CS2040S
Data Structures and Algorithms

BFS, DFS, and Directed Graphs!

Roadmap

Last time: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs: BFS

What is a graph?

Graph $G = \langle V, E \rangle$

- V is a set of nodes
  - At least one: $|V| > 0$.
- E is a set of edges:
  - $E \subseteq \{ (v,w) : (v \in V), (w \in V) \}$
  - $e = (v,w)$
  - For all $e_1, e_2 \in E : e_1 \neq e_2$

2 x 2 x 2 Rubik's Cube

Configuration Graph

- Vertex for each possible state
- Edge for each basic move
  - 90 degree turn
  - 180 degree turn

Puzzle: given initial state, find a path to the solved state.

Trade-offs

Adjacency Matrix:

- Fast query: are v and w neighbors?
- Slow query: find me any neighbor of v.
- Slow query: enumerate all neighbors.

Adjacency List:

- Fast query: find me any neighbor.
- Fast query: enumerate all neighbors.
- Slower query: are v and w neighbors?

Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

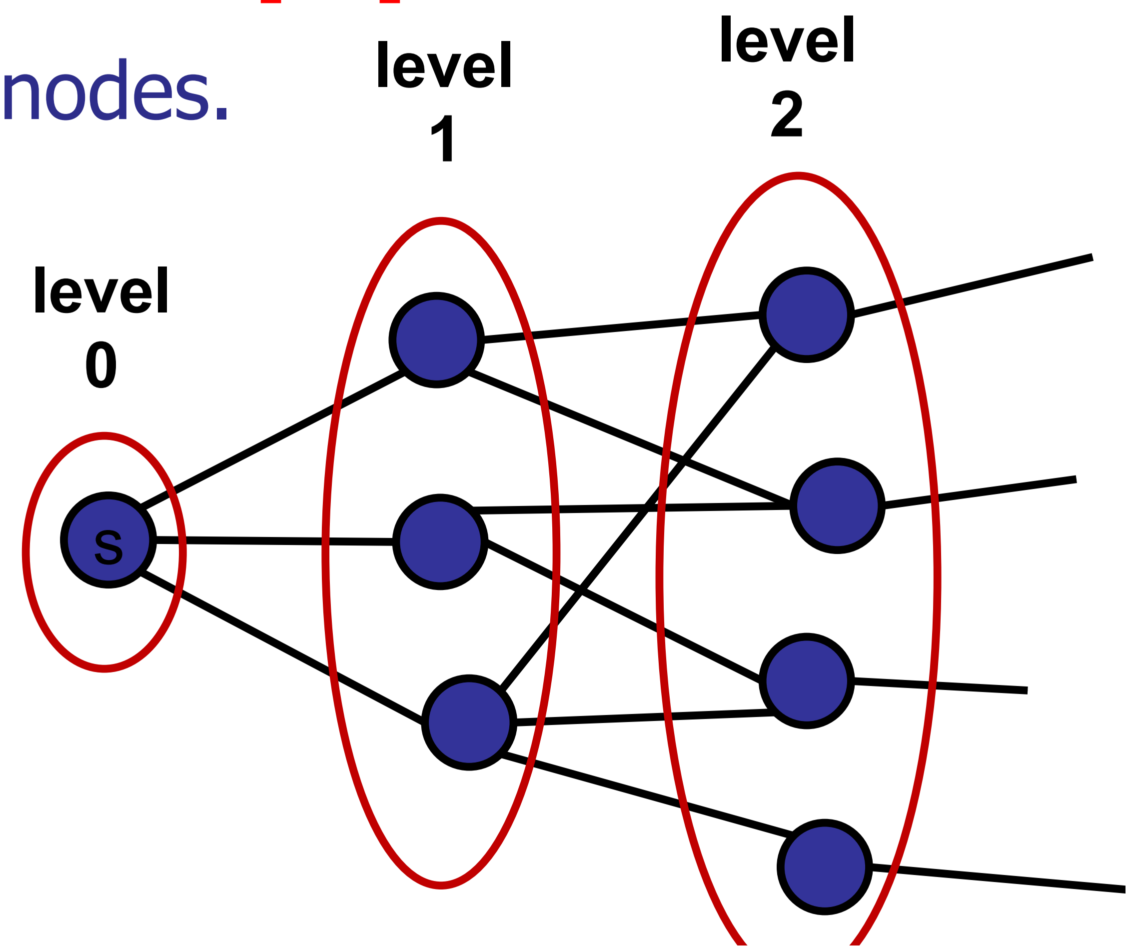
Graph representation:

- Adjacency list

Searching a graph

Breadth-First Search:

- Explore graph level by level.
- Calculate level[i] from level[i-1]
- Skip already visited nodes.

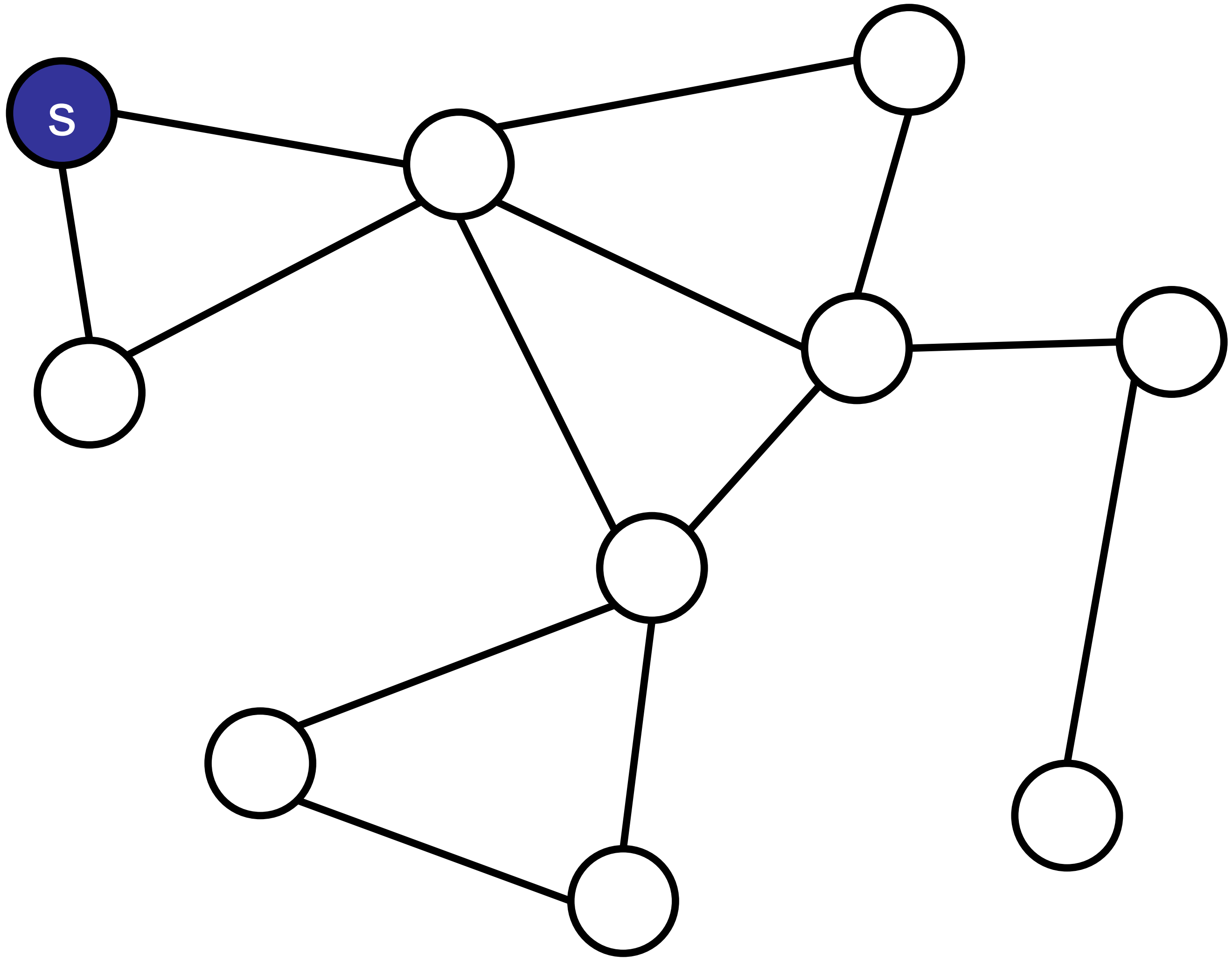


Searching a graph

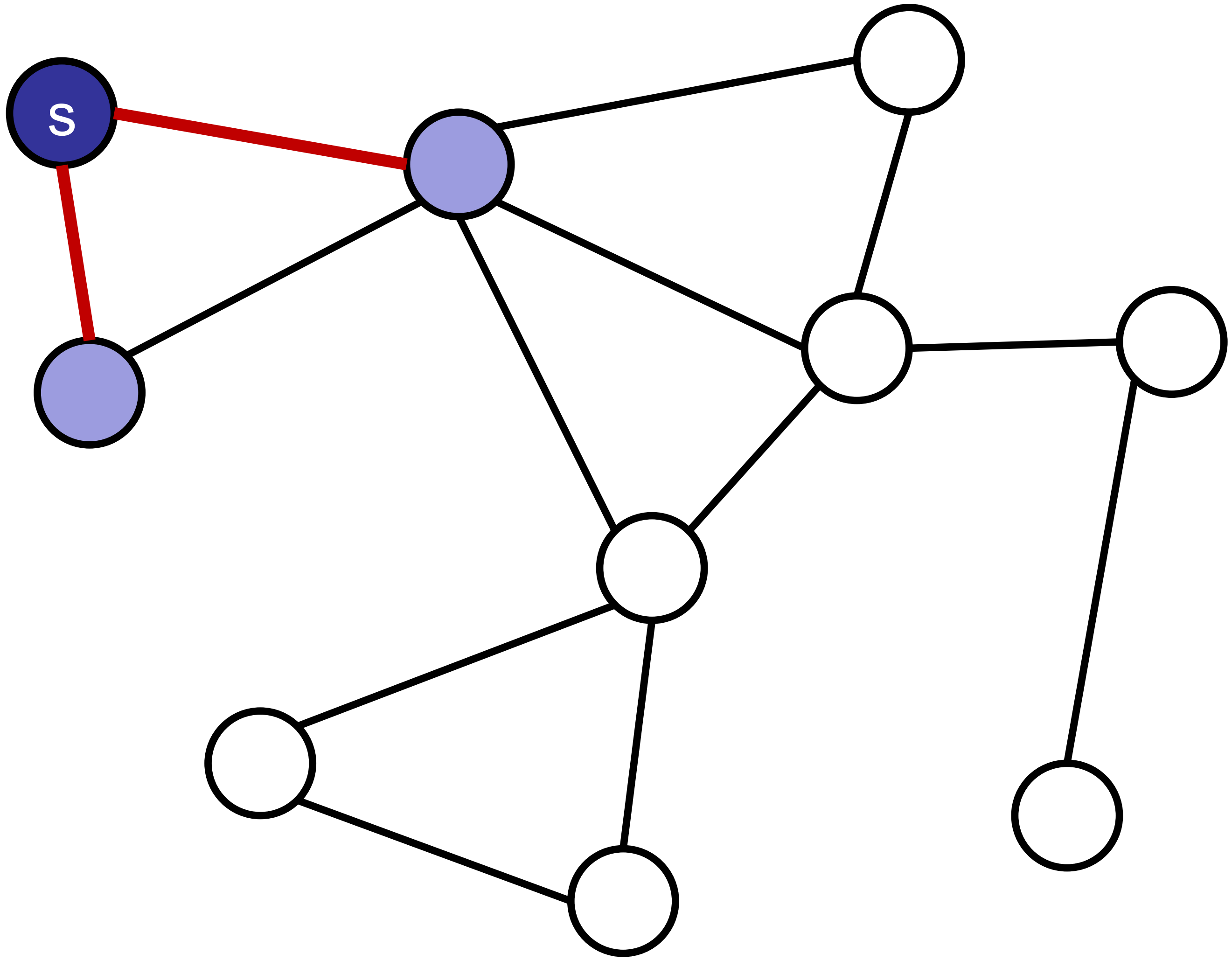
Breadth-First Search:

```
frontier = {s}
while frontier is not empty:
    next-frontier = {}
    for each node u in the frontier:
       for each edge (u,v) in the graph:
           if v is not marked visited, add v to next-frontier
           mark v as visited.
    frontier = next-frontier
```

Breadth First Search



Breadth First Search



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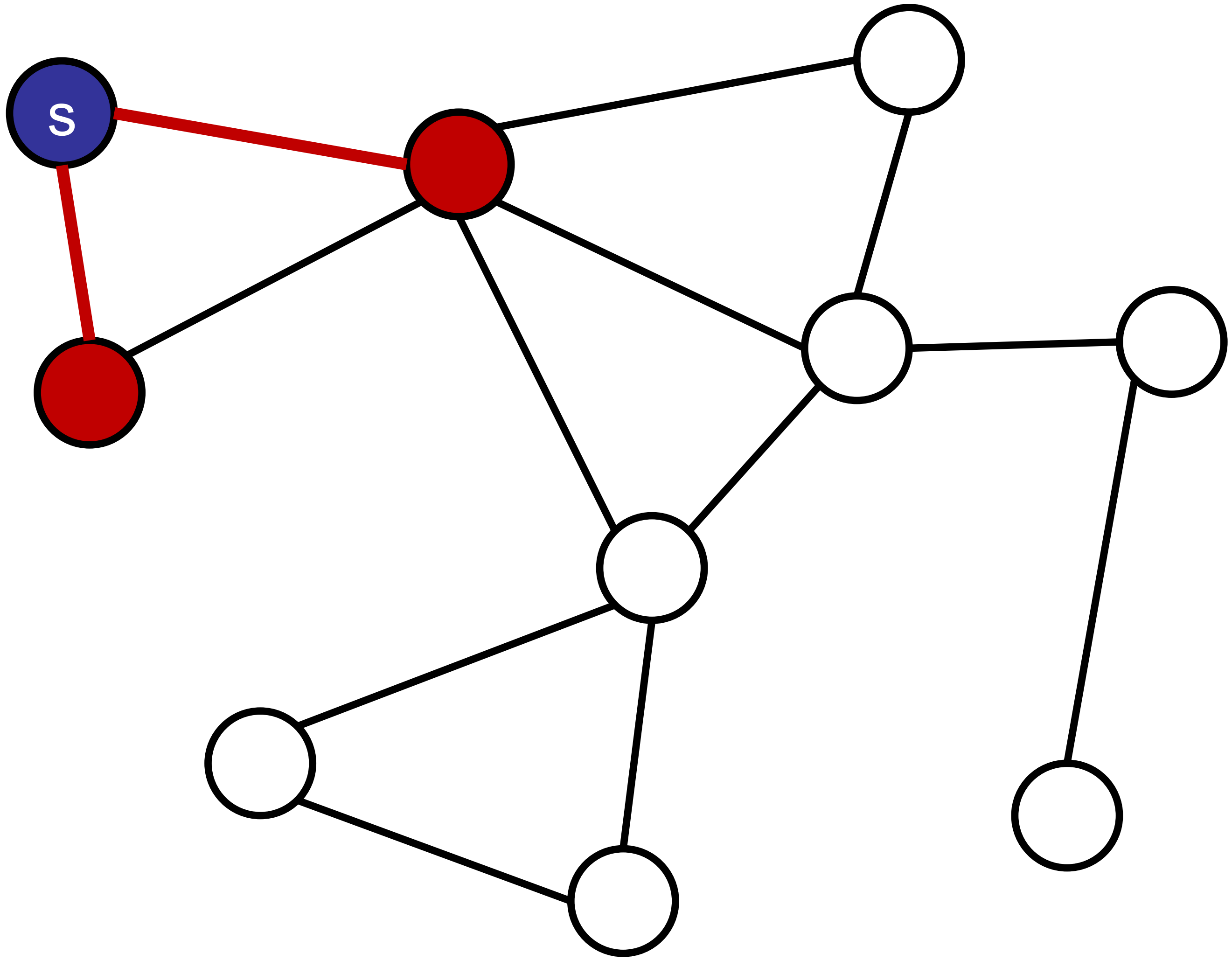

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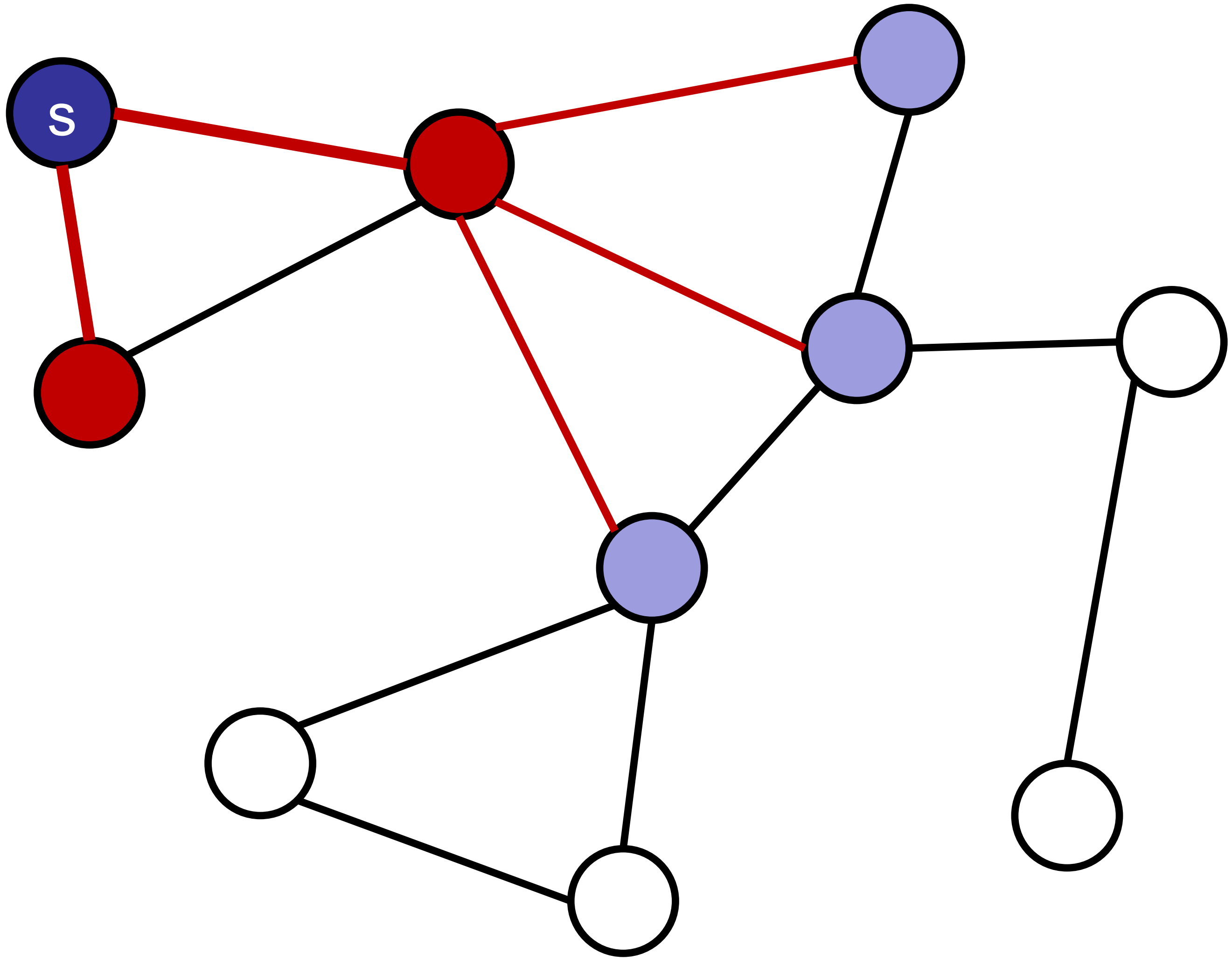

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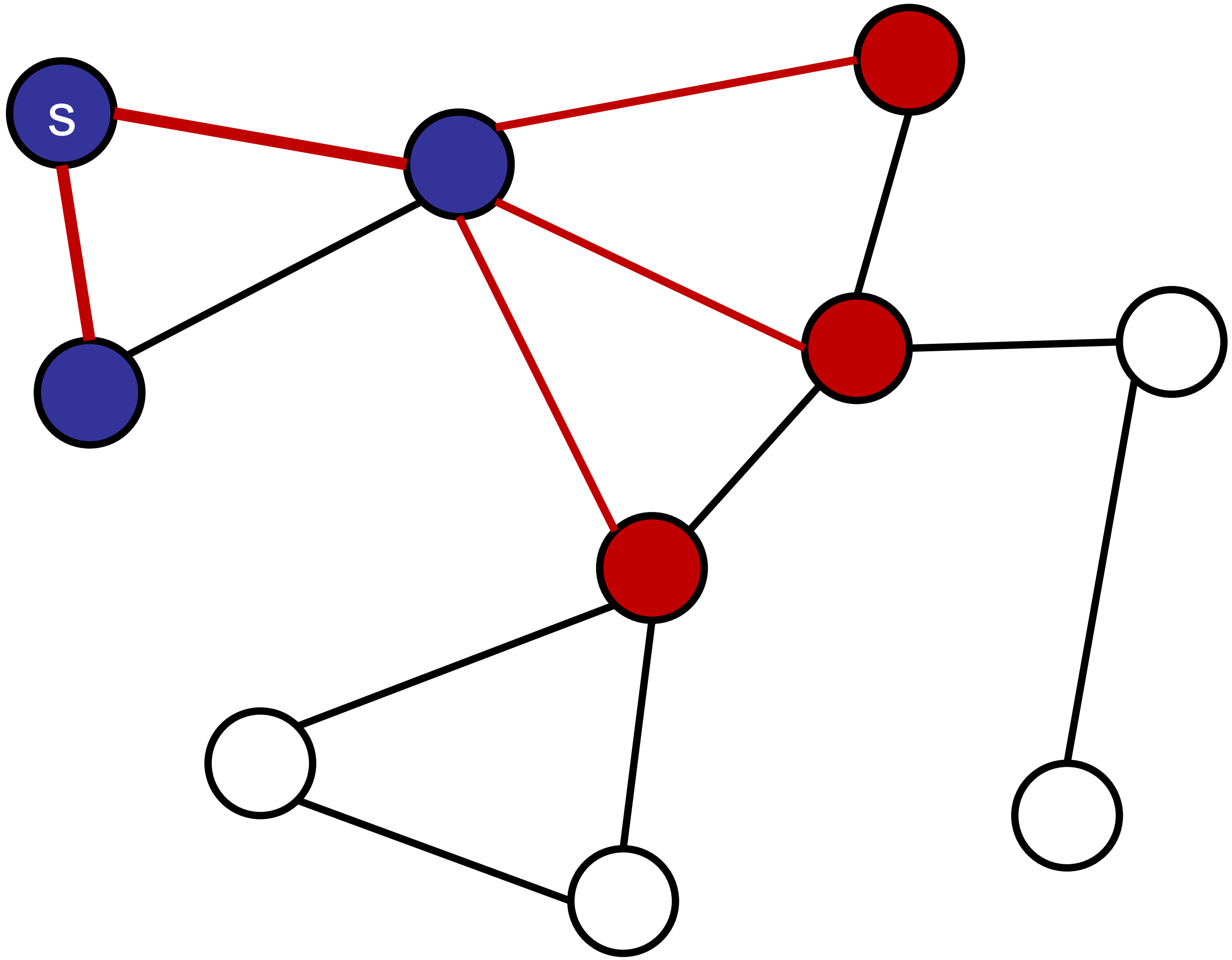

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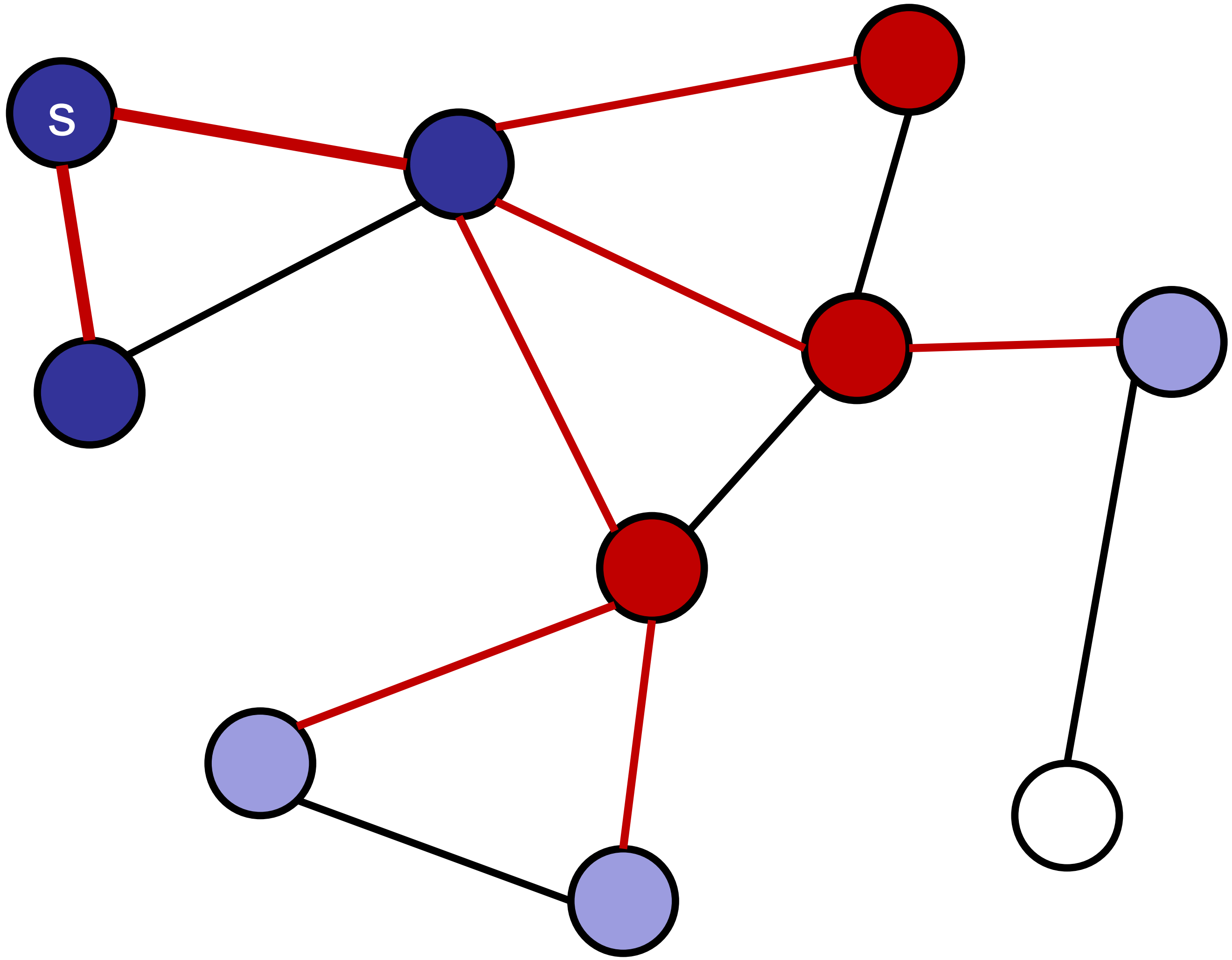

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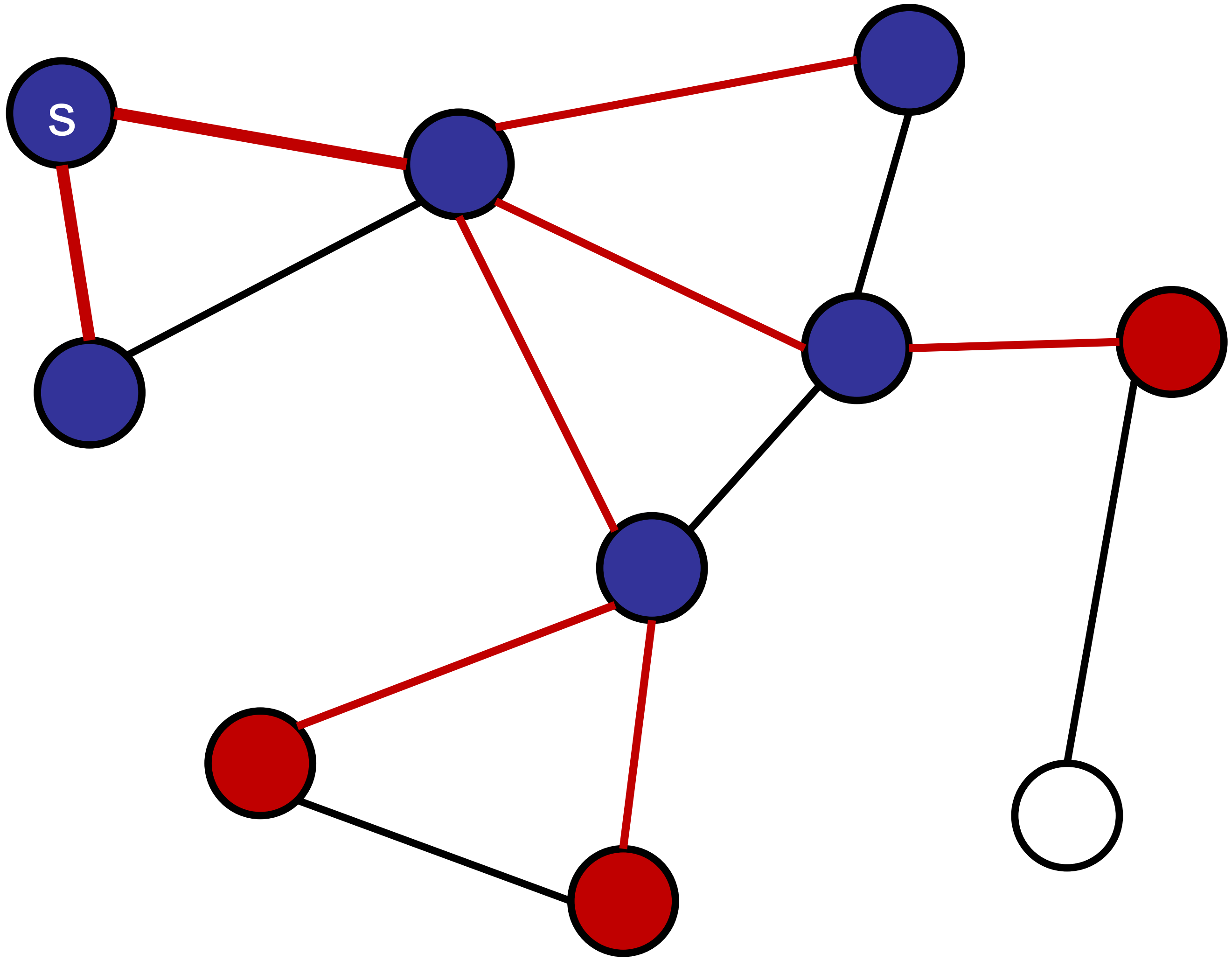

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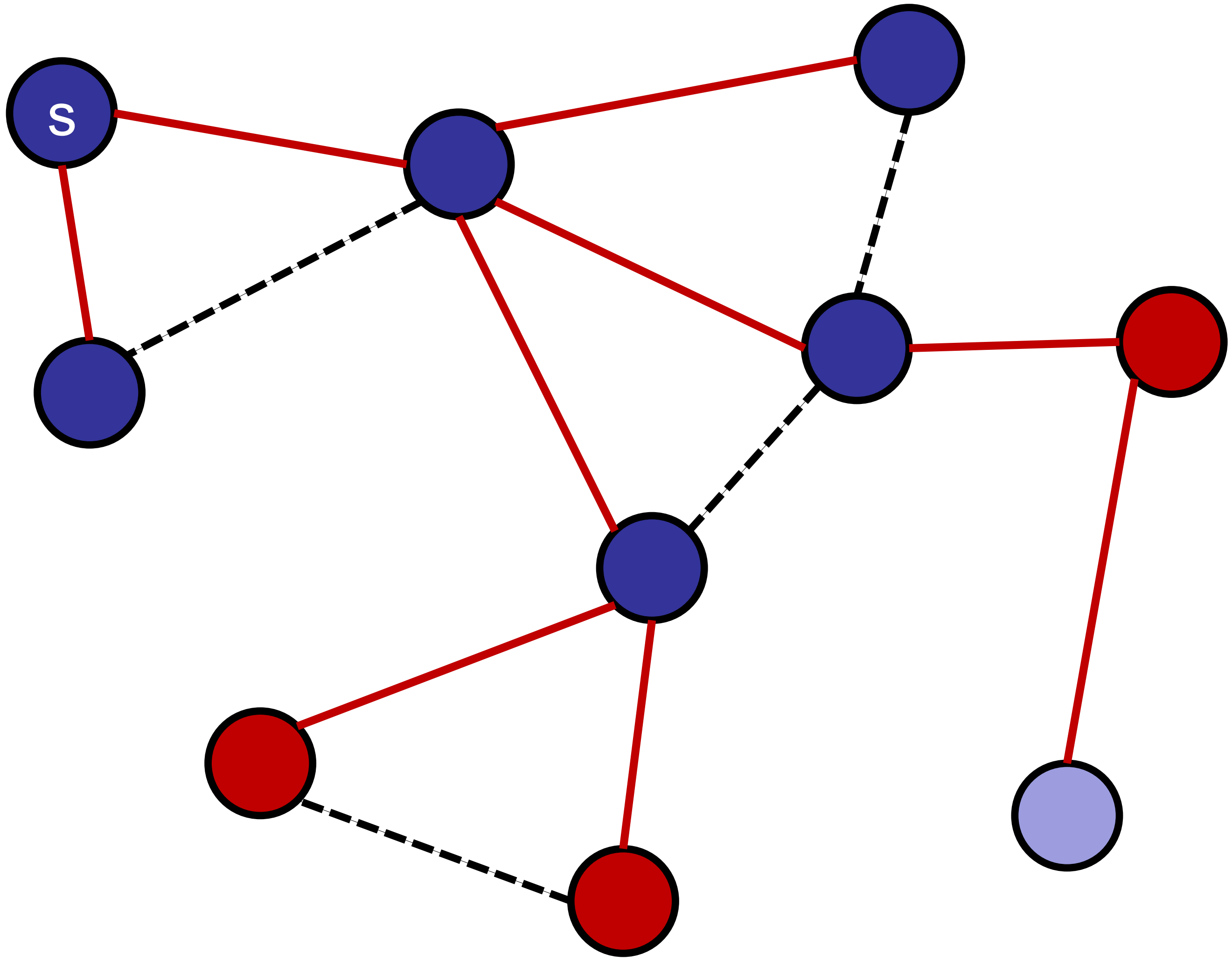

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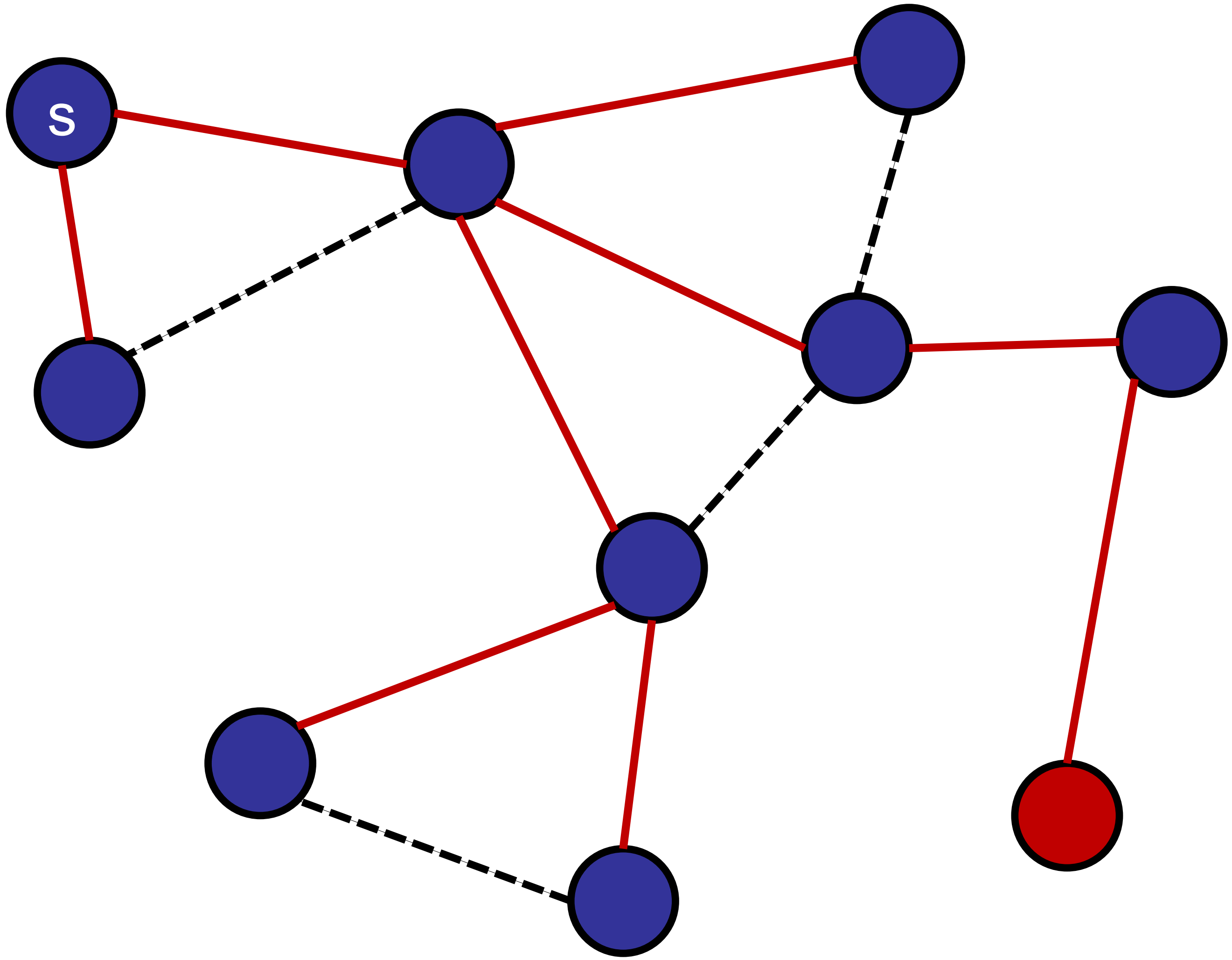

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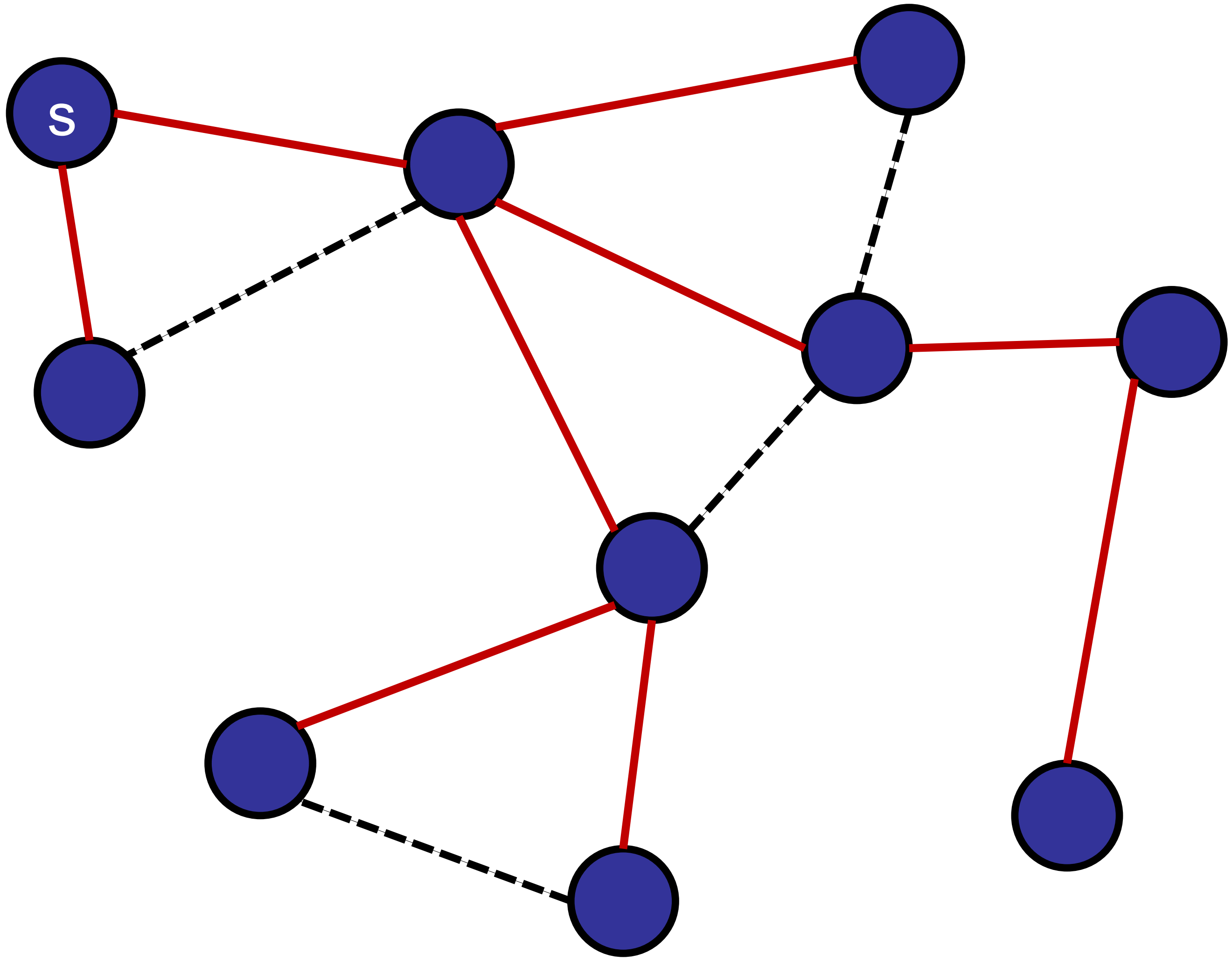

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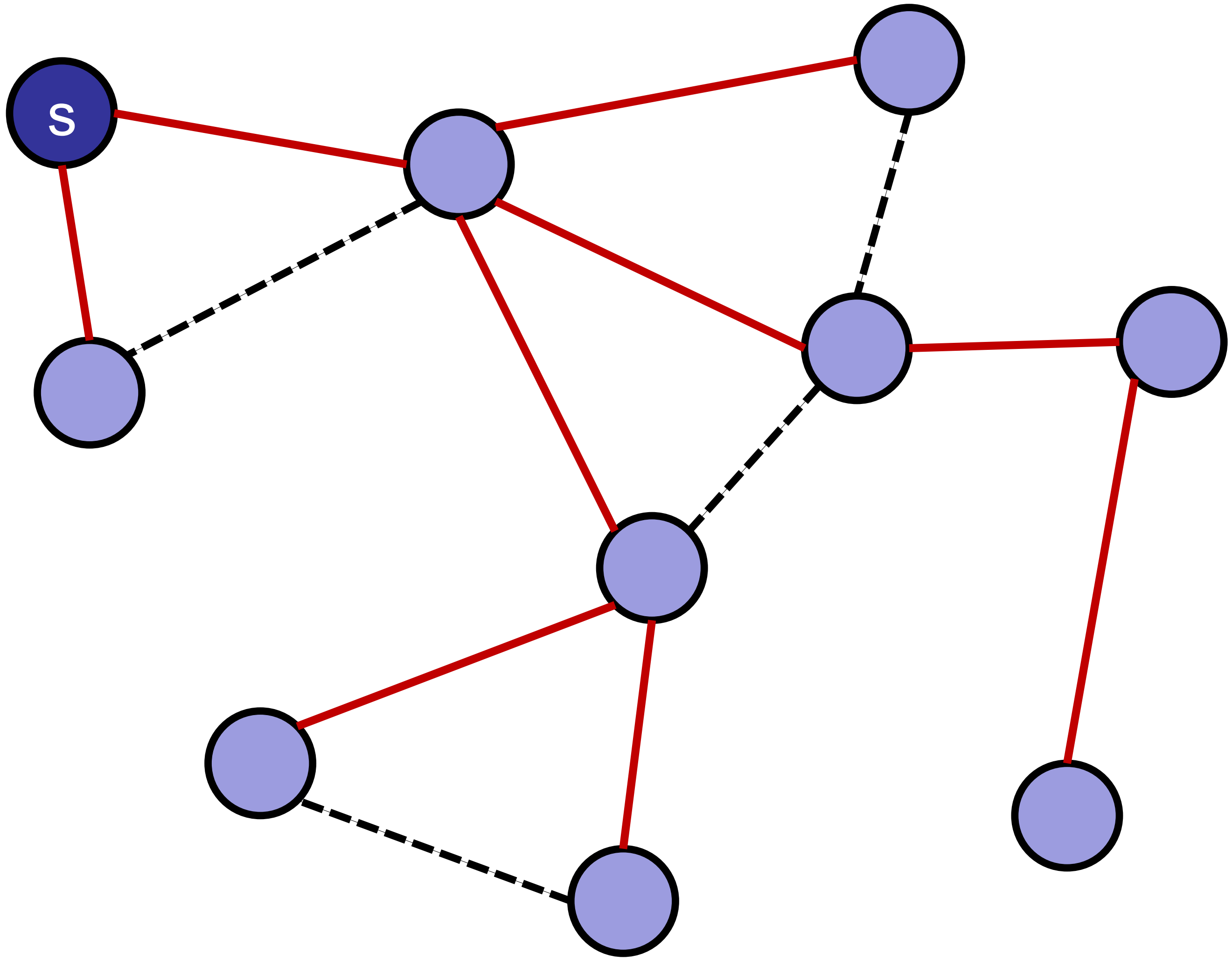

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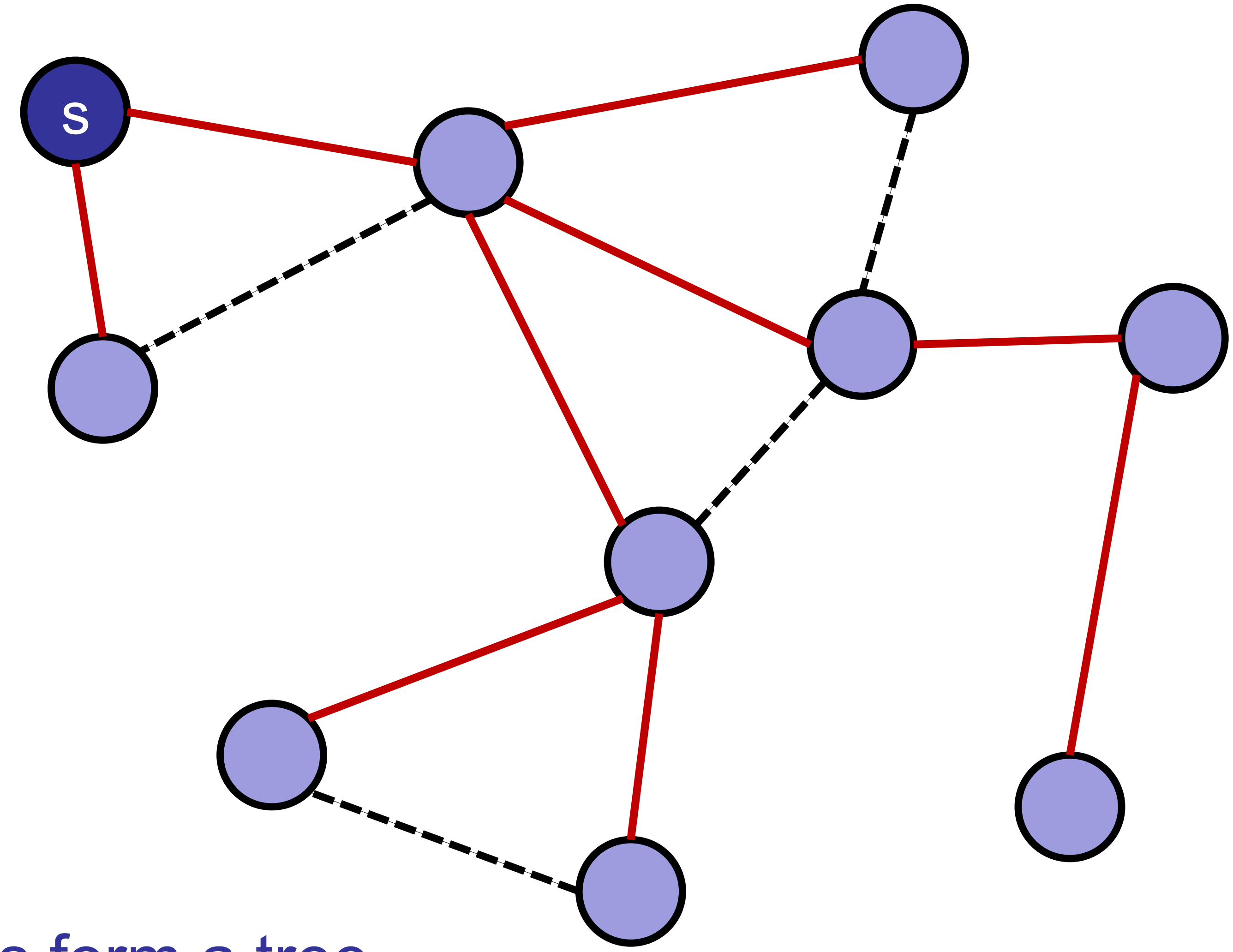
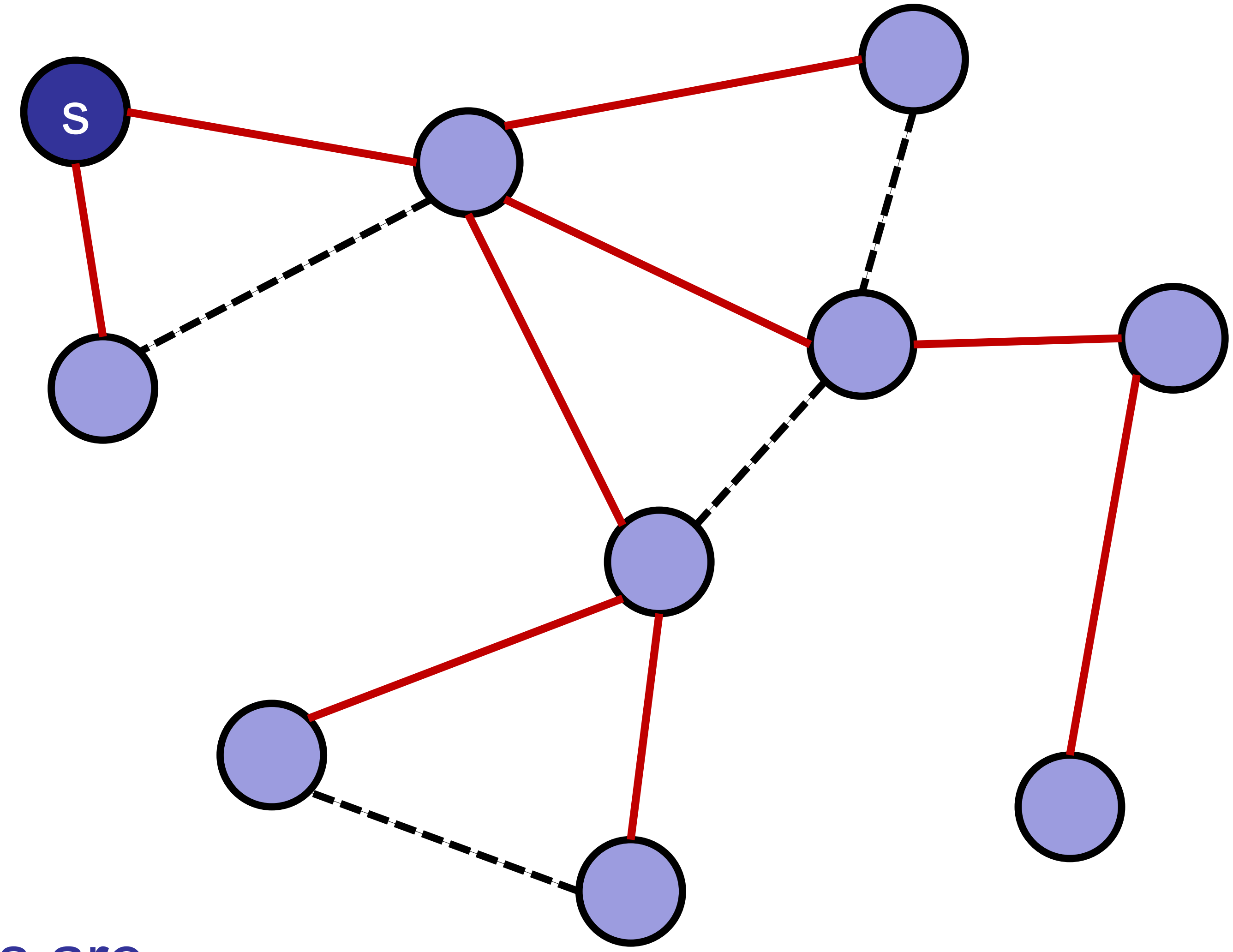
What are the properties of the parent edges?

Breadth First Search



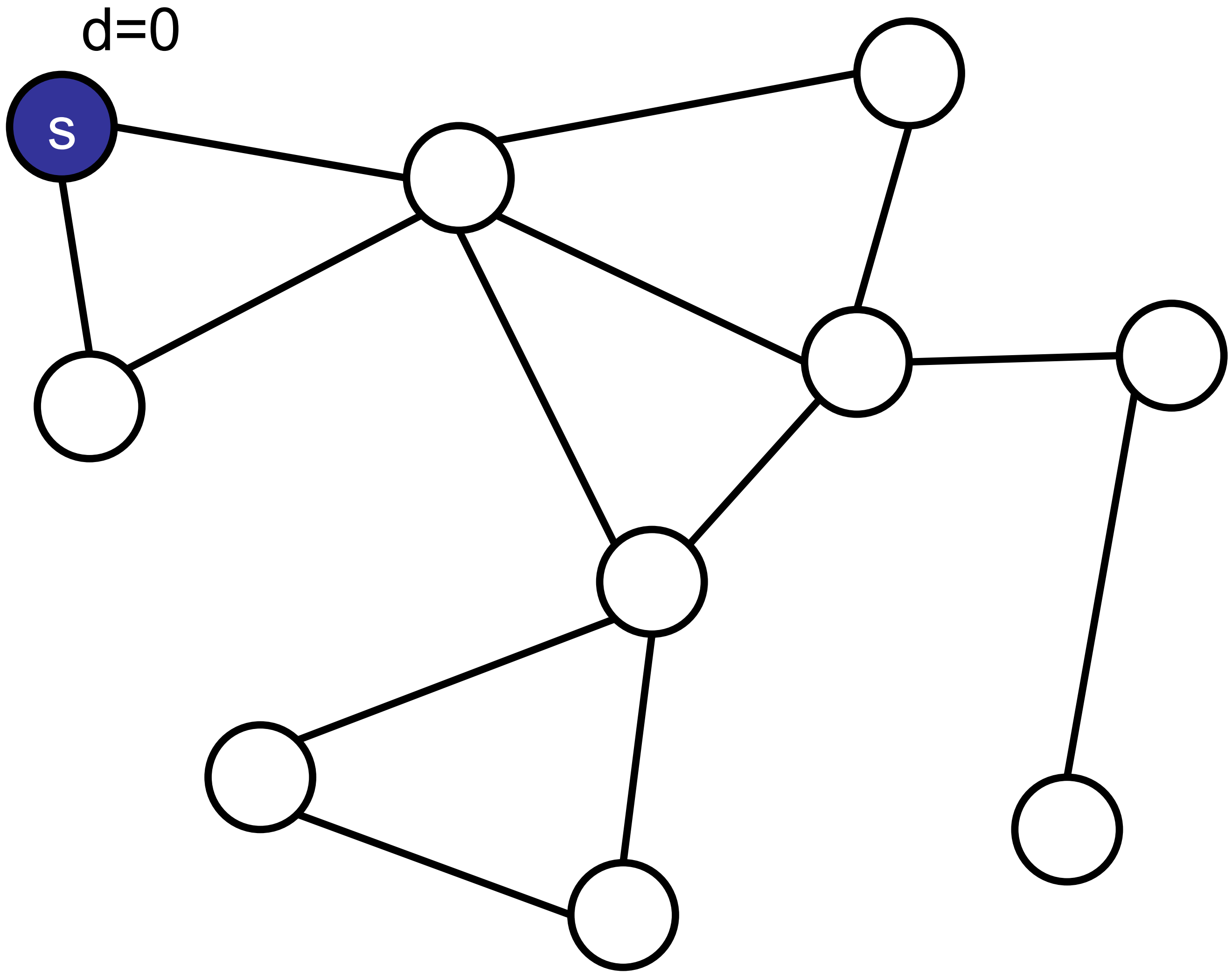
1. Parent edges form a tree
   (No cycles.)

Breadth First Search



2. Parent edges are shortest paths from s.

Breadth First Search



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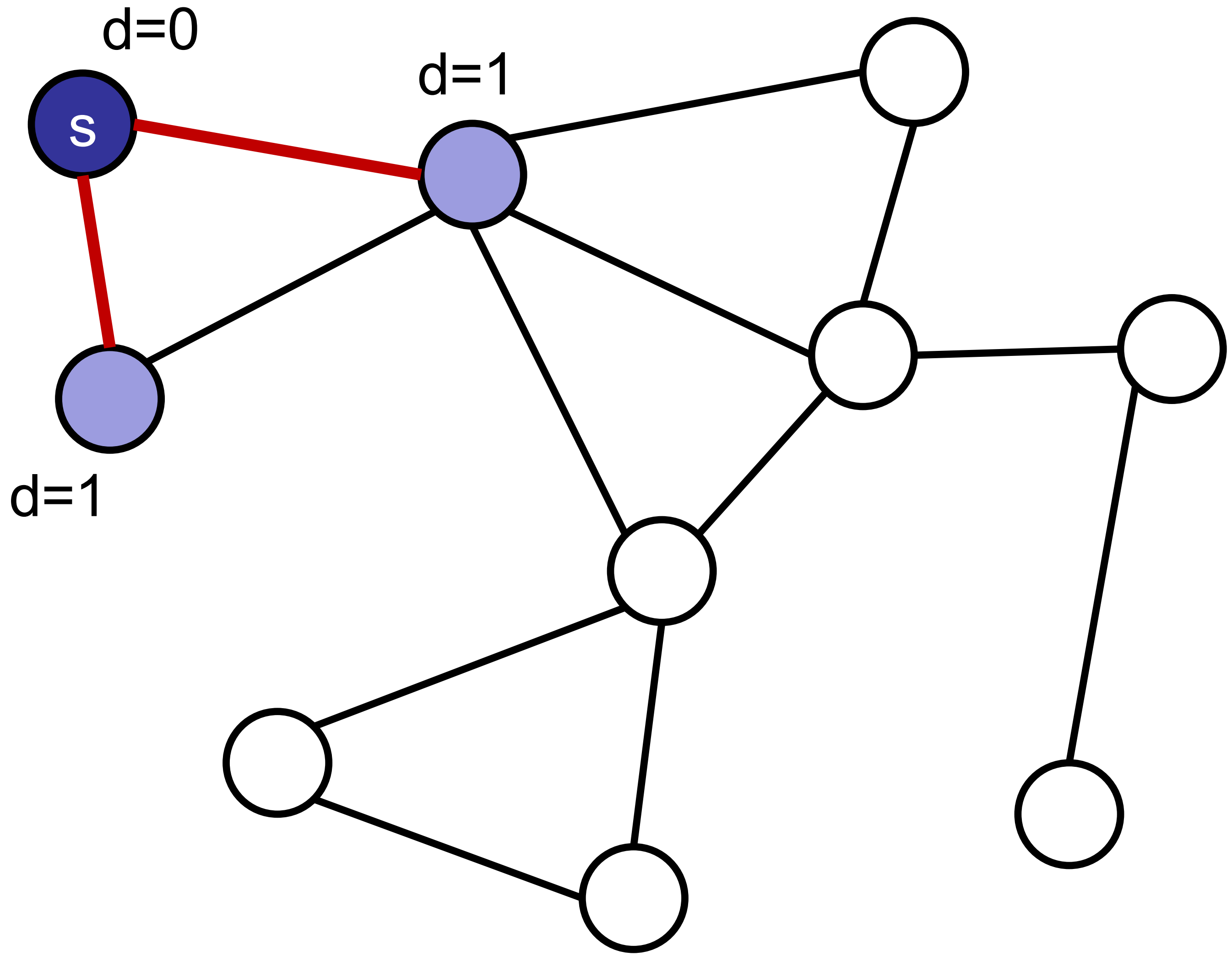

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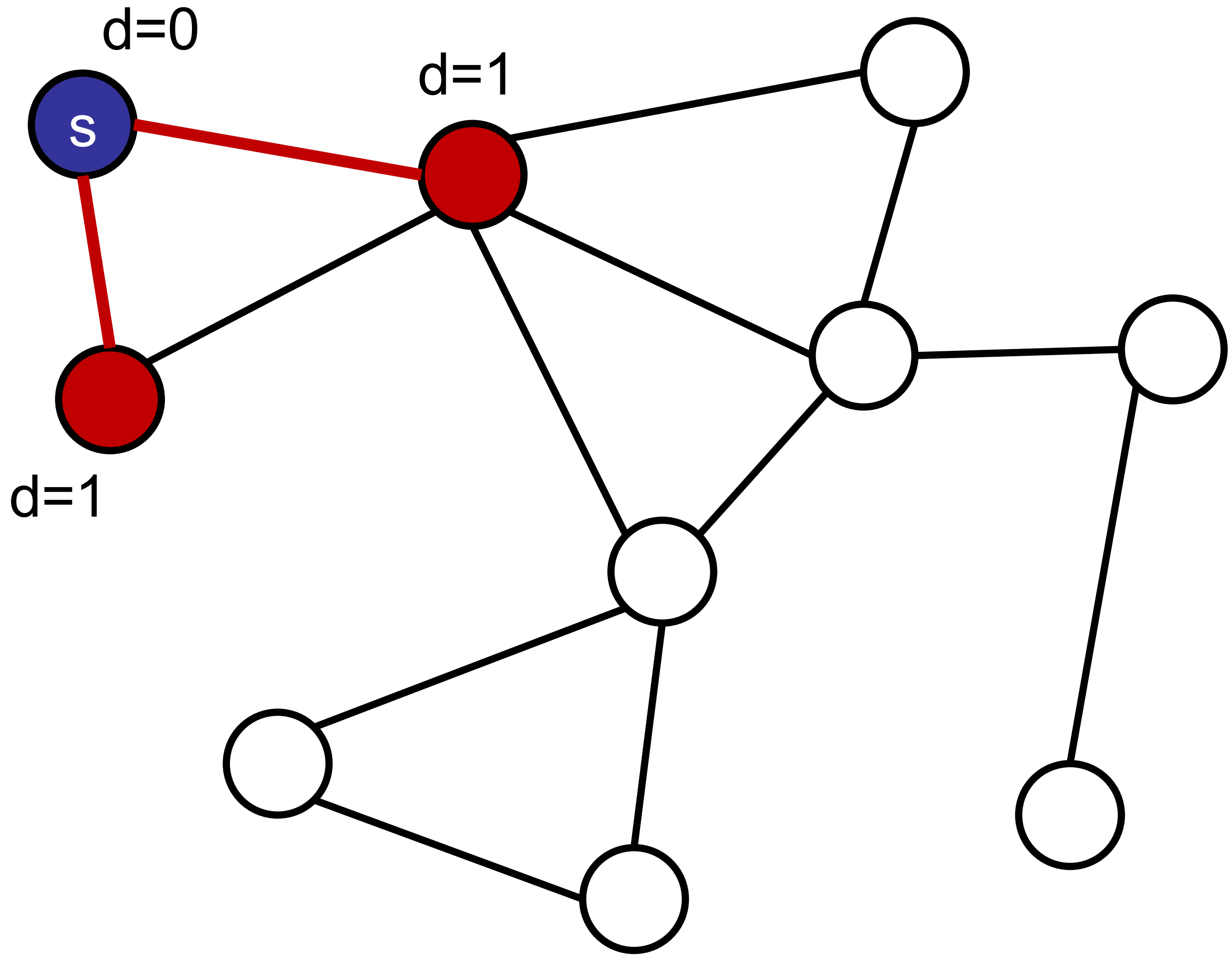

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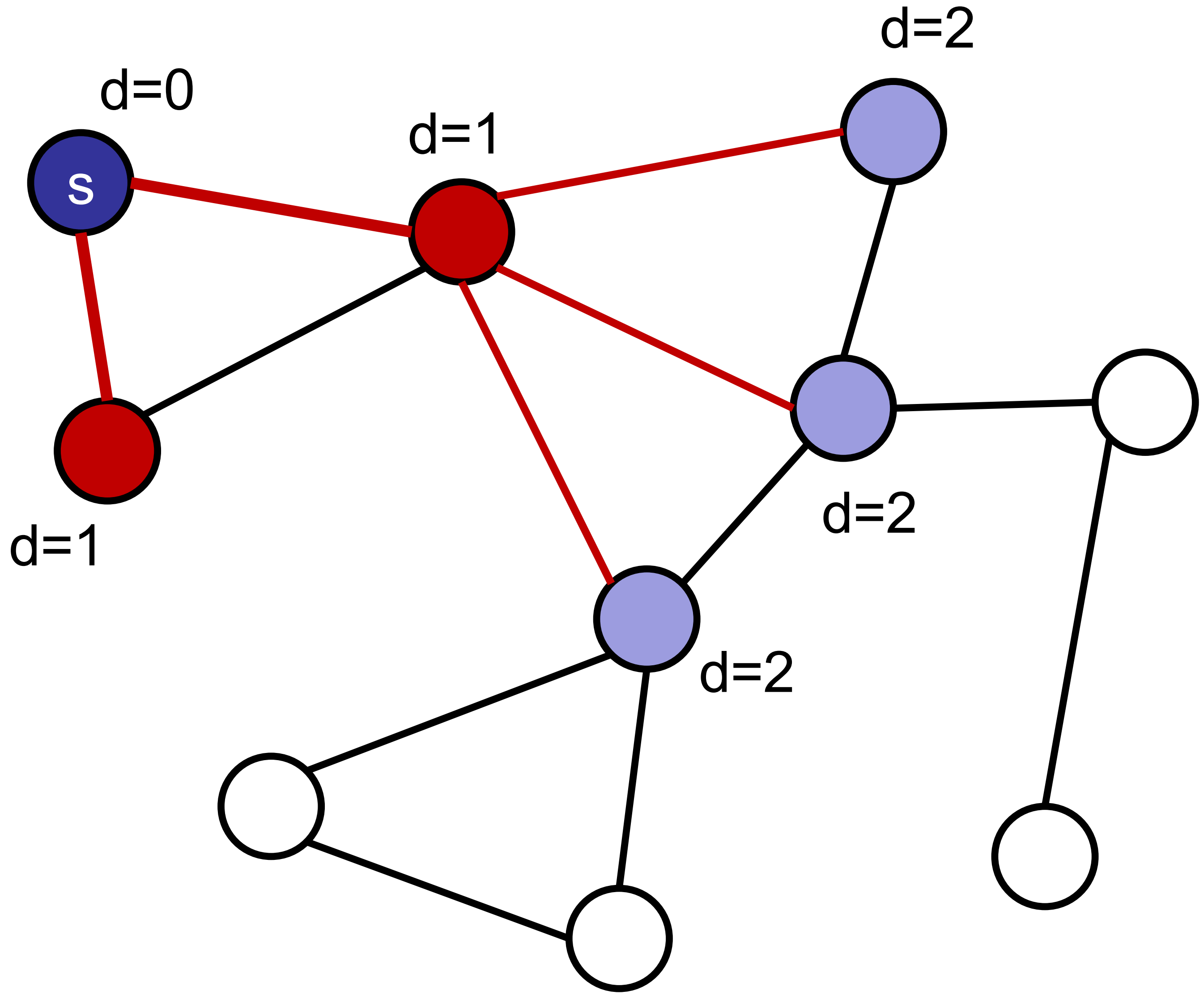

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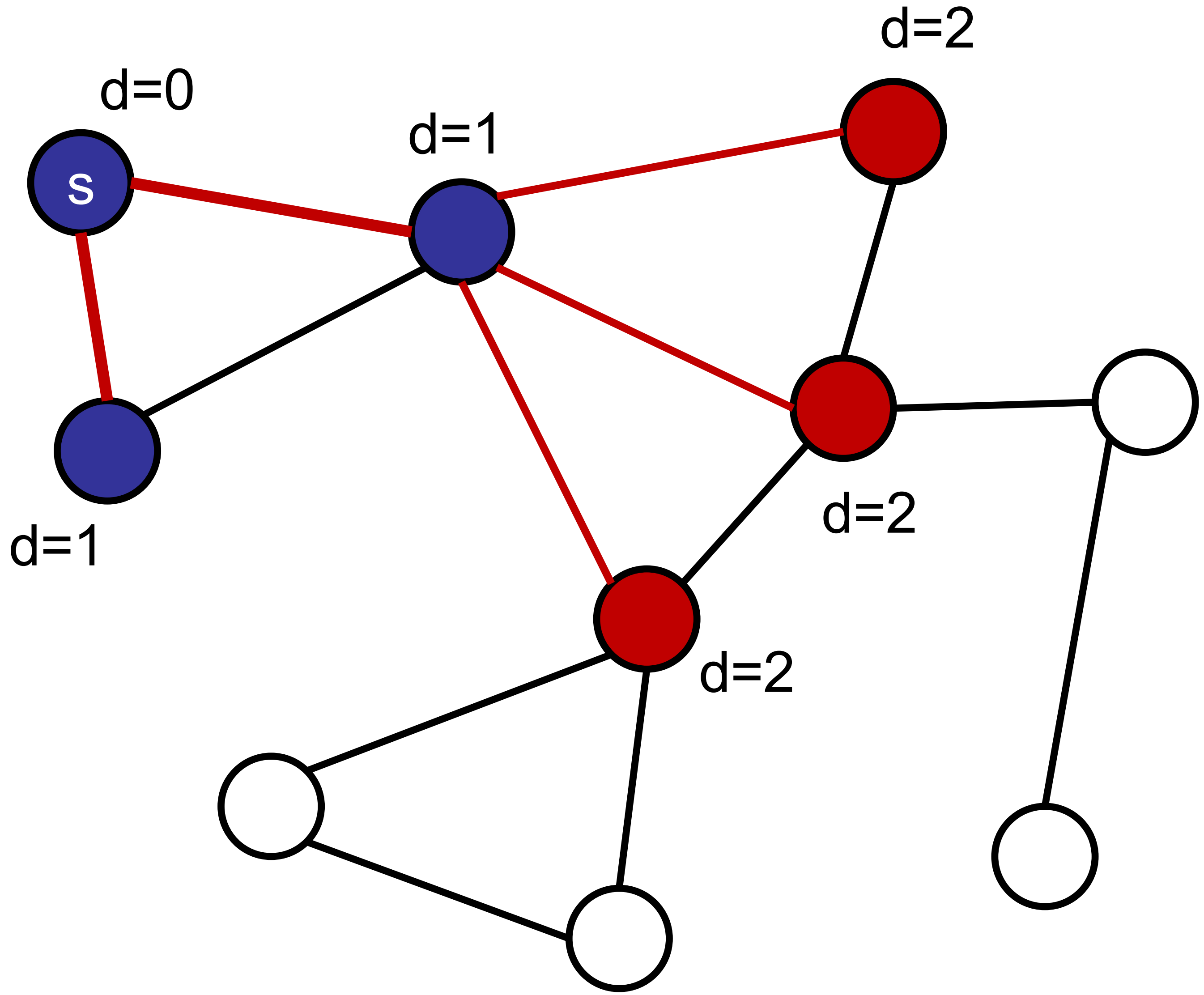

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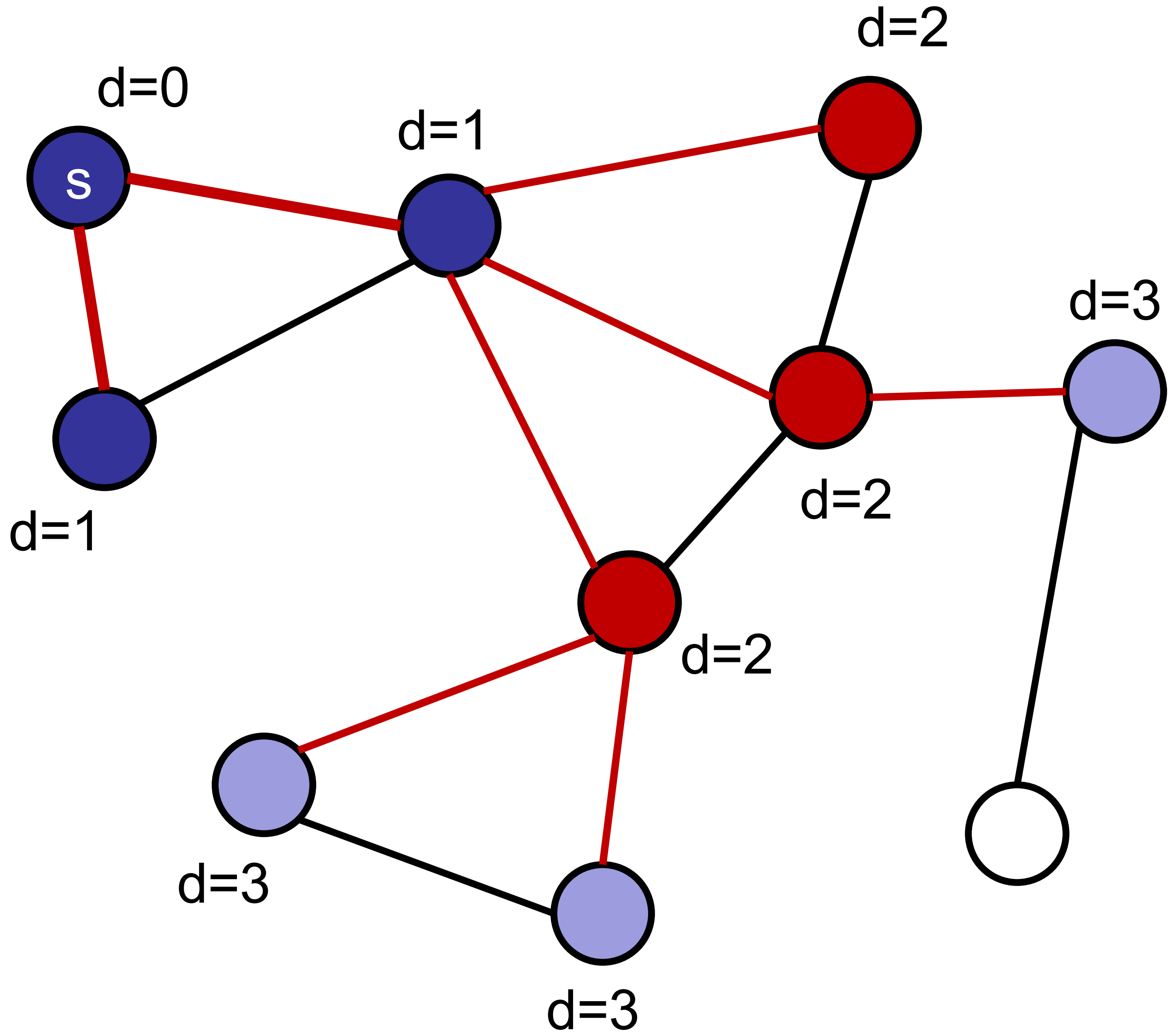

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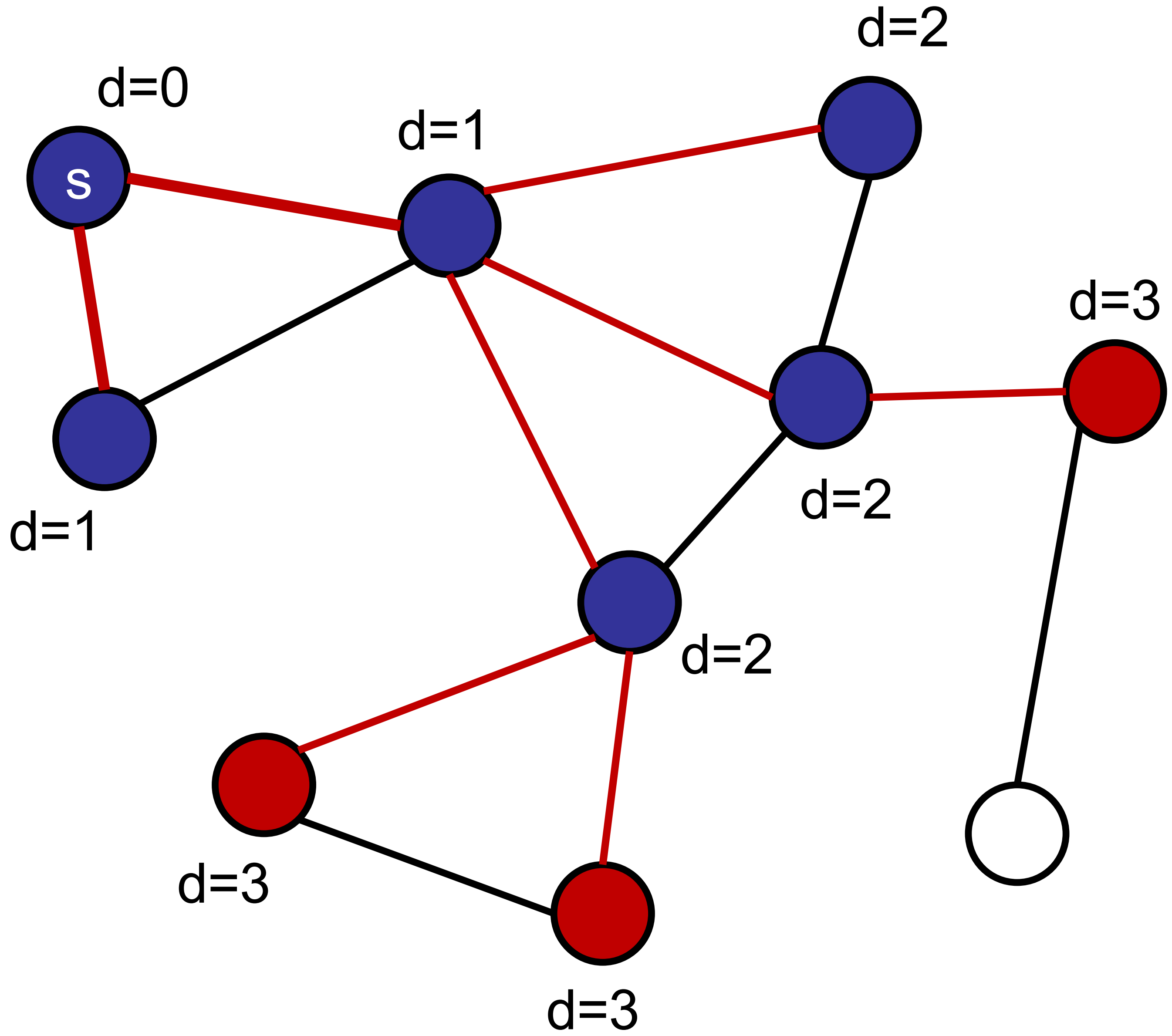

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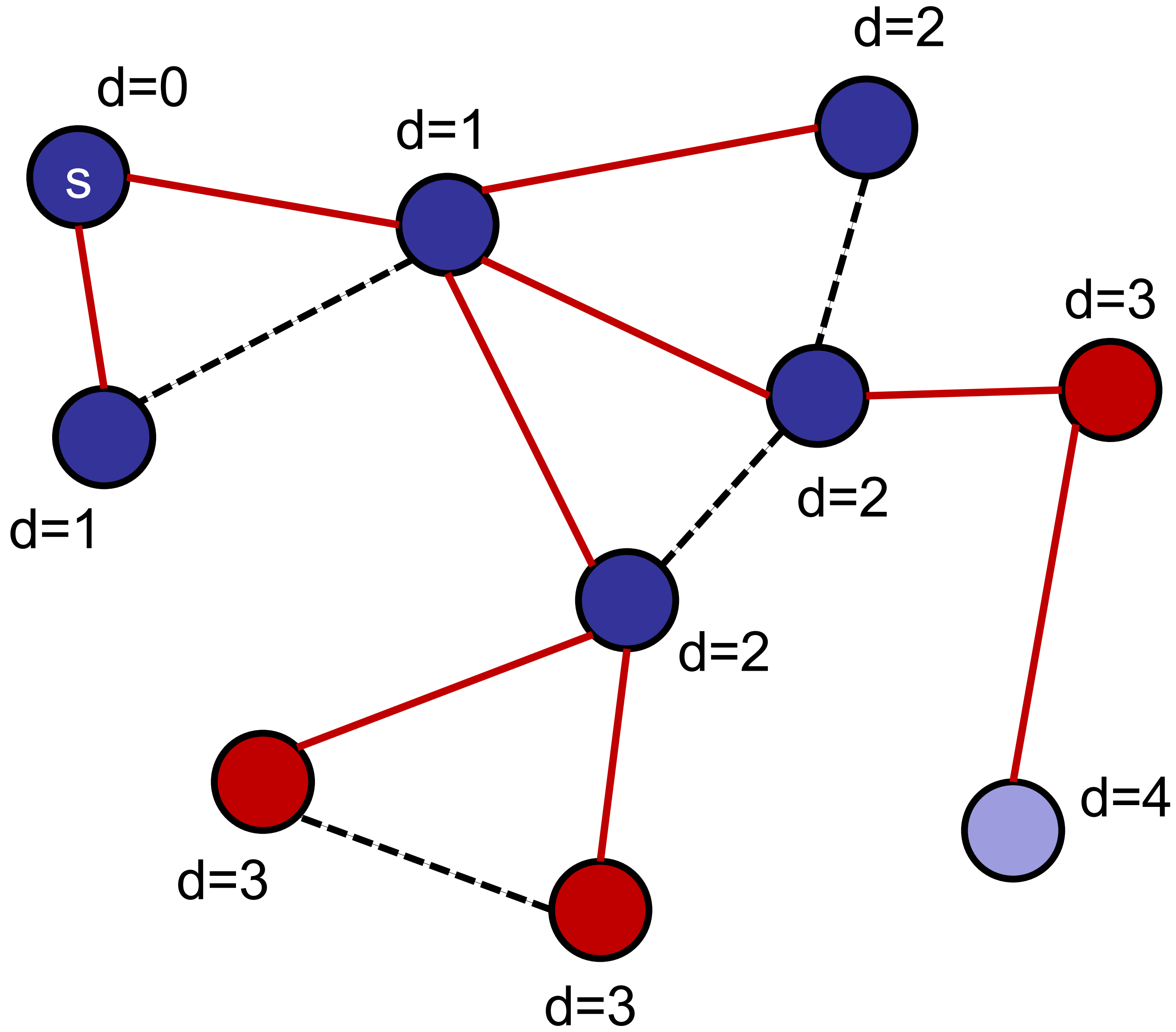

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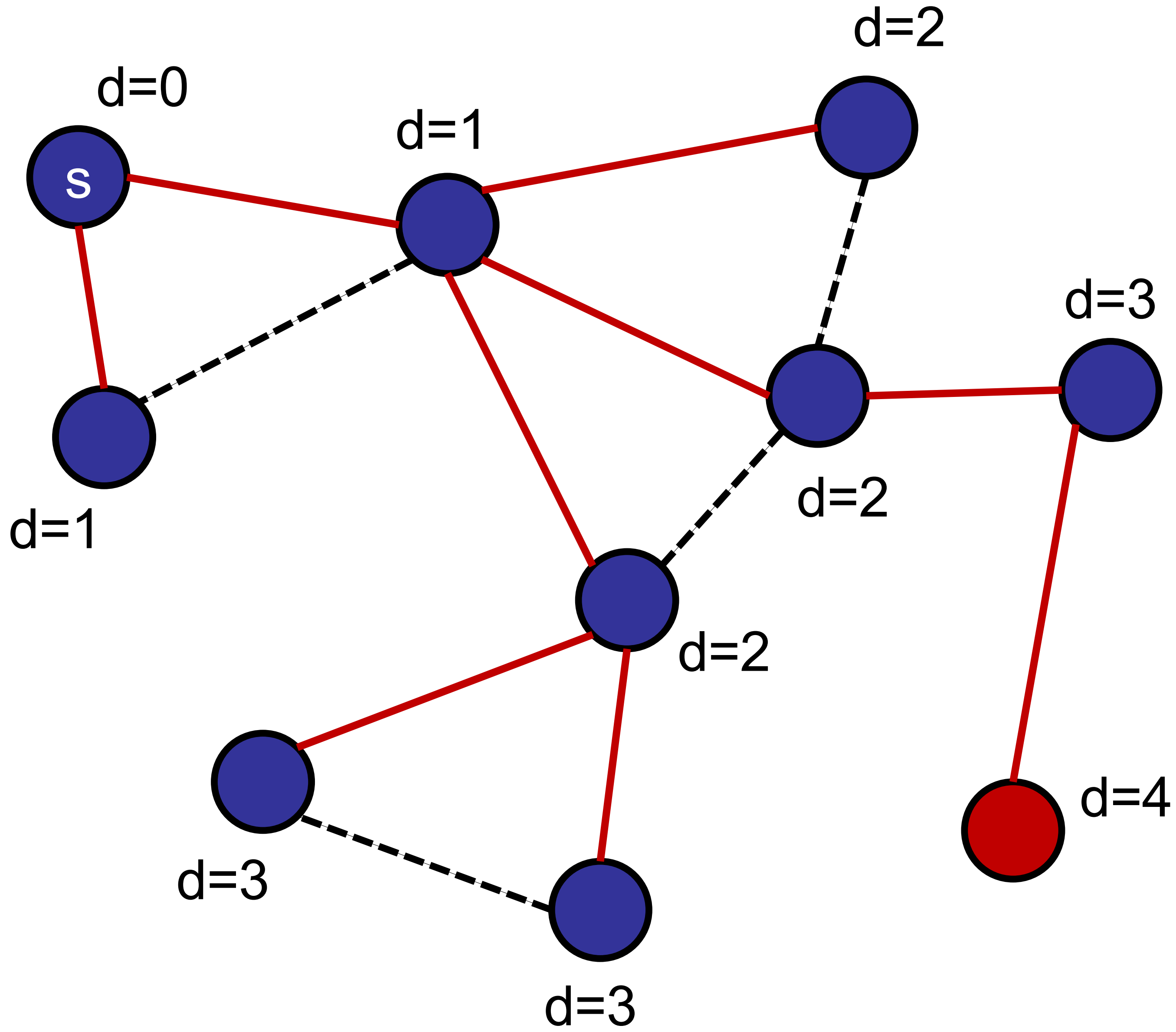

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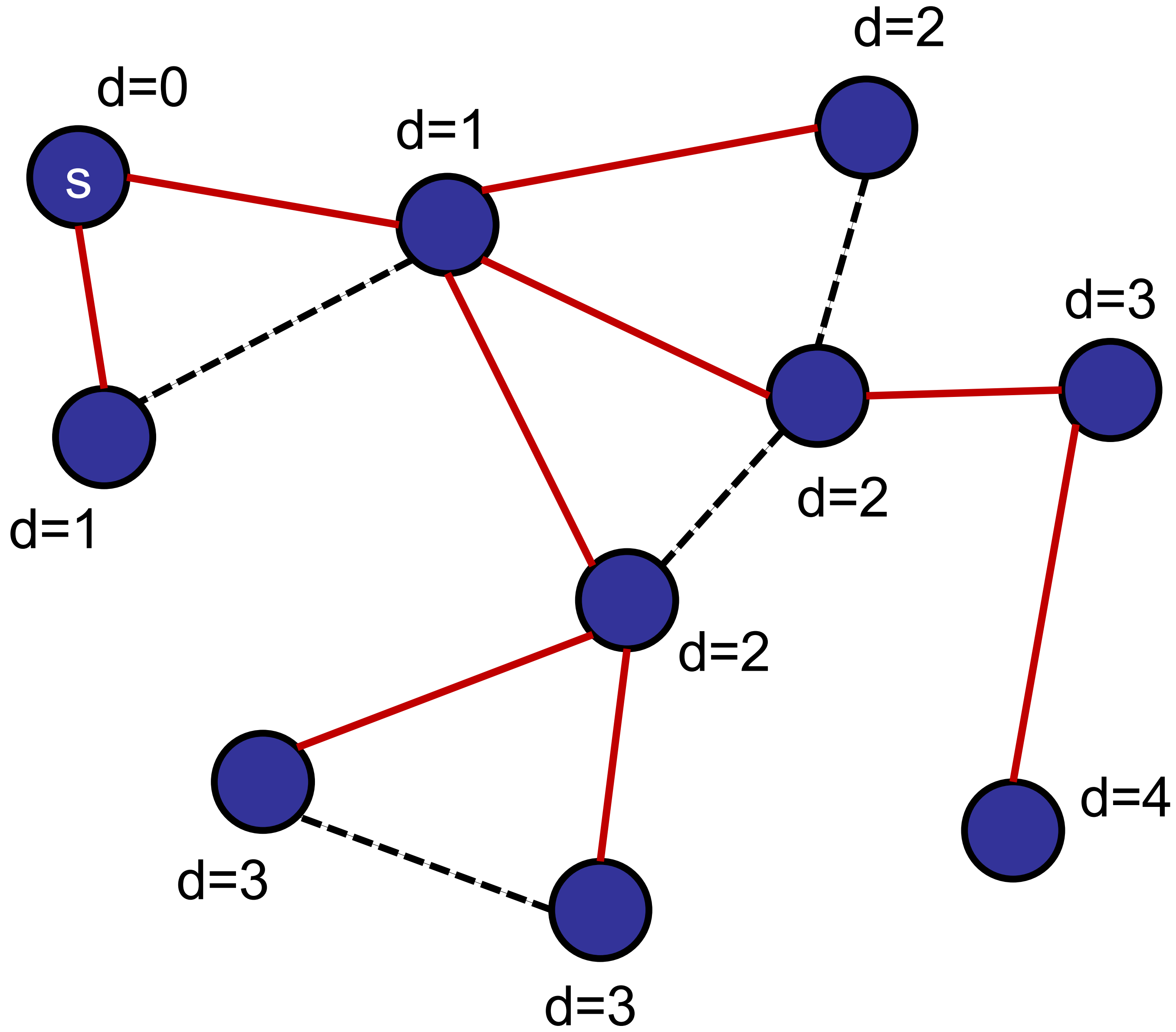
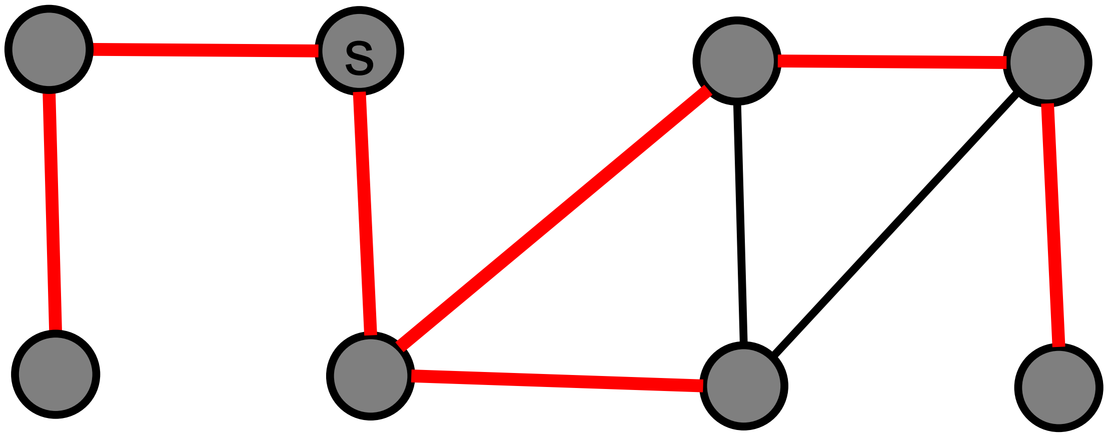


Breadth-First Search

Shortest paths:

- Parent pointers store shortest path.
- Shortest path is a tree.
- (Possibly high degree; possibly high diameter.)



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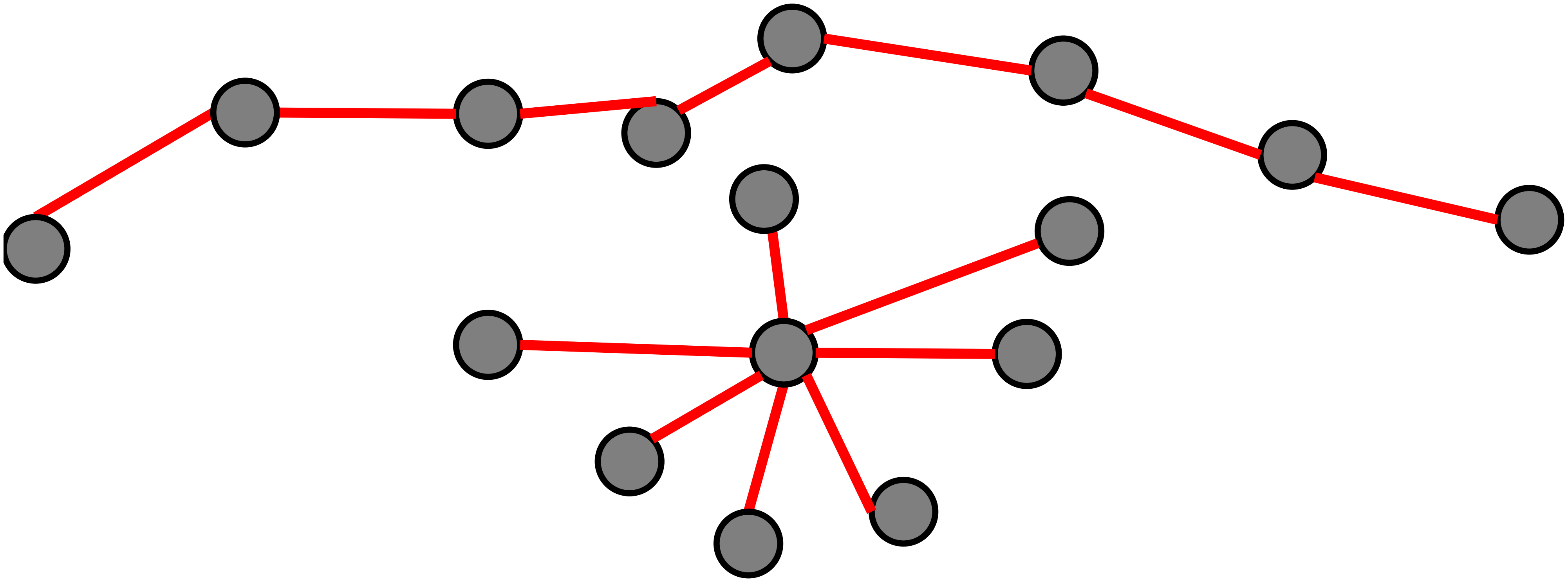
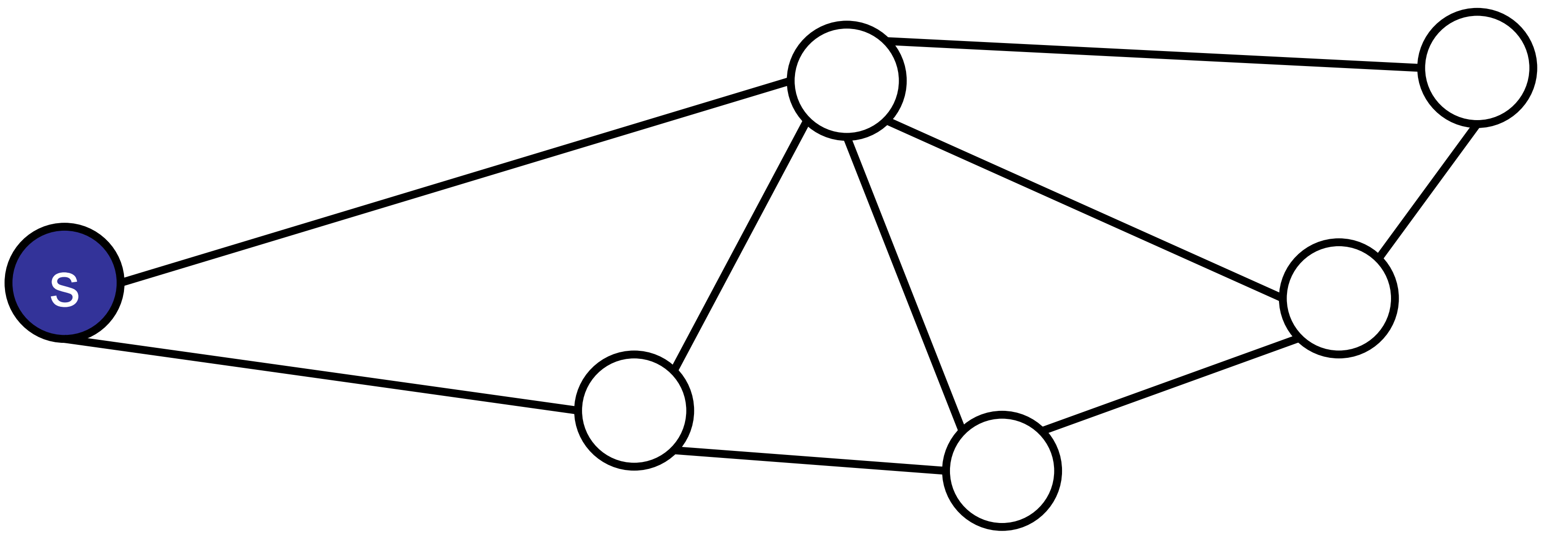


Breadth-First Search

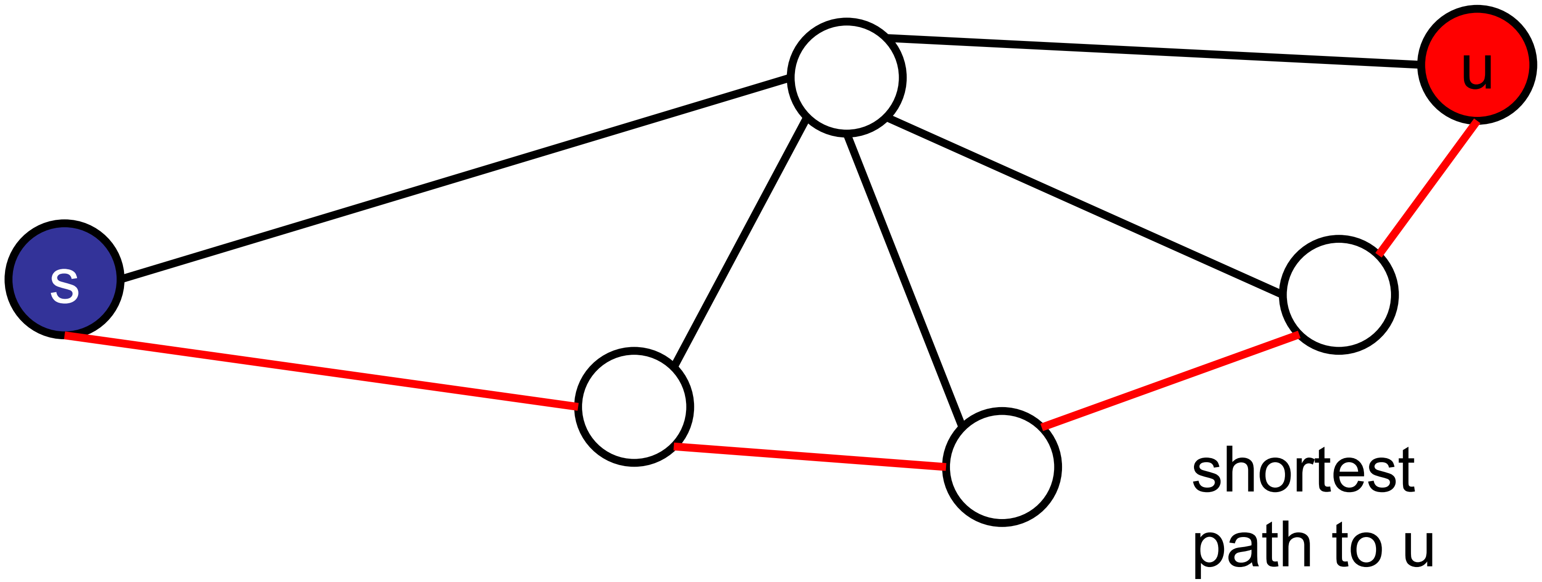
Claim:

- Let P be the set of "shortest paths from s to each node u."
- Shortest paths *always* form a tree
- Can never have a cycle of shortest paths



Breadth-First Search

Assume we have a cycle…



Breadth-First Search

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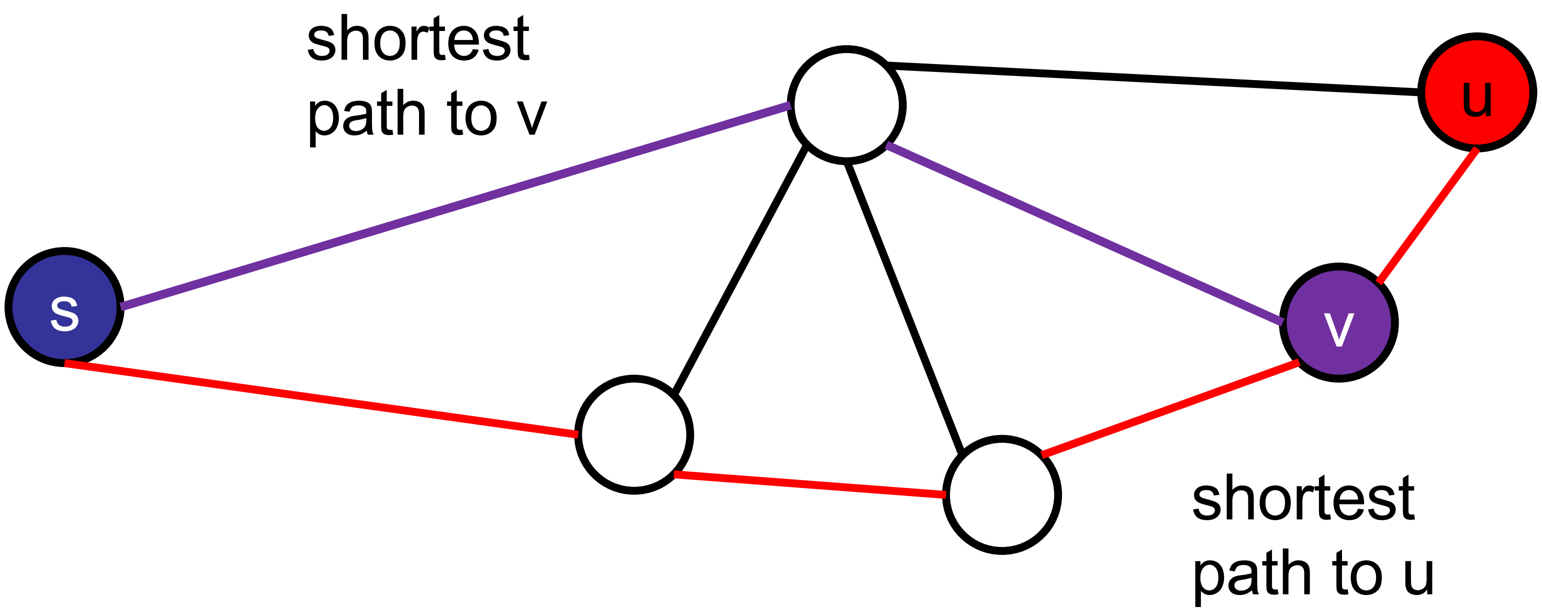
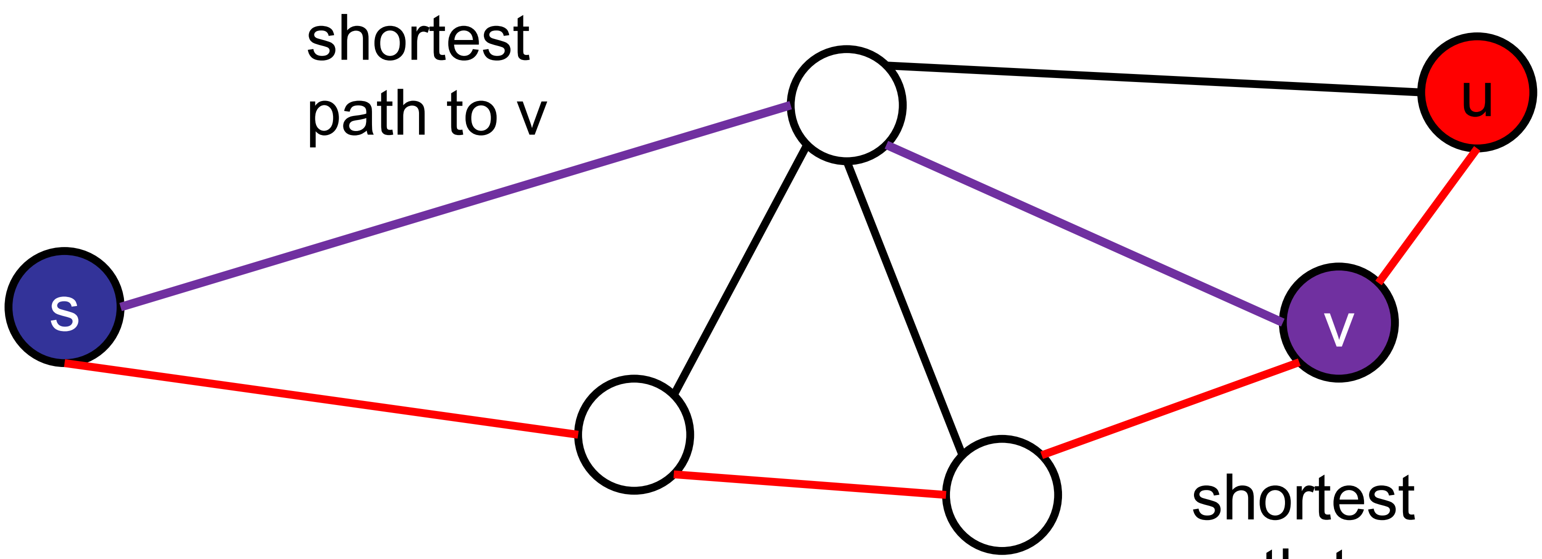


Breadth-First Search

Assume we have a cycle…

Purple path to v is shorter than red path to v.

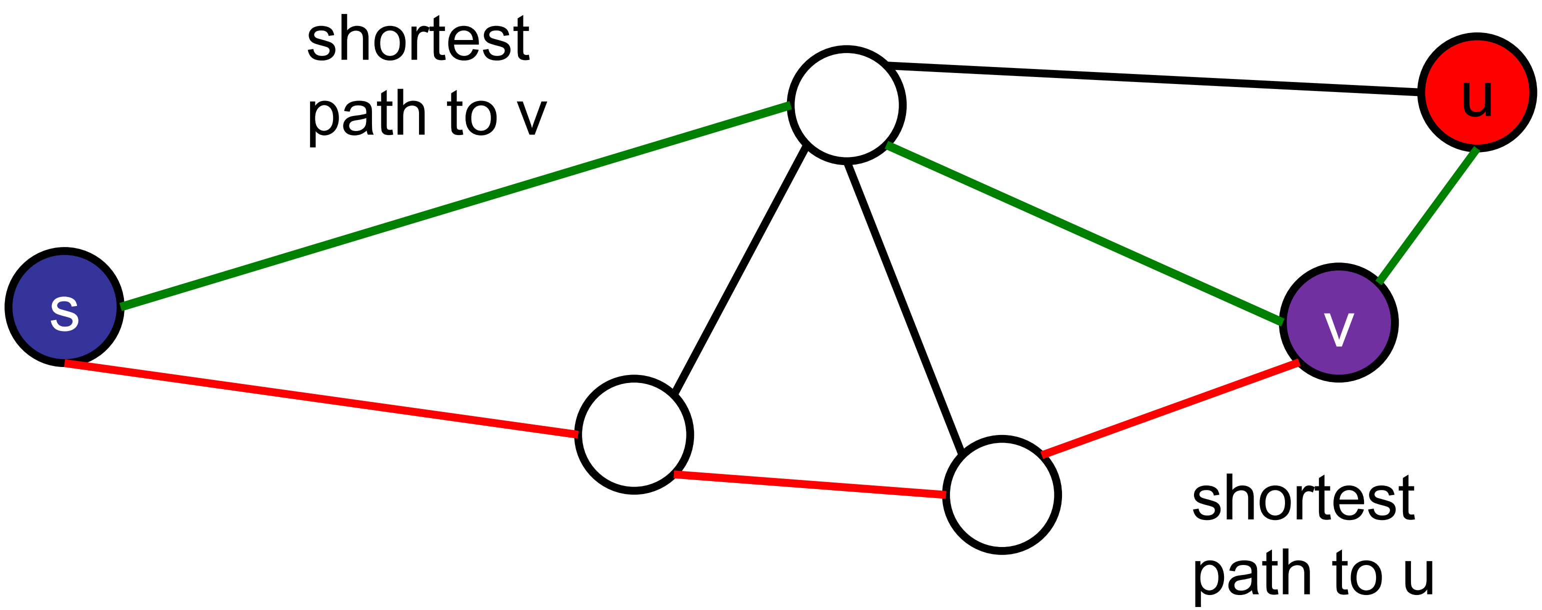


Breadth-First Search

Assume we have a cycle…

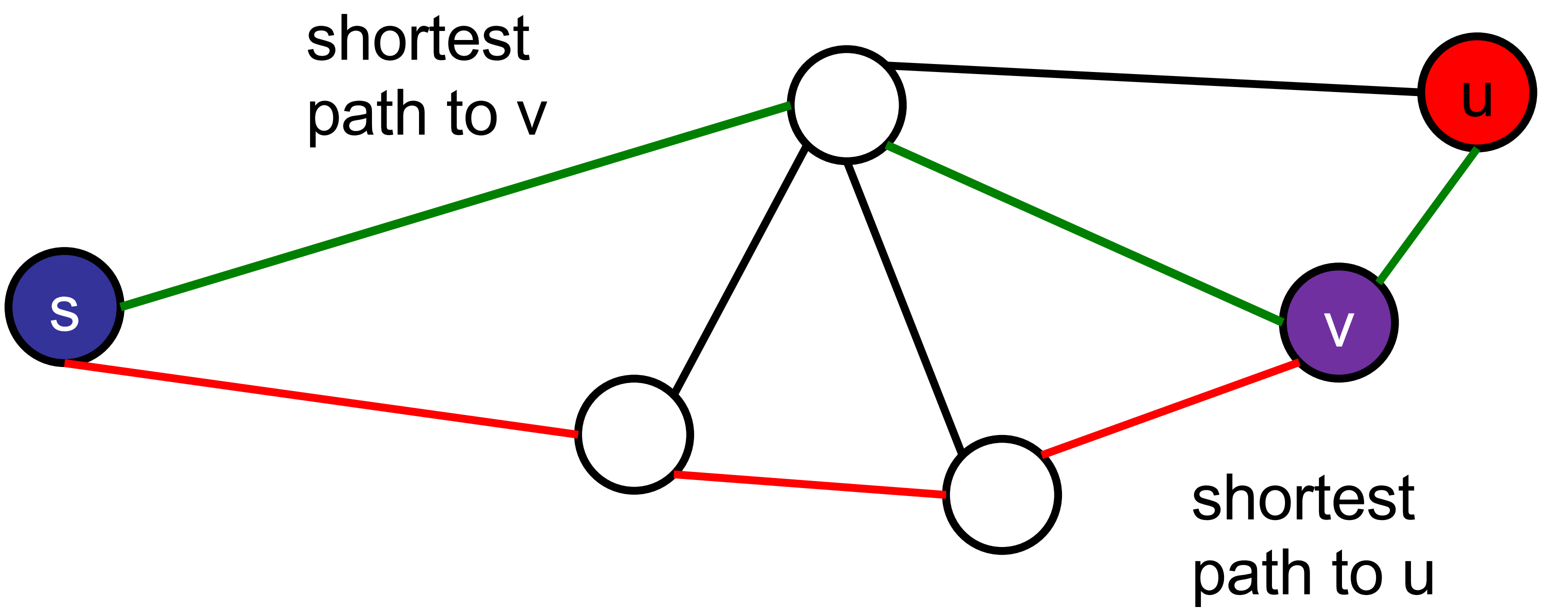
Purple path to v is shorter than red path to v.

So green path to u is shorter than red path to u.



Breadth-First Search

Problem: what if two paths have the same length?



Breadth-First Search

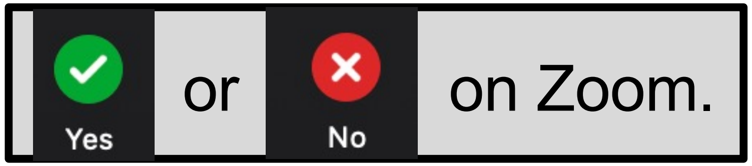
Claim:

- Assume each node has a unique shortest path.
- Let P be the set of “shortest paths from s to each node u.”
- Shortest paths *always* form a tree
- Can never have a cycle of shortest paths

Breadth-First Search

What if non-unique shortest paths?

Will BFS ever construct a cycle?



Breadth-First Search

What if non-unique shortest paths?

Will BFS ever construct a cycle?

NO!   Because of the ”visited” check.

```
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while frontier is not empty:
   next-frontier = {}
   for each node u in the frontier:
     for each edge (u,v) in the graph:
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   frontier = next-frontier
```
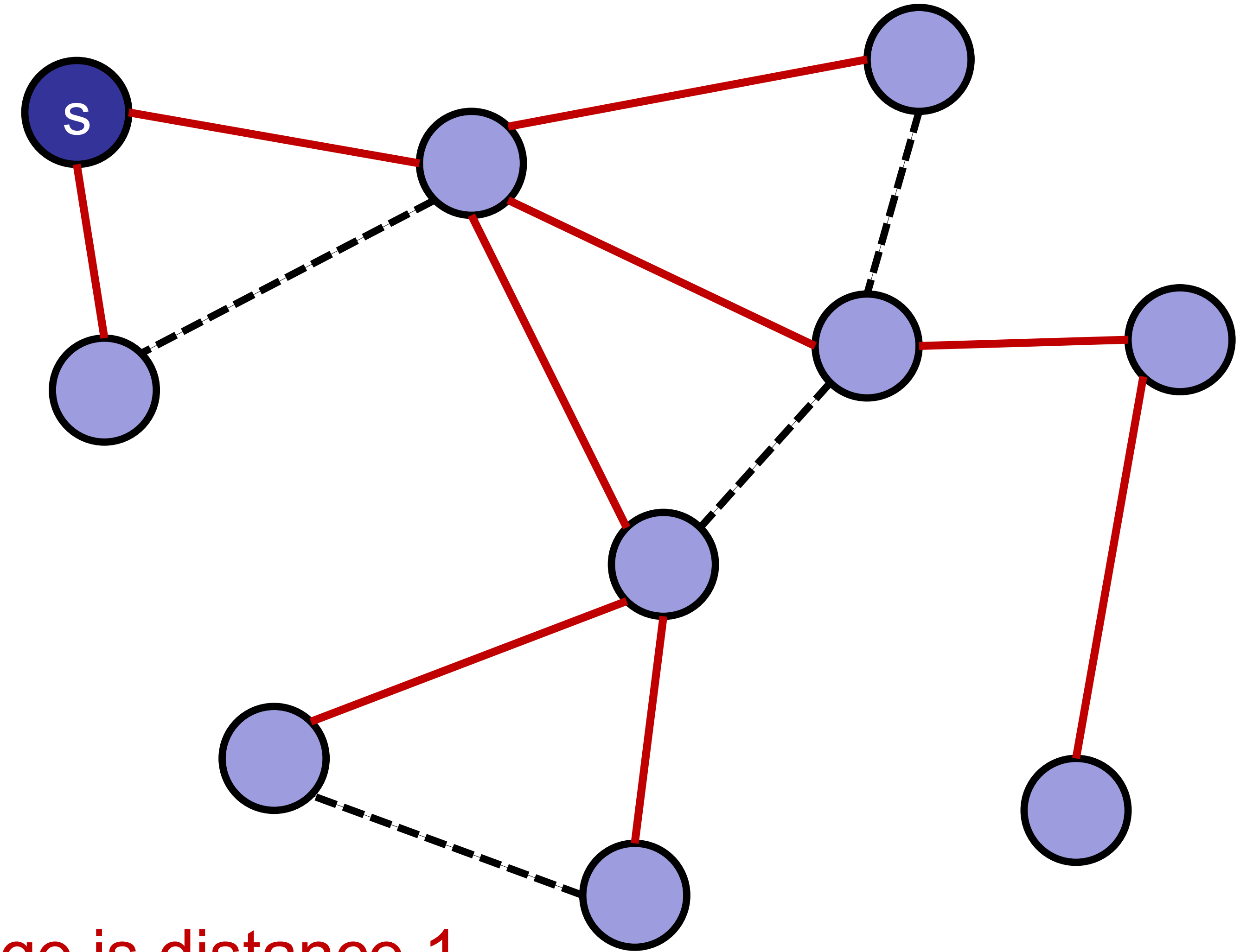

Breadth First Search



Beware: each edge is distance 1.

Next week: graphs with distances on the edges.

When does BFS fail to visit every node?

1. In a clique.
2. In a cycle.
3. In a graph with two components.
4. In a sparse graph.
5. In a dense graph.
6. Never.

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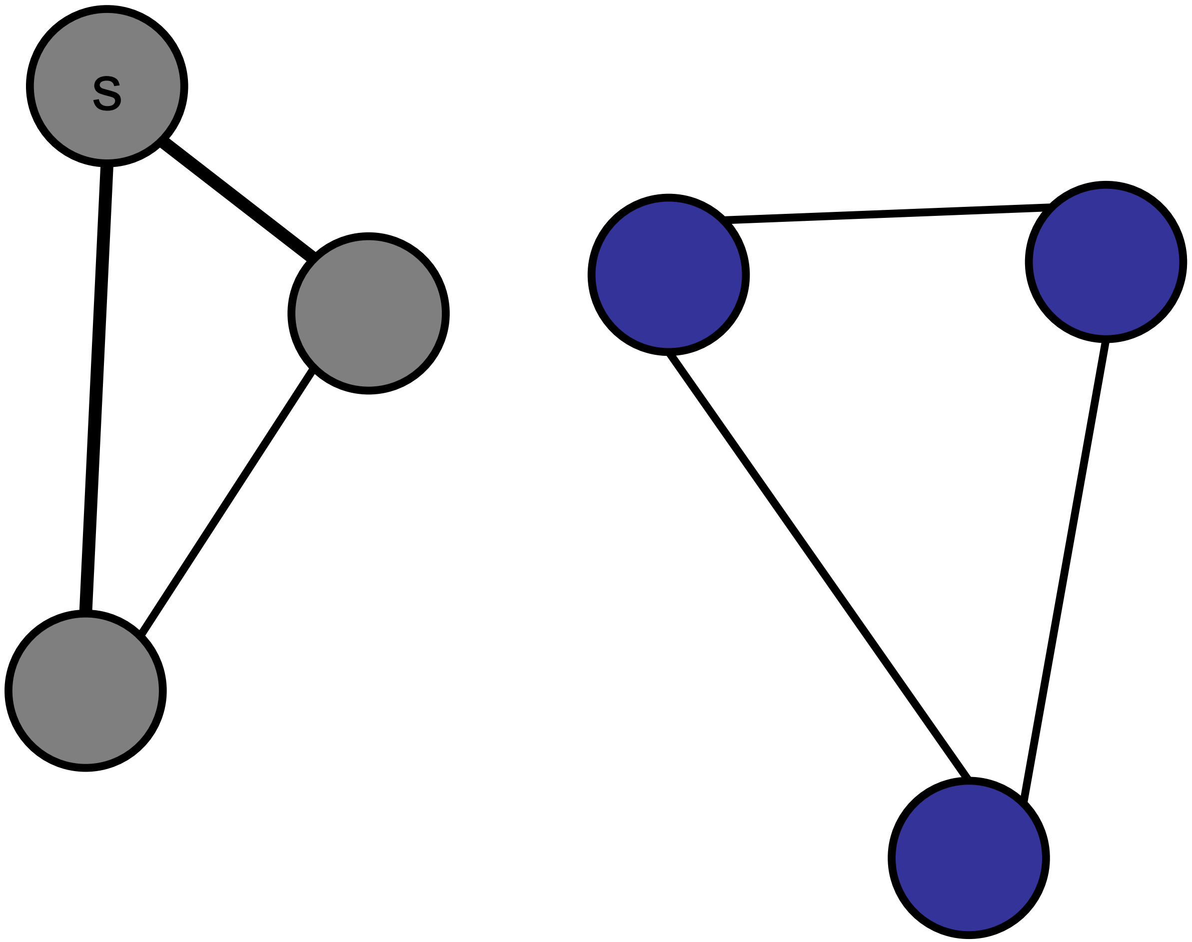
is open

BFS on Disconnected Graph

Example:



Visiting every component

Breadth-First Search:

```
for each node u in the graph:
    if u is not marked visited:
        frontier = {u}
        while frontier is not empty:
            next-frontier = {}
            for each node u in the frontier:
                for each edge (u,v) in the graph:
                    if v is not marked visited, add v to next-frontier
                    mark v as visited.
            frontier = next-frontier
```

Important if need to visit *every* node.
Important if searching for something.
NOT good for measuring distances.

The running time of BFS (using adjacency list) is:

1. O(V)
2. O(E)
3. O(V+E)
4. O(VE)
5. (V^2)
6. I have no idea.

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The running time of BFS (using adjacency list) is:

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Depends on adjacency list vs. adjacency matrix.

Here: assume adjacency list.

Breadth-First Search

Analysis:

– Vertex v = "start" once.

O(V)

– Vertex v added to nextFrontier (and frontier) once.

  - After visited, never re-added.

– Each v.nbrlist is enumerated once.

  - When v is removed from frontier.

O(E)

Running time

Breadth-First Search:

```
for each node u in the graph:
    if u is not marked visited:
        frontier = {u}
        while frontier is not empty:
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            frontier = next-frontier
```

Each node is only in ONE frontier.

Each edge only has two endpoints and so is examined only twice.

Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

Graph representation:

- Adjacency list

Depth-First Search

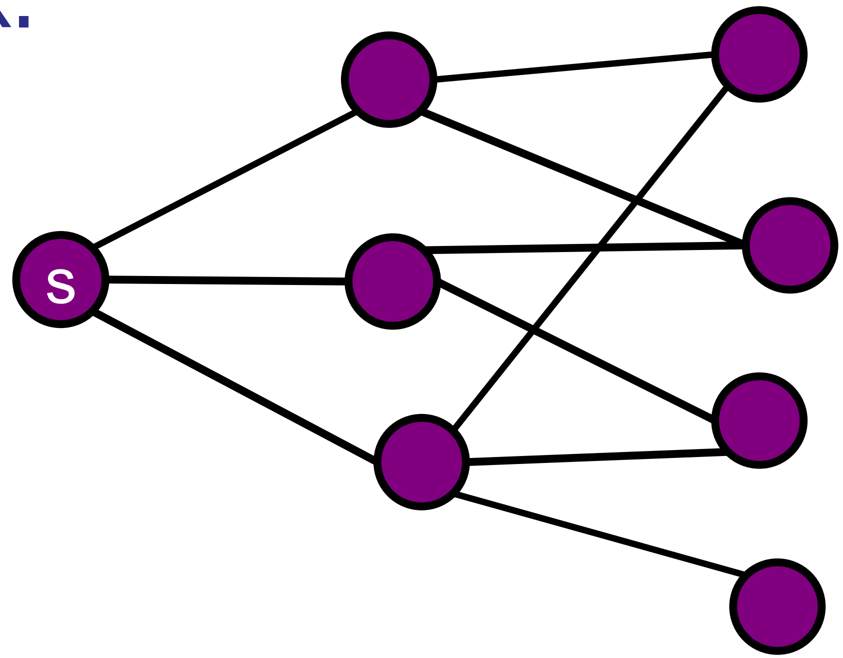
Exploring a maze:

- Follow path until stuck.
- Backtrack along breadcrumbs until reach unexplored neighbor.
- Recursively explore.

Searching a graph

Depth-First Search:

- Follow path until you get stuck
- Backtrack until you find a new edge
- Recursively explore it
- Don't repeat a vertex.



Searching a graph

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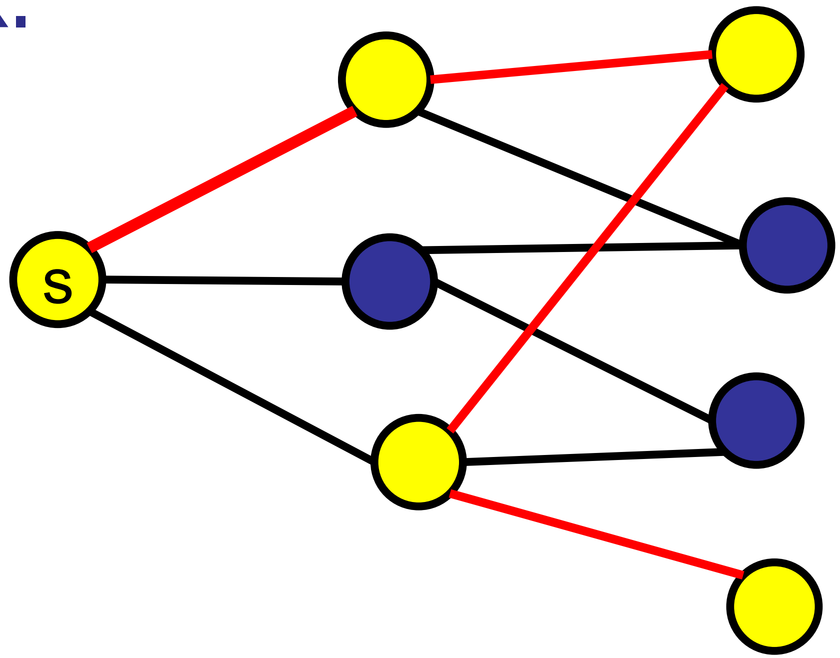

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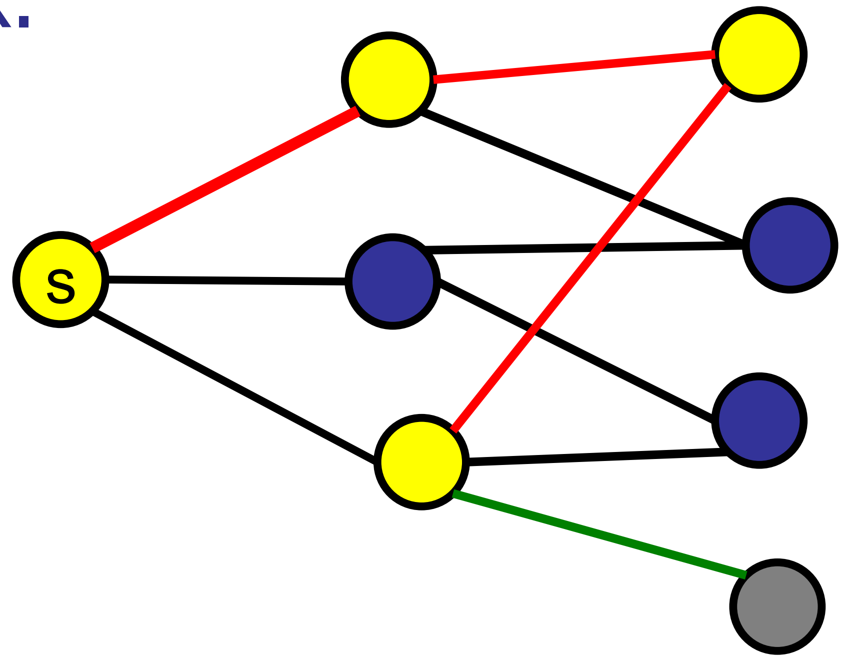

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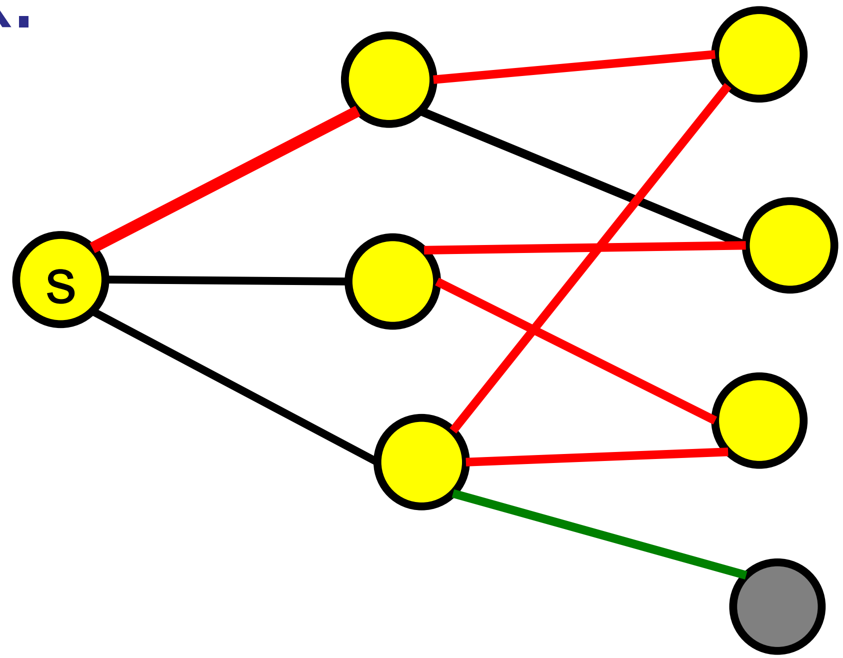

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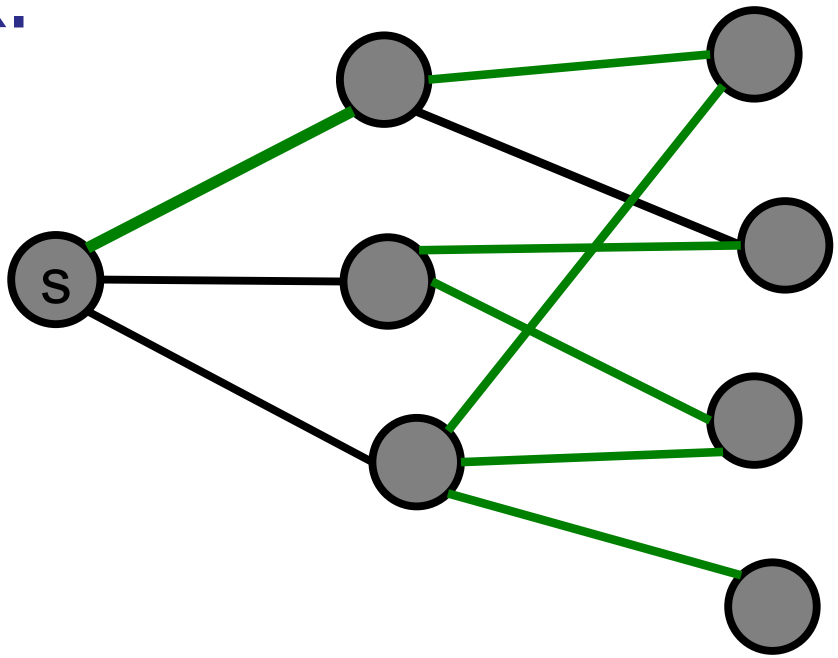


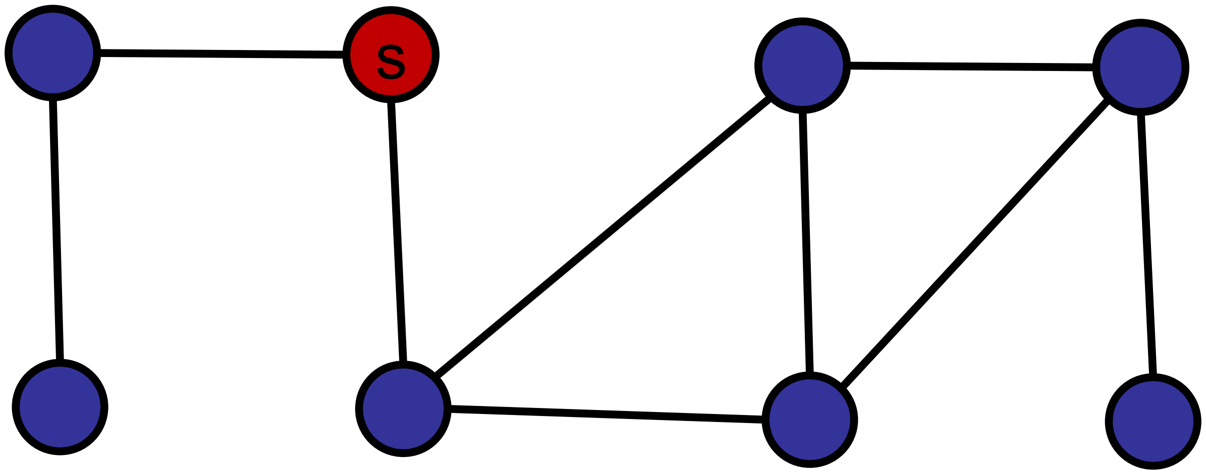
Depth-First Search

```
DFS-visit(Node[] nodeList, boolean[] visited, int startId){
  for (Integer v : nodeList[startId].nbrList) {
     if (!visited[v]){
          visited[v] = true;
          DFS-visit(nodeList, visited, v);
     }
  }
}
```

Depth-First Search

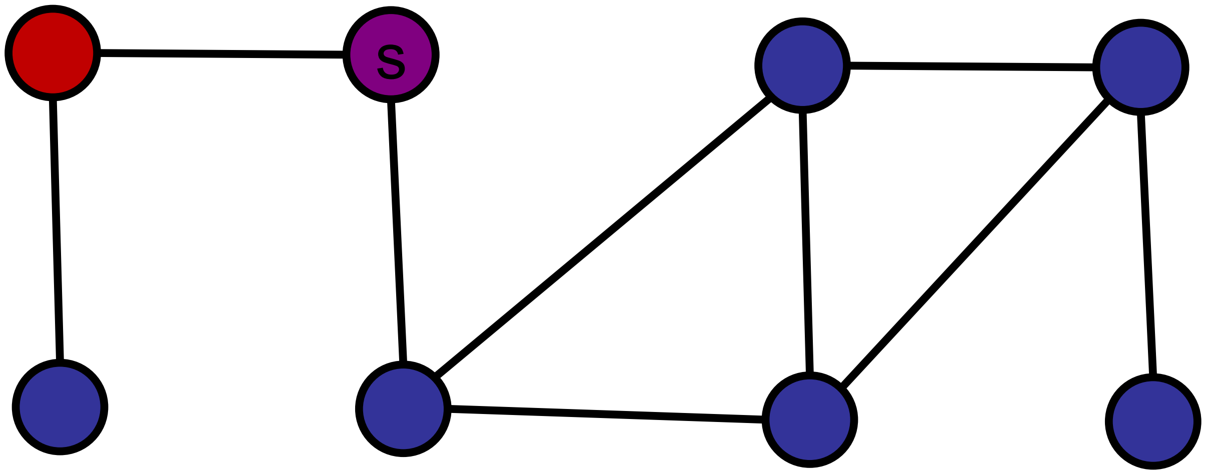
```
DFS(Node[] nodeList){
  boolean[] visited = new boolean[nodeList.length];
  Arrays.fill(visited, false);


  for (start = i; start<nodeList.length; start++) {
     if (!visited[start]){
           visited[start] = true;
           DFS-visit(nodeList, visited, start);
     }
  }
}
```

Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Depth-First Search Example



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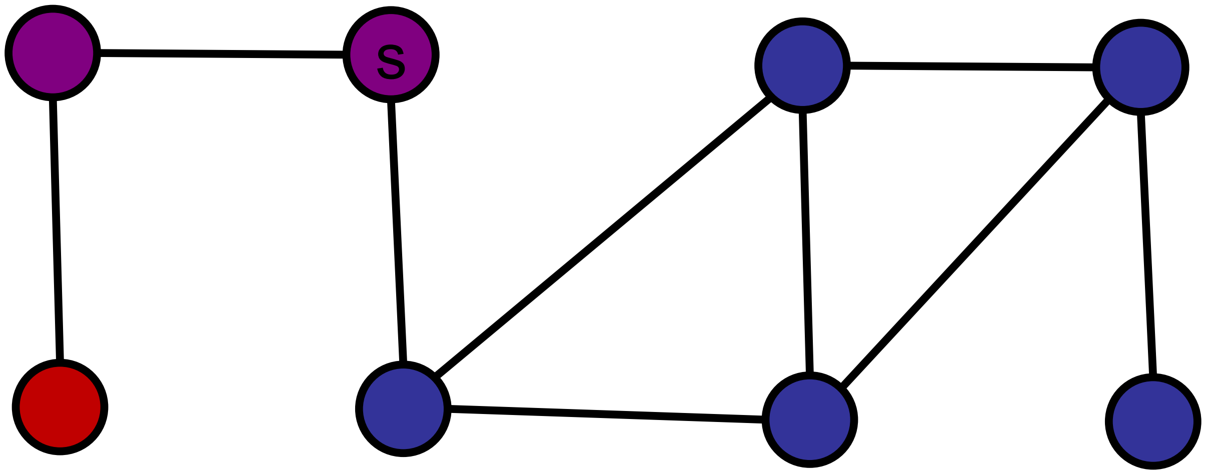

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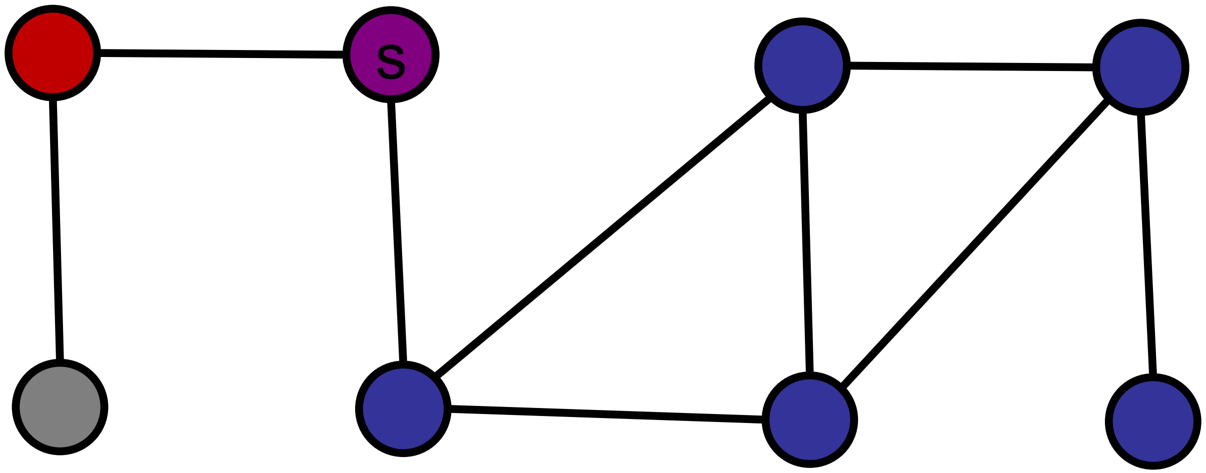

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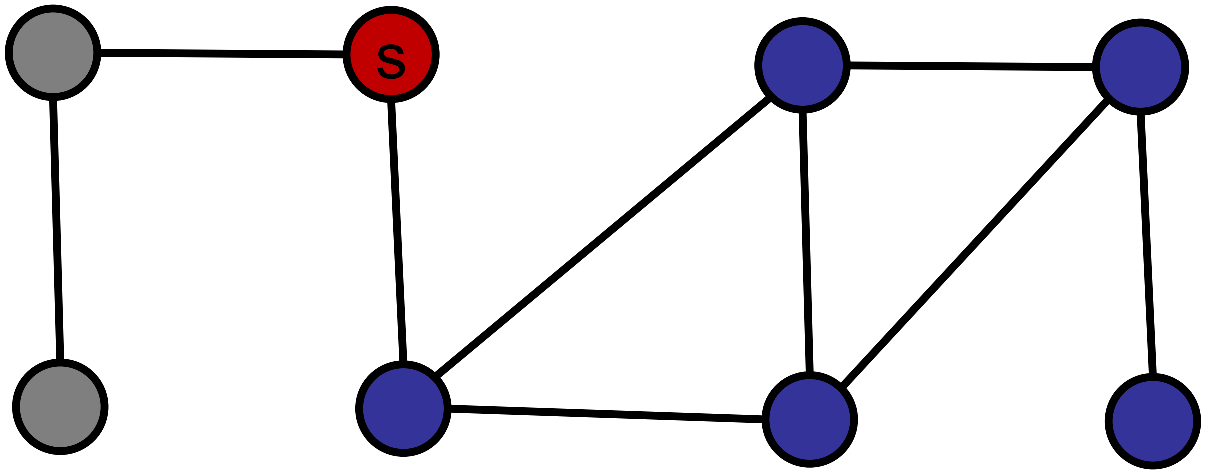

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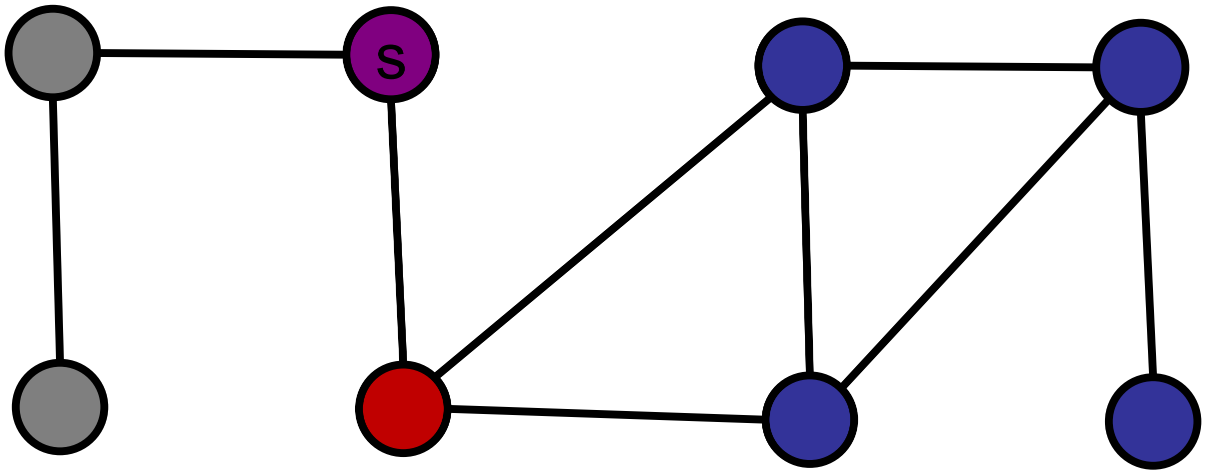

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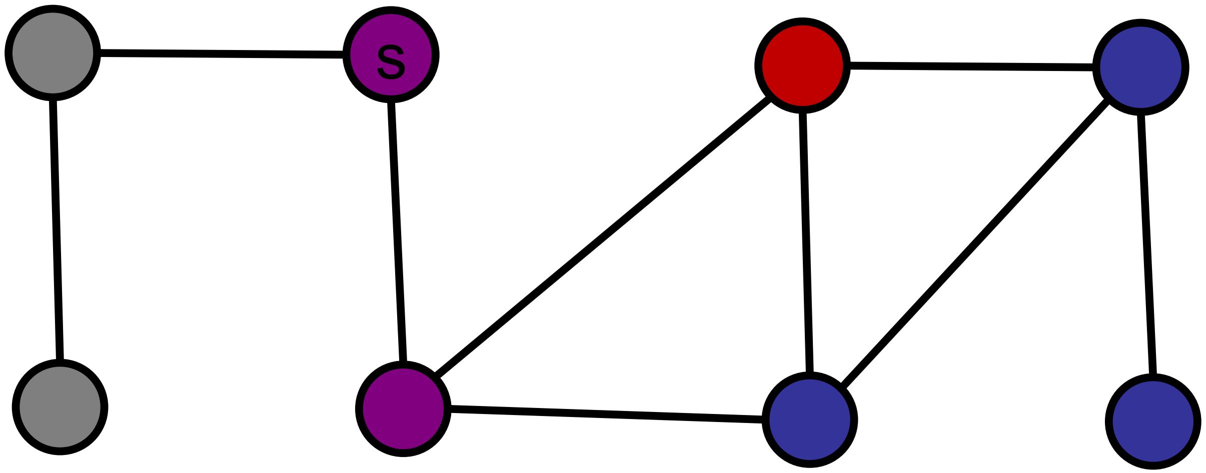

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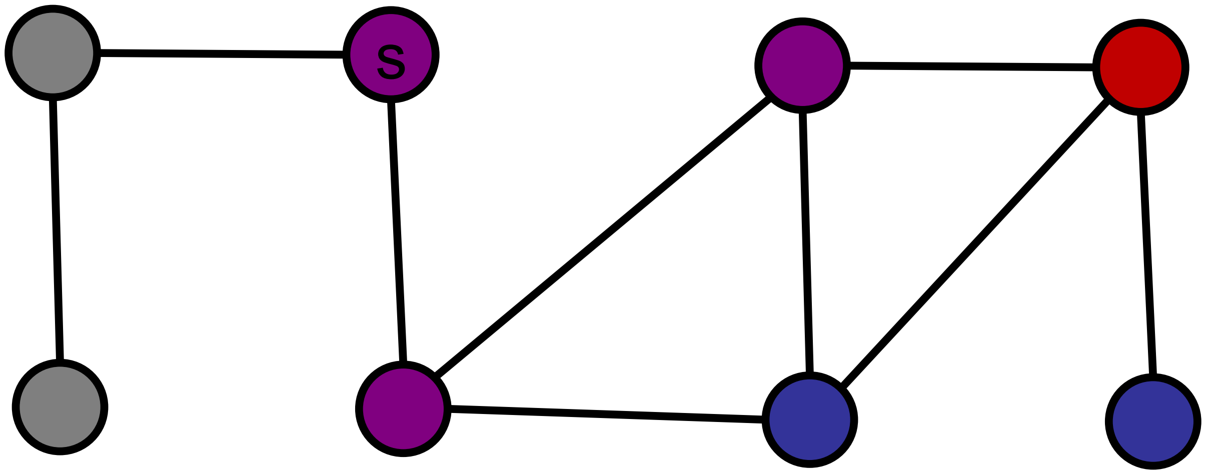

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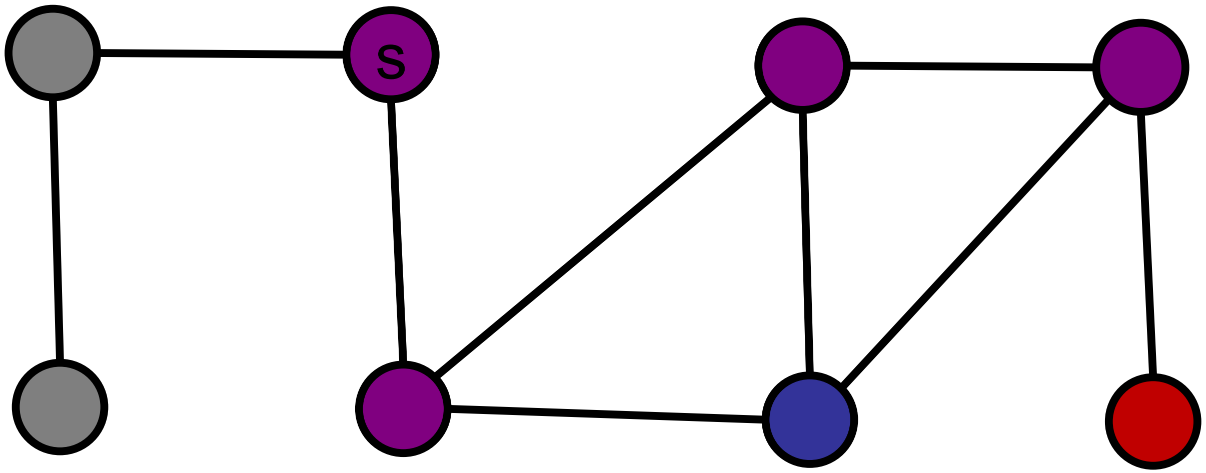

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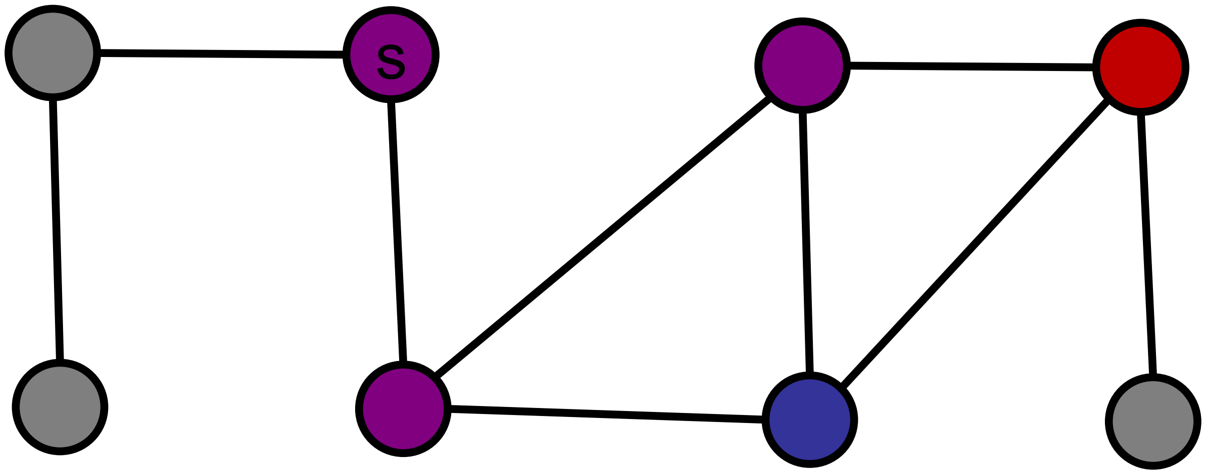

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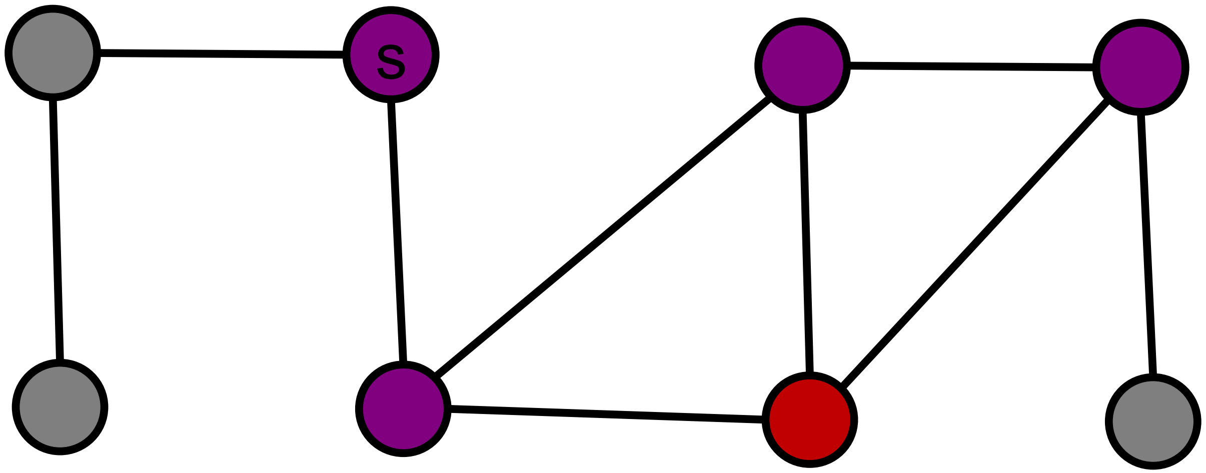
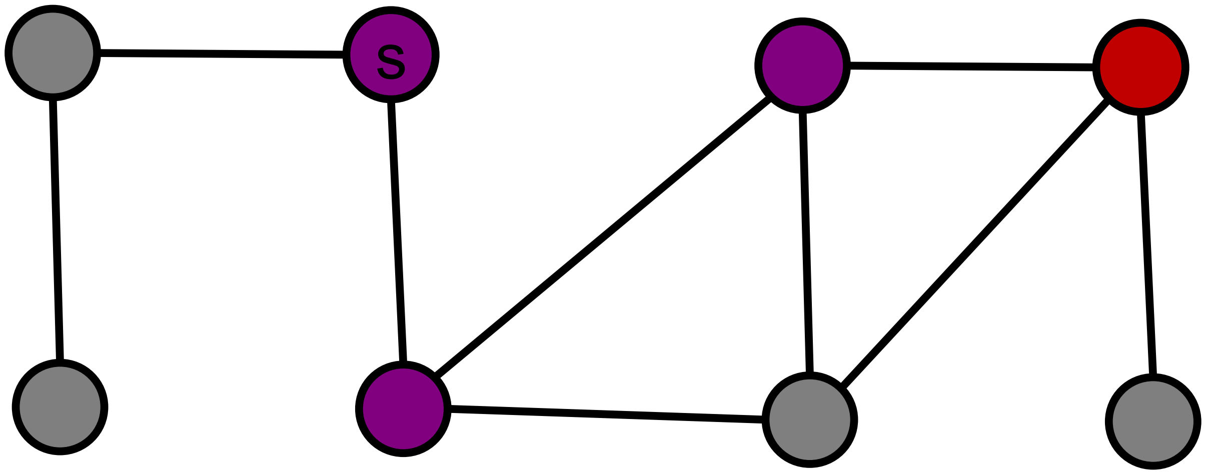
Blue = unvisited

Depth-First Search Example



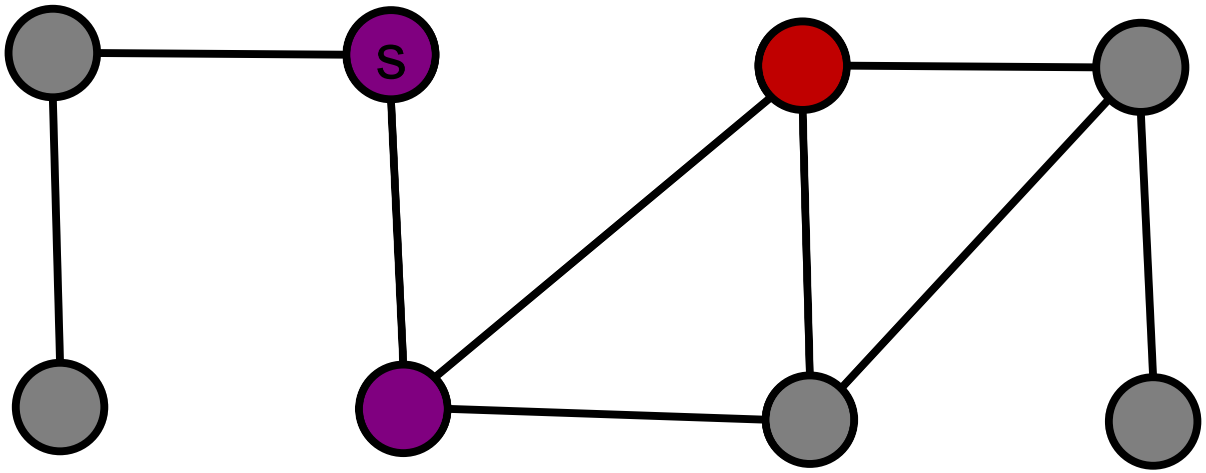
Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
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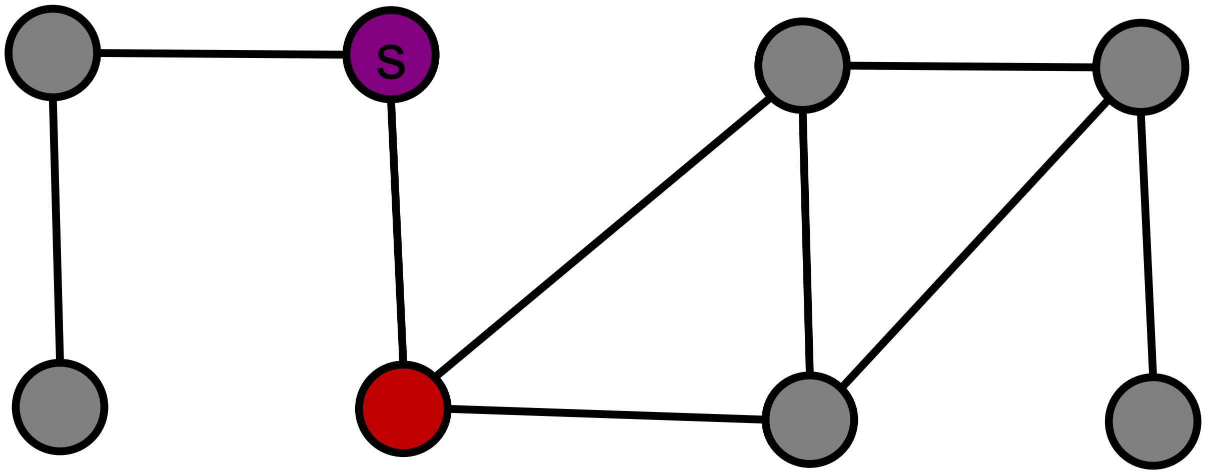
Depth-First Search Example



Red = active frontier
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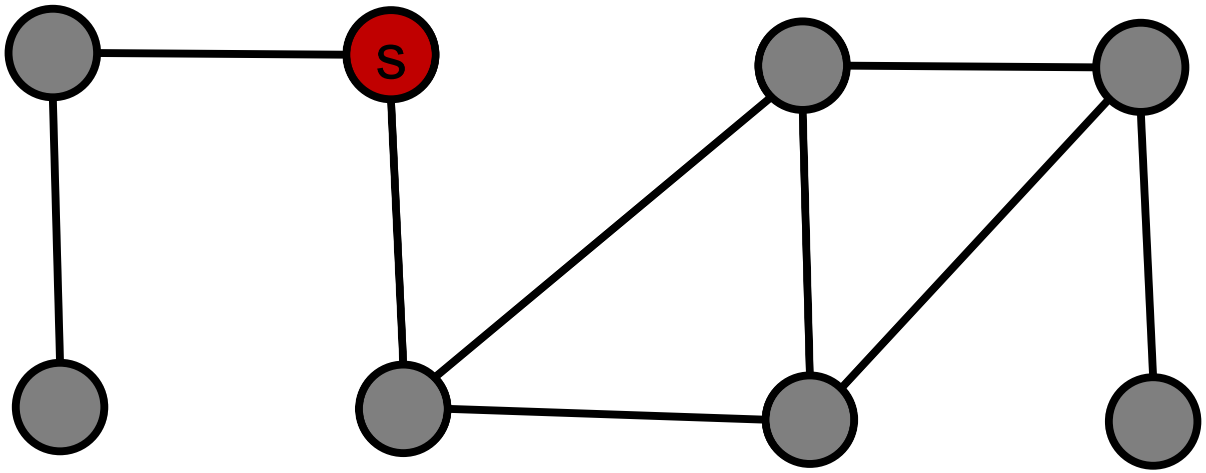
Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
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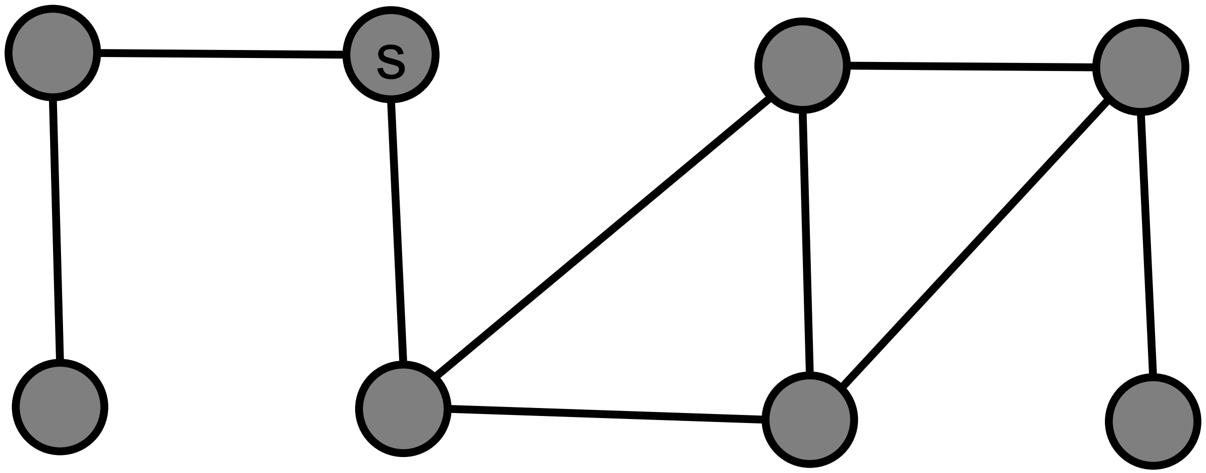
Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
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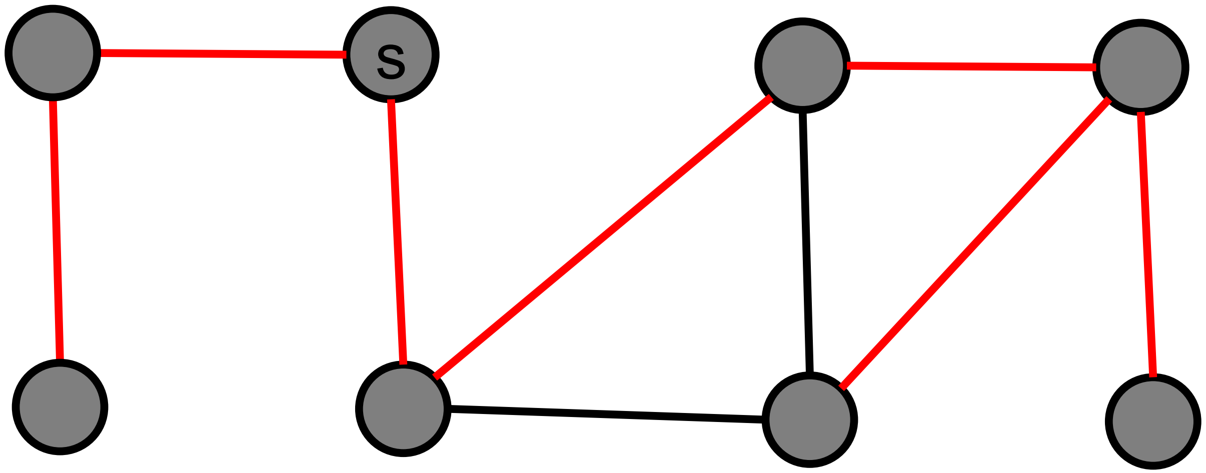
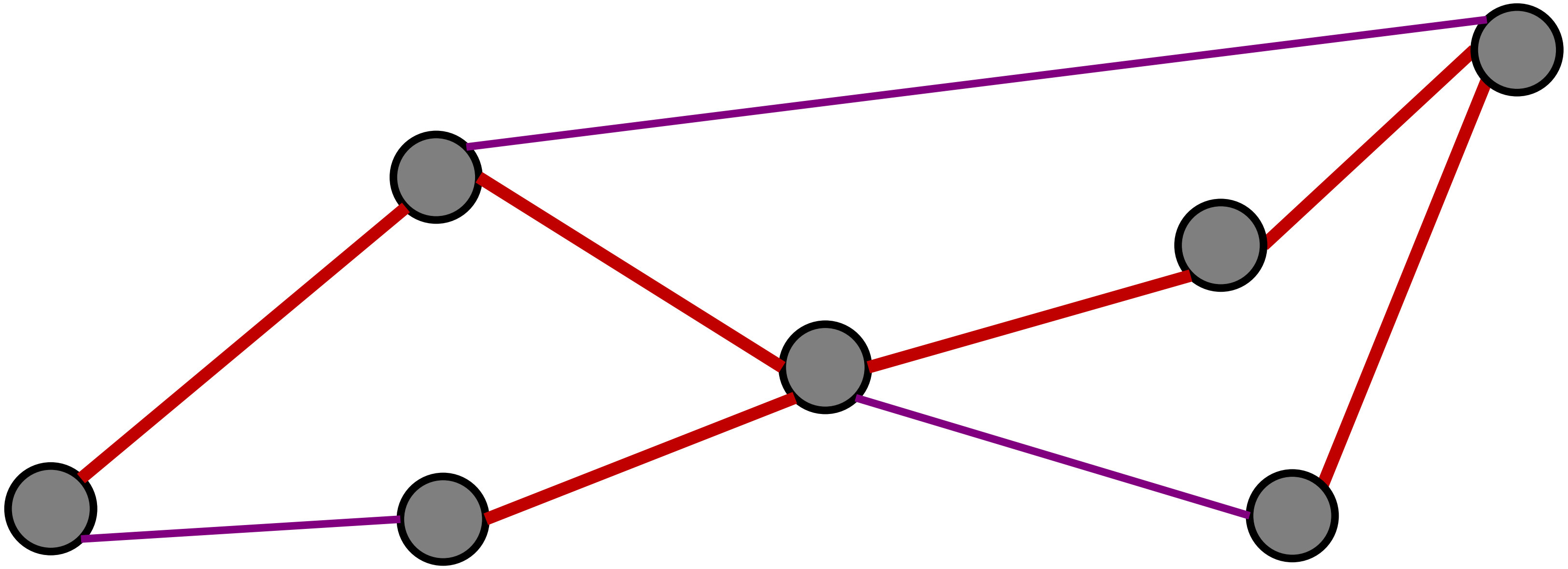
Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

DFS parent edges

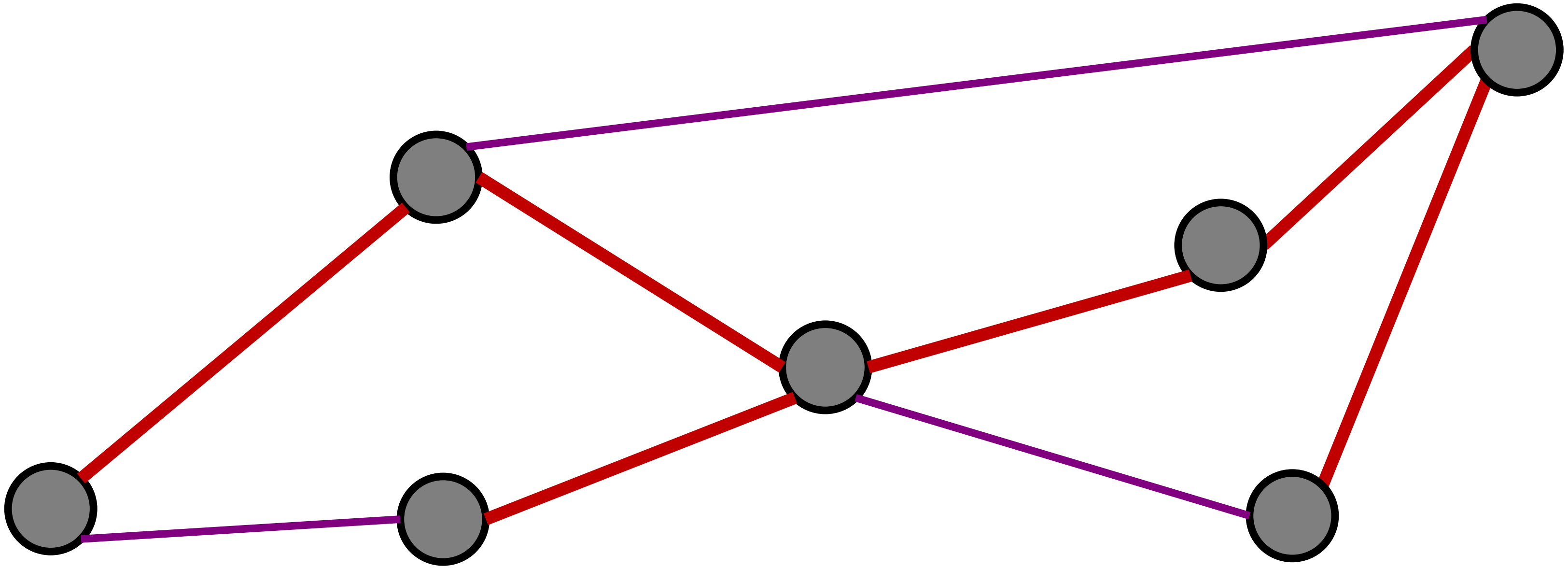


Red = Parent Edges
Purple = Non-parent edges

Which is true? (More than one may apply.)

1. DFS parent graph is a cycle.
2. DFS parent graph is a tree.
3. DFS parent graph has low-degree.
4. DFS parent graph has low diameter.
5. None of the above.

DFS parent edges = tree



Red = Parent Edges
Purple = Non-parent edges

True or false:
DFS parent graph contains shortest paths.

1. True
2. False

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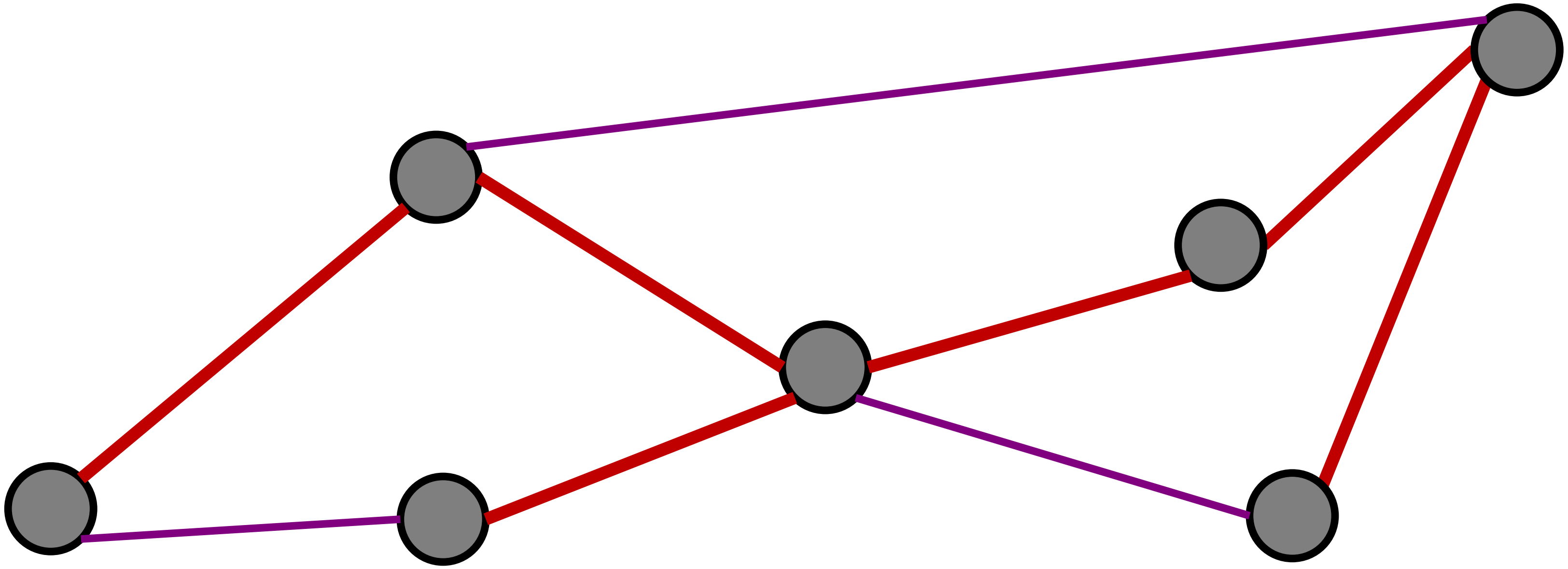
DFS parent edges = tree



Red = Parent Edges
Purple = Non-parent edges

Note: not shortest paths!

The running time of DFS is:

1. O(V)
2. O(E)
3. O(V+E)
4. O(VE)
5. $O(V^2)$
6. I have no idea.

ARCHIPELAGO

is open

Depth-First Search

$O(V)$

Analysis:

- DFS-visit called only once per node.
  - After visited, never call DFS-visit again.

- In DFS-visit, each neighbor is enumerated.

$O(E)$

If the graph is stored as an adjacency matrix, what is the running time of DFS?

1. $O(V)$
2. $O(E)$
3. $(V+E)$
4. $O(VE)$
5. $O(V^2)$
6. $O(E^2)$

Depth-First Search

Analysis:

- DFS-visit called only once per node.
  - After visited, never call DFS-visit again.

- In DFS-visit, each neighbor is enumerated.

O(V)

O(V)
per
node