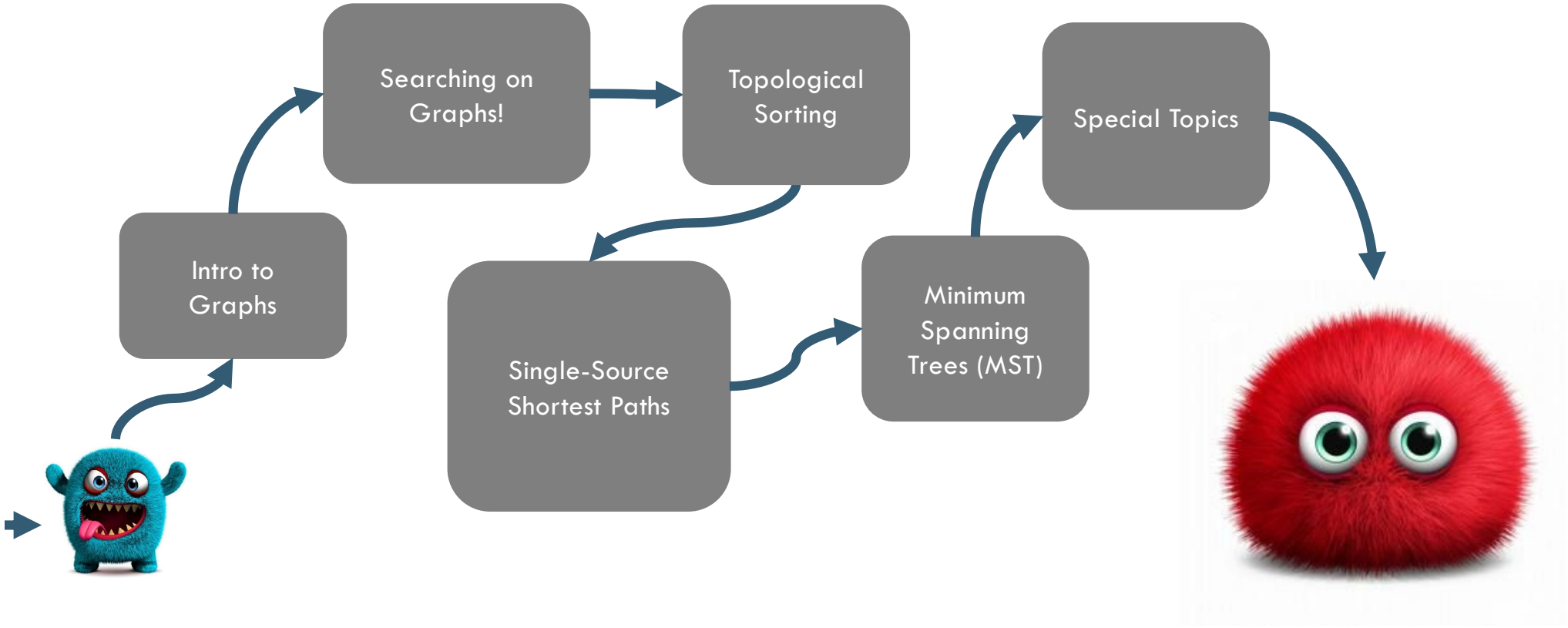
CS2040S
Data Structures and Algorithms

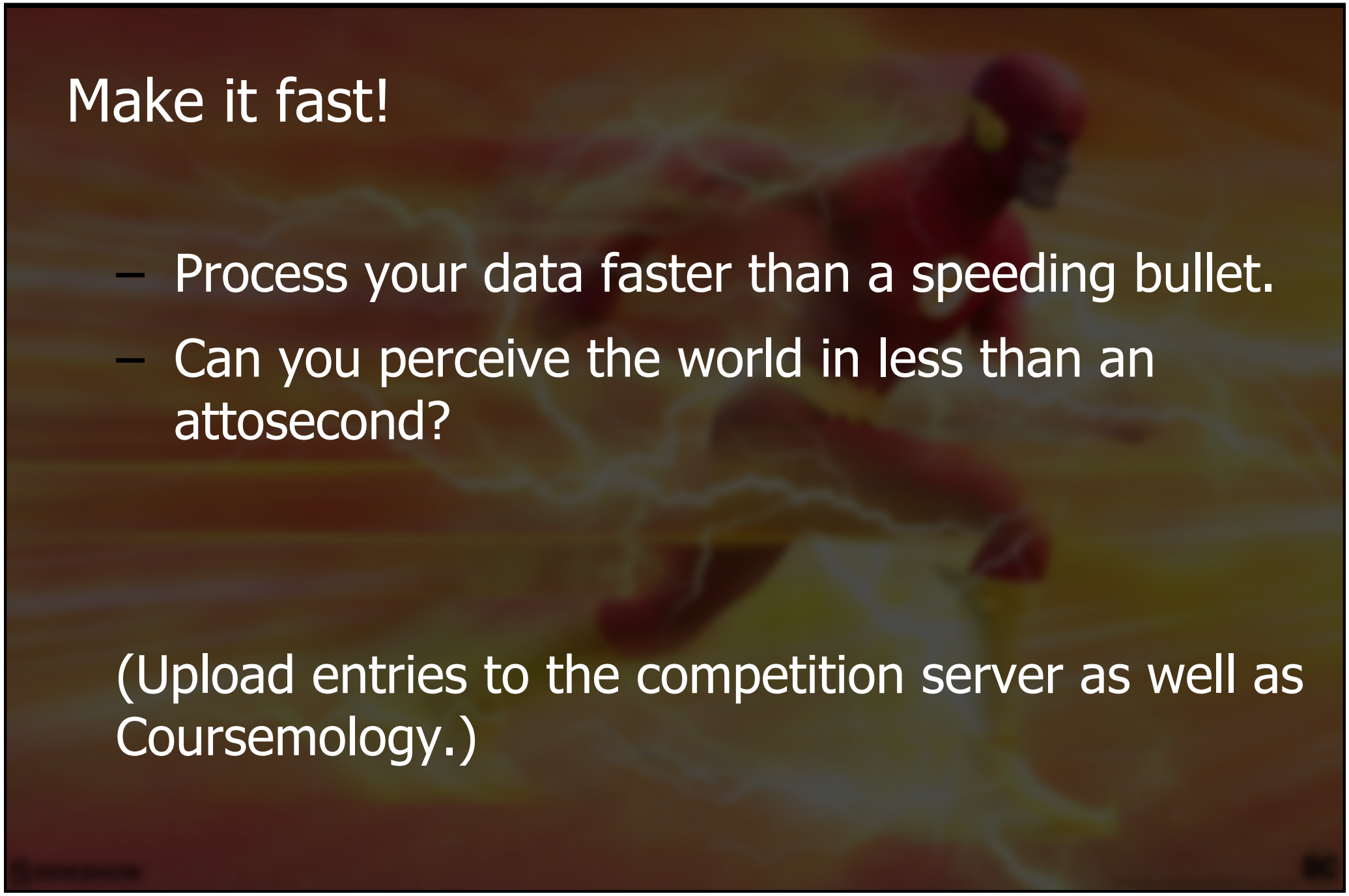
Graphs!

COURSE STRUCTURE



POST-MIDTERM: MODELLING AND SOLVING PROBLEMS

Contest: Speed Demon



Make it fast!

- Process your data faster than a speeding bullet.
- Can you perceive the world in less than an attosecond?

(Upload entries to the competition server as well as Coursemology.)

Make-up Midterm

Saturday: 11am-3pm

- Two start times: 11am, 1pm
- Location TBA.
- I will send an e-mail to everyone who signed up on Coursemology for the make-up.

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

Roadmap

Next: Searching Graphs

- Searching graphs
- Shortest path problem
- Bellman-Ford Algorithm
- Dijkstra's Algorithm

Roadmap

Next next:

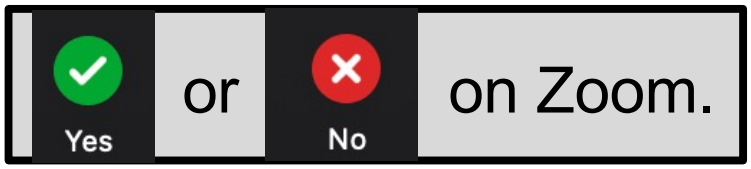
- Connected component problem
  - Union-Find data structure
- The Minimum Spanning Tree Problem
  - Kruskal's Algorithm
  - Prim's Algorithm

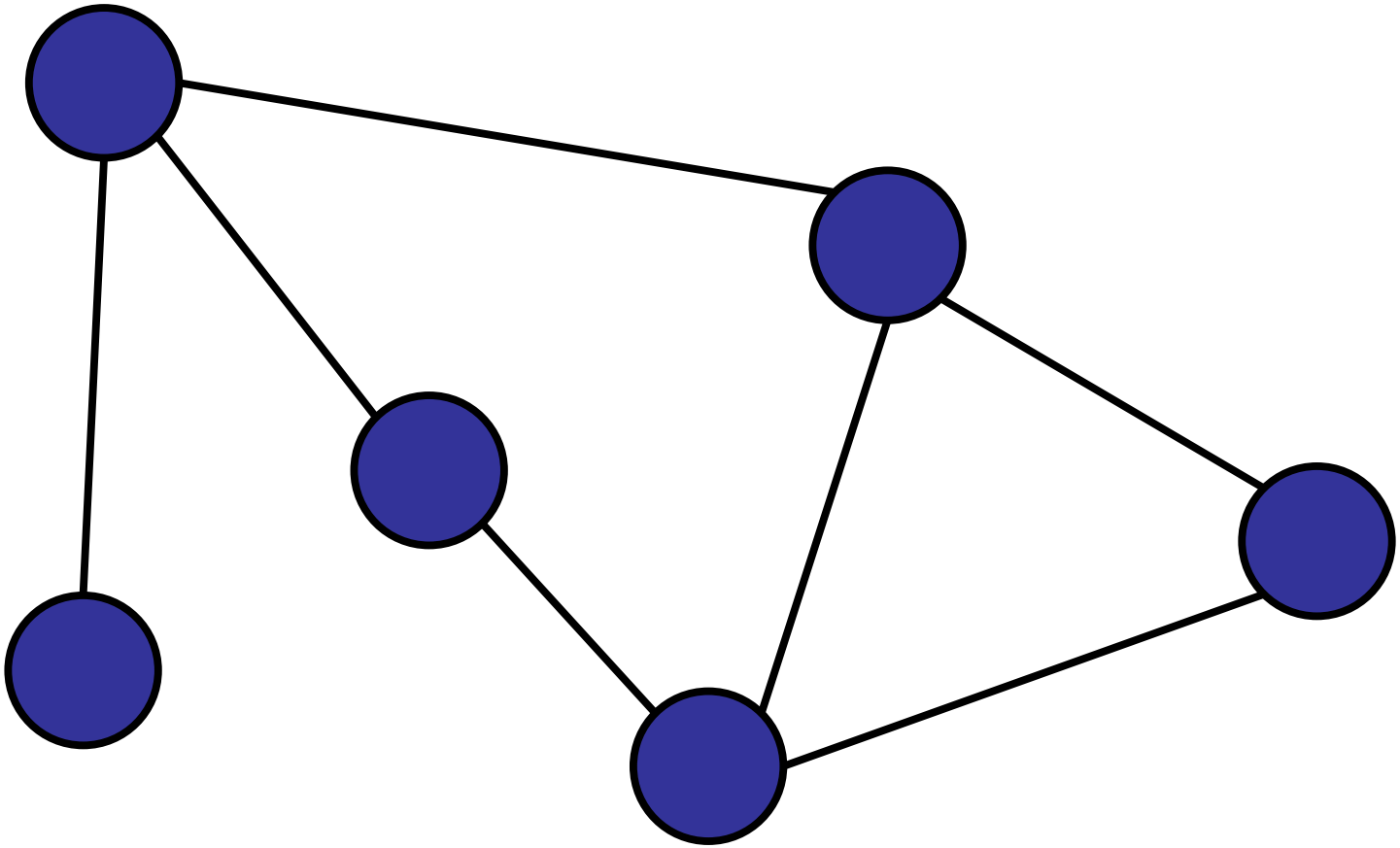
What is a graph?

Is it a graph?

✓1. Yes

2. No.





Is it a graph?

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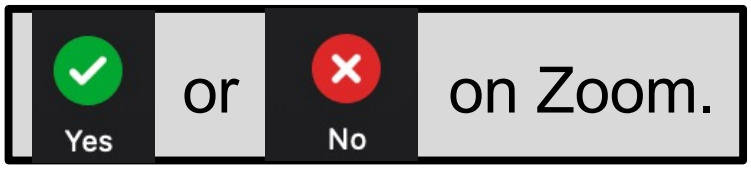
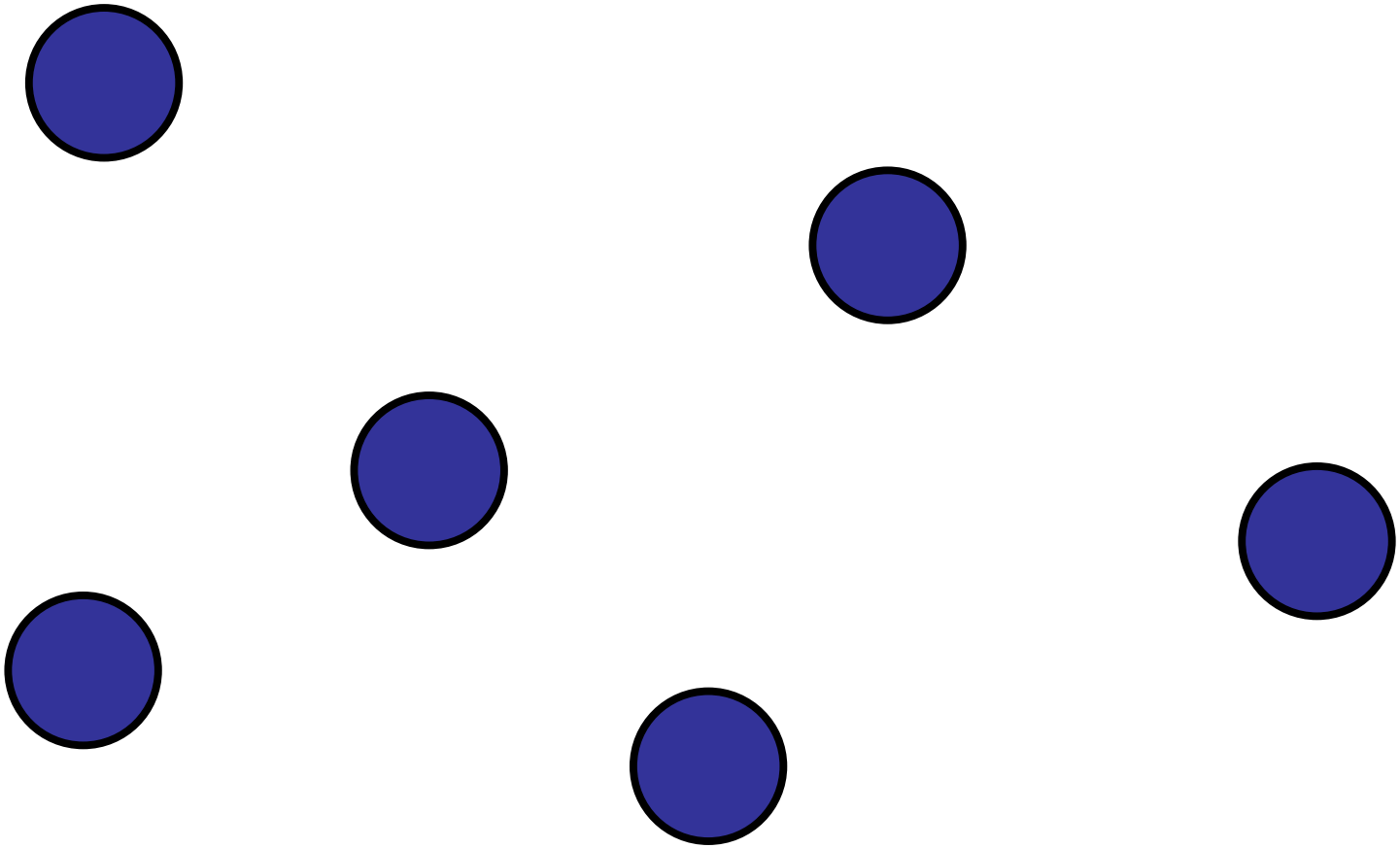
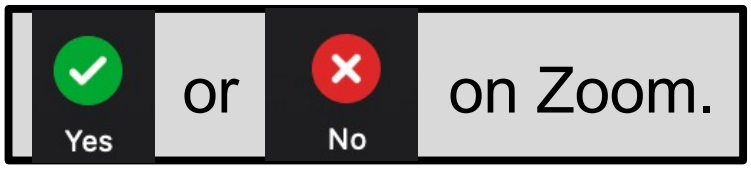
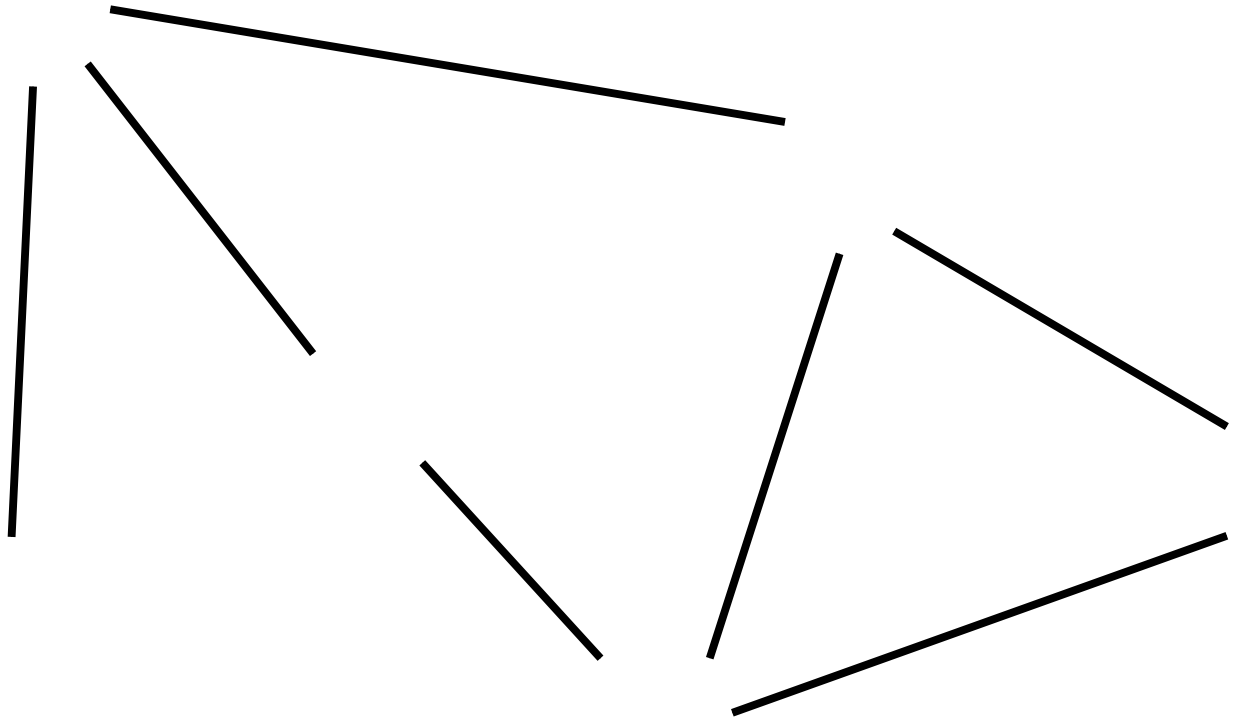




Is it a graph?

1. Yes
2. ✓ No.





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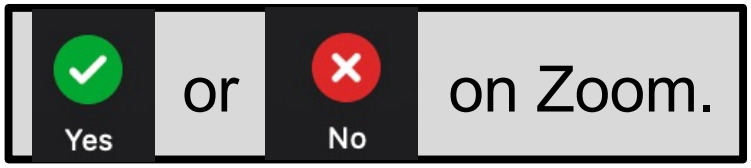

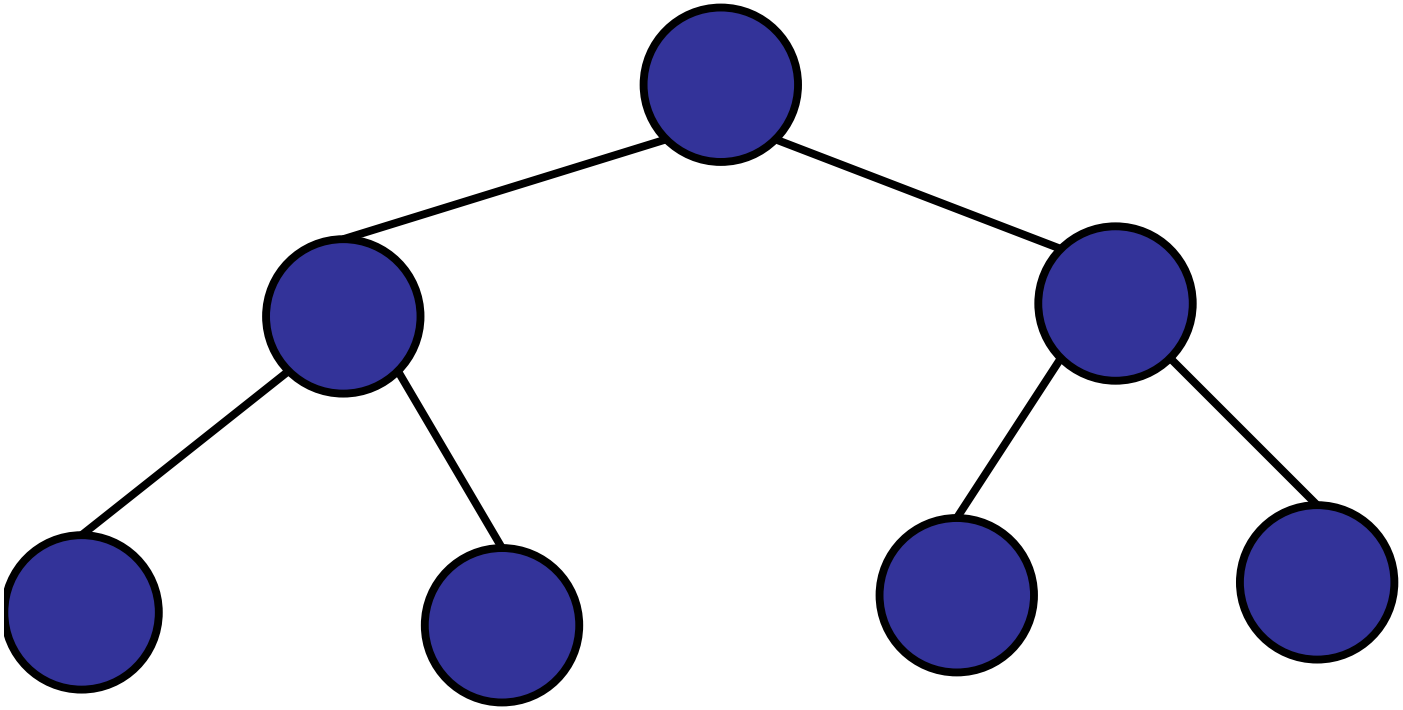

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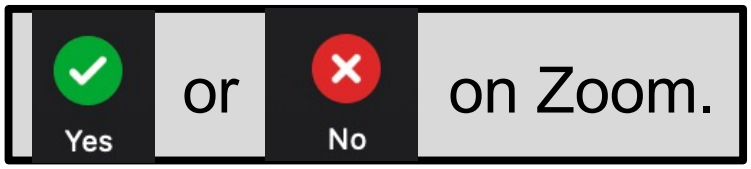

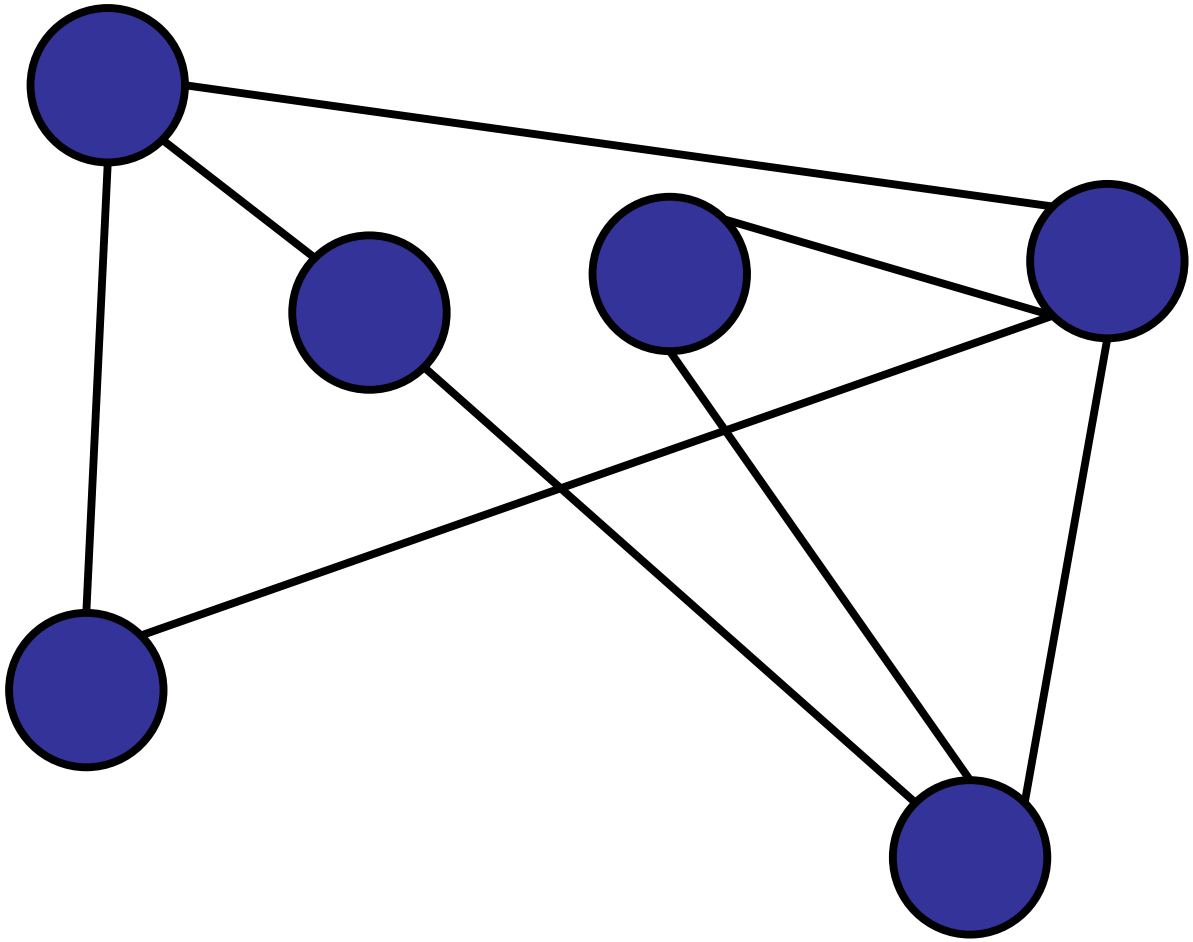

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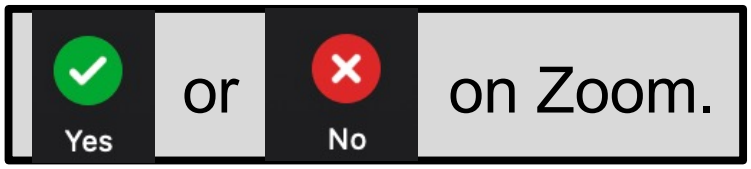
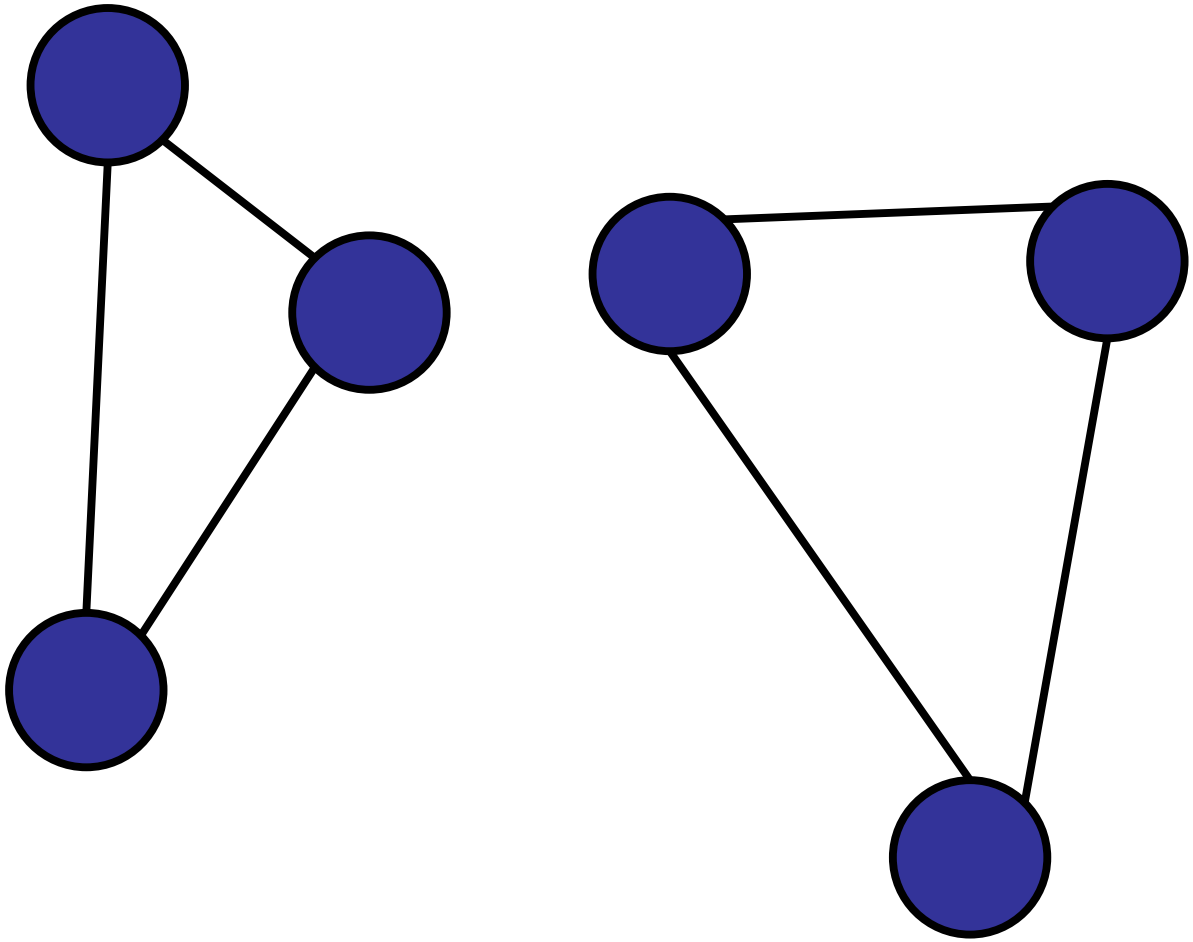
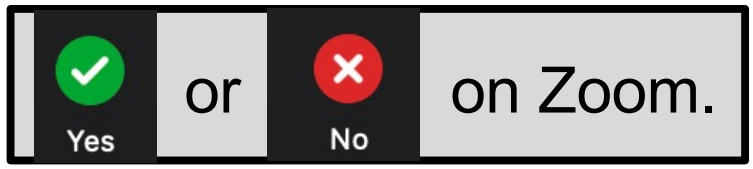
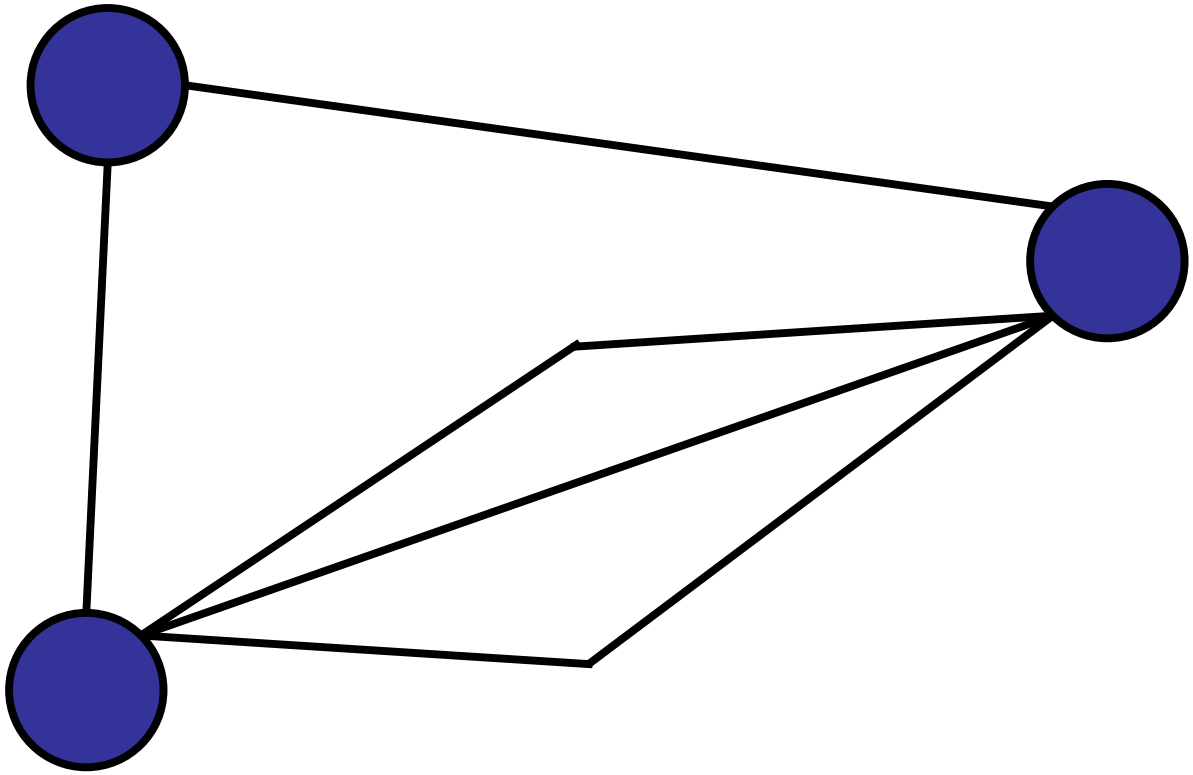




Is it a graph?

1. Yes
2. ✓ No.





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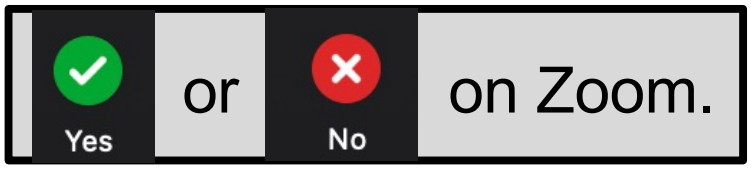


What is a graph?

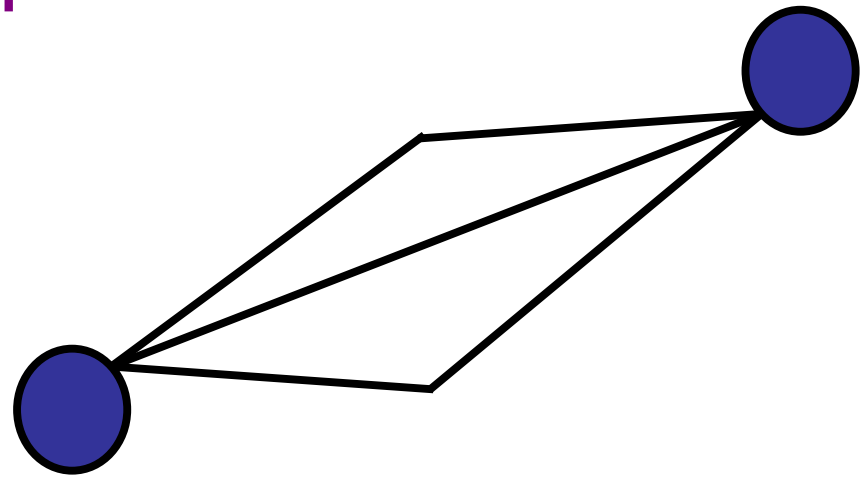
Graph consists of two types of elements:

- Nodes (or vertices)
  - At least one.
- Edges (or arcs)
  - Each edge connects two nodes in the graph
  - Each edge is unique.

What is a multigraph?

Graph consists of two types of elements:

- Nodes (or vertices)
  - At least one.



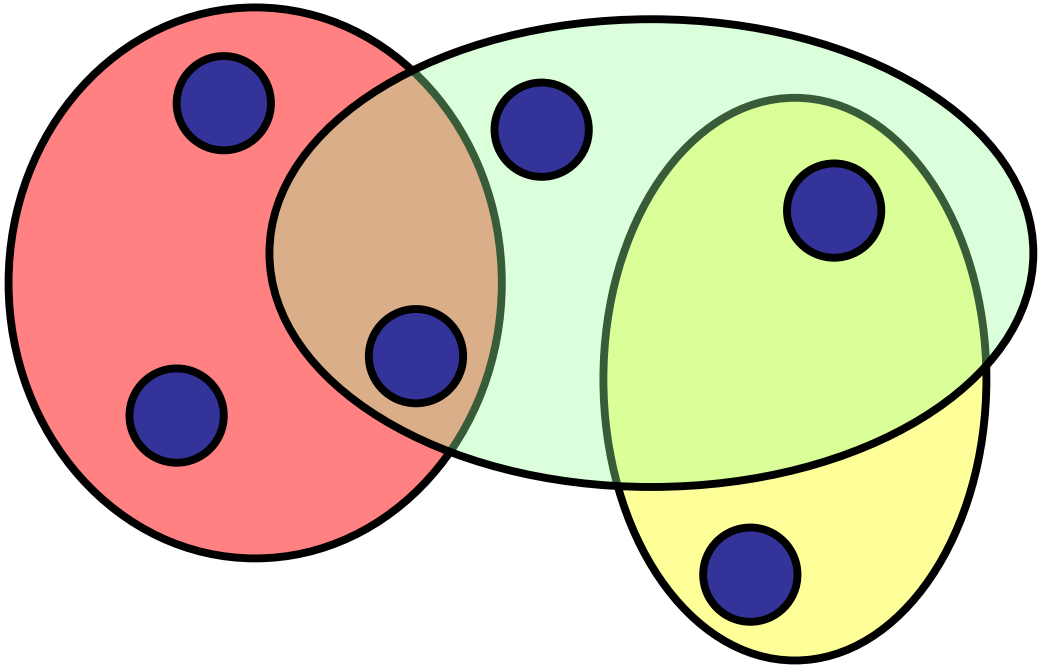
- Edges (or arcs)
  - Each edge connects two nodes in the graph
  - Two nodes may be connected by more than one edge.

(Rare in CS2040S.)

What is a hypergraph?

Graph consists of two types of elements:

- Nodes (or vertices)
  - At least one.



- Edges (or arcs)
  - Each edge connects >= 2 nodes in the graph
  - Each edge is unique.

(Not common in CS2040S)

What is a graph?

Graph $G = \langle V, E \rangle$

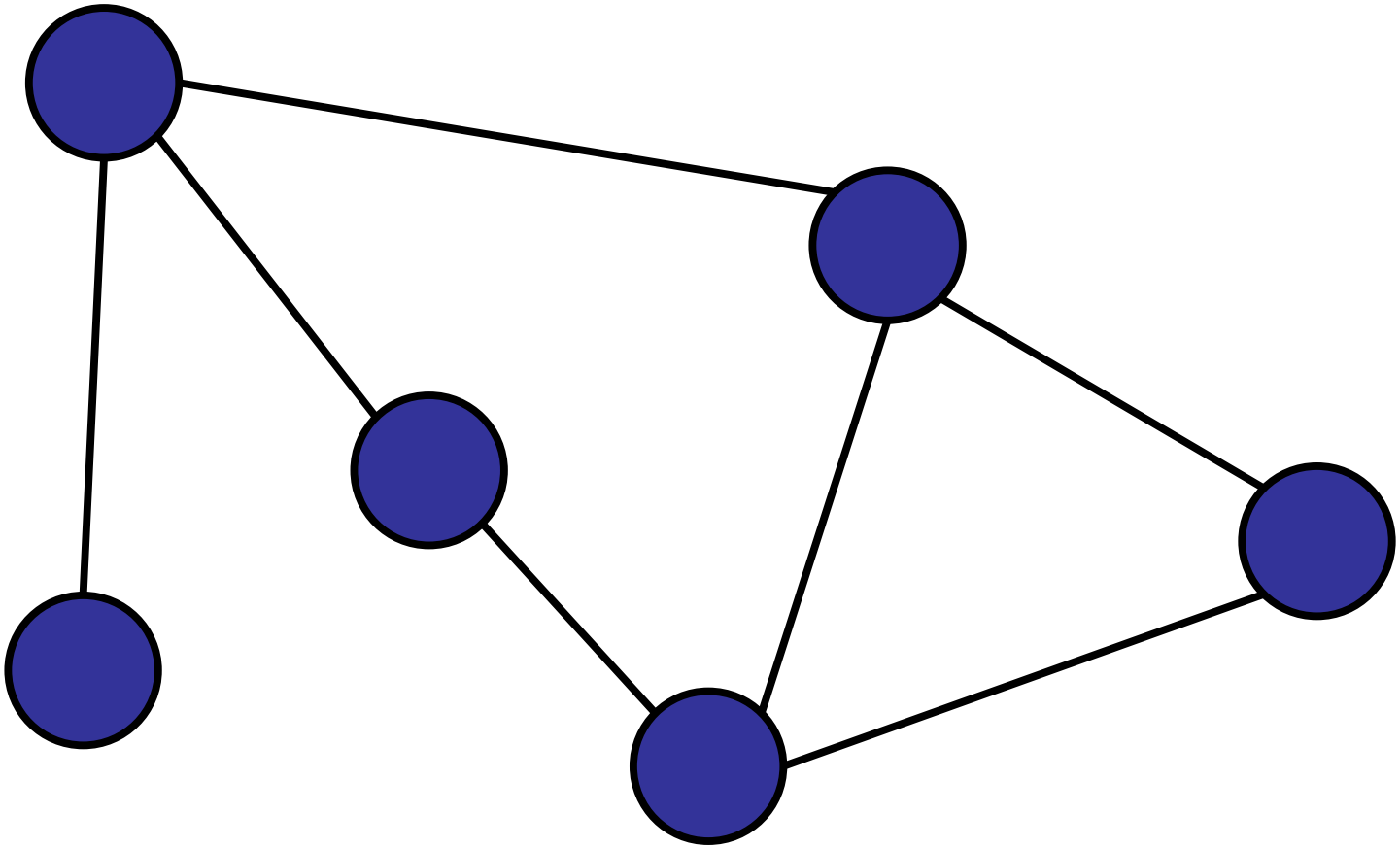
V is a set of nodes

- At least one: $|V| > 0$.

E is a set of edges:

- $E \subseteq \{ (v,w) : (v \in V), (w \in V) \}$
- $e = (v,w)$
- For all $e_1, e_2 \in E : e_1 \neq e_2$

Graph Terminology

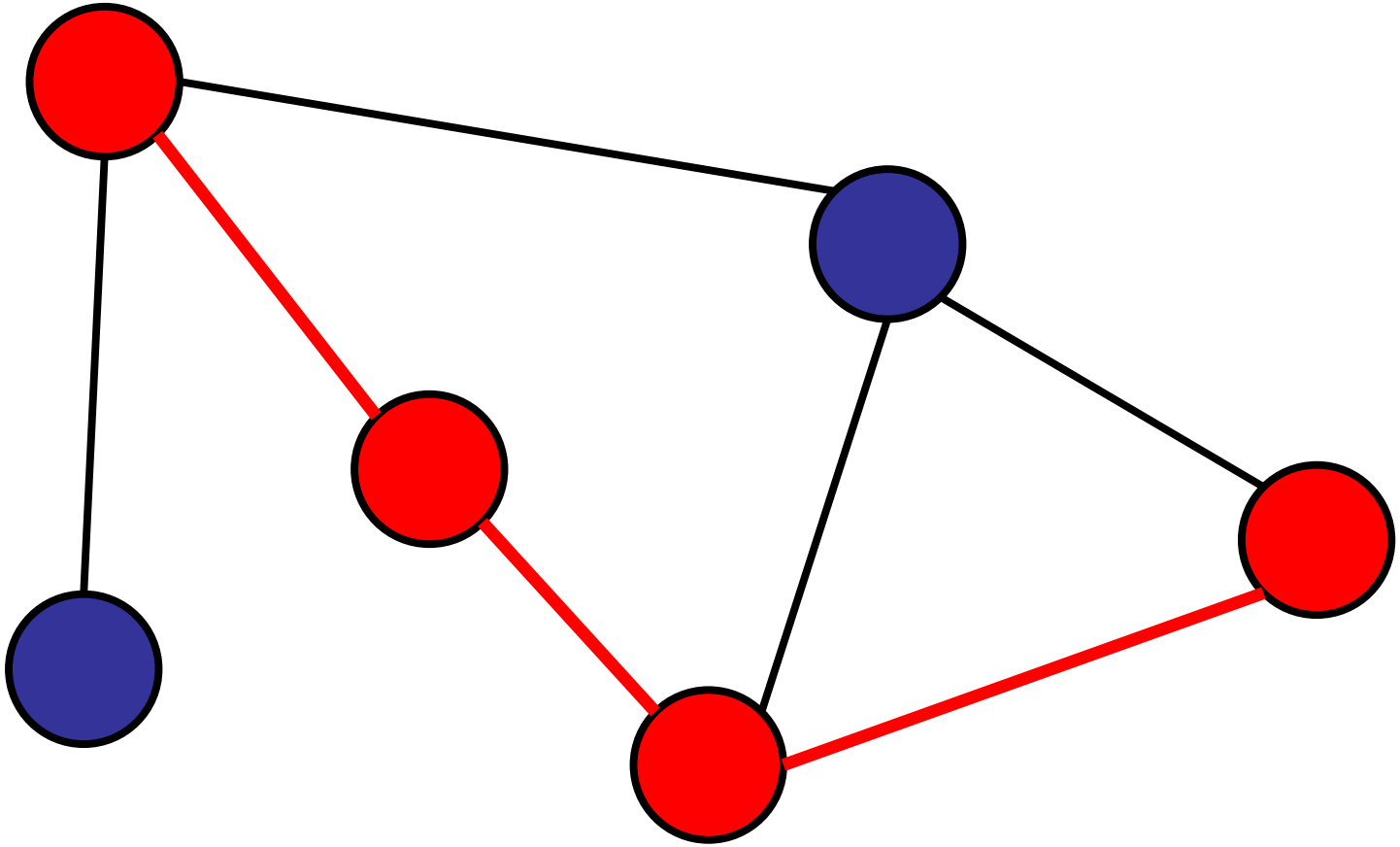


Graph Terminology

(Simple) Path:

Set of edges connecting two nodes.

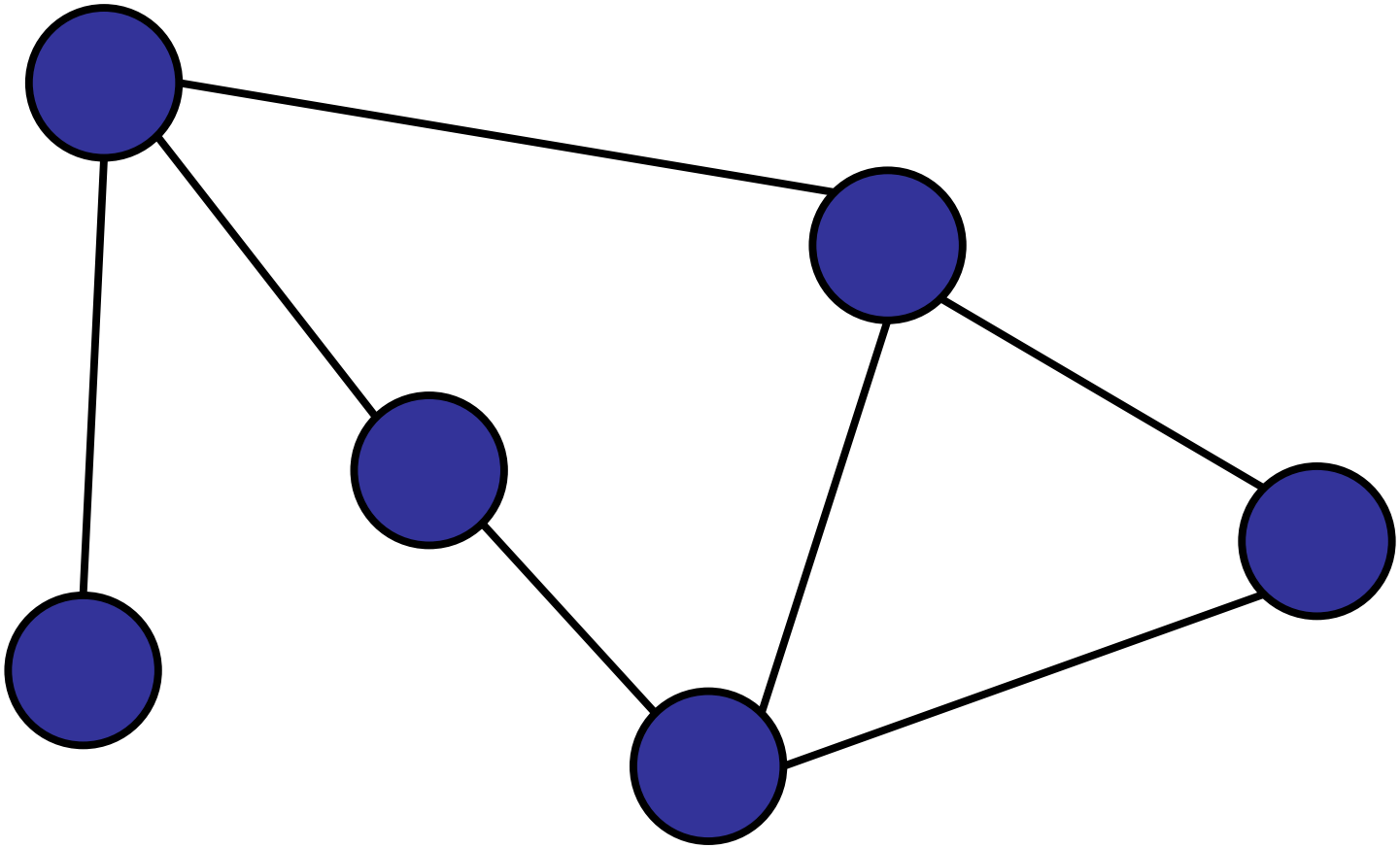
Path intersects each node at most once.



Graph Terminology

Connected:

Every pair of nodes is connected by a path.



Graph Terminology

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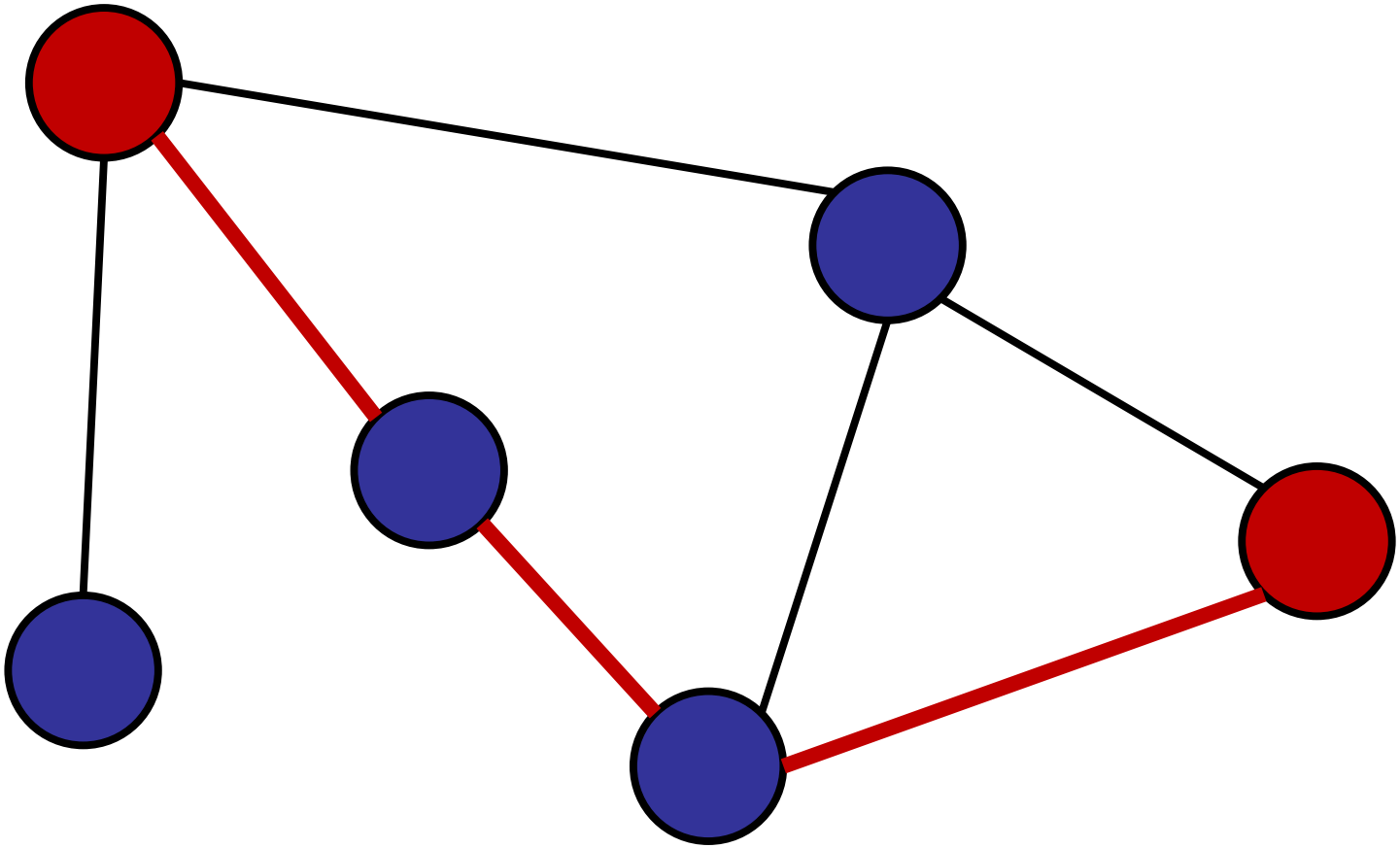
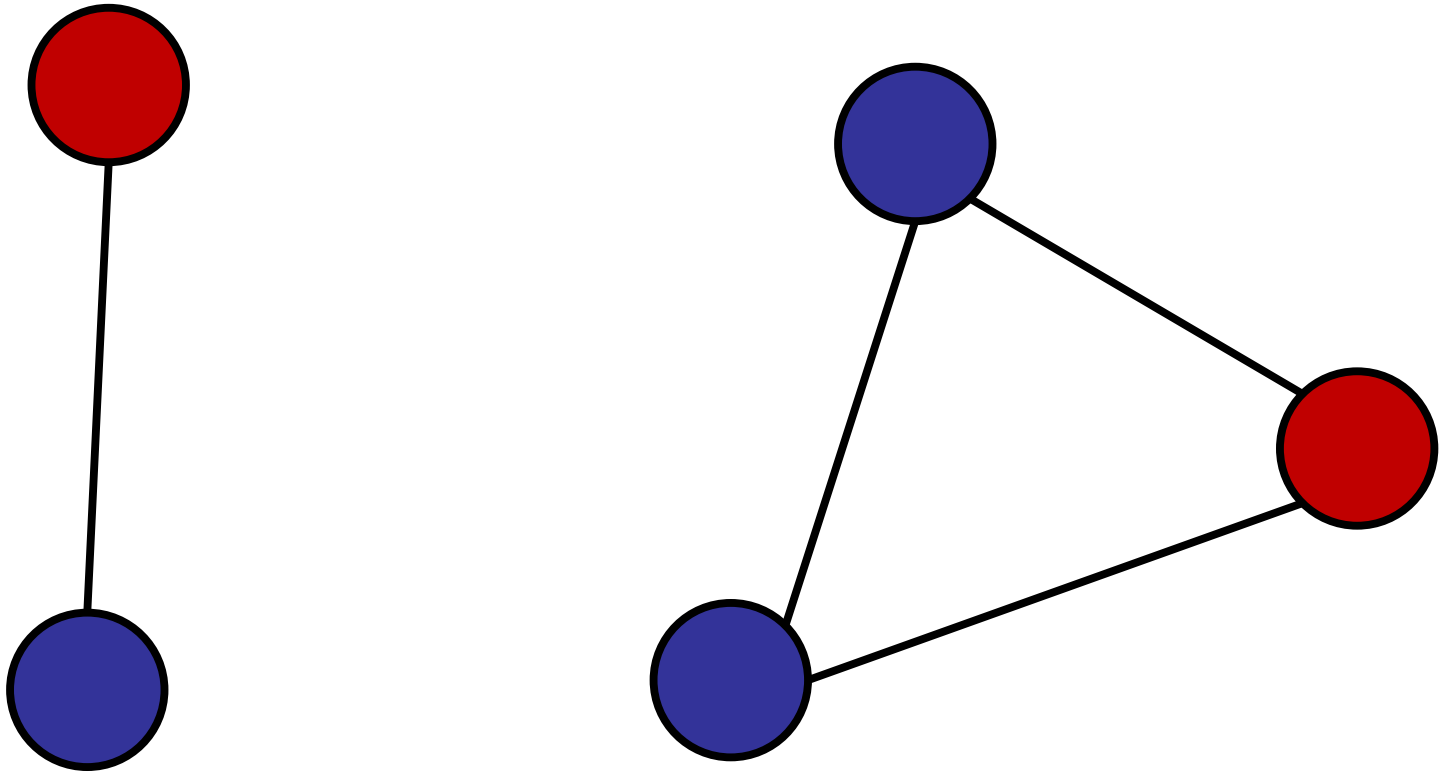


Graph Terminology

Disconnected:

- Some pair of nodes is not connected by a path.

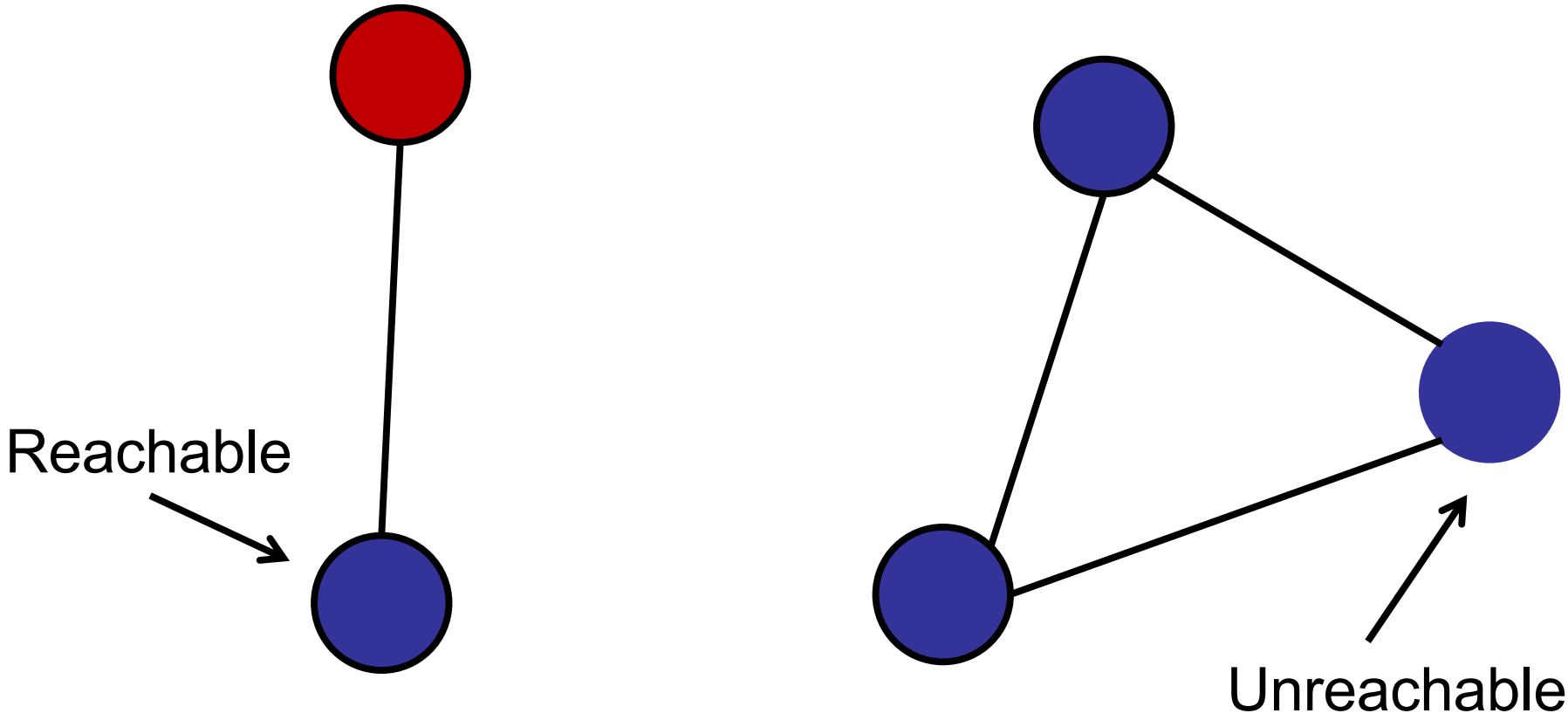


- Two **connected components**.

Graph Terminology

Disconnected:

- Some pair of nodes is not connected by a path.

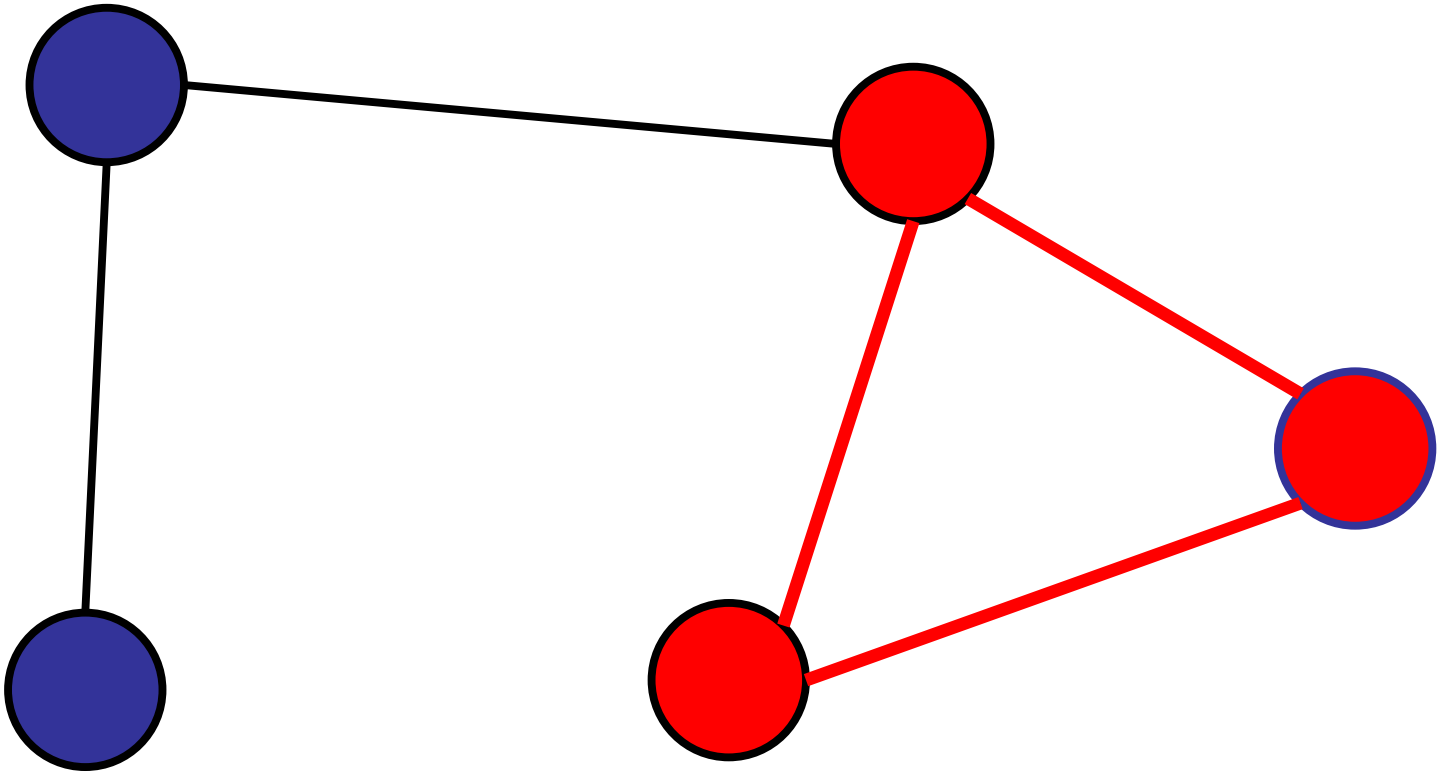


- Two **connected components**.

Graph Terminology

Cycle:

"Path" where first and last node are the same.

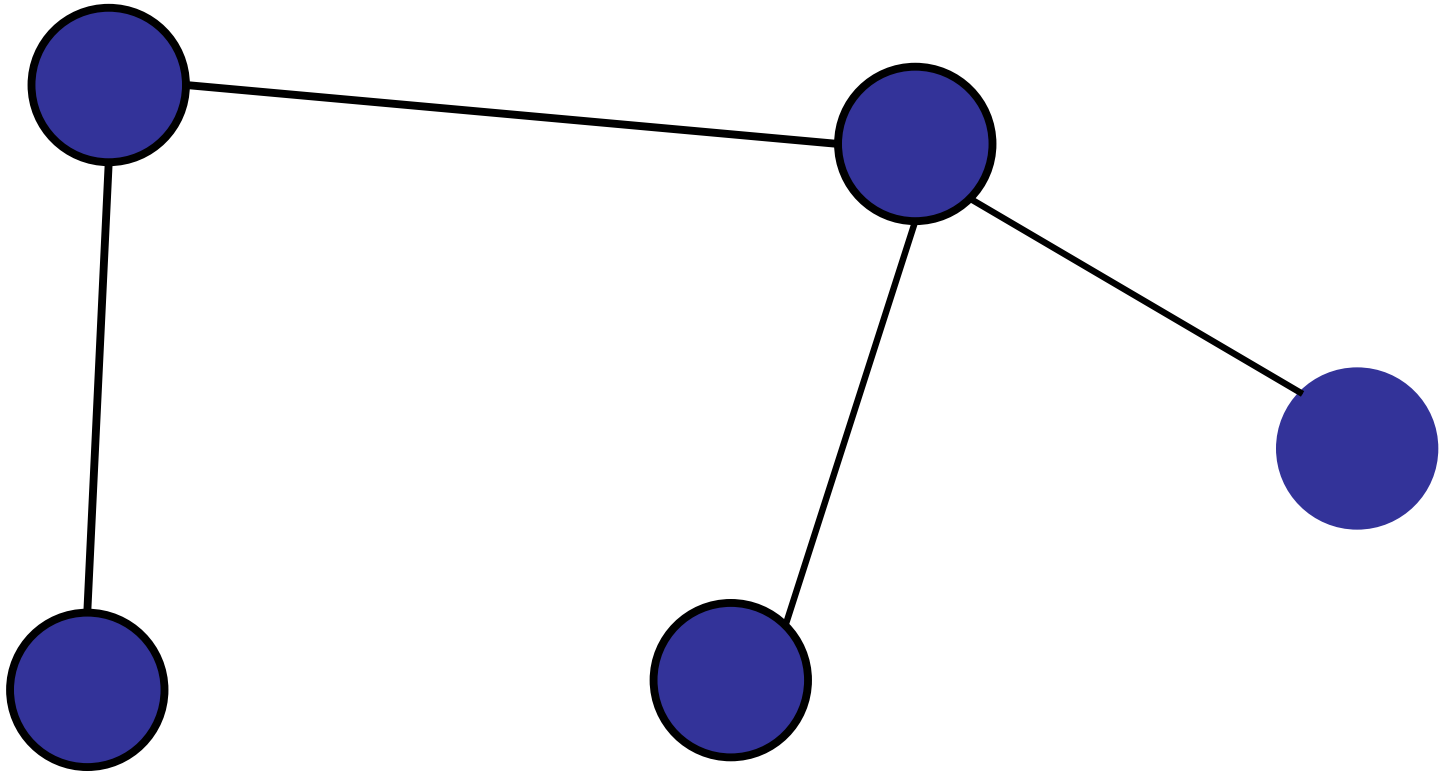


(**Not actually a path, since one node appears twice.)

Graph Terminology

(Unrooted) Tree:

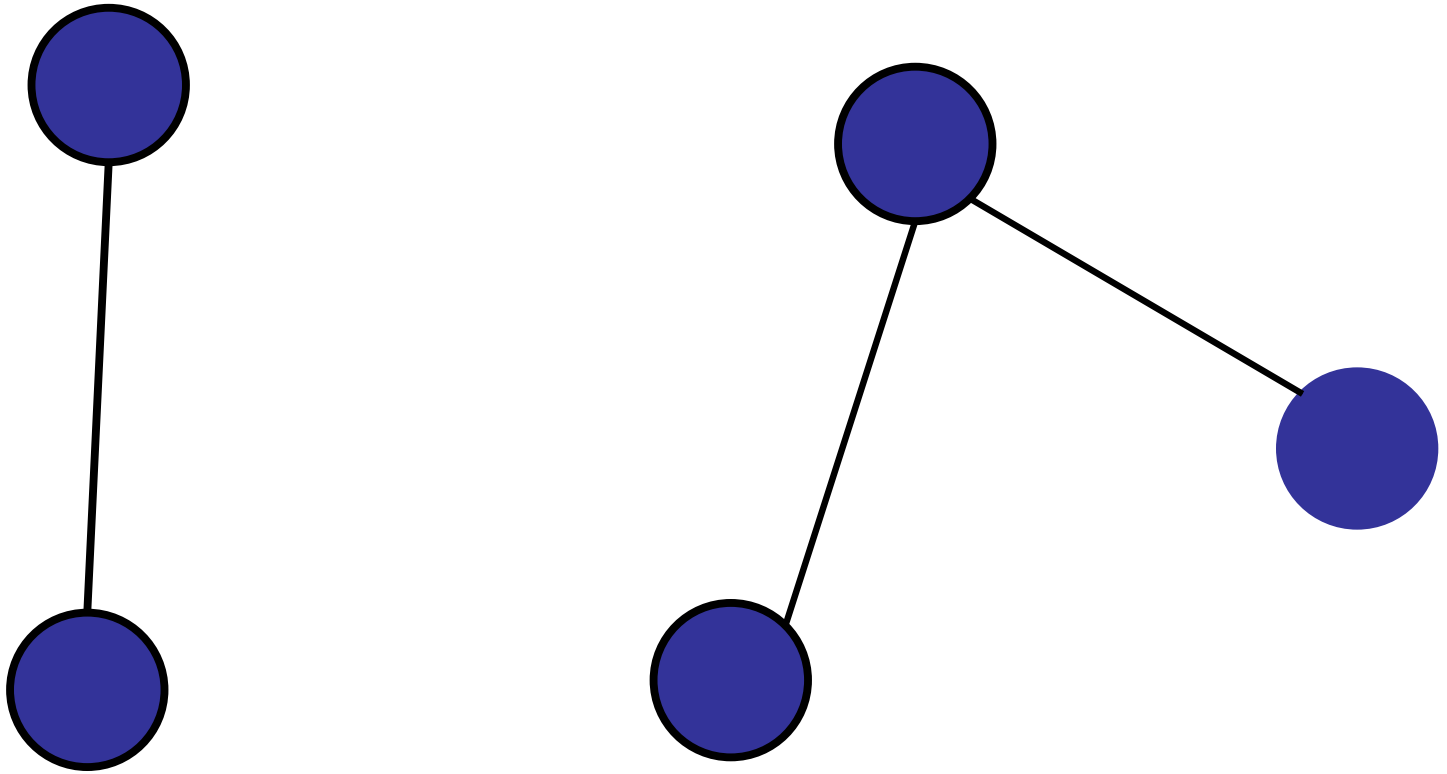
Connected graph with no cycles.



Graph Terminology

Forest:

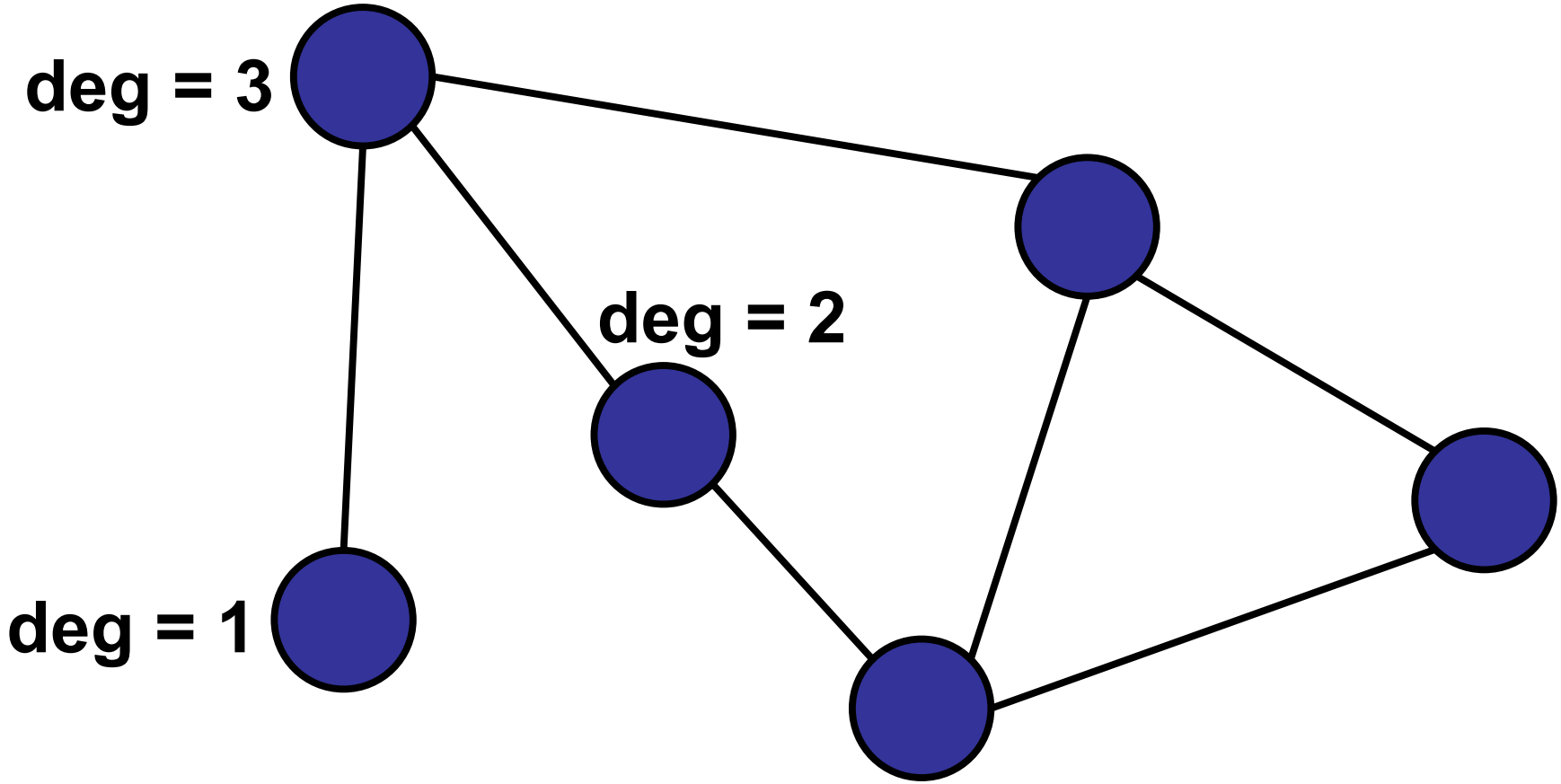
Graph with no cycles.



Graph Terminology

Degree of a node:

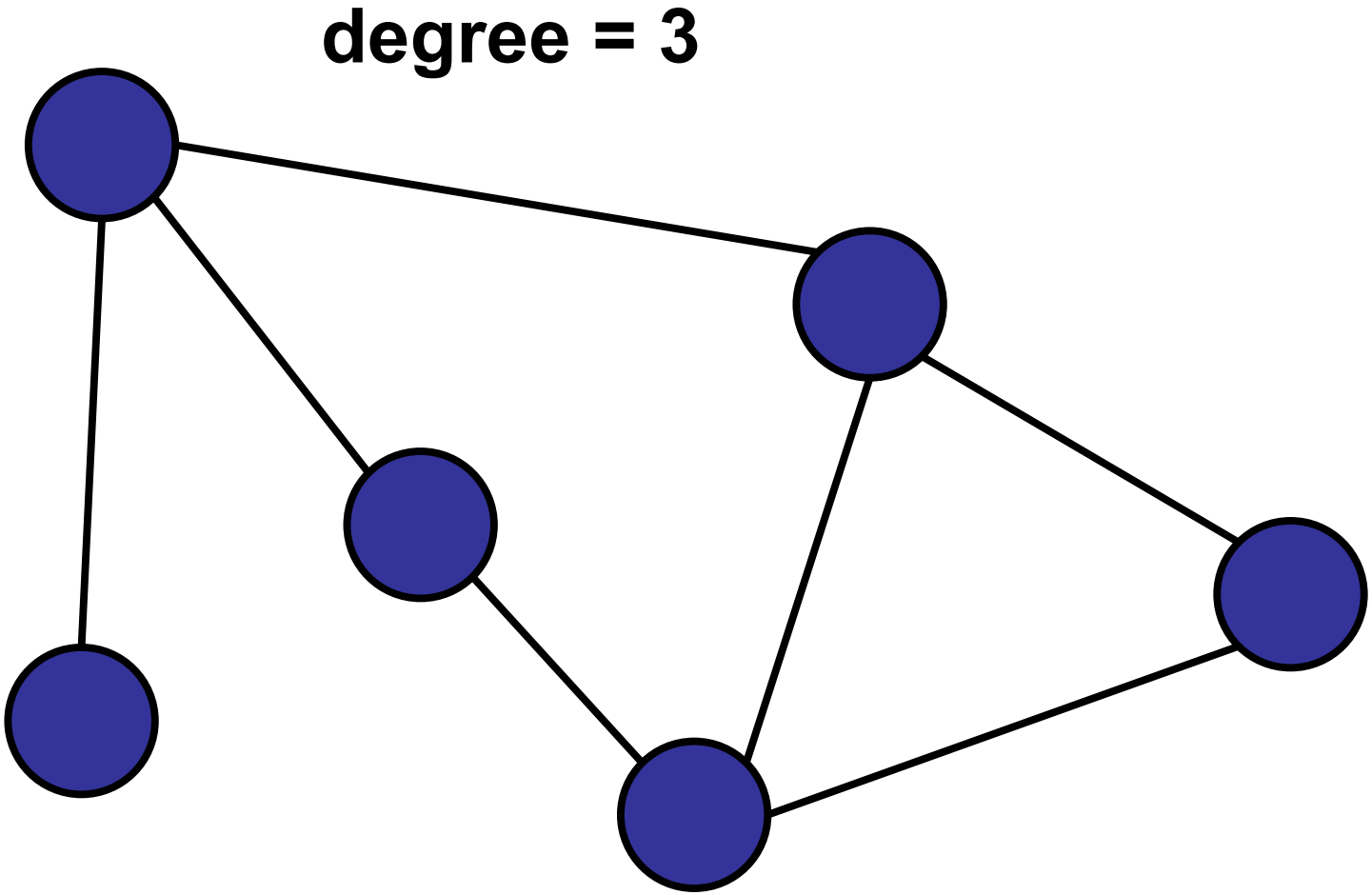
- Number of **adjacent** edges.



Graph Terminology

Degree of a graph:

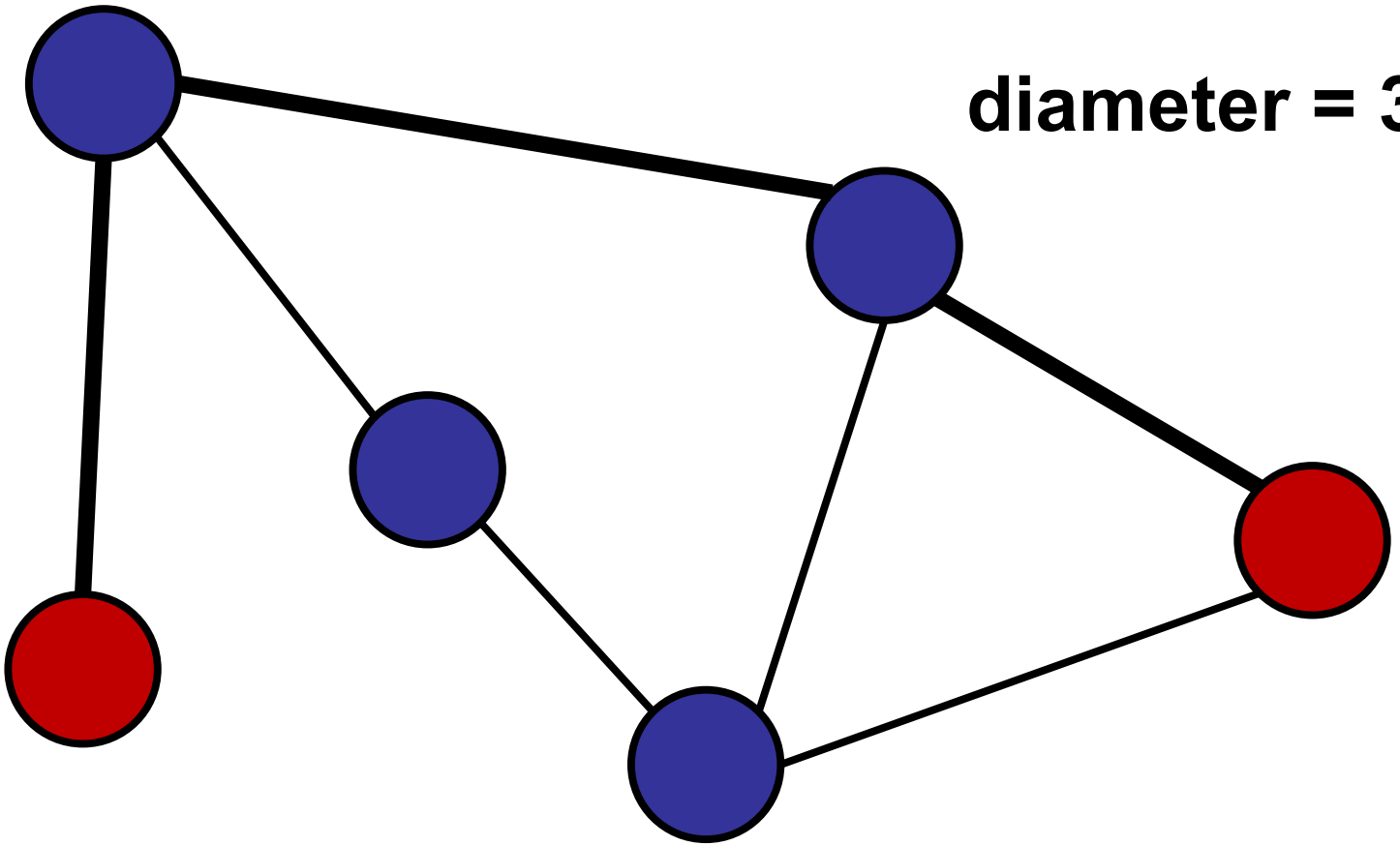
- Maximum number of **adjacent** edges.



Graph Terminology

Diameter:

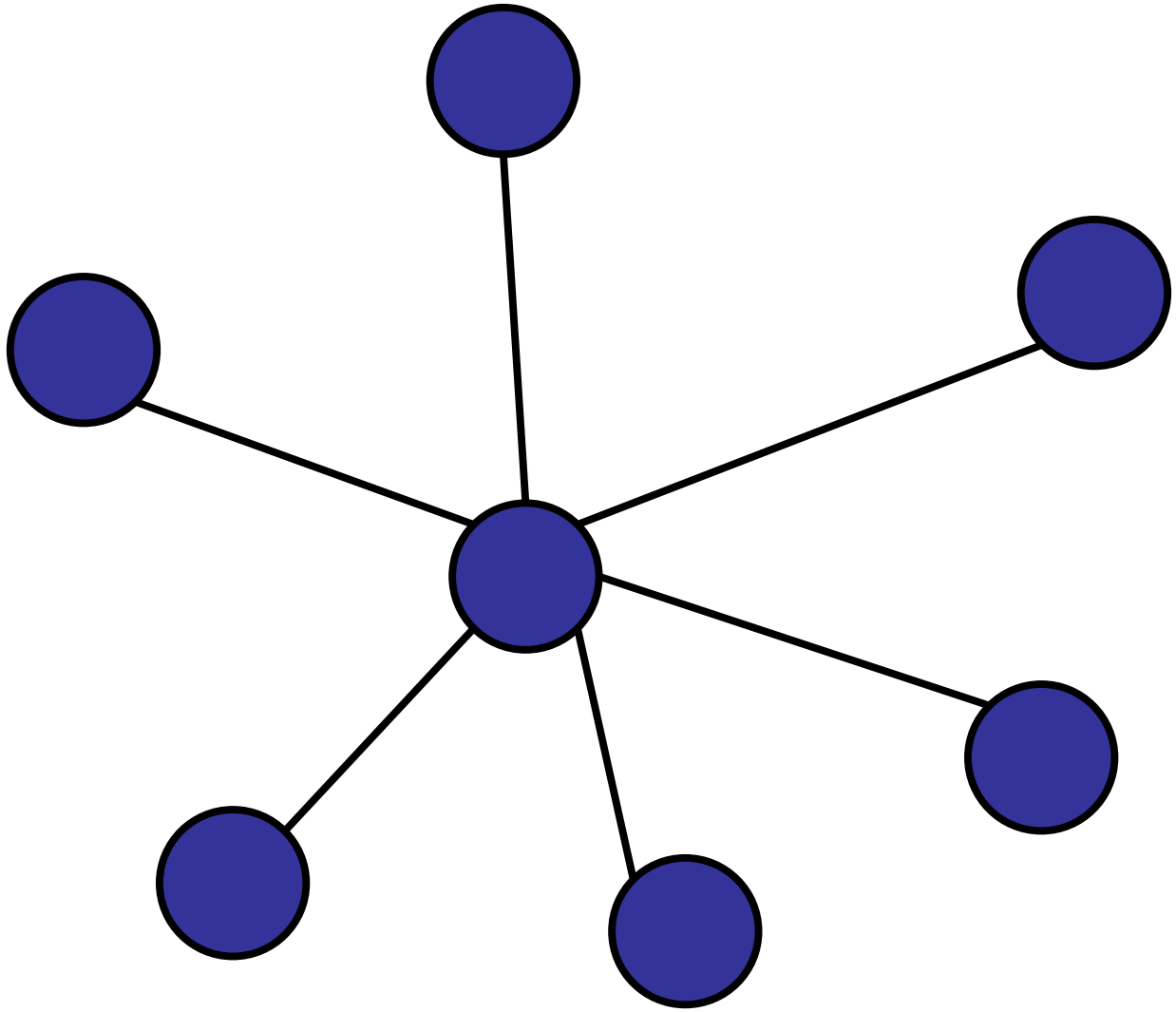
- Maximum distance between two nodes, following the shortest path.



Special Graphs

Special Graphs

Star



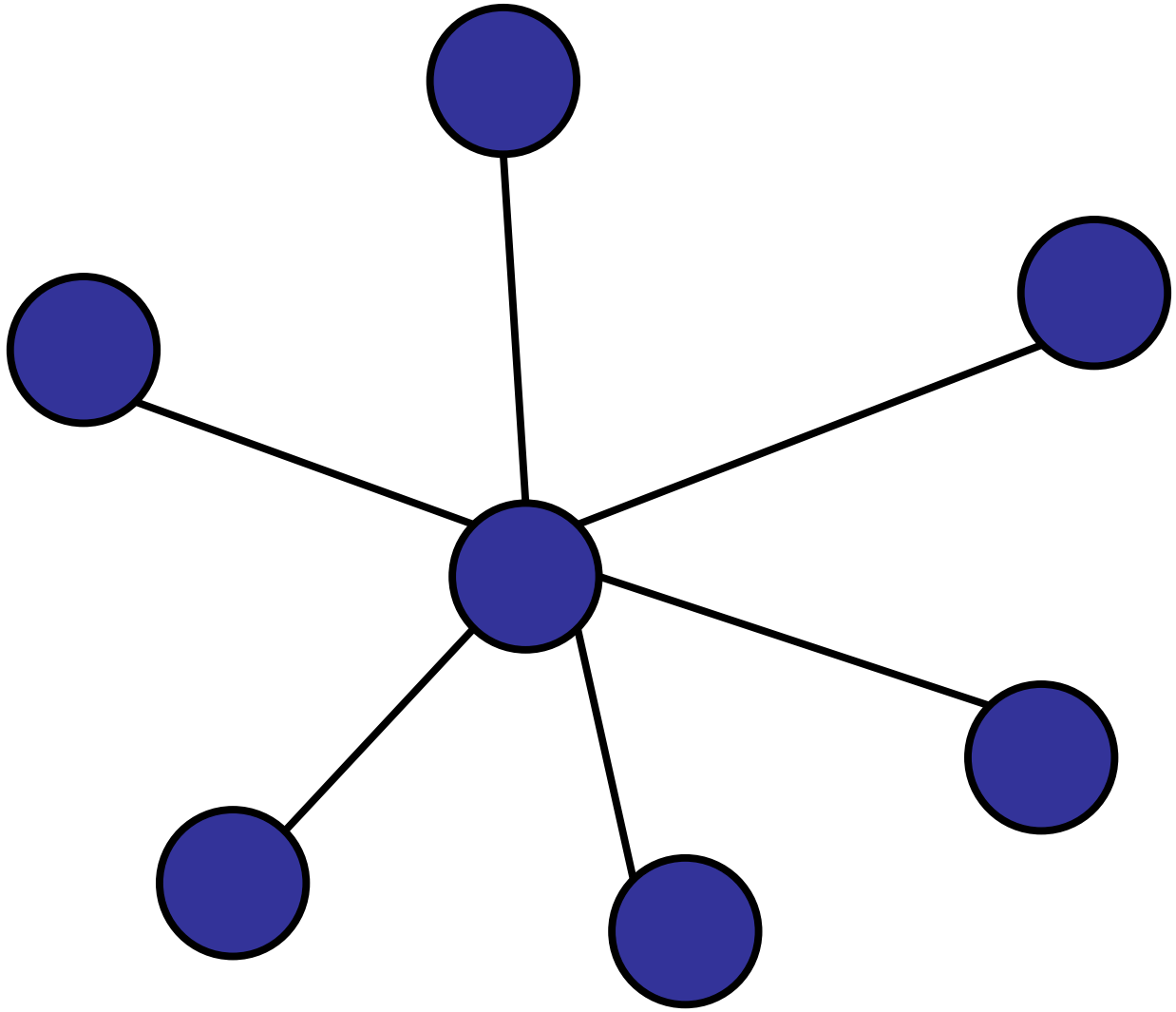
One central node, all edges connect center to edges.

Degree of n-node star is:

1. 1
2. 2
3. n/2
4. n-2
5. ✓ n-1
6. n

Special Graphs

Star



One central node, all edges connect center to edges.

Diameter of n-node star:

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2. ✓ 2
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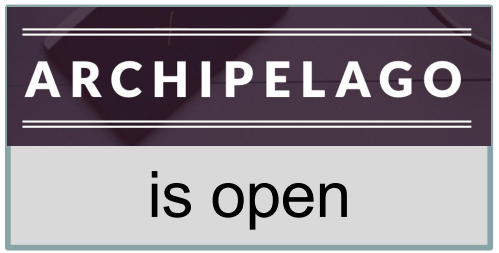
4. n-2
5. n-1
6. n



Special Graphs

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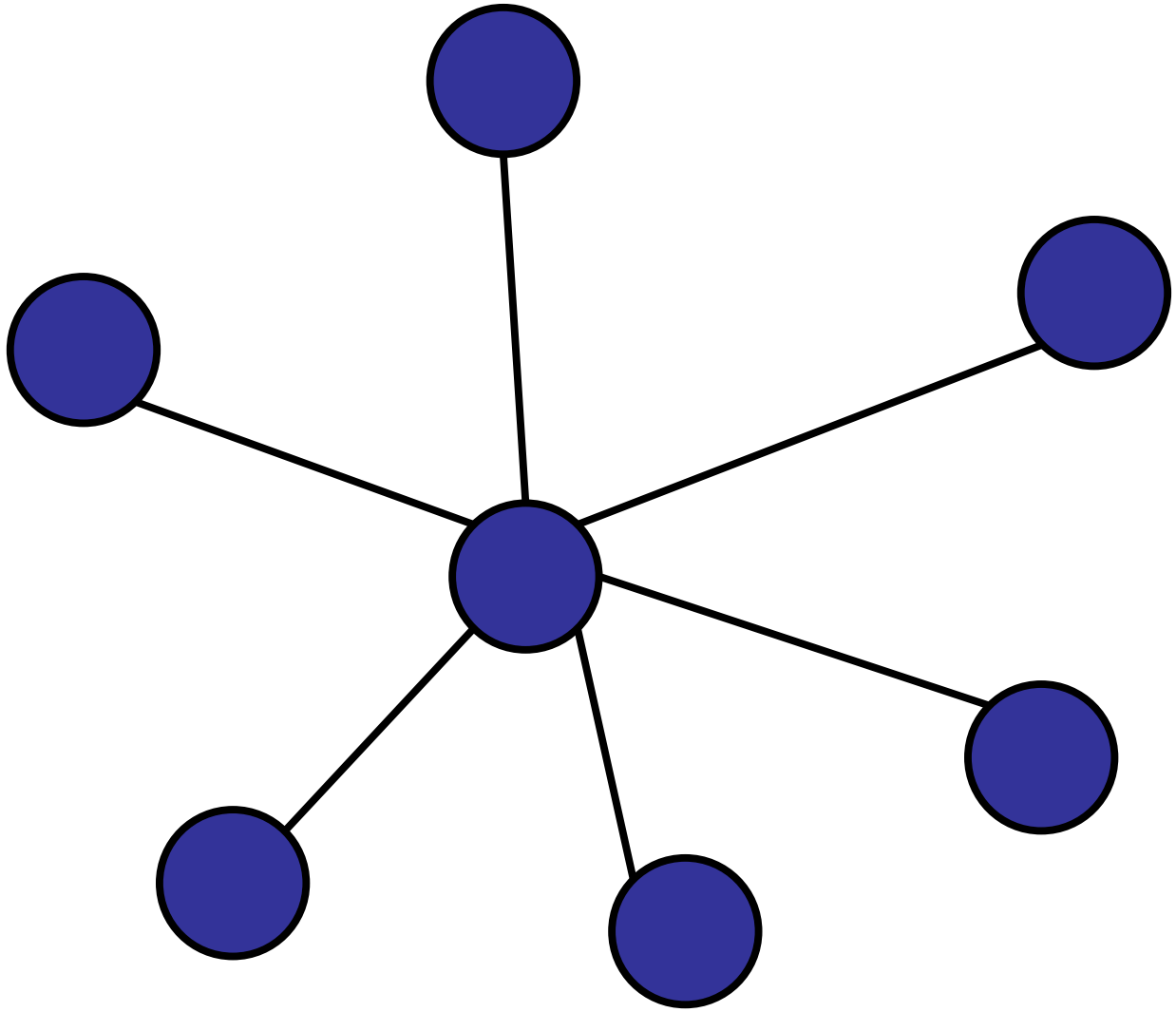


One central node, all edges connect center to edges.

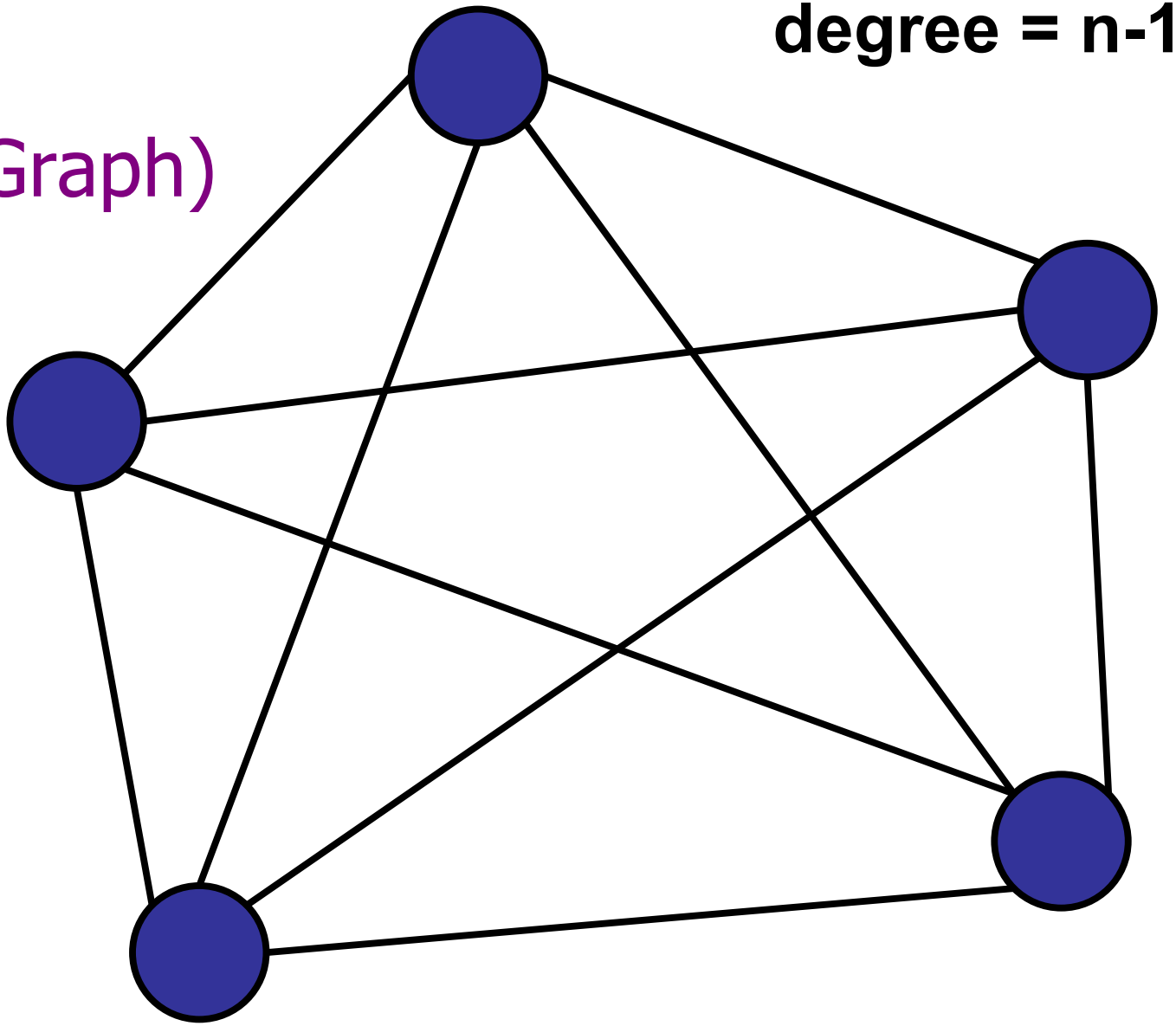
Special Graphs

diameter = 1

degree = n-1

Clique

(Complete Graph)



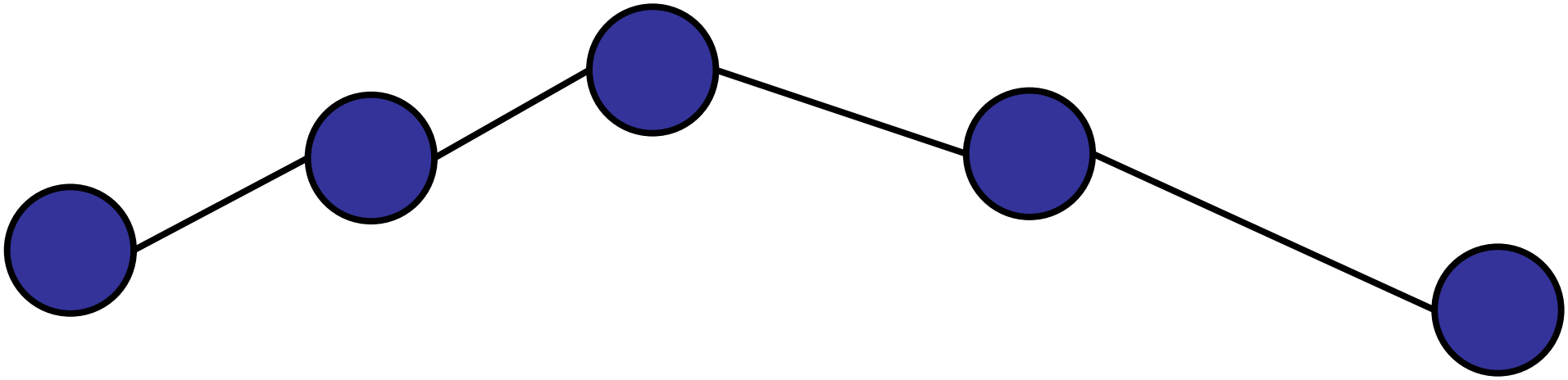
All pairs connected by edges.

Special Graphs

Line (or path)

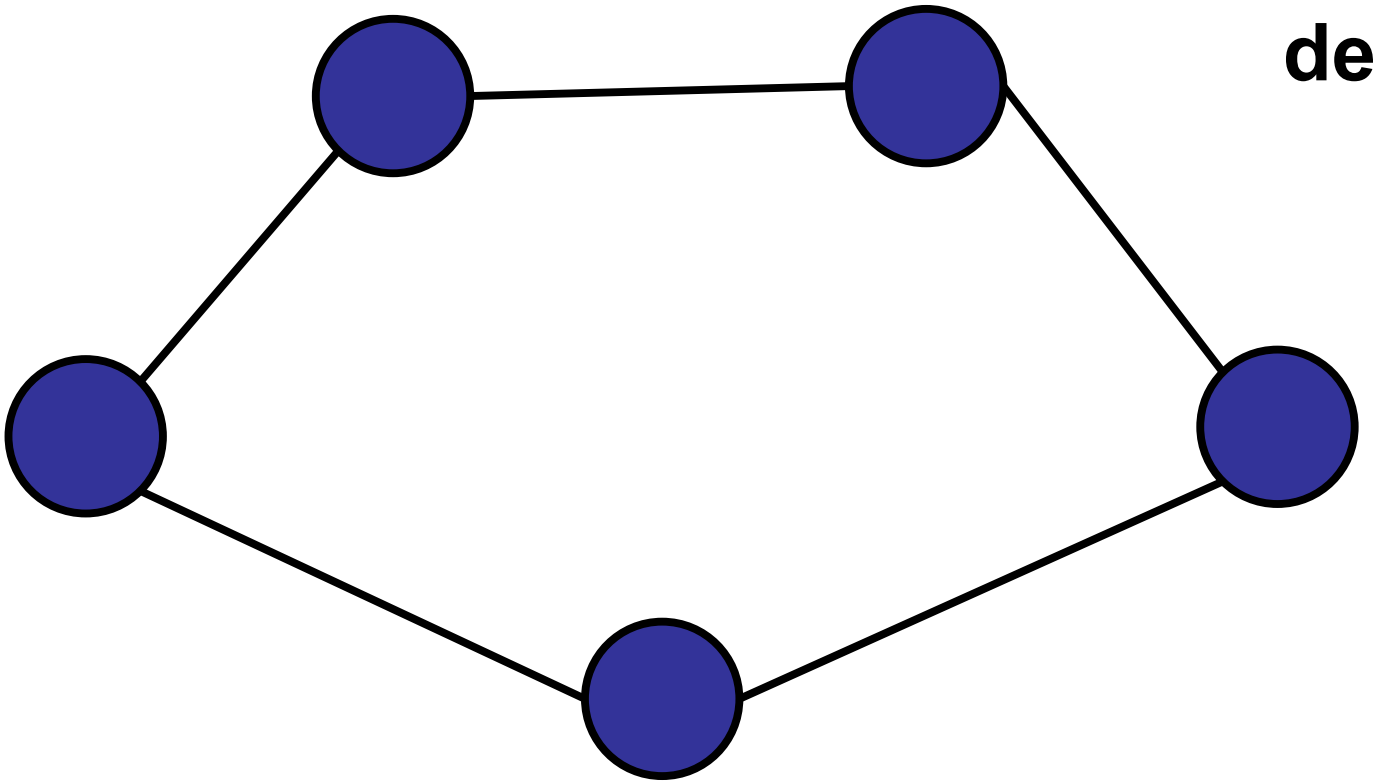
diameter = n-1

degree = 2



Special Graphs

Cycle



diameter = n/2

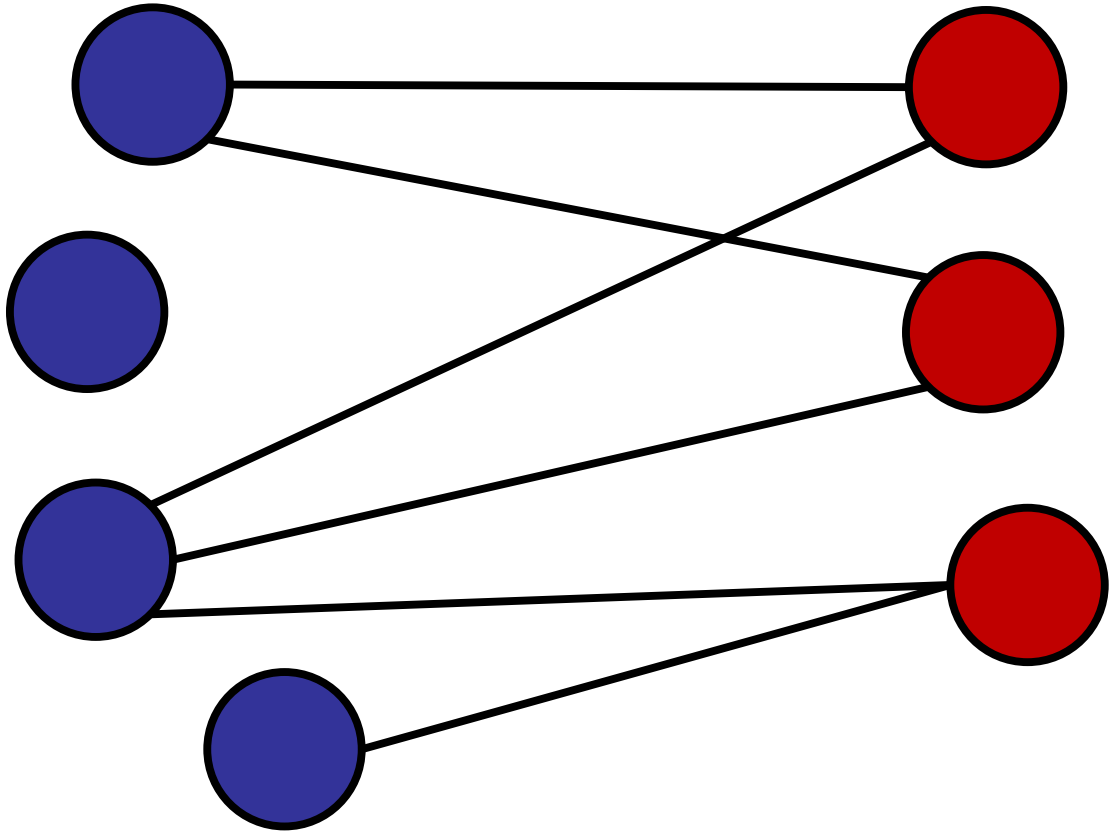
or

diameter = n/2-1

degree = 2

Special Graphs

Bipartite Graph



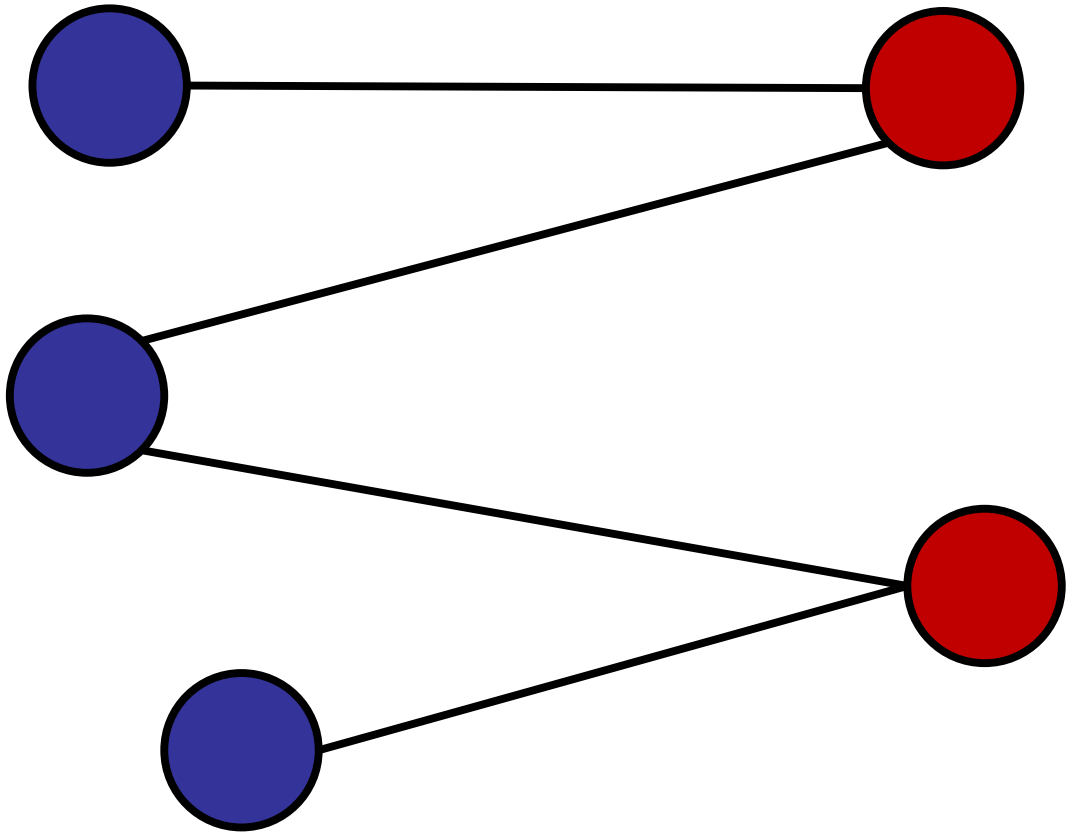
Nodes divided into two sets with no edges between nodes in the same set.

Max. diameter of n-node bipartite graph is:

1. 1
2. 2
3. n/2-1
4. n/2
5. ✓ n-1
6. n

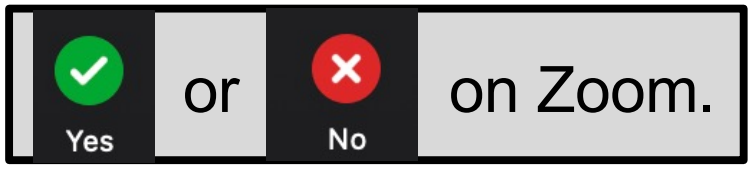
Special Graphs

Bipartite Graph

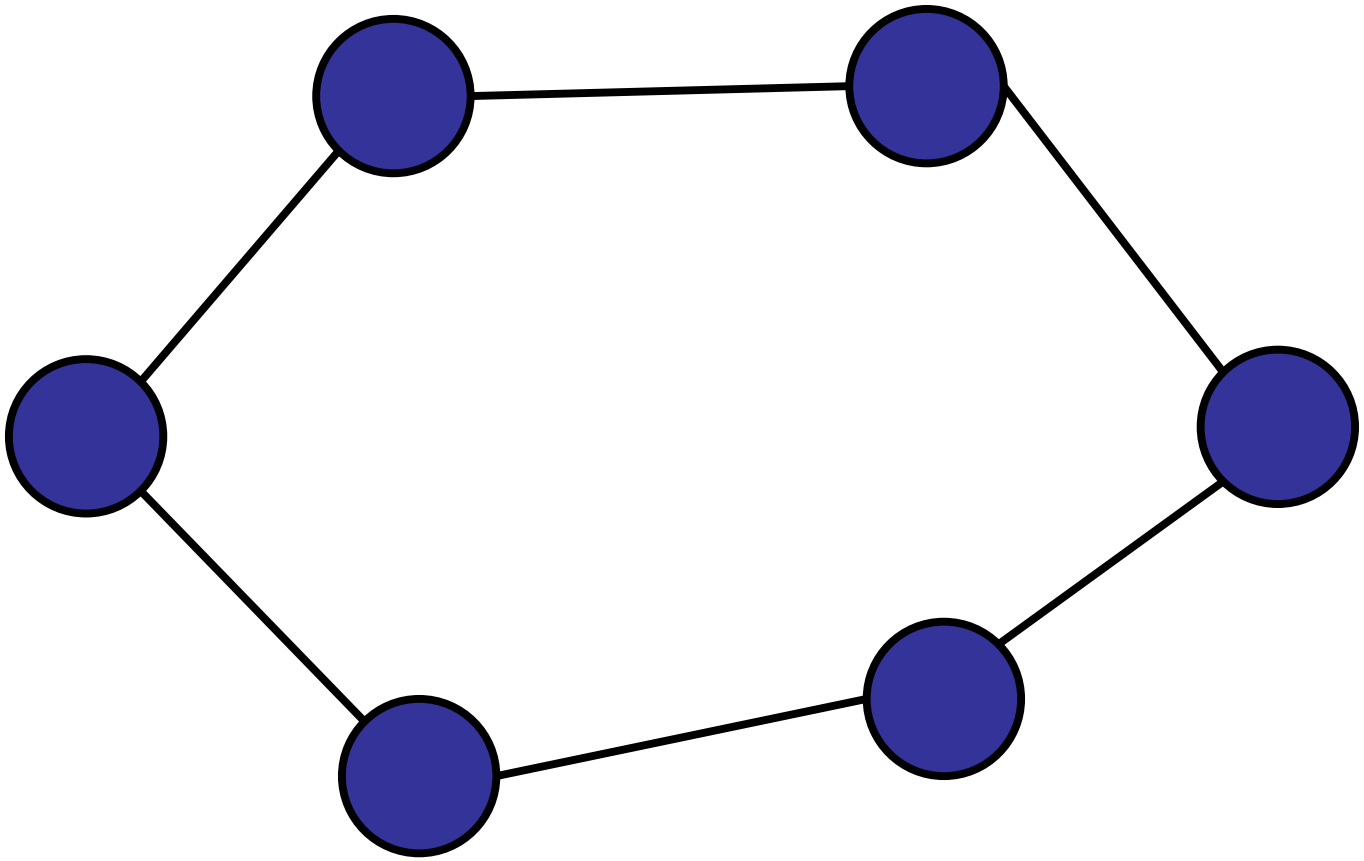


Nodes divided into two sets with no edges between nodes in the same set.

Is it bipartite?

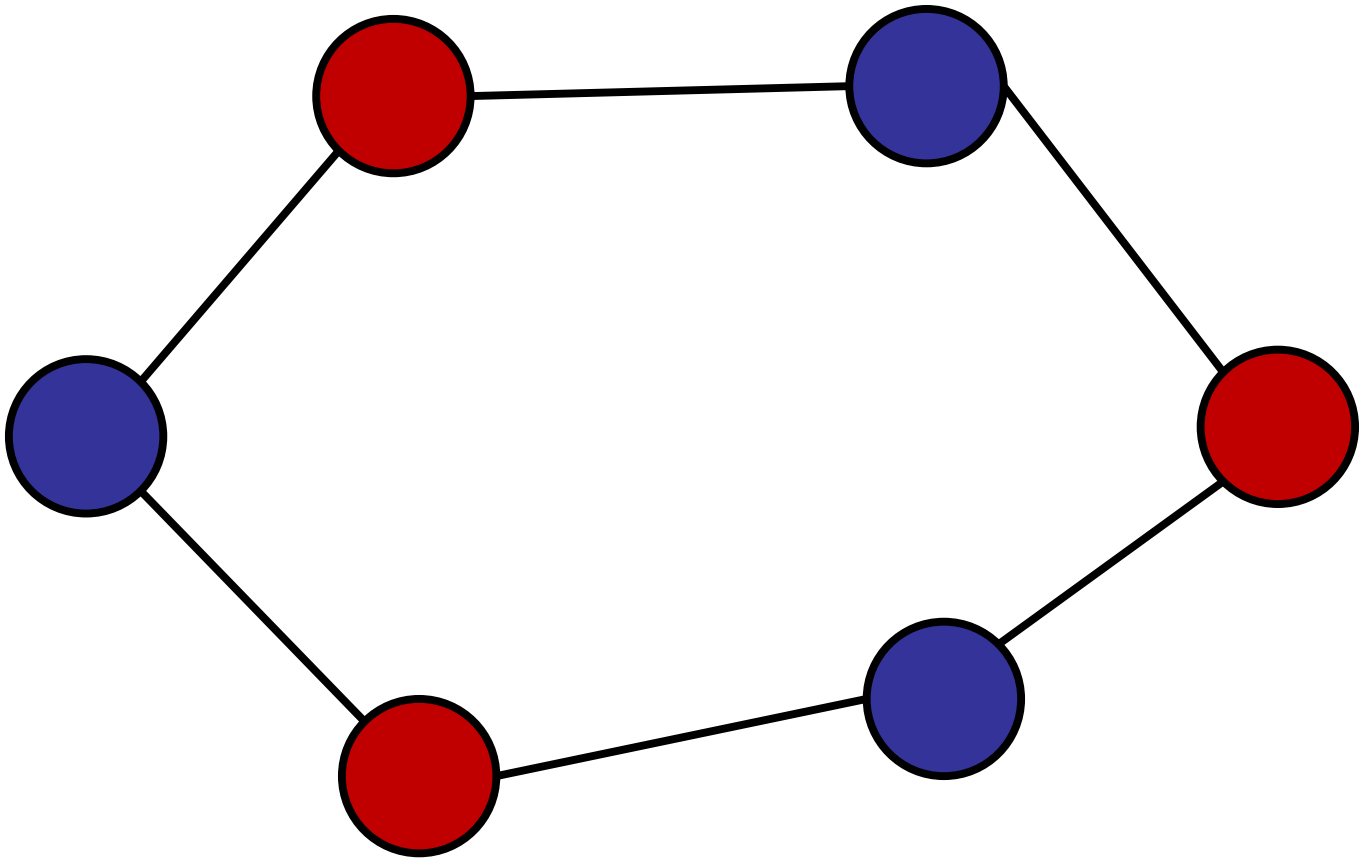


1. Yes
2. No

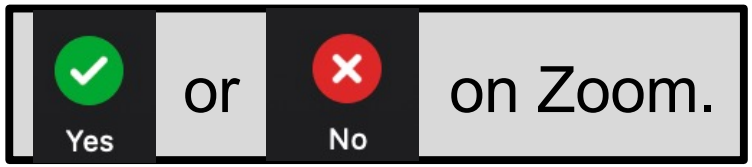


Is it bipartite?

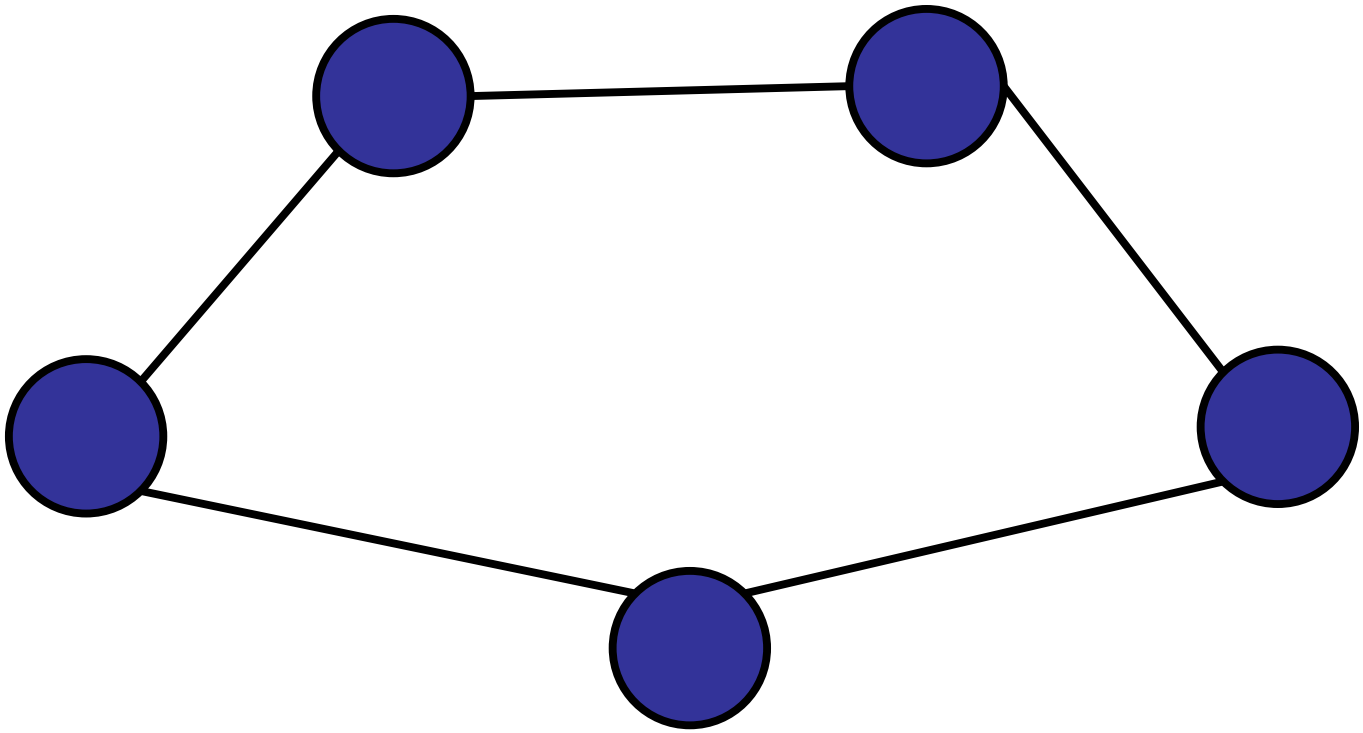
✔1. Yes
2. No



Is it bipartite?

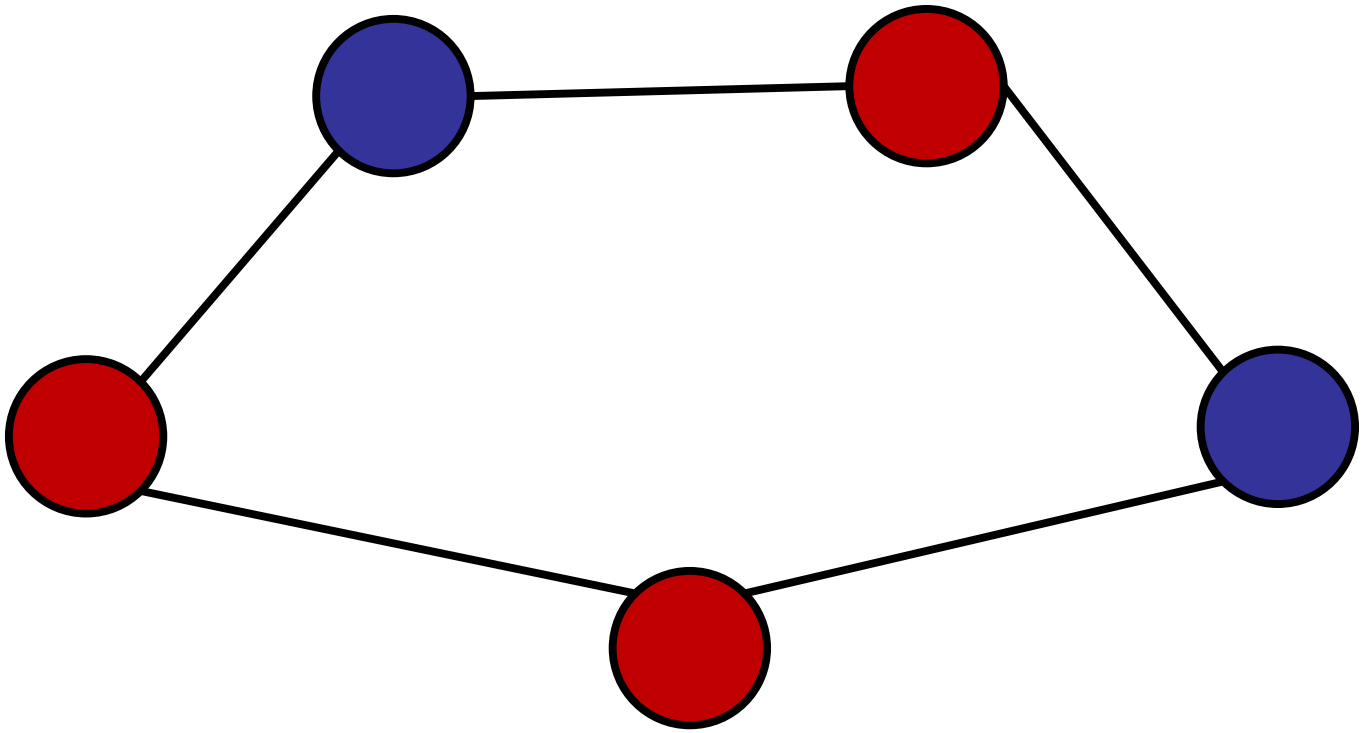


1. Yes
2. No



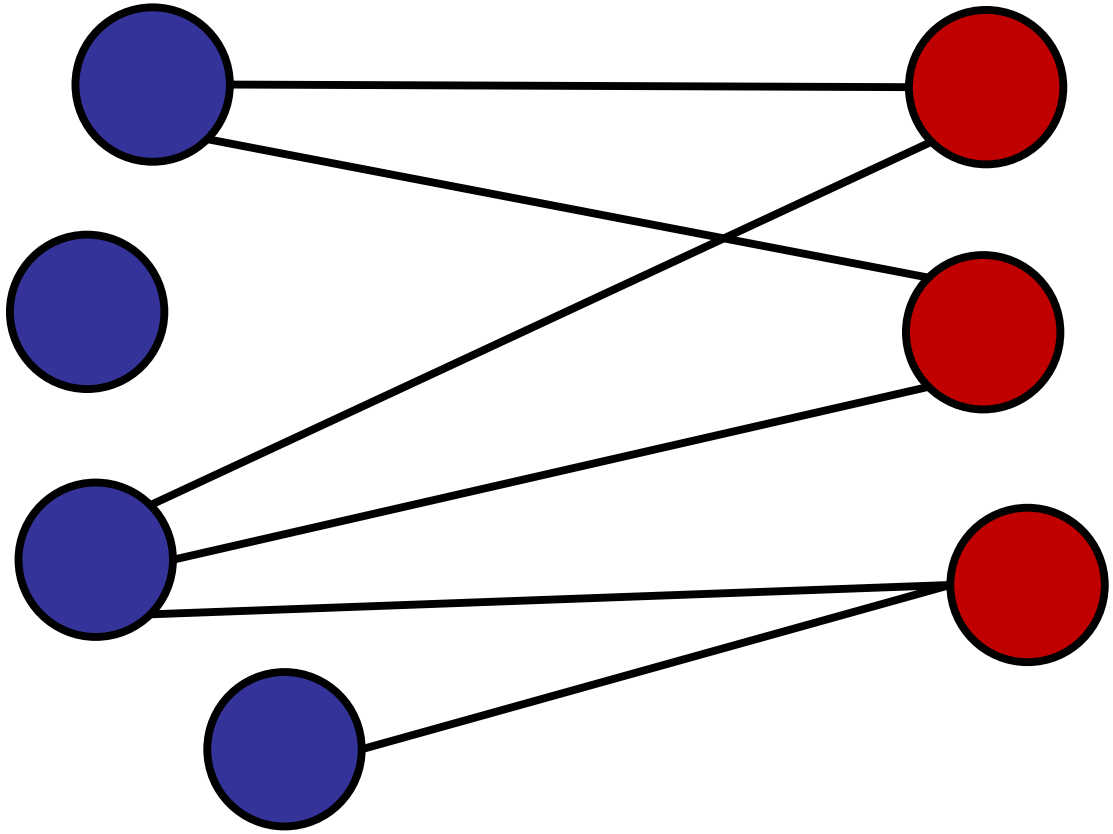
Is it bipartite?

1. Yes
2. ✓ No



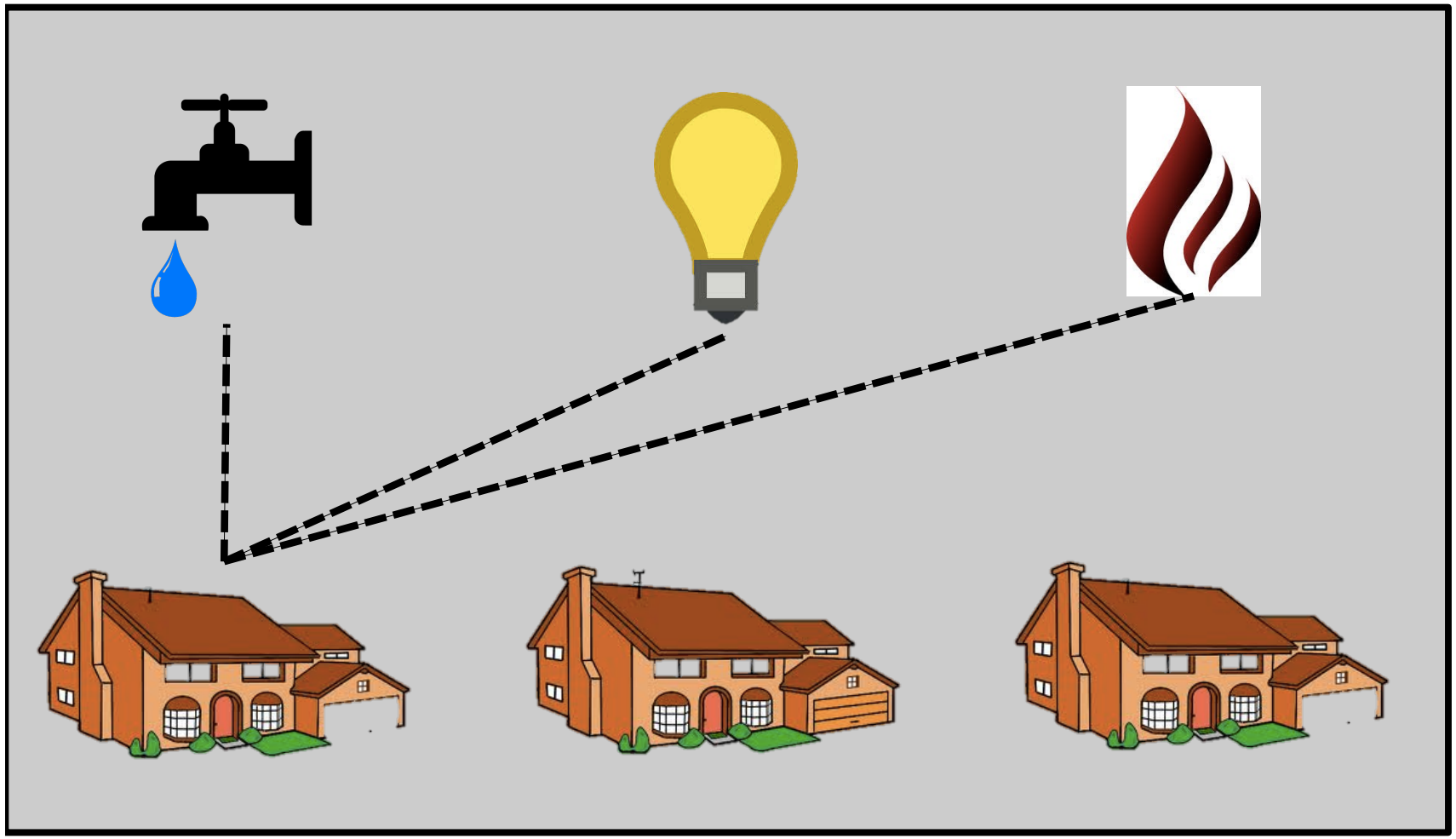
Special Graphs

Bipartite Graph



Nodes divided into two sets with no edges between nodes in the same set.

Puzzle of the Week



Connect each house to all three utilities (water, electricity, gas).
Do not let any of the cables or pipes cross.
(Or show that it is impossible.)

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

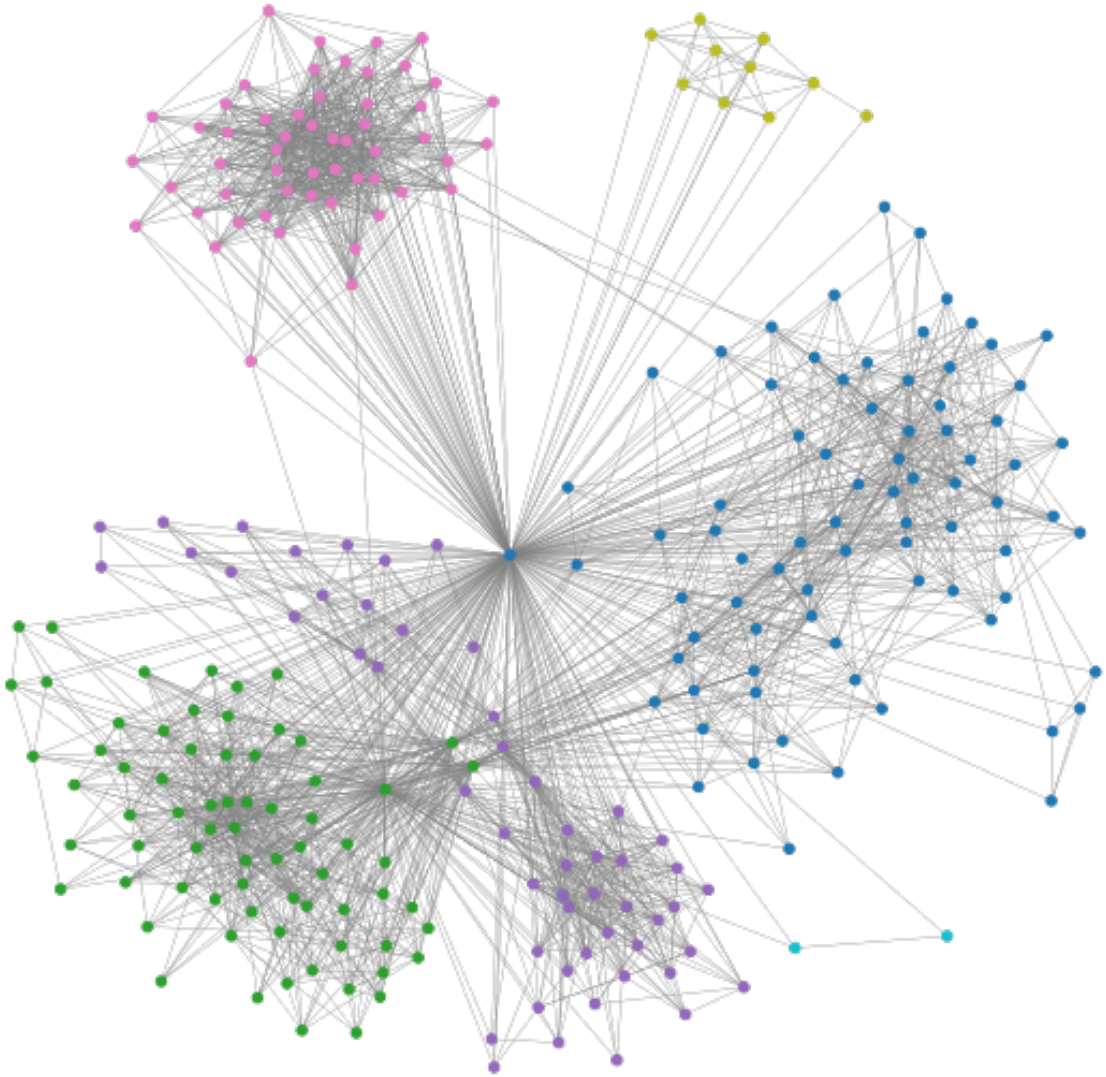
Where do we find graphs?

(How to model real problems as a graph!)

Where do we find graphs?

Social network:

- Nodes are people
- Edge = friendship



facebook

Source: https://www.databentobox.com/2019/07/28/facebook-friend-graph/

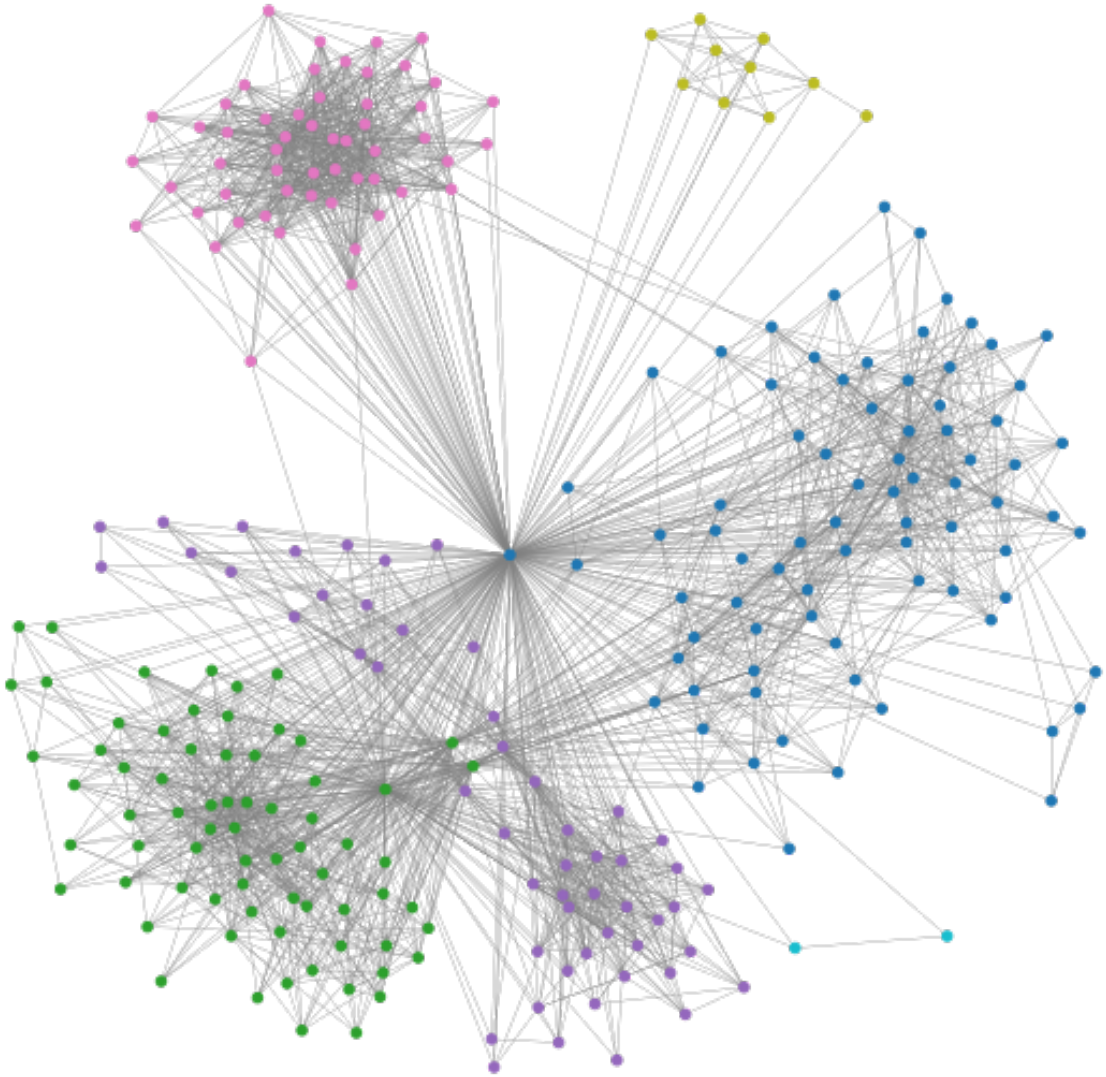
Where do we find graphs?

Social network:

- Nodes are people
- Edge = friendship

Questions:

- Connected?
- Diameter?
- Degree?



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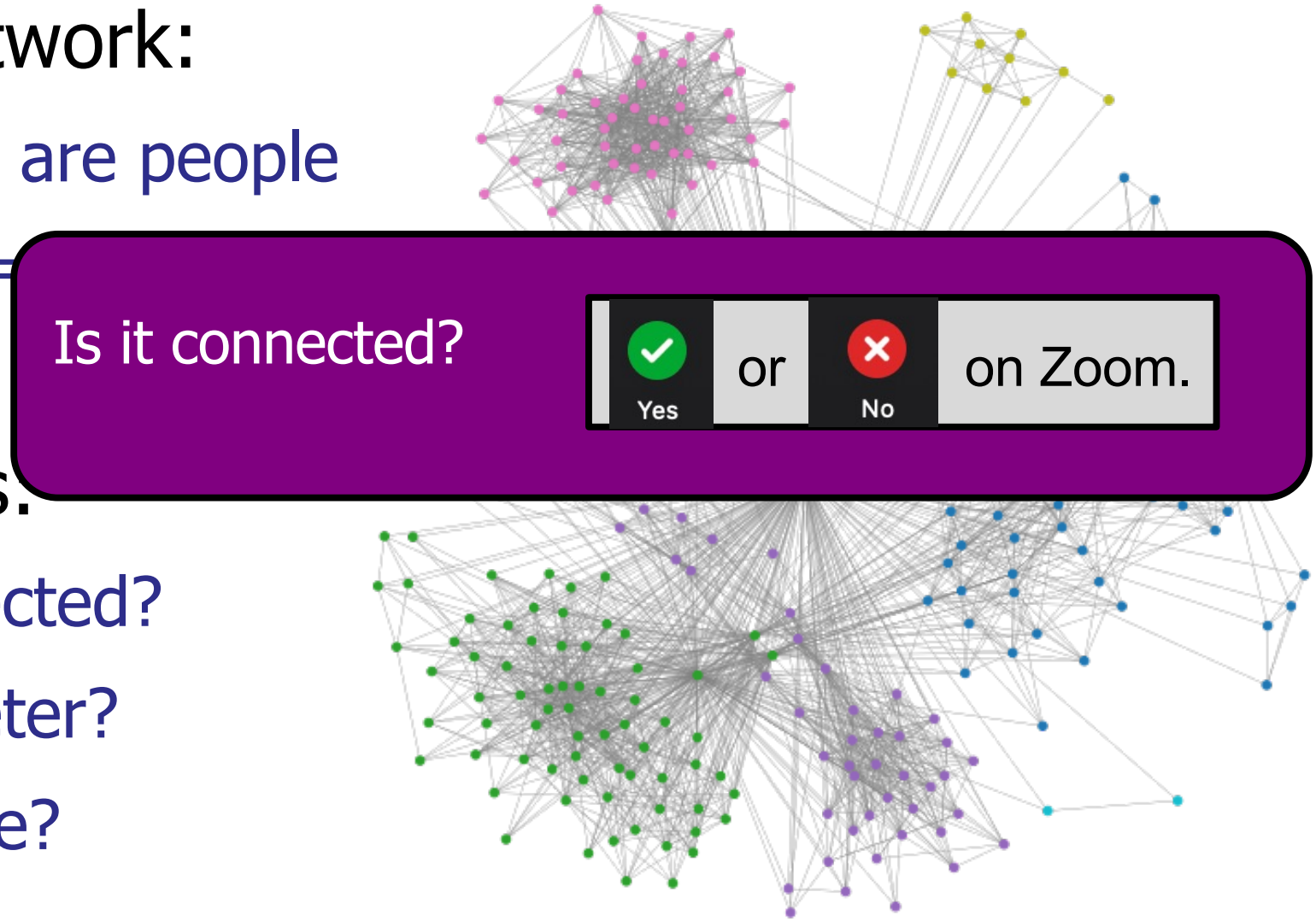

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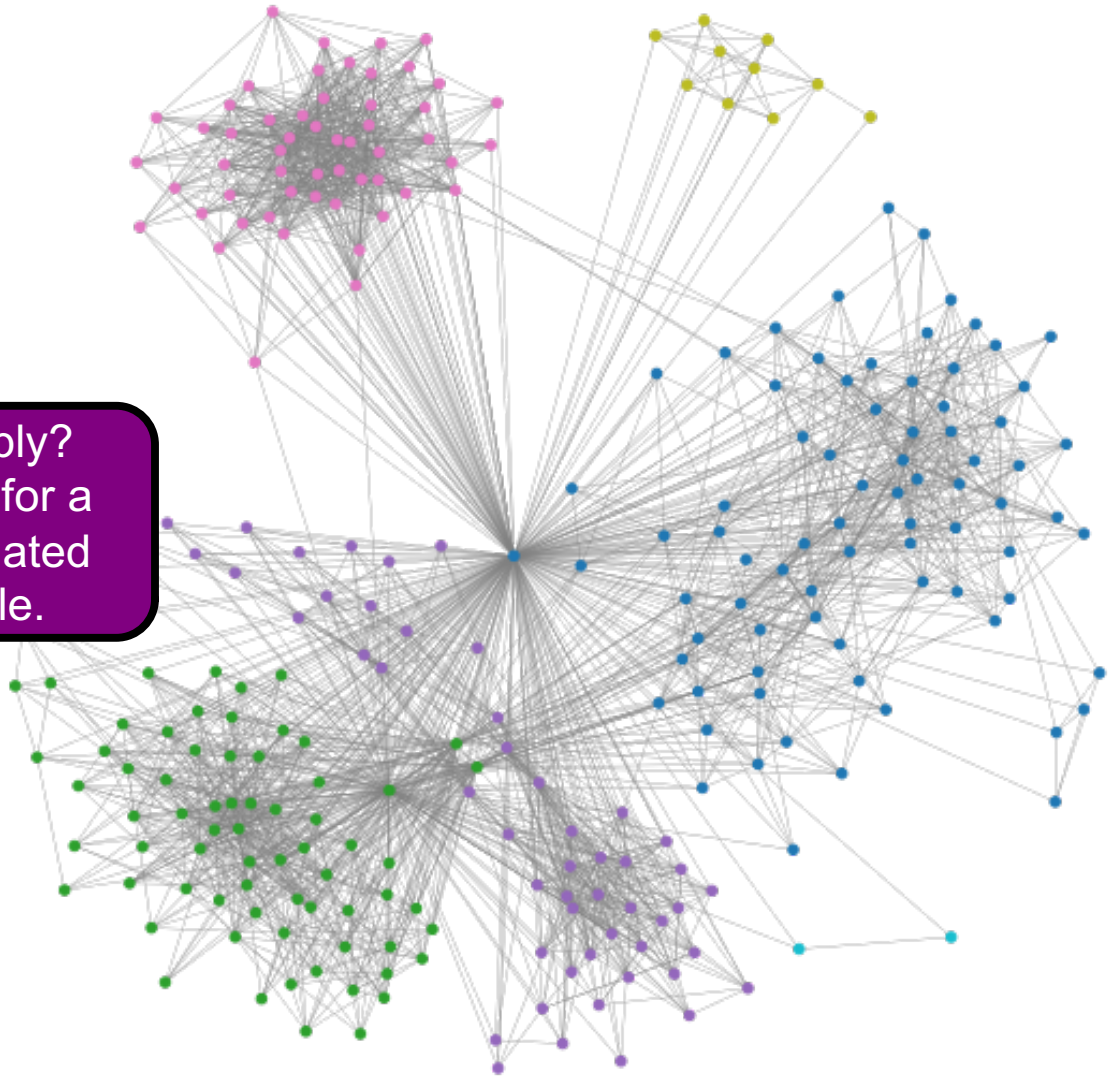

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Questions:

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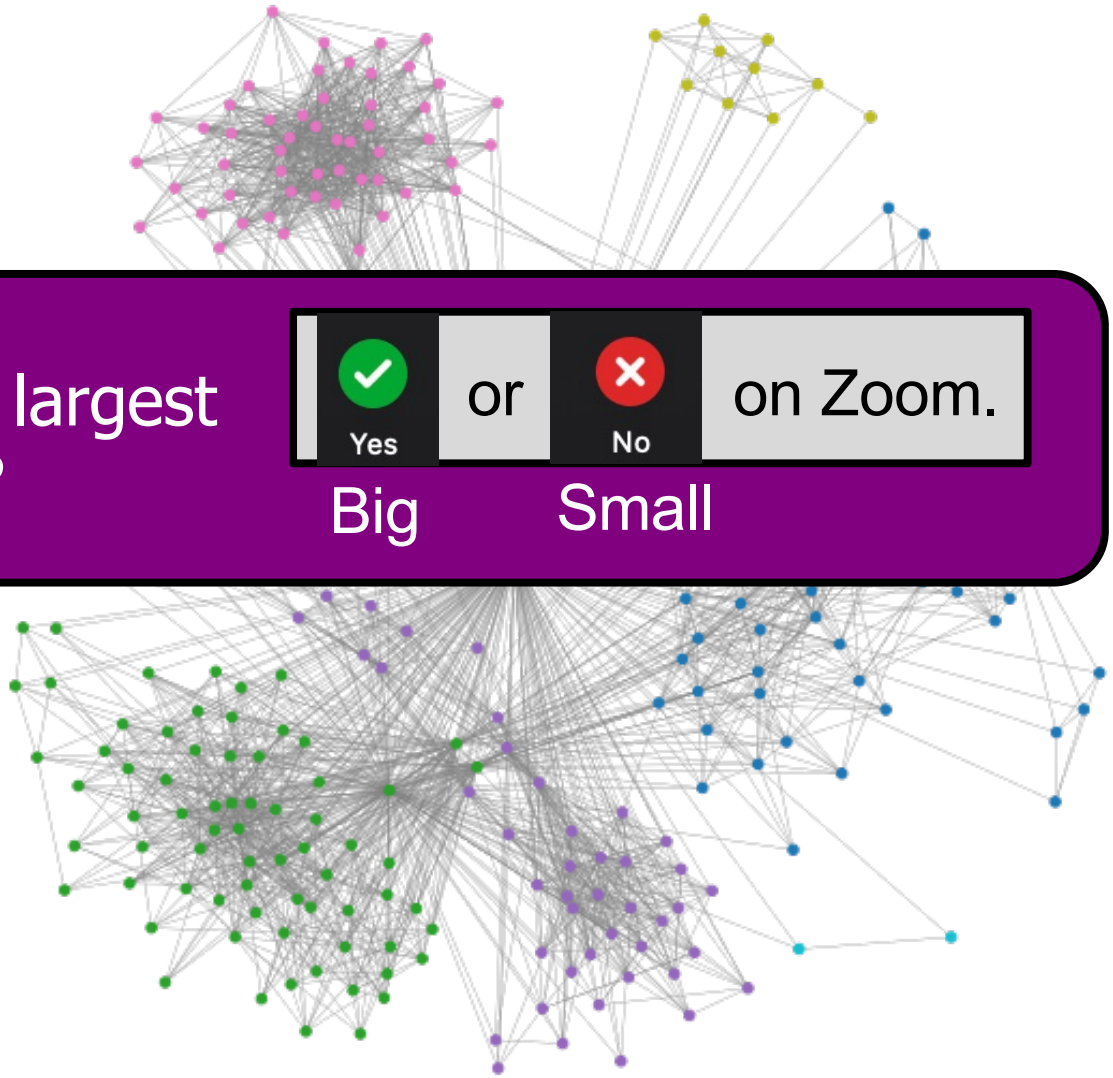

Diameter of largest component?

Yes or No on Zoom.

Big Small

Source: https://www.databentobox.com/2019/07/28/facebook-friend-graph/

Where do we find graphs?

Social network:

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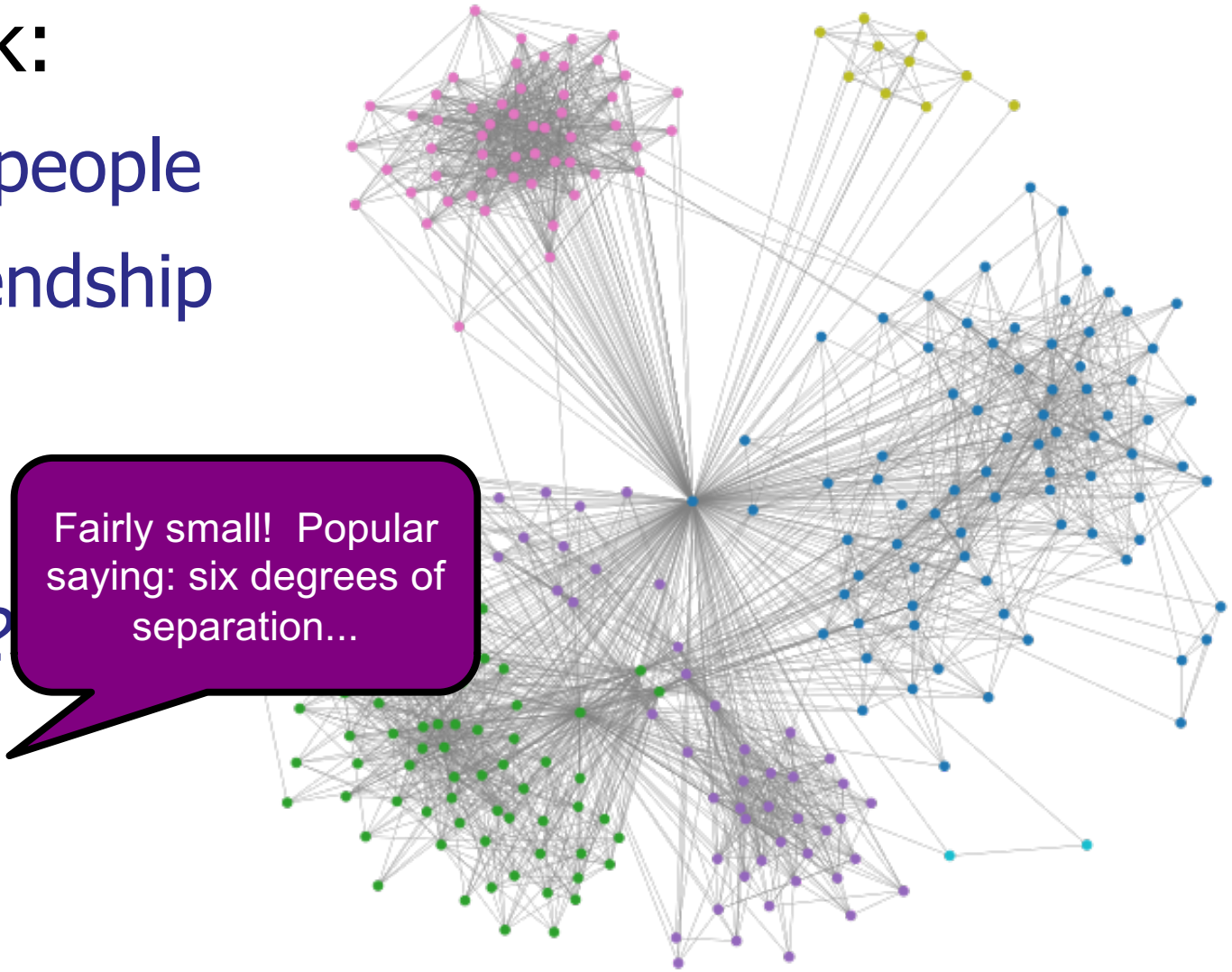
- Edge = friendship

Questions:

- Connected?
- Diameter?
- Degree?



Source: https://www.databentobox.com/2019/07/28/facebook-friend-graph/

Where do we find graphs?

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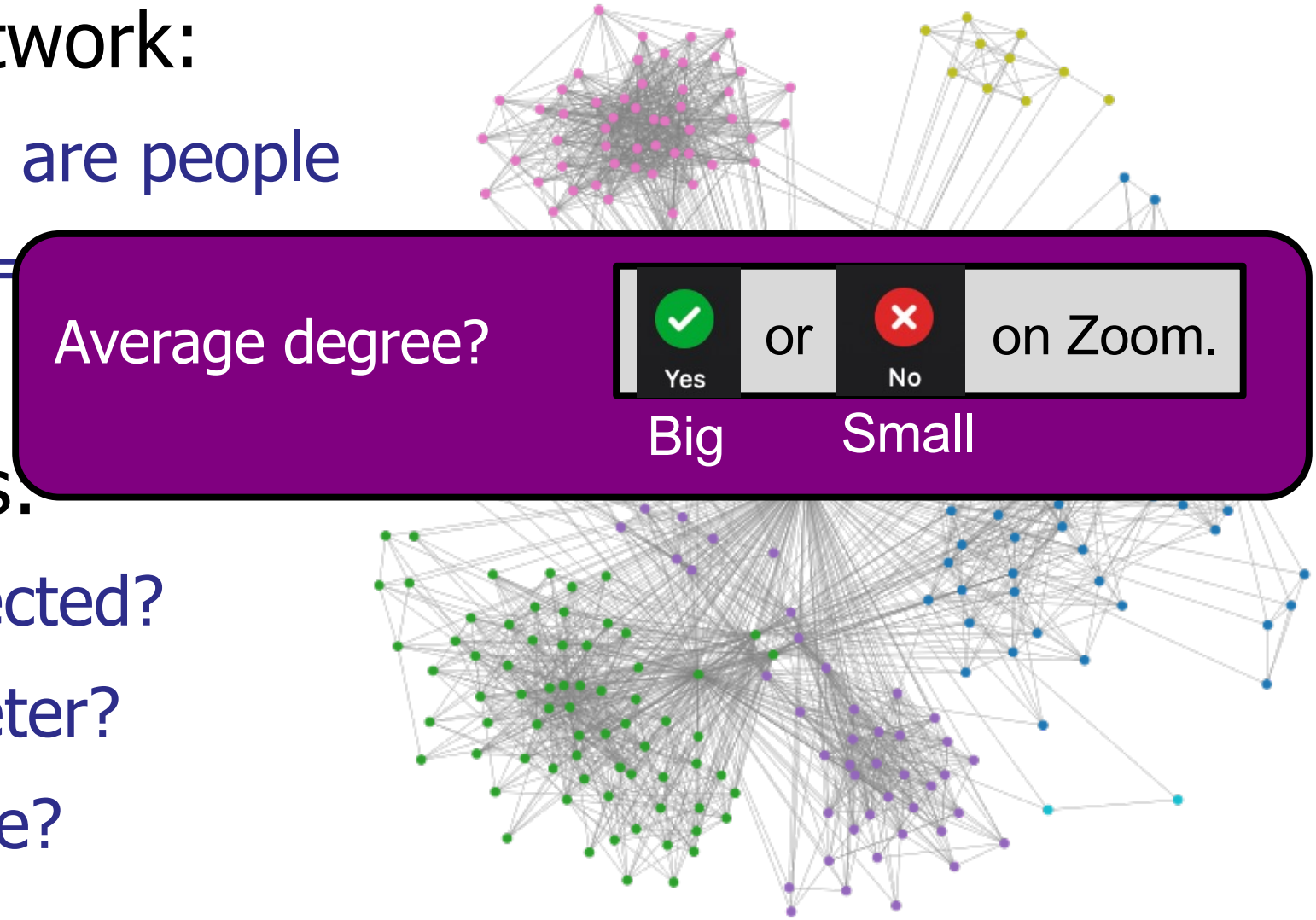


Source: https://www.databentobox.com/2019/07/28/facebook-friend-graph/

Where do we find graphs?

Social network:

- Nodes are people
- Edge = friendship

Questions:

- Connected
- Diameter?
- Degree?

What does big mean? Probably very small compared to number of nodes. (Graph is sparse.)

Source: https://www.databentobox.com/2019/07/28/facebook-friend-graph/

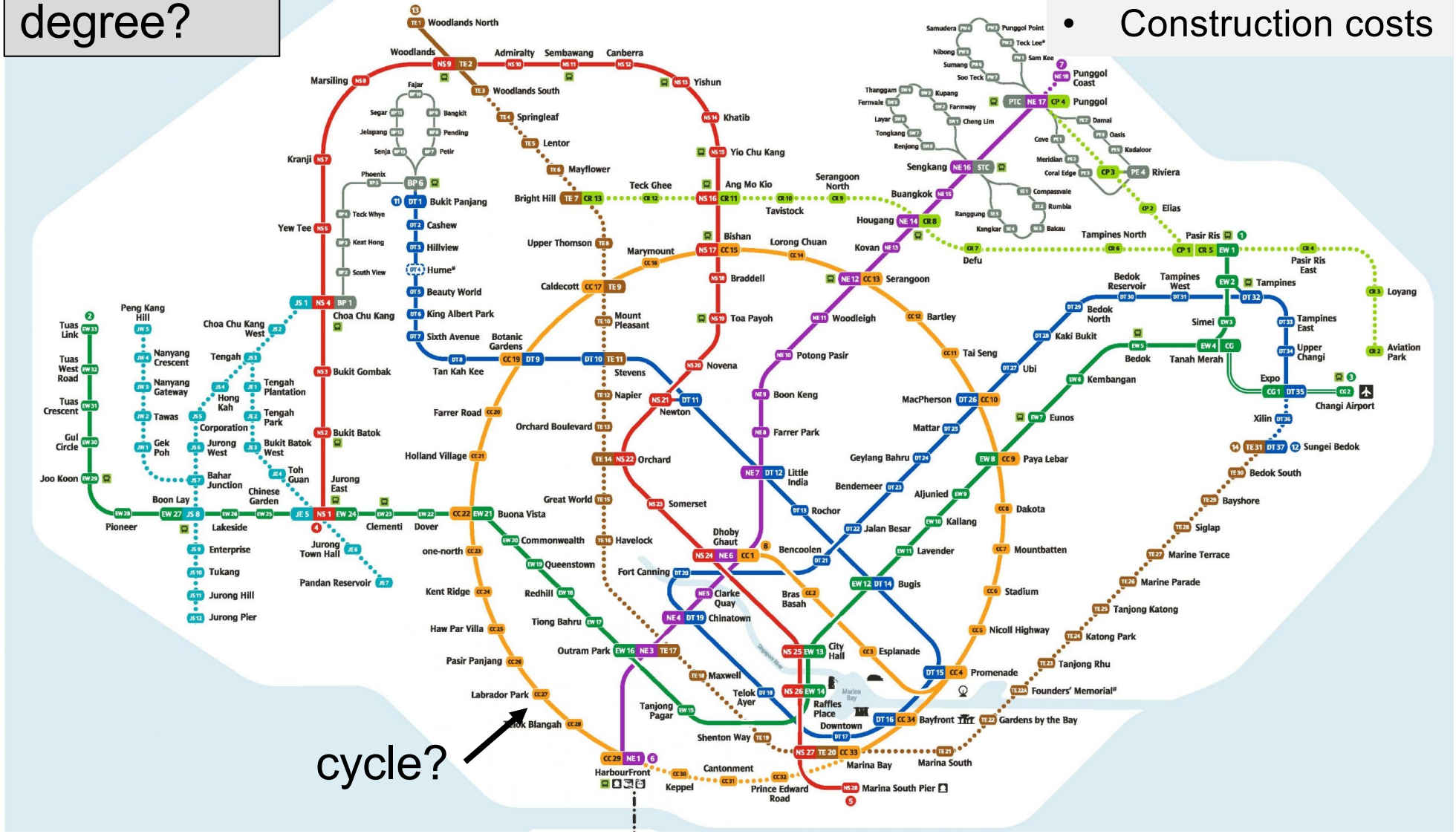
Transportation Network

connected?

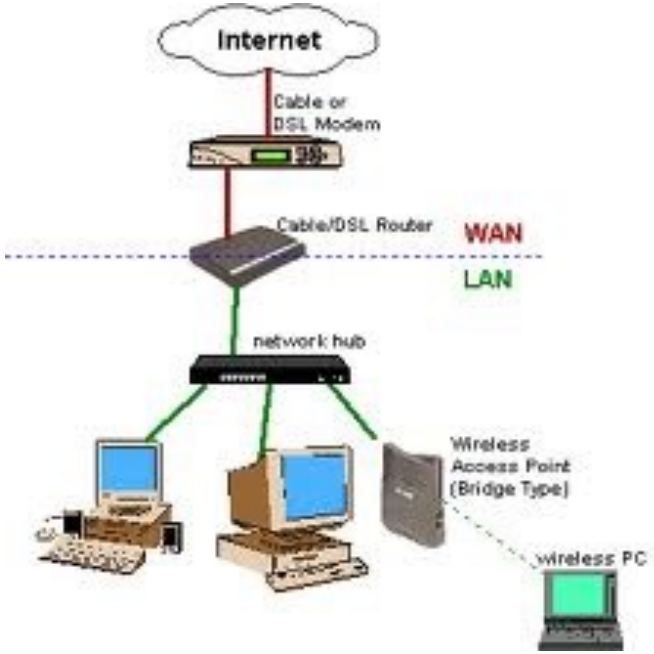
degree?

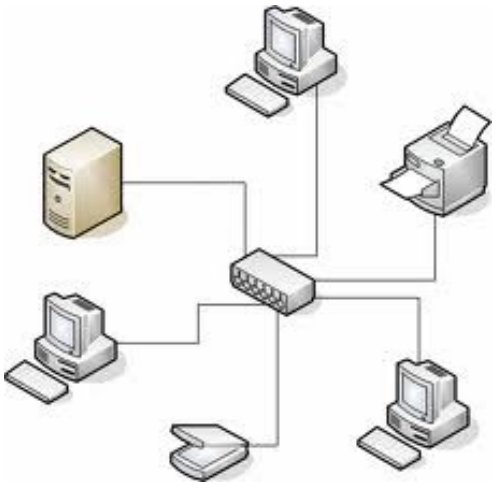
Questions:

- Shortest path
- Capacity
- Bottlenecks
- Construction costs



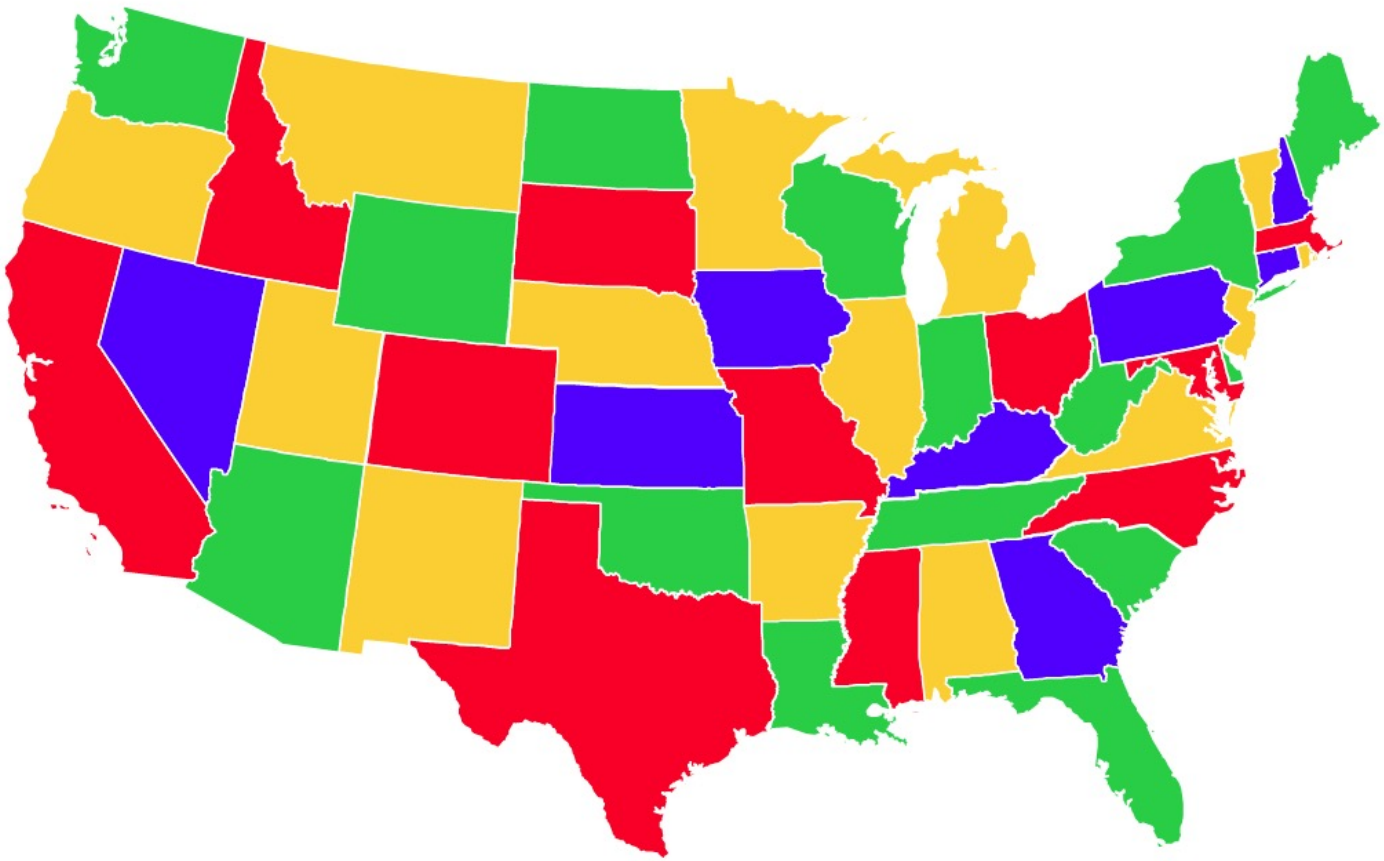
Internet / Computer Networks





Communication Network

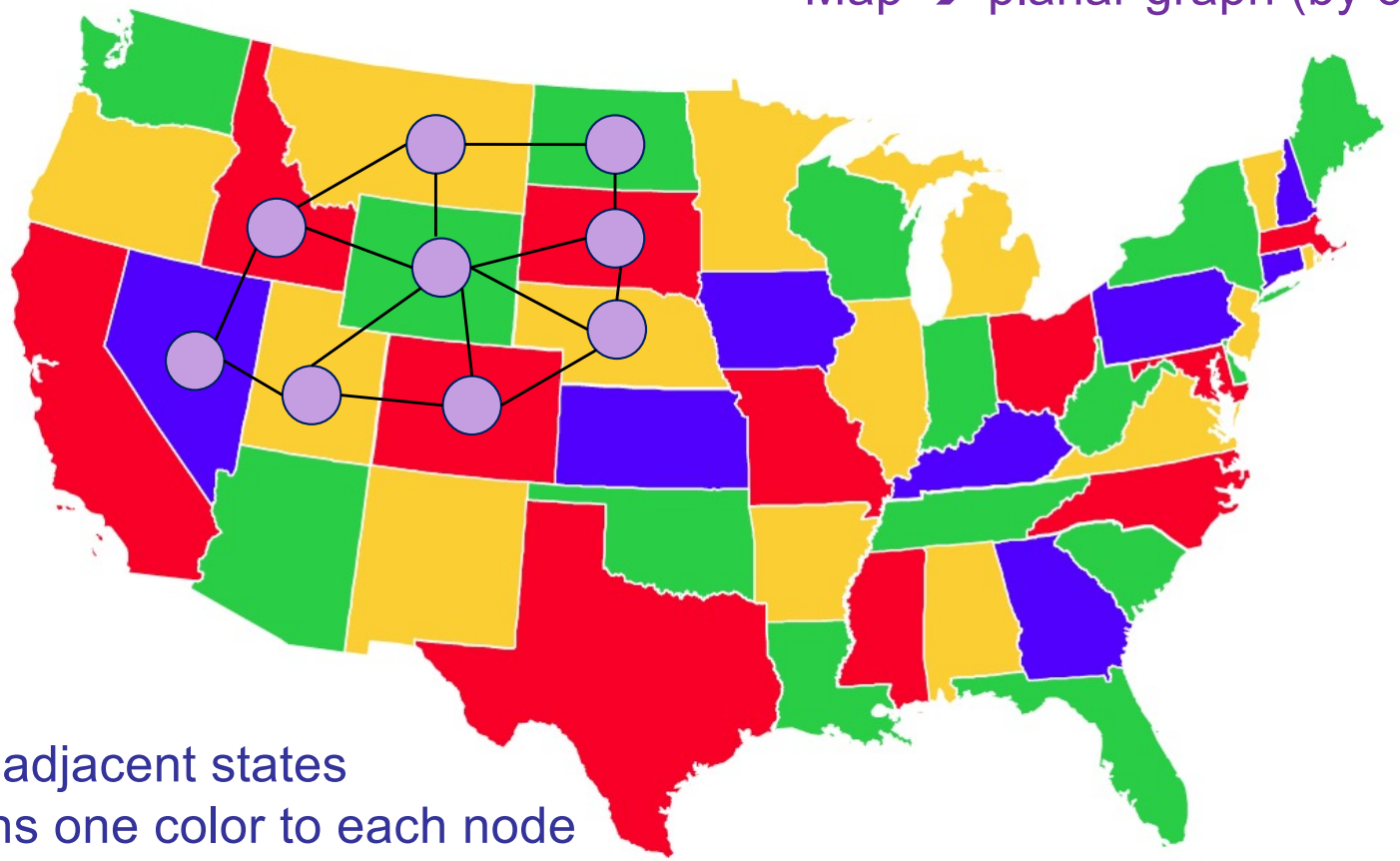
Optimization: 4-Coloring



Can you color a map using only 4-colors so that no two adjacent countries/states have the same color?

4-Coloring Theorem:

Map ➔ planar graph (by construction)

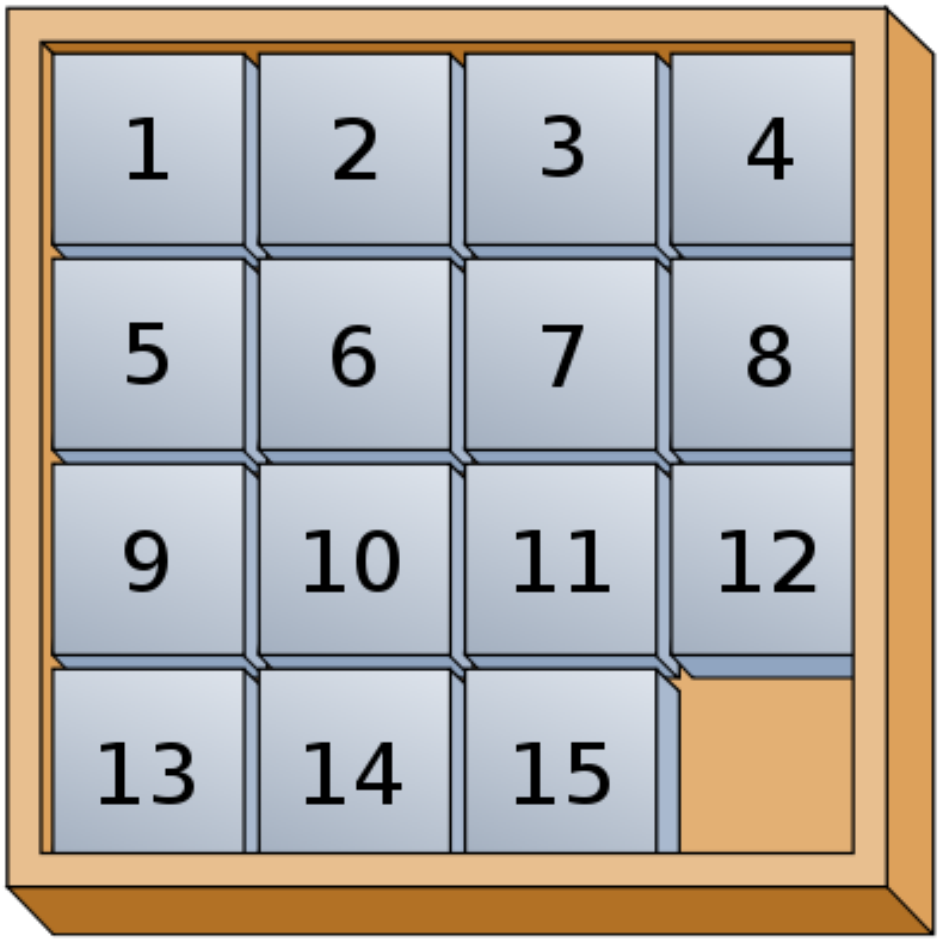


Nodes: states
Edges: pairs of adjacent states
Coloring: assigns one color to each node

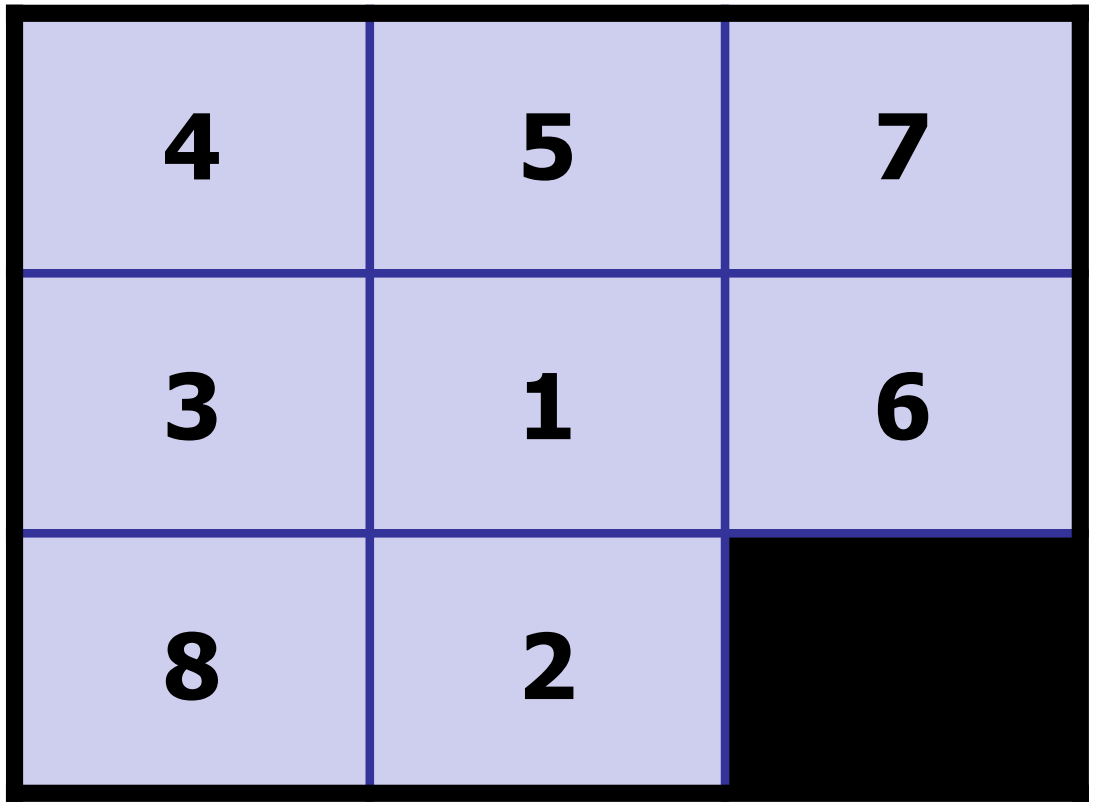
For any planar graph, you can color it using only 4-colors
so that no two adjacent countries/states have the same color?

Can be drawn on a 2d-plane with no crossing edges.

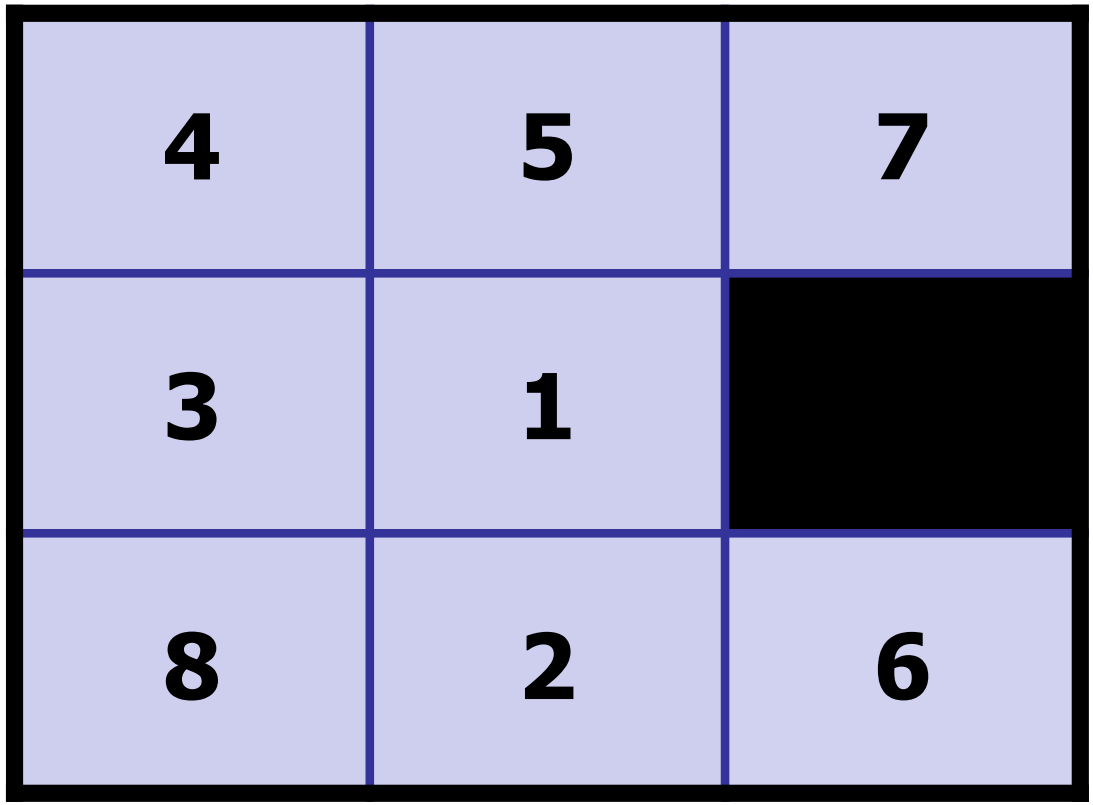
Sliding Puzzle



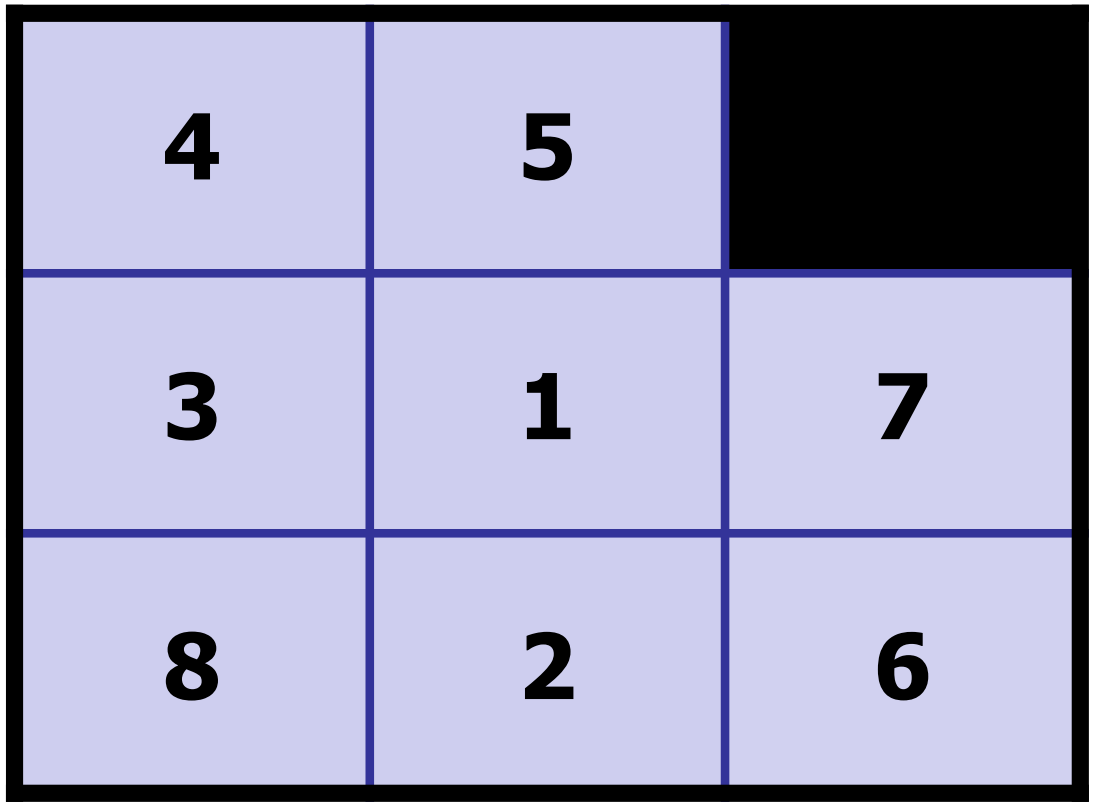
Sliding Puzzle



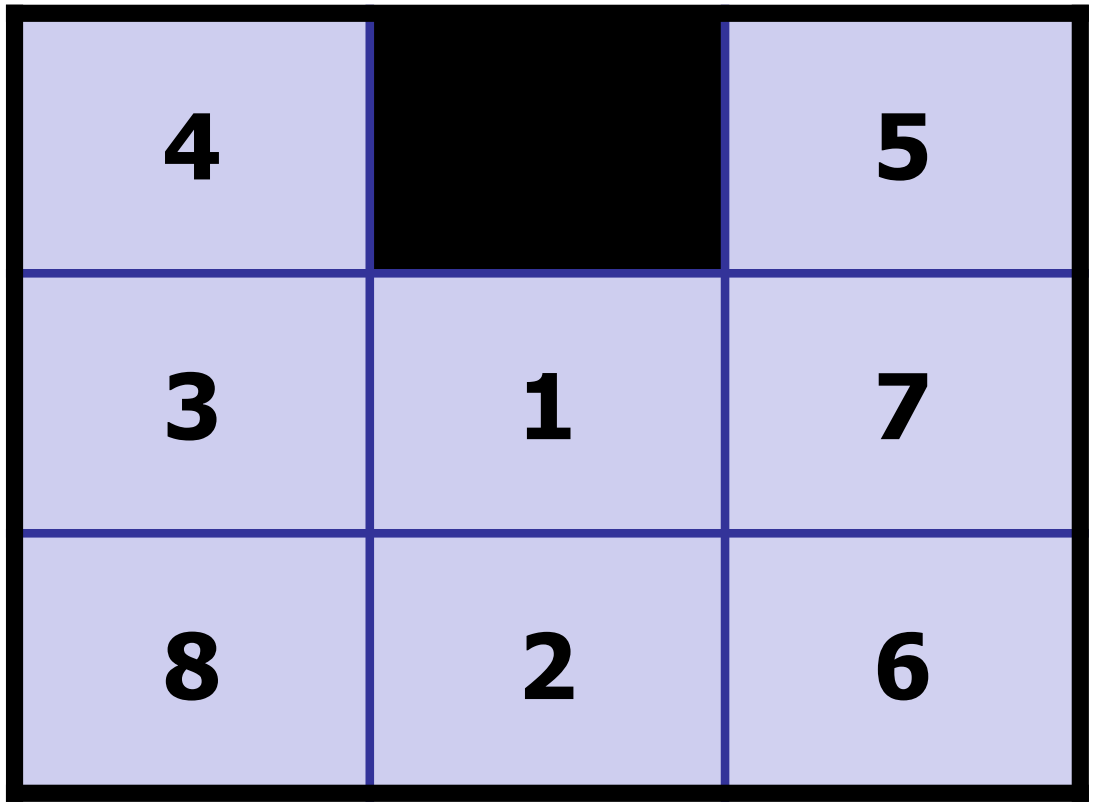
Sliding Puzzle



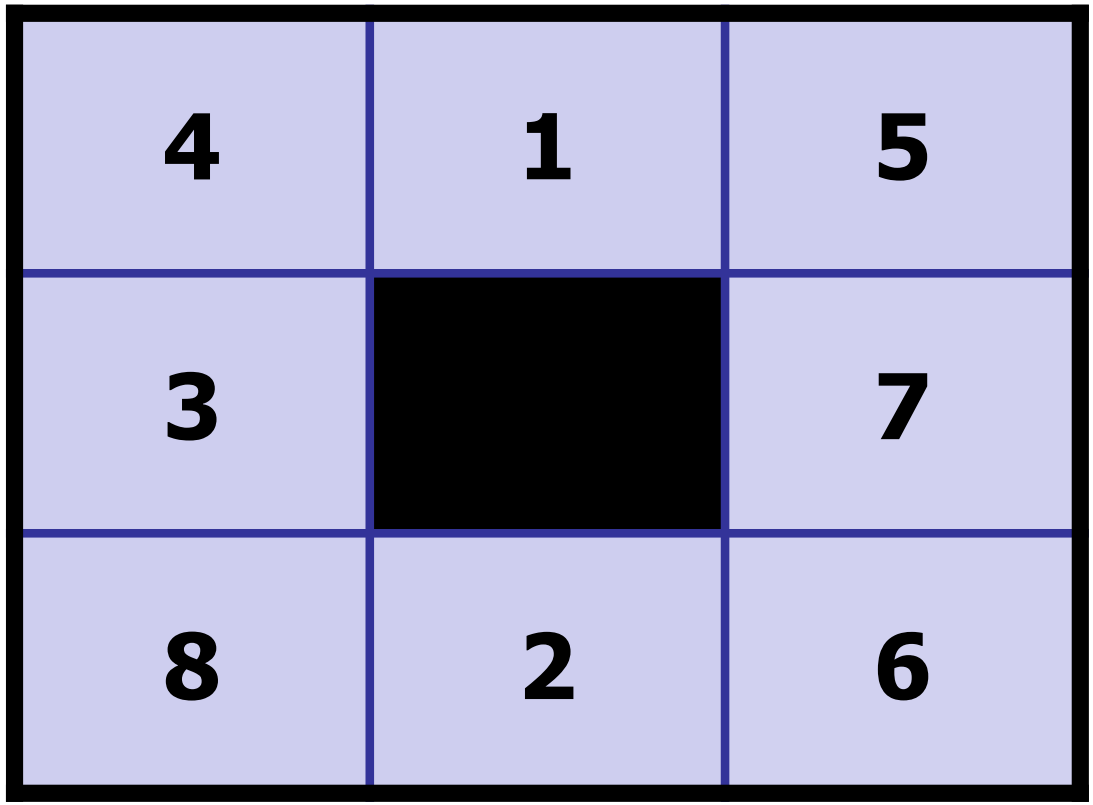
Sliding Puzzle



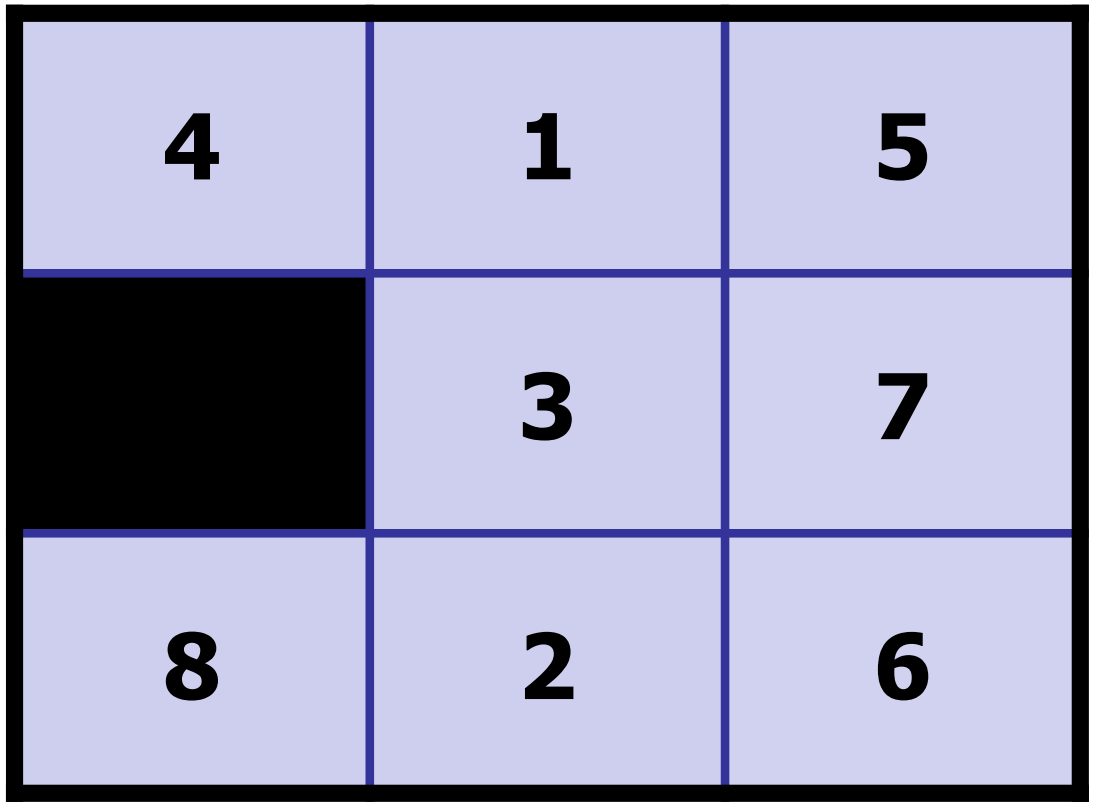
Sliding Puzzle



Sliding Puzzle



Sliding Puzzle



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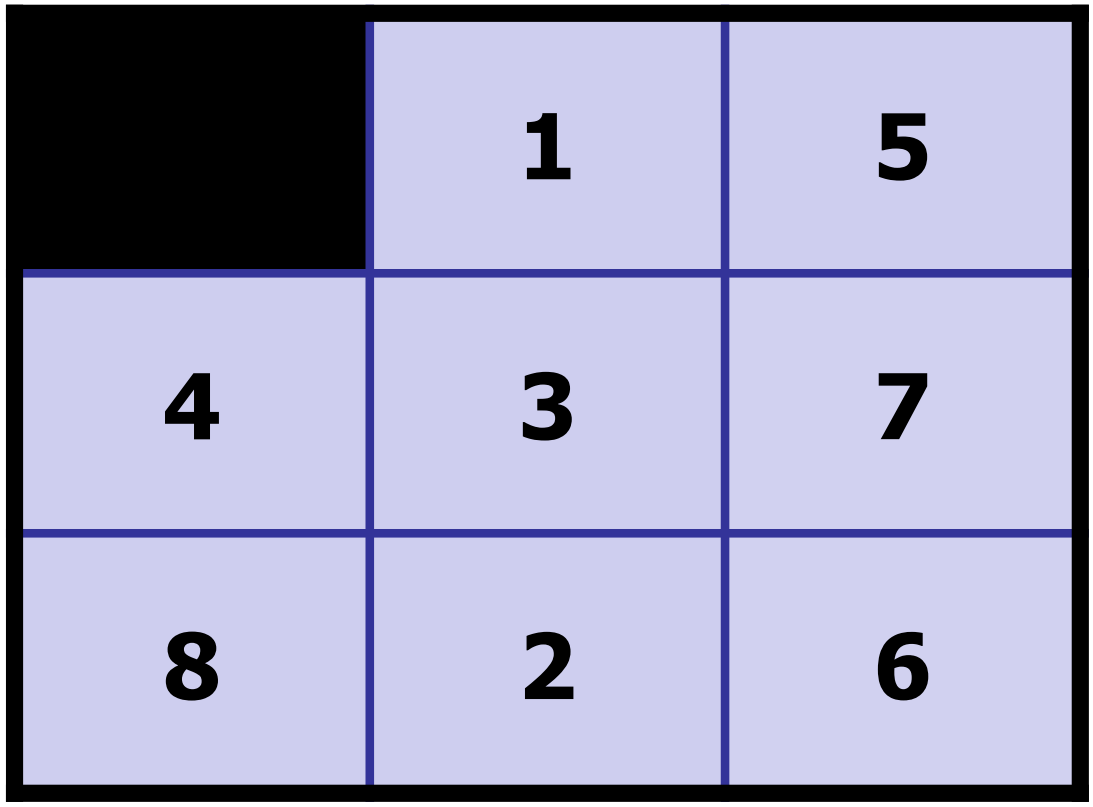
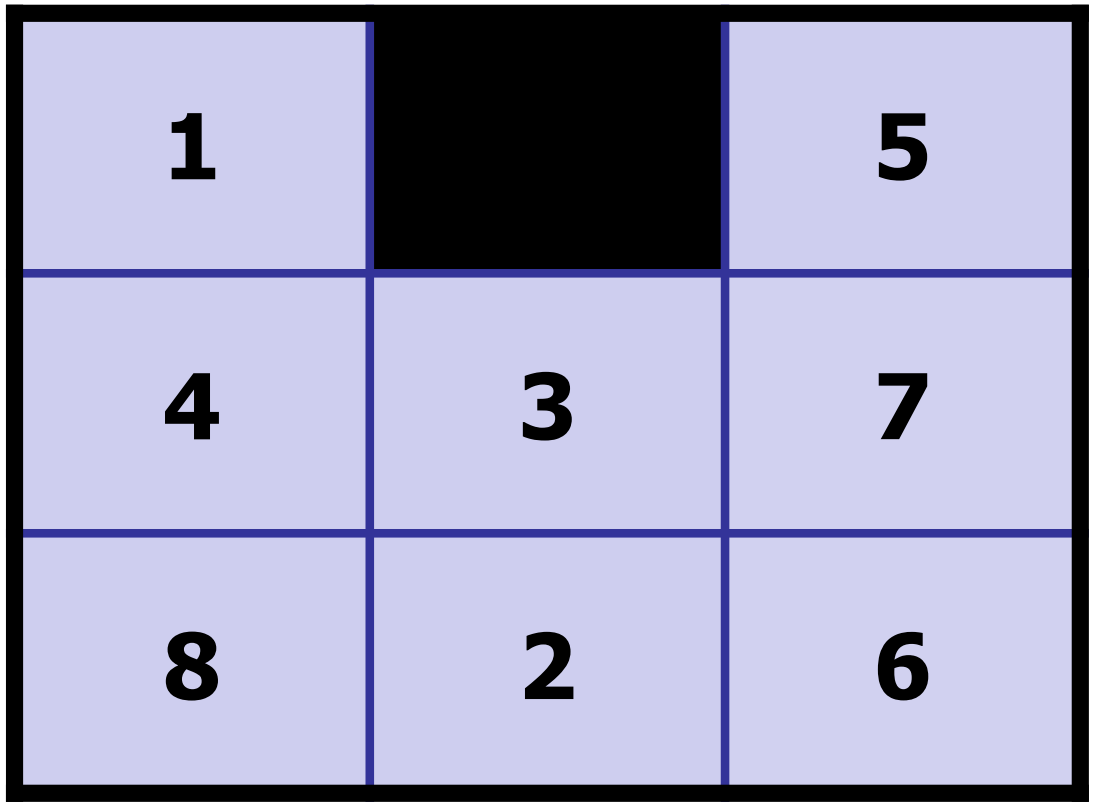
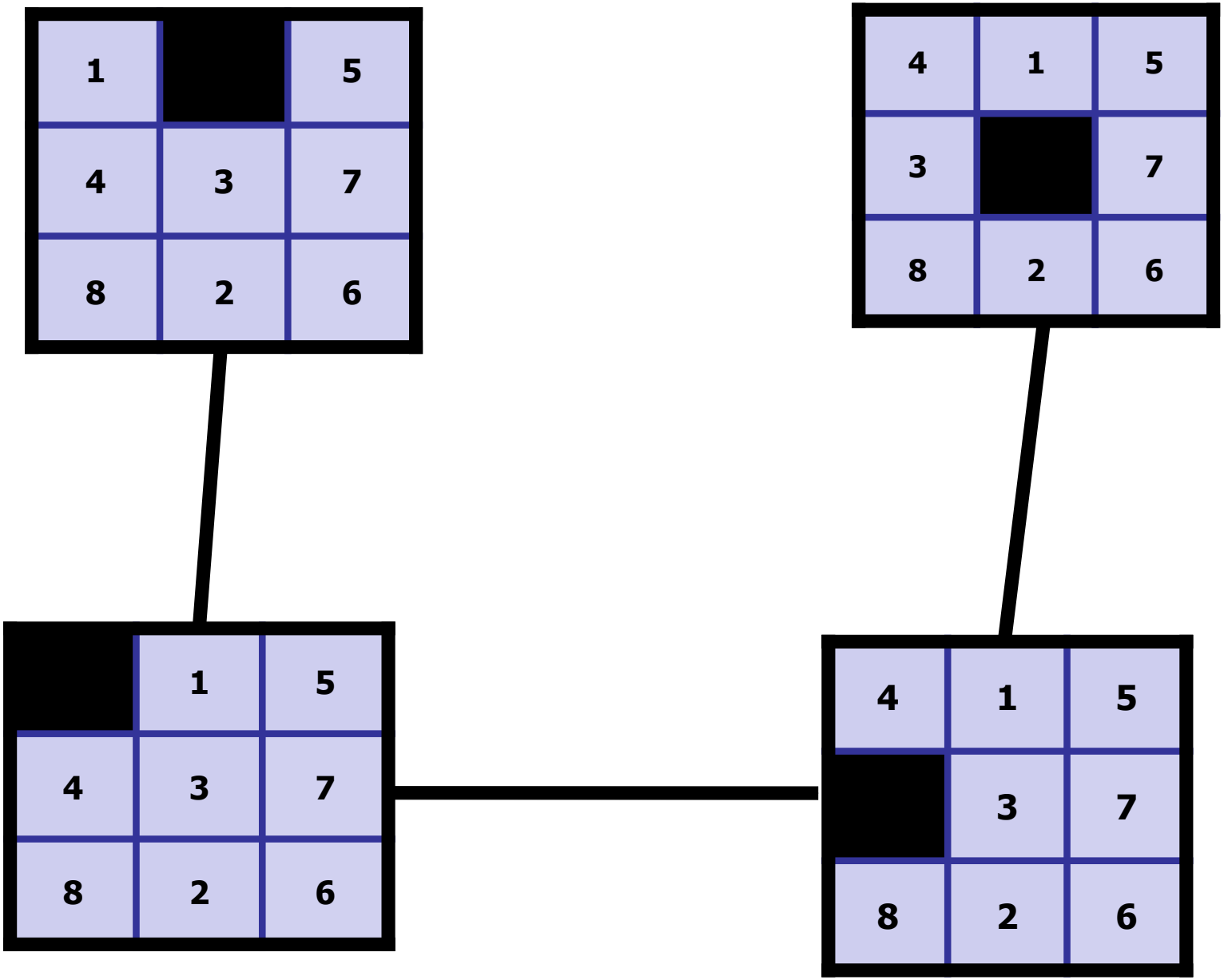


Sliding Puzzle



Sliding Puzzle is a Graph



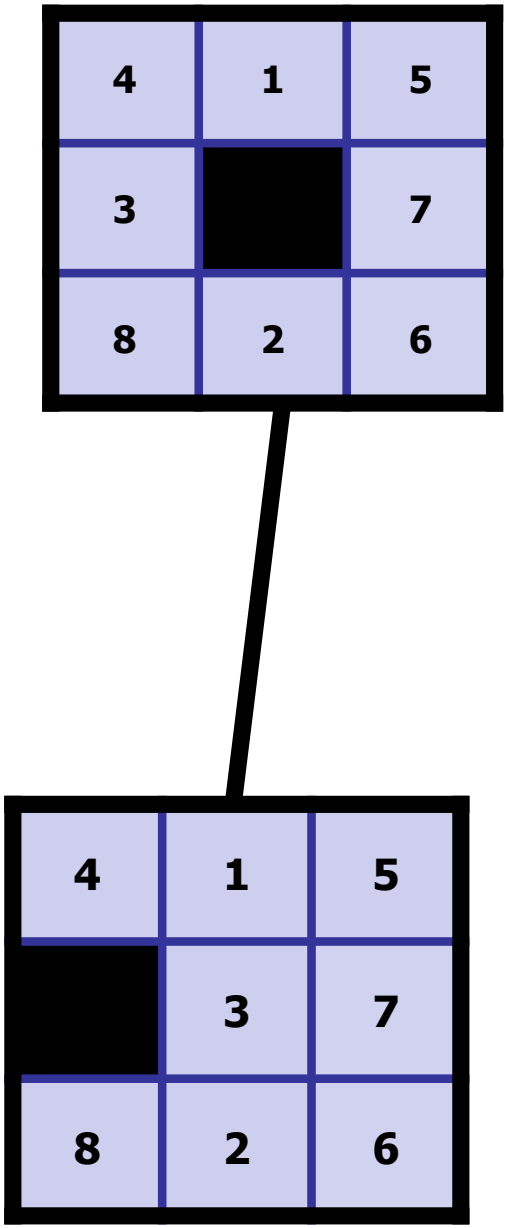
Sliding Puzzle

Nodes:

- State of the puzzle
- Permutation of nine tiles

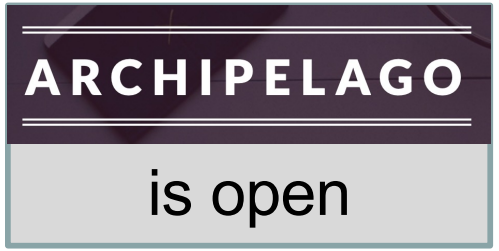
Edges:

- Two states are edges if they differ by only one move.



What is the maximum degree of the Sliding Puzzle graph?

1. 1
2. 2
3. 3
4. ✓ 4
5. n/2
6. n
7. n!



Sliding Puzzle

Nodes:

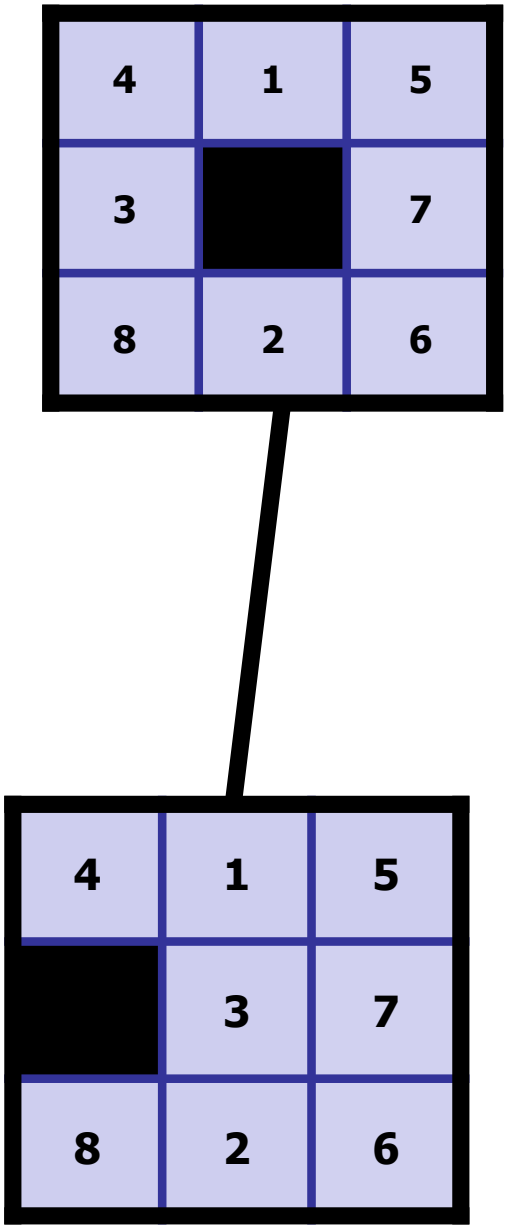
- State of the puzzle
- Permutation of nine tiles

Edges:

- Two states are edges if they differ by only one move.

Nodes = 9! = 362,880

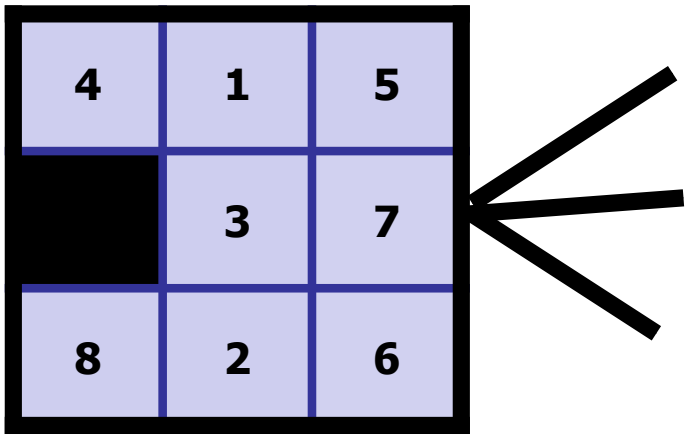
Edges < 4*9! < 1,451,520



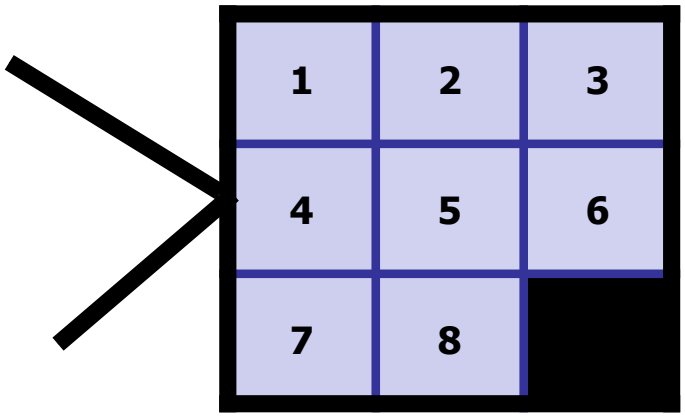
Sliding Puzzle

Number of moves to solve the puzzle?

Initial, scrambled state:



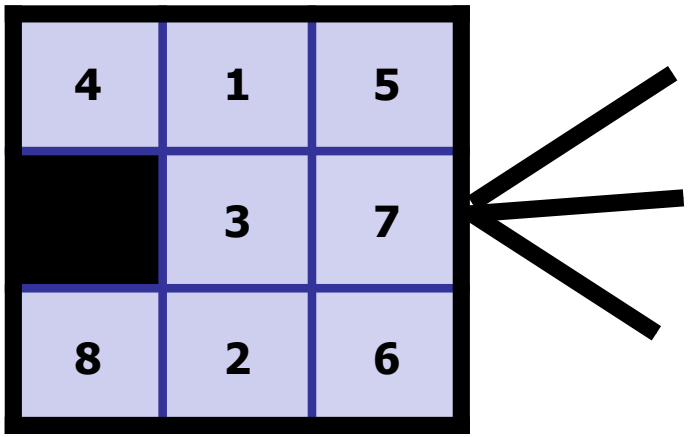
Final, unscrambled state:



Sliding Puzzle

Number of moves <= Diameter

Initial, scrambled state:



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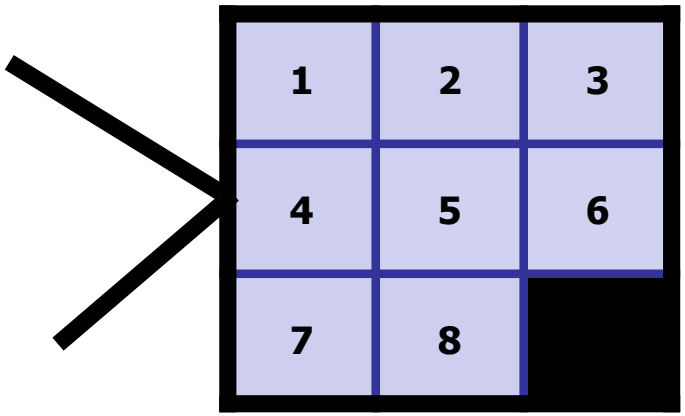


2 x 2 x 2 Rubik's Cube

2 x 2 x 2 Rubik's Cube

Record solve time: 0.69 seconds

2 x 2 x 2 Rubik's Cube

Configuration Graph

- Vertex for each possible state
- Edge for each basic move
  - 90 degree turn
  - 180 degree turn

Puzzle: given initial state, find a path to the solved state.

2 x 2 x 2 Rubik's Cube

How many vertices?

$$8! \cdot 3^8 = 264{,}539{,}520$$

cubelets

Each cubelet is
in one of 8 positions.

Each of the 8 cubelets
can be in one of three
orientations

2 x 2 x 2 Rubik's Cube

How many vertices?

$$7! \cdot 3^7 = 11{,}022{,}480$$

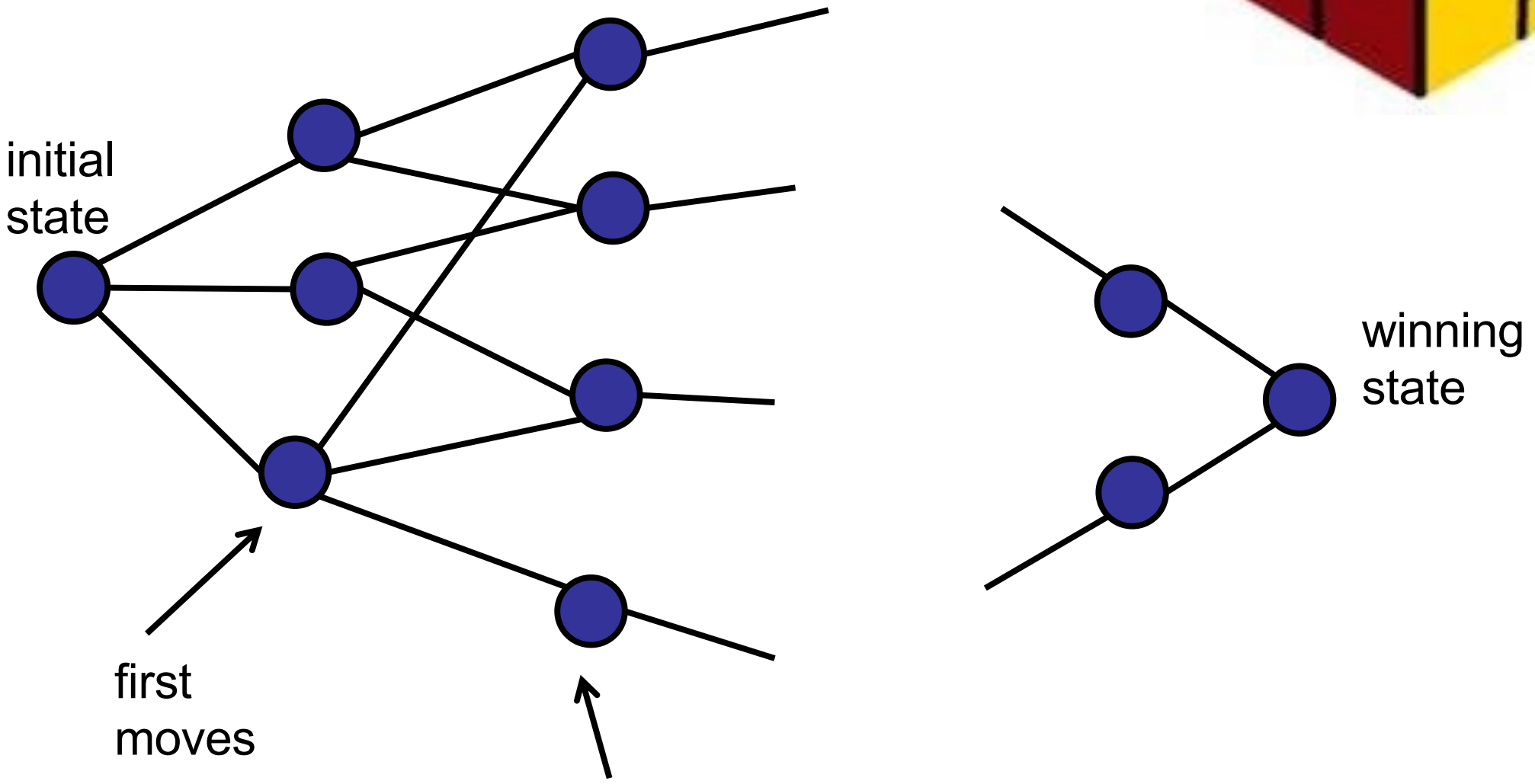
Symmetry:

Fix one cubelet.

Each of the 8 cubelets can be in one of three orientations

2 x 2 x 2 Rubik's Cube

Geography of Rubik's configurations:



Reachable configurations

Distance	90 deg. turns	90/180 deg. turns
0	1	1
1	6	9
2	27	54
3	120	321
4	534	1,847
5	2,256	9,992
6	8,969	50,136
7	33,058	227,536
8	114,149	870,072
9	360,508	1,887,748
0	930,588	623,800
11	1,350,852	2,644
12	782,536	
13	90,280	
14	276	

diameter

Reachable configurations

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Challenge:
How do you generate this table?

diameter

3 x 3 x 3 Rubik's Cube

Configuration Graph

- 43 quintillion vertices (approximately)
- Diameter: 20
  - 1995: require at least 20 moves.
  - 2008: 20 moves is enough from every position.
  - Using Google server farm.
  - 35 CPU-years of computation.
  - 20 seconds / set of 19.5 billion positions.
  - Lots of mathematical and programming tricks.

3 x 3 x 3 Rubik's Cube

What is the diameter of an (n x n x n) cube?

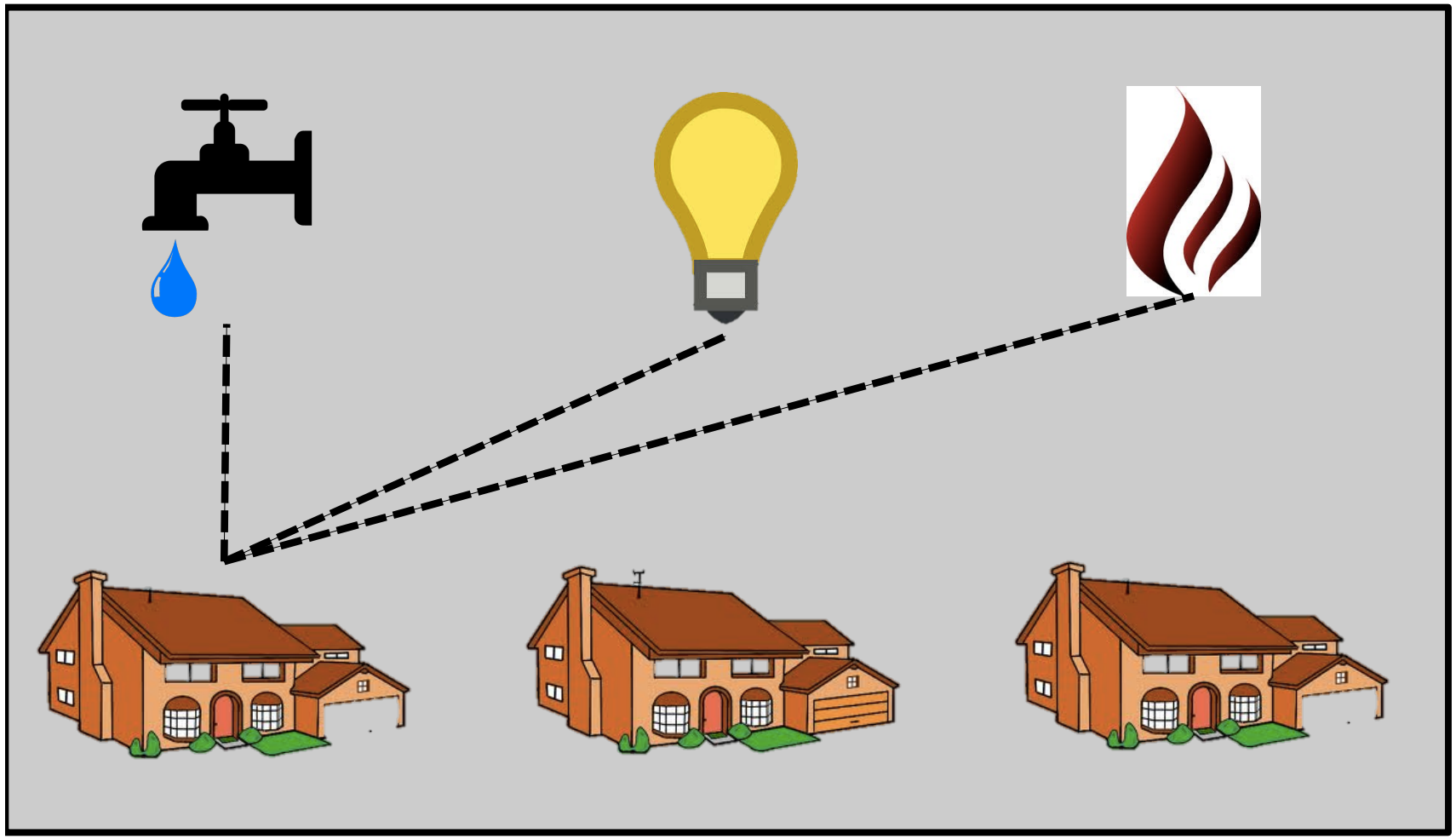
$\theta(n^2 / \log n)$

Roadmap

Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

Puzzle



Connect each house to all three utilities (water, electricity, gas).
Do not let any of the cables or pipes cross.
(Or show that it is impossible.)

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Today: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs (DFS / BFS)

Representing a Graph

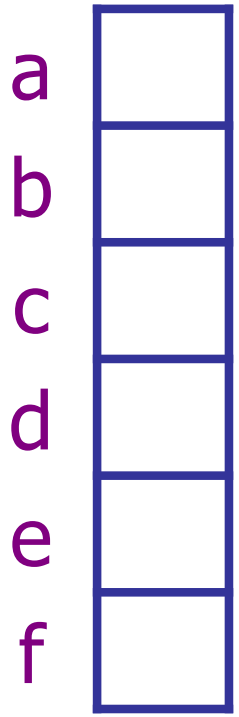
Graph consists of:

- Nodes
- Edges

Representing a Graph

Graph consists of:

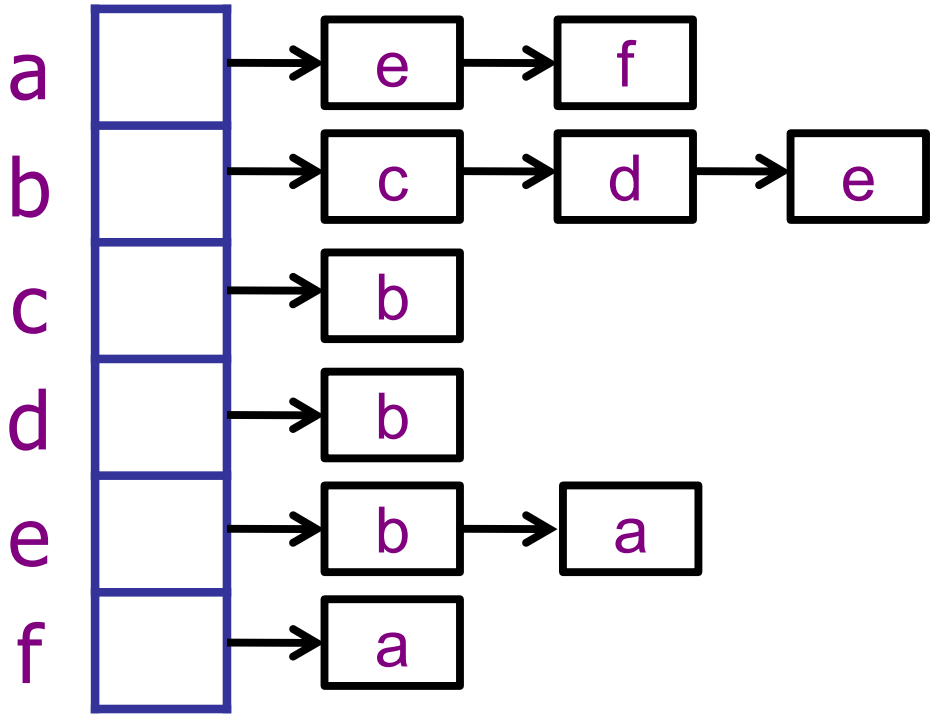
- Nodes: stored in an array
- Edges

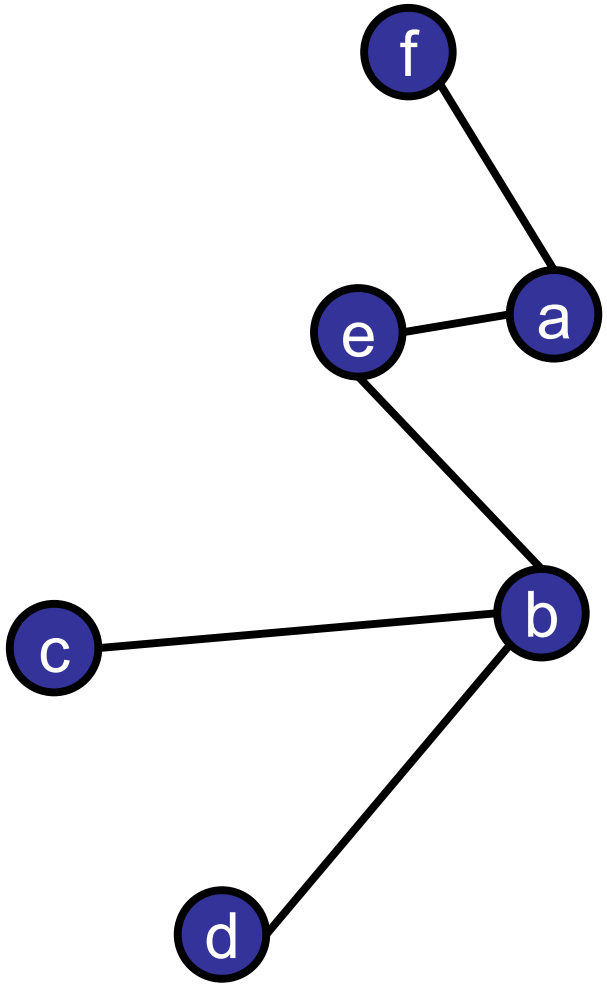


Adjacency List

Graph consists of:

- Nodes: stored in an array
- Edges: linked list per node





Adjacency List in Java

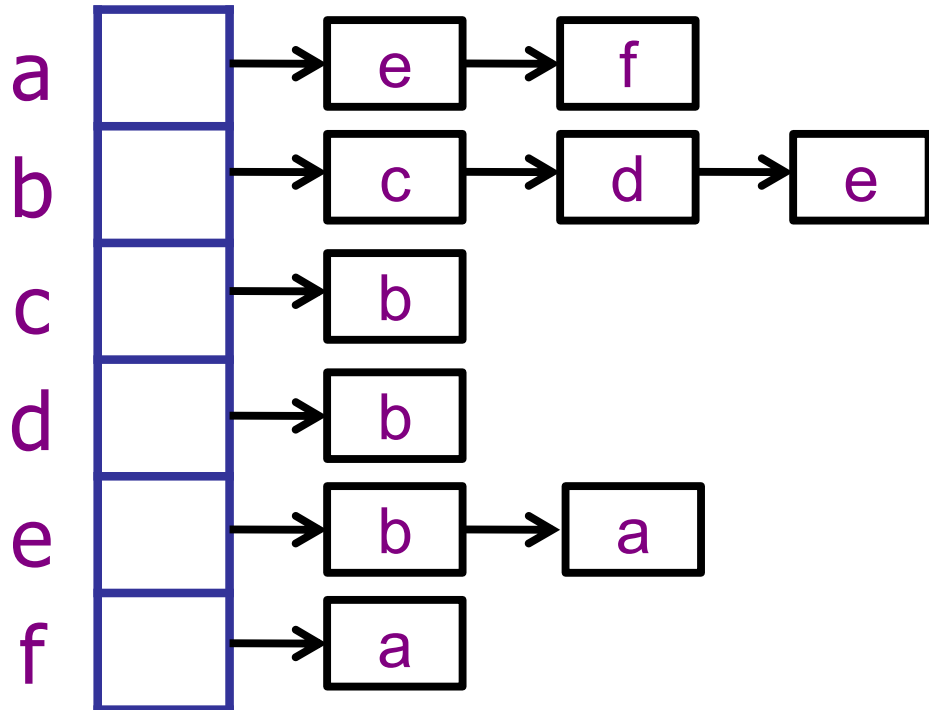
```
class NeighborList extends LinkedList<Integer> {
}


class Node {
  int key;
  NeighborList nbrs;
}

class Graph {
  Node[] nodeList;



}
```



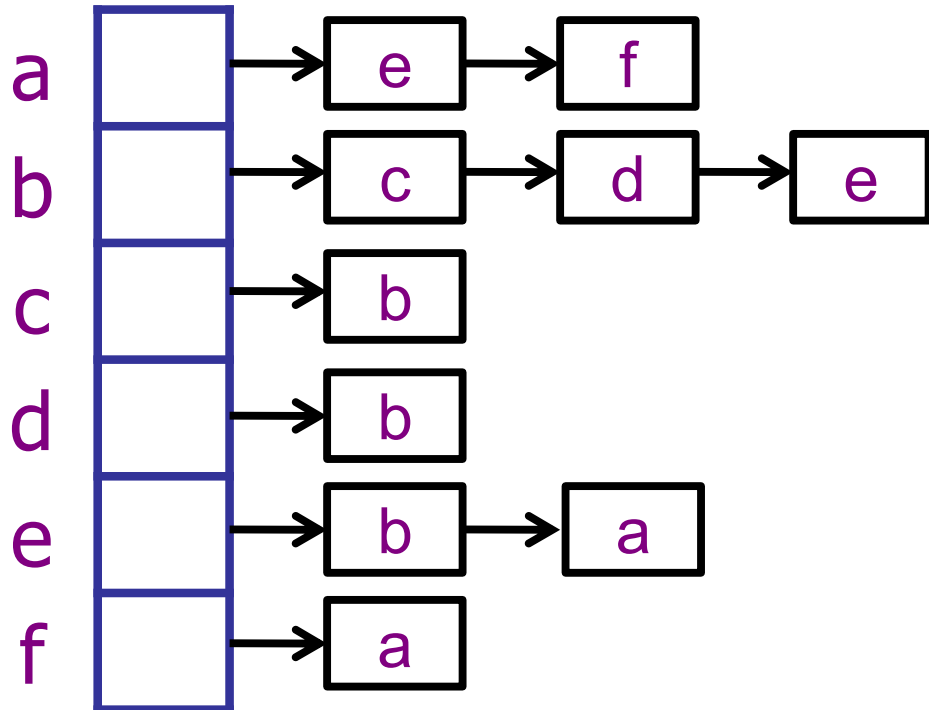
Adjacency List in Java

```
class Graph{
    List<List<Integer>> nodes;

}
```

More concise code is not *always* better…

- Harder to read
- Harder to debug
- Harder to extend



Representing a Graph

Graph consists of:

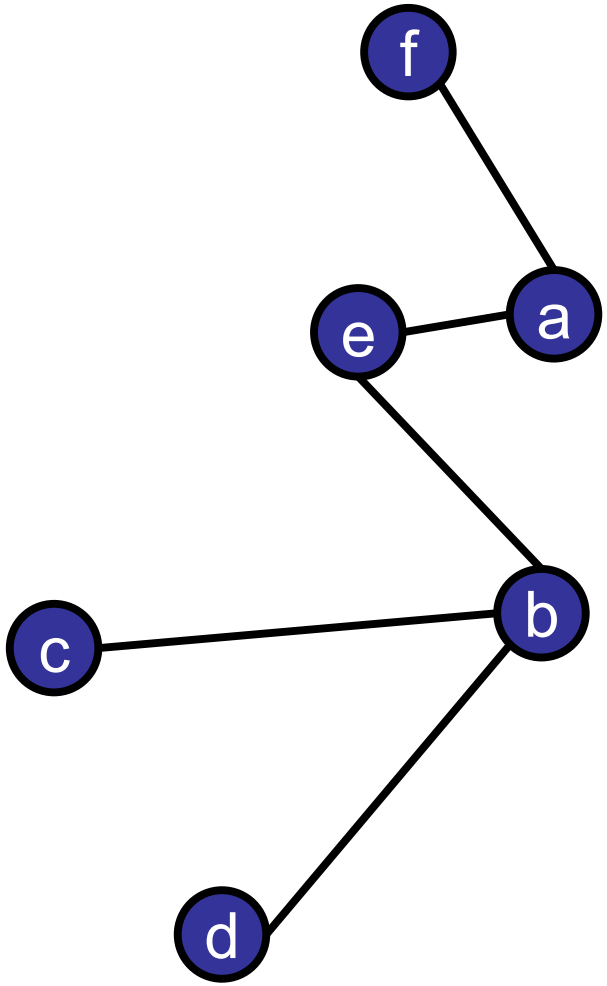
- Nodes
- Edges = pairs of nodes

Adjacency Matrix

Graph consists of:

- Nodes
- Edges = pairs of nodes

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0



Adjacency Matrix

Graph represented as:

A[v][w] = 1 iff (v,w) $\in$ E

Neat property:

- A^2 = length 2 paths

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

Adjacency Matrix

To find out if c and d are 2-hop neighbors:

- Let $B = A^2$
- $B[c, d] = A[c, .] \bullet A[., d]$

- $B[c, d] >= 1$ iff

  $A[c, x] == A[x, d]$

  for some x.

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
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Adjacency Matrix

Graph represented as:

A[v][w] >= 1 iff (v,w) $\in$ E

Neat properties:

- A^2 = length 2 paths
- A^4 = length 4 paths

Neat way to figure out connectivity…
Neat way to figure out diameter…
Not always the most efficient…
Parallelizes well….

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

Adjacency Matrix

Graph represented as:

$A[v][w] = 1$ iff $(v,w) \in E$

Neat properties:

- A^2 = length 2 paths
- A^4 = length 4 paths
- $A^\infty \approx$ Google pagerank?

(Simulate random walk by replacing '1' with probability.)

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

Adjacency Matrix in Java

Graph represented as:

A[v][w] = 1 iff (v,w) $\in$ E

```
class Graph {

  boolean[][] adjMatrix;


}
```

	a	b	c	d	
a	0	0	0	0	
b	0	0	**1**	**1**	
c	0	**1**	0	0	
d	0	**1**	0	0	
e	**1**	**1**	0	0	
f	**1**	0	0	0	

Adjacency Matrix in Java

Graph represented as:

A[v][w] = 1 iff (v,w) $\in$ E

```
class Graph {

  Node[][] adjMatrix;


}
```

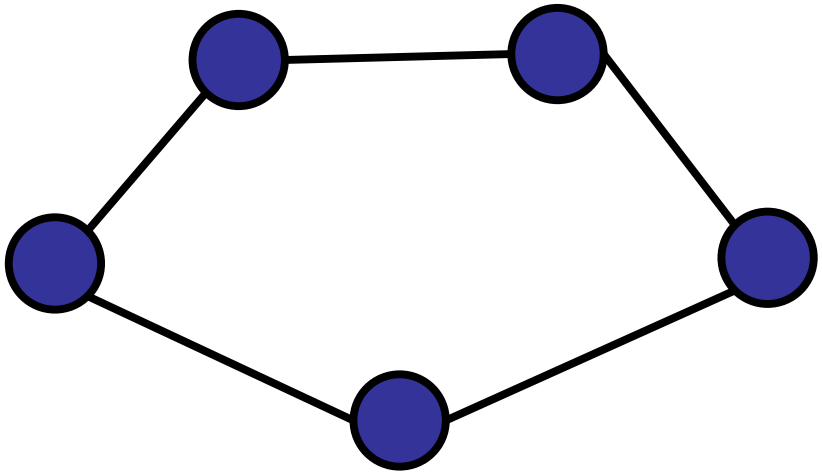
	a	b	c	d	
a	0	0	0	0	
b	0	0	**1**	**1**	
c	0	**1**	0	0	
d	0	**1**	0	0	
e	**1**	**1**	0	0	
f	**1**	0	0	0	

Trade-offs

Adjacency Matrix vs. Array?

For a cycle, which representation is better?

✓1. Adjacency list
2. Adjacency matrix
3. Equivalent



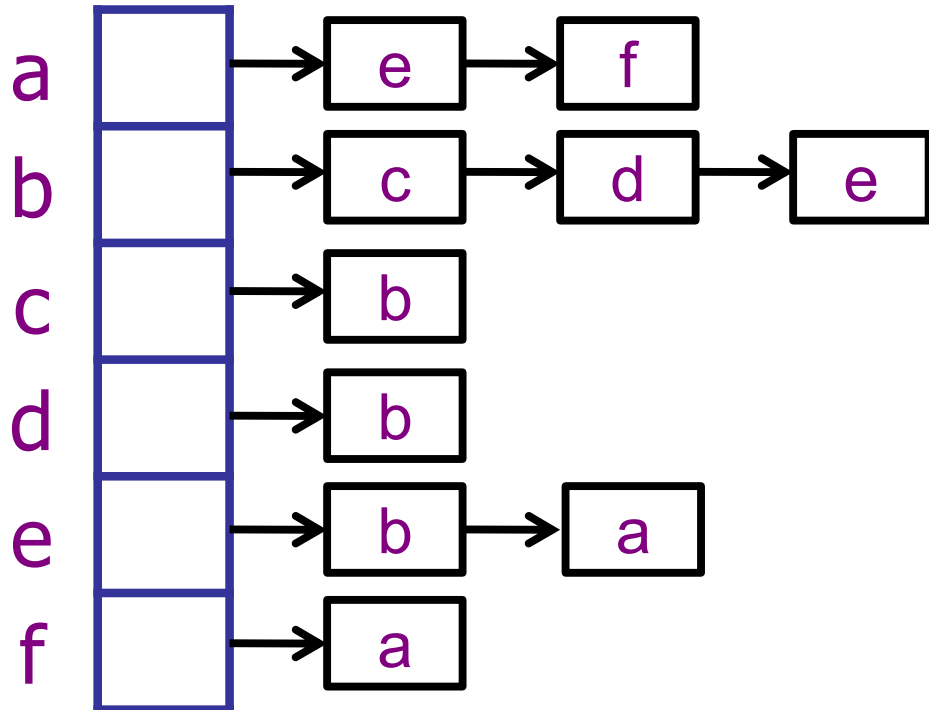
Adjacency List

Memory usage for graph G = (V, E):

- array of size |V|
- linked lists of size |E|

Total: O(V + E)

For a cycle: O(V)



Adjacency Matrix

Memory usage for graph G = (V, E):

- array of size |V|*|V|

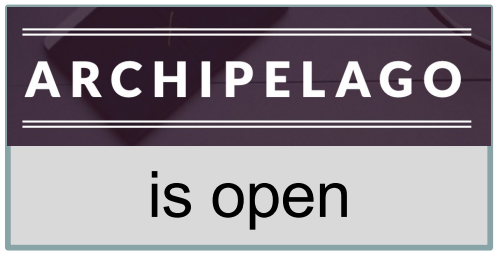
Total: $O(V^2)$

For a cycle: $O(V^2)$

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

For a clique, which representation is better?

1. Adjacency matrix
2. Adjacency list
3. Equivalent



Adjacency List vs. Matrix

Memory usage for graph $G = (V, E)$:

- Adjacency List: $O(V + E)$
- Adjacency Matrix: $O(V^2)$

For a cycle: $O(V)$ vs. $O(V^2)$

For a clique: $O(V + E) = O(V^2)$ vs. $O(V^2)$

Adjacency List vs. Matrix

Memory usage for graph G = (V, E):

- Adjacency List: $O(V + E)$
- Adjacency Matrix: $O(V^2)$

For a cycle: $O(V)$ vs. $O(V^2)$

For a clique: $O(V + E) = O(V^2)$ vs. $O(V^2)$

Base rule: if graph is dense then use an adjacency matrix; else use an adjacency list.

dense: $|E| = \theta(V^2)$

Which representation for Facebook Graph?
Query: Are Bob and Joe friends?

1. Adjacency List
2. ✓ Adjacency Matrix
3. Equivalent

List: (much) better space.
Matrix: somewhat faster

Which representation for Facebook Graph?
Query: List all my friends?

✓1. Adjacency List
2. Adjacency Matrix
3. Equivalent

Trade-offs

Adjacency Matrix:

- Fast query: are v and w neighbors?
- Slow query: find me any neighbor of v.
- Slow query: enumerate all neighbors.

Adjacency List:

- Fast query: find me any neighbor.
- Fast query: enumerate all neighbors.
- Slower query: are v and w neighbors?

Graph Representations

Key questions to ask:

- Space usage: is graph dense or sparse?
- Queries: what type of queries do I need?
  - Enumerate neighbors?
  - Query relationship?

Roadmap

Today: Graph Basics

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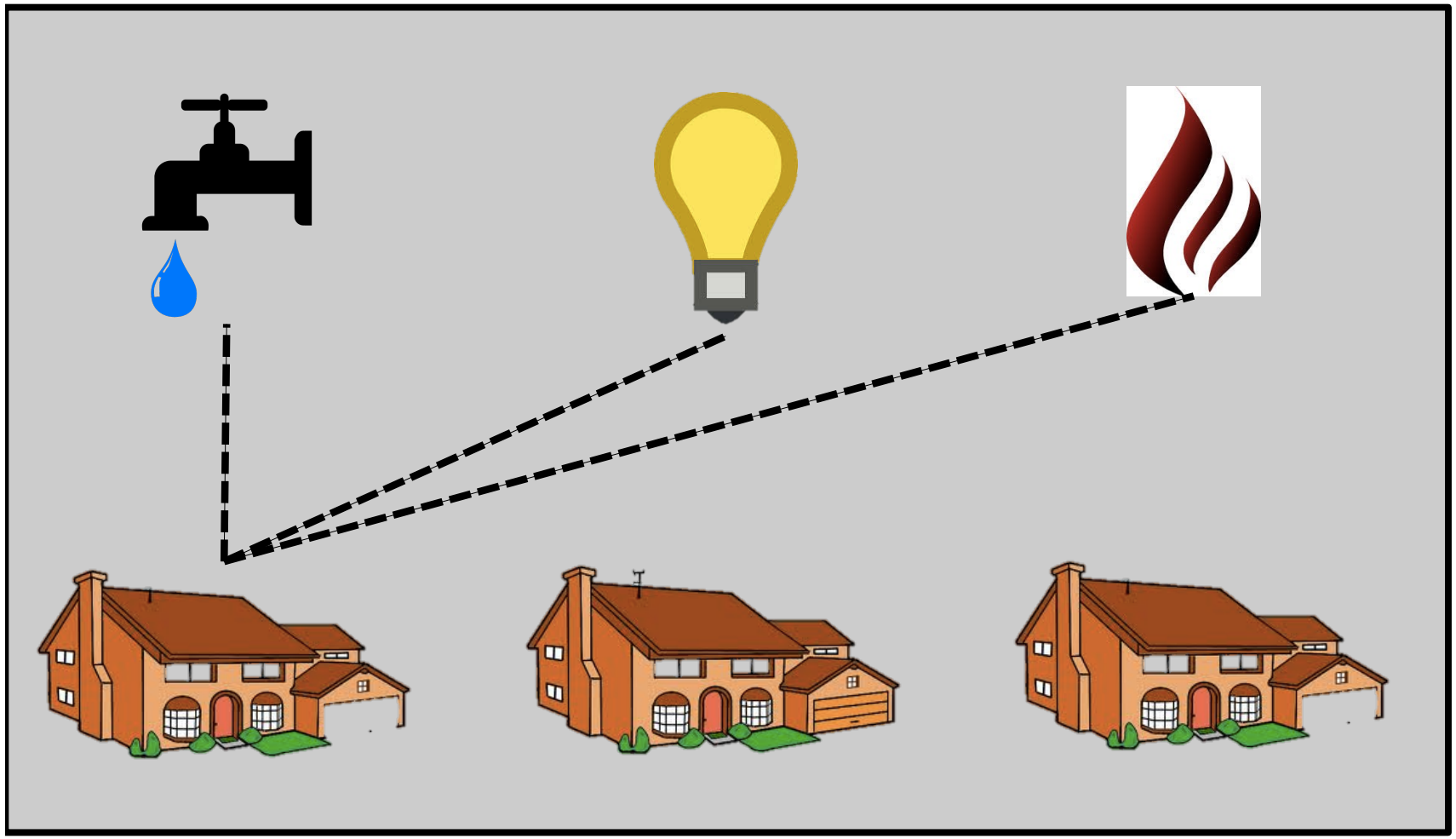
- Searching graphs (DFS / BFS)

Puzzle



Connect each house to all three utilities (water, electricity, gas).
Do not let any of the cables or pipes cross.
(Or show that it is impossible.)

Roadmap

Today: Graph Basics

- What is a graph?
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Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

Graph representation:

- Adjacency list

BFS & DFS

Simple, but very, very important:

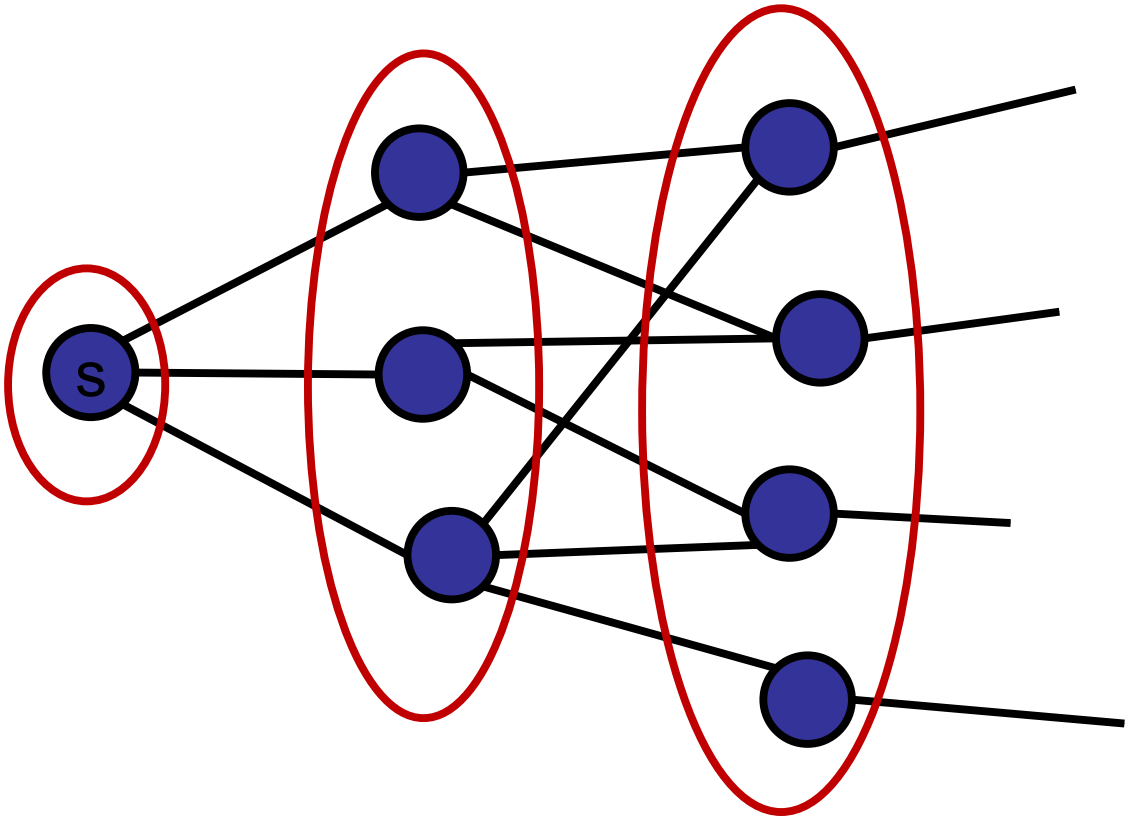
Recent conversation with CS2109S instructor (Introduction to AI and Machine Learning):

"It is really, really important that every student who take CS2109S can easily code up a simple BFS or DFS. Critical foundation before we can do anything more interesting."

Searching a graph

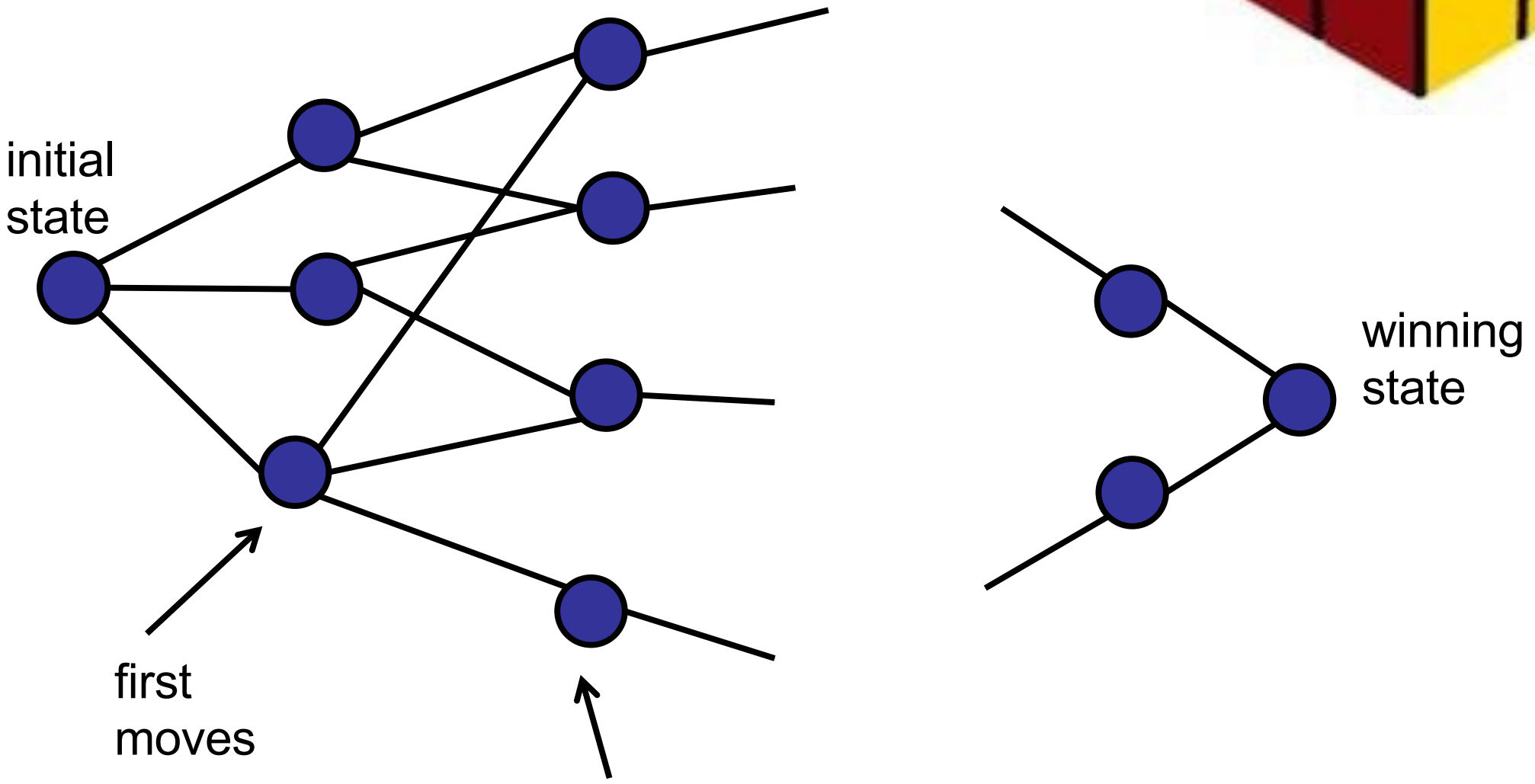
Breadth-First Search:

- Explore level by level



2 x 2 x 2 Rubik's Cube

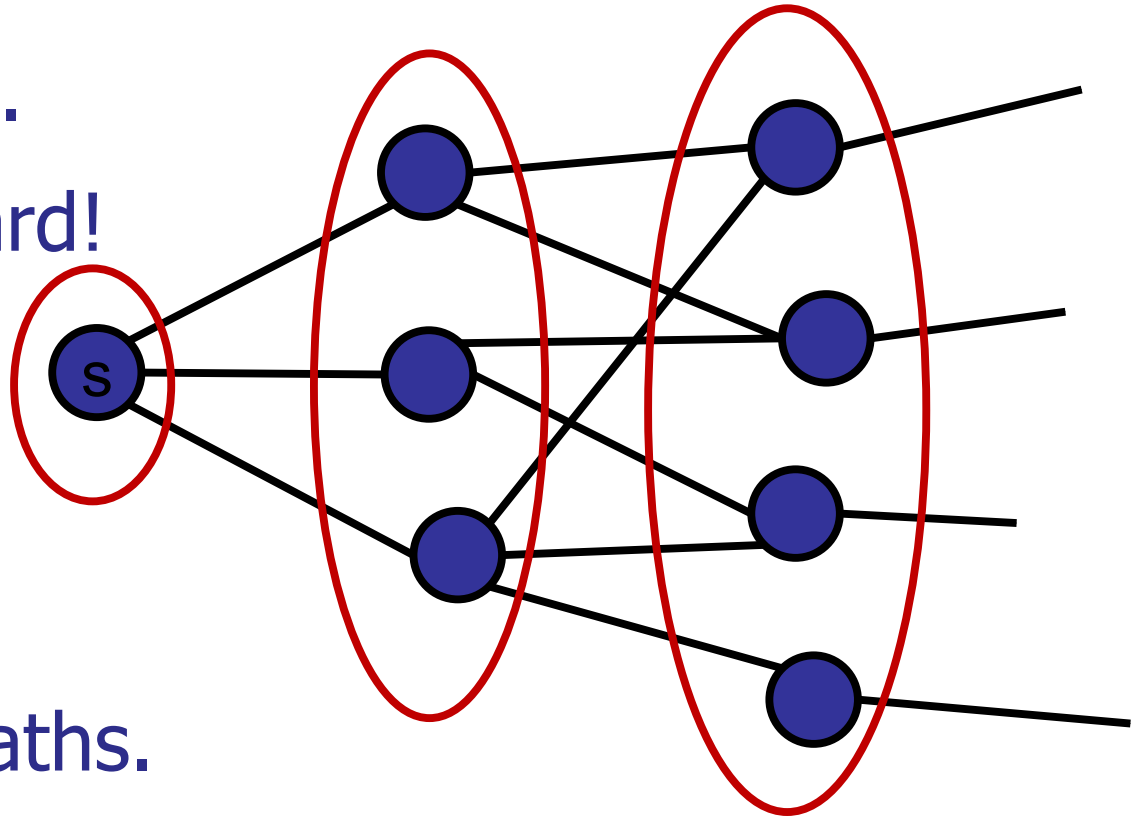
Geography of Rubik's configurations:



Searching a graph

Breadth-First Search:

- Explore level by level
- Frontier: current level
- Initially: {s}
- Advance frontier.
- Don't go backward!

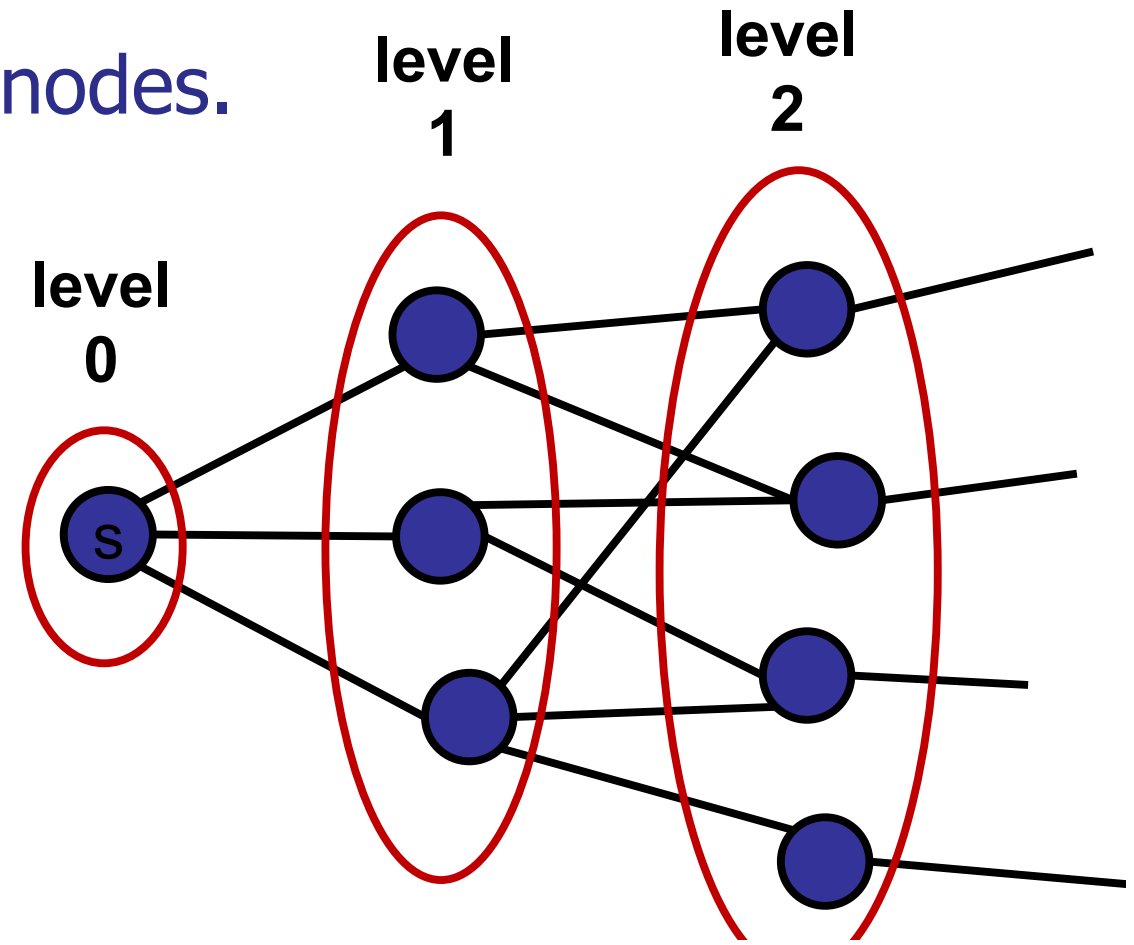


- Finds shortest paths.

Searching a graph

Breadth-First Search:

- Build levels.
- Calculate level[i] from level[i-1]
- Skip already visited nodes.



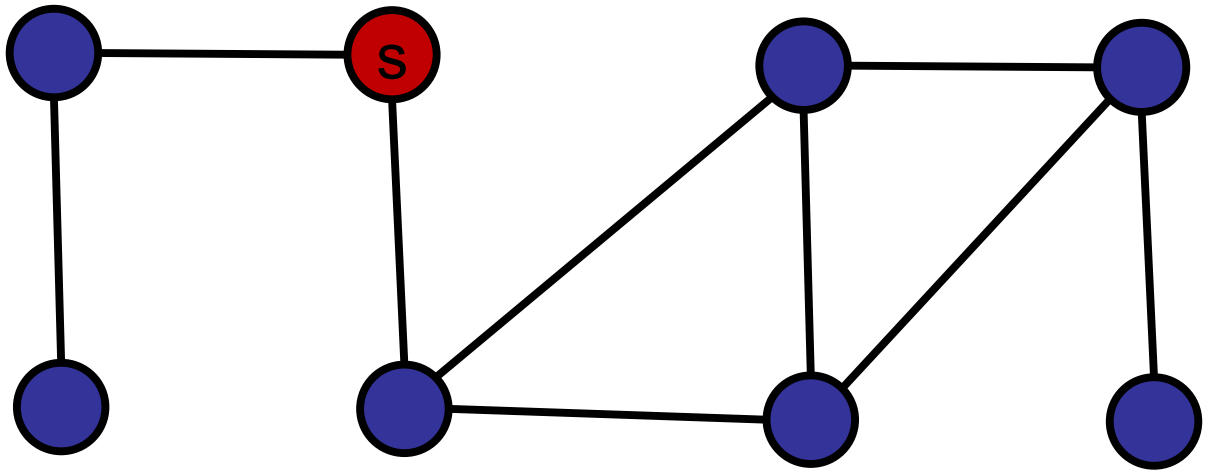
Breadth-First Search

```
BFS(Node[] nodeList, int startId) {
  boolean[] visited = new boolean[nodeList.length];
  Arrays.fill(visited, false);

  int[] parent = new int[nodelist.length];
  Arrays.fill(parent, -1);

  Collection<Integer> frontier = new Collection<Integer>;
  frontier.add(startId);

  // Main code goes here!

}
```

Breadth-First Search

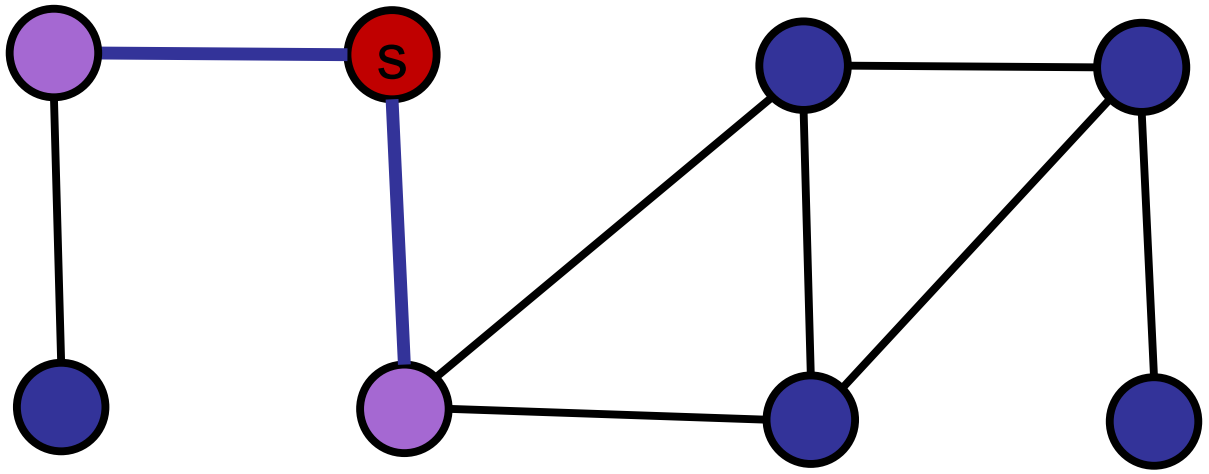
```
while (!frontier.isEmpty()){
    Collection<Integer> nextFrontier = new … ;
    for (Integer v : frontier) {
          for (Integer w : nodeList[v].nbrList) {
                if (!visited[w]) {
                      visited[w] = true;
                      parent[w] = v;
                      nextFrontier.add(w);
                }
          }
    }
    frontier = nextFrontier;
}
```

Breadth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Breadth-First Search Example



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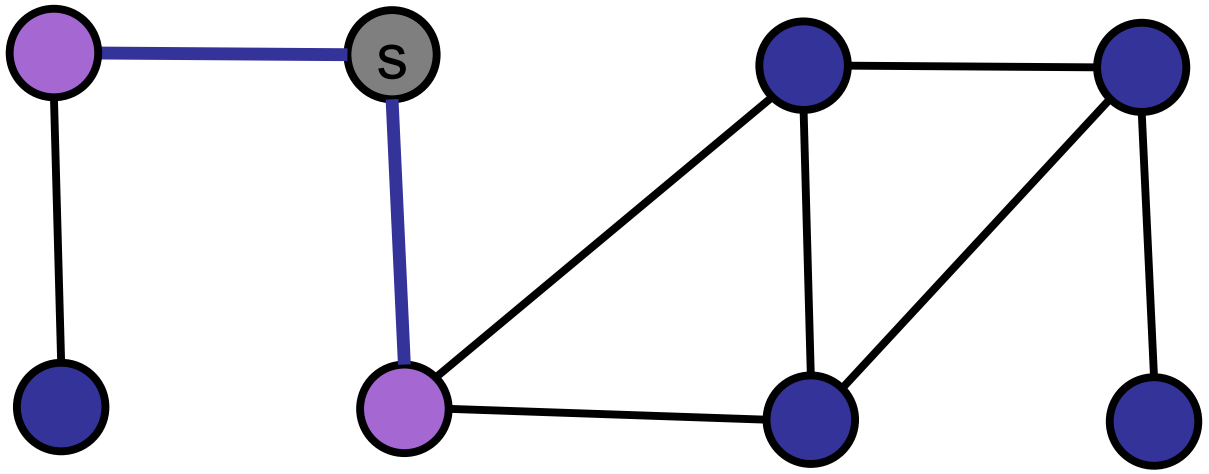

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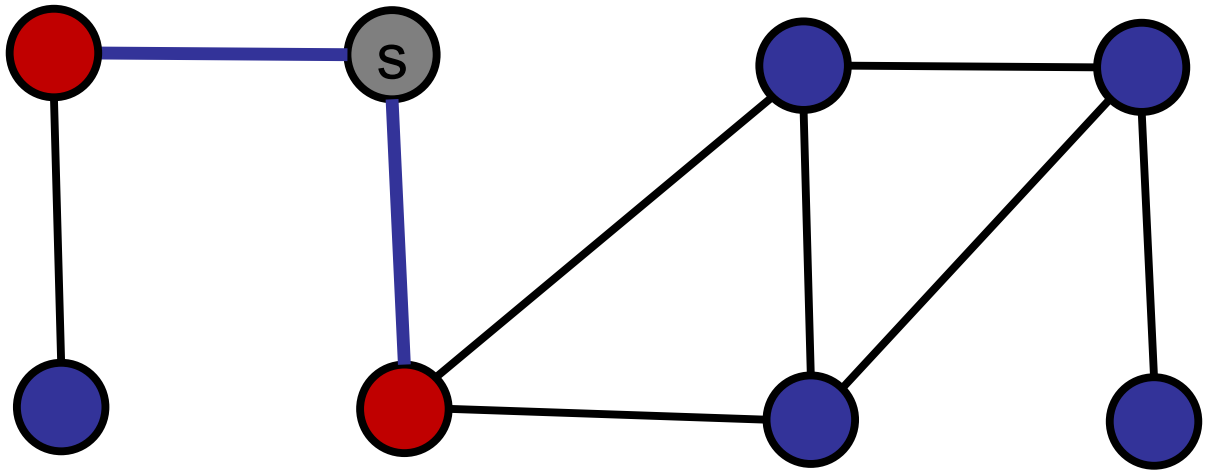

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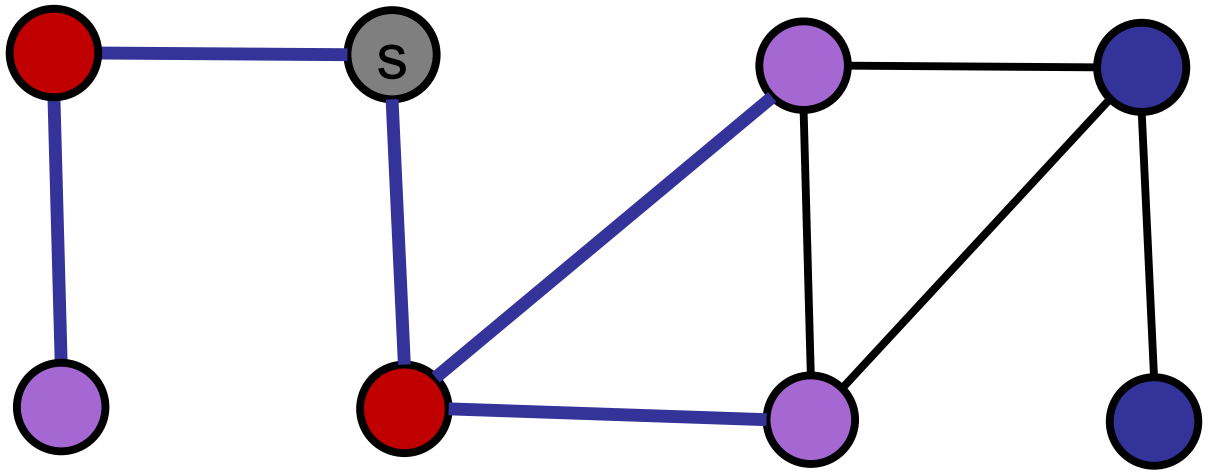

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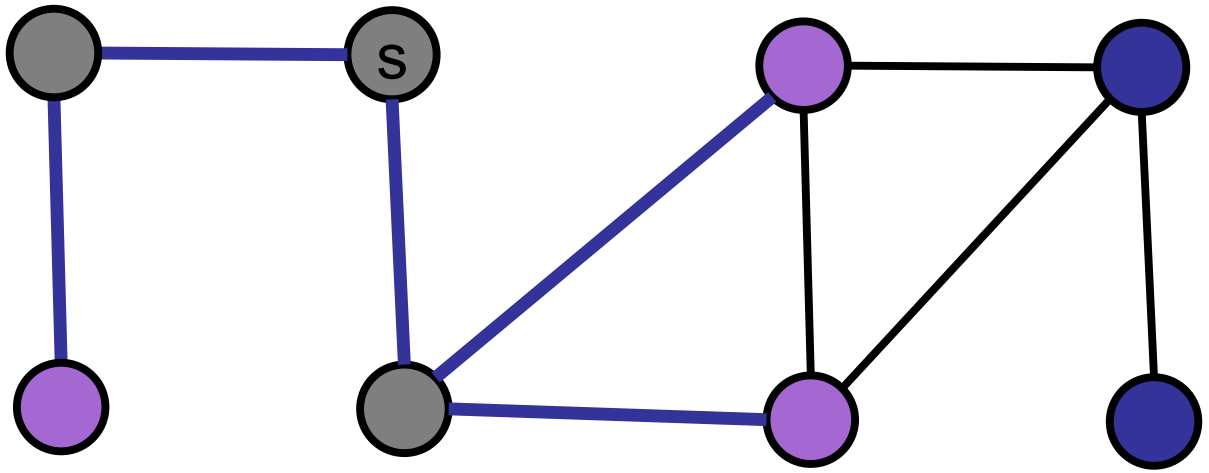

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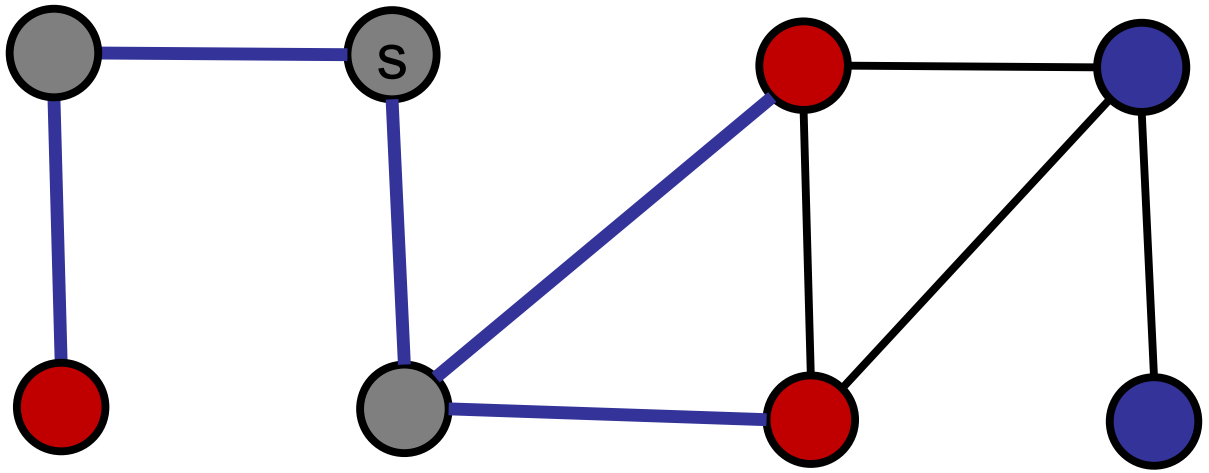

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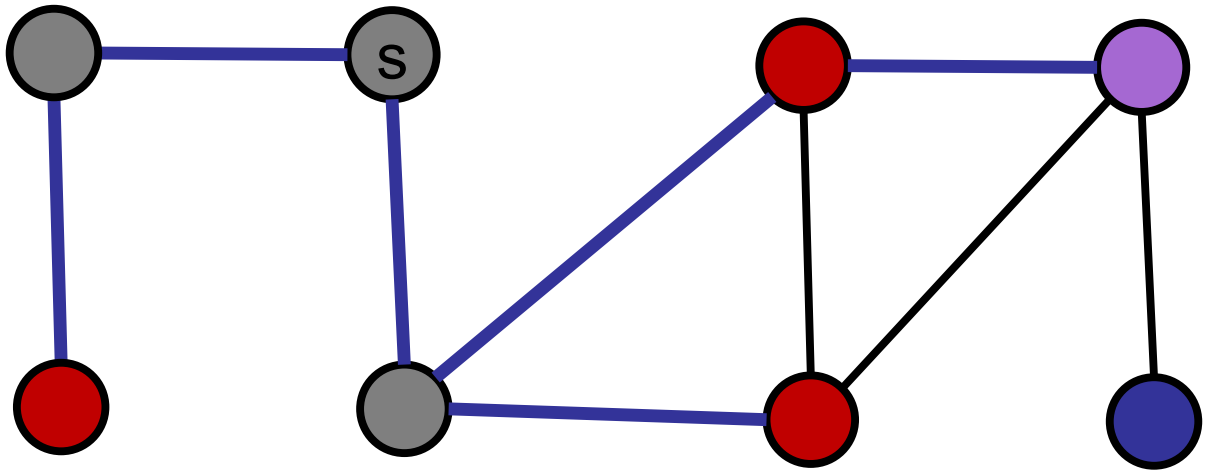

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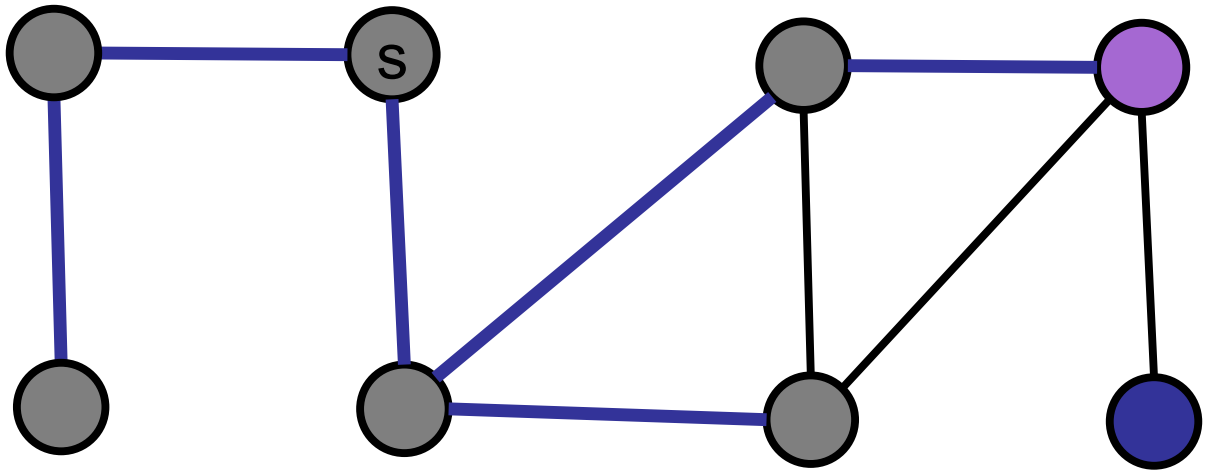

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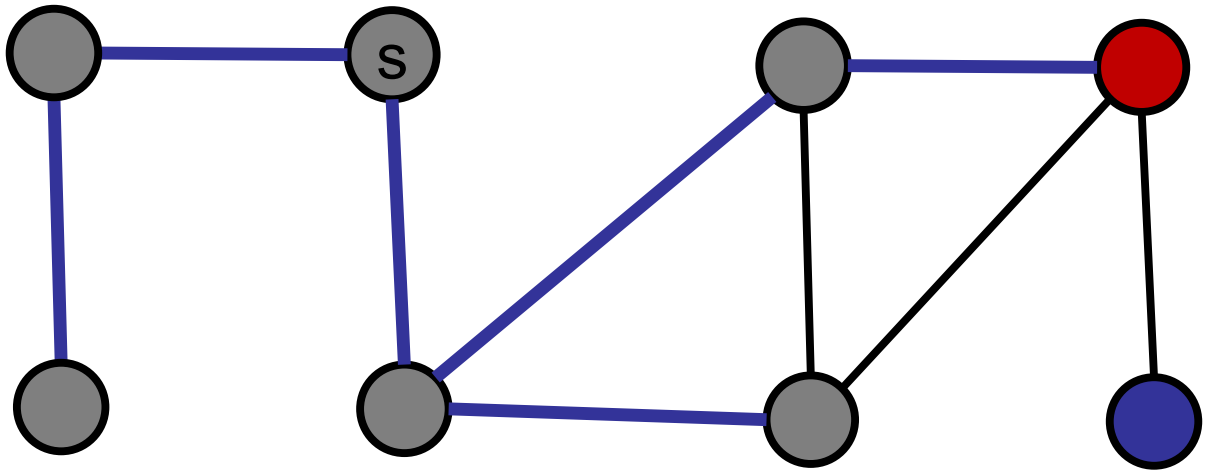

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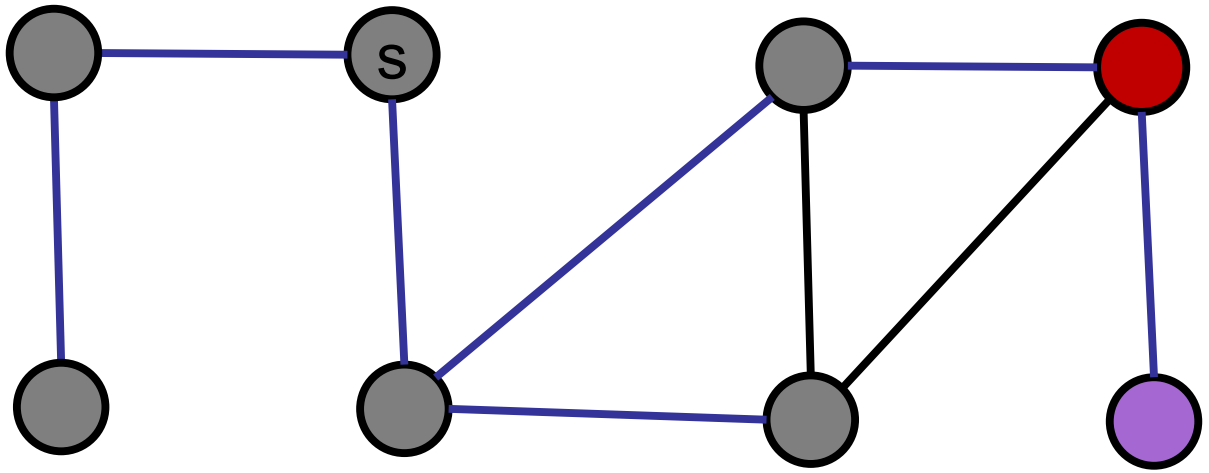

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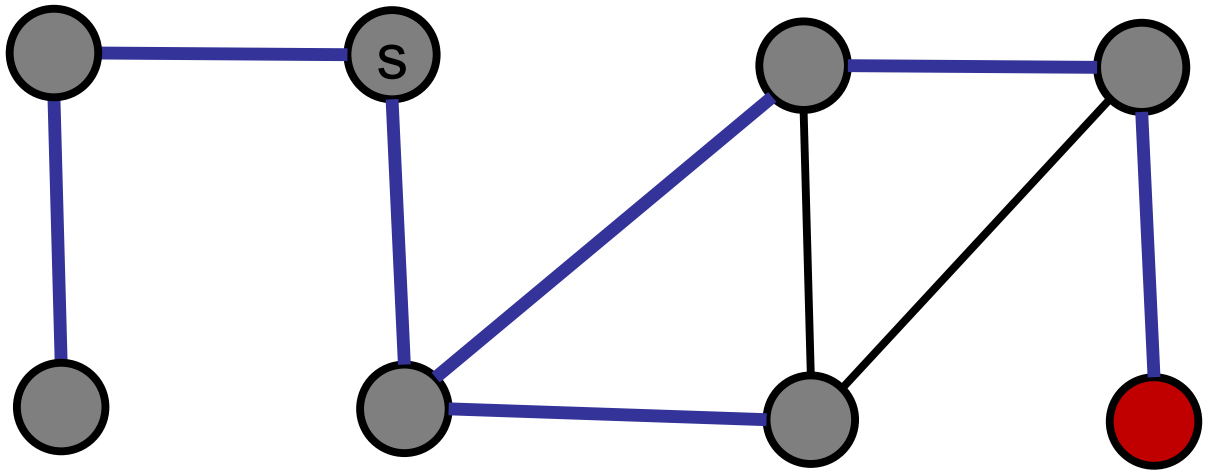

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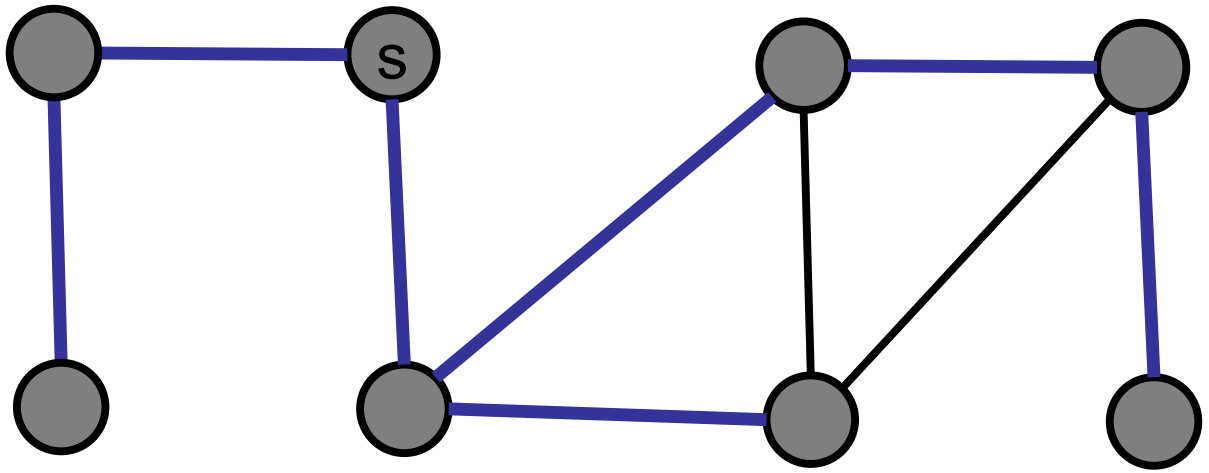

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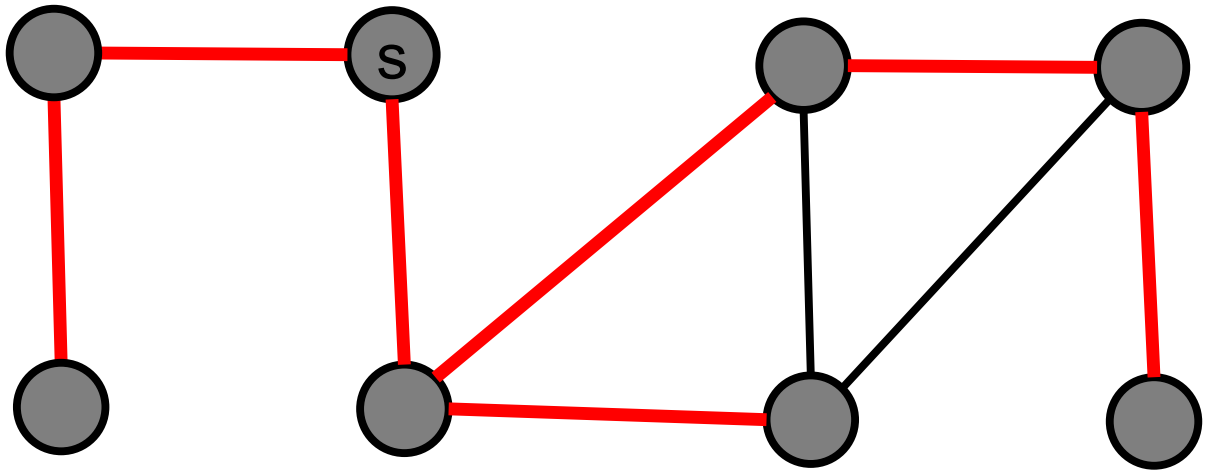

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