2.4. Regime Switching

Occasionally Binding Constraints in DSGE Models

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[1]The views expressed are those of the authors and should not be interpreted as reflecting the views of the Bank of Canada.

The concept

An OBC represents two regimes:

1. In one regime the constraint is slack
2. In the other, the constraint binds

In each regime, the model is approximated via perturbation.

There is an endogenous ‘switching’ between regimes.

The transition probabilities may or may not be endogenous.

Can generalize to a higher number of constraints (and regimes)

Piecewise-linear

Piecewise-linear

This method makes the futher assumptions:

- There is a ‘reference regime’ that holds in the long-run
- The reference regime must be locally stable
- The second regime does not need to be stable
- The model is linearized in each regime around the deterministic steady state
- The transition probability in the reference regime is always $[1 \quad 0]$
- In the second regime, agents expect no future shocks

Algorithm

The basic algorithm:

- The model is simulated using the standard, linear policy function with in regime 1.

$$b_t = g(b_{t-1}, z_t) \tag{1}$$

- Suppose a shock causes b_t to fall below $\underline{b}$.
- Guess an initial horizon T for the point b_t will again be unconstrained.
- For $t \geq T$, use the regime 1 policy function.
- Solve backwards for b_{T-1}, b_{T-2}, using the linearized, forward-looking model under regime 2, with no shocks.

$$\mathcal{A}x_{t+1} + \mathcal{B}x_t + \mathcal{C}x_{t-1} + \mathcal{D} = 0 \tag{2}$$

  where $x_t = [b_t \quad z_t]$.
- Update T if too high/low, and repeat.

Borrowing constraints model

In the borrowing constraints model, the sufficient conditions to satisfy optimal household savings are:

$$\frac{1}{c_t} - r\beta\mathbb{E}_t\left[\frac{1}{c_{t+1}}\right] - \mu_t + 2\delta b_t = 0 \tag{3}$$

$$\min\left\{\mu_t, b_t - \underline{b}\right\} = 0. \tag{4}$$

We saw earlier that these satisfy the Kuhn-Tucker conditions. These are replaced with the following regimes:

- Regime 1 ($\mu_t = 0$):

$$\frac{1}{c_t} = r\beta\mathbb{E}_t\left[\frac{1}{c_{t+1}}\right] - 2\delta b_t \tag{5}$$

- Regime 2 ($\mu_t > 0$):

$$b_t = \underline{b} \tag{6}$$

Simple implementation

We can solve this easily in matlab: **borrowing_constraints/piecewise-linear/soe_obc_simulate.m**

Here we simulate over S periods:

1. Use dynare to solve the linear policy function for regime 1, $b_t = g(b_{t-1}, z_{t-1}, \epsilon_t)$.
2. Draw the shocks to compute a path for the exogenous variables for $t = 1 : S$
3. Simulate the model period-by-period using $b_t = g(b_{t-1}, z_{t-1}, \epsilon_t)$
4. Each period t, check if $b_t < \underline{b}$
5. If not, move onto next period (4). Otherwise guess how long before b_t is unconstrained, T
6. It follows that $\{b_{t+s}\}_0^{T-1} = \underline{b}$, and $b_{t+T} = g(\underline{b}, z_{t+T-1})$
   - If $b_{t+T} = g(\underline{b}, z_{t-1}) < \underline{b}$, increase T and try again
   - If $g(\underline{b}, z_{t+T-2}) > \underline{b}$, decrease T and try again
7. Continue simulation from period $t + T + 1$ (4).

OccBin: implementation

- Code solving the above example can be found at **borrowing_constraints/piecewise-linear/do.m**
- OccBin code is in **/functions/occbin/**
- You set up with two .mod files: 1) is for the reference regime, e.g., with $\mu = 0$ and b unconstrained, 2) is the alternative regime
  - Note that regime 1 must hold in steady state
- You simulate the model using:

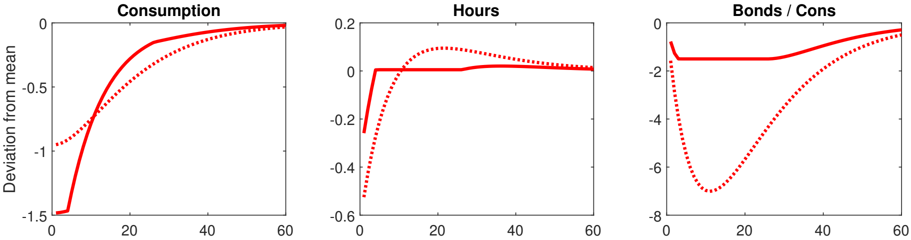
```
solve_one_constraint(modnam,modnamstar,constraint,...
constraint_relax,shockssequence,irfshock,nperiods);
```

  - `modnam` (string) regime 1 mod filename
  - `modnamstar` (string) regime 2 mod filename
  - `constraint` (string) returns true when switching to regime 2, e.g., `'b<b_limit'`
  - `constraint_relax` (string) returns true when returning to regime 1, e.g., `'b>b_limit'`
  - `nperiods` number of periods, T
  - `shockssequence` $T \times n$ numerical array, n is number of shocks
  - `irfshock` char array, names of shocks

See Guerrieri & Iacoviello (2015) and dynare OccBin package at:
https://www2.bc.edu/matteo-iacoviello/research_files/occbin_20140630.zip

Comparing IRFs: OBC vs linear

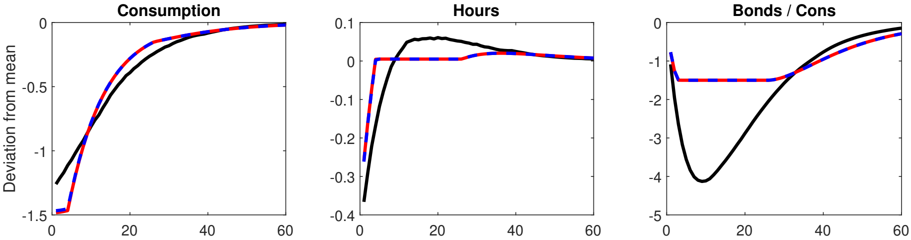
OccBin: constraint vs no constraint



- Red dash = no constraint (linear), red solid = borrowing constraints model (occbin)

Comparing IRFs: projection vs OccBin

Projection vs Perfect Foresight vs OccBin



- Red = OccBin, Black = projection, blue = Perfect foresight

Impact of OBC on IRFs

There are two main reasons simulating a single IRF in a piecewise-linear model is incorrect:

1. The starting point can be misleading as there can be more ‘room’ to the OBC on average
   - The ergodic mean of a linear model (asymptotically) equals the deterministic steady state
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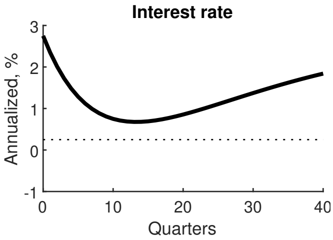
Illustration:



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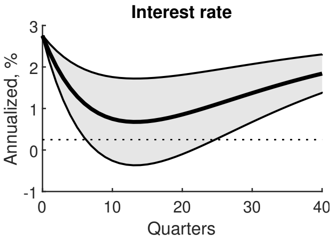

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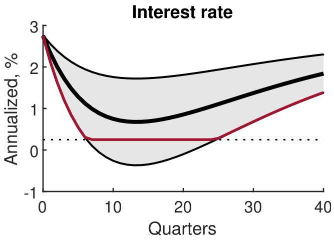

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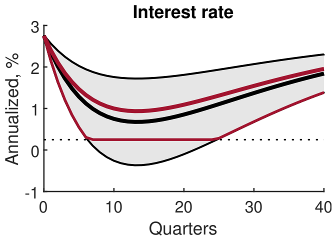
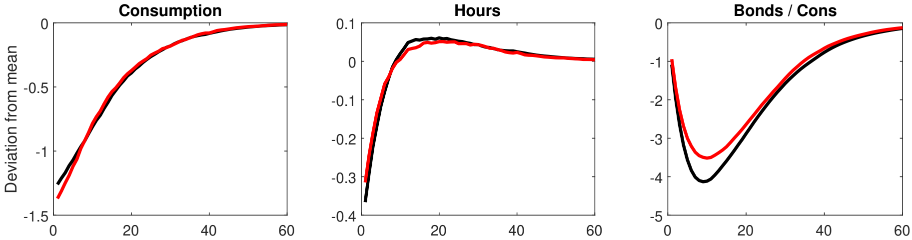


Comparing GIRFs: projection vs OccBin

Projection vs OccBin (GIRF)



- Red = OccBin (GIRF), Black = projection

Simulated Moments

Projection results:

	Mean	Standard deviation	Skewness	
	Relative to no constraint	Relative to no constraint	Baseline	No constraint
Consumption	+0.03%	+15%	-0.22	0.09
Hours	-0.01%	-43%	-0.09	-0.04
Bonds / $\bar{c}$	0.3% $\longrightarrow$ 5%	-59%	1.18	0.007

All methods:

		Mean	Standard deviation	**Skewness** (projection)
		Relative to projection	Relative to projection	
VFI	Consumption	$\approx$ 0	+0.02%	-0.23 (-0.22)
	Hours	$\approx$ 0	+0.6%	-0.09 (-0.09)
	Bonds / $\bar{c}$	5.0% $\longrightarrow$ 4.9%	<1%	1.18 (1.18)
Extended-path	Consumption	-0.03%	+0.5%	-0.29 (-0.22)
	Hours	+0.003%	+1%	-0.1 (-0.09)
	Bonds / $\bar{c}$	5.0% $\longrightarrow$ 4.0%	-1.4%	1.3 (1.18)
Penalty function	Consumption	-0.02%	+25%	0.01 (-0.22)
	Hours	-0.002%	-99%	-0.1 (-0.09)
	Bonds / $\bar{c}$	5.0% $\longrightarrow$ 4.4%	$\approx$-100%	-0.19 (1.18)
Piecewise linear	Consumption	-0.006%	+0.8%	-0.29 (-0.22)
	Hours	+0.003%	+5%	-0.4 (-0.09)
	Bonds / $\bar{c}$	5.0% $\longrightarrow$ 4.0%	+0.9%	1.3 (1.18)

Regime-switching

Regime switching: overview

Junior Maih's (Norges Bank) RISE toolkit improves on piecewise-linear

- See: `https://github.com/jmaih/RISE_toolbox`
- Similar idea – treats the problem as multiple regimes and uses local approximation, but:
  - Agents anticipate the constraint binding in the future
  - Still act under uncertainty when the constraint is binding
  - Can handle multiple OBCs
  - Can be solved to higher order approximations

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Using the policy function notation from above, the model can be written:

$$\mathbb{E}_t \left[\sum_{j=1}^{h} p_{i,j} f_i \left(g_j \left(g_i \left(x_{t-1}, \varepsilon_t \right), \varepsilon_{t+1} \right), g_i \left(x_{t-1}, \varepsilon_t \right), x_{t-1}, \varepsilon_t \right) \right] = 0 \tag{7}$$

- $x_t = g_j \left(x_{t-1}, \varepsilon_t \right)$ is the choice conditional being in regime $j \in (1, h)$ at time t.
- $p_{i,j}$ is the probability of transitioning from regime i to j

Regime switching: key assumptions

To solve

$$\mathbb{E}_t \left[\sum_{j=1}^{h} p_{i,j} f_i \left(g_j \left(g_i \left(x_{t-1}, \varepsilon_t \right), \varepsilon_{t+1} \right), g_i \left(x_{t-1}, \varepsilon_t \right), x_{t-1}, \varepsilon_t \right) \right] = 0, \quad \forall i \tag{8}$$

we rely on a couple of key assumptions:

1. Approximate each regime around steady state *as if* the regime will hold forever, i.e., for a single OBC for example:

$$P = \begin{bmatrix} 1 - p_{1,2} & p_{1,2} \\ p_{2,1} & 1 - p_{2,1} \end{bmatrix} = I_2 \tag{9}$$

$$f_i \left(\bar{x}_i, \bar{x}_i, \varepsilon \right) = 0, \quad \text{for } i = 1, 2 \tag{10}$$

2. P is determined exogenously by the user, although can be a function of the state. e.g.,:

$$p_{1,2} = \theta_{1,2} + \psi_{1,2} b_t \tag{11}$$

$$p_{2,1} = \theta_{2,1} + \psi_{2,1} \mu_t \tag{12}$$

Regime-switching: Borrowing constraints model

Recall:

$$\frac{1}{c_t} - r\beta\mathbb{E}_t\left[\frac{1}{c_{t+1}}\right] - \mu_t + 2\delta b_t = 0 \tag{13}$$

$$c_t \equiv \frac{1}{1+\chi}\left(\exp\left(z_t\right) + rb_{t-1} - b_t\right) \tag{14}$$

For current regime $i = \{1, 2\}$, we can write:

$$\frac{1}{c_t} - r\beta\mathbb{E}_t\left[p_{i,1}\frac{1}{c_{t+1}(1)} + (1-p_{i,1})\frac{1}{c_{t+1}(2)}\right] - \mu_t + 2\delta b_t = 0 \tag{15}$$

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- Regime 1:

$$b_t = \frac{r\beta}{2\delta}\left(p_{1,1}\mathbb{E}_t\left[\frac{1+\chi}{\exp\left(z_{t+1}\right) + rb_t - b_{t+1}}\right] + (1-p_{1,1})\mathbb{E}_t\left[\frac{1+\chi}{\exp\left(z_{t+1}\right) + rb_t - \underline{b}}\right]\right) - \frac{1}{2\delta c_t} \tag{16}$$

- Regime 2:

$$\mu_t = \frac{1}{c_t} - r\beta\left(p_{2,1}\mathbb{E}_t\left[\frac{1+\chi}{\exp\left(z_{t+1}\right) + r\underline{b} - b_{t+1}}\right] + (1-p_{2,1})\mathbb{E}_t\left[\frac{1+\chi}{\exp\left(z_{t+1}\right) + (r-1)\underline{b}}\right]\right) + 2\delta\underline{b} \tag{17}$$

See code: **borrowing_constraints/regime-switching/do_sim_1.m**

Regime switching: example code 1/2

- Example code using RISE can be found at **borrowing_constraints/regime-switching/do_rise.m**
- The RISE toolkit is in **/functions/RISE_toolbox-master/**
- You write the model in a **.rs** file, similar to a **.mod** file:
  - Declare parameters and endogenous/exogenous variables

```
endogenous c h b muu z

exogenous epsz

parameters betta R delta rho chi siggma b_limit theta_1_2 theta_2_1 pssi
```

  - Declare a regime flag to indicate the regime

```
parameters(borrcon,2) conflag
```

`borrcon` labels the constraint and `2` indicates the number of regimes

Regime switching: example code 2/2

- In the model block you need transition probabilities.
  - These could be constant in an exogenous Markov-switching model. In a OBC model, the probability of regime switch depends on the state. So, we define these:

```
        ! borrcon_tp_1_2 = theta_1_2/(theta_1_2+exp(pssi*(b-b_limit)));
        ! borrcon_tp_2_1 = theta_2_1/(theta_2_1+exp(pssi*muu));
```

  The `!` sets it as a model-local variable and you must follow the naming convention (e.g., *contraint-label*_tp_1_2 for moving from regime 1 to 2).
- You write the constraint using the switching parameter:

```
 (1-conflag) * muu + conflag * (b-b_limit) = 0;
```

and to help RISE impose the constraint in simulations, you use

```
 ? b-b_limit>=0;
```

- See the example code for more detail.

References I

Guerrieri, L. & Iacoviello, M. (2015), 'OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily', *Journal of Monetary Economics* **70**, 22–38.