (the *category of sets*), constructivism (the *free topos*), intuitionism (the *topos of sheaves over the Baire space*) and vulgar constructivism (the *effective topos*).

Toposes have both a *geometric* and a *logical* character; the geometric aspect was the first to be developed, in the form of *Grothendieck toposes*, which are universes of sets which vary continuously over some (generalized) form of space. More generally, the *logical* aspect of topos theory is developed in Lawvere and Tierney's notion of an elementary topos, an abstract and axiomatic generalization of Grothendieck's concept.

These two aspects of topos theory go hand-in-hand: whilst the laws of an elementary topos are often justified by appealing to their realization in a Grothendieck topos, it is often easier understand the complicated and fully analytic definitions of objects in a Grothendieck topos by relating them to their logical counterparts. We will try and appeal to both the geometric and the logical intuitions in this tutorial.

2 Presheaves and presheaf toposes

Presheaves are the simplest way to capture mathematical objects which vary over a category $\mathbb{C} : \mathbf{Cat}$.

Definition 2.1 (Presheaf)**.** A presheaf on $\mathbb{C} : \mathbf{Cat}$ is a functor $\mathsf{P} : \mathbb{C}^{\mathsf{op}} \longrightarrow \mathbf{Set}$.

Unfolding definitions, this means that for every object $\Psi : \mathbb{C}$, we have a set $\mathsf{P}(\Psi) : \mathbf{Set}$; moreover, for any morphism $\psi : \mathsf{Hom}_{\mathbb{C}}(\Phi, \Psi)$, we have an induced *restriction* map $\mathsf{P}(\psi) : \mathsf{P}(\Psi) \longrightarrow \mathsf{P}(\Phi)$ which preserves identities and composition. The presheaves on $\mathbb{C}$ form a category, which we will call $\widehat{\mathbb{C}} : \mathbf{Cat}$. When the presheaf $\mathsf{P} : \widehat{\mathbb{C}}$ is understood from context, we will write $\mathfrak{m} \cdot \psi : \mathsf{P}(\Phi)$ for $\mathsf{P}(\psi)(\mathfrak{m})$.

Definition 2.2 (Yoneda embedding)**.** We have a full and faithful functor $\mathcal{H} : \mathbb{C} \to \widehat{\mathbb{C}}$ called the *Yoneda embedding*, which is defined as follows:

$$\mathcal{H}_{\Psi} \triangleq \mathsf{Hom}_{\mathbb{C}}(-, \Psi)$$

A functor which is isomorphic to $\mathcal{H}_{\Psi}$ is called *representable* by Ψ.

[2] That is, if we are content to temporarily ignore the matter of *predicativity*; in practice, this can be dealt with using various notions of *pretopos*.

Definition 2.3 (Sieve)**.** A sieve on an object $\Psi : \mathbb{C}$ is a subfunctor of the presheaf represented by Ψ, i.e. $S \sqsubseteq \mathcal{H}_\Psi$.[3] In particular, a sieve on Ψ picks out *functorially* a collection of arrows ending in Ψ; the *maximal sieve* is $\mathcal{H}_\Psi$ itself, which chooses all arrows ending in Ψ.

Accordingly, will write $S\ \mathit{sieve}_\Psi$ to mean $S \sqsubseteq \mathcal{H}_\Psi$.

2.1 Sieves and subobjects

In classical set theory, every subset $A \subseteq B$ has a *characteristic function* $\chi_A : B \longrightarrow \mathbf{2}$, defined as follows:

$$\chi_A(b) \triangleq \begin{cases} 0 & \textbf{if } b \in A \\ 1 & \textbf{if } b \notin A \end{cases}$$

In toposes, there is always an object analogous to $\mathbf{2}$, called the *subobject classifier*; this object is always written Ω. Following [6], a subobject classifier is a monomorphism $\mathsf{true} : \mathbf{1} \rightarrowtail \Omega$ which for any monomorphism $f : A \rightarrowtail B$ induces a characteristic morphism $\chi_f : B \longrightarrow \Omega$ such that the following diagram is a pullback:

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "f"'] & \mathbf{1} \arrow[d, tail, "\mathsf{true}"] \\ B \arrow[r, dashed, "\chi_f"'] & \Omega \end{tikzcd}$$

In the category of sets, the subobject classifier is simply the two-element set; its construction in a presheaf topos is more complicated, essentially because it must be made to respect the fact that the objects under consideration are “in motion”.

The subobject classifier in a presheaf topos The subobject classifier in a presheaf topos is defined using *sieves*:

$$\Omega(\Psi) \triangleq \{ S \mid S\ \mathit{sieve}_\Psi \}$$
$$\mathsf{true}_\Psi \triangleq \mathcal{H}_\Psi$$

Remark 2.4*.* It may not be immediately clear why the subobject classifier is defined in this way: what does the collection of sieves have to do with $\mathbf{2}$ in (classical) set theory? One way to understand what is happening is to observe how Ω behaves when our base category $\mathbb{C}$ is *discrete*, in the sense that the only arrows are identities. If $\mathbb{C}$ is discrete, then $\mathcal{H}_\Psi(\Phi) = \mathbf{1}$ for every $\Phi : \mathbb{C}$; therefore, the judgment $S\ \mathit{sieve}_\Psi$ comes out to mean simply $S(\Psi) \subseteq \mathbf{1}$, i.e. $S(\Psi) \in \mathcal{P}(\mathbf{1})$. Therefore, $\Omega(\Psi) = \mathcal{P}(\mathbf{1}) = \mathbf{2}$. An analogous argument can be made in case $\mathbb{C}$ is *chaotic*, or even a groupoid.

[3]Usually, an alternative definition is given in terms of “sets of arrows closed under precomposition”, but we prefer a definition with fewer moving parts. In practice it will be useful to use the alternative definition when reasoning.

Understanding Ω using Yoneda's Lemma as a weapon Much like how a candidate construction of the exponential in a functor category can be hypothesized using an insight from the Yoneda Lemma, it is also possible to apply the same technique to the construction of the subobject classifier in a presheaf category.

Based on our intention that Ω shall be a construction of the subobject classifier, we want to identity maps $\phi : X \to \Omega$ with subobjects of $X : \widehat{\mathbb{C}}$, i.e. we intend to exhibit a bijection $[X, \Omega] \cong \mathsf{Sub}(X)$. Now, cleverly choose $X \triangleq \mathcal{H}_U$: then we have $[U, \Omega] \cong \mathsf{Sub}(X)$. But the Yoneda Lemma says that $[\mathcal{H}_U, \Omega] \cong \Omega(U)$! Therefore, we may take as a scientific hypothesis the definition $\Omega(U) \triangleq \mathsf{Sub}(\mathcal{H}_U)$. It remains to show that this definition exhibits the correct properties (exercise for the reader).

3 Generalized topologies

Let us remark that so far, we have described via presheaves a notion of *variable set* which requires only functoriality; in case we are varying over a poset, this corresponds to *monotonicity* in Kripke models. We will now consider a notion of set which varies *continuously*, a property which corresponds to *local character* in Beth models.

The definition of a Grothendieck topology is quite complicated, but we will show how to understand it conceptually using the logical perspective that we alluded to in the introduction.

Definition 3.1 (Grothendieck Topology [6])**.** A Grothendieck topology is, for each object $\Psi : \mathbb{C}$, a collection $J(\Psi)$ of sieves on Ψ; a sieve $S \in J(\Psi)$ is called a *covering sieve*. To be called a topology, the predicate J must be closed under the following rules:

$$\frac{}{\mathcal{H}_\Psi \in J(\Psi)}\ \textit{maximality} \qquad \frac{S \in J(\Psi) \quad \psi : \mathsf{Hom}_{\mathbb{C}}(\Phi, \Psi)}{S \cdot \psi \in J(\Phi)}\ \textit{stability}$$

$$\frac{S \in J(\Psi) \quad R\ \textit{sieve}_\Psi \quad \forall \psi \in S(\Phi).\ R \cdot \psi \in J(\Phi)}{R \in J(\Psi)}\ \textit{transitivity}$$

The above rules seem fairly poorly-motivated at first; however, it is easy to understand their purpose when one considers the *logical* perspective. First, one should recognize that the *stability* law above is a disguised form of functoriality for J: that is, it ensures that J itself be a presheaf, namely, a subobject of Ω.

Now, every subobject induces a characteristic map into Ω, and it turns out that it will be far more informative to ignore the geometric aspects of J and focus only on the properties of its characteristic map $j : \Omega \to \Omega \triangleq \chi_J$:

$$\begin{tikzcd}
J \arrow[r, "!"] \arrow[d, tail] & \mathbf{1} \arrow[d, tail, "\mathsf{true}"] \\
\Omega \arrow[r, dashed, "j \triangleq \chi_J"'] & \Omega
\end{tikzcd}$$

3.1 The logical view

Definition 3.2 (Lawvere-Tierney Operator)**.** The characteristic map $j : \Omega \to \Omega$ of the Grothendieck Topology $J \rightarrowtail \Omega$ is called a *Lawvere-Tierney topology / local operator* [6] or a *nucleus* [5], and exhibits the following characteristics:

$$j \circ \mathsf{true} = \mathsf{true} \tag{3.1}$$

$$j \circ j = j \tag{3.2}$$

$$j \circ \wedge = \wedge \circ (j \times j) \tag{3.3}$$

In other words, j is an idempotent cartesian monad in the internal logic of the topos. There is, however, a better-motivated way to state these laws which makes more sense from a logical perspective; in particular, axiom 3.3 can be replaced with the more intuitive *monotonicity* condition that j shall preserve entailment:

$$p \leq q \Rightarrow j(p) \leq j(q) \tag{3.3*}$$

Proof. ($\Rightarrow$) Suppose $j \circ \wedge = \wedge \circ (j \times j)$ and $p \leq q$. We need to show that $j(p) \leq j(q)$; first, observe that it suffices to show that $j(p) \wedge j(q) = j(p)$. Because j preserves meets, this is the same as $j(p \wedge q) = j(p)$; but $p \wedge q = p$, because $p \leq q$.[4]

($\Leftarrow$) Suppose that j preserves entailment; we need to show that it preserves meets. To show that $j(p \wedge q) \leq j(p) \wedge j(q)$, it suffices to show both $j(p \wedge q) \leq j(p)$ and $j(p \wedge q) \leq j(q)$; these obtain respectively from the fact that $p \wedge q \leq p$ and $p \wedge q \leq q$. Next, we have to show that $j(p) \wedge j(q) \leq j(p \wedge q)$; this follows from idempotency and monotonicity. ☐

Relating the logical and the geometric views Each of the rules for a Grothendieck topology corresponds to an intuitive logical requirement: *maximality*, i.e. the inclusion of represented functors as covers, corresponds to the requirement that our local operator shall preserve truth; *stability* corresponds to the requirement that J shall in fact be a presheaf; the *transitivity* law corresponds exactly to axioms 3.2 and 3.3*, composed to form the Kleisli extension for the monad j.

Remark 3.3 (Pretopologies and coverages). There are several other ways to define some form of topology on a category, including *coverages* and *pretopologies*. In some contexts, these are allegedly easier to work with, but they tend to impose extra requirements on the category $\mathbb{C}$, and end up obscuring the crisp logical character of topologies and their correspondence with modal operators. From a logical perspective, the concept of a "pretopology" is essentially meaningless, so we prefer to avoid it.

4 Sheaves on a site

A *site* is a category $\mathbb{C}$ together with a topology $J \rightarrowtail \Omega : \widehat{\mathbb{C}}$. We will now proceed to give perspicuous definitions of what it means for a presheaf $P : \widehat{\mathbb{C}}$ to be *separated* and a *sheaf* respectively.

[4]Thanks to Danny Gratzer for suggesting this argument.

There are many different definitions of separated presheaves and sheaves, most of which involve a number of complicated analytic conditions, or that some diagram shall commute; we prefer to give an equivalent, simpler definition (which is usually presented as a *theorem*).

First, observe that for any S *sieve*$_{\Psi}$, we have a canonical function between hom-sets $i_S^* : [\mathcal{H}_\Psi, P] \longrightarrow [S, P]$ as follows:

$$\begin{tikzcd} \mathcal{H}_\Psi \arrow[r, "m"] & P \\ S \arrow[u, tail, "i_S"] \arrow[ur, dashed, "i_S^* m"'] & \end{tikzcd}$$

Definition 4.1. The presheaf P is *separated* iff for every S *sieve*$_{\Psi}$, the induced map between hom-sets $i_S^* : [\mathcal{H}_\Psi, P] \longrightarrow [S, P]$ is a monomorphism. P is a *sheaf* iff this map is also an isomorphism.

It is worth taking a moment to cultivate some insight as to what is going on here. First, recall that by the Yoneda lemma, we can identify elements $P(\Psi)$ with natural transformations from the maximal sieve, i.e. $[\mathcal{H}_\Psi, P]$. So, sheafhood for P is really saying that as far as P is concerned, the elements of $P(\Psi)$ can be identified with natural transformations from *any* sieve that covers Ψ, not just the maximal one.

Definition 4.2 (Matching families and amalgamations)**.** A natural transformation $m : [S, P]$ for $S \in J(\Psi)$ is usually called a *matching family* for S; then, the member of $P(\Psi)$ which is determined by the sheaf-induced isomorphism (and the Yoneda lemma) is called an *amalgamation*.

5 The Schanuel Topos

The purpose of this tutorial was originally to introduce the Schanuel topos, a presentation of the category of nominal sets [7] as a Grothendieck topos.

The syntax and semantics of *local names* is most popularly dealt with via nominal sets. In short, an ambient set of names $\mathbb{A}$ is fixed, letting **Perm** be the group of permutations of subsets of $\mathbb{A}$; then, the topos of nominal sets is got by taking the subcategory of the category of **Perm**-sets which obey a condition called *finite support*.

A more intuitive way to work with objects indexed in local names is to consider an alternative presentation of nominal sets as a particular Grothendieck topos, called the Schanuel topos. This presentation has the advantage of being similar in form to standard methods for developing syntax with binding; when *combined* with the latter, the sheaf-based presentation of nominal sets leads to greater harmony.

Definition 5.1 (The category of nominal contexts)**.** We will begin by defining a category of *nominal contexts and renamings*; typically a covariant presentation based on finite sets and injections is given, but it will be clearer and more suitable for generalization to work with a contravariant presentation. Let $\mathbb{I}$ be the *free strict semi-cartesian category* generated by a single object $\mathbf{nom} : \mathbb{I}$; that is to say, every object of $\mathbb{I}$ is a finite power of **nom** with $\mathbf{1} = \mathbf{nom}^0$, and we have a unique map $! : \mathbf{nom}^n \longrightarrow \mathbf{1}$.

Remark 5.2. The opposite category $\mathbb{I}^{\mathrm{op}}$ is the same as **Inj**, the category of finite sets and injective maps.

Definition 5.3 (Ore condition)**.** A category $\mathbb{C}$ satisfies the *right Ore condition* if every pair of morphisms $\psi_0 : \Phi_0 \to \Psi$ and $\psi_1 : \Phi_1 \to \Psi$ can be completed into a commutative square:

$$\begin{tikzcd} \Phi \arrow[r, dashed] \arrow[d, dashed] & \Phi_0 \arrow[d, "\psi_0"] \\ \Phi_1 \arrow[r, "\psi_1"'] & \Psi \end{tikzcd}$$

Having pullbacks is a sufficient condition, but not a necessary one.

Definition 5.4 (Atomic topology)**.** If $\mathbb{C}$ satisfies the right Ore condition from Definition 5.3, then it is possible to impose the *atomic topology* on $\mathbb{C}$, where all inhabited sieves[5] are covering:

$$J_{\mathrm{at}}(\Psi) \triangleq \left\{ S\ \mathit{sieve}_\Psi \mid \cup_\Phi S(\Phi)\ \mathit{inhabited} \right\}$$

Definition 5.5 (Schanuel topos)**.** The Schanuel topos is the topos of sheaves on the atomic site $(\mathbb{I}, J_{\mathrm{at}})$.

We will see that the atomic topology is a special case of something called the *dense topology* or, classically, the *double-negation topology*, which will give us some insight into the logical characteristics of the Schanuel topos when formulated in a classical metatheory.

Definition 5.6 (Dense topology)**.** The *dense topology* is defined as follows:

$$J_{\mathrm{dense}}(\Psi) \triangleq \left\{ S\ \mathit{sieve}_\Psi \mid \forall \psi : \Phi \to \Psi.\ \exists \phi : X \to \Phi.\ \psi \circ \phi \in S(X) \right\}$$

Lemma 5.7. *When distilled into its pure essence as a Lawvere-Tierney local operator (Definition 3.2), the dense topology corresponds classically to the double-negation modality* $\neg\neg$.[6]

Proof. It suffices to "compile" double-negations from the internal language of the topos into statements about sieves. Recall that $S \in J(\Psi)$ iff $\Psi \Vdash S = \top$, where $\top \triangleq \mathcal{H}_\Psi$ is the maximal sieve. First, we unfold the meaning of $\Psi \Vdash \neg\neg\, S = \top$ using Beth-Kripke-

[5] In classical sheaf theory, the covering sieves for the atomic topology are the *non-empty* ones; however, in a constructive metatheory, this is not enough to develop the necessary results, including Lemma 5.9 and Theorem 5.13.

[6] The definition of the dense topology that we have assumed is *not*, however, equivalent to the double negation topology in a constructive metatheory, since the equivalence relies on De Morgan duality [8, 2].

Joyal semantics of the topos as a weapon:

$$\Psi \Vdash \neg\neg S = \top \tag{5.1}$$

$$\forall \psi : \Phi \to \Psi.\ \neg(\Phi \Vdash \neg\, S \cdot \psi = \top\,) \tag{5.2}$$

$$\forall \psi : \Phi \to \Psi.\ \neg(\forall \phi : X \to \Phi.\ \neg(X \Vdash\ S \cdot \psi \cdot \phi = \top\,)) \tag{5.3}$$

$$\forall \psi : \Phi \to \Psi.\ \exists \phi : X \to \Phi.\ X \Vdash\ S \cdot \psi \cdot \phi = \top \tag{5.4}$$

$$\forall \psi : \Phi \to \Psi.\ \exists \phi : X \to \Phi.\ X \Vdash\ S \cdot (\psi \circ \phi) = \top \tag{5.5}$$

Now it suffices to show that $X \Vdash\ S \cdot (\psi \circ \phi) = \top$ iff $\psi \circ \phi \in S(X)$. ($\Rightarrow$) Unfolding the meaning of our assumption, we have for all $\rho : Y \to X$ that $\psi \circ \phi \circ \rho \in S(Y)$. Now choose $Y \triangleq X$ and $\rho \triangleq \mathrm{id}_X$; therefore $\psi \circ \phi \in S(X)$. ($\Leftarrow$) Now suppose $\psi \circ \phi \in S(X)$. We have to show that for all $\rho : Y \to X$, then $\psi \circ \phi \circ \rho \in S(Y)$. This follows because sieves are closed under precomposition. $\square$

Remark 5.8. Observe the essential use of De Morgan's duality in the passage between 5.3 and 5.4 above. The dense topology does not correspond to the double negation topology in a constructive metatheory; moreover, the version of the dense topology which *does* correspond to double negation cannot suffice for our purposes, as the author observed in [9].

Lemma 5.9. *The atomic topology coincides with the dense topology.*

Proof. ($\Rightarrow$) Suppose $S \in J_{\mathsf{at}}(\Psi)$, i.e. S *sieve*$_\Psi$ and $\cup_\Phi S(\Phi)$ *inhabited*. Fix $\psi : \Phi \to \Psi$; we have to exhibit some $\phi : X \to \Phi$ such that $\psi \circ \phi \in S(X)$. By assumption, there is some $\Upsilon : \mathbb{I}$ for which we have some $\psi' : \Upsilon \to \Psi$ such that $\psi' \in S(\Upsilon)$; by the right Ore condition (Definition 5.3), we have some $X : \mathbb{I}$ with the following property:

$$\begin{tikzcd}
X \arrow[r, dashed, "\phi"] \arrow[d, dashed, "\upsilon"'] & \Phi \arrow[d, "\psi"] \\
\Upsilon \arrow[r, "\psi'"'] & \Psi
\end{tikzcd}$$

Because sieves are closed under precomposition, we have $\psi' \circ \upsilon \in S(X)$; because the diagram above commutes, we therefore have $\psi \circ \phi \in S(X)$.

($\Leftarrow$) Suppose $S \in J_{\mathsf{dense}}(\Psi)$, i.e. S *sieve*$_\Psi$ and for any $\psi : \Phi \to \Psi$ there exists some $\phi : X \to \Phi$ such that $\psi \circ \phi \in S(X)$. We have to exhibit some $\Upsilon : \mathbb{I}$ together with some $\psi' : \Upsilon \to \Psi$ such that $\psi' \in S(\Upsilon)$. Choose $\Phi \triangleq \Psi$ and $\psi \triangleq 1$; then, we have some $\phi : X \to \Psi$ such that $\phi \in S(X)$. Then choose $\Upsilon \triangleq X$ and $\psi' \triangleq \phi$. $\square$

Remark on constructivity One should be cautious about the numerous results in topos theory of the form "Any topos with property X is boolean" (e.g. $X \triangleq$ *"well-pointed"*), which, far from elucidating an essential consequence of the property X, merely expose a leakage of information from the (Platonistic) external world into the topos. This kind of glitch serves only to underscore the essentially Tarskian deviation [3] which classical topos theory has inherited from old-fashioned mathematics and semantics.

To resist this deviation amounts to adopting Bishop's dictum that meaningful distinctions must be preserved; in doing so, we enter a profoundly alien world in which, for instance, the Schanuel topos is *not* boolean. We do not take a strident position on this here; our remarks are meant only to provide hope to the radical constructivist that it is possible to develop a semantics for names which does not incidentally commit one to a classical ontology.

5.1 Understanding nominal indexing

The mathematical specification of concepts which vary over nominal atoms is most clearly captured in the Schanuel topos; such a concept is a sheaf on the site $(\mathbb{I}, J_{\mathsf{at}})$. Someone more familiar with the syntactic aspects of nominal indexing may have a few questions, however, which we will try to address in this section.

5.1.1 What is $\mathbb{I}$ really?

We have defined $\mathbb{I}$ as the free strict semicartesian category on a single object $\mathbf{nom} : \mathbb{I}$. What is this really, and how does this correspond to contexts of names? It will be more familiar to consider a locative [3] notation for $\mathbb{I}$, in which objects $\mathbf{nom} \otimes \mathbf{nom} \otimes \mathbf{nom} : \mathbb{I}$ are written as (meta)-named contexts $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} : \mathbb{I}$; then, contexts and renamings are identified up to permutations of metavariables.

This presentation corresponds to the usual view of contexts in syntax and programming languages; it is equivalent to the combinatorial version, which is less convenient in practice but also less complicated from a mathematical perspective.

5.1.2 Why sheaves instead of presheaves?

A first-cut attempt to define concepts which are indexed in nominal atoms would be to define them as presheaves on $\mathbb{I}$: that is, sets which are defined over name contexts $\Psi : \mathbb{I}$ and implement the appropriate renamings from $\mathbb{I}$ (weakening and exchange).

Presheaves on $\mathbb{I}$ justify comparing names Because $\mathbb{I}$ has only projections and permutations (which correspond respectively to weakening and exchange), and not diagonals (which correspond to contraction), natural transformations between presheaves on $\mathbb{I}$ are able to proceed by comparing the identity of different atoms. This is because the provenience and distinctness of names are preserved, whereas they would not be preserved in the presence of diagonals, in which two names could become identified.

Sheaves justify support and strengthening When developing a language that is indexed in some form of name or variable, there is usually a well-defined way to talk about which names are free in a term. If the category of name contexts includes projections, then it is always possible that a term in context $\Phi \otimes \Psi$ may only "use" variables from Φ.

In syntax, the question of which variables are "used" can be answered by traversing a term; however, this notion is not sufficient for dealing with mathematical objects in general which vary over names, especially when these objects may not be finitary or syntactically presentable in any sense. The semantical analogue to the "use" of names is that of *support*; necessarily, support has a behavioral definition rather than a structural one. Fixing a presheaf $P : \widehat{\mathbb{I}}$, we can define what it means for a renaming $\psi : \Phi \to \Psi$ to support a term $m : P(\Phi)$:

Definition 5.10 (Support)**.** An arrow $\psi : \Phi \to \Psi$ *supports* a term $m \in P(\Phi)$ when, for any $\phi_0, \phi_1 : X \to \Phi$ such that the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, shift left, "\phi_0"] \arrow[r, shift right, "\phi_1"'] & \Phi \arrow[r, "\psi"] & \Psi \end{tikzcd} \tag{5.6}$$

we have $m \cdot \phi_0 = m \cdot \phi_1$. We will write $m \lhd \psi : \Psi$ when ψ supports m.

Remark 5.11 (Understanding support). How can we relate the above to an intuitive notion of the positive "use" of names? It may help to consider what the statement $m \lhd \psi : \Psi$ is saying about the renaming $\psi : \Phi \to \Psi$. Suppose $\Phi \equiv \Psi \otimes \Phi'$ and $\psi \equiv \pi_1$; then, Diagram 5.6 says that ϕ_0 and ϕ_1 can only differ in their behavior on the names in Φ'. So, the fact that ψ supports m means that m is insensitive to this difference, i.e. m is oblivious to the names in Φ'.

Definition 5.12 (Strengthening)**.** A presheaf $P : \widehat{\mathbb{I}}$ has the *strengthening* property when for any $m \in P(\Phi)$, if $m \lhd \psi : \Psi$, then there is a unique $m_\Psi \in P(\Psi)$ such that $m = m_\Psi \cdot \psi$. We call m_Ψ the strengthening of m along ψ.

Theorem 5.13. *A presheaf $P : \widehat{\mathbb{I}}$ has strengthening if and only if P is a sheaf for the atomic topology, i.e. $P : \mathbf{Sh}(\mathbb{I}, J_{at})$.*

Proof. ($\Rightarrow$) Suppose that $P : \mathbf{Sh}(\mathbb{I}, J_{at})$ and $m \lhd \psi : \Psi$ for some $m \in P(\Phi)$; we have to exhibit a unique $m_\Psi \in P(\Psi)$ such that $m = m_\Psi \cdot \psi$. By the Yoneda lemma, we can regard m as a natural transformation $\lfloor m \rfloor : [\mathcal{H}_\Phi, P]$ and it suffices to exhibit some unique $\lfloor m_\Psi \rfloor : [\mathcal{H}_\Psi, P]$ such that $\lfloor m \rfloor = \lfloor m_\Psi \rfloor \circ \mathcal{H}_\psi : [\mathcal{H}_\Phi, \mathcal{H}_\Psi]$, i.e.

$$\begin{tikzcd} \mathcal{H}_\Phi \arrow[r, "\mathcal{H}_\psi"] \arrow[d, "\lfloor m \rfloor"'] & \mathcal{H}_\Psi \arrow[dl, dashed, "\lfloor m_\Psi \rfloor"] \\ P & \end{tikzcd}$$

Because P is a sheaf, if we exhibit a covering sieve $S \in J_{at}(\Psi)$ and a matching family $f : [S, P]$, we can get a unique natural transformation $\lfloor m_\Psi \rfloor : [\mathcal{H}_\Psi, P]$ such that $f = \lfloor m_\Psi \rfloor \circ i_S$ where $i_S : S \rightarrowtail \mathcal{H}_\Psi$. For $S(X)$, choose the set of all composites $\Psi_X : X \to \Psi$ factoring through ψ, i.e. arrows of the following form:

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] \arrow[rr, bend left, "\psi_X \in S(X)"] & \Phi \arrow[r, "\psi"] & \Psi \end{tikzcd}$$

For the matching family $\mathfrak{f}$, for any $\psi_X : X \to \Psi \in S(X)$ we have to functionally exhibit some $\mathfrak{f}_X(\psi_X) \in P(X)$, naturally in X. Because $\psi_X \in S(X)$, there exists some (not necessarily unique) $\phi : X \to \Phi$ such that $\psi_X = \psi \circ \phi$; therefore, choose $\mathfrak{f}_X(\psi_X) \triangleq m \cdot \phi$. This assignment is functional in ψ_X because m is invariant under the action of any ϕ which is precomposed with ψ (this is a consequence of $m \lhd \psi : \Psi$ and Diagram 5.6). It remains to show that $\mathfrak{f}_X$ is natural in X, which we establish by means of a diagram chase, fixing $\chi : \Upsilon \to X$:

$$\begin{tikzcd} S(X) \arrow[r, "S(\chi)"] \arrow[d, "\mathfrak{f}_X"'] & S(\Upsilon) \arrow[d, "\mathfrak{f}_\Upsilon"] & \psi \circ \phi \arrow[r, mapsto, "S(\chi)"] \arrow[d, mapsto, "\mathfrak{f}_X"'] & \psi \circ \phi \circ \chi \arrow[d, mapsto, "\mathfrak{f}_\Upsilon"] \\ P(X) \arrow[r, "P(\chi)"'] & P(\Upsilon) & m \cdot \phi \arrow[r, mapsto, "P(\chi)"'] & m \cdot \phi \cdot \chi = m \cdot (\phi \circ \chi) \end{tikzcd}$$

Therefore, because P is a sheaf, we have a unique $\lfloor m_\Psi \rfloor : [\mathcal{H}_\Psi, P]$ such that $\mathfrak{f} = \lfloor m_\Psi \rfloor \circ i_S$; it remains to show that $\lfloor m \rfloor = \lfloor m_\Psi \rfloor \circ \mathcal{H}_\psi$, i.e. for any $X : \mathbb{I}$ the following diagram commutes in **Set**:

$$\begin{tikzcd} {[X, \Phi]} \arrow[r, "\lfloor m \rfloor_X"] \arrow[d, "\psi \circ -"'] & P(X) \\ {[X, \Psi]} \arrow[ur, "\lfloor m_\Psi \rfloor_X"'] & \end{tikzcd}$$

To make the above diagram commute, we need to show that for any $\phi : X \to \Phi$, we have $\lfloor m \rfloor_X(\phi) = \lfloor m_\Psi \rfloor_X(\psi \circ \phi)$, i.e. $m \cdot \phi = \lfloor m_\Psi \rfloor_X(\psi \circ \phi)$; by the definition of $\mathfrak{f}$, it suffices to show $\mathfrak{f}_X(\psi \circ \phi) = \lfloor m_\Psi \rfloor_X(\psi \circ \phi)$, which is clearly the case because we already have $\mathfrak{f} = \lfloor m_\Psi \rfloor \circ i_S$ and $\psi \circ \phi \in S(X)$.

($\Leftarrow$) Now suppose that we have a presheaf $P : \widehat{\mathbb{I}}$ which is equipped with strengthening. Fixing a covering sieve $S \in J_{at}(\Psi)$, recall that we have a canonical restriction $i_S^* \triangleq - \circ i_S : [\mathcal{H}_\Psi, P] \to [S, P]$ where $i_S : S \rightarrowtail \mathcal{H}_\Psi$; to show that P is a sheaf, we need to show that i_S^* is an isomorphism.

Given a matching family $\mathfrak{f} : [S, P]$, we need to exhibit a unique extension $\lfloor m \rfloor : [\mathcal{H}_\Psi, P]$ such that $\mathfrak{f} = \lfloor m \rfloor \circ i_S$. Because $S \in J_{at}(\Psi)$, there exists some $\psi : \Phi \to \Psi$ such that $\psi \in S(\Phi)$, whence we have $\mathfrak{f}_\Phi(\psi) \in P(\Phi)$. We want to show that $\mathfrak{f}_\Phi(\psi) \lhd \psi : \Psi$; that is, fixing $\phi_0, \phi_1 : X \to \Phi$ such that $\psi \circ \phi_0 = \psi \circ \phi_1$, we need to show that $\mathfrak{f}_\Phi(\psi) \cdot \phi_0 = \mathfrak{f}_\Phi(\psi) \cdot \phi_1$.

By naturality, we have $P(\phi_i) \circ \mathfrak{f}_\Phi = \mathfrak{f}_X \circ P(\phi_i)$ for $i \in \{0, 1\}$; specifically, we have $\mathfrak{f}_\Phi(\psi) \cdot \phi_i = \mathfrak{f}_X(\psi \circ \phi_i)$, so it suffices to show that $\mathfrak{f}_X(\psi \circ \phi_0) = \mathfrak{f}_X(\psi \circ \phi_1)$, which is clearly the case because $\psi \circ \phi_0 = \psi \circ \phi_1$. Therefore, $\mathfrak{f}_\Phi(\psi) \lhd \psi : \Psi$. By assumption, we have a unique strengthening $m \in P(\Psi)$ such that $\mathfrak{f}_\Phi(\psi) = m \cdot \psi$, i.e. $\lfloor m \rfloor : [\mathcal{H}_\Psi, P]$ such that $\lfloor \mathfrak{f}_\Phi(\psi) \rfloor = \lfloor m \rfloor \circ \mathcal{H}_\psi$.

It remains only to show that $\mathfrak{f} = \lfloor m \rfloor \circ i_S$, i.e. for any $\psi \circ \phi \in S(X)$ we have $\mathfrak{f}_X(\psi \circ \phi) = m \cdot (\psi \circ \phi)$, or by naturality, $\mathfrak{f}_\Phi(\psi) \cdot \phi = m \cdot (\psi \circ \phi) = m \cdot \psi \cdot \phi$. Canceling ϕ, it suffices to show $\mathfrak{f}_\Phi(\psi) = m \cdot \psi$, which holds by definition.

□

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