CS2040S
Data Structures and Algorithms

All about minimum spanning trees…

Riddle of the Week: The Travelling SalesPeople

Three travelers show up at a hotel where a room costs $300. They each pay $100 and go to their room.

The manager realizes there is a special sale and the room only costs $250. He gives his assistant $50 to return to the travelers. The assistant only has tens for change, and so gives each traveler $10 in change, keeping $20 for himself.

So each traveler paid $90, and the assistant kept $20, leading to a total of 3*90+20 = 290 dollars. What happened to the remaining 10 dollars?

Roadmap

Minimum Spanning Trees

- Background
- Prim’s Algorithm
- Kruskal’s Algorithm
- (Boruvka’s Algorithm)

Variations:

- Constant weight edges
- Bounded integer edge weights
- Directed graphs
- Maximum Spanning Tree

Roadmap

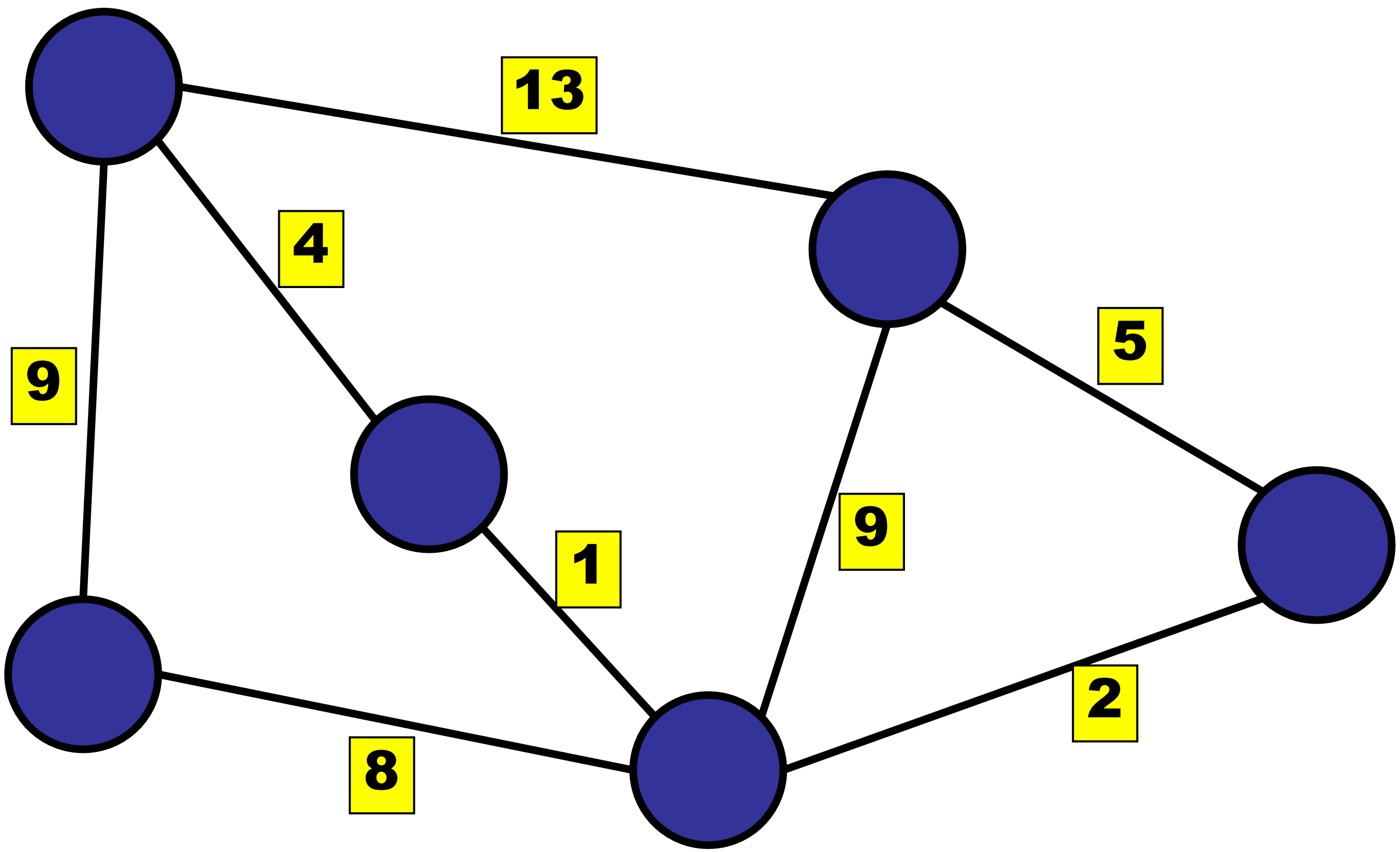
Minimum Spanning Trees

- **The MST Problem**
- Basic Properties of an MST
- Generic MST Algorithm
- Prim’s Algorithm
- Kruskal’s Algorithm
- Variations
- (Boruvka’s Algorithm)

Spanning Tree

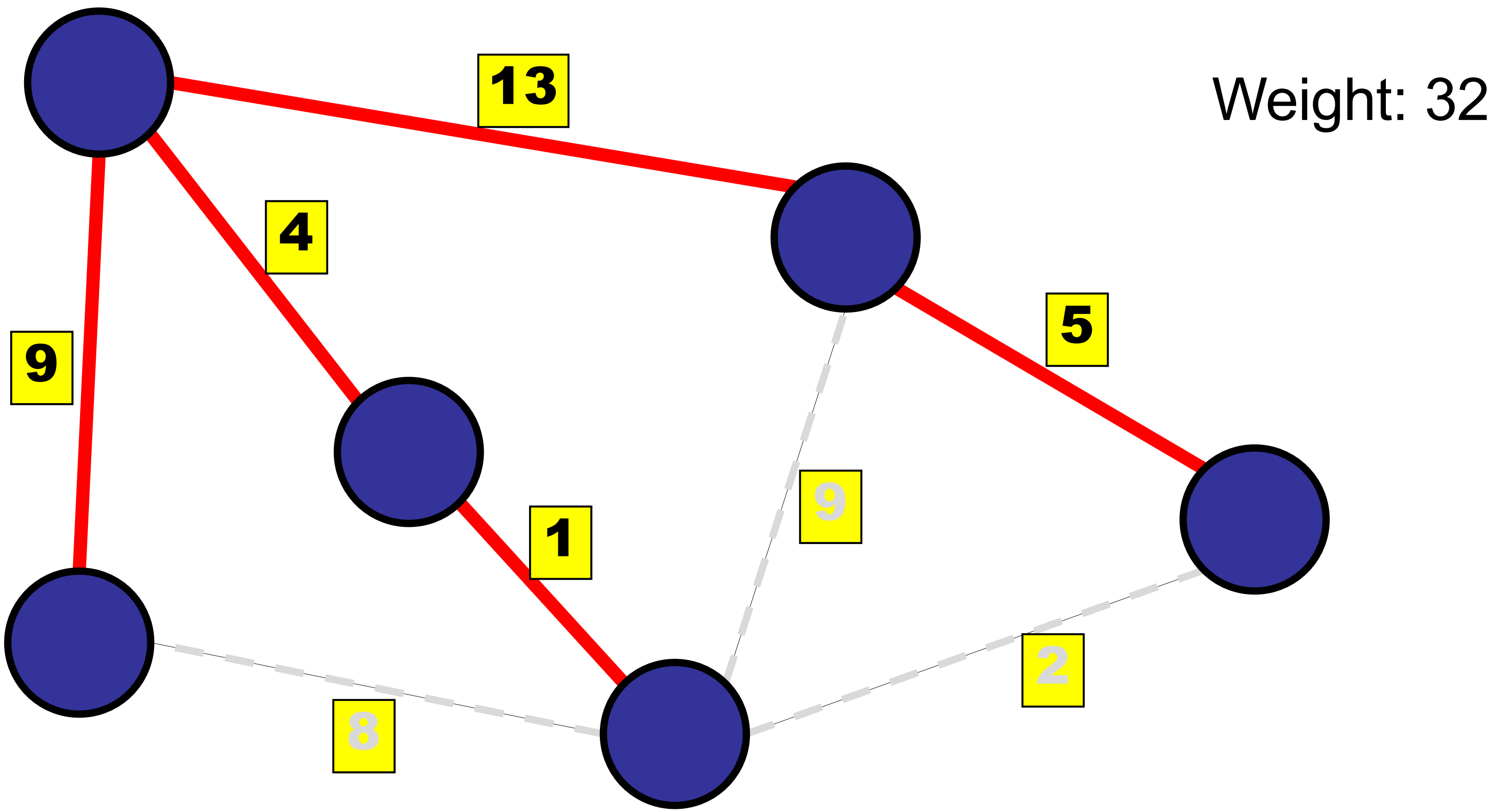
To think about:
Why is this more complicated with directed graphs?

Weighted, undirected graph:



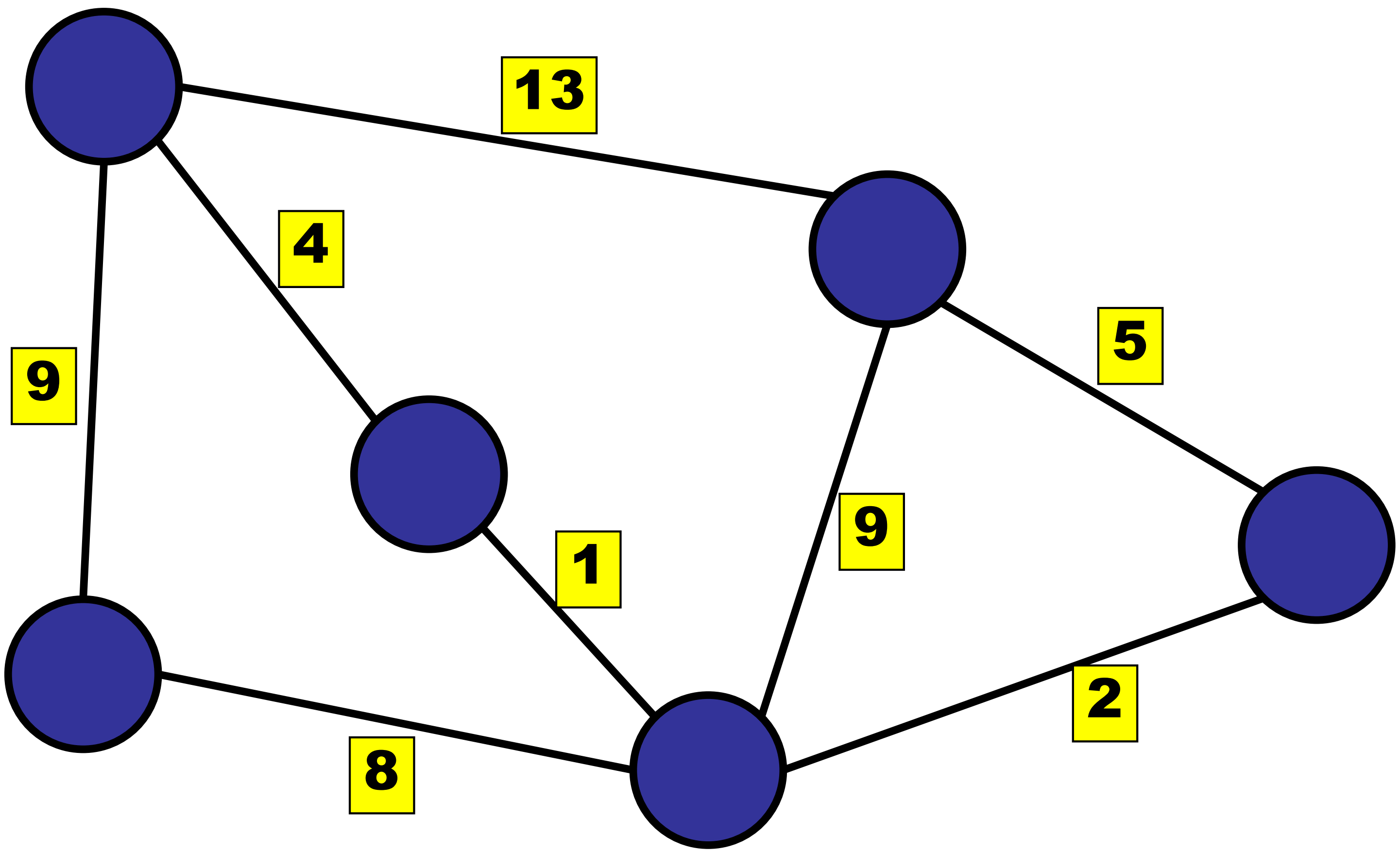
Spanning Tree

Definition: a spanning tree is an acyclic subset of the edges that connects all nodes



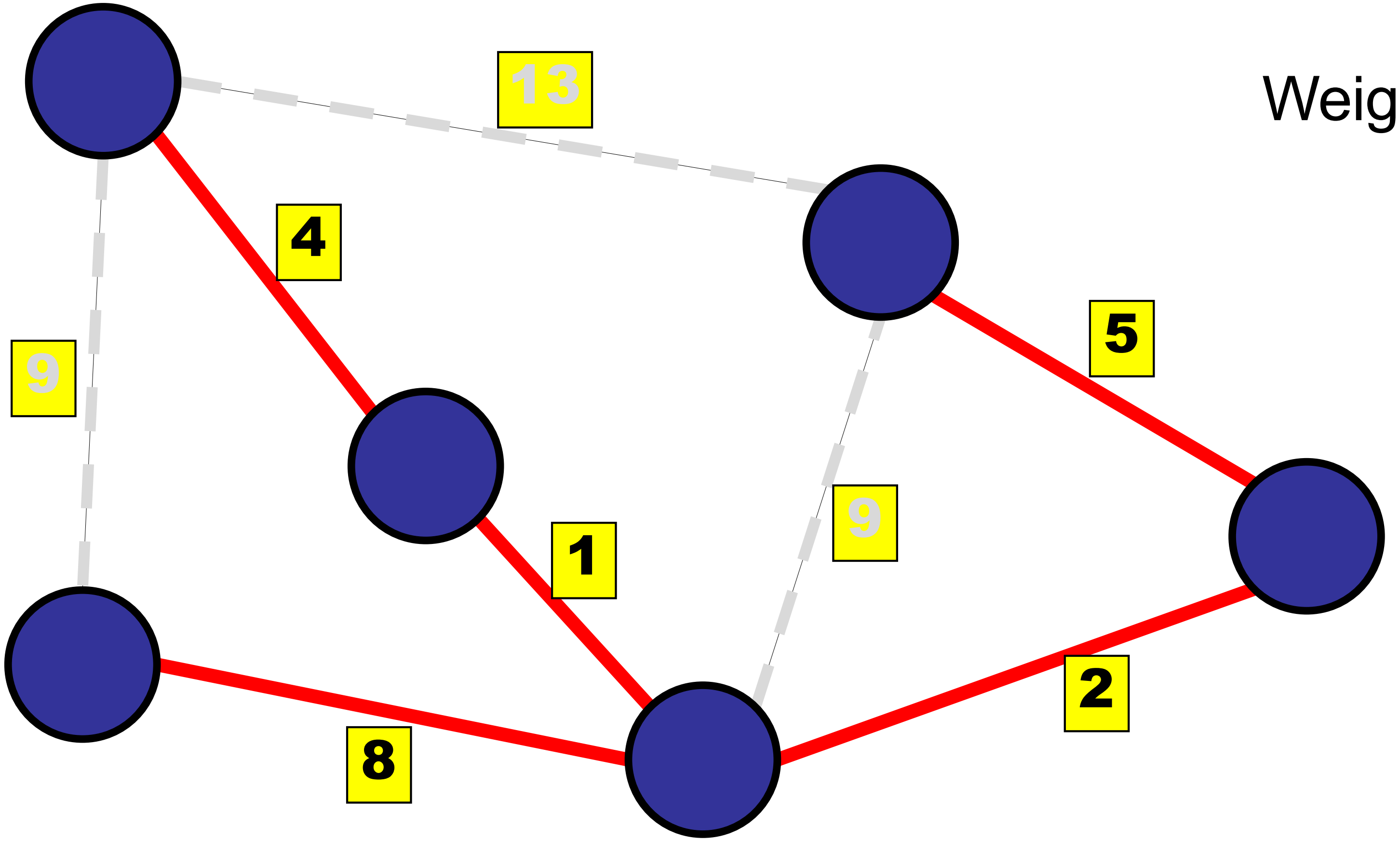
Minimum Spanning Tree

Definition: a spanning tree with minimum weight



Minimum Spanning Tree

Definition: a spanning tree with minimum weight

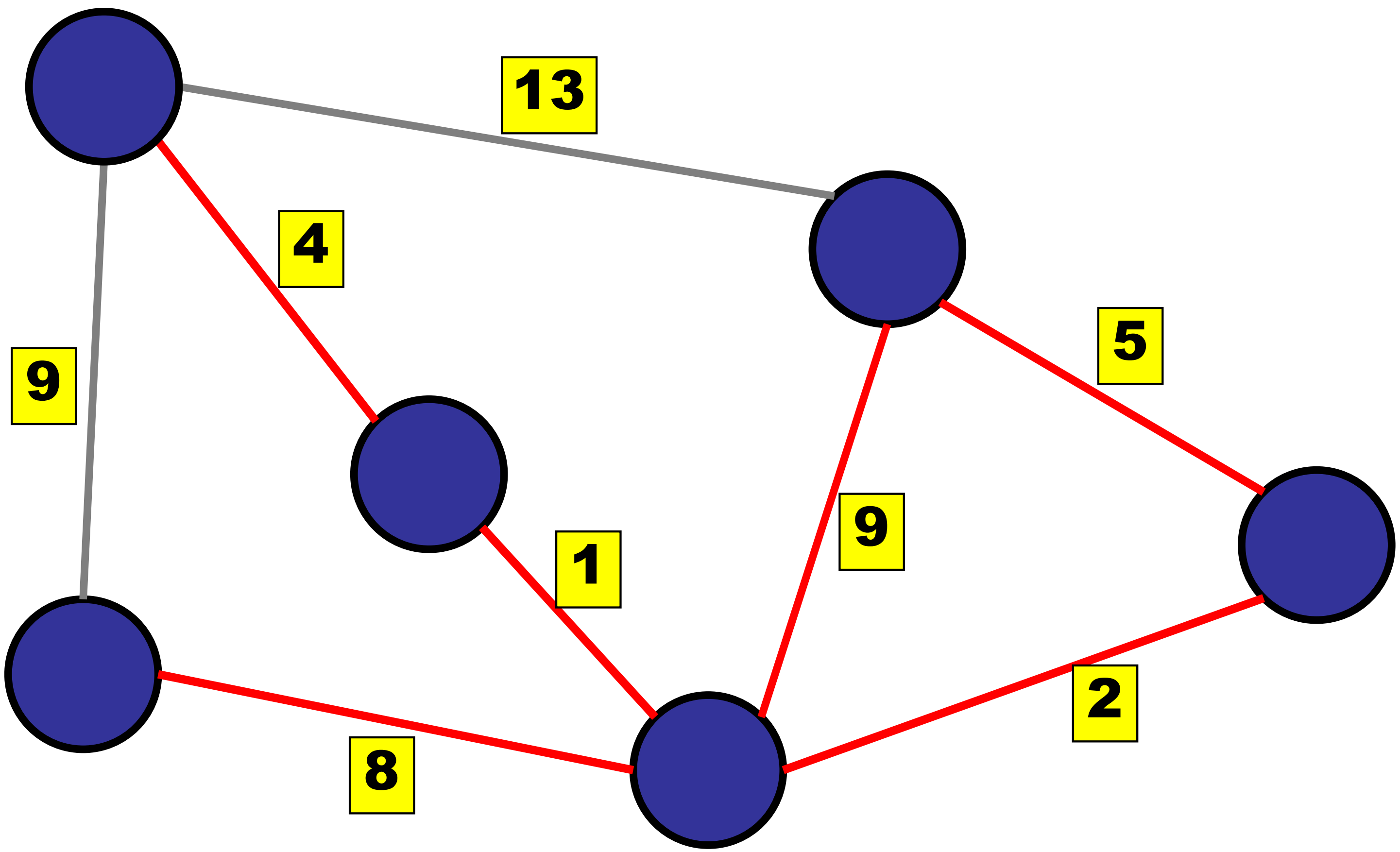


Weight: 20

Minimum Spanning Tree

Note: no cycles

Why? If there were cycles, we could remove one edge and reduce the weight!



Can we use MST to find shortest paths?

1. Yes
2. Only on connected graphs.
3. Only on dense graphs.
4. No.
5. I need to see a picture.

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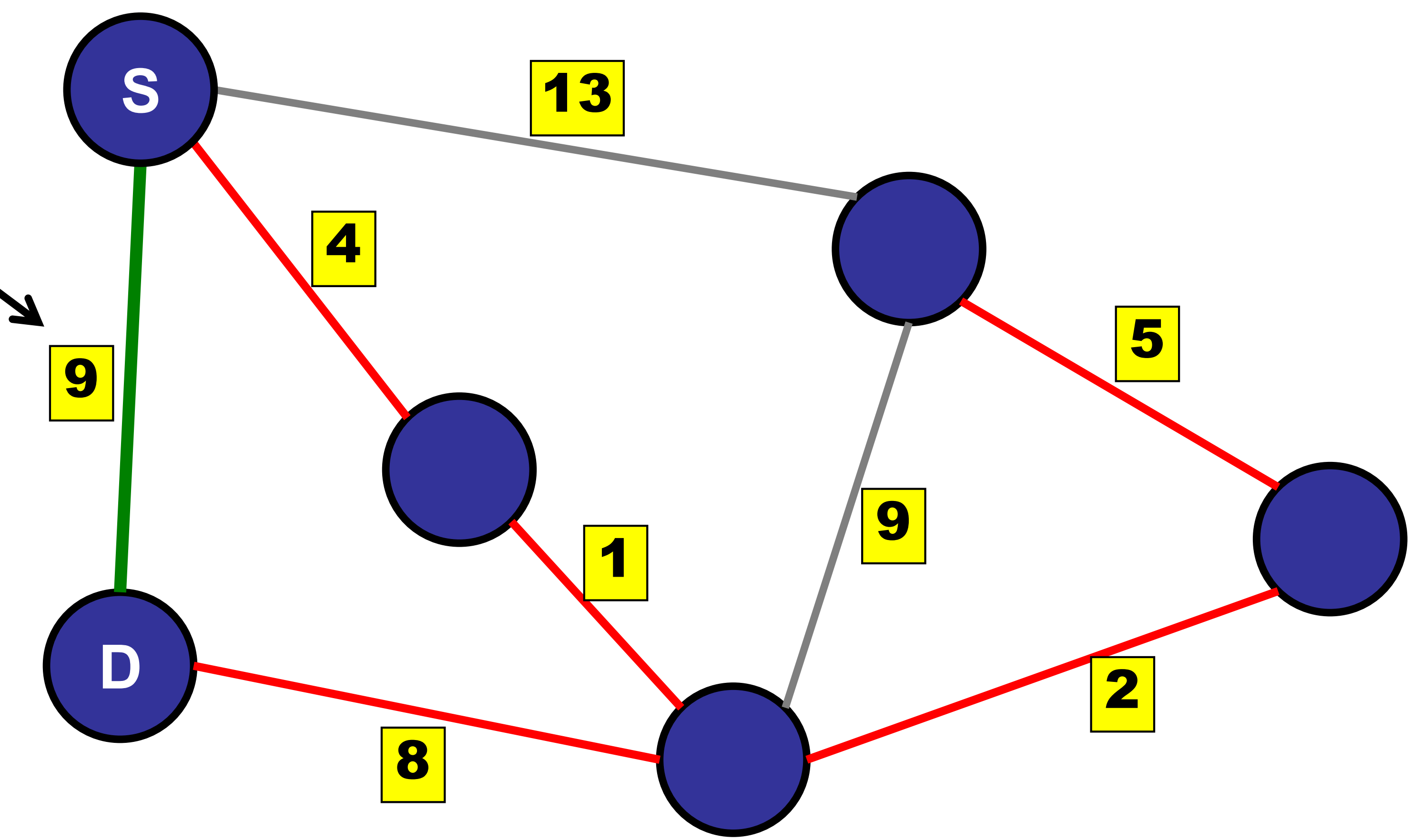
is open

Minimum Spanning Tree

Not the same a shortest paths:

Shortest path
from S->D



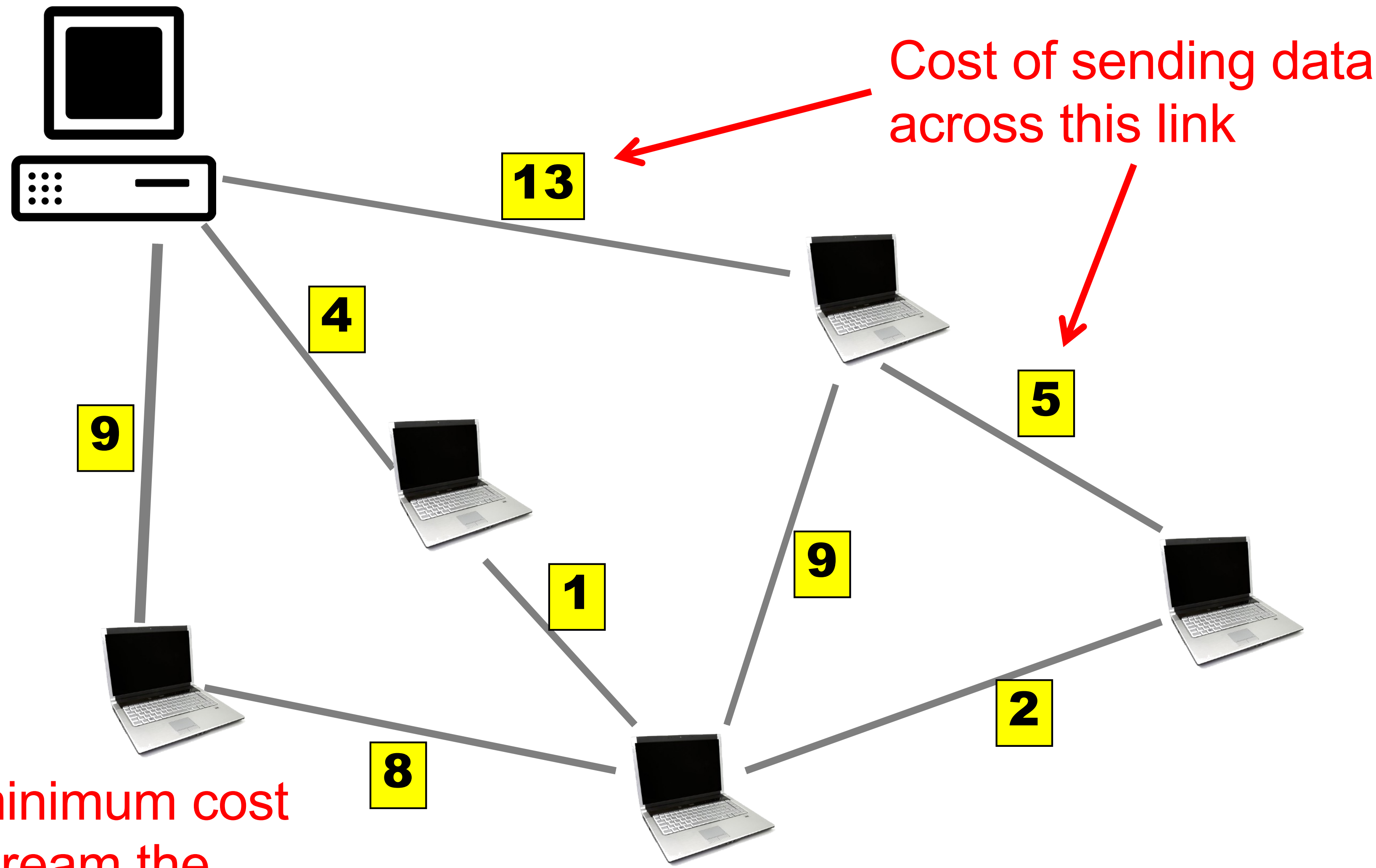
Applications of MST

Network design problems:

- Telephone networks
- Electrical networks
- Computer networks
- Ethernet autoconfig
- Road networks
- Bottleneck paths

Data distribution

Stream a movie over the internet:



What is the minimum cost **network** to stream the movie to all the users?

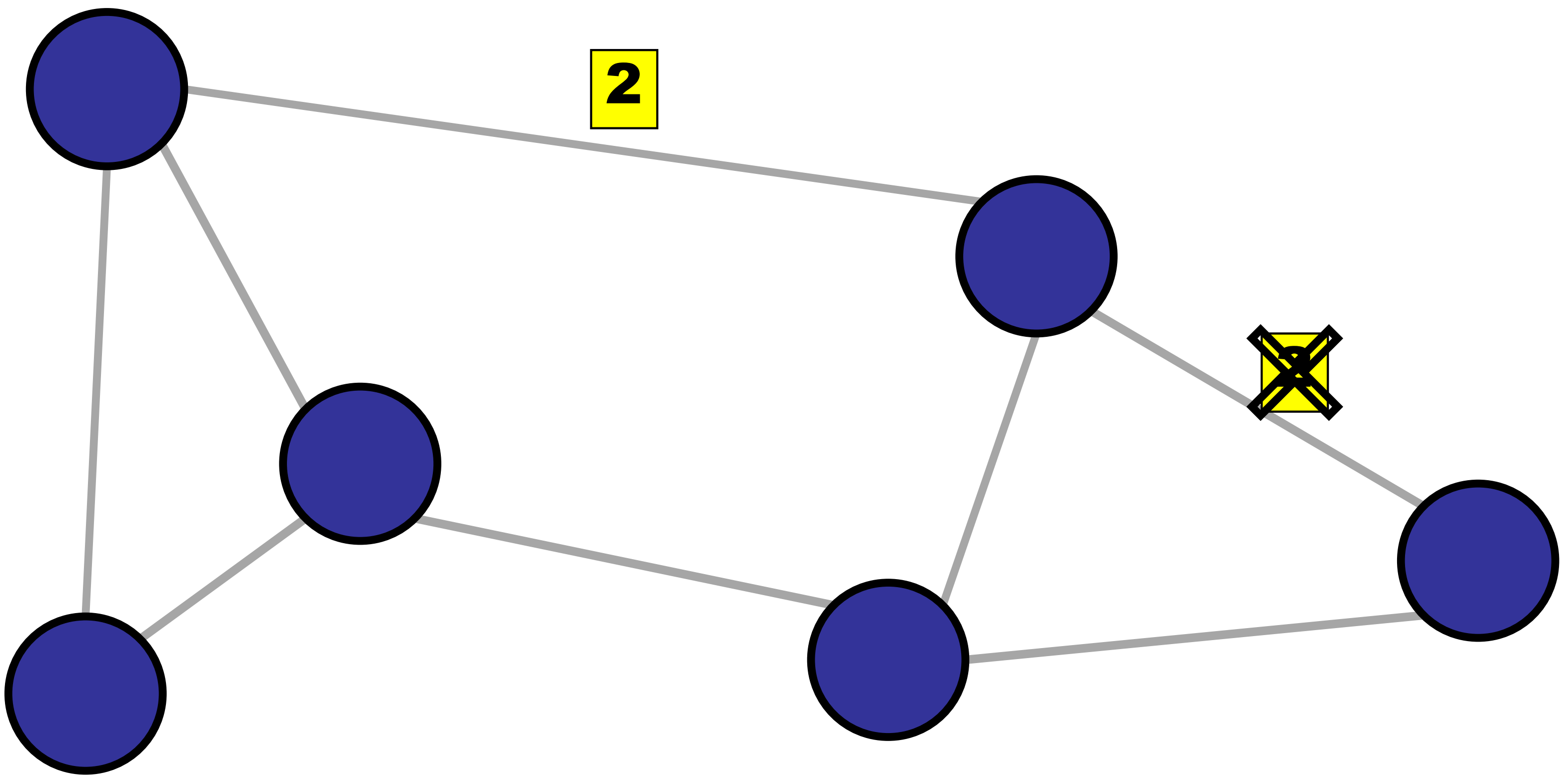
Applications of MST

Less obvious applications:

- Error correcting codes
- Face verification
- Cluster analysis
- Image registration

Assumption

All edge weights are distinct. (Simplification…)



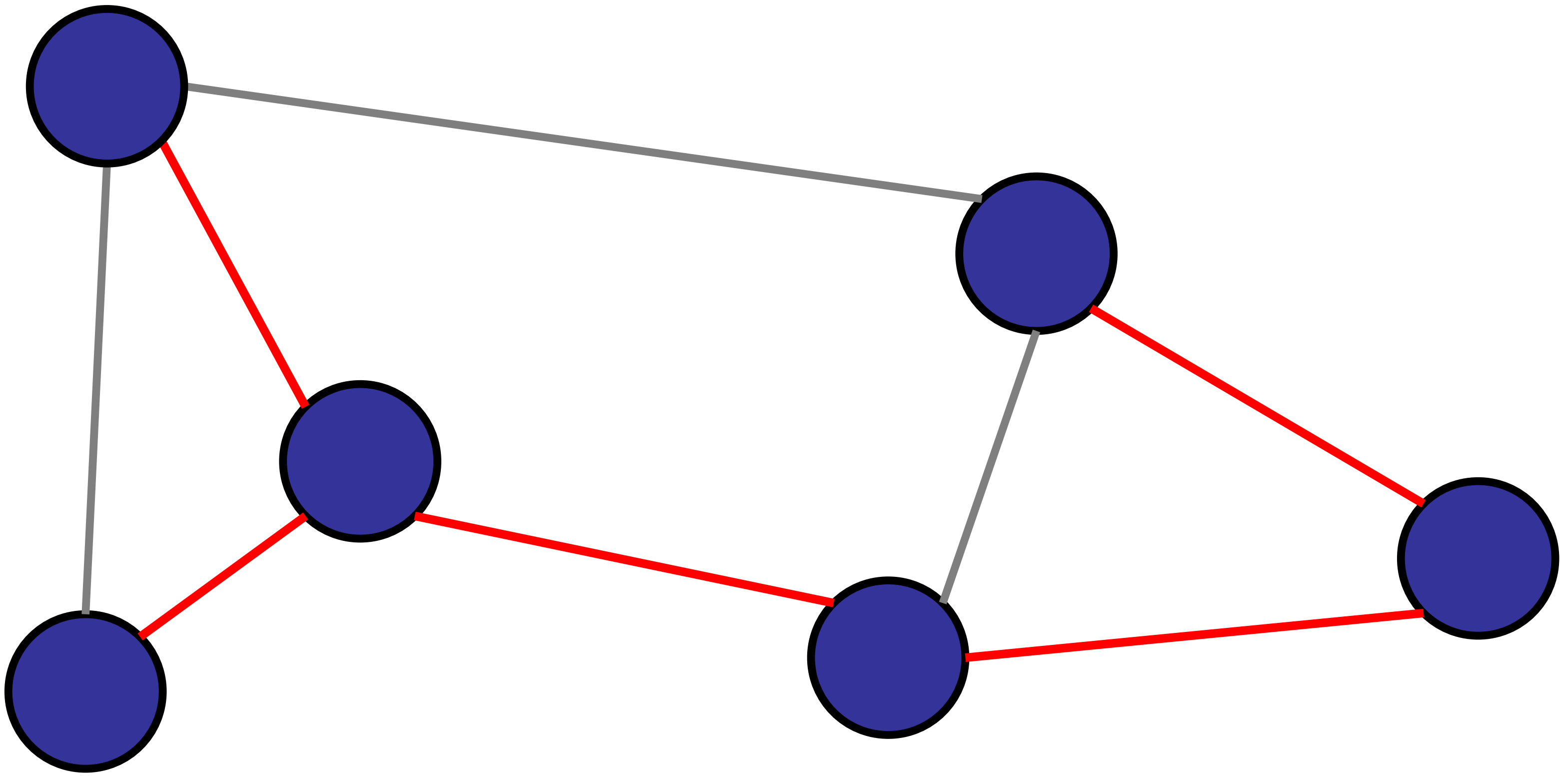
Roadmap

Minimum Spanning Trees

- The MST Problem
- **Basic Properties of an MST**
- Generic MST Algorithm
- Prim's Algorithm
- Kruskal's Algorithm
- Boruvka's Algorithm
- Variations

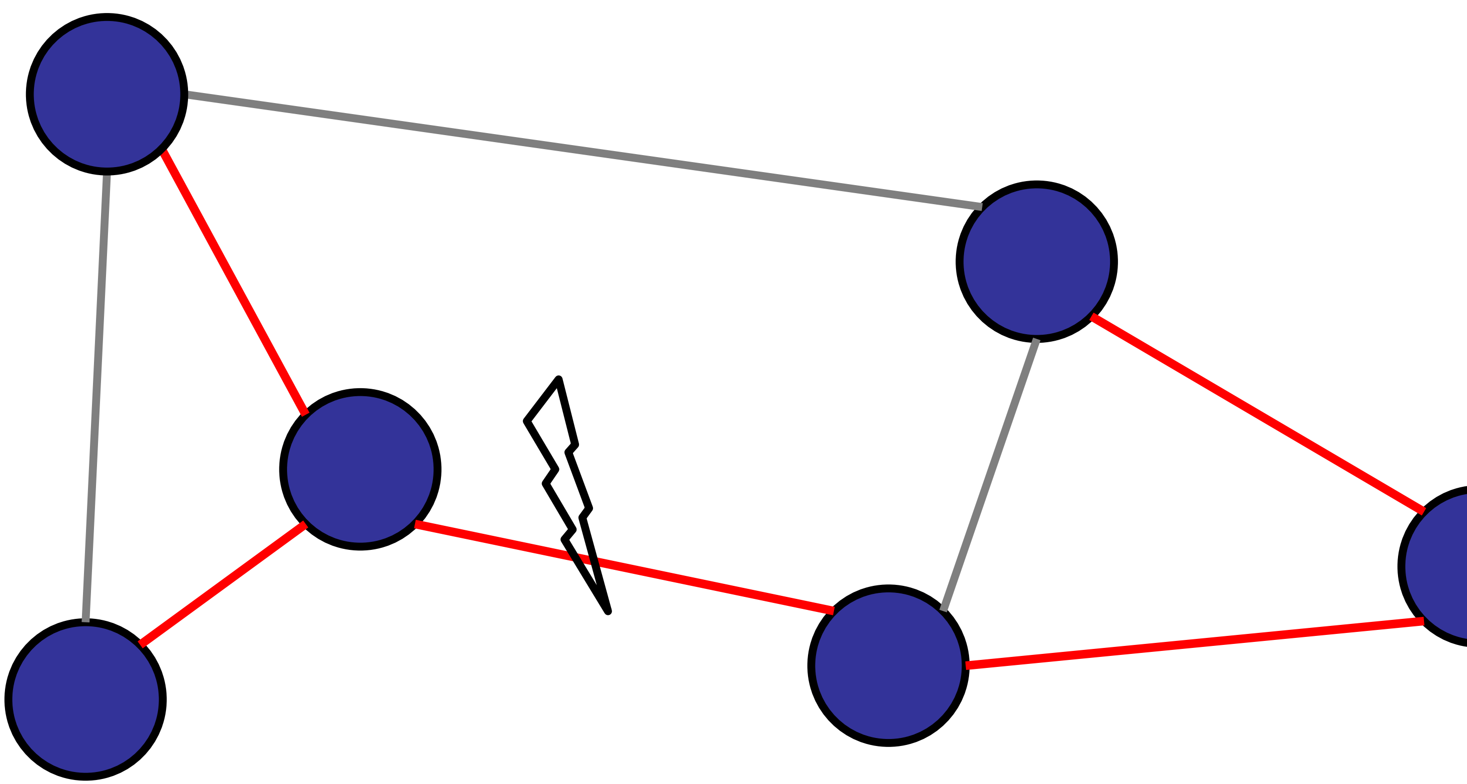
Properties of MST

Property 1: No cycles



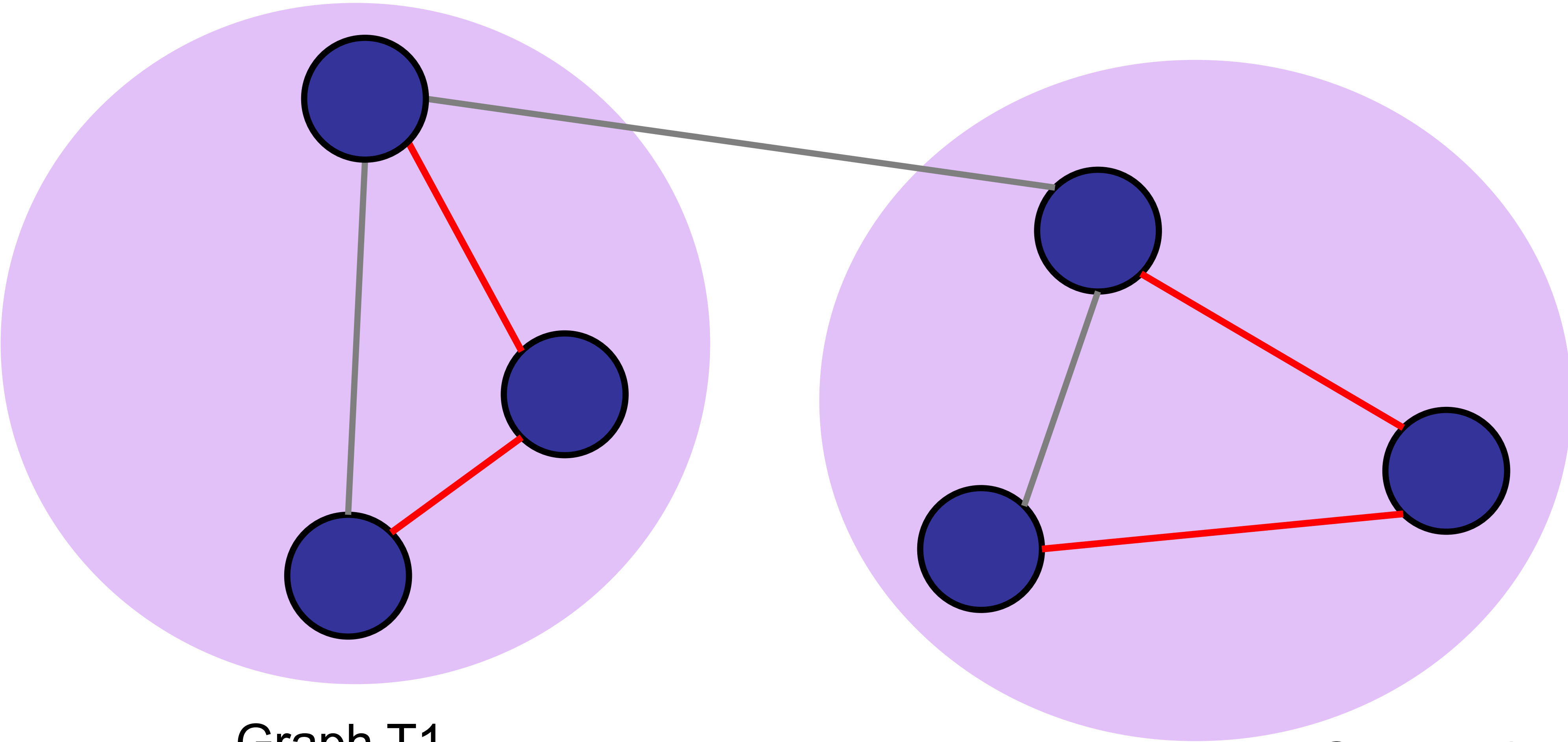
Properties of MST

What happens if you cut an MST?



Properties of MST

What happens if you cut an MST?

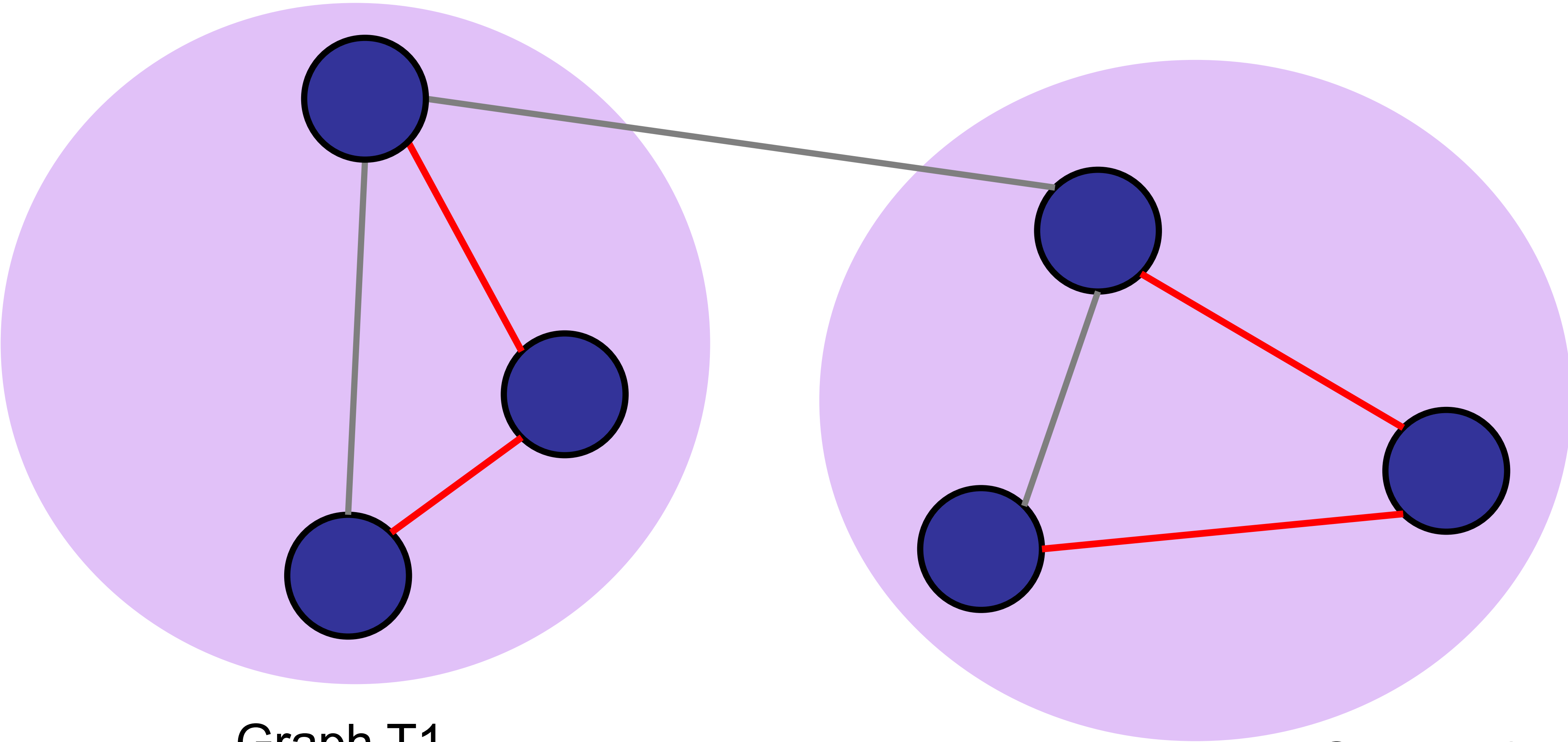


Graph T1

Graph T2

Properties of MST

Theorem: T1 is an MST and T2 is an MST.

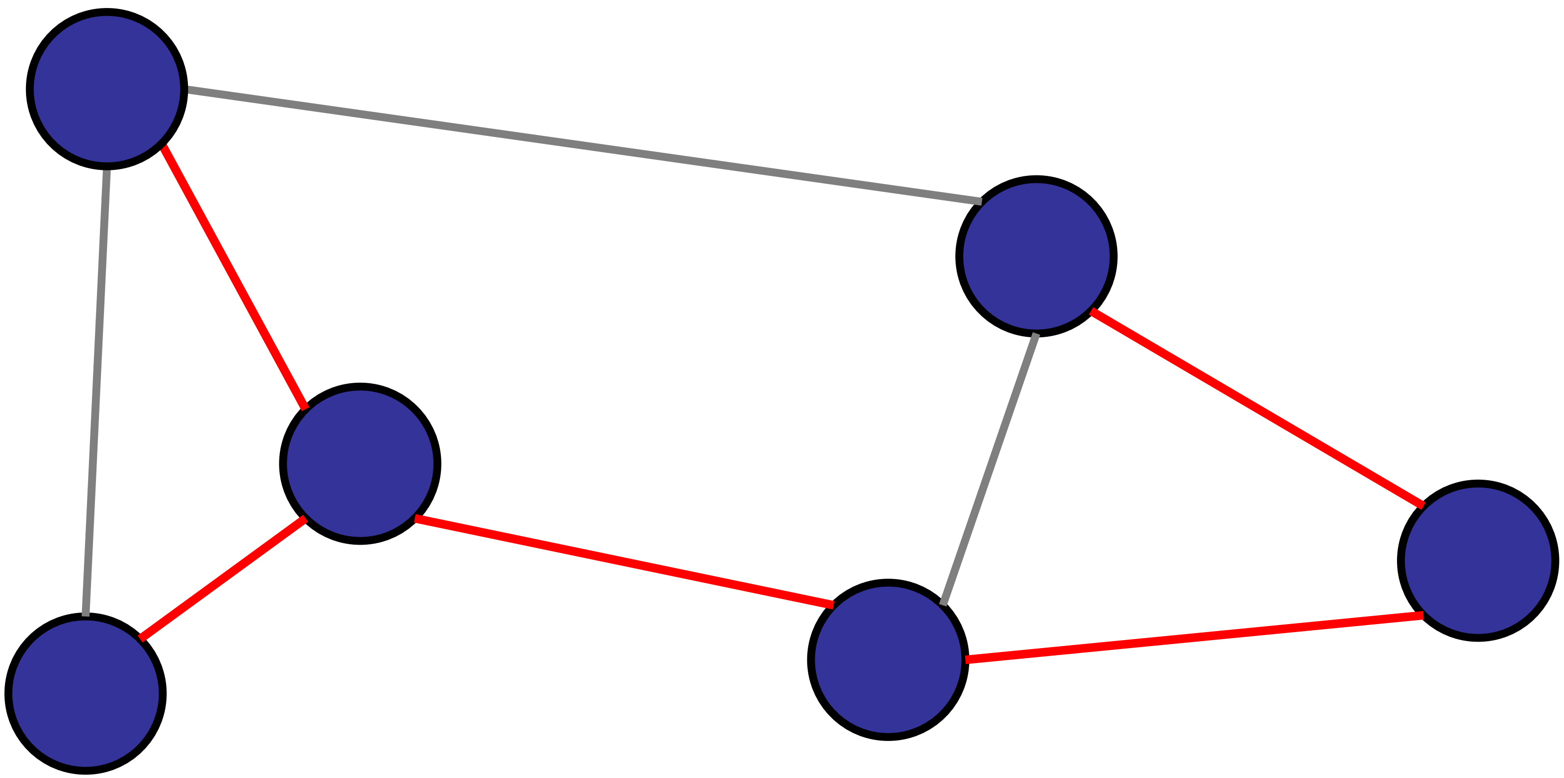


Graph T1

Graph T2

Properties of MST

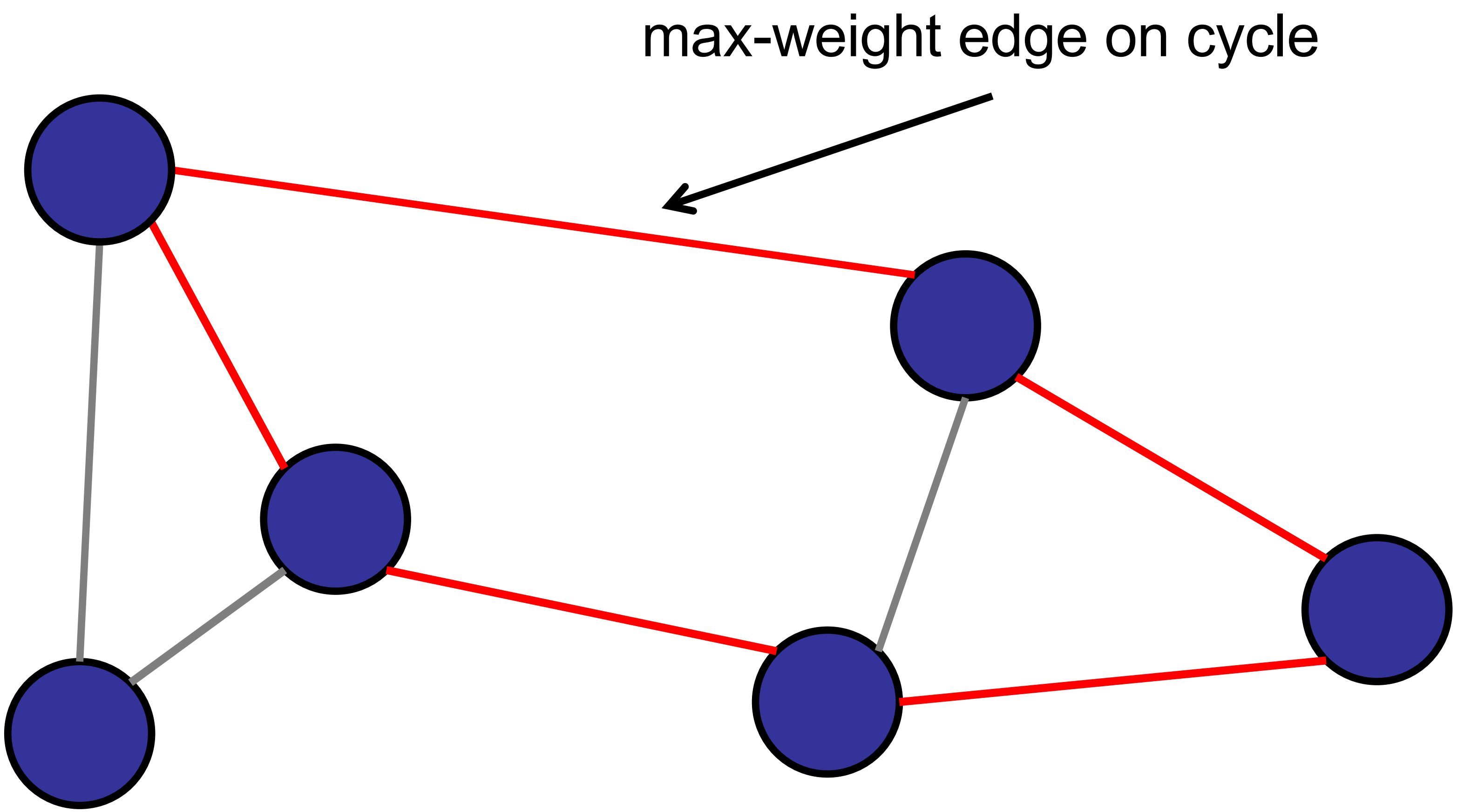
Property 2: If you cut an MST, the two pieces are both MSTs.



Overlapping sub-problems! Dynamic programming? Yes, but better…

Properties of MST

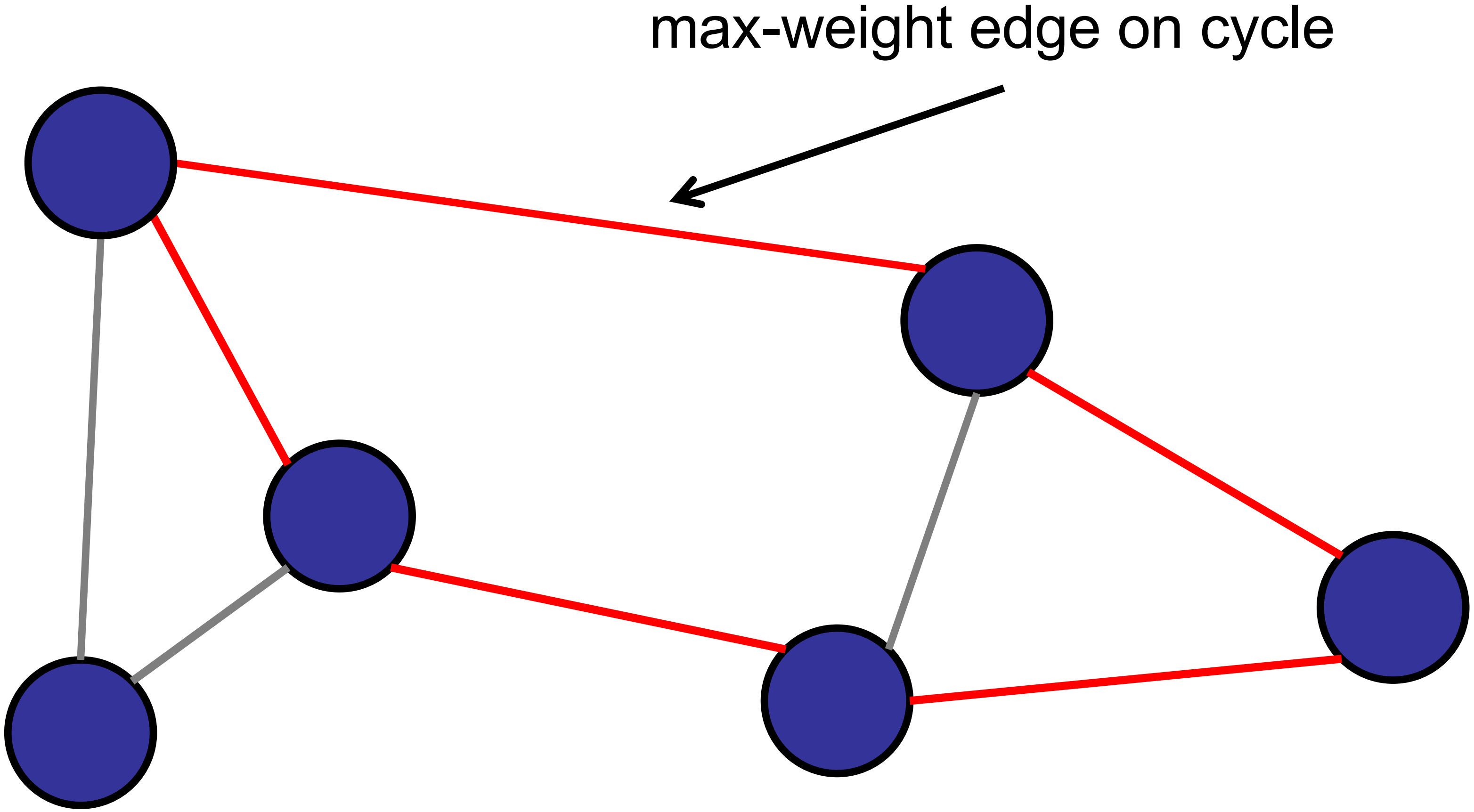
Property 3: Cycle property



Properties of MST

Property 3: Cycle property

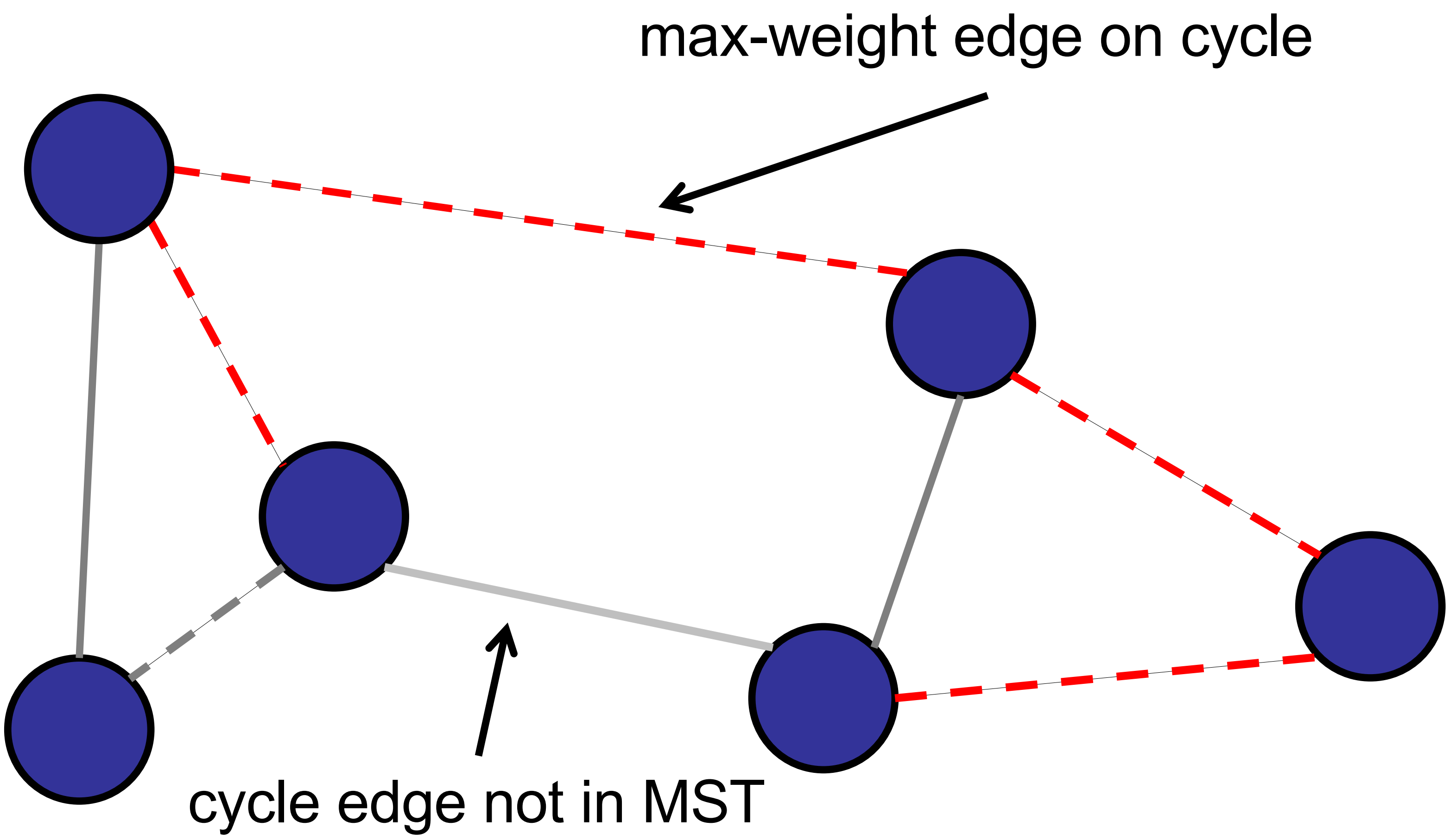
For every cycle, the maximum weight edge is ***not*** in the MST.



Properties of MST

Proof: Cut-and-paste

Assume heavy edge is in the MST.

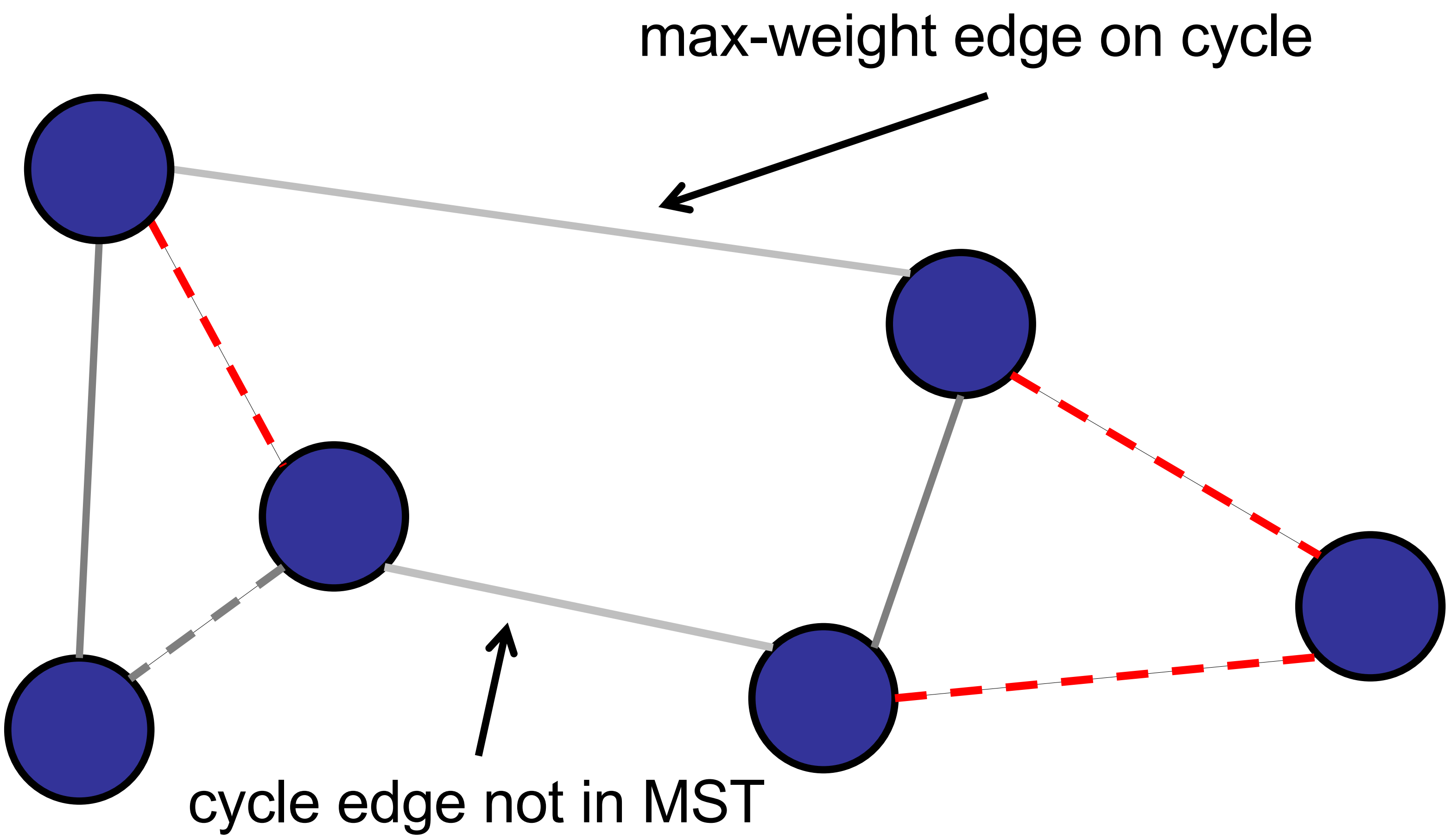


Properties of MST

Proof: Cut-and-paste

Assume heavy edge is in the MST.

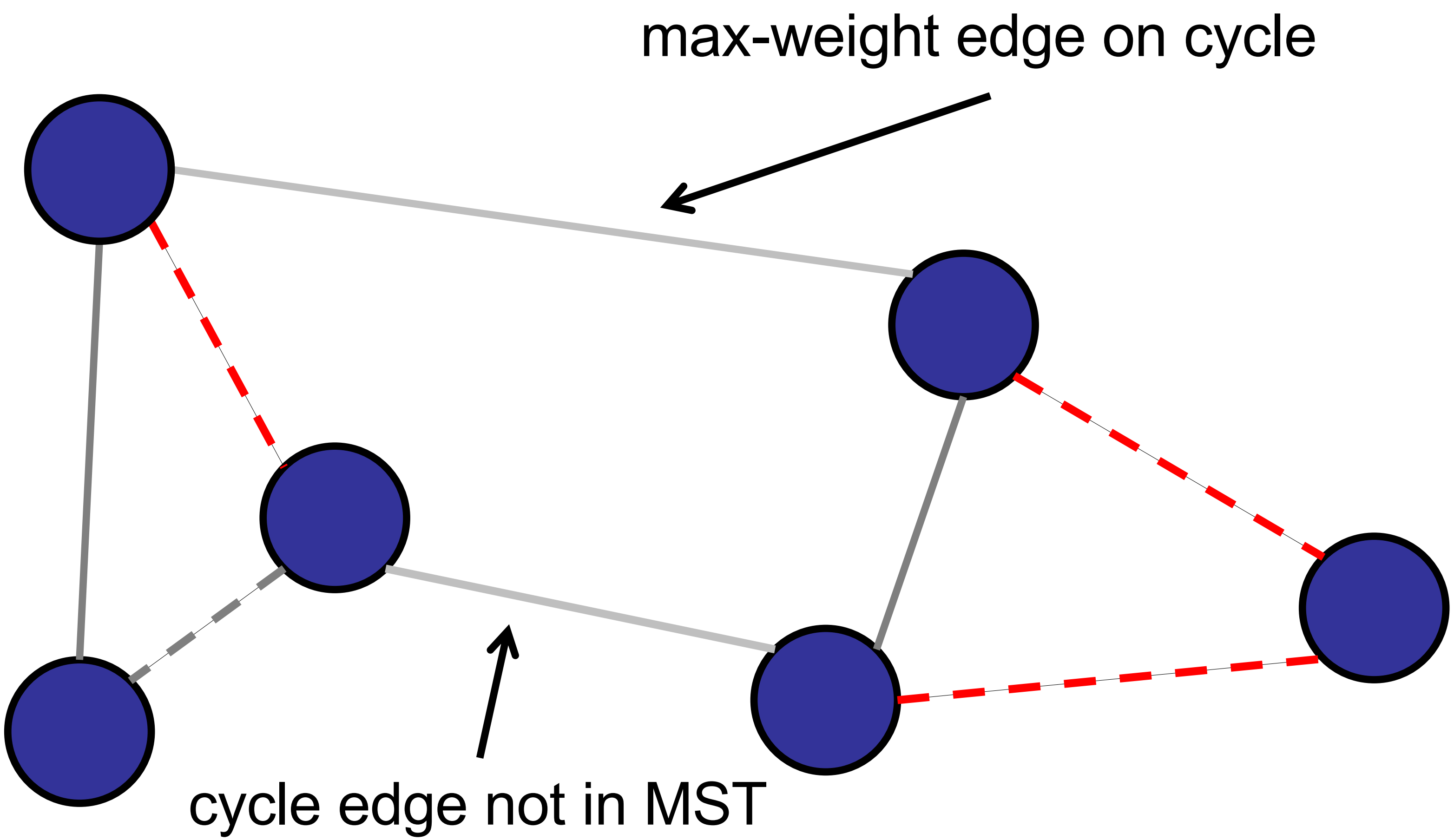
Remove max-weight edge; cuts graph.



Properties of MST

Proof: Cut-and-paste

There exists another cycle edge that crosses the cut. (Even # of cycle edges across cut.)

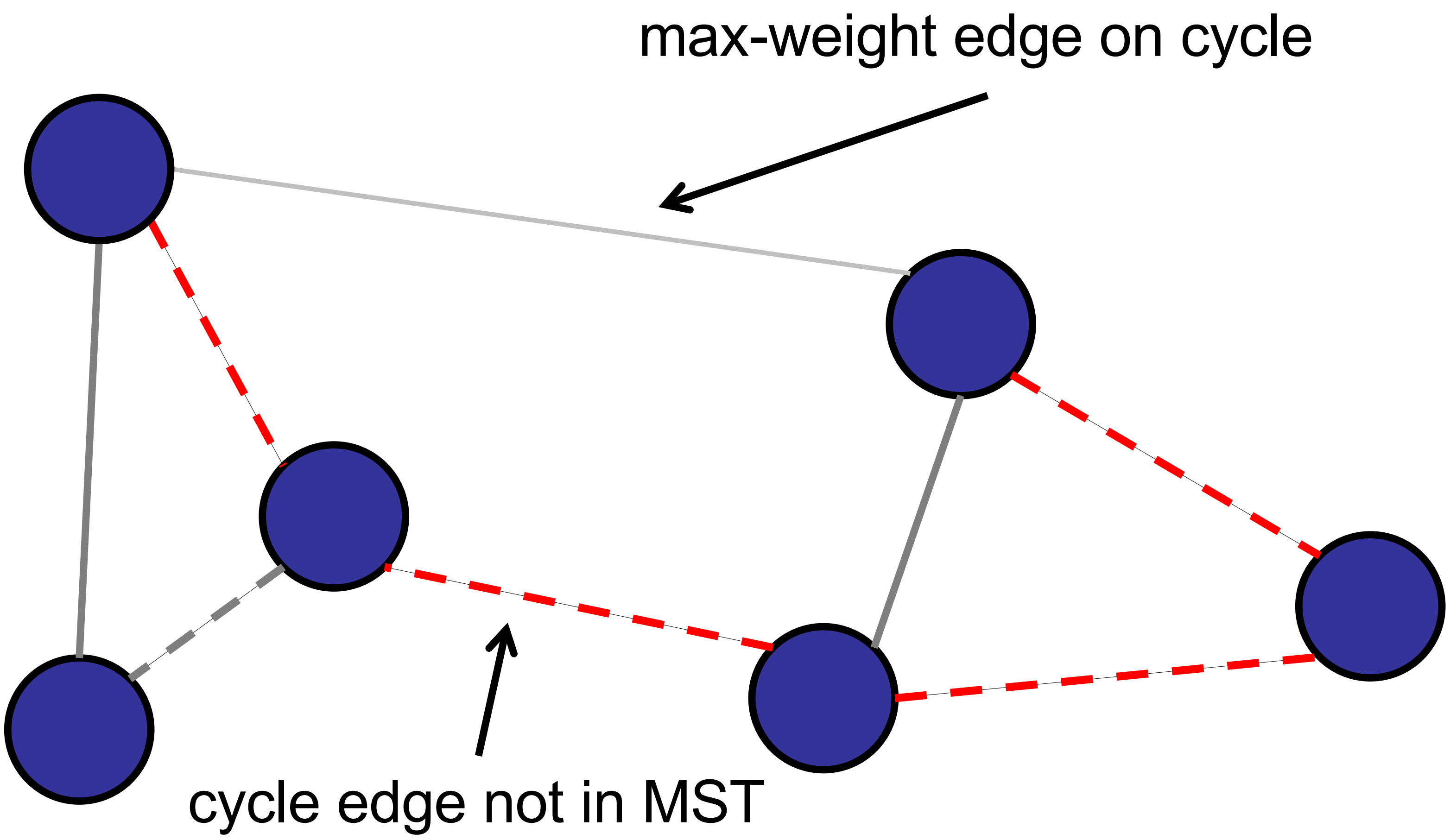


Properties of MST

Proof: Cut-and-paste

Replace heavy edge with lighter edge.

Still a spanning tree: Property 2.

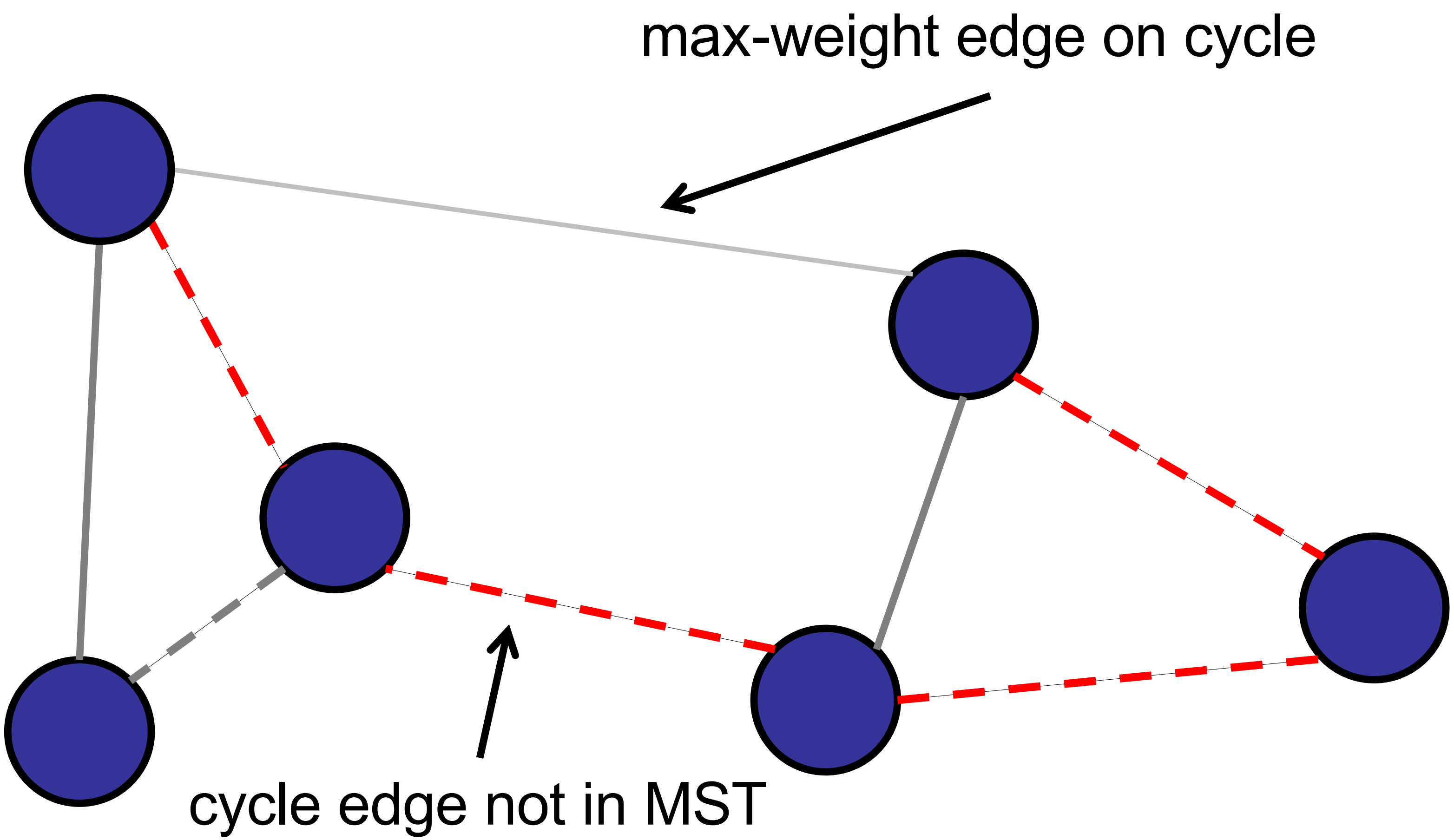


Properties of MST

Proof: Cut-and-paste

Replace heavy edge with lighter edge.

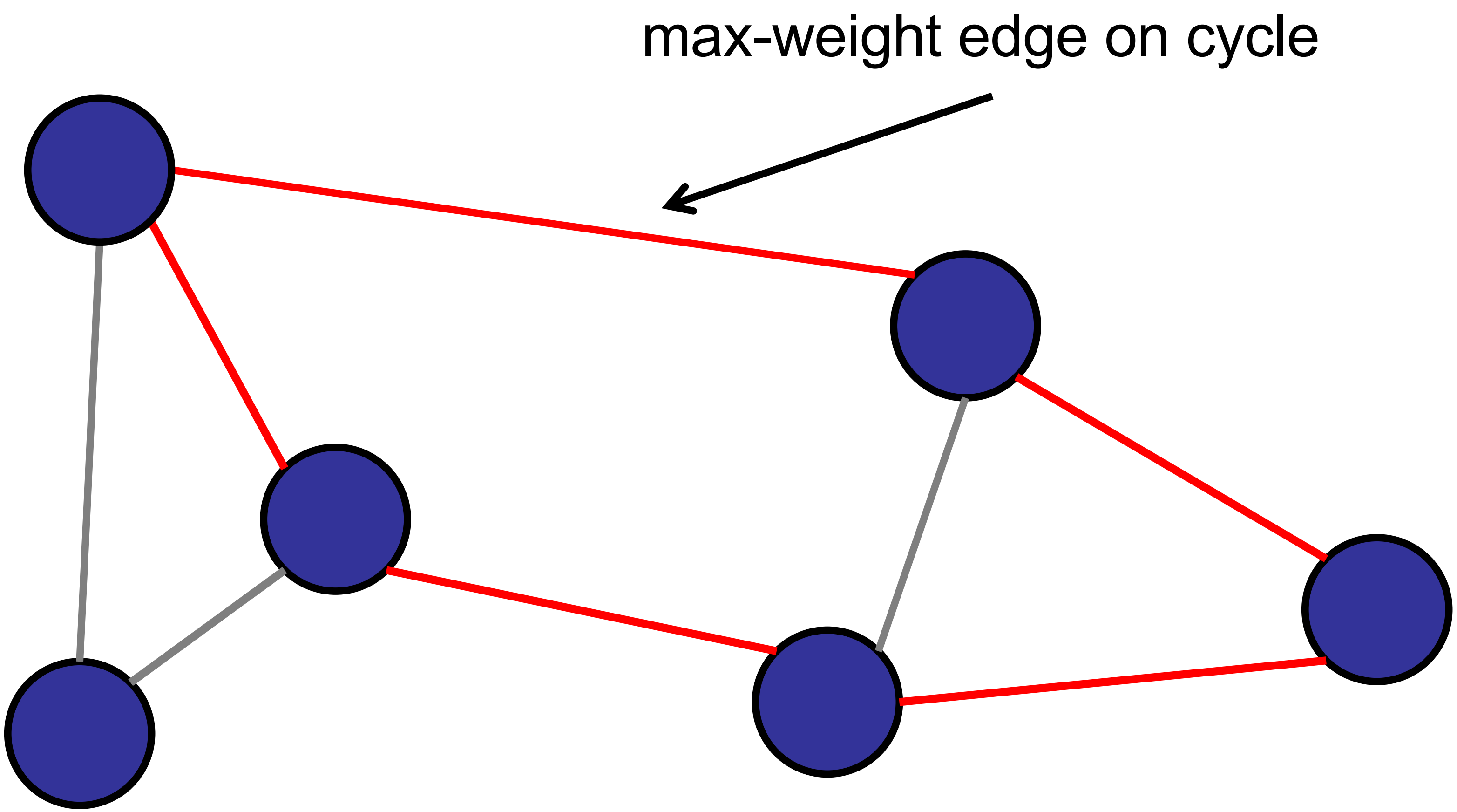
Less weight! Contradiction…



Properties of MST

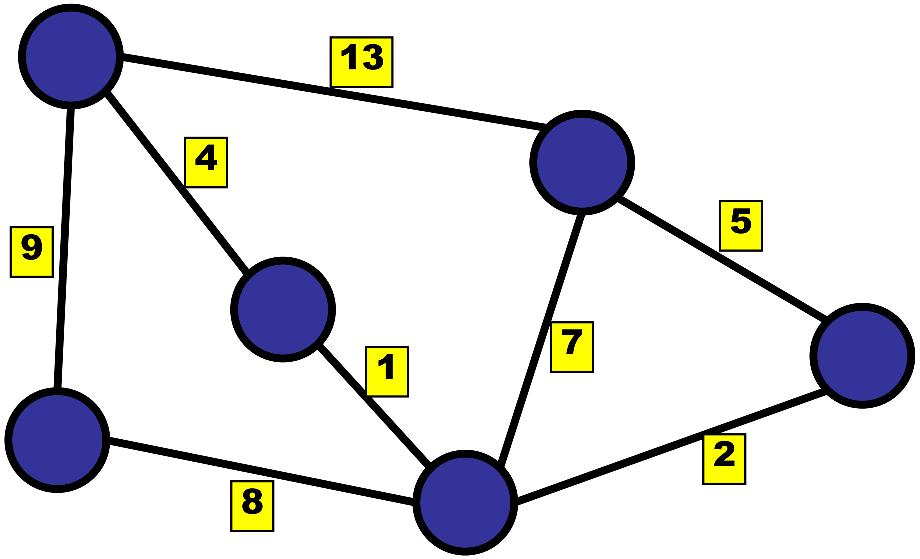
Property 3: Cycle property

For every cycle, the maximum weight edge is ***not*** in the MST.



True or False:
For every cycle, the minimum weight edge is always in the MST.

1. True
2. False
3. I don't know.



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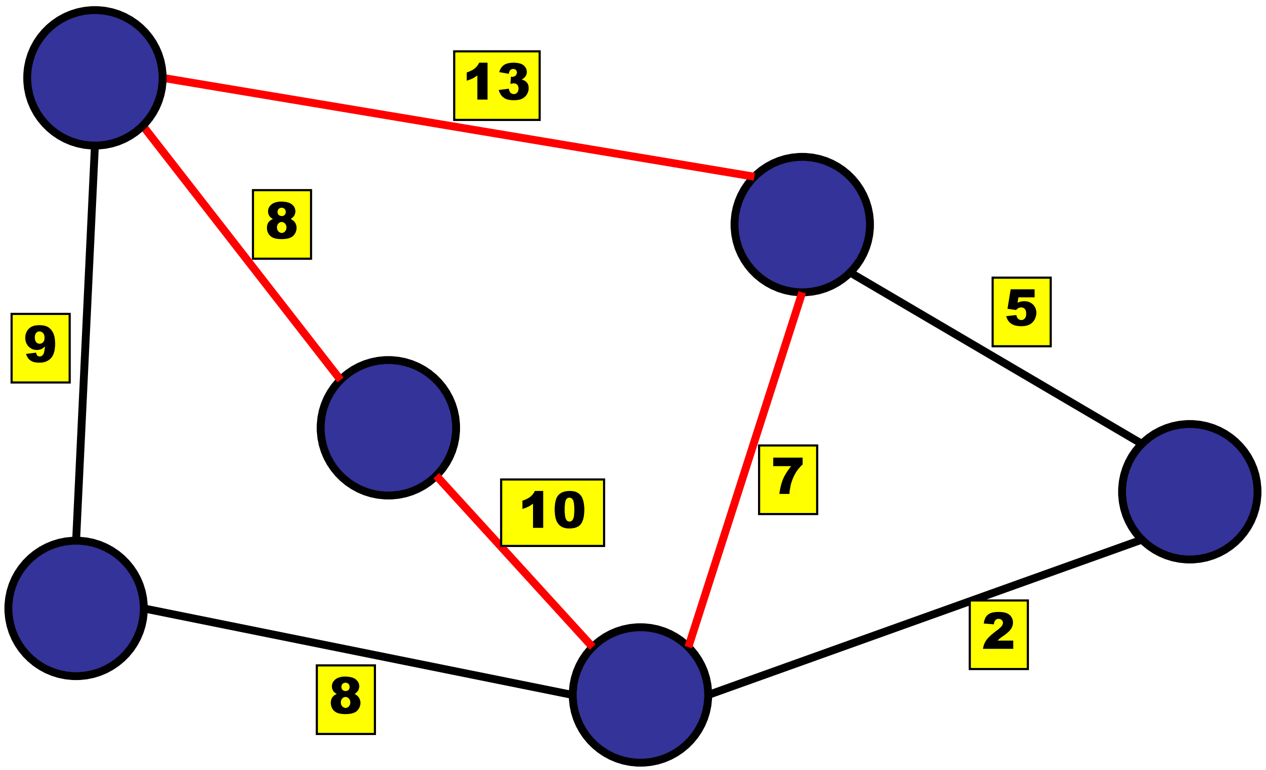
Properties of MST

Property 3: False Cycle property

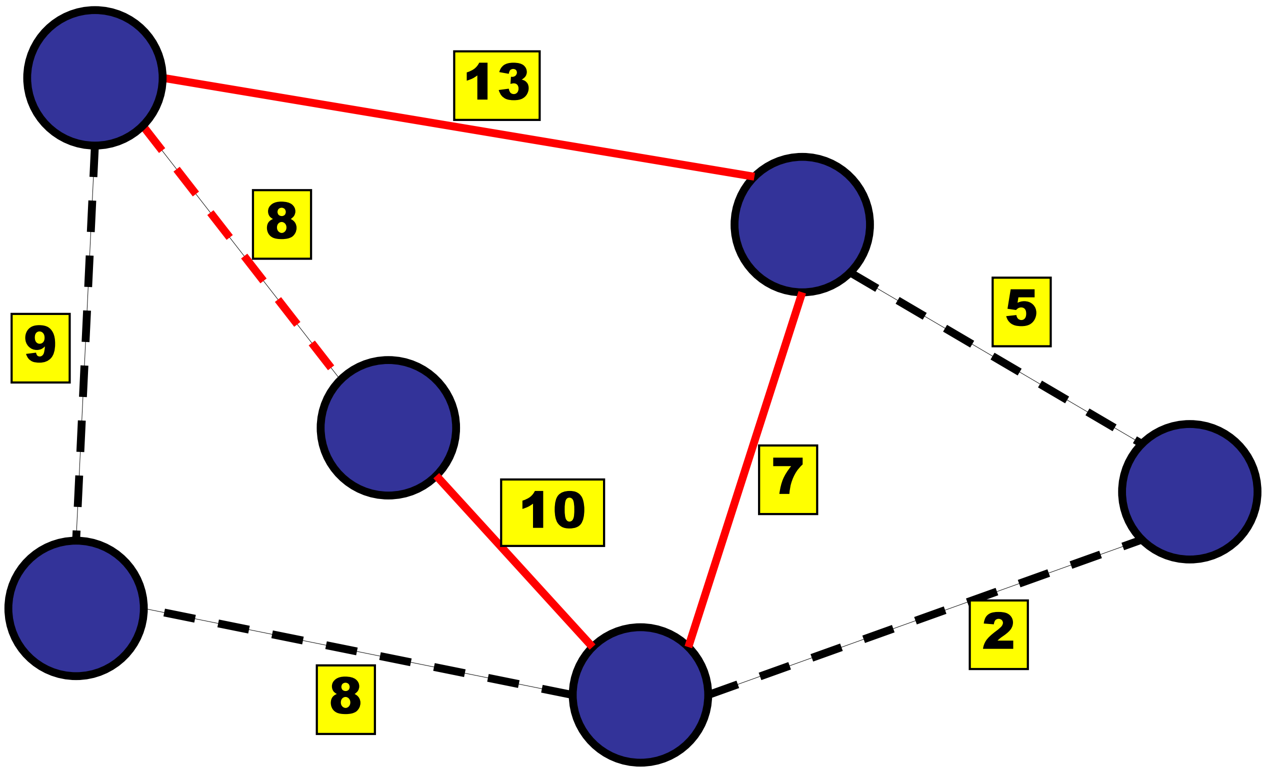
For every cycle, the minimum weight edge may or may *not* be in the MST.



Properties of MST

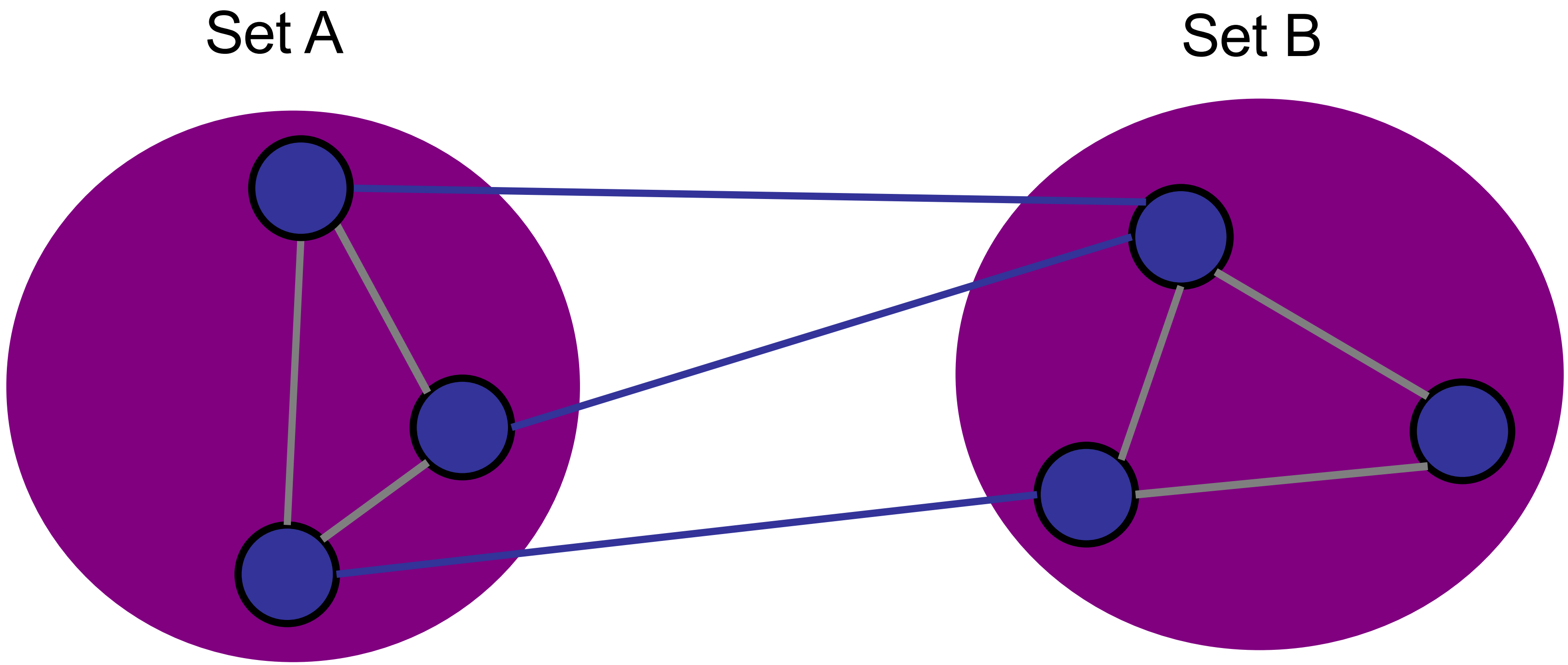
Property 3: False Cycle property

For every cycle, the minimum weight edge may or may *not* be in the MST.



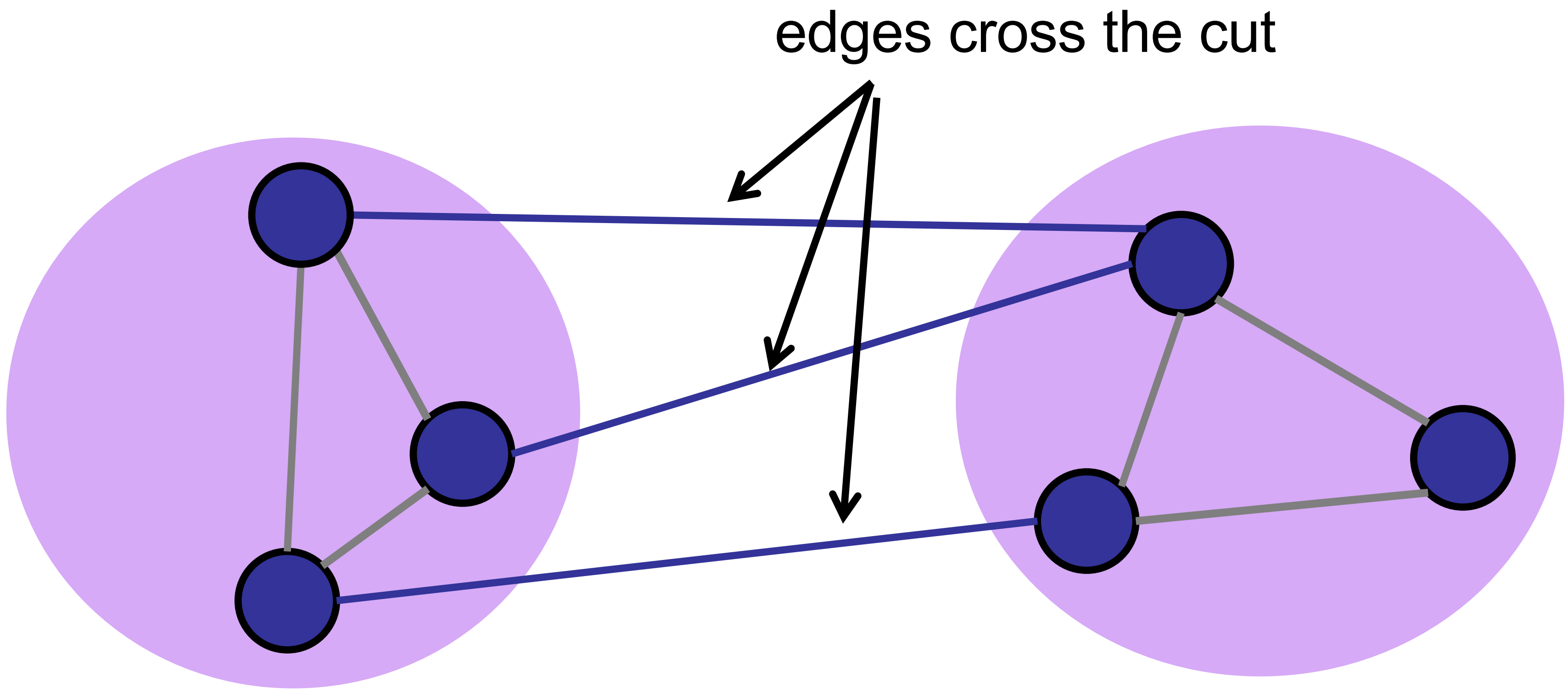
Properties of MST

Definition: A *cut* of a graph G=(V,E) is a partition of the vertices V into two disjoint subsets.



Properties of MST

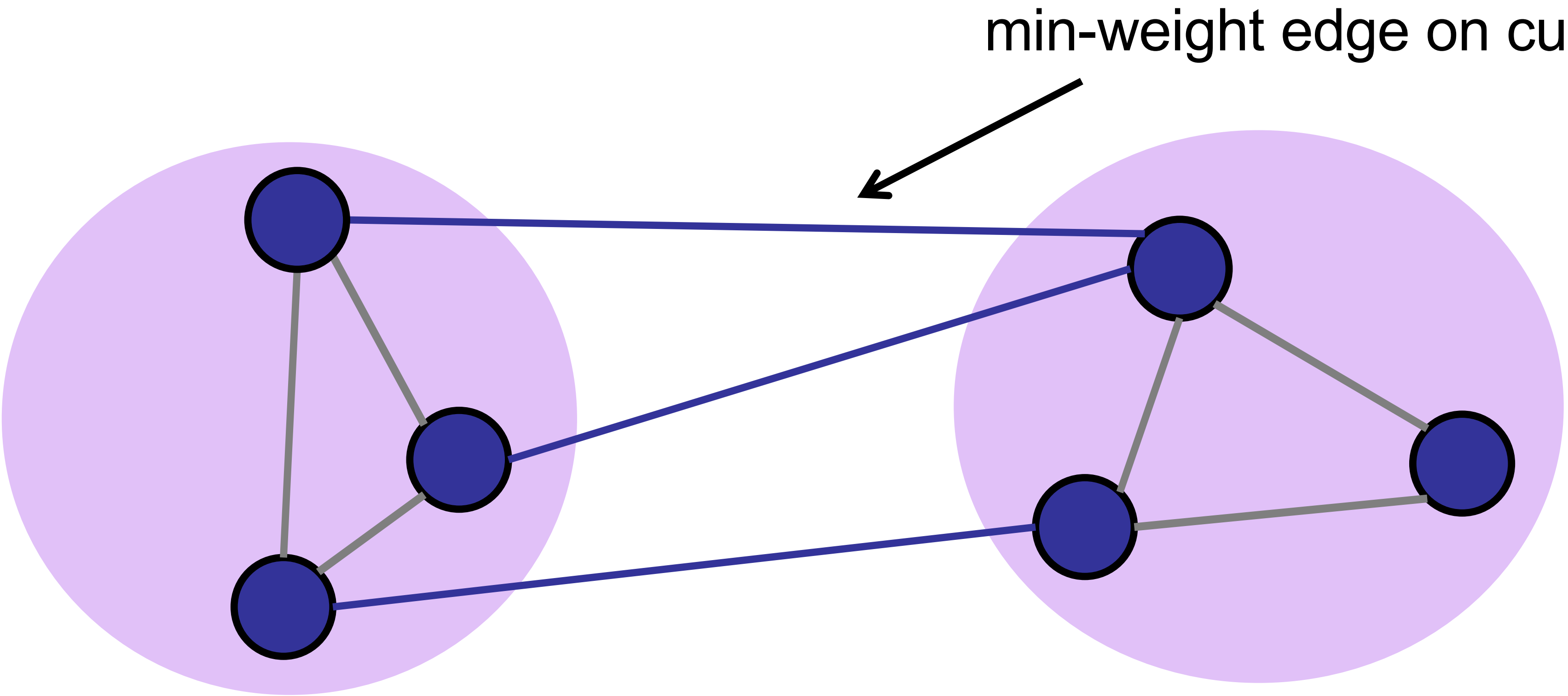
Definition: An edge *crosses a cut* if it has one vertex in each of the two sets.



Properties of MST

Property 4: Cut property

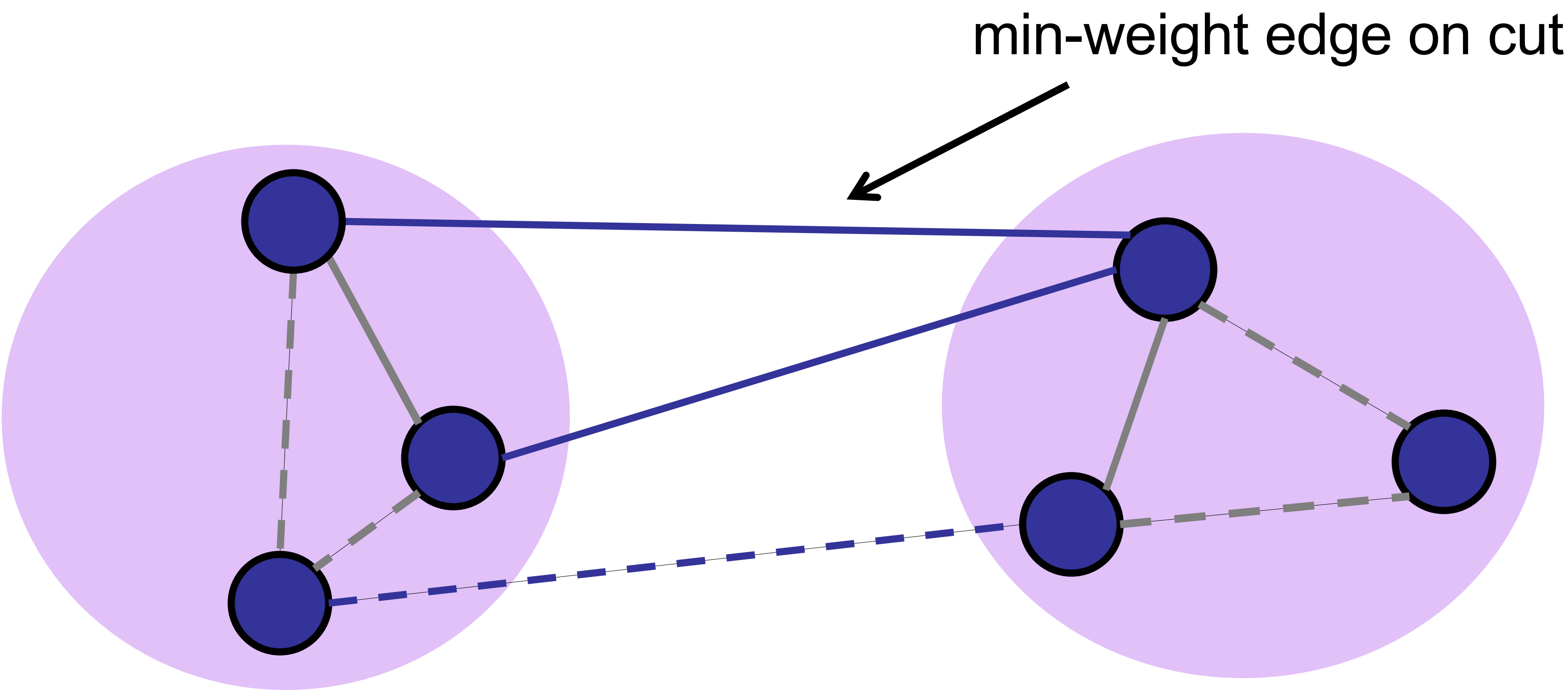
For every partition of the nodes, the minimum weight edge across the cut *is* in the MST.



Properties of MST

Proof: Cut-and-paste

Assume not.

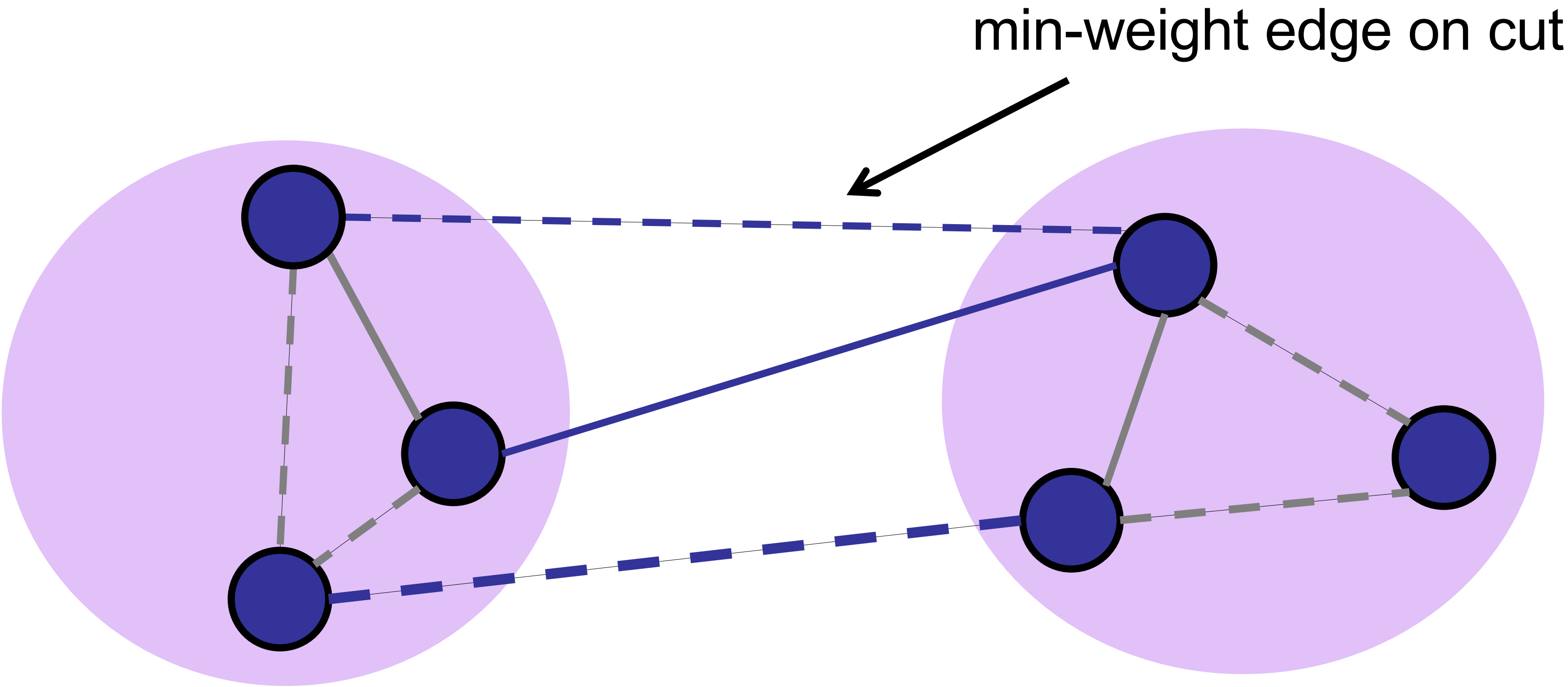


Properties of MST

Proof: Cut-and-paste

Assume not.

Add min-weight edge on cut.



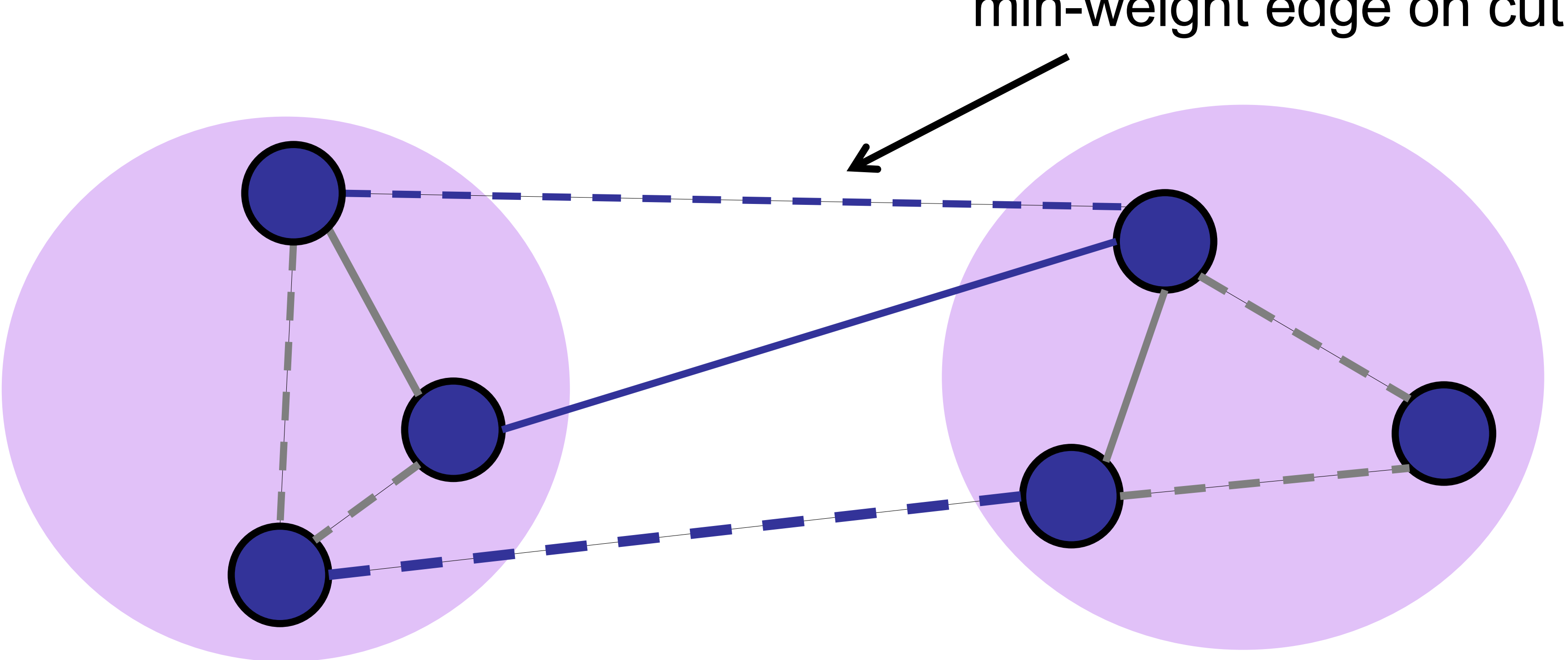
Properties of MST

Proof: Cut-and-paste

Assume not.

Add min-weight edge on cut.

Oops, creates a cycle!



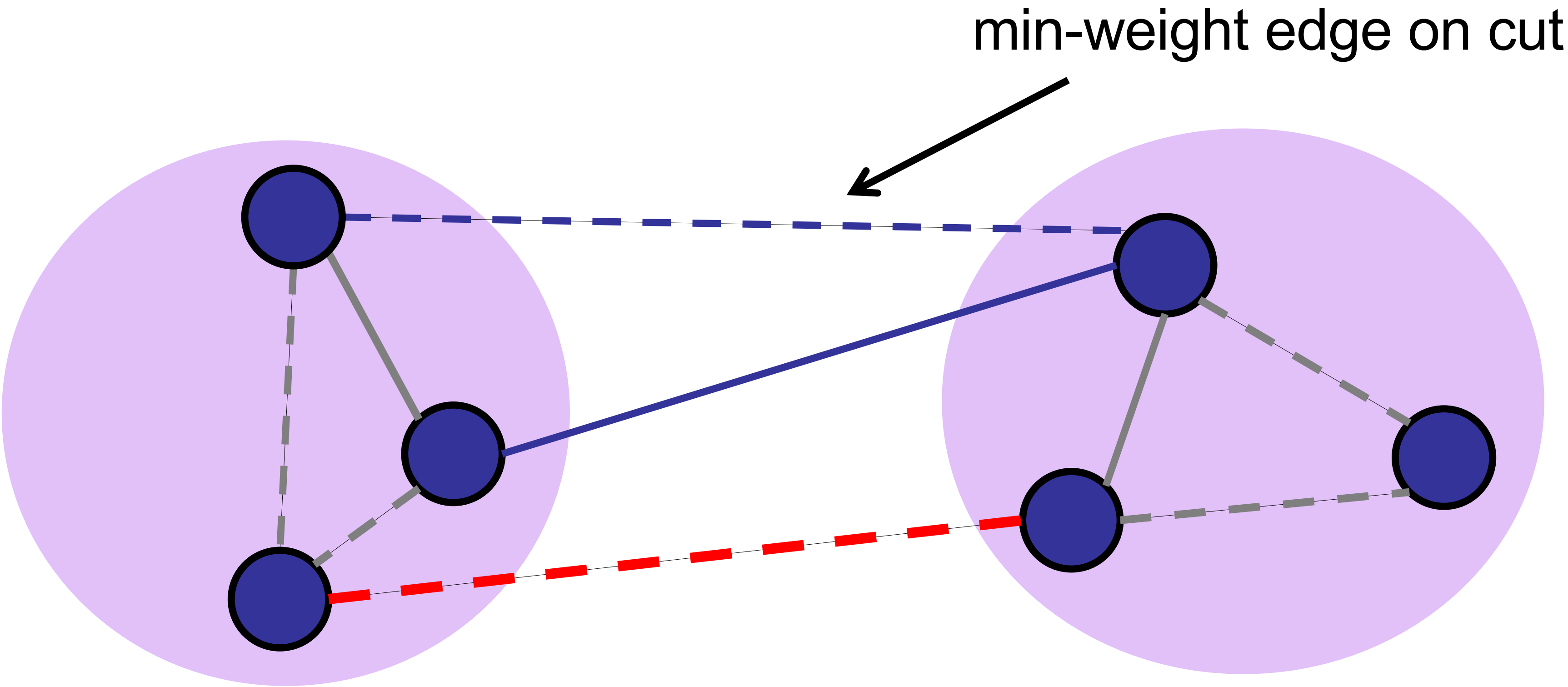
Properties of MST

Proof: Cut-and-paste

Assume not.

Add min-weight edge on cut.

Remove heaviest edge on cycle.



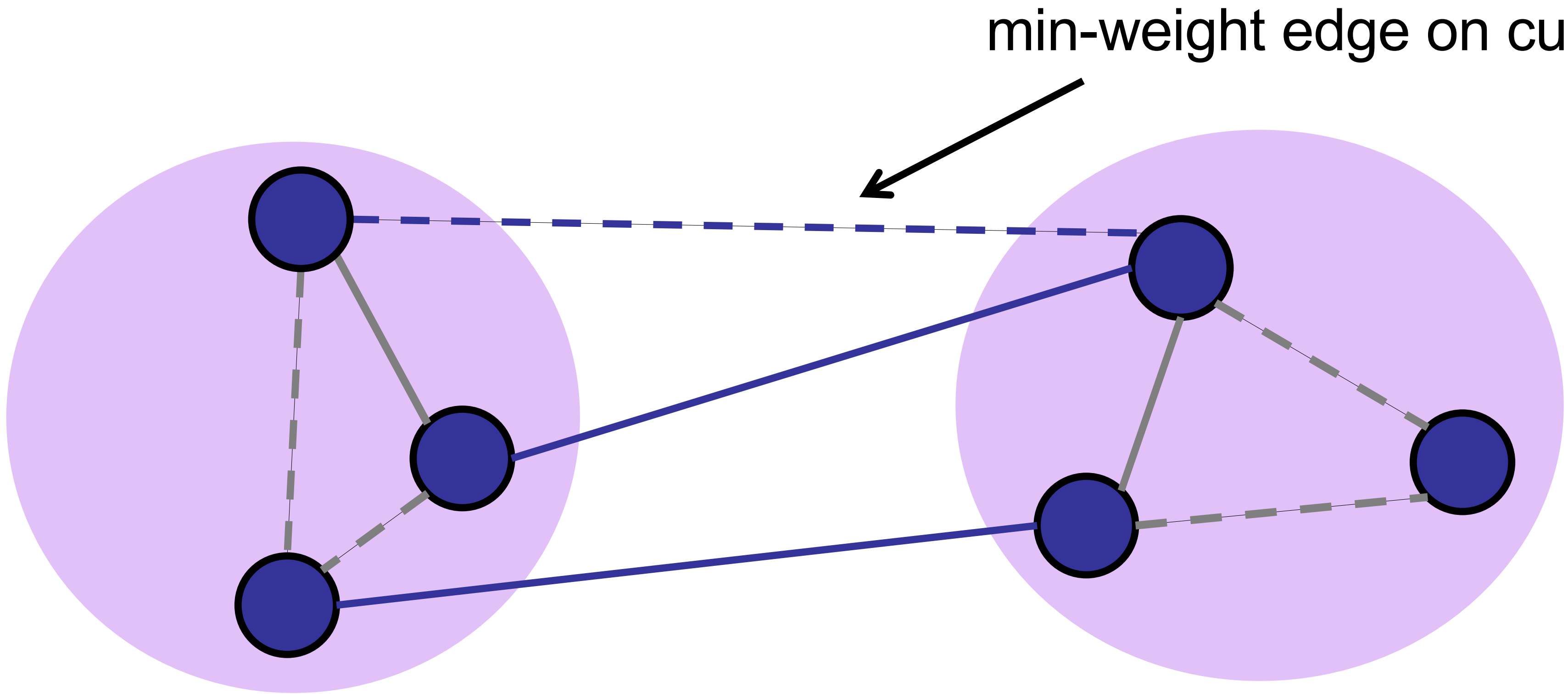
Properties of MST

Proof: Cut-and-paste

Assume not.

Add min-weight edge on cut.

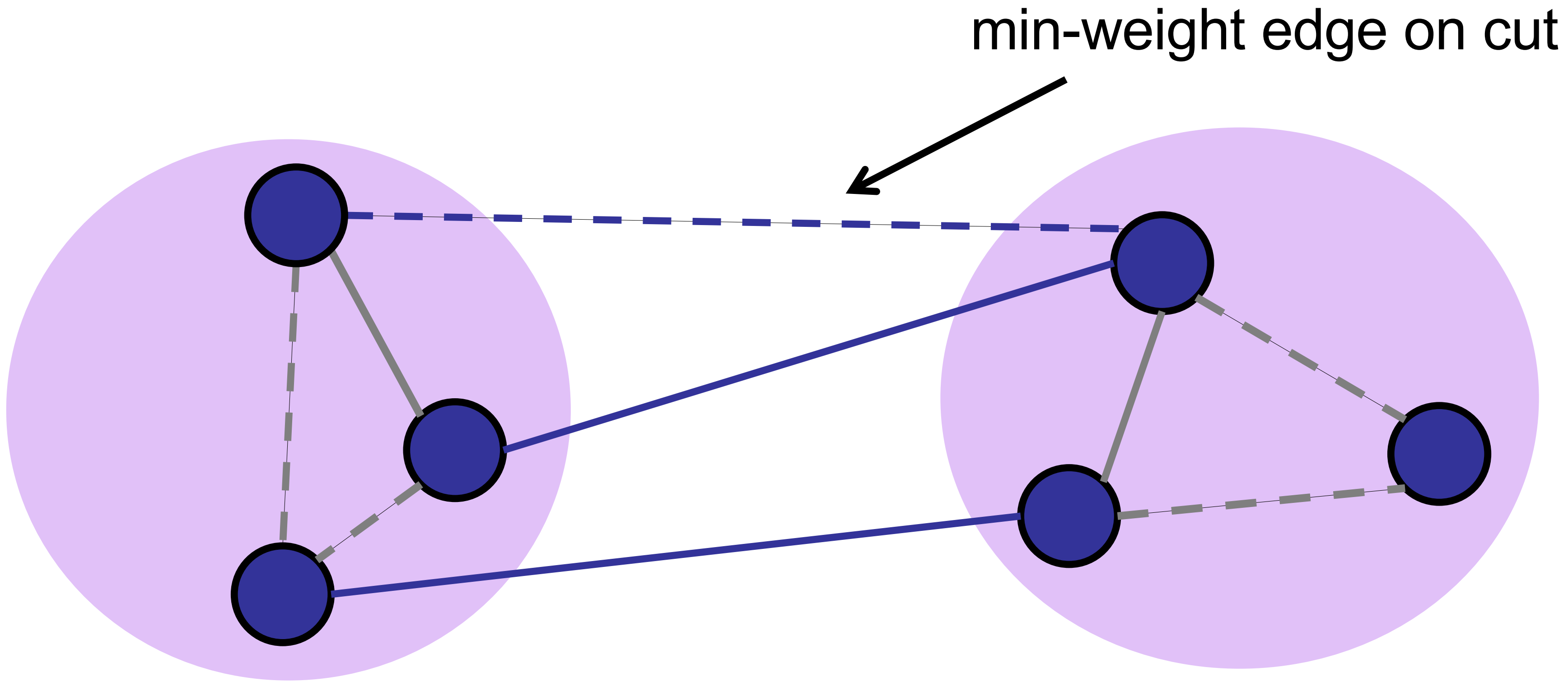
Remove heaviest edge on cycle.



Properties of MST

Proof: Cut-and-paste

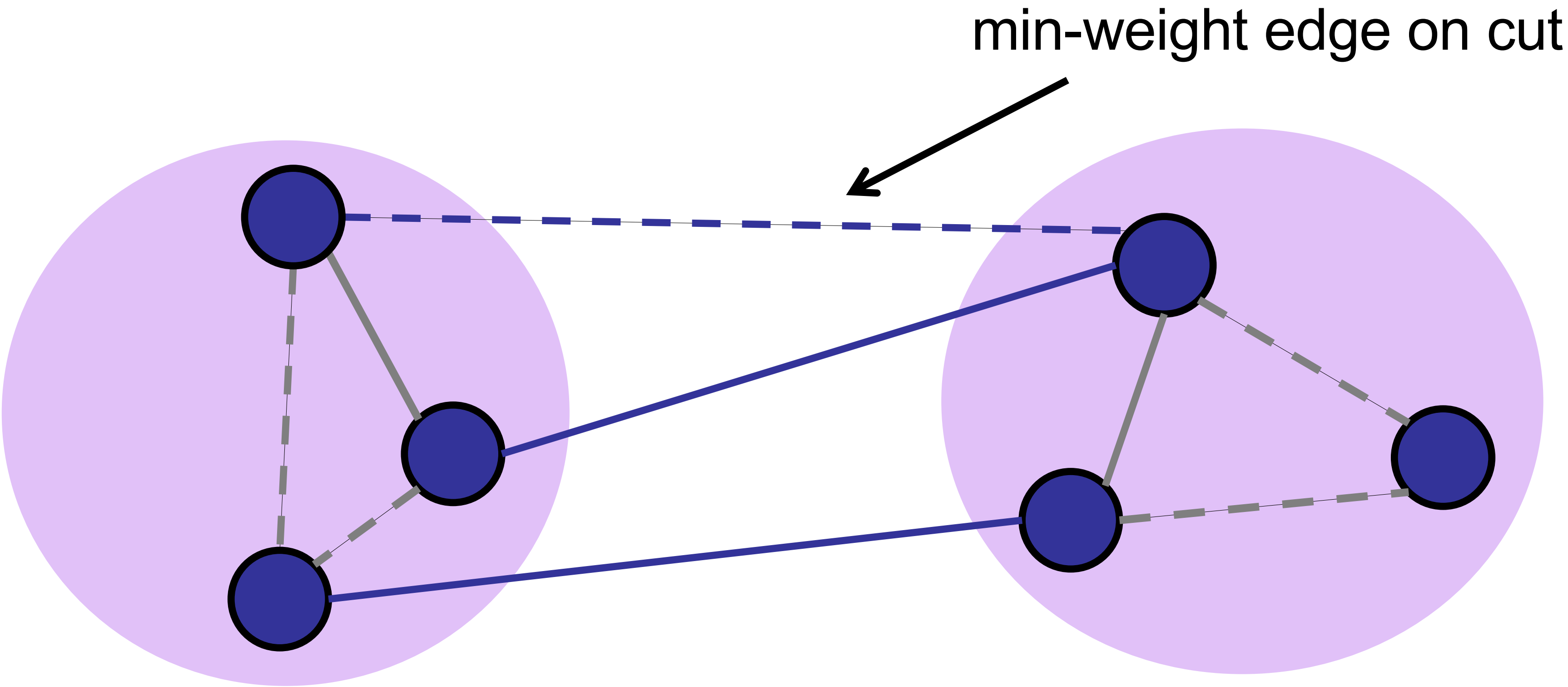
Result: a new spanning tree.



Properties of MST

Proof: Cut-and-paste

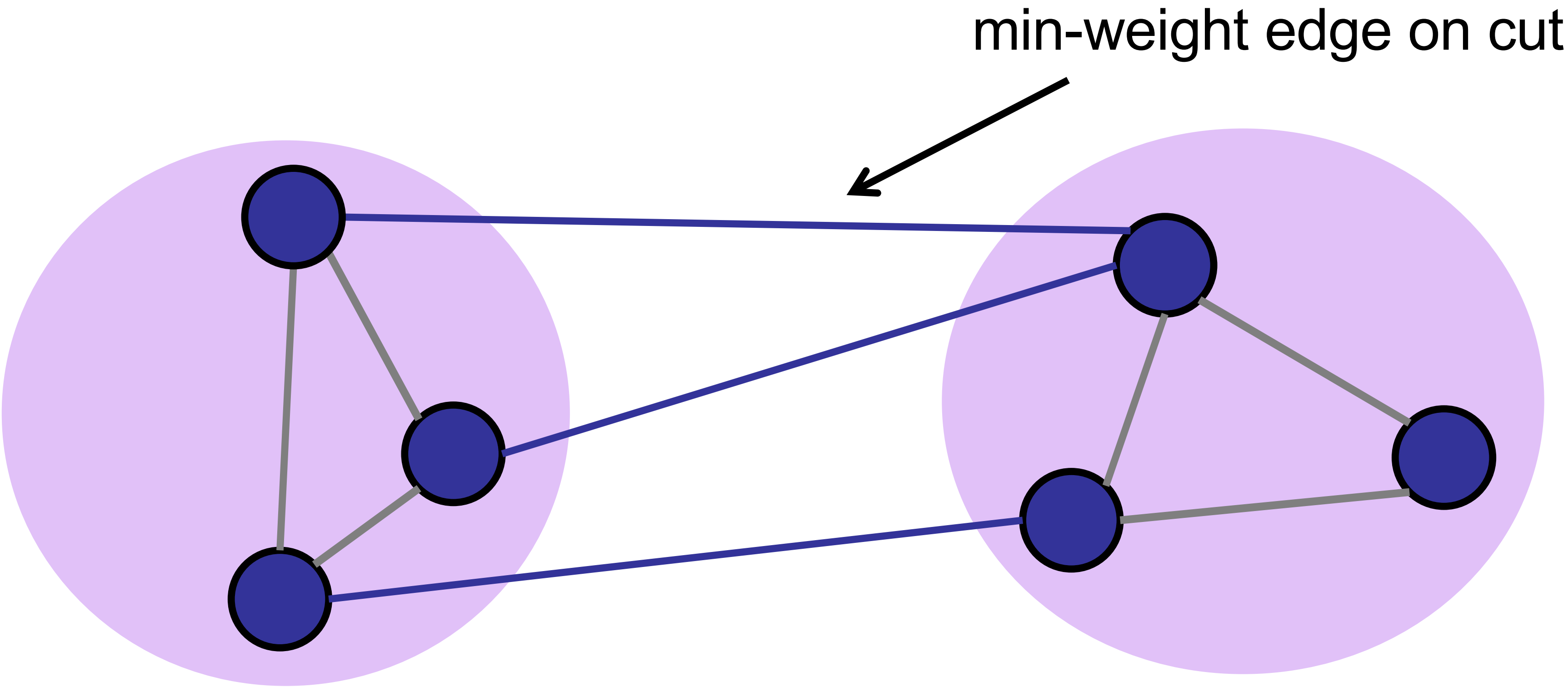
Less weight: replaced heavier edge with lighter edge.



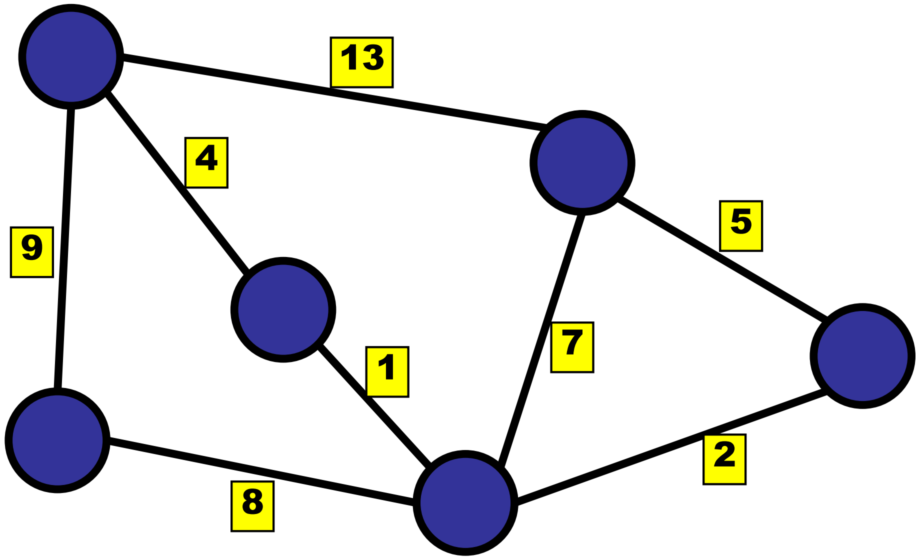
Properties of MST

Property 4: Cut property

For every partition of the nodes, the minimum weight edge across the cut *is* in the MST.



True or False:
For every vertex, the minimum outgoing edge is always part of the MST.



1. True
2. False
3. I don't know.

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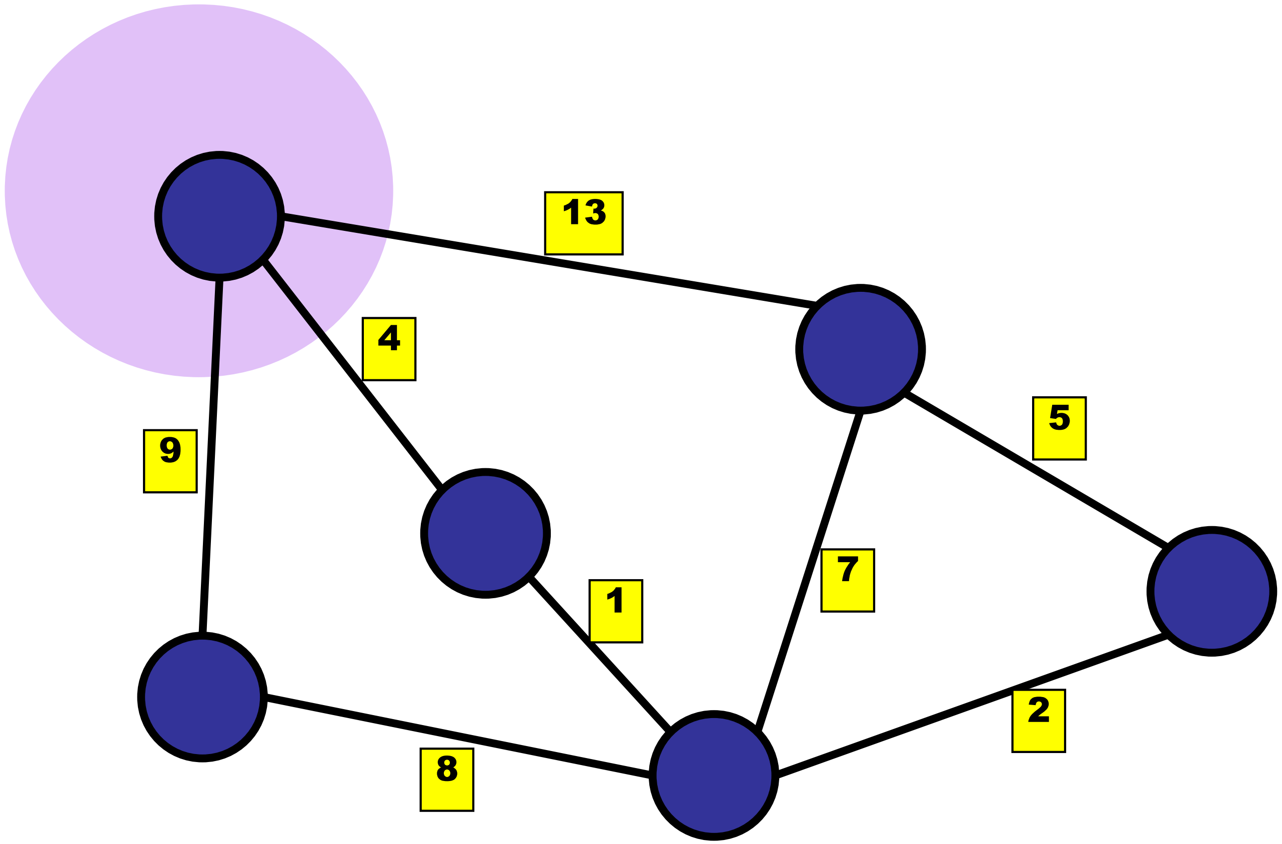
Properties of MST

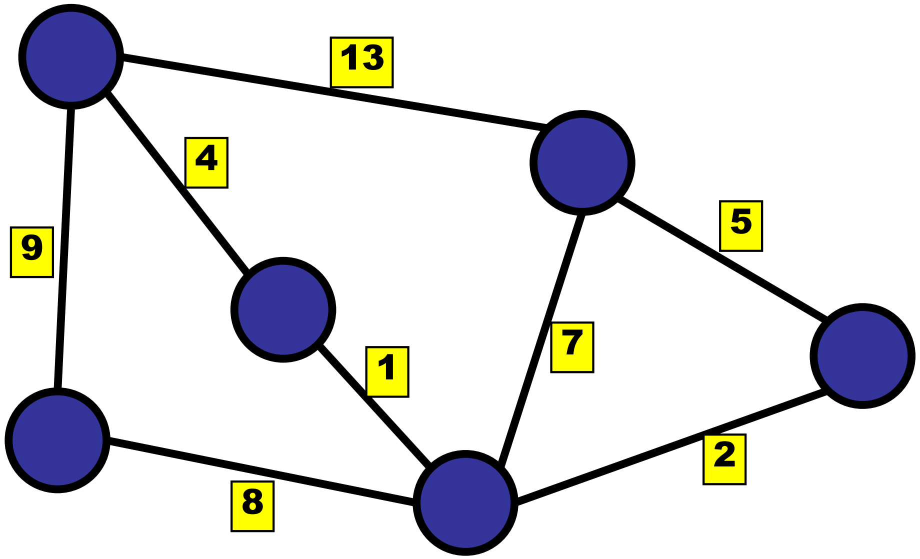
Property 4: Cut property

For every partition of the nodes, the minimum weight edge across the cut *is* in the MST.



True or False:
For every vertex, the maximum outgoing edge is never part of the MST.

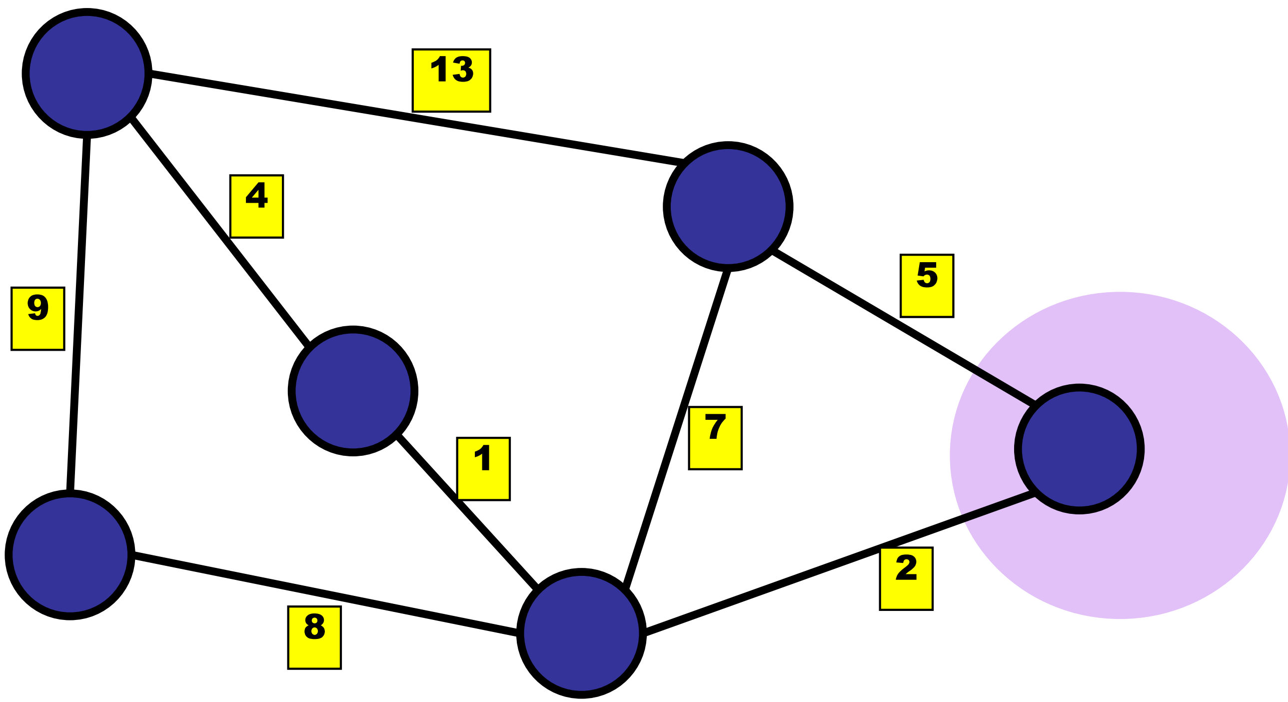
1. True
2. False
3. I don't know.



Properties of MST

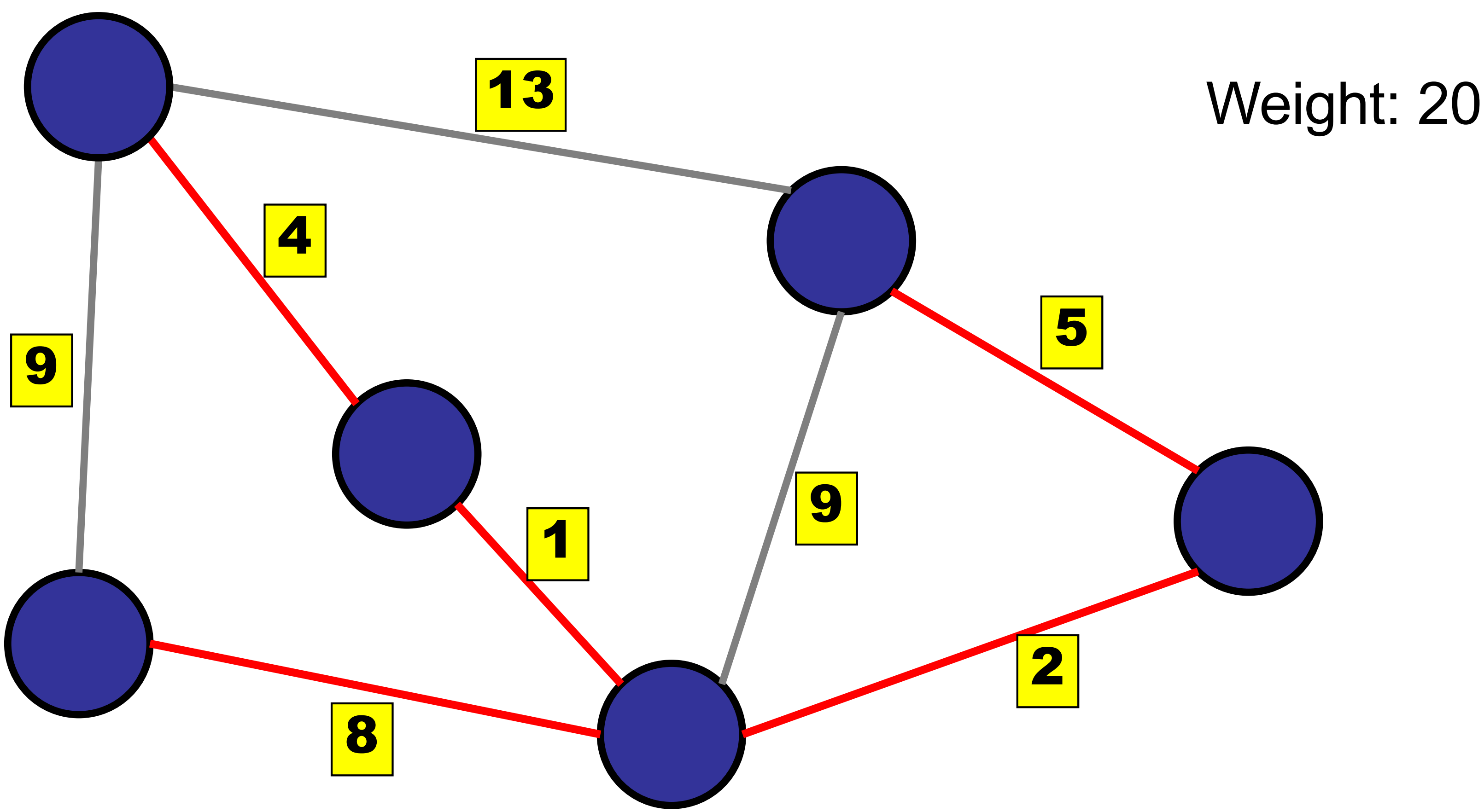
Property 4: Cut property

For every partition of the nodes, the minimum weight edge across the cut *is* in the MST.



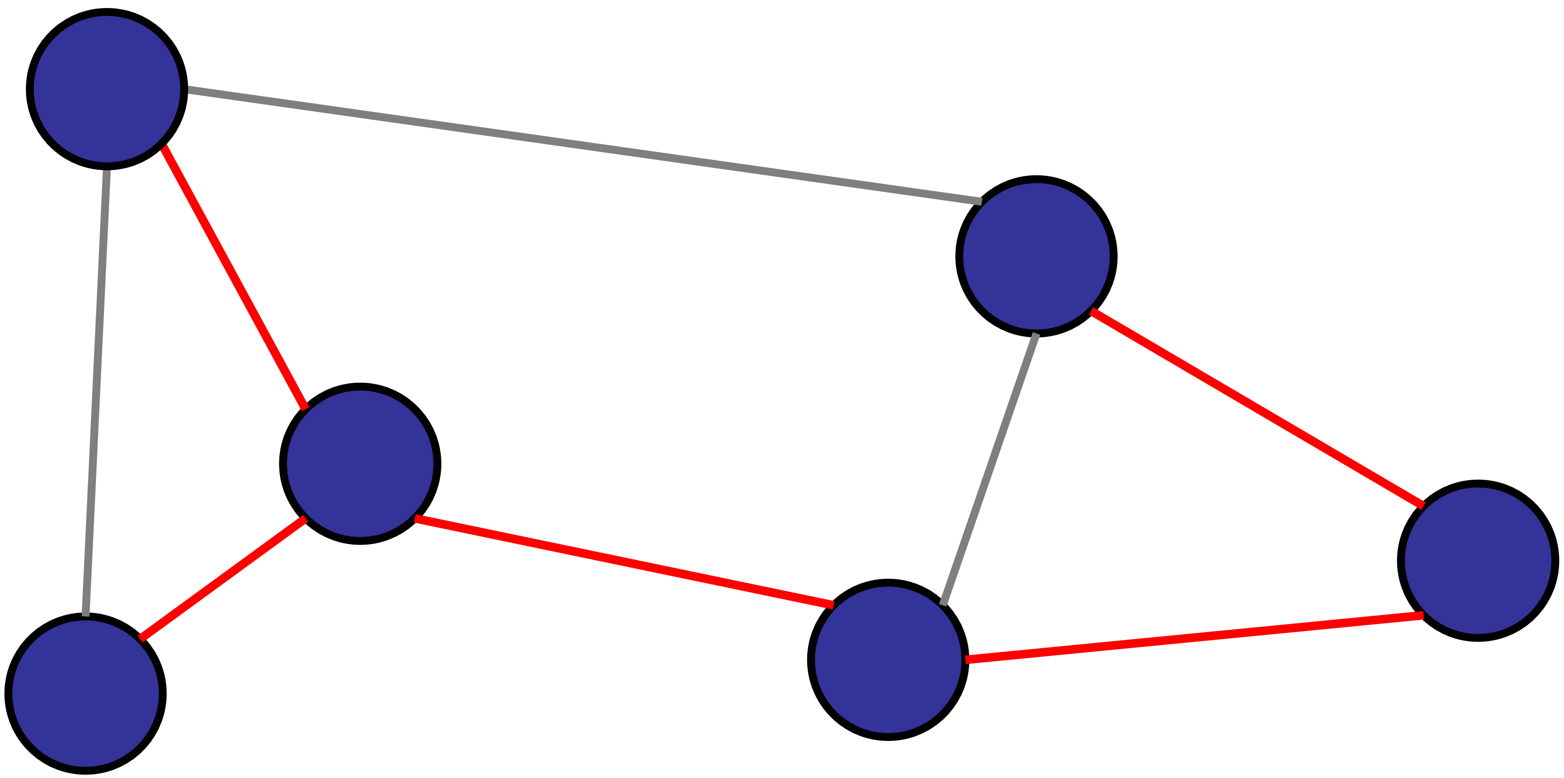
Minimum Spanning Tree

Definition: a spanning tree with minimum weight



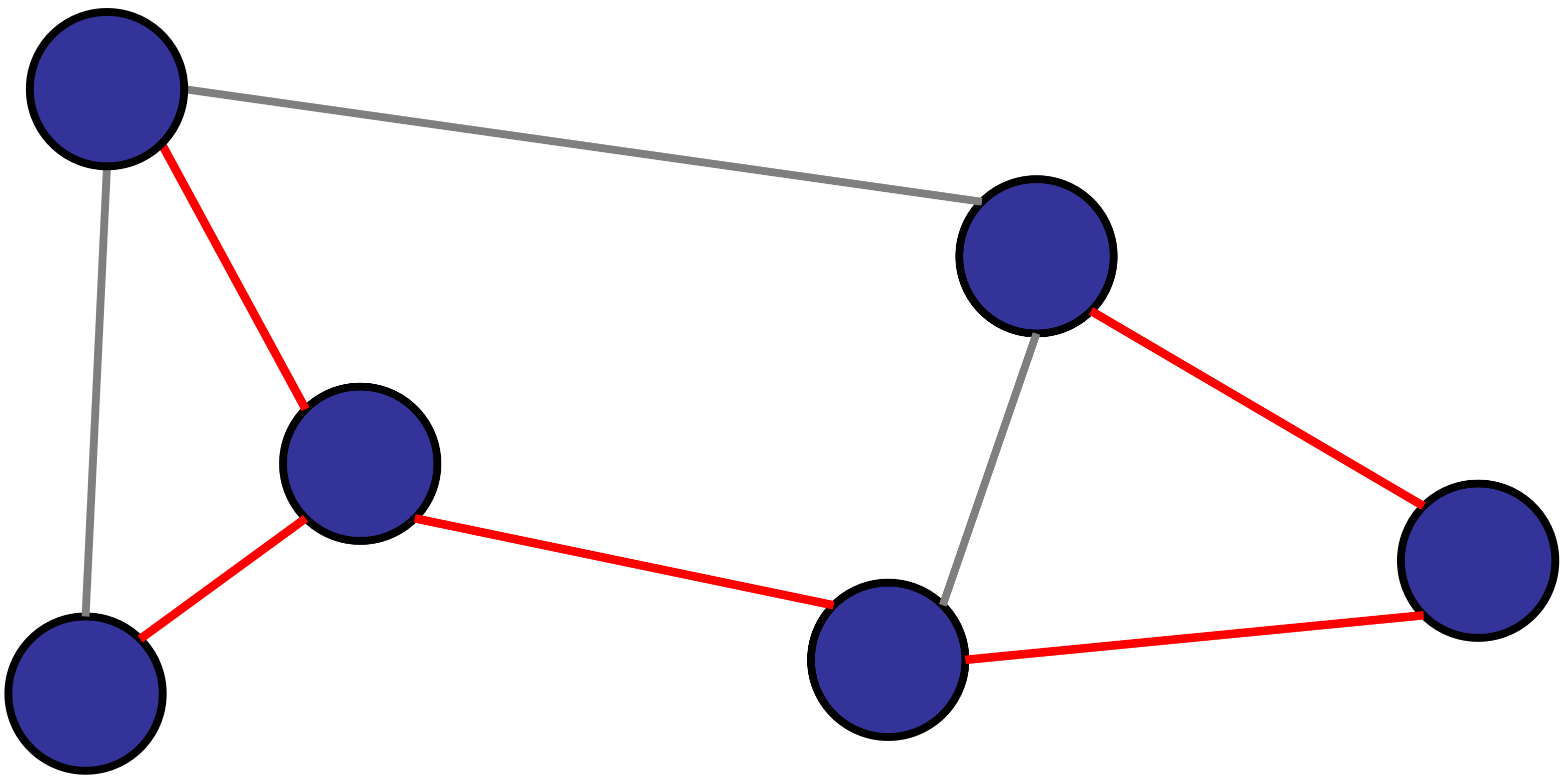
Properties of MST

Property 1: No cycles



Properties of MST

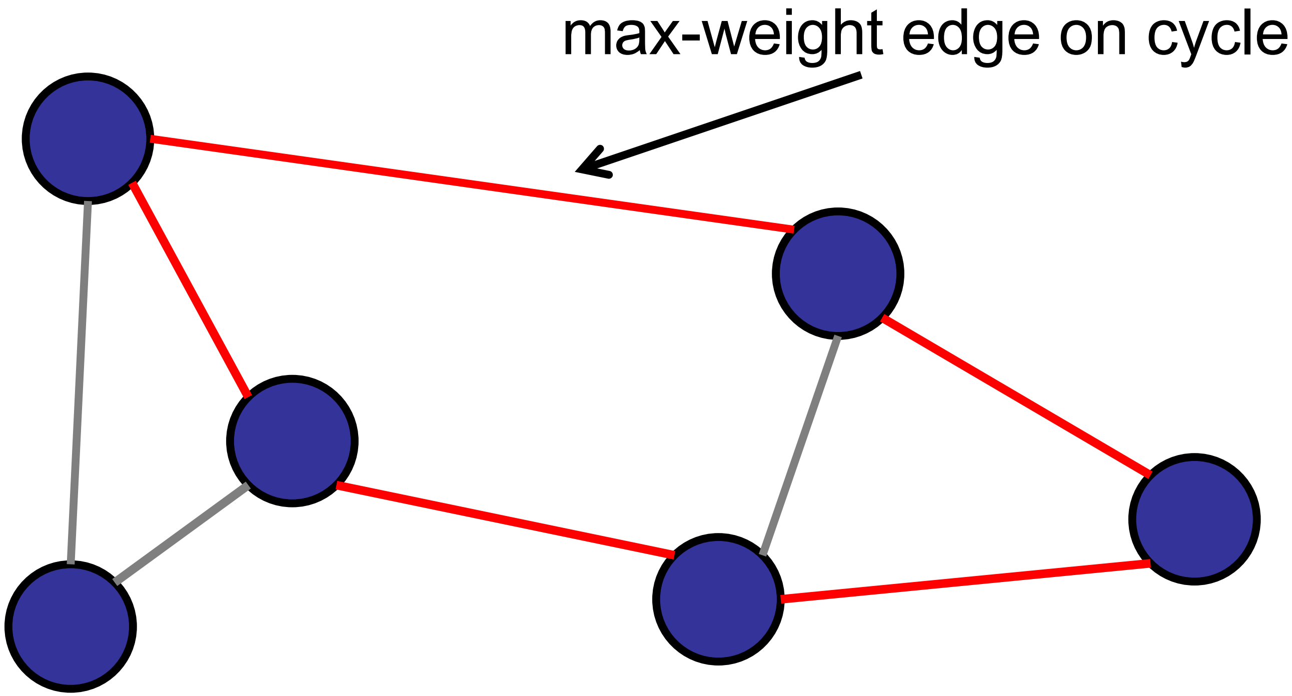
Property 2: If you cut an MST, the two pieces are both MSTs.



Properties of MST

Property 3: Cycle property

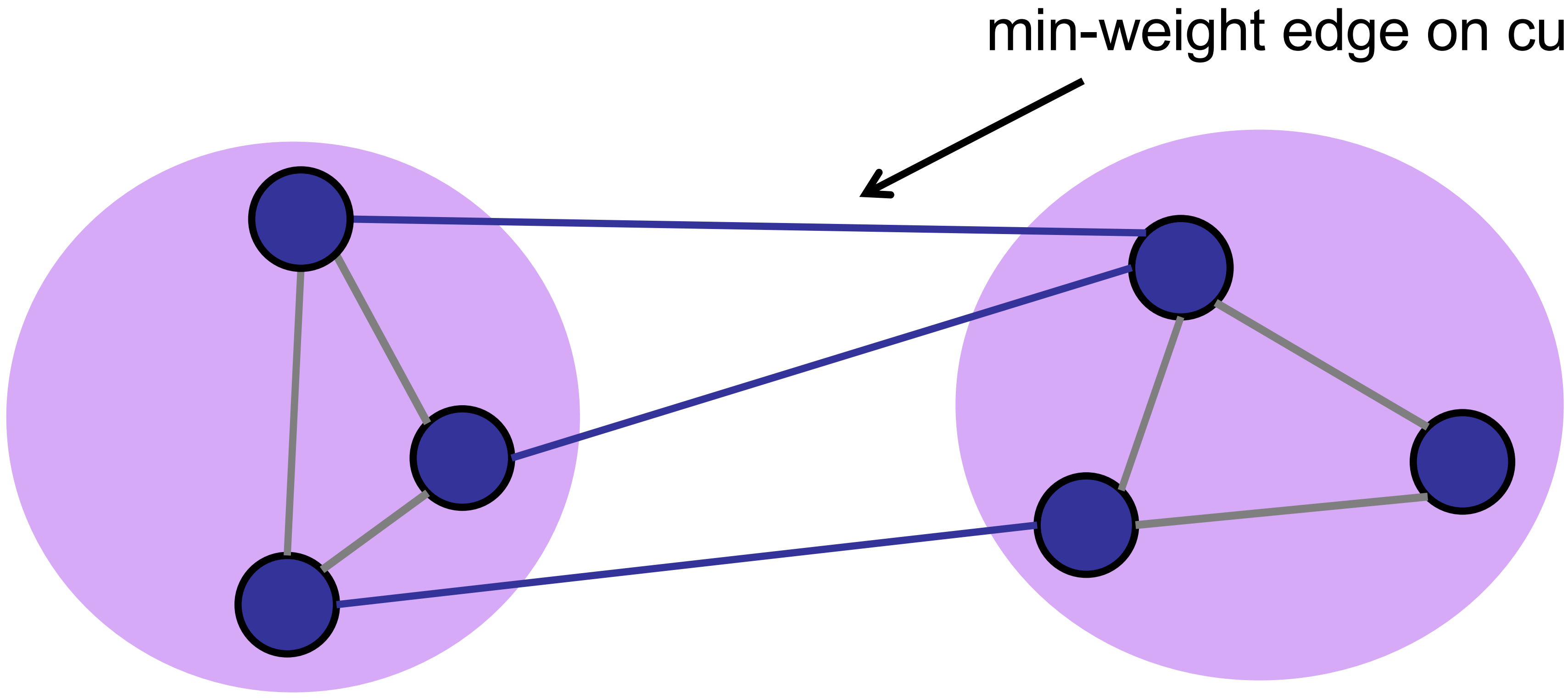
For every cycle, the maximum weight edge is *not* in the MST.



Properties of MST

Property 4: Cut property

For every cut D, the minimum weight edge that crosses the cut *is* in the MST.



Property of MST

- No cycles
- If you cut an MST, the two pieces are both MSTs.
- Cycle property
  - For every cycle, the maximum weight edge is not in the MST.
- Cut property
  - For every cut D, the minimum weight edge that crosses the cut is in the MST.

Roadmap

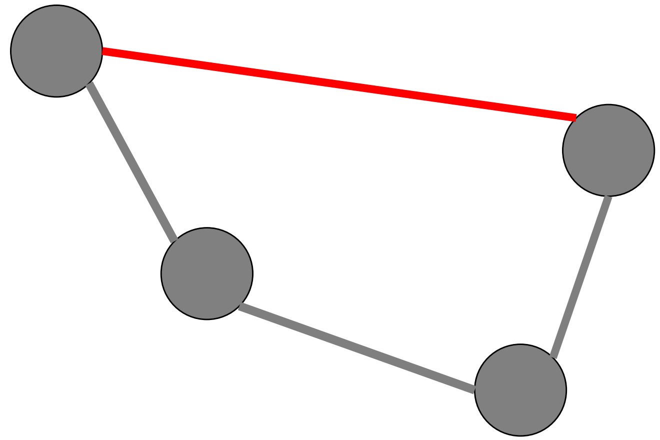
Minimum Spanning Trees

- The MST Problem
- Basic Properties of an MST
- **Generic MST Algorithm**
- Prim’s Algorithm
- Kruskal’s Algorithm
- Boruvka’s Algorithm
- Variations

Generic MST Algorithm

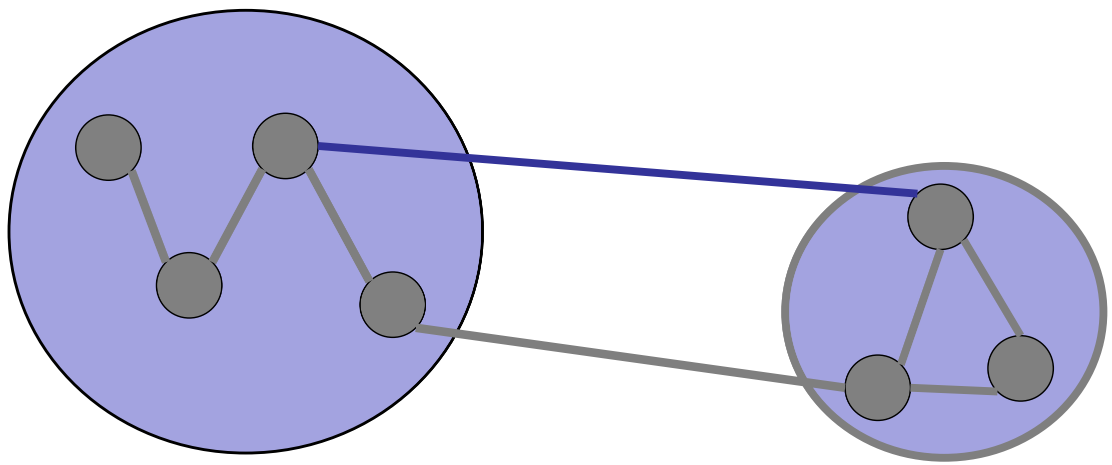
Red rule:

If C is a cycle with no red arcs, then color the max-weight edge in C red.



Blue rule:

If D is a cut with no blue arcs, then color the min-weight edge in D blue.



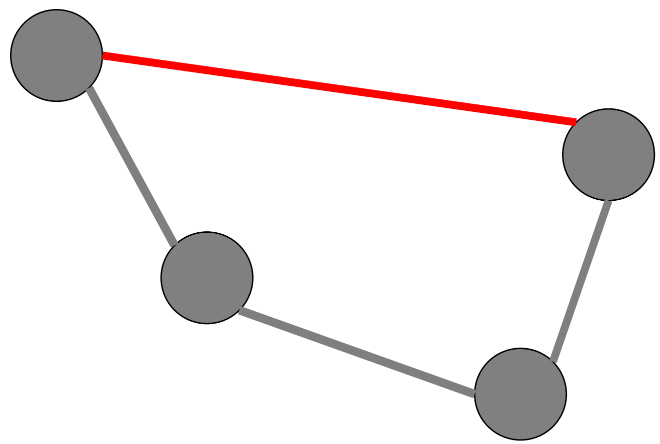
Generic MST Algorithm

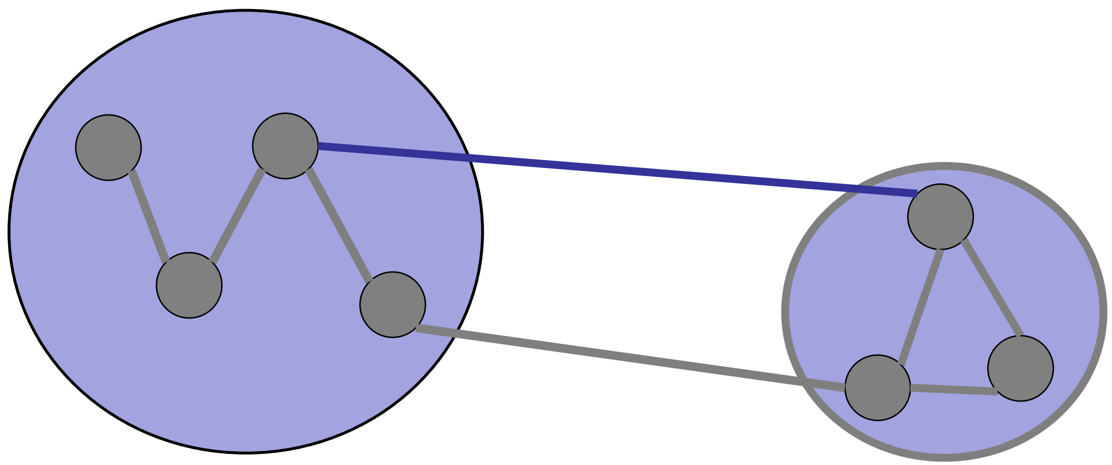
Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

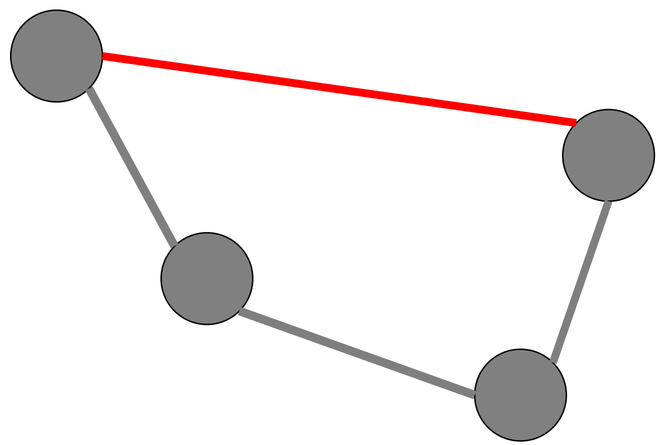
until no more edges can be colored.

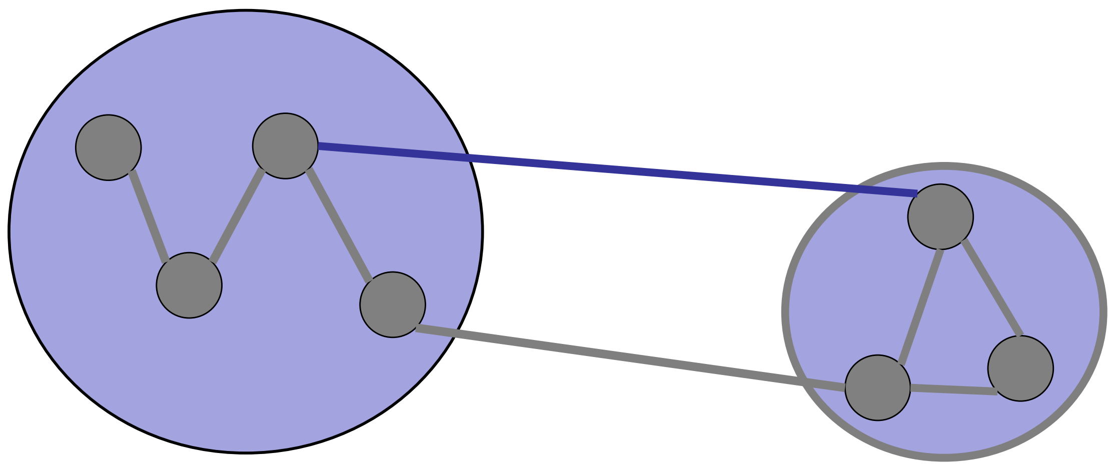




Generic MST Algorithm

Claim: On termination, the blue edges are an MST.



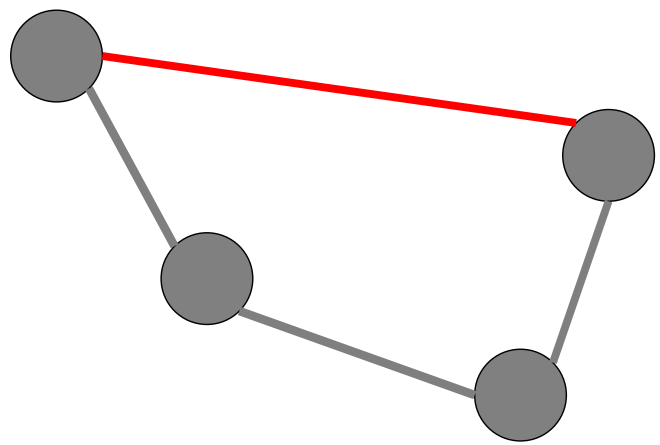


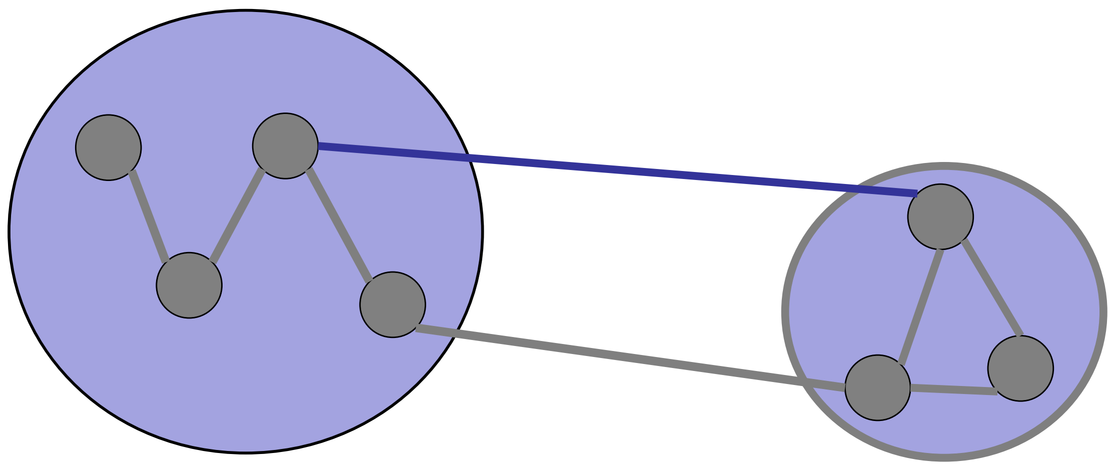
Generic MST Algorithm

Claim: On termination, the blue edges are an MST.

On termination:

1. Every cycle has a red edge.
   No blue cycles → forest.





Generic MST Algorithm

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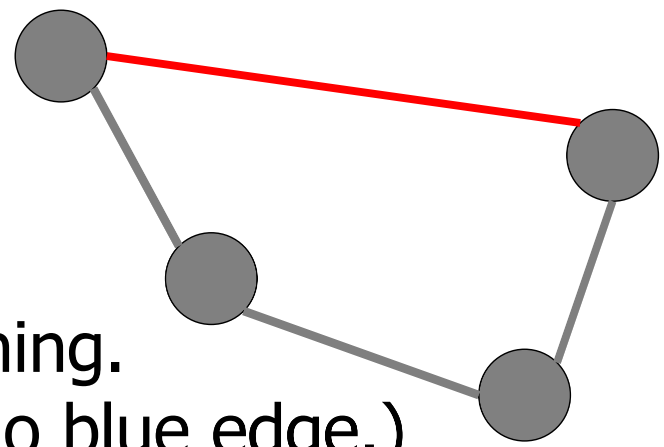
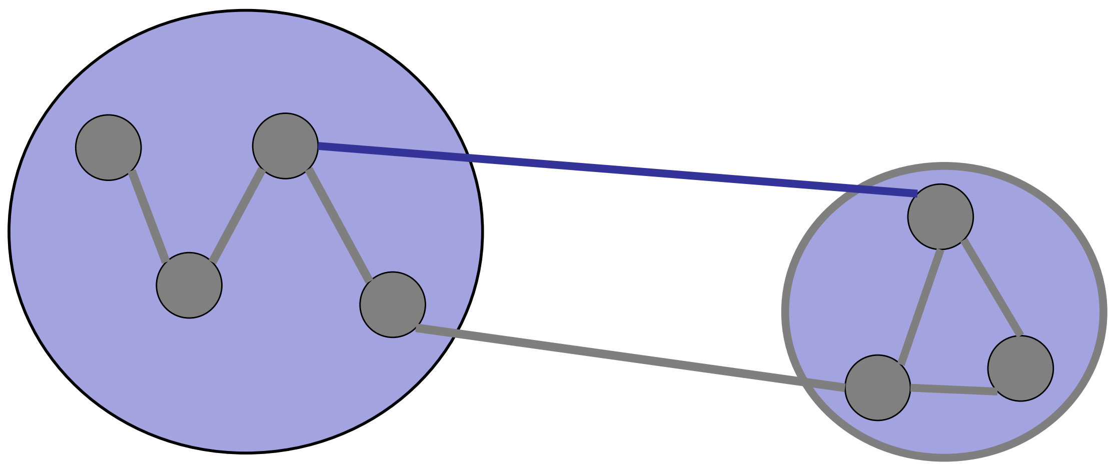


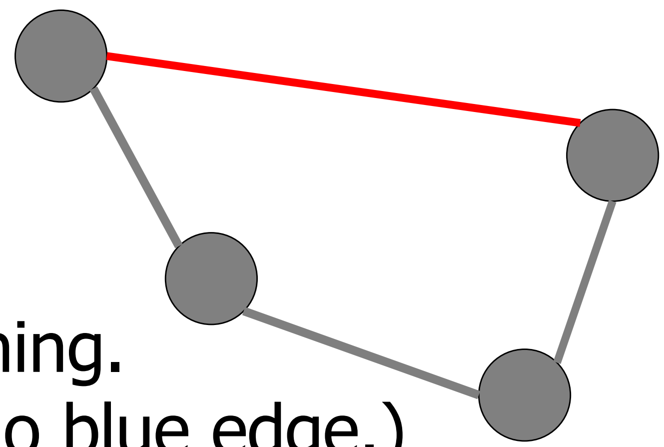
On termination:

1. Every cycle has a red edge.
   No blue cycles ➔ forest.
2. Blue edges form a tree ➔ spanning.
   (Otherwise, there is a cut with no blue edge.)



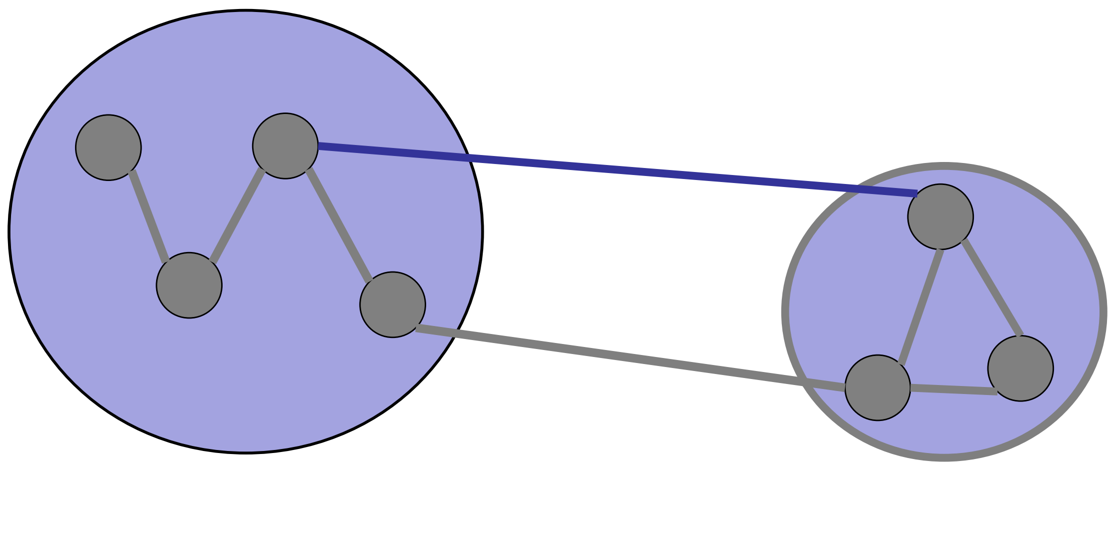
Generic MST Algorithm

Claim: On termination, the blue edges are an MST.



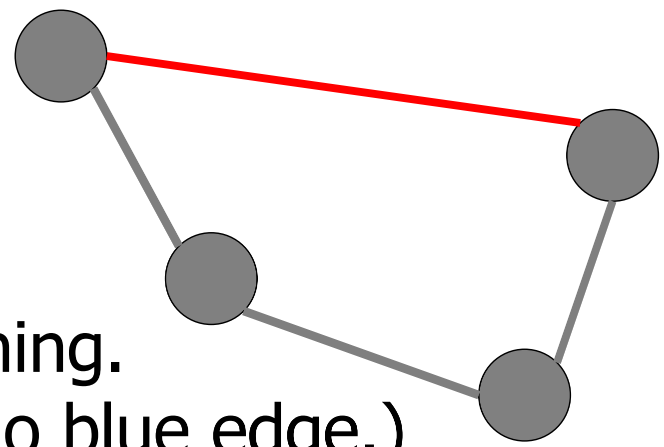
On termination:

1. Every cycle has a red edge.
   No blue cycles ➔ forest.
2. Blue edges form a tree ➔ spanning.
   (Otherwise, there is a cut with no blue edge.)
3. Every edge is colored.



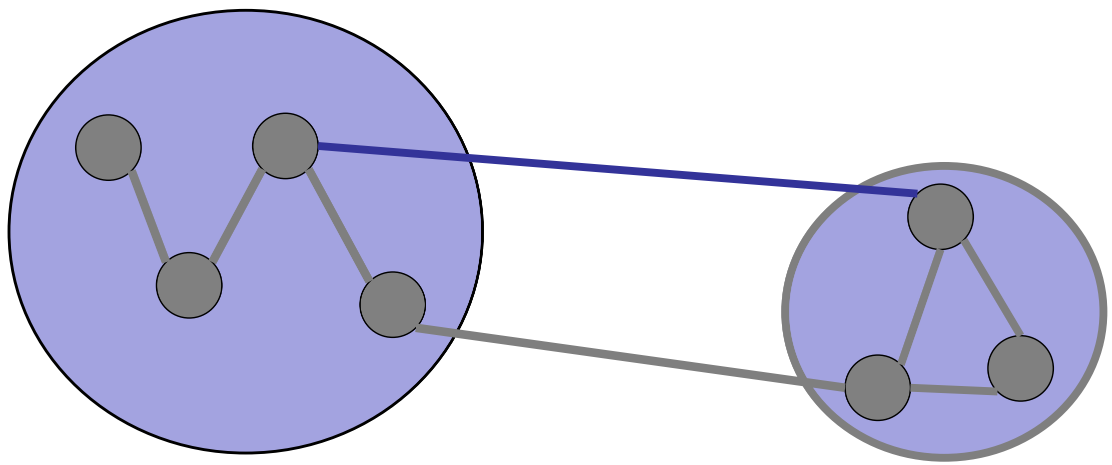
Generic MST Algorithm

Claim: On termination, the blue edges are an MST.



On termination:

1. Every cycle has a red edge.
   No blue cycles ➔ forest.
2. Blue edges form a tree ➔ spanning.
   (Otherwise, there is a cut with no blue edge.)
3. Every edge is colored.
4. Every blue edge is in
   the MST (Property 4).



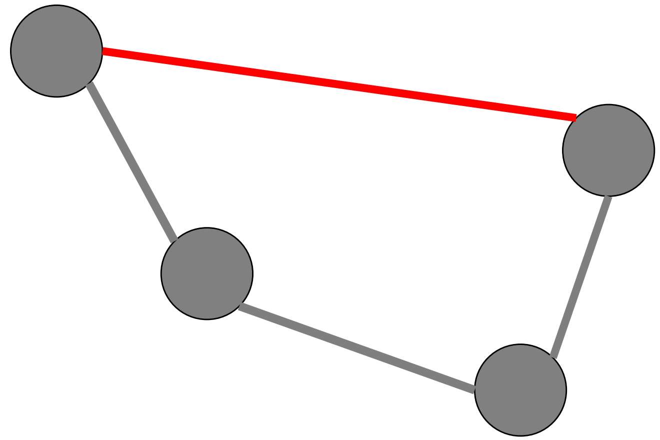
Generic MST Algorithm

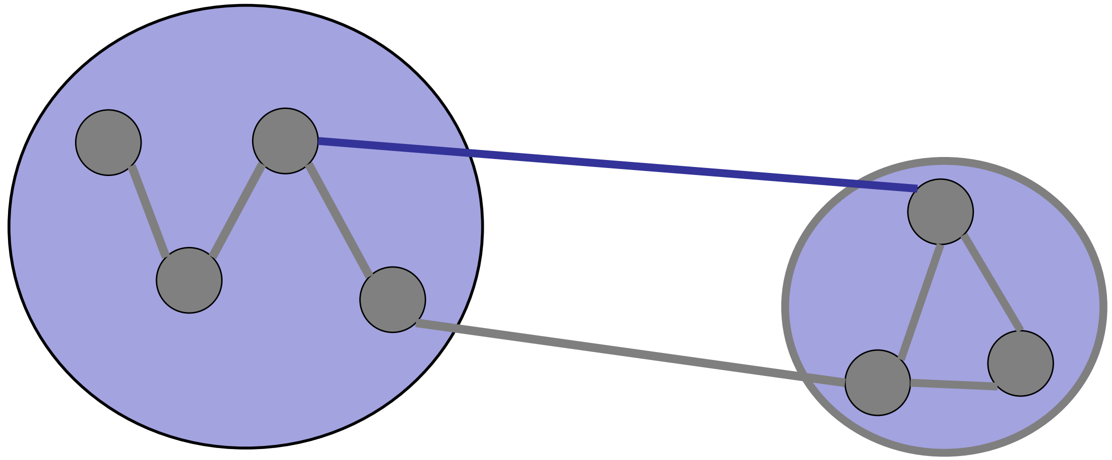
Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

until no more edges can be colored.





Candidate MST Algorithm

Divide-and-Conquer:

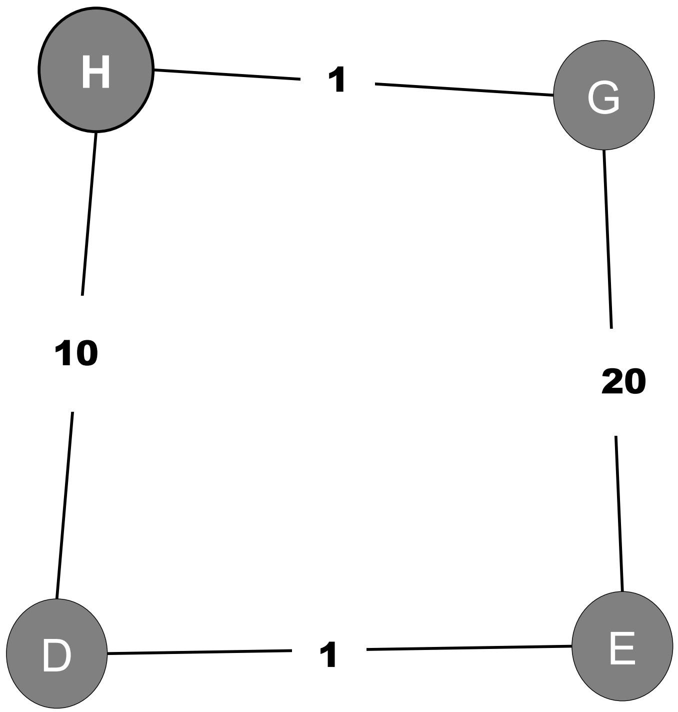
1. If the number of vertices is 1, then return.
2. Divide the nodes into two sets.
3. Recursively calculate the MST of each set.
4. Find the lightest edge the connects the two sets and add it to the MST.
5. Return.

The problem with this algorithm is?

1. Nothing. It efficiently implements the red-blue strategy.
2. It is too expensive to implement because finding the lightest edge is hard.
3. It is too expensive to implement because partitioning the nodes is expensive.
4. It returns the wrong answer.

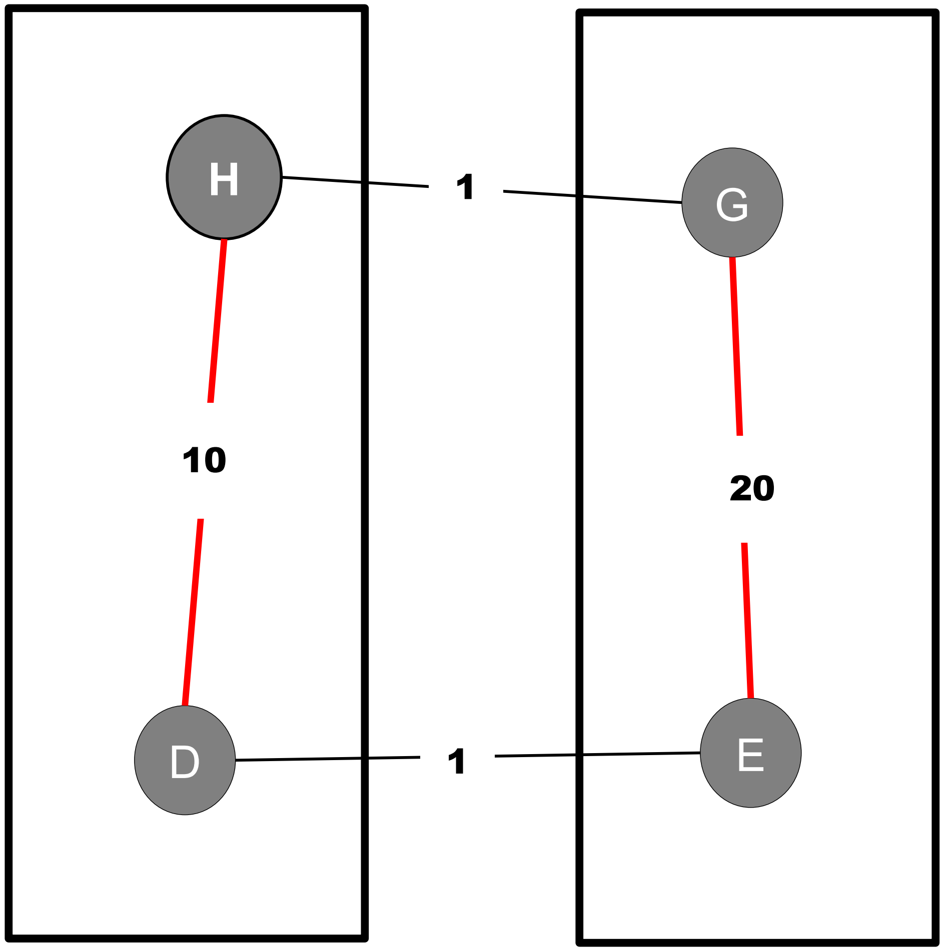
Candidate MST Algorithm

Example:



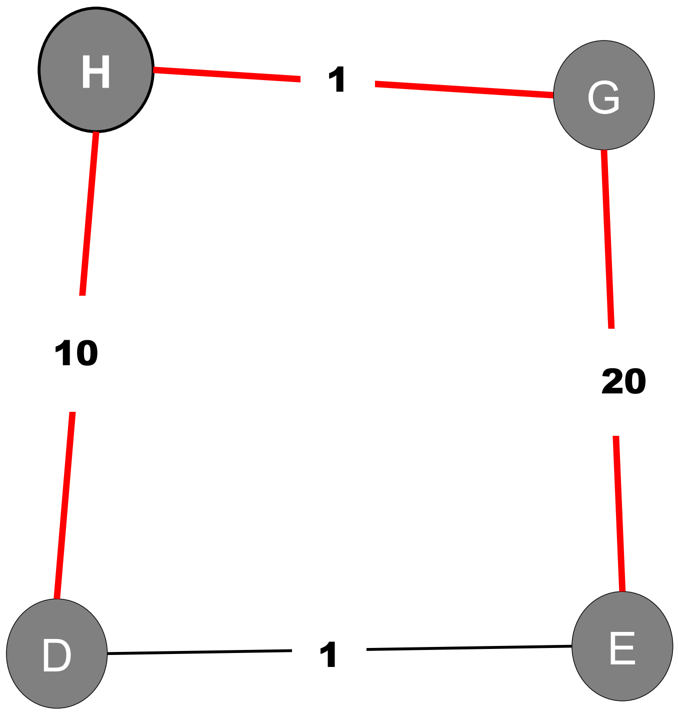
Candidate MST Algorithm

Example: Divide-and-Conquer



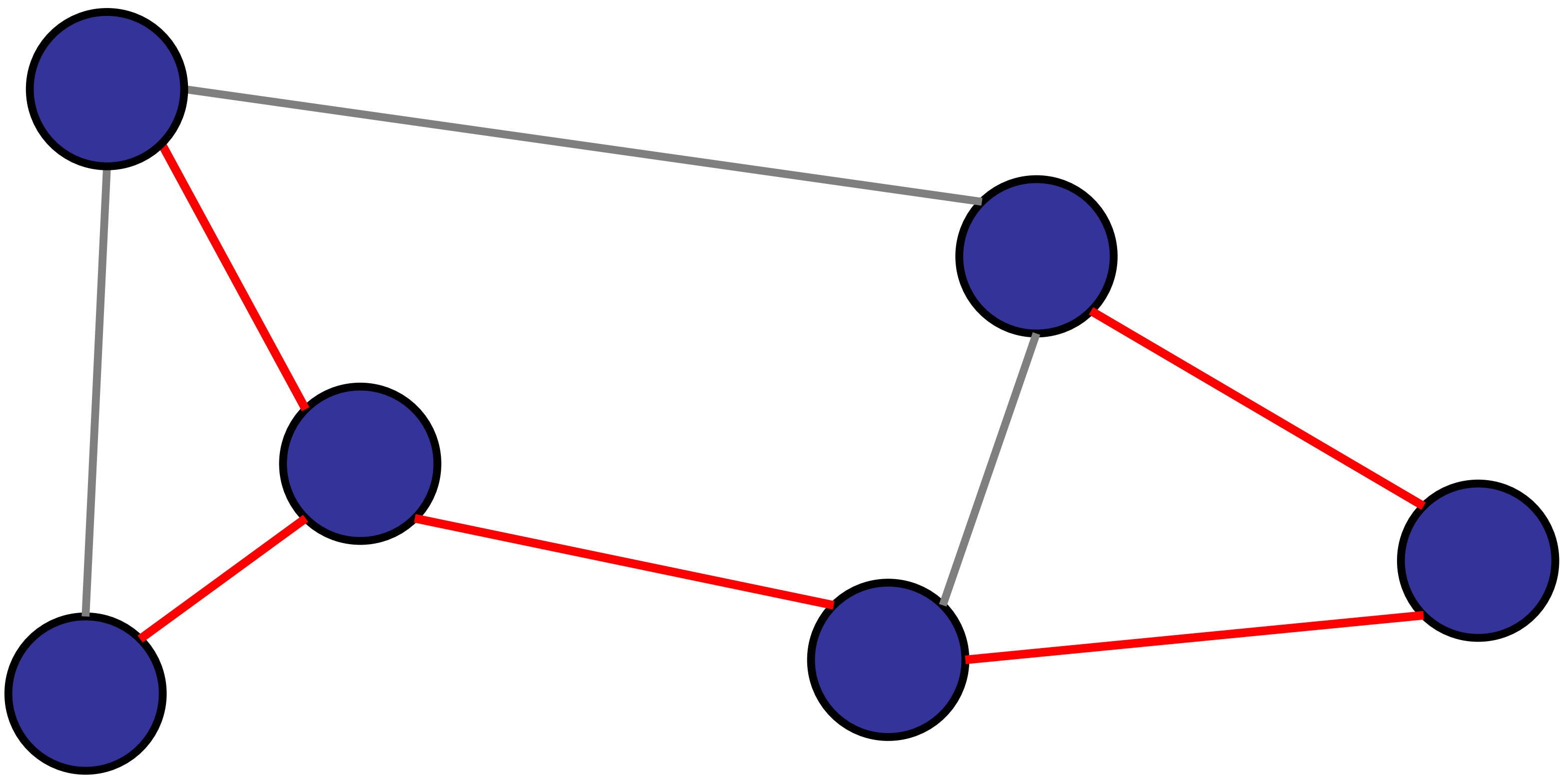
Candidate MST Algorithm

Example: Divide-and-Conquer



Properties of MST

Property 2: If you cut an MST, the two pieces are both MSTs.



BAD MST Algorithm

Divide-and-Conquer:

1. If the number of vertices is 1, then return.
2. Divide the nodes into two sets.
3. Recursively calculate the MST of each set.
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Generic MST Algorithm

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Repeat:

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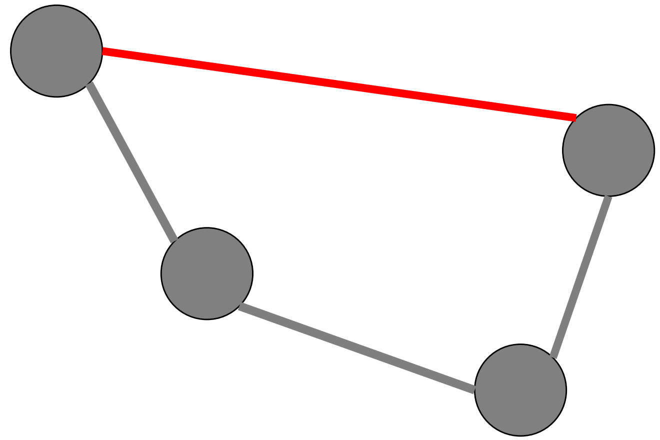
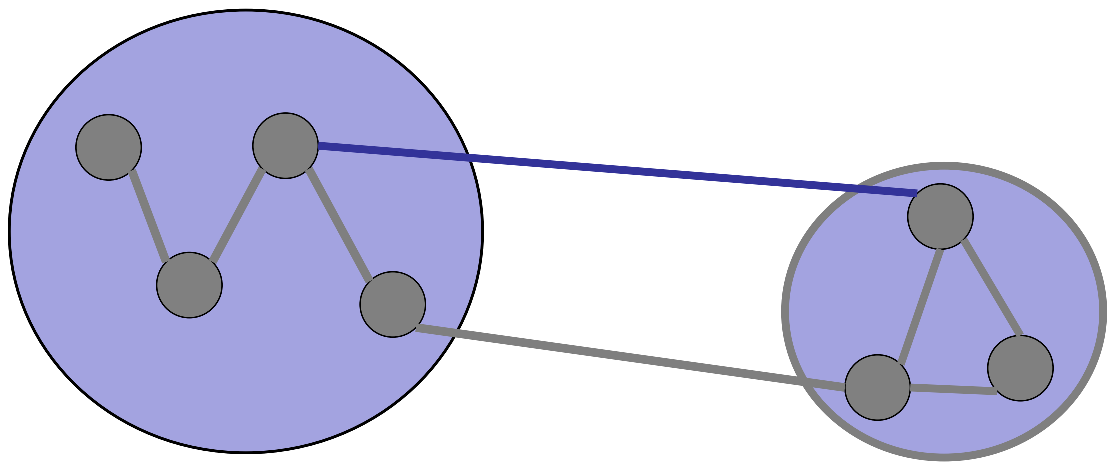




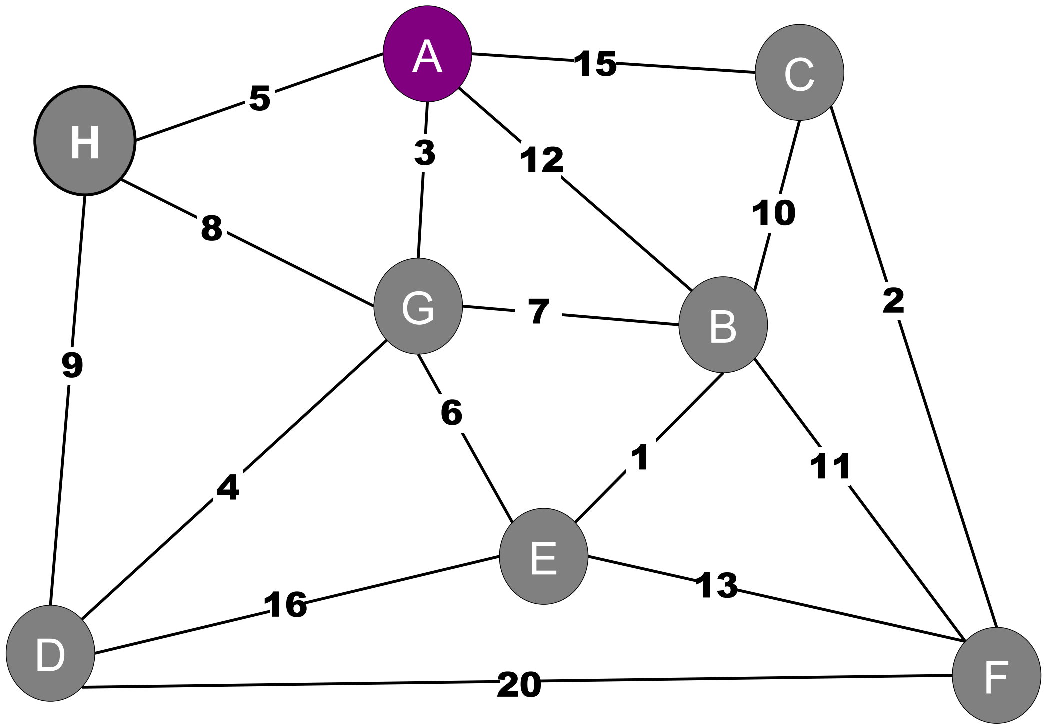
Roadmap

Minimum Spanning Trees

- The MST Problem
- Basic Properties of an MST
- Generic MST Algorithm
- **Prim's Algorithm**
- Kruskal's Algorithm
- Boruvka's Algorithm
- Variations

Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

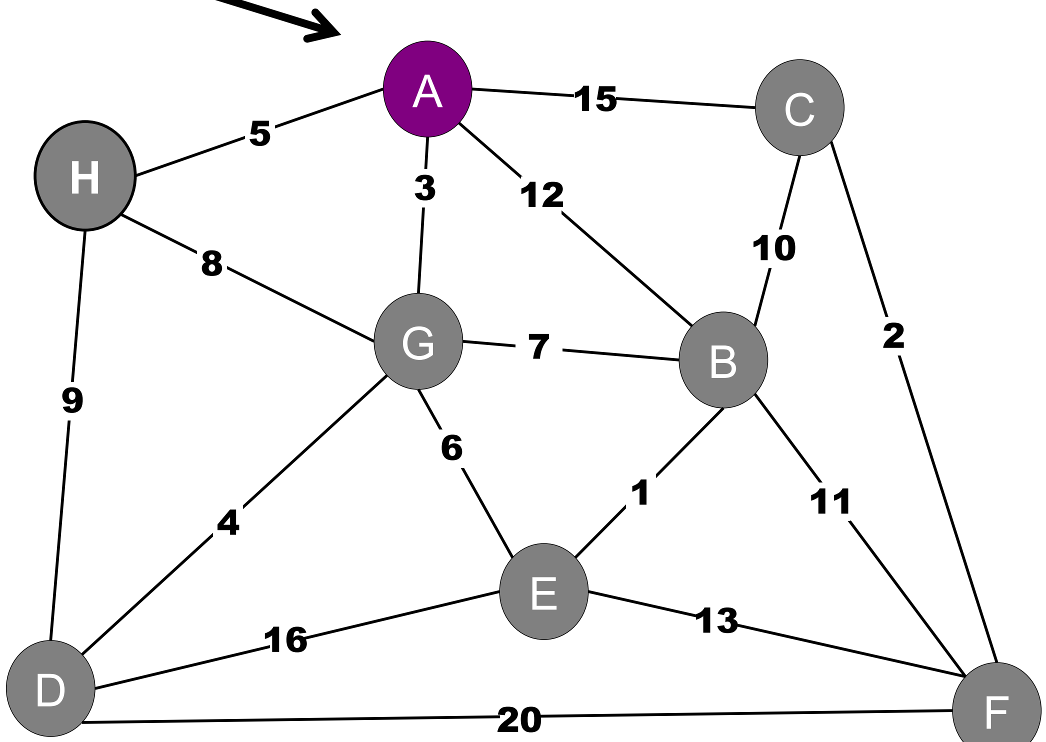


Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges.
- Initially: S = {A}

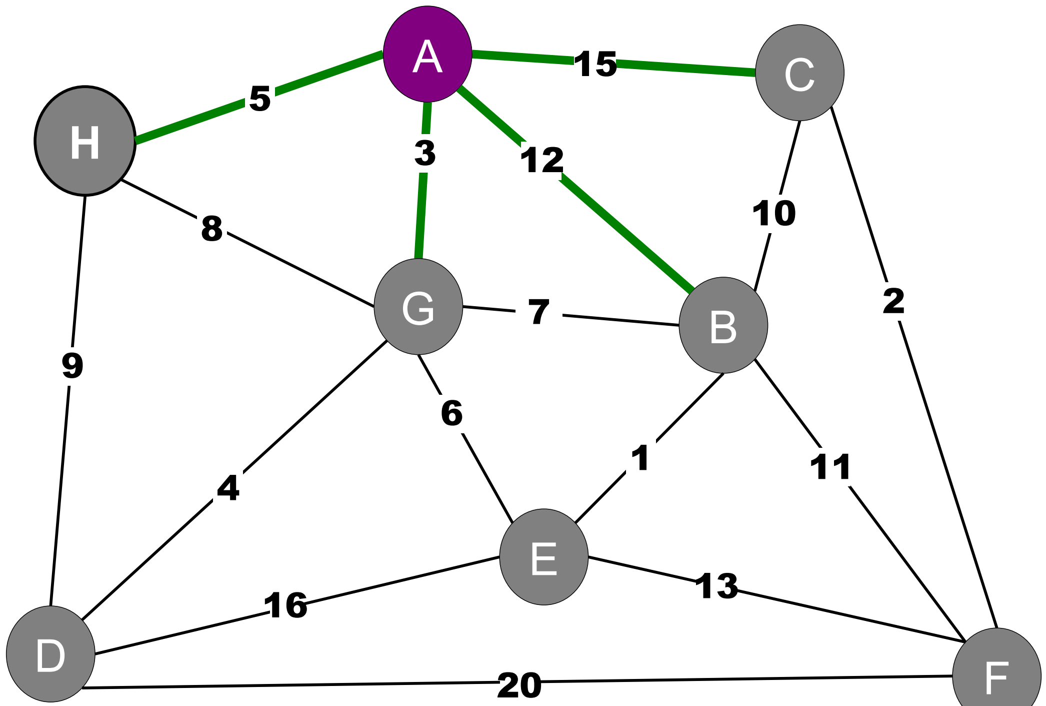


Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges.
- Initially: S = {A}
- Identify cut: {S, V–S}

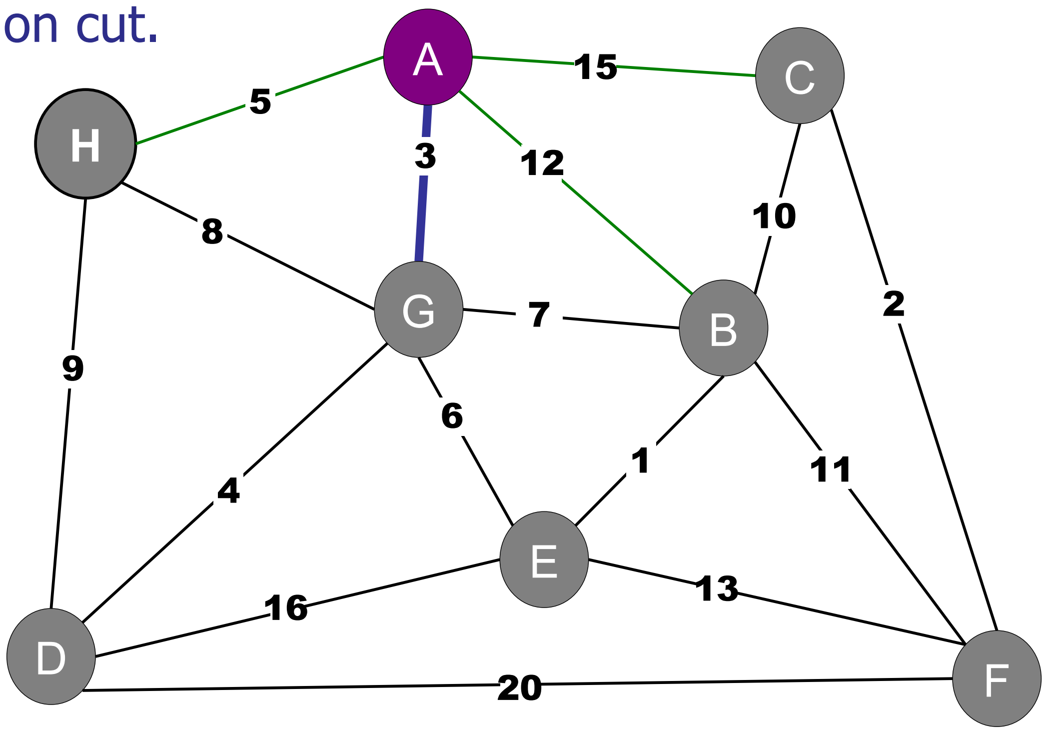


Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges.
- Initially: S = {A}
- Identify cut: {S, V–S}
- Find minimum weight edge on cut.

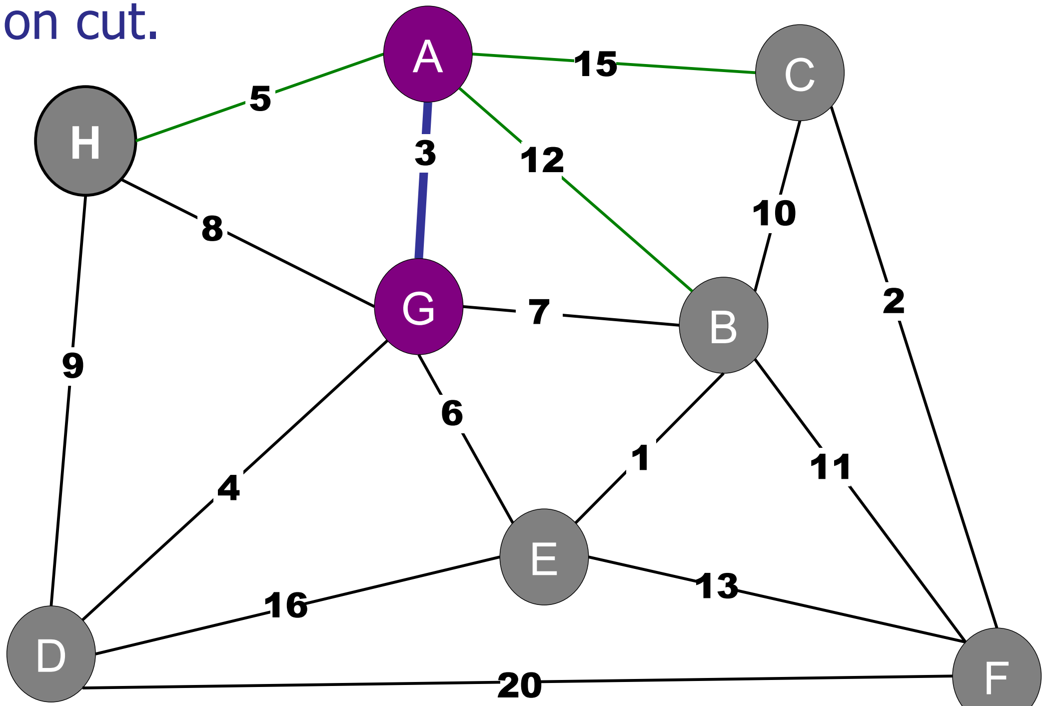


Prim's Algorithm

Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges.
- Initially: S = {A}
- Identify cut: {S, V–S}
- Find minimum weight edge on cut.
- Add new node to S.



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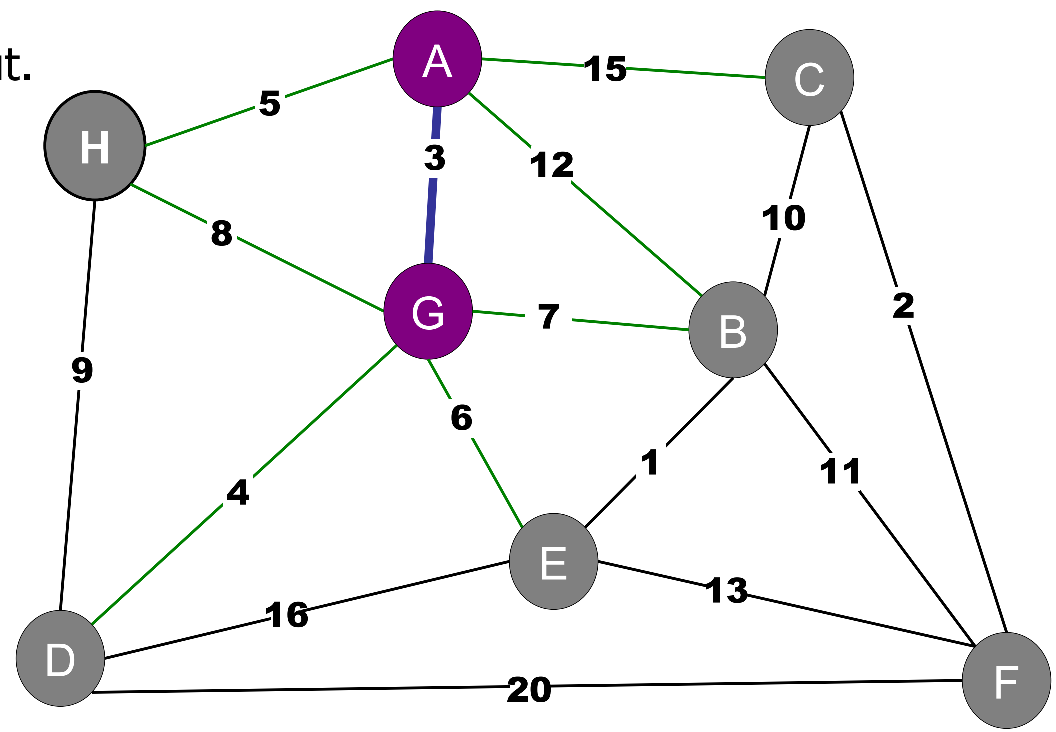

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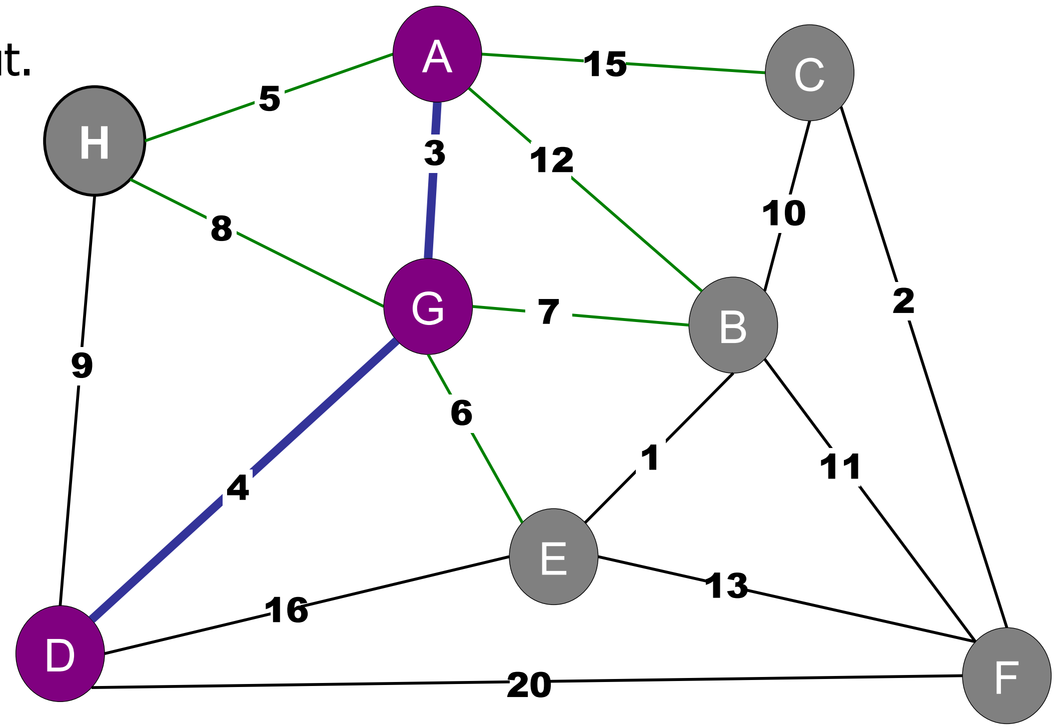

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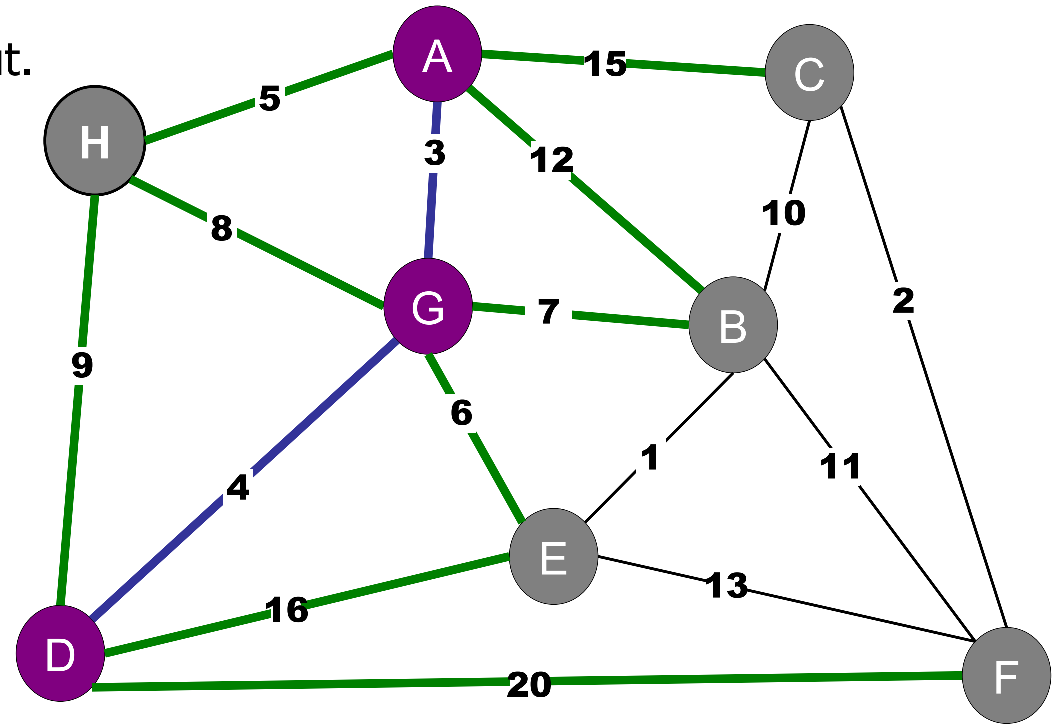
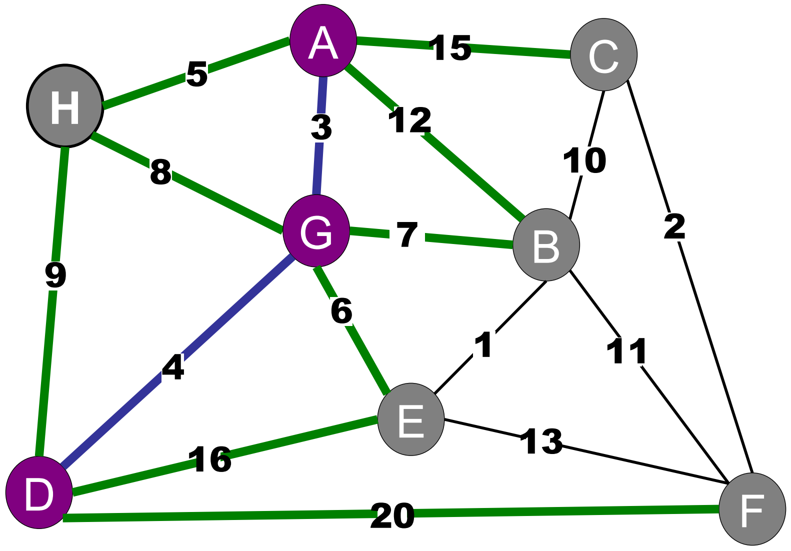
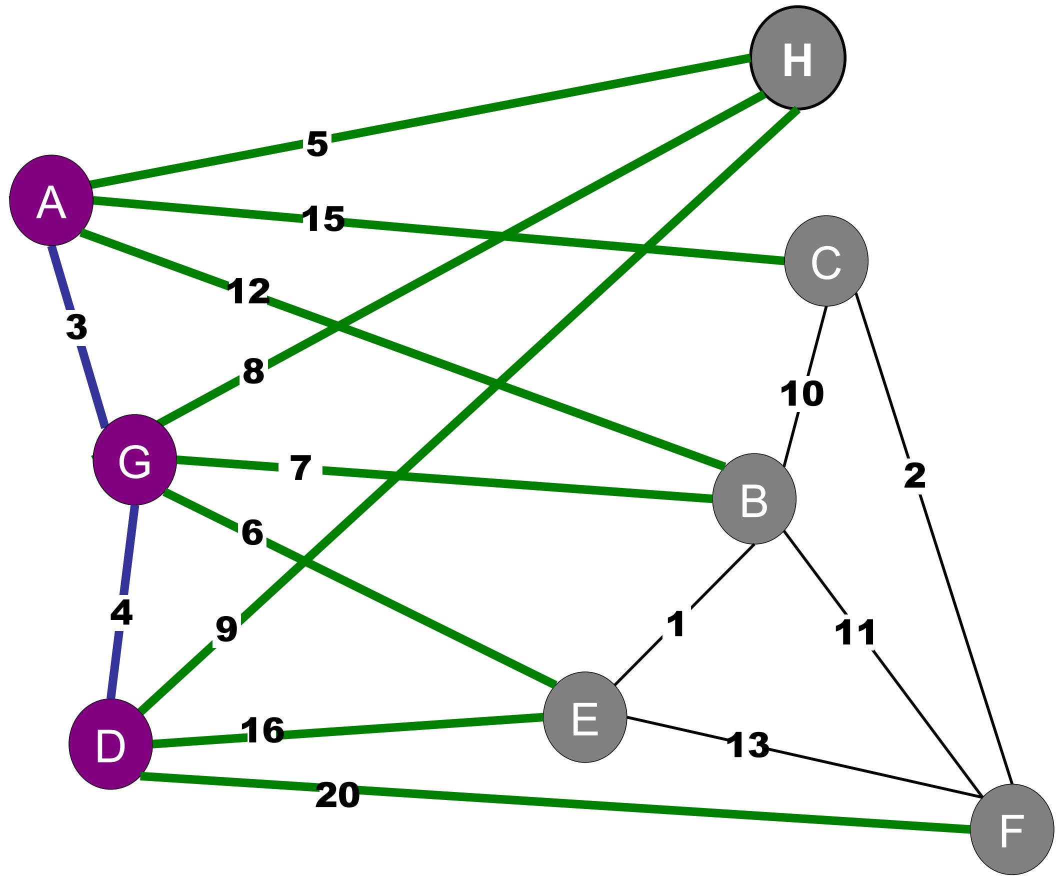


Prim's Algorithm





How do we find the lightest edge on a cut?

1. Priority Queue
2. Union-Find
3. Max-flow / Min-cut
4. BFS
5. DFS

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Prim's Algorithm: Initialization

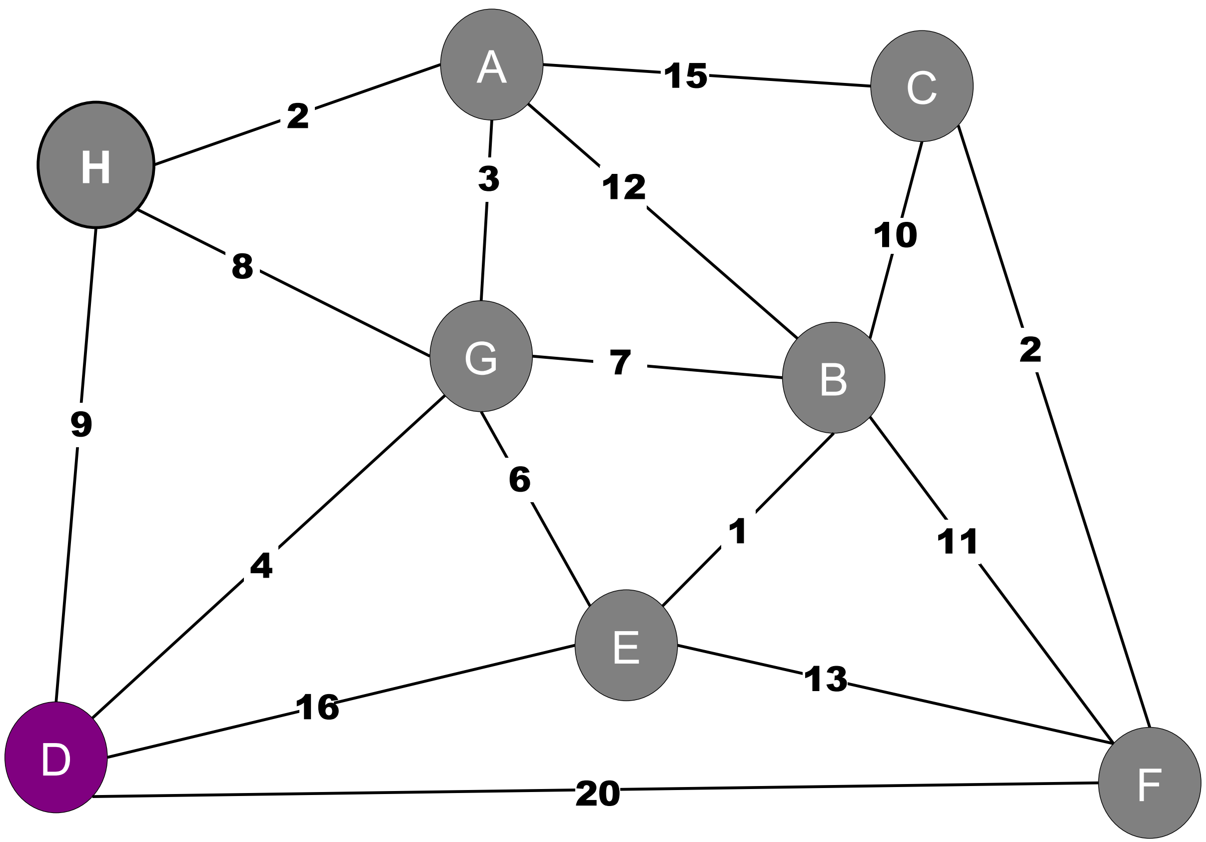
```
// Initialize priority queue
PriorityQueue pq = new PriorityQueue();
for (Node v : G.V()) {
        pq.insert(v, INFTY);
}
pq.decreaseKey(start, 0);

// Initialize set S
HashSet<Node> S = new HashSet<Node>();
S.put(start);

// Initialize parent hash table
HashMap<Node,Node> parent = new HashMap<Node,Node>();
parent.put(start, null);
```

Prim's Algorithm

```
while (!pq.isEmpty()){
    Node v = pq.deleteMin();
    S.put(v);
    for each (Edge e : v.edgeList()){
        Node w = e.otherNode(v);
        if (!S.get(w)) {
            pq.decreaseKey(w, e.getWeight());
            parent.put(w, v);
        }
    }
}
```

Assume:
decreaseKey does nothing if new weight is larger than old weight

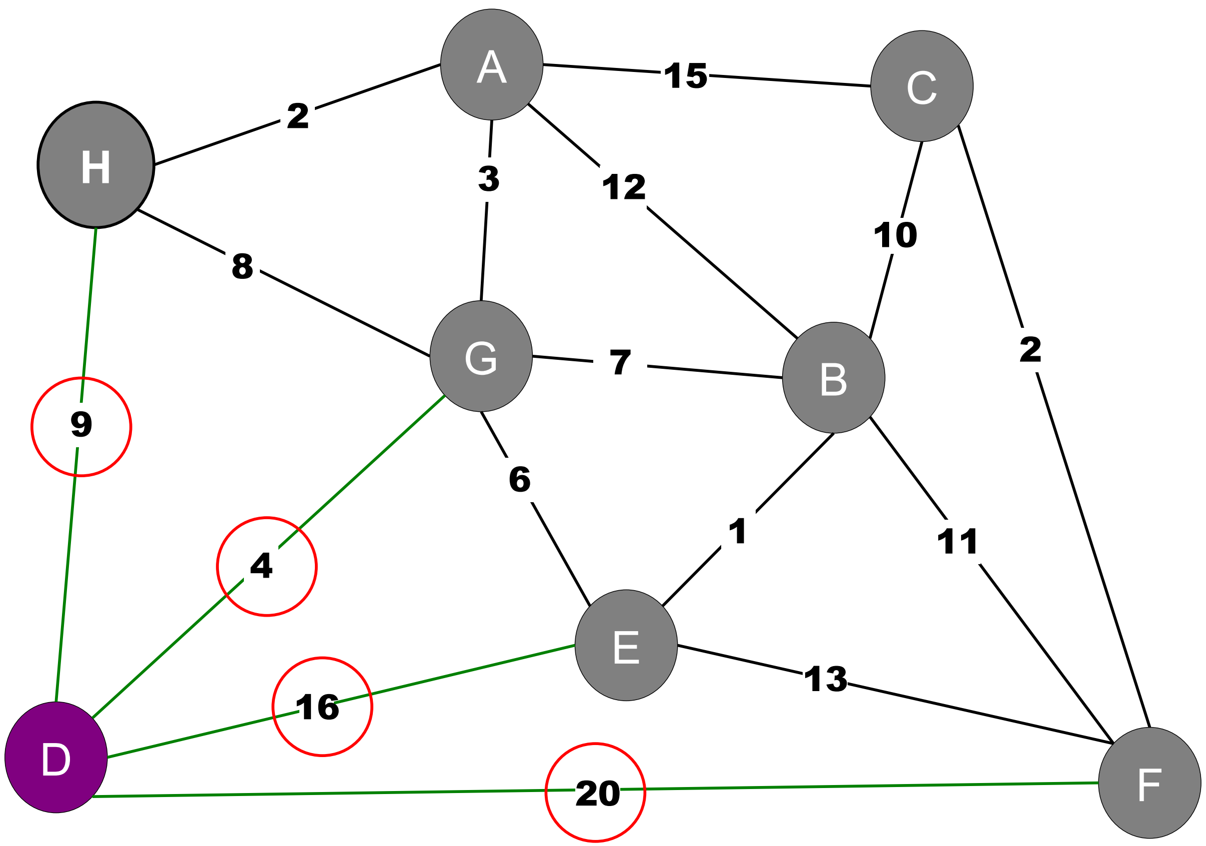
Prim's Example



Vertex	Weight
D	0

Not drawing infinite weight nodes in PQ.

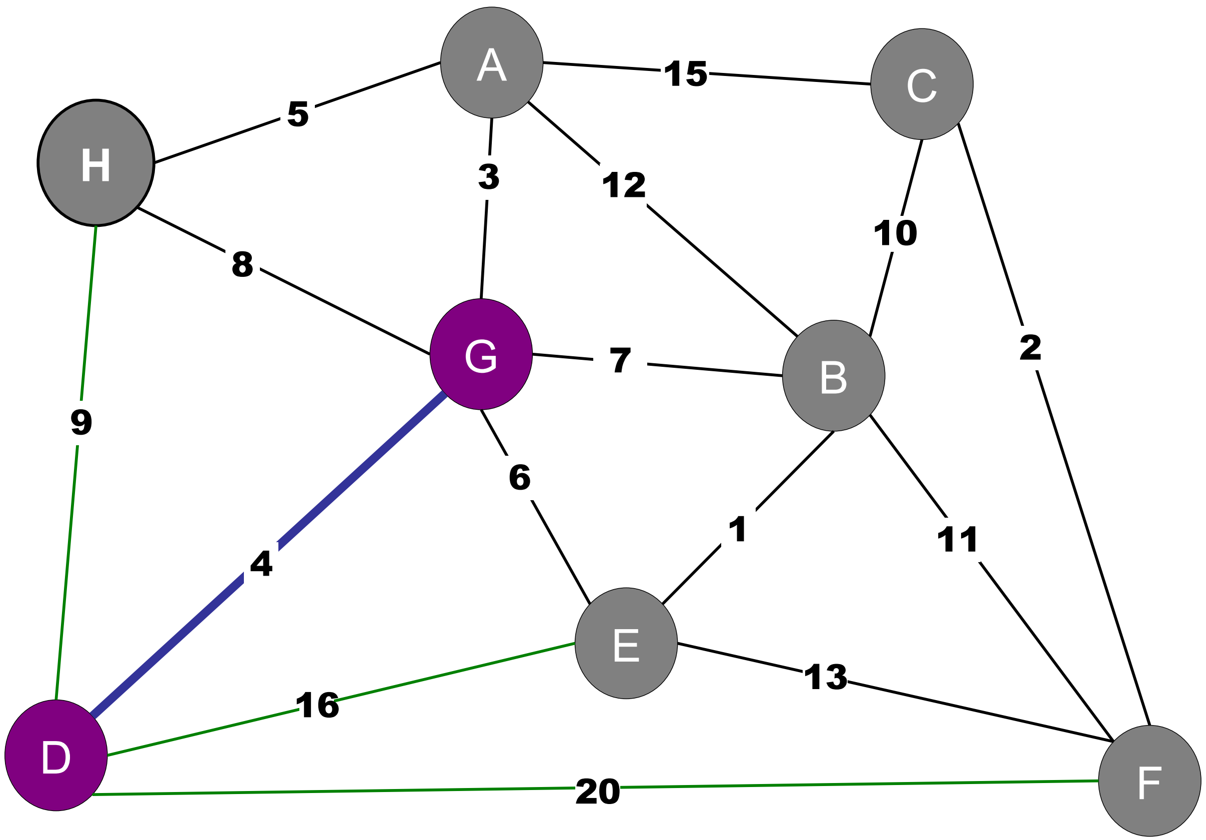
Prim’s Example



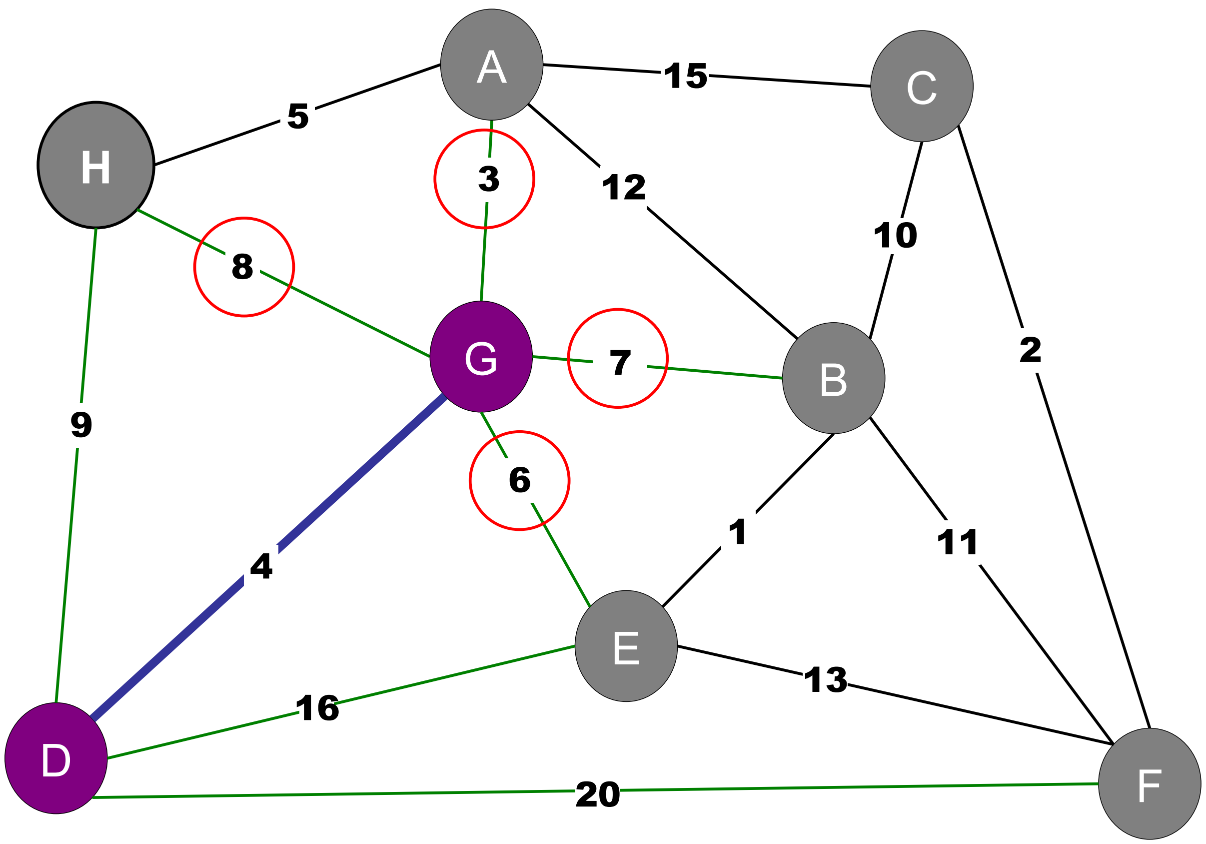
Vertex	Weight
G	4
H	9
E	16
F	20

Prim's Example

Vertex	Weight
H	9
E	16
F	20

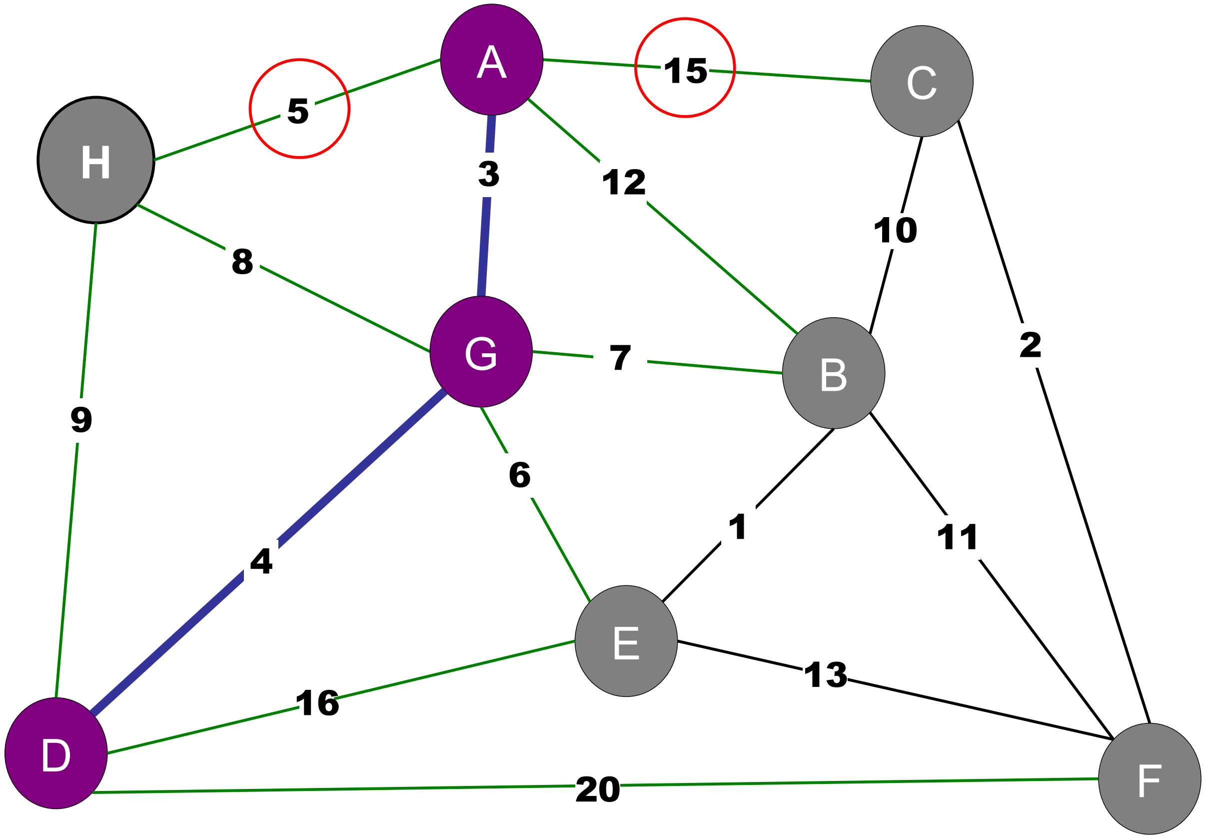


Prim’s Example



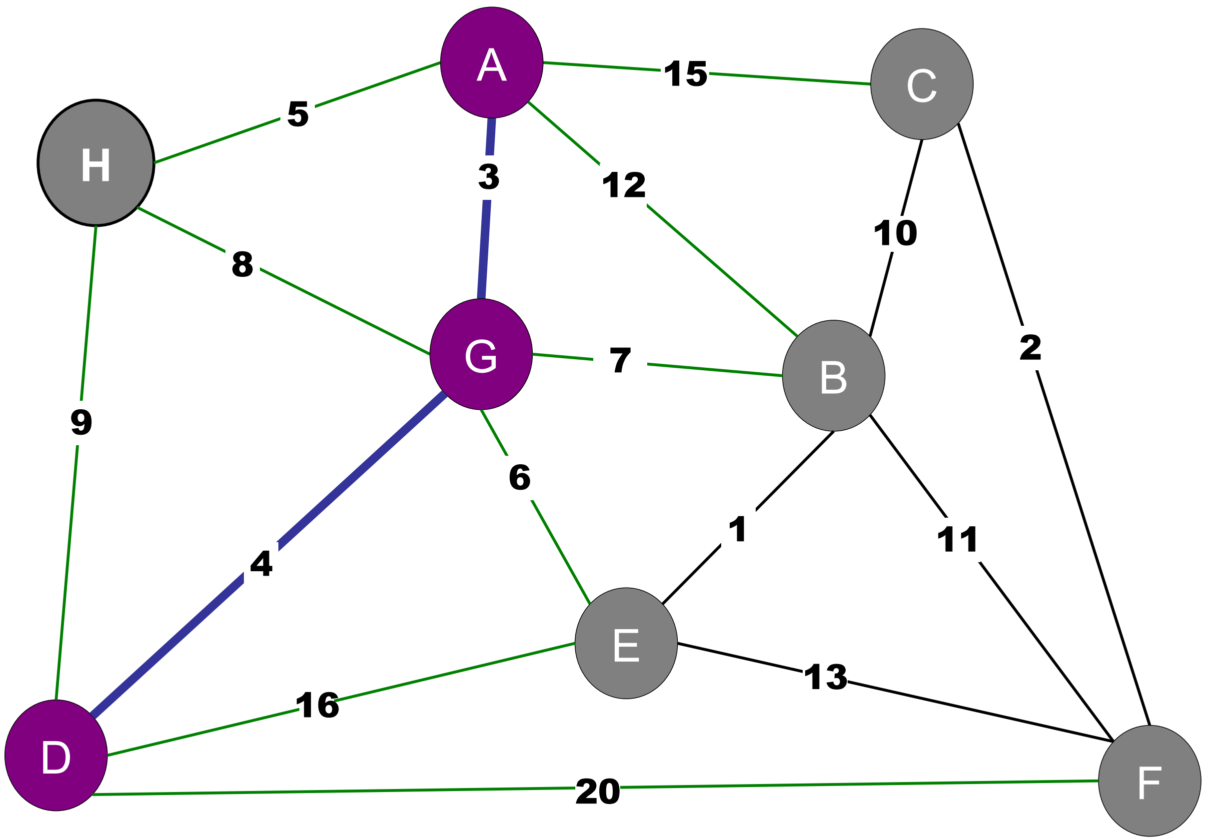
Vertex	Weight
A	3
E	16->6
B	7
H	9->8
F	20

Prim’s Example



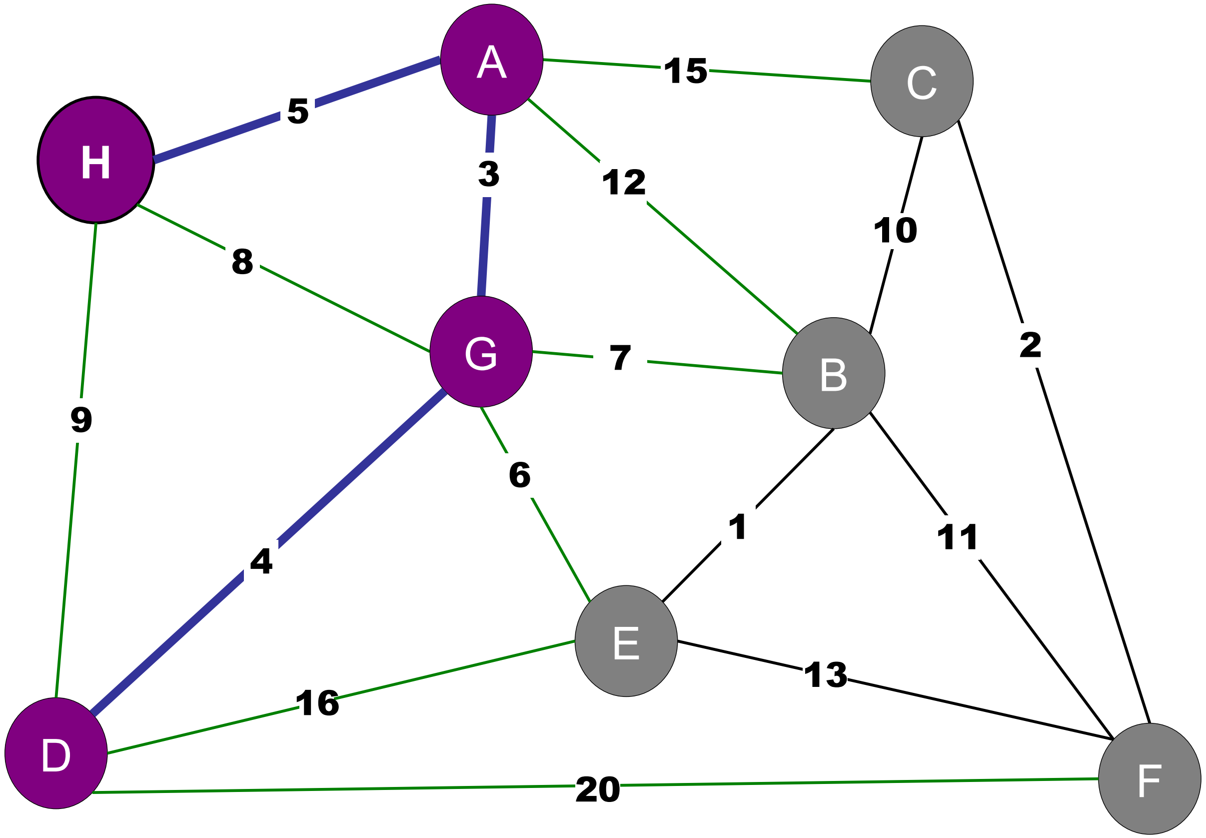
Vertex	Weight
H	**8->5**
E	6
B	7
C	**15**
F	20

Prim's Example



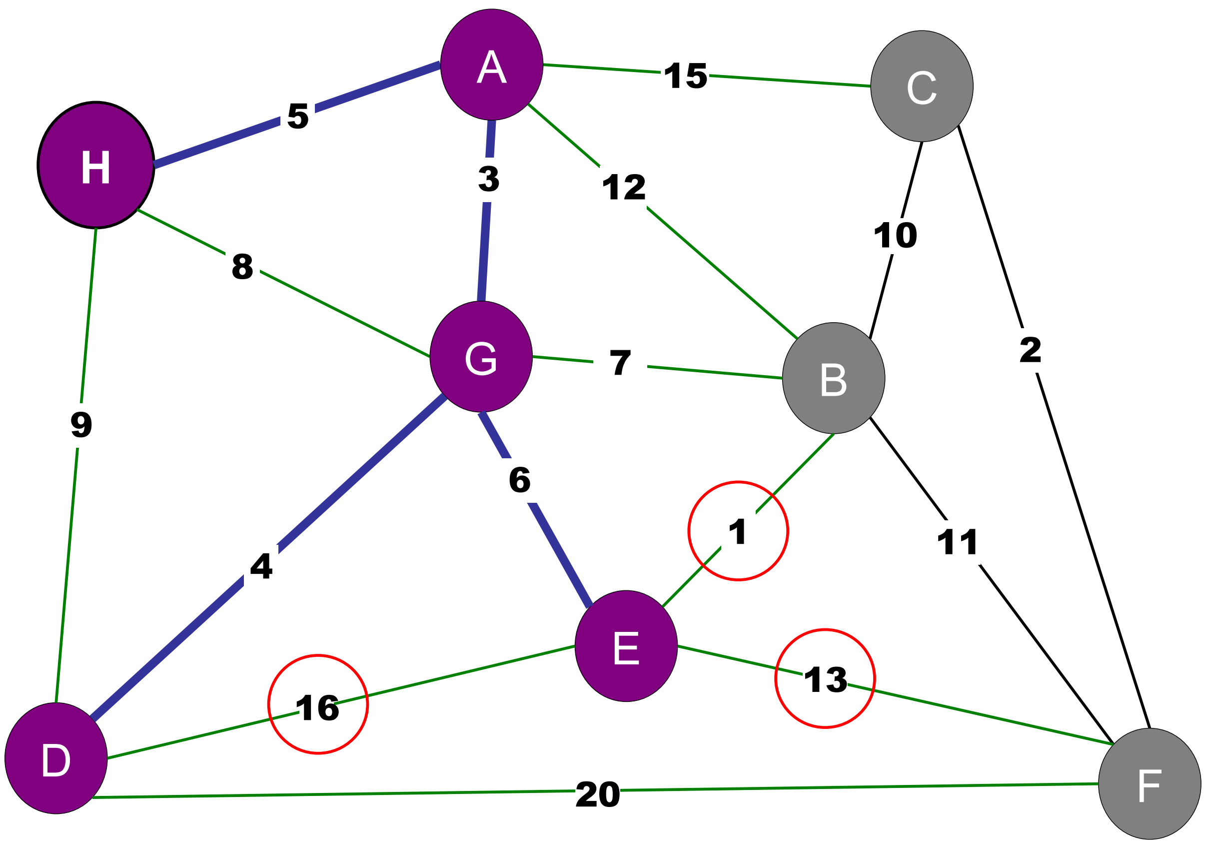
Vertex	Weight
H	5
E	6
B	7
C	15
F	20

Prim's Example



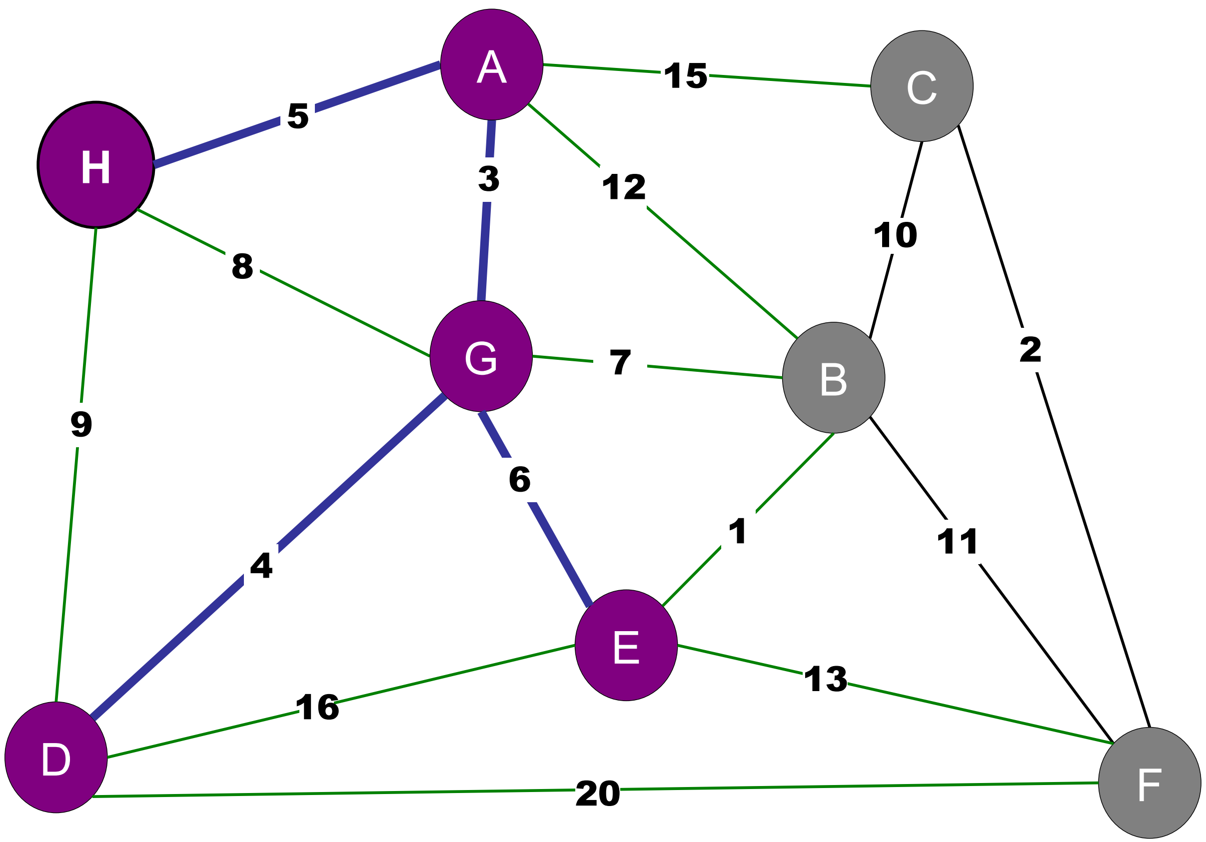
Vertex	Weight
E	6
B	7
C	15
F	20

Prim's Example



Vertex	Weight
B	**7->1**
C	15
F	**20->13**

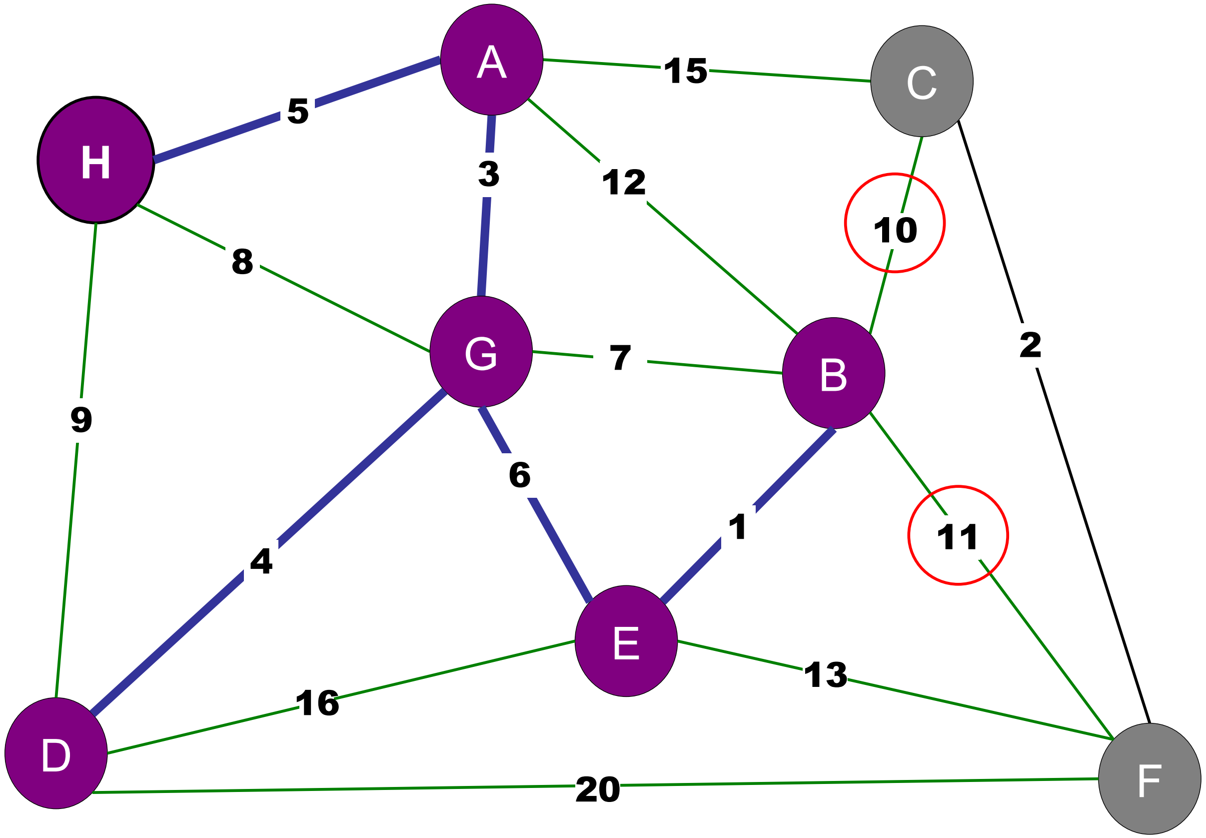
Prim's Example



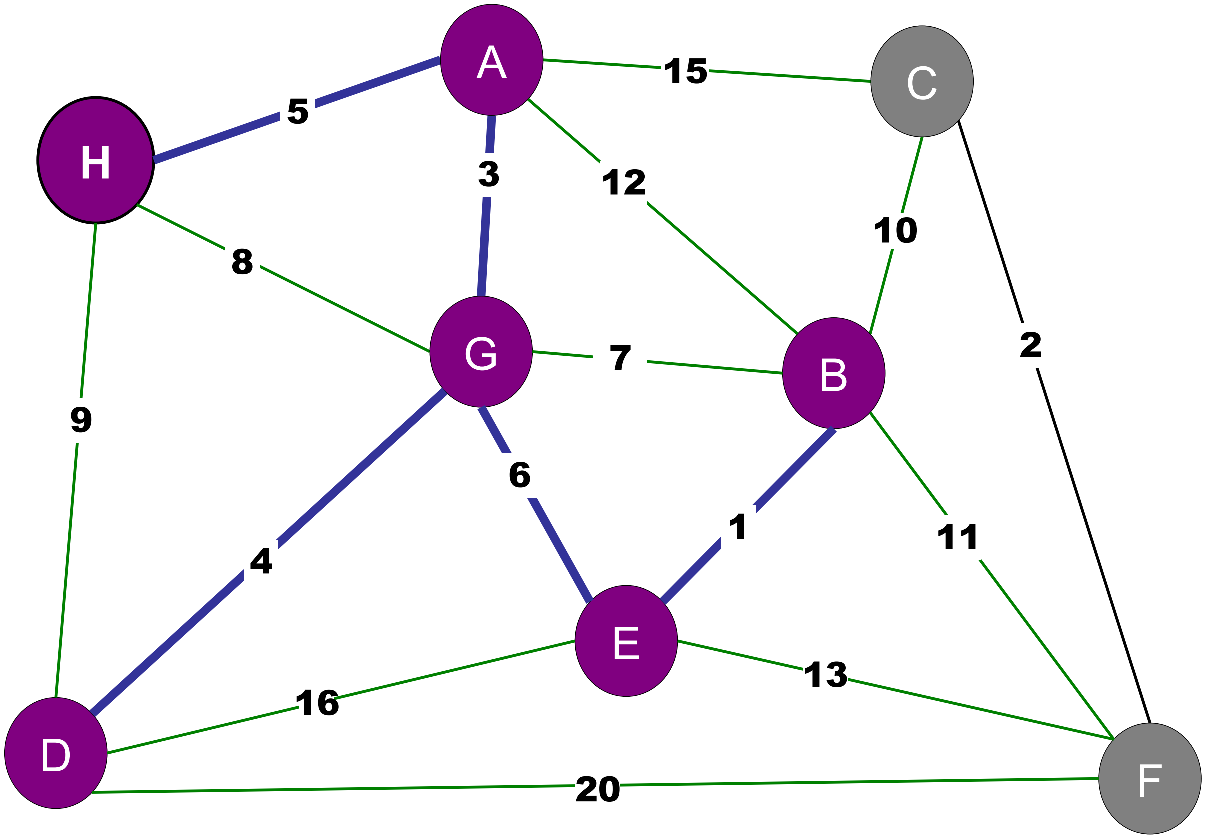
Vertex	Weight
B	1
C	15
F	13

Prim's Example

Vertex	Weight
C	15->10
F	13->11



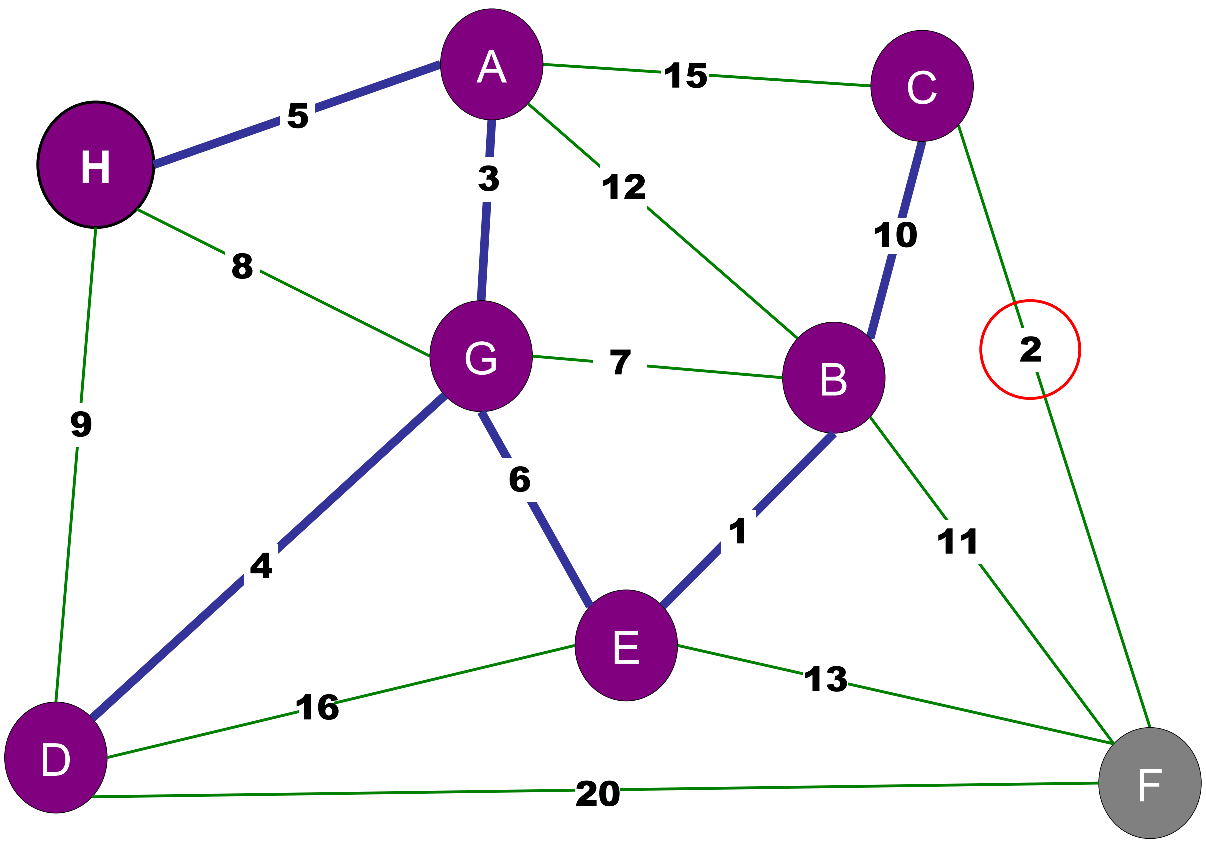
Prim's Example



Vertex	Weight
C	10
F	11

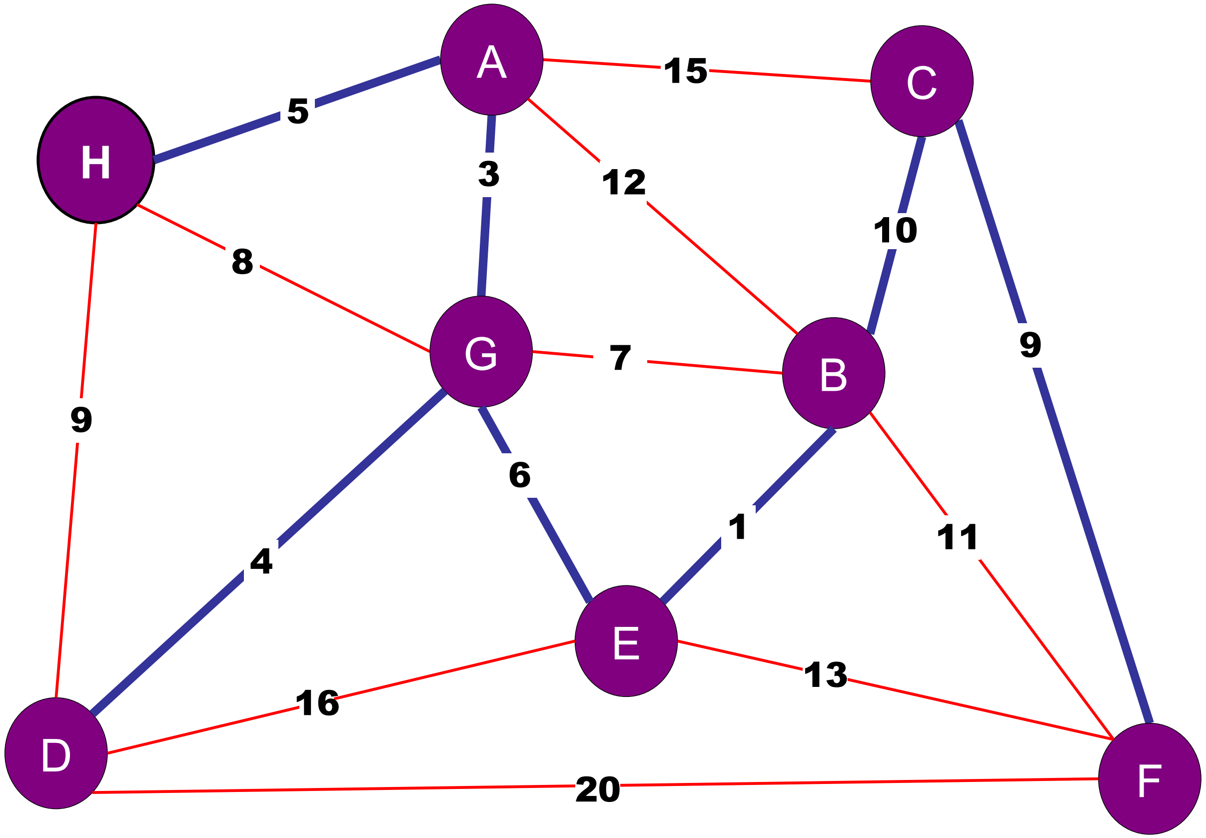
Prim’s Example

Vertex	Weight
F	11->2



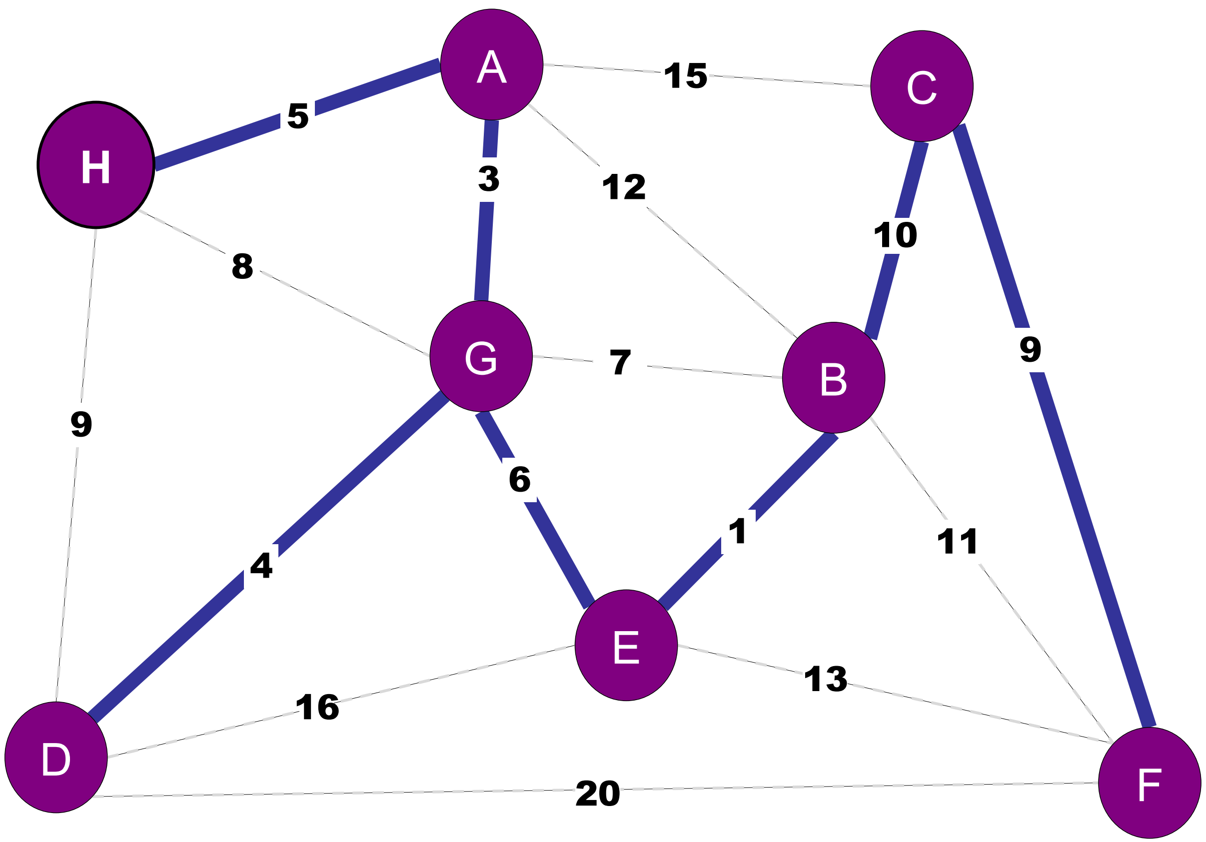
Prim's Example

Vertex	Weight



Prim's Example

Vertex	Weight



Prim's Algorithm

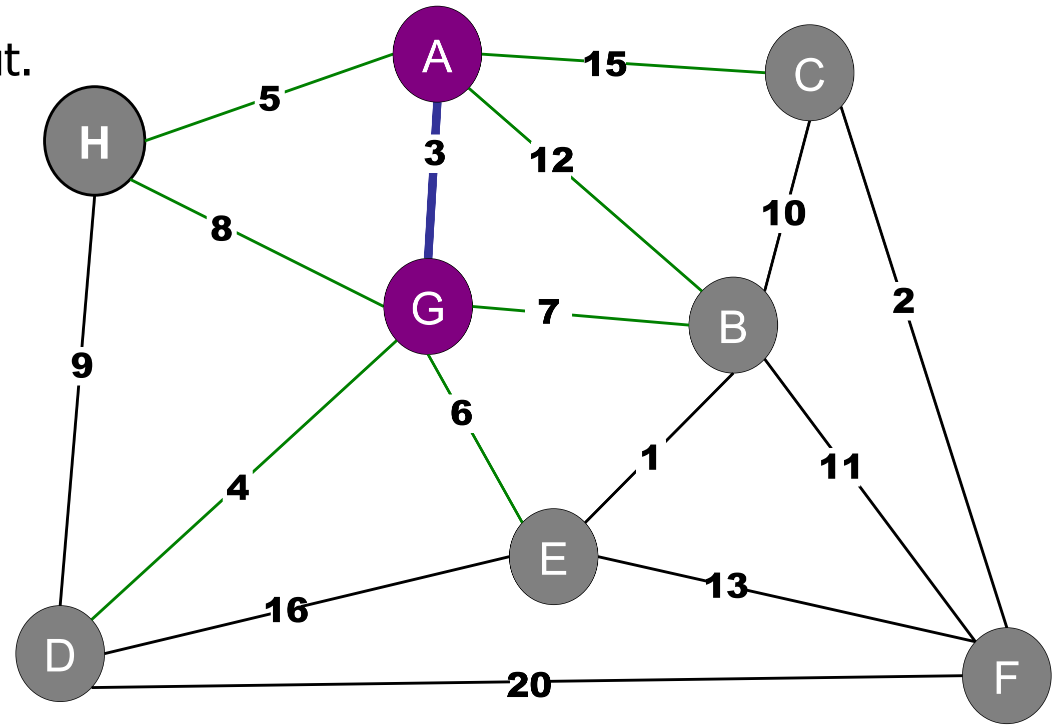
Prim's Algorithm. (Jarnik 1930, Dijkstra 1957, Prim 1959)

Basic idea:

- S : set of nodes connected by blue edges.
- Initially: S = {A}
- Repeat:
  - Identify cut: {S, V–S}
  - Find minimum weight edge on cut.
  - Add new node to S.

Proof:

- Each added edge is the lightest on some cut.
- Hence each edge is in the MST.



What is the running time of Prim's Algorithm, using an AVL tree for PQ?

1. O(V)
2. O(E)
3. O(E log V)
4. O(V log E)
5. O(EV)

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Prim's Algorithm

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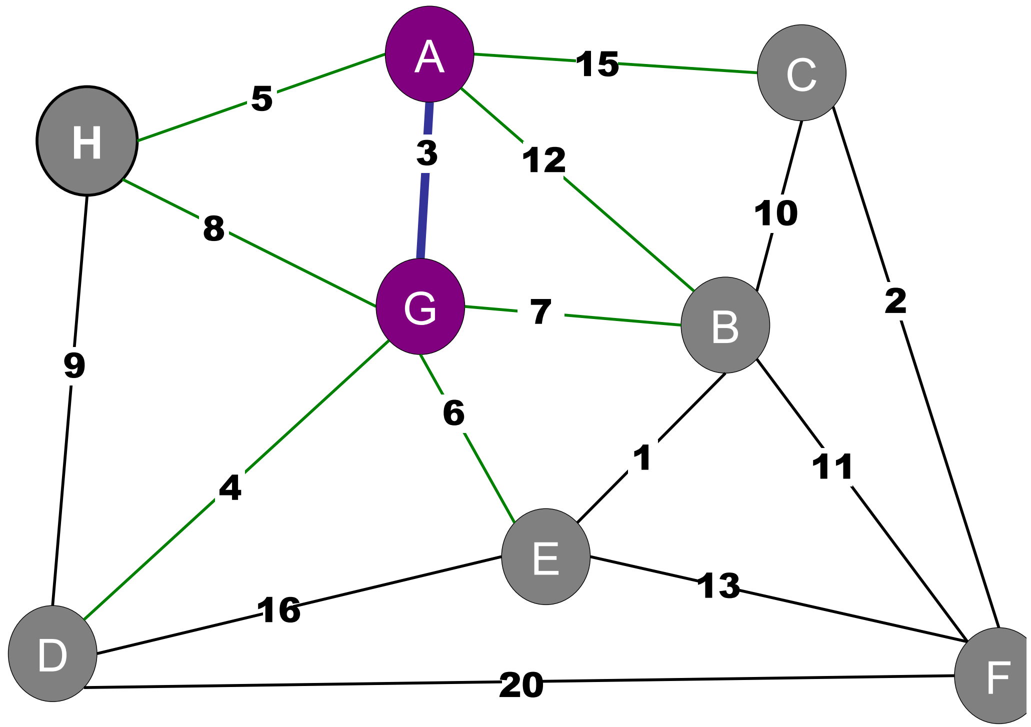

Analysis:

- Each vertex added/removed once from the priority queue: O(V log V)
- Each edge => one decreaseKey: O(E log V).

Two Algorithms

Prim’s Algorithm.

Basic idea:

- Maintain a set of visited nodes.
- Greedily grow the set by adding node connected via the lightest edge.
- Use Priority Queue to order nodes by edge weight.

Dijkstra’s Algorithm.

Basic idea:

- Maintain a set of visited nodes.
- Greedily grow the set by adding neighboring node that is closest to the source.
- Use Priority Queue to order nodes by distance.

Roadmap

Minimum Spanning Trees

- The MST Problem
- Basic Properties of an MST
- Generic MST Algorithm
- Prim’s Algorithm
- **Kruskal’s Algorithm**
- Boruvka’s Algorithm
- Variations

Generic MST Algorithm

Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

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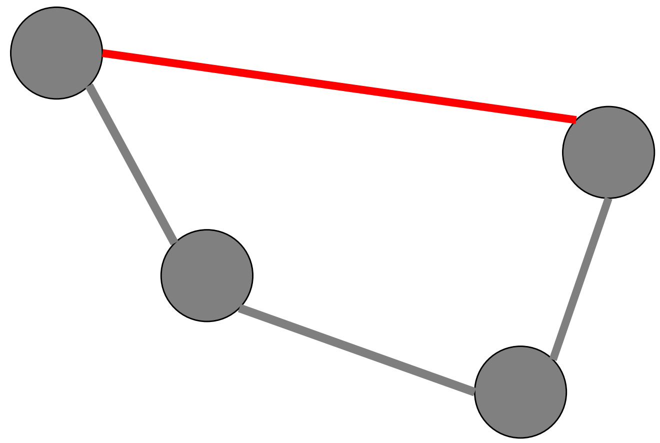
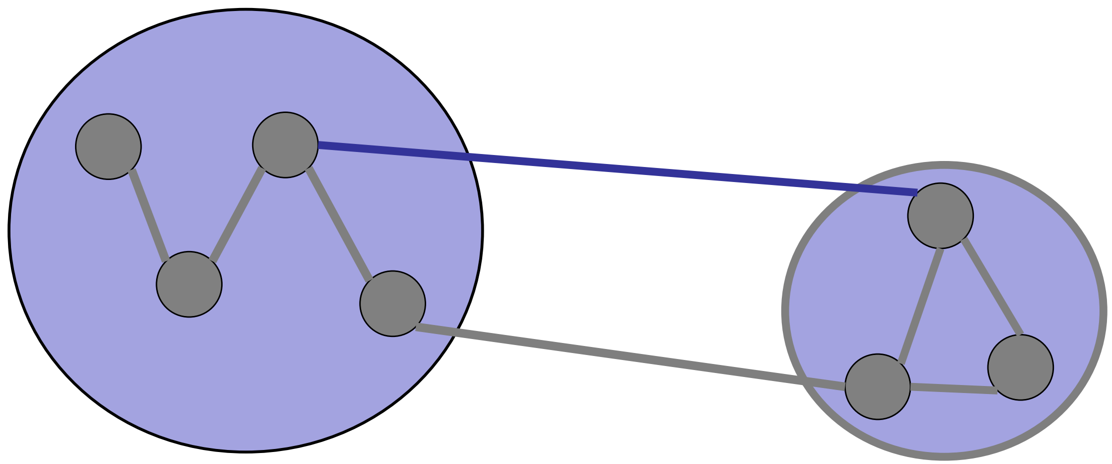
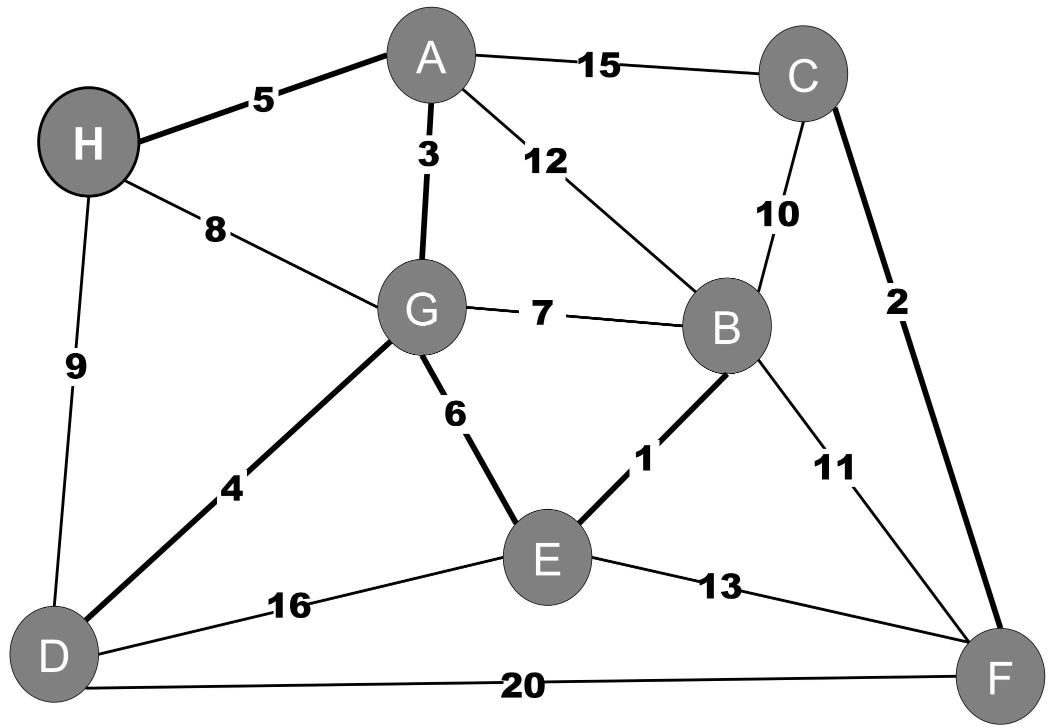




Kruskal's Algorithm

Kruskal's Algorithm. (Kruskal 1956)

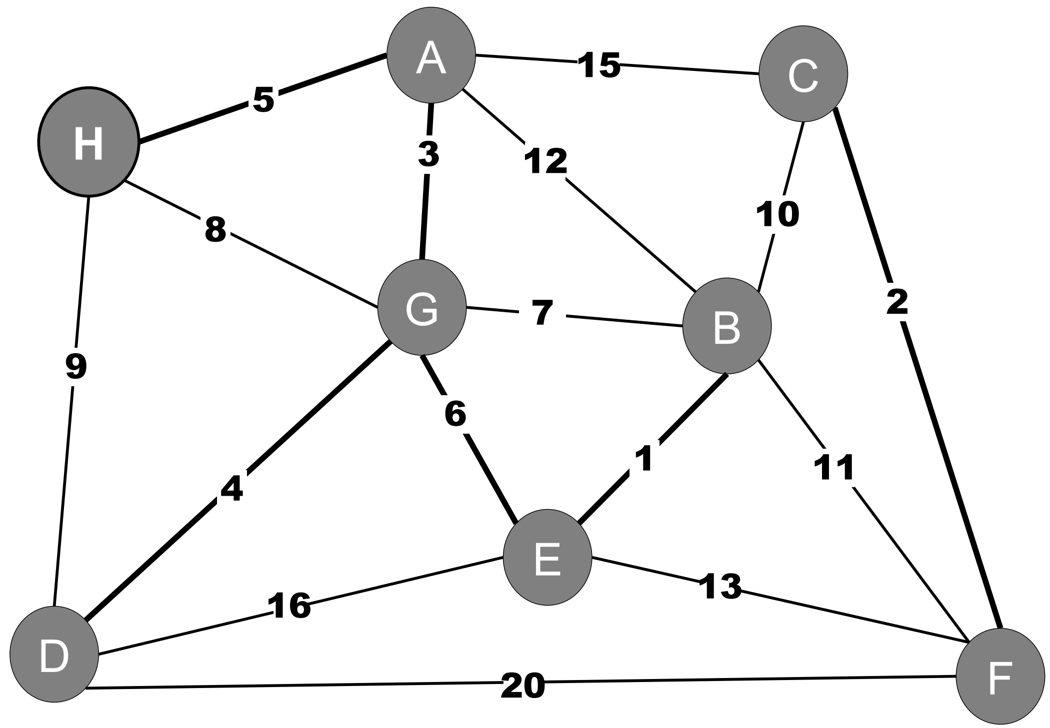


Kruskal's Algorithm

Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

- Sort edges by weight from smallest to biggest.
- Consider edges in ascending order:
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.



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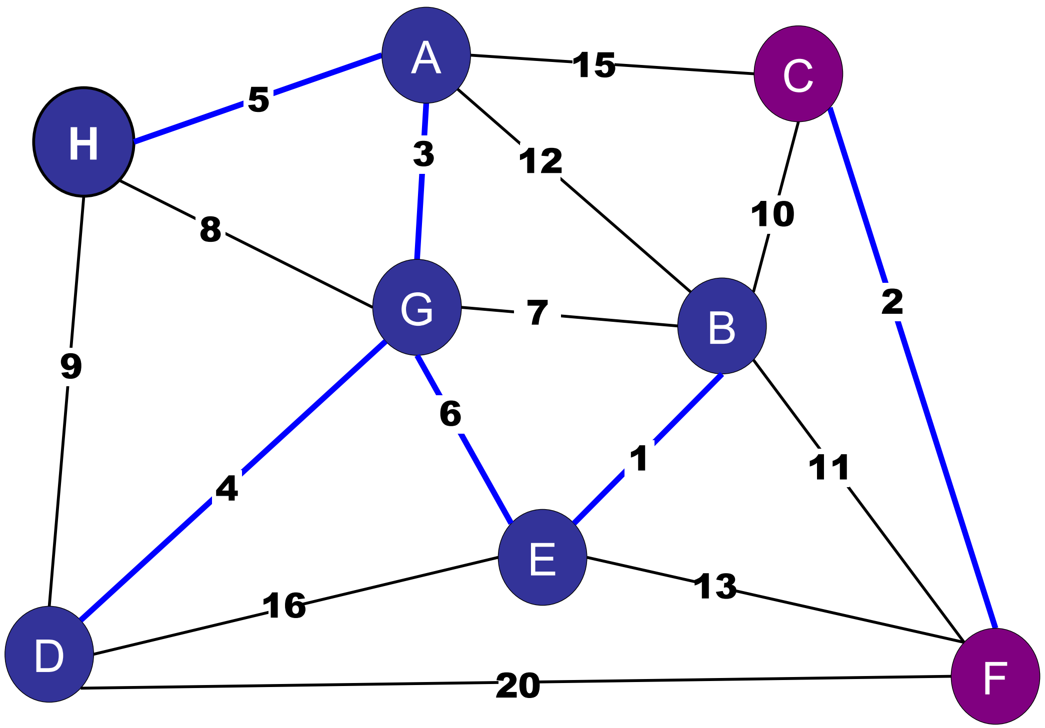
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.

Must be the heaviest edge on the cycle!



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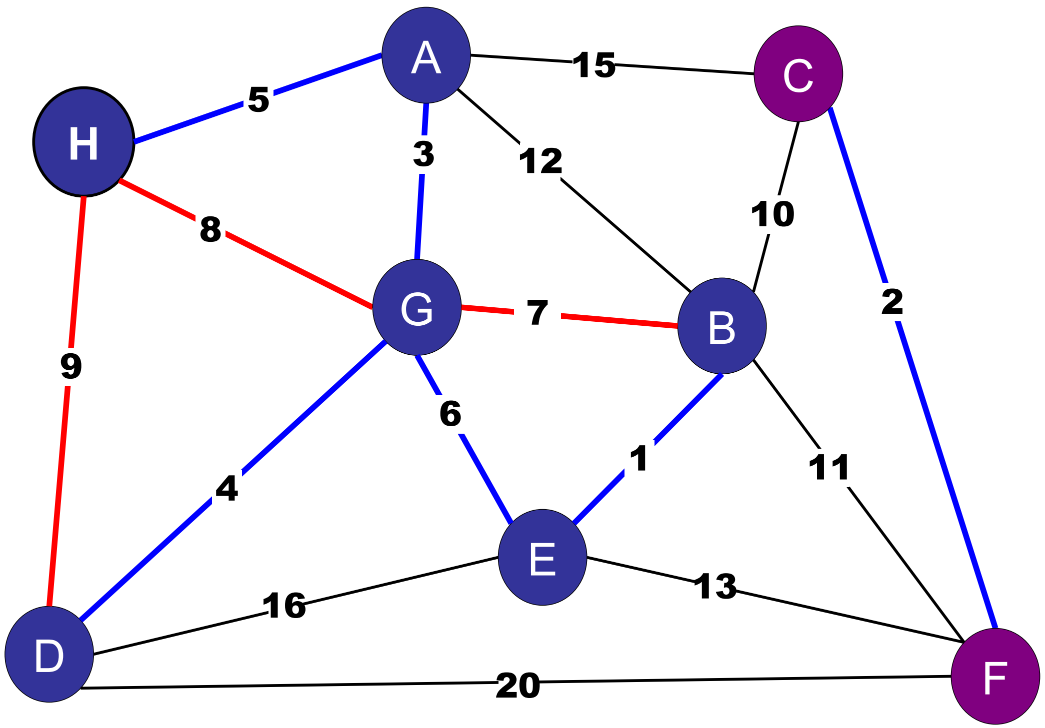

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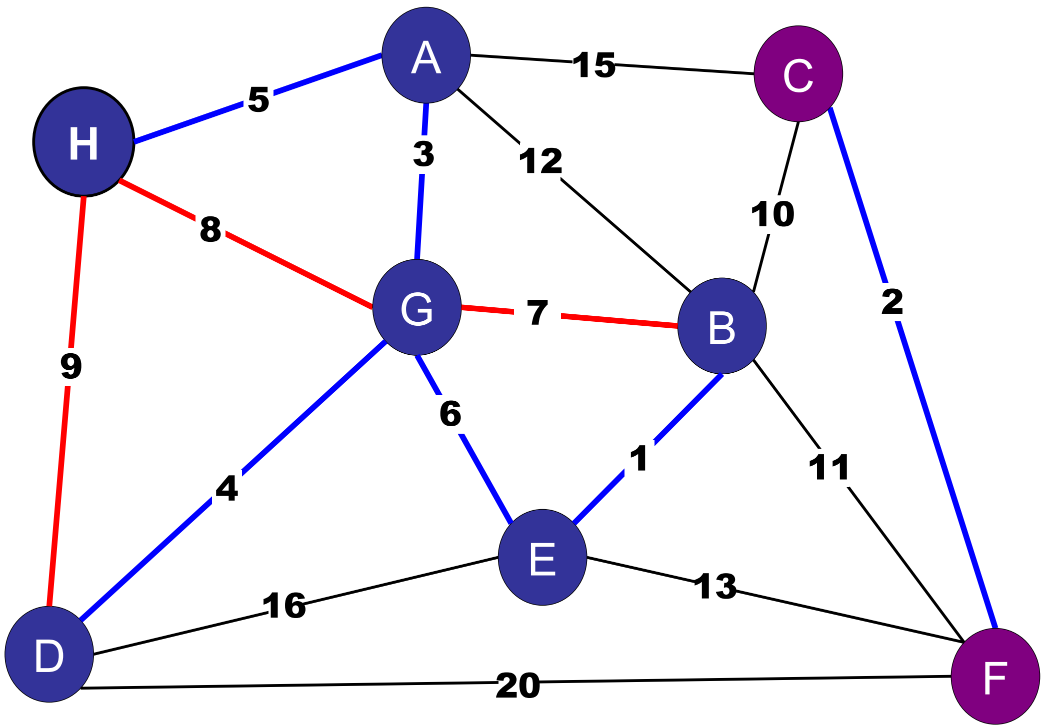
  - Otherwise, color the edge blue.

Data structure:

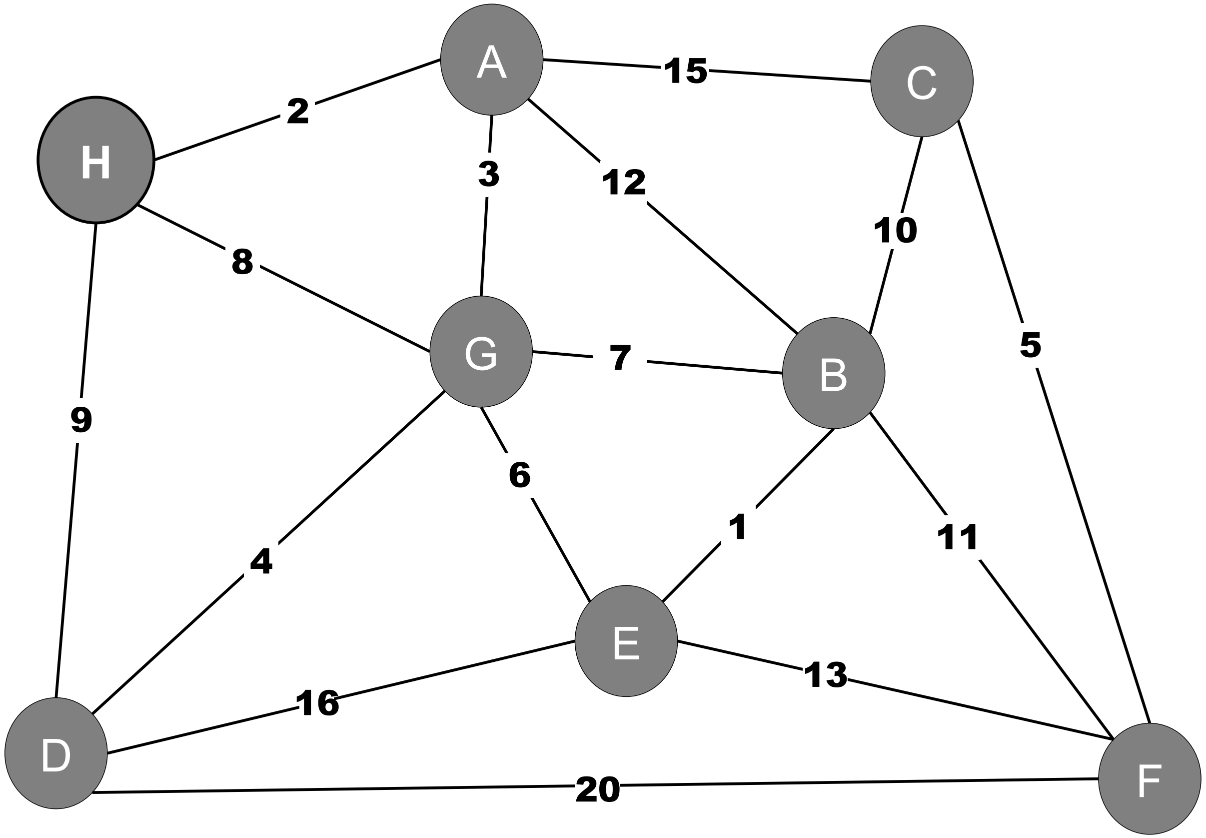
- Union-Find
- Connect two nodes if they are in the same blue tree.



Kruskal's Algorithm

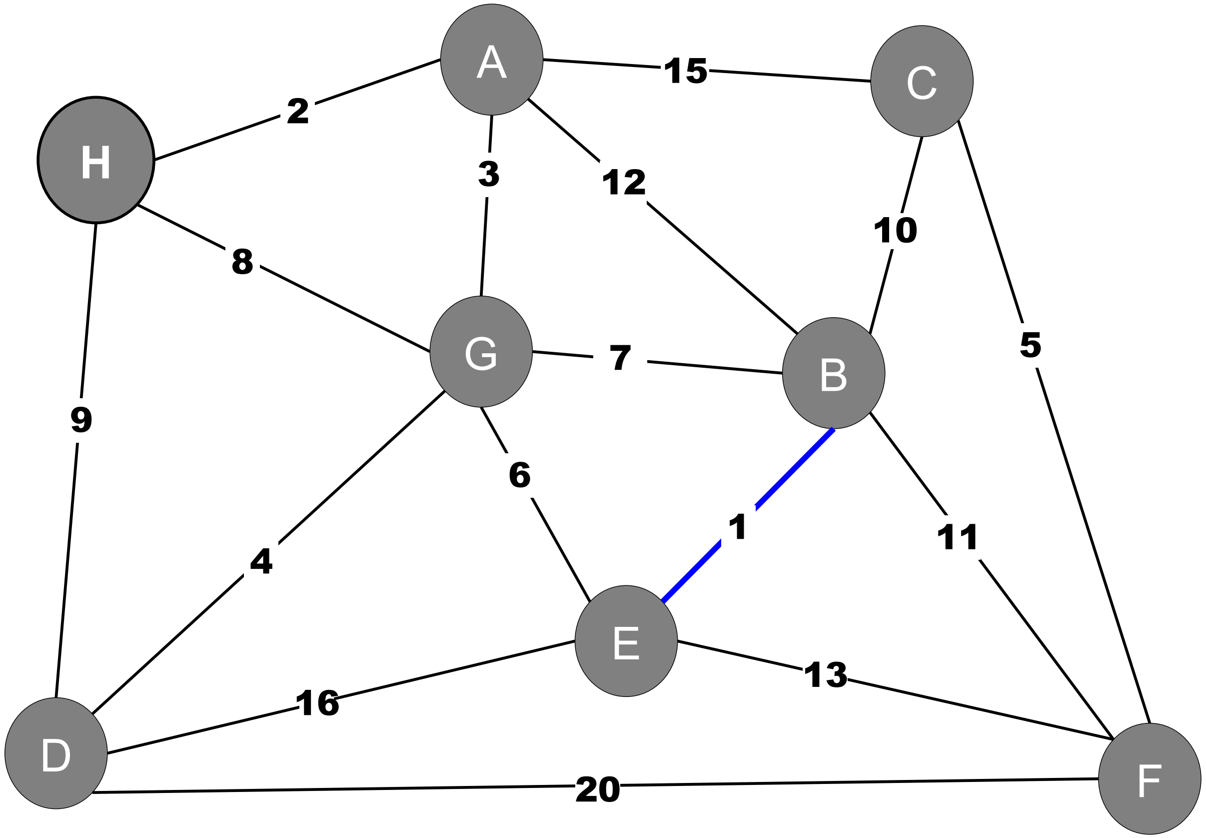
```
// Sort edges and initialize
Edge[] sortedEdges = sort(G.E());
ArrayList<Edge> mstEdges = new ArrayList<Edge>();
UnionFind uf = new UnionFind(G.V());

// Iterate through all the edges, in order
for (int i=0; i<sortedEdges.length; i++) {
        Edge e = sortedEdges[i]; // get edge
        Node v = e.one(); // get node endpoints
        Node w = e.two();

        if (!uf.find(v,w)) { // in the same tree?
            mstEdges.add(e); // save edge
            uf.union(v,w); // combine trees
        }
}
```

Kruskal's Example



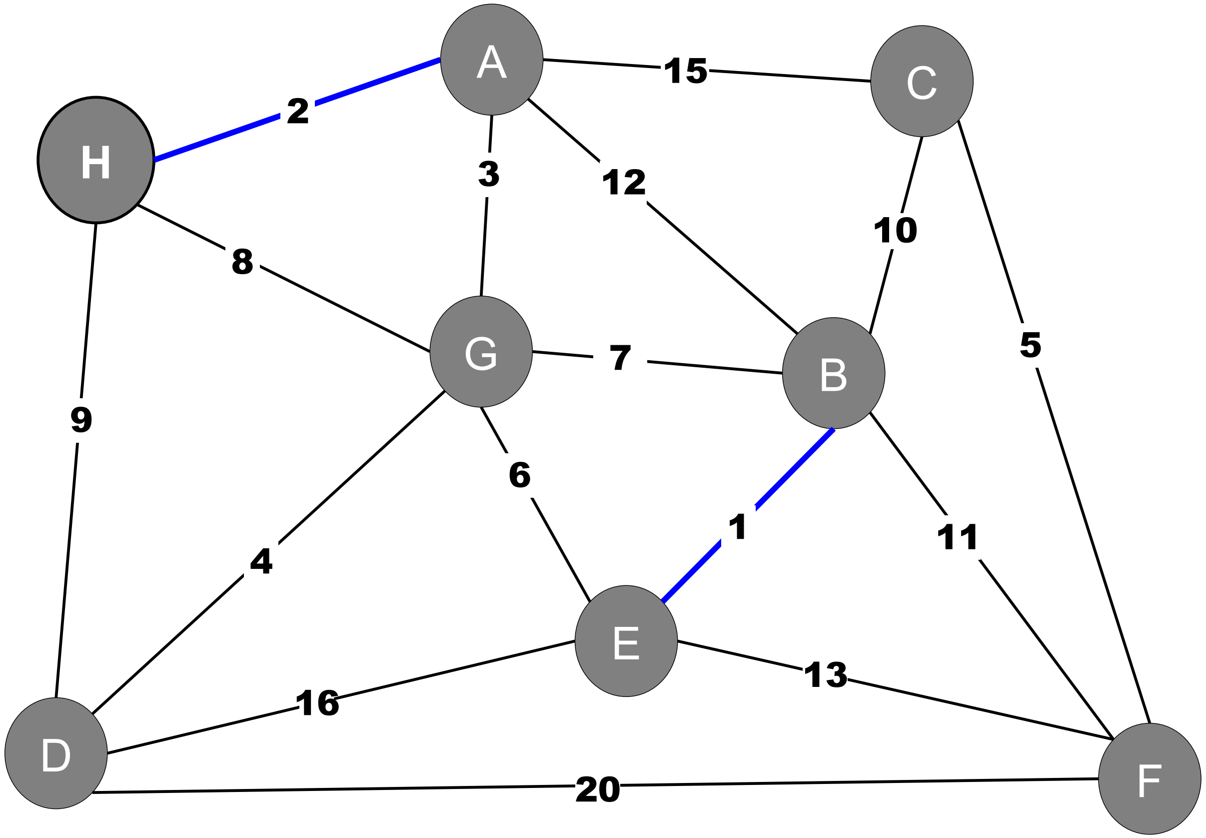
Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Example



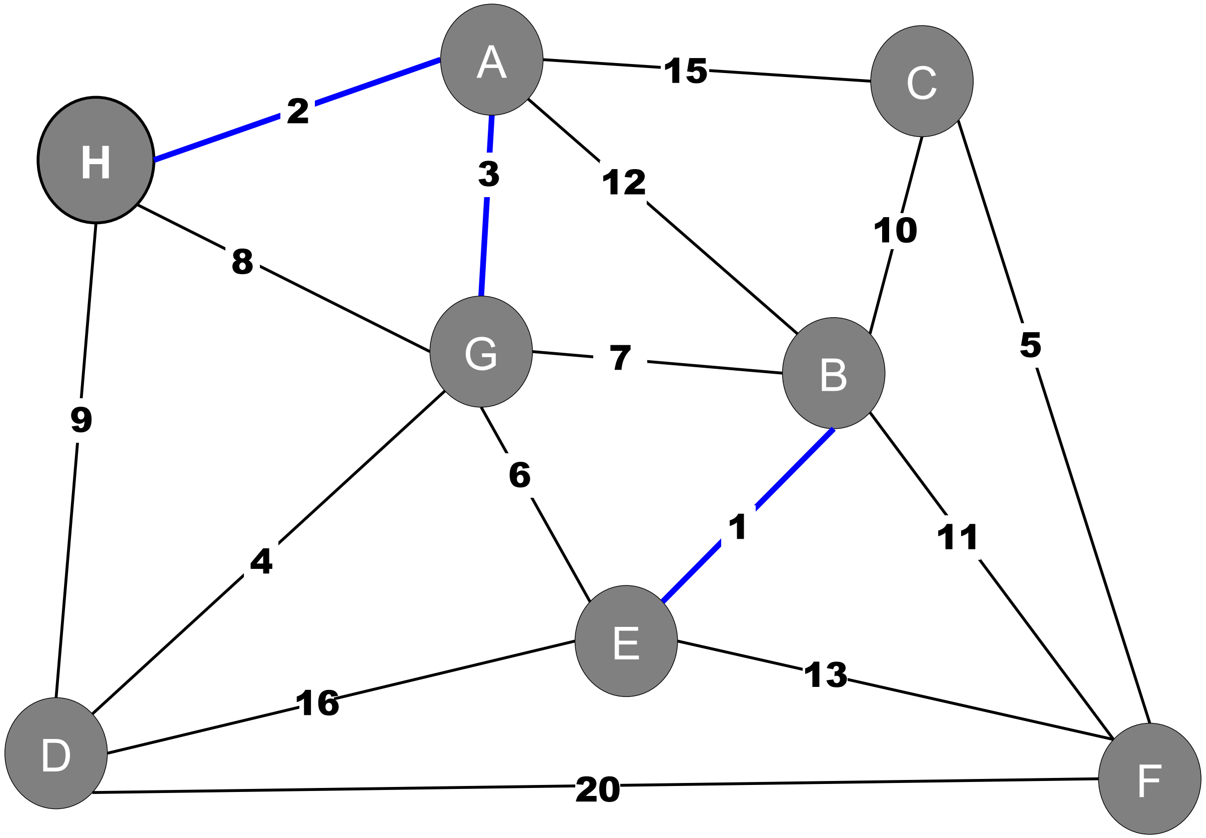
Weight	Edge
1	**(E,B)**
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Example



Weight	Edge
1	(E,B)
2	**(C,F)**
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Example



Weight	Edge
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3	**(A,G)**
4	(D,G)
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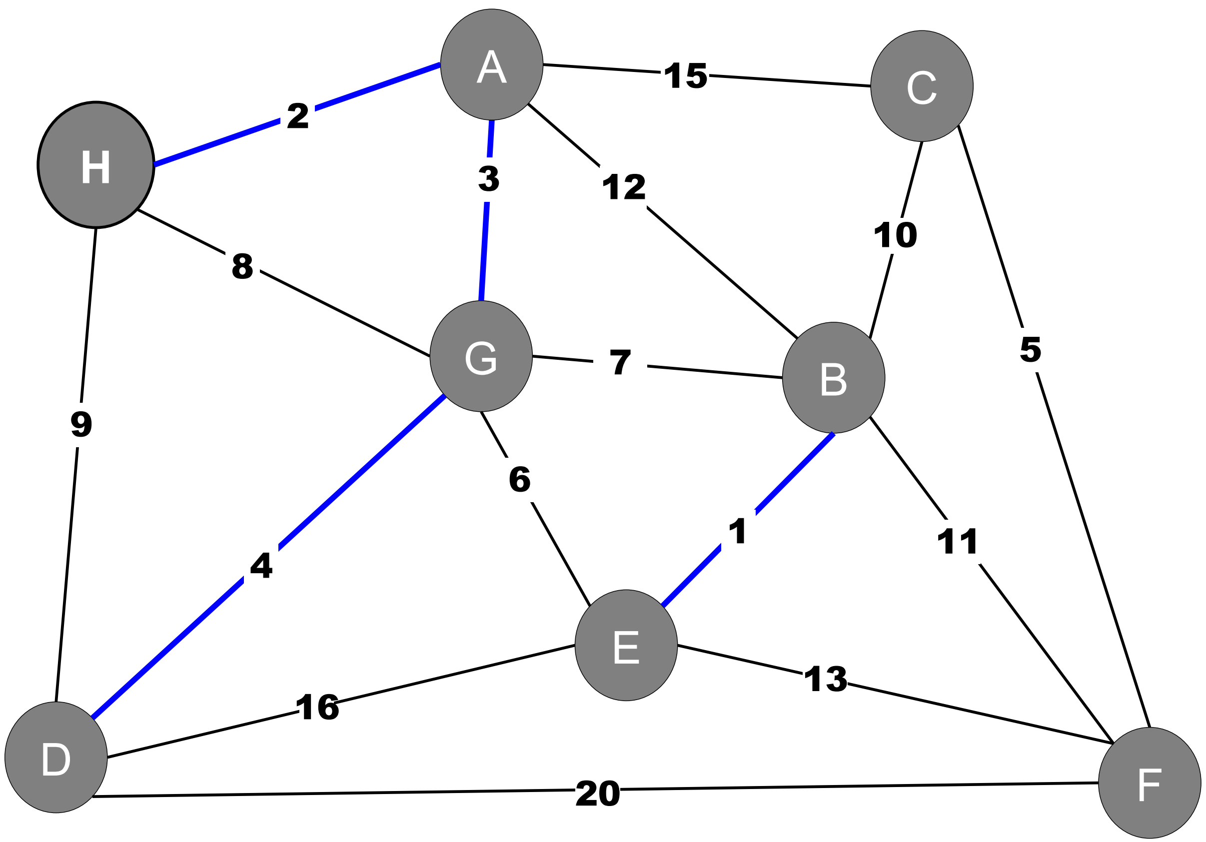
16	(D,E)
20	(D,F)

Kruskal's Example



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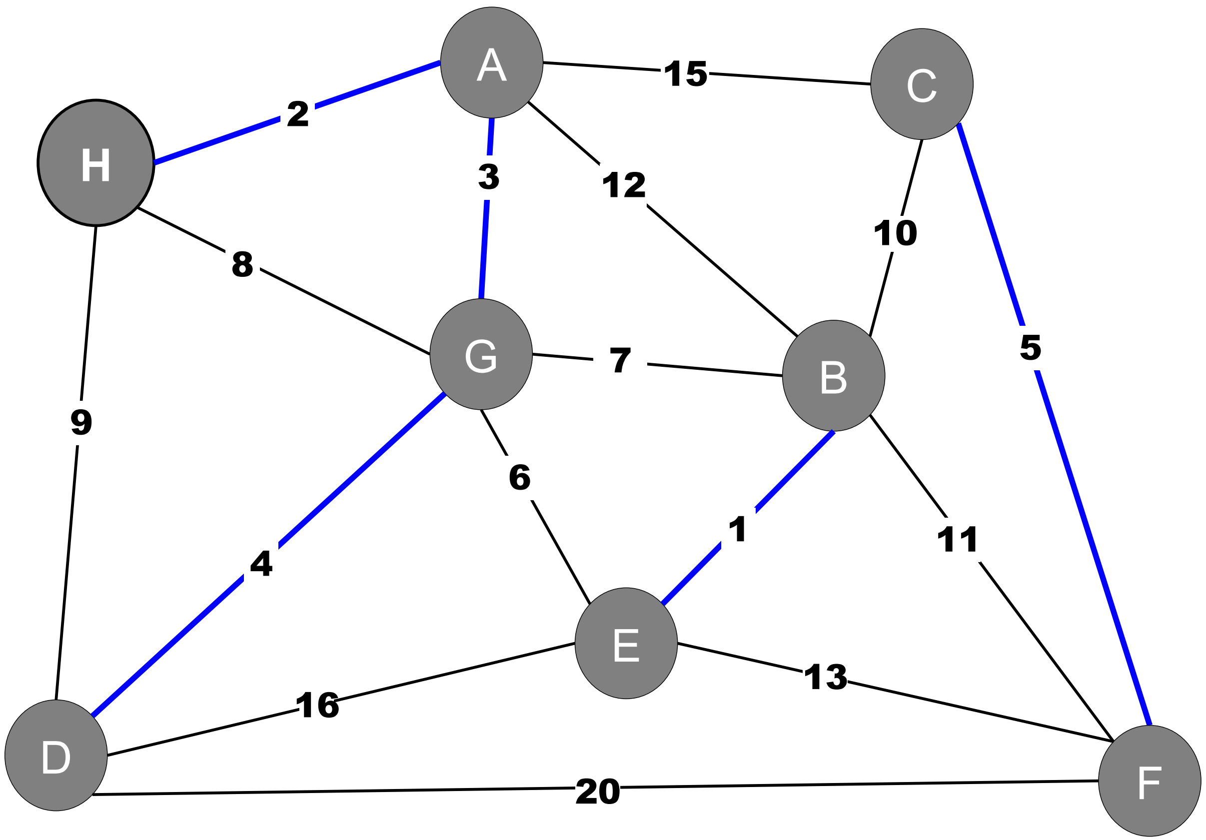
Kruskal’s Example



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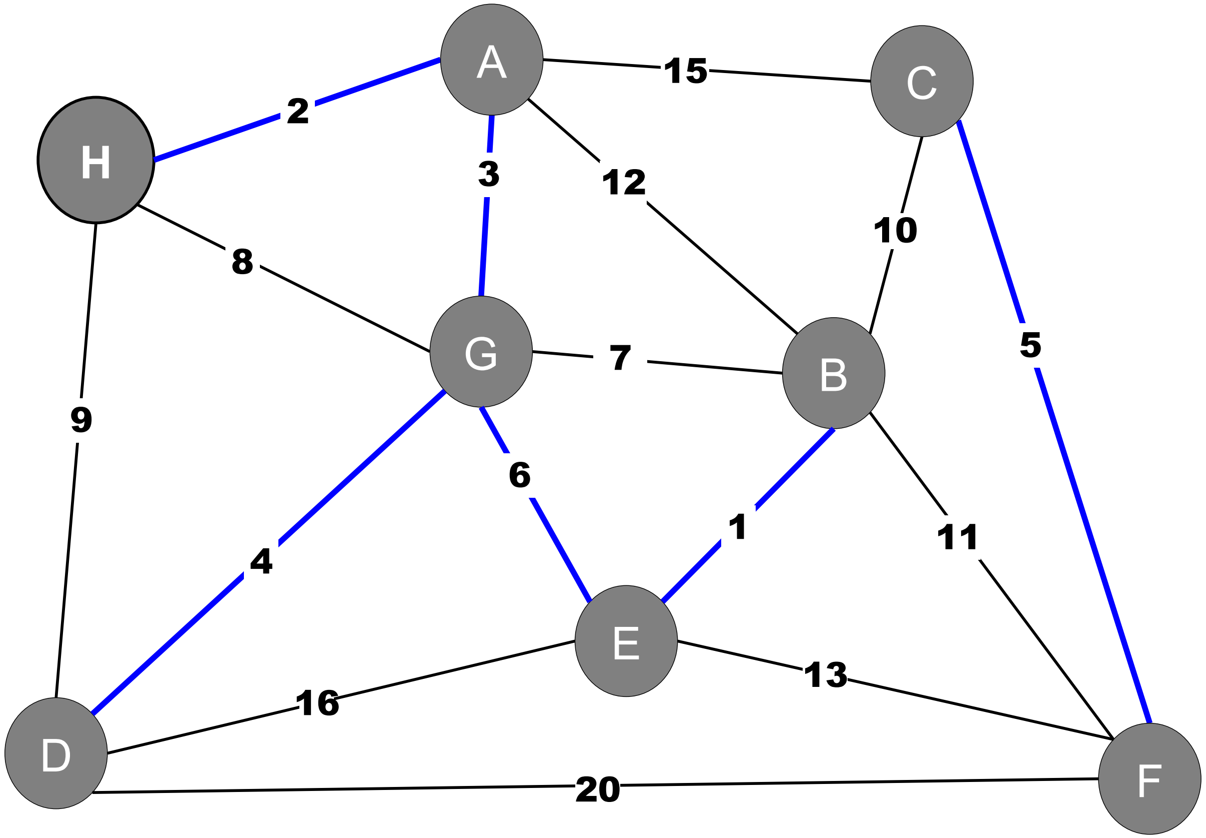
Kruskal's Example



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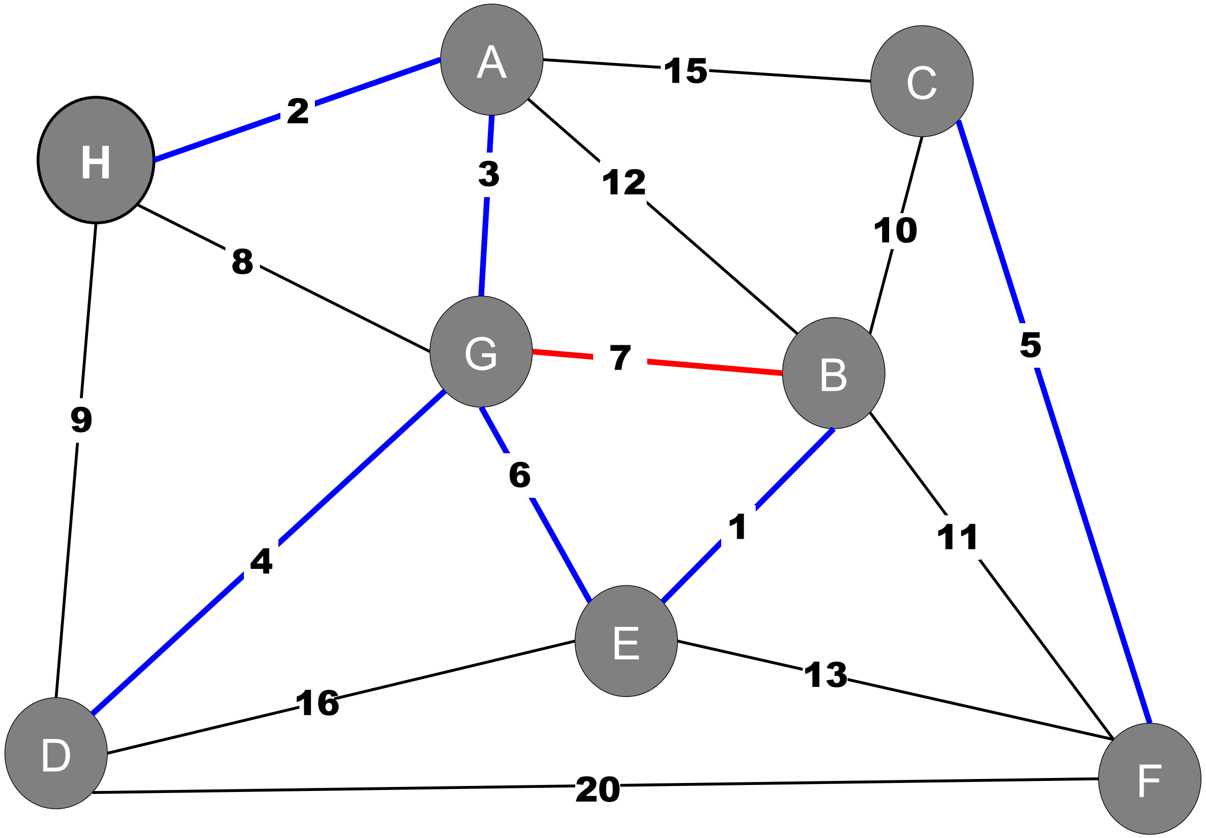
Kruskal's Example



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4	(D,G)
5	(C,F)
6	(E,G)
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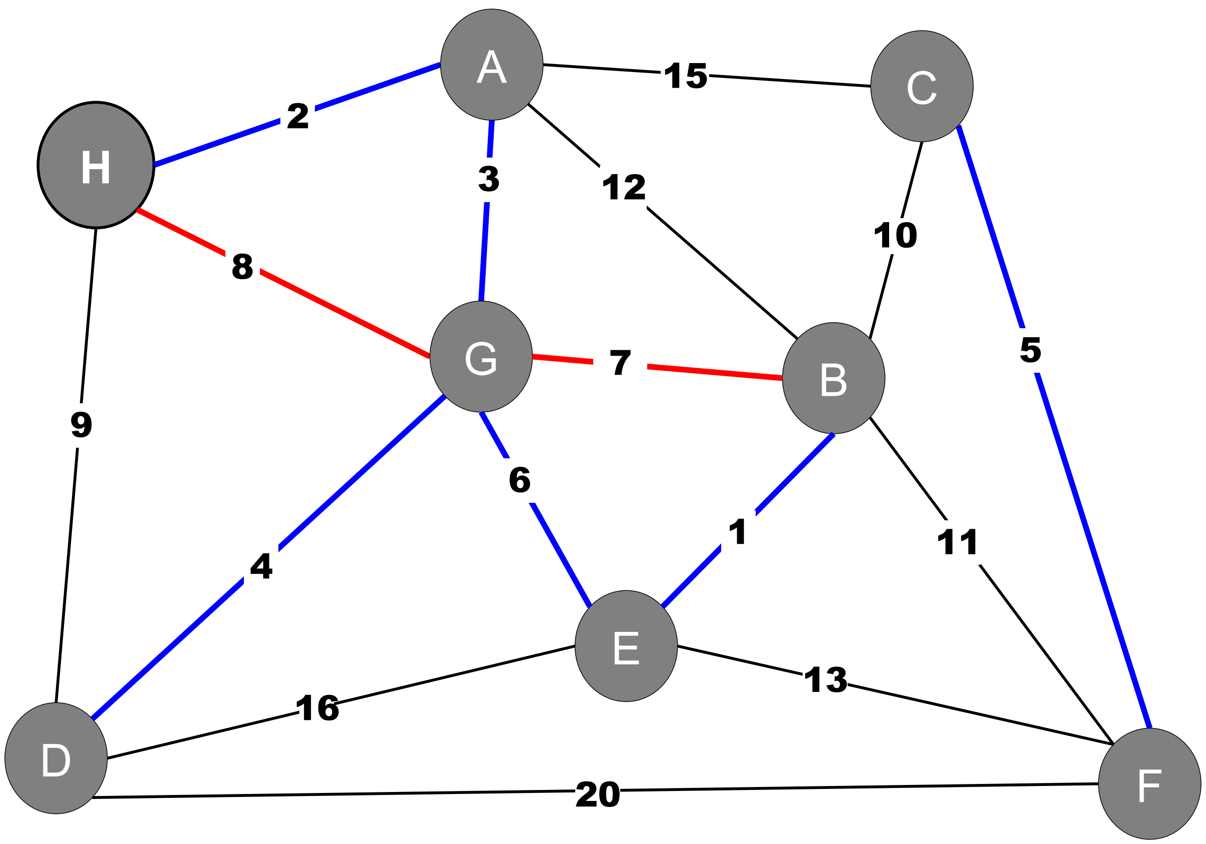
Kruskal’s Example



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5	(C,F)
6	(E,G)
7	(B,G)
8	**(G,H)**
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10	(B,C)
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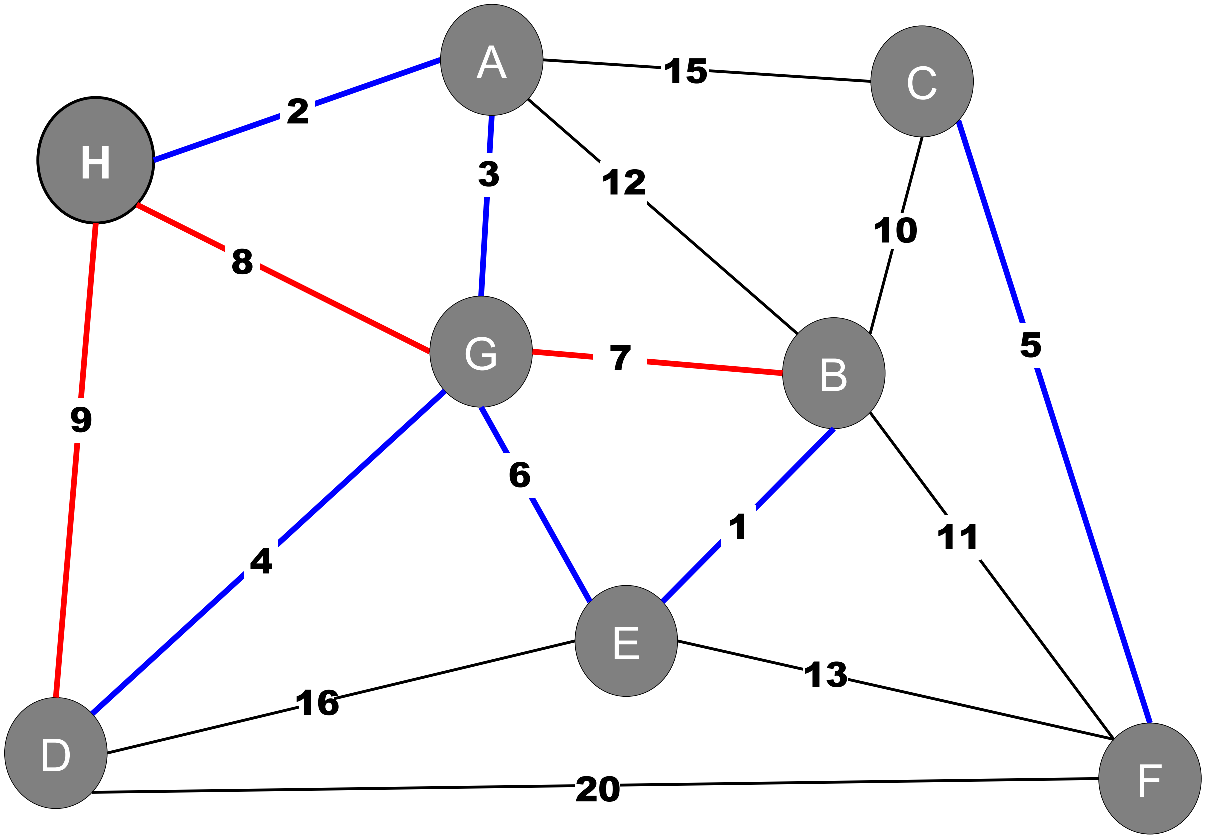
Kruskal's Example



Weight	Edge
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4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
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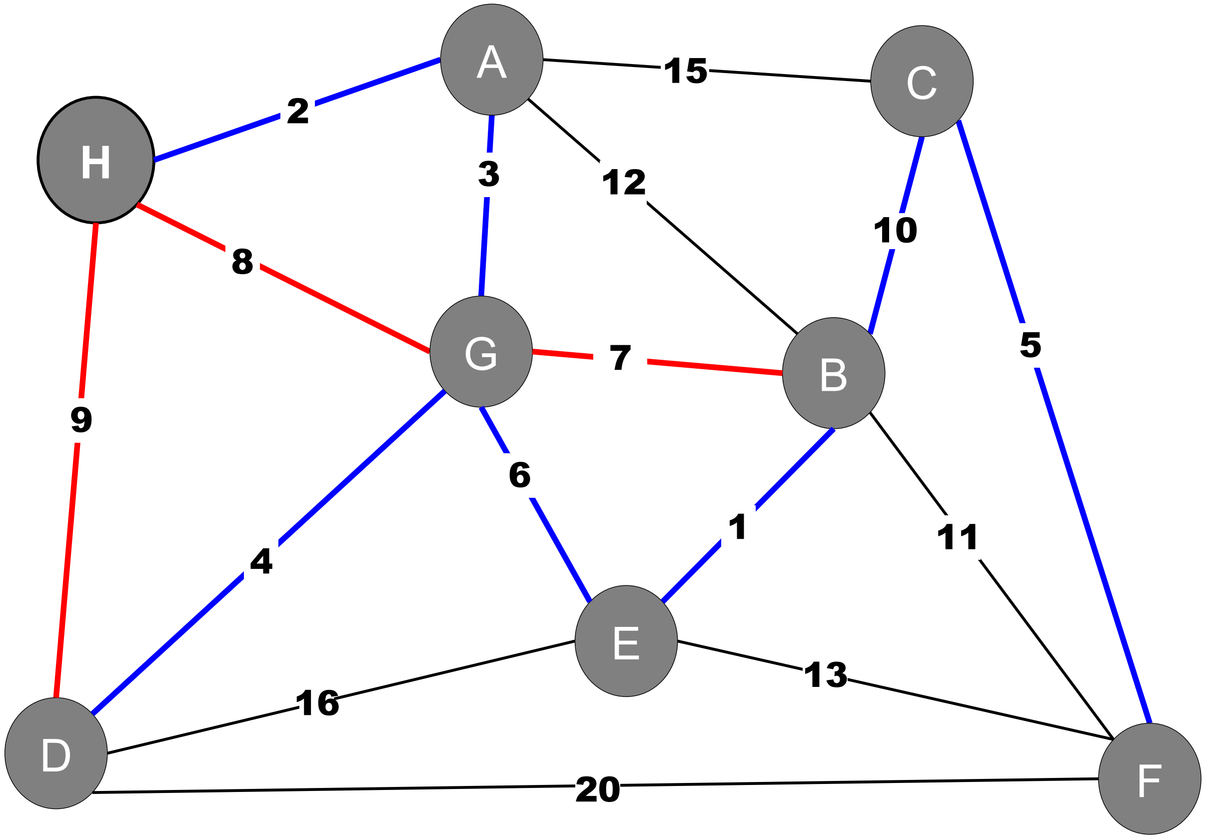
Kruskal's Example



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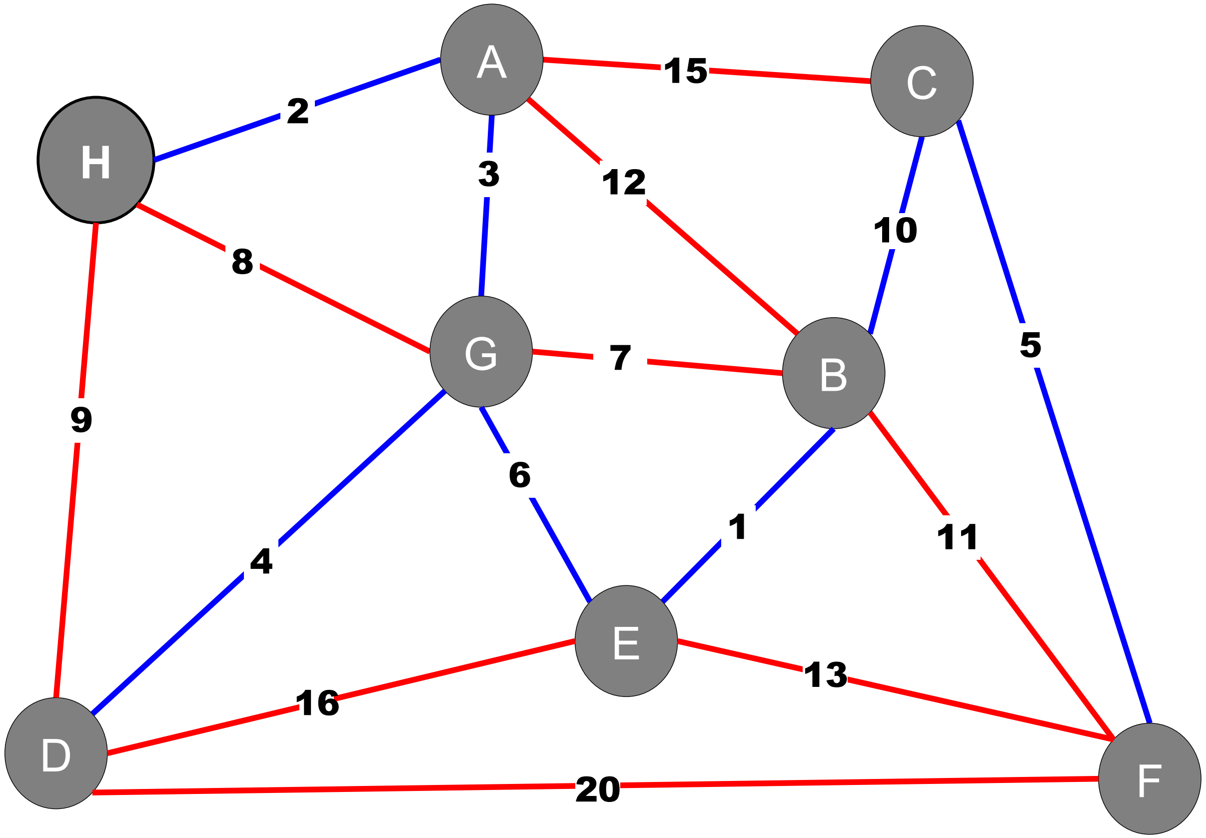
Kruskal's Example



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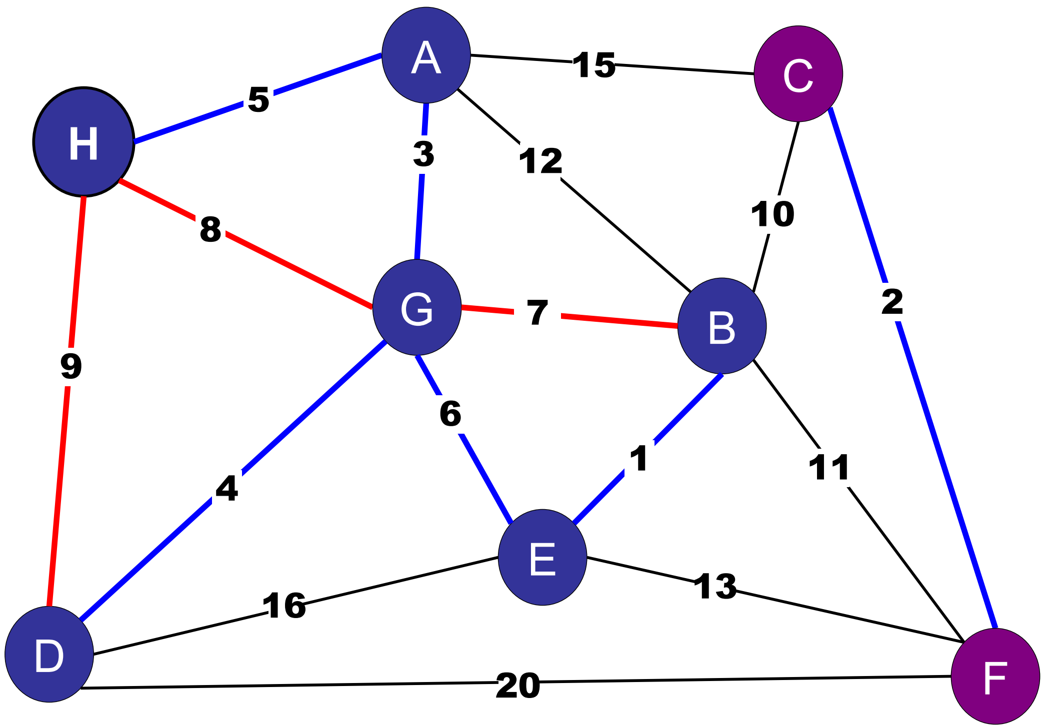
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Kruskal's Algorithm

Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

- Sort edges by weight.
- Consider edges in ascending order:
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.



Proof:

- Each added edge crosses a cut.
- Each edge is the lightest edge across the cut: all other lighter edges across the cut have already been considered.

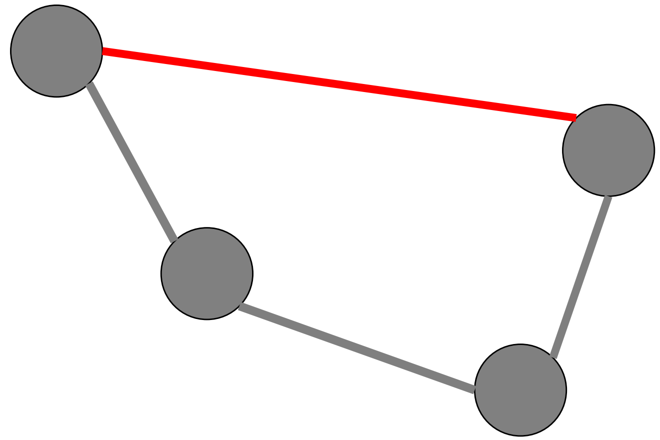
Generic MST Algorithm

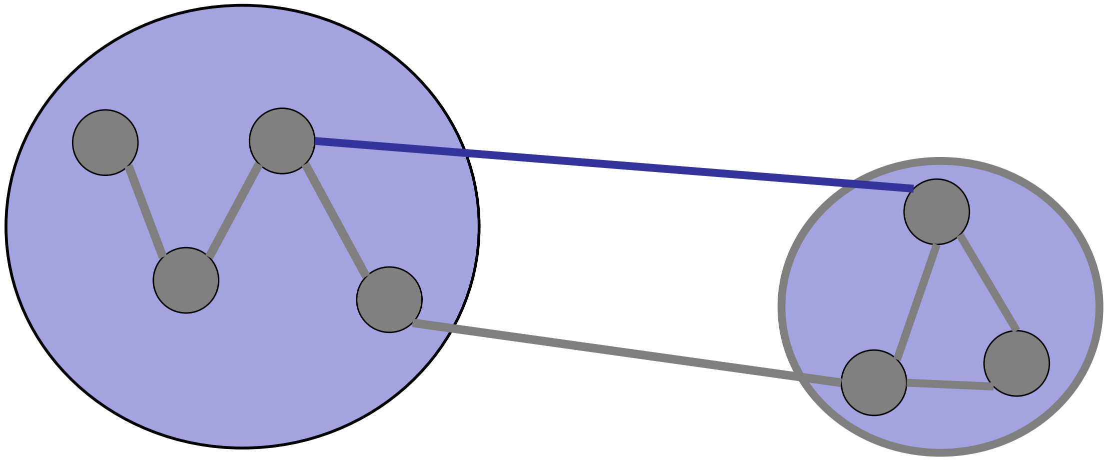
Greedy Algorithm:

Repeat:

Apply red rule or blue rule to an arbitrary edge.

until no more edges can be colored.





What is the running time of Kruskal's Algorithm on a connected graph?

1. $O(V)$
2. $O(E)$
3. $O(E\,\alpha)$
4. $O(V\,\alpha)$
5. $O(E \log V)$
6. $O(V \log E)$

Kruskal's Algorithm

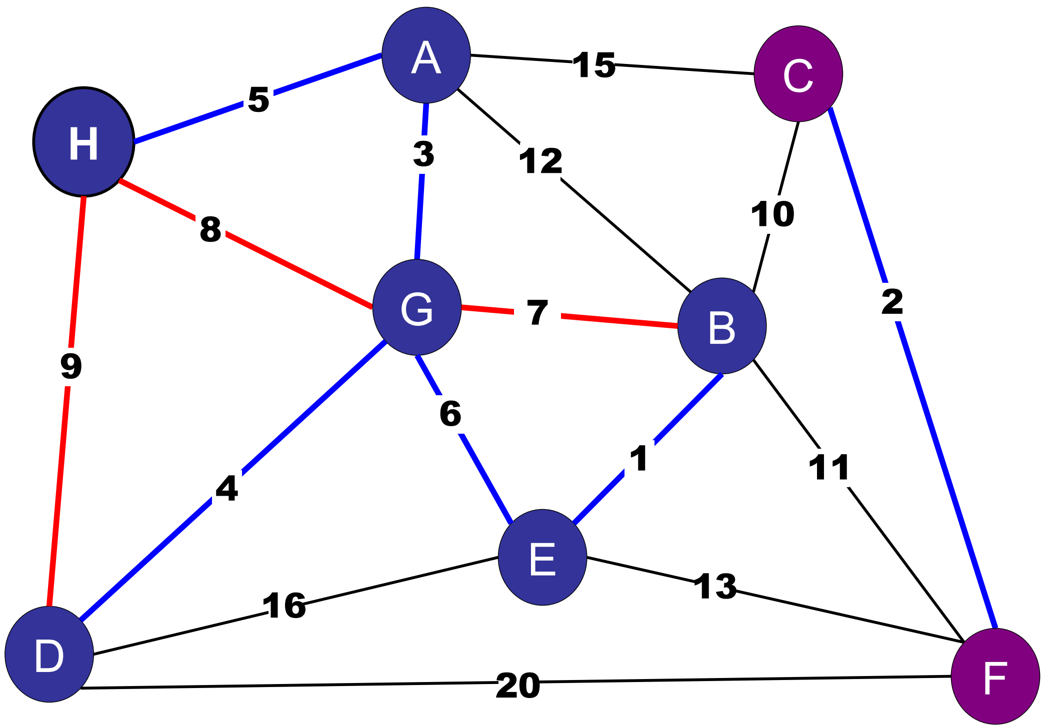
Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

- Sort edges by weight.
- Consider edges in ascending order:
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.

Performance:

- Sorting: O(E log E) = O(E log V)
- For E edges:
  - Find: $O(\alpha(n))$ or O(log V)
  - Union: $O(\alpha(n))$ or O(log V)



Roadmap

Minimum Spanning Trees

- Prim’s Algorithm
- Kruskal’s Algorithm
- (Boruvka’s Algorithm)

Variations

- Constant weight edges
- Bounded integer edge weights
- Directed graphs
- Maximum Spanning Tree

MST Variants

What if all the edges have the same weight?

How fast can you find an MST?

1. $O(V)$
2. $O(E)$
3. $O(E \log V)$
4. $O(V \log E)$
5. $O(VE)$

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MST Variants

What if all the edges have the same weight?

Depth-First-Search or Breadth-First-Search

If all edge-weights are 2, what is the **cost** of a MST?

1. V-1
2. V
3. 2(V-1)
4. 2V
5. E-V
6. E

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MST Variants

What if all the edges have the same weight?

- Depth-First-Search or Breadth-First-Search
- An MST contains exactly (V-1) edges.
- Every spanning tree contains (V-1) edges!
- Thus, any spanning tree you find with DFS/BFS is a minimum spanning tree.

Kruskal's Variants

What if all the edges have weights from {1..10}?

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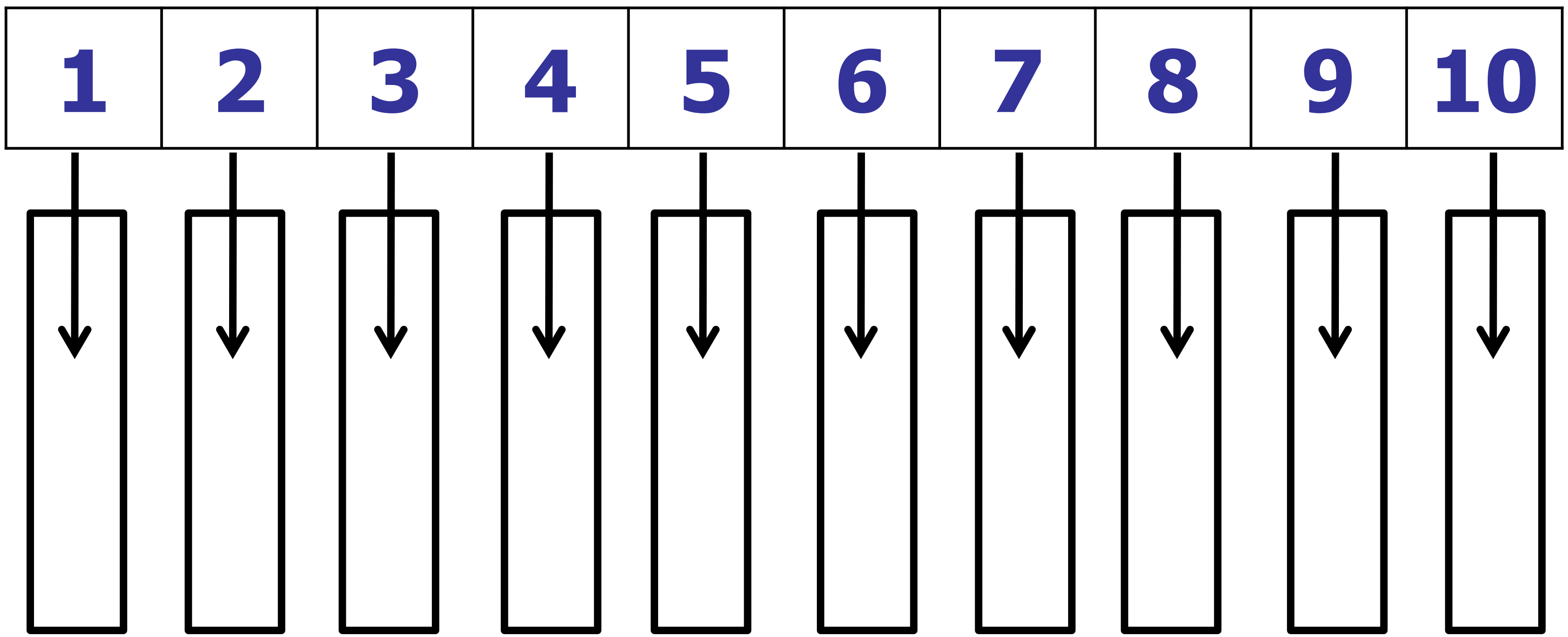
Kruskal's Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10 to sort



slot A[j] holds a linked list of edges of weight j

Kruskal's Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10

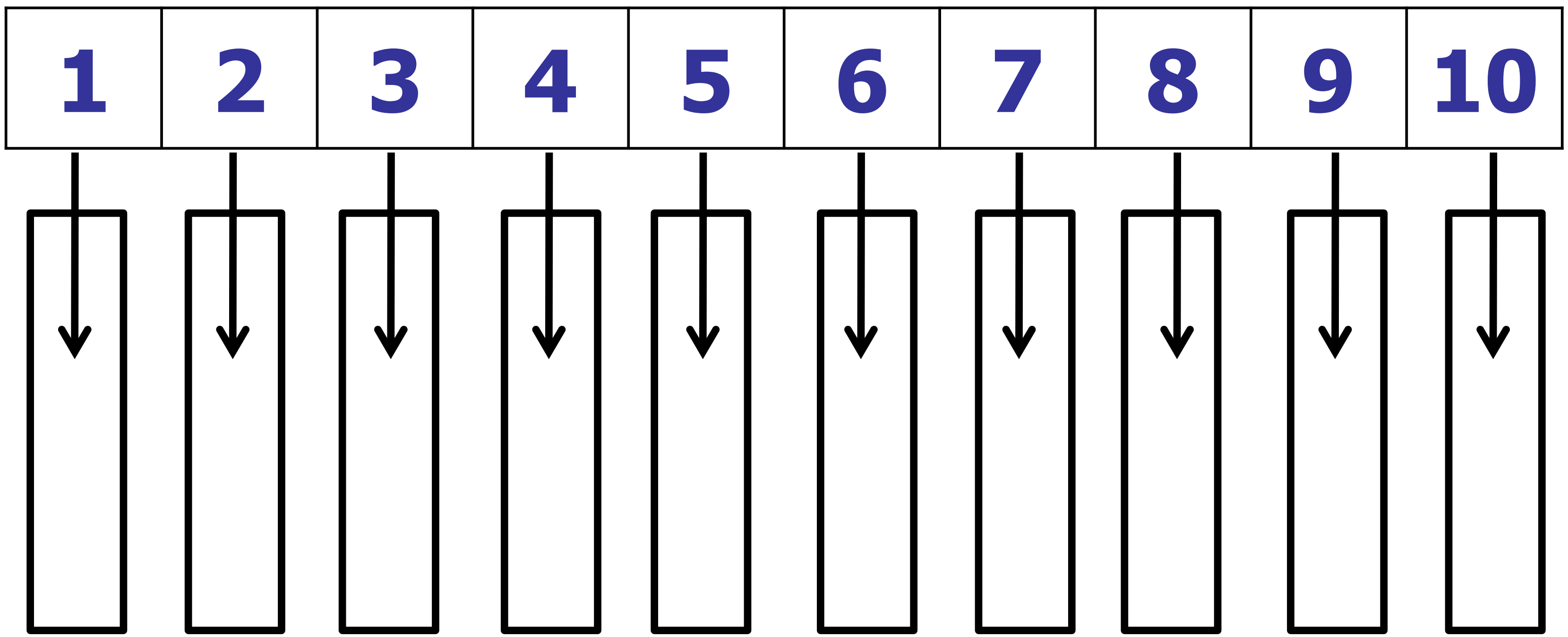
- Putting edges in array of linked lists: O(E)
- Iterating over all edges in ascending order: O(E)
- For each edge:
  - Checking whether to add an edge: O(α)
  - Union two components: O(α)

Total: O(αE)

Prim's Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10 as a Priority Queue

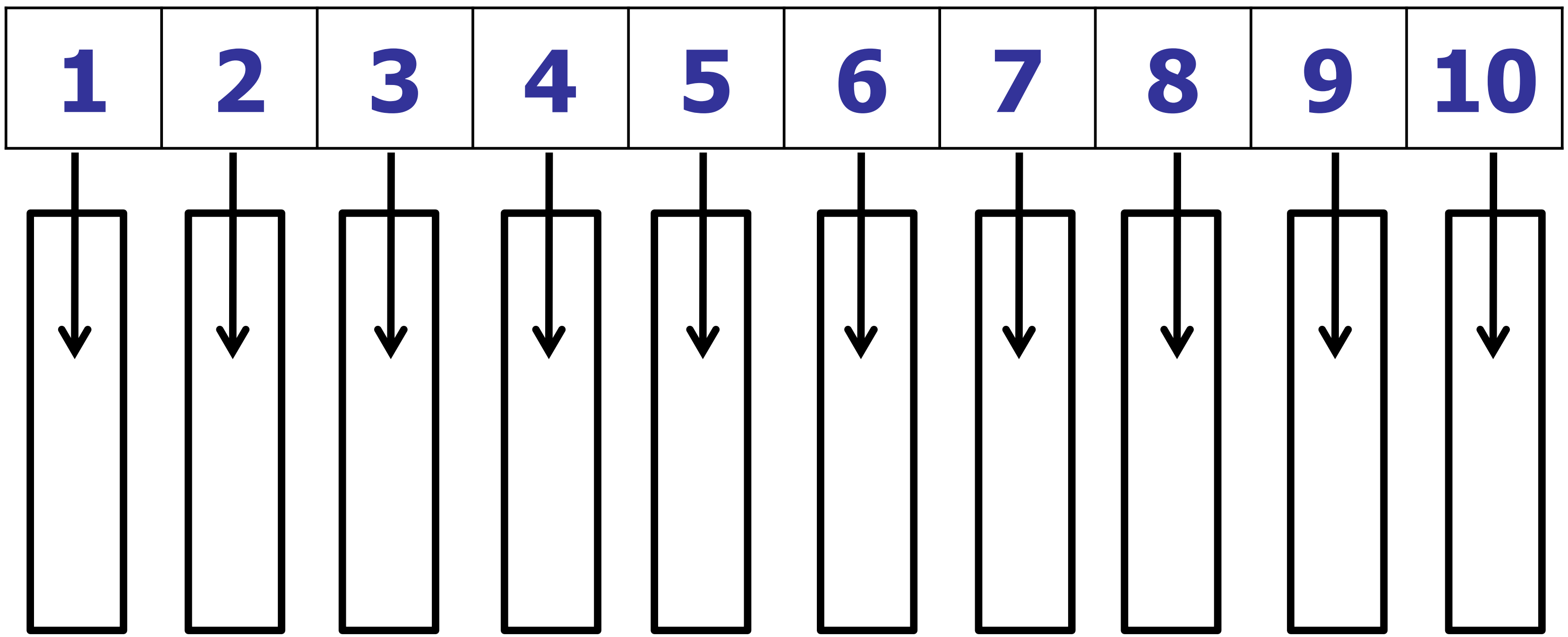


slot A[j] holds a linked list of **nodes** of weight j

Prim's Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10 as a Priority Queue



decreaseKey: move node to new linked list

What is the running time of (modified) Prim's if all the edge weights are in {1..10}?

1. O(V)
2. O(E)
3. O(E log V)
4. O(V log E)
5. O(EV)

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Prim's Variants

What if all the edges have weights from {1..10}?

Implement Priority Queue:

- Use an array of size 10 to implement
- Insert: put node in correct list
- Remove: lookup node (e.g., in hash table) and remove from liked list.
- ExtractMin: Remove from the minimum bucket.
- DecreaseKey: lookup node (e.g., in hash table) and move to correct liked list.

Prim's Variants

What if all the edges have weights from {1..10}?

Idea: Use an array of size 10

- Inserting/Removing nodes from PQ: $O(V)$
- decreaseKey: $O(E)$

Total: $O(V + E) = O(E)$

Prim's Variants

What if all the edges have weights from {1..10}?

Implement Priority Queue….

Why does this fail for Dijkstra's Algorithm?

Roadmap

Minimum Spanning Trees

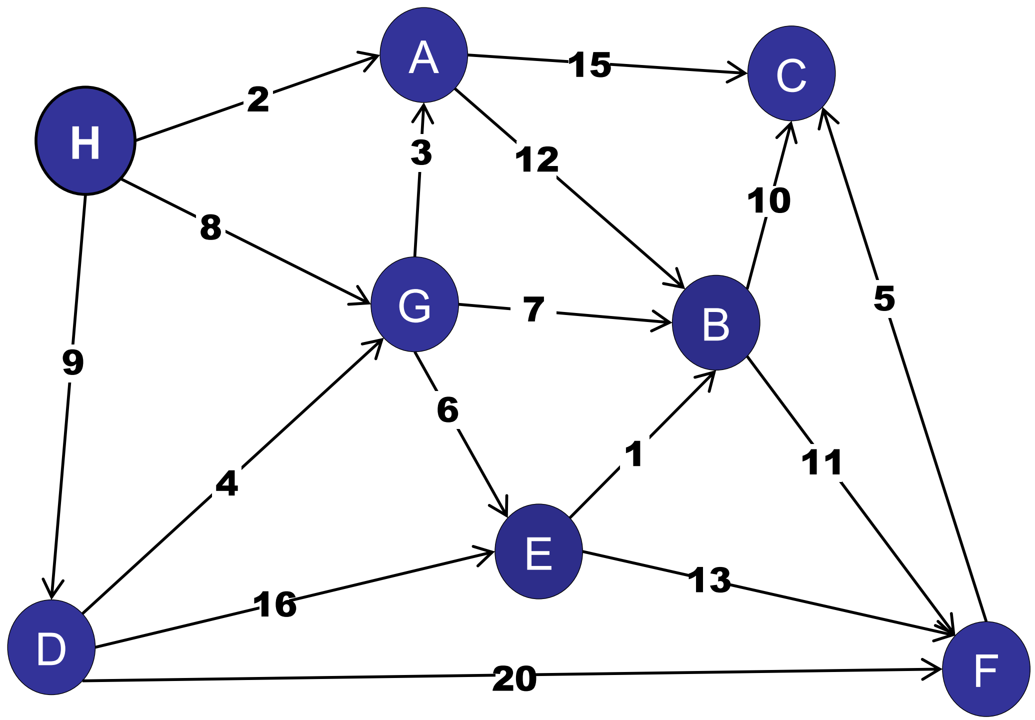
- Prim’s Algorithm
- Kruskal’s Algorithm
- (Boruvka’s Algorithm)

Variations:

- Constant weight edges
- Bounded integer edge weights
- Directed graphs
- Maximum Spanning Tree

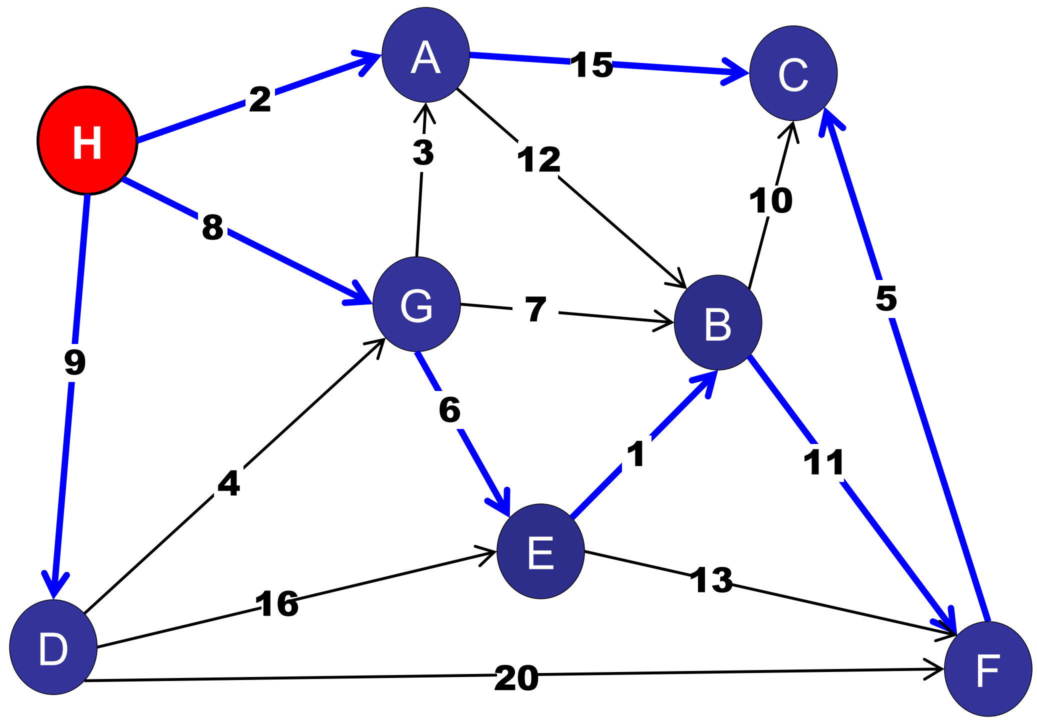
Directed Minimum Spanning Tree

What if the edges are directed?



Directed Minimum Spanning Tree

A rooted spanning tree:



Every node is reachable on a path from the root.

No cycles.

Directed Minimum Spanning Tree

Harder problem:

- Cut property does not hold.
- Cycle property does not hold.
- Generic MST algorithm does not work.

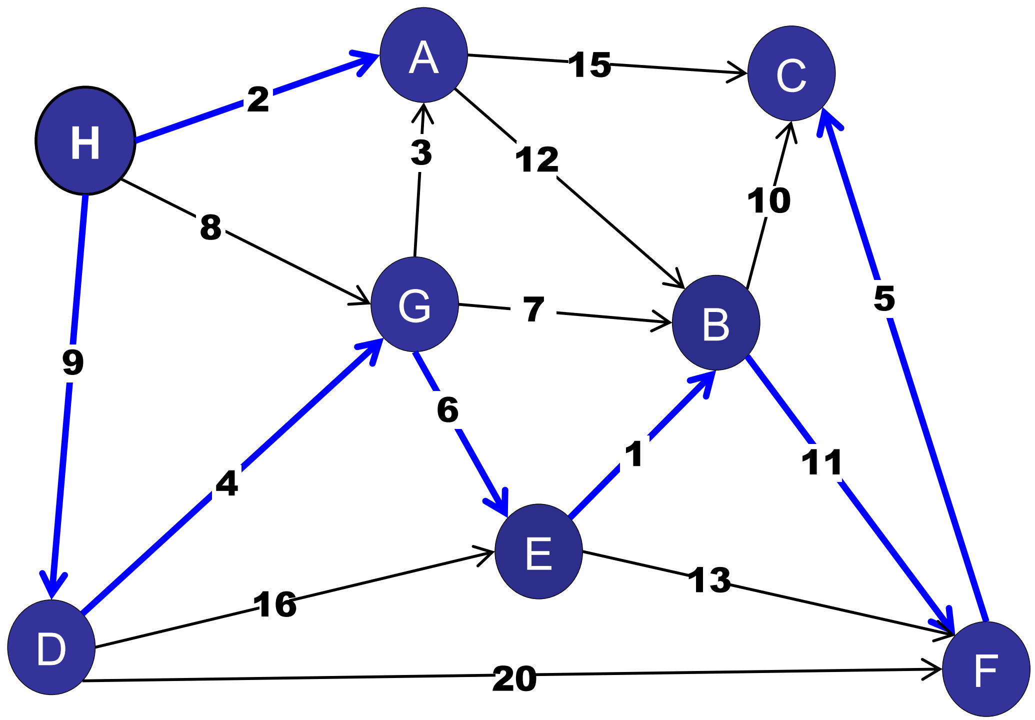
Prim's, Kruskal's, Boruvka's do not work.

See CS3230 / CS4234 for more details…

Exercise: Draw a directed graph where the cut property or cycle property is violated.

Directed Minimum Spanning Tree

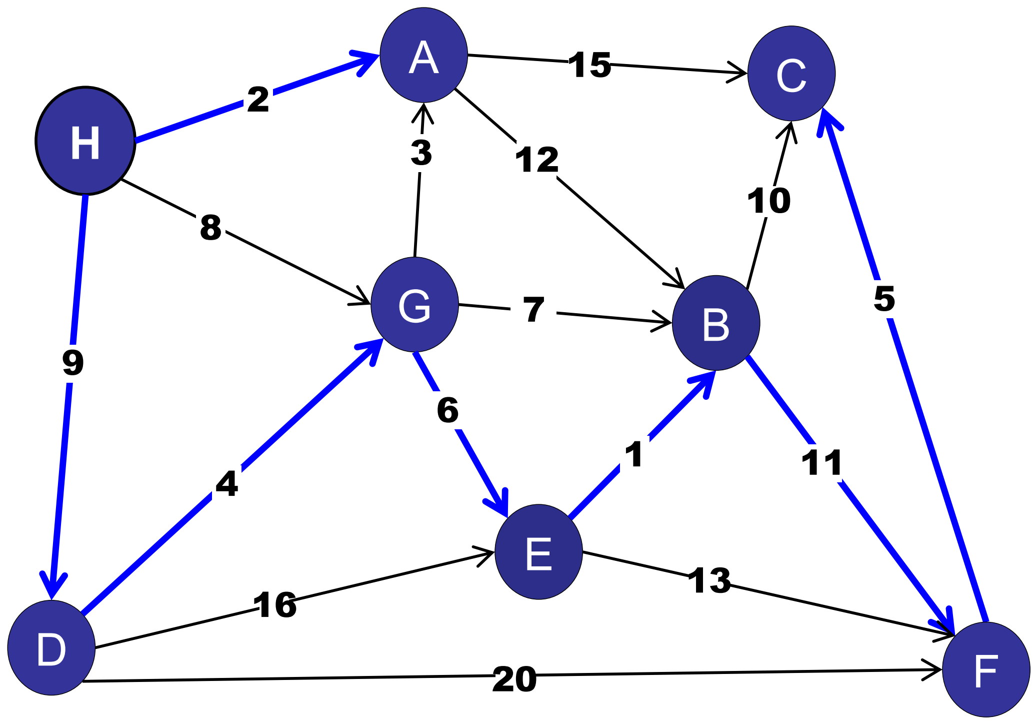
Special case: a directed acyclic graph with one “root”:



Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.



Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.

Observations:

- No cycles (since acyclic graph).
- Each edge is chosen only once.

Tree:
V nodes
V – 1 edges
No cycles

Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.

Observations:

- No cycles (since acyclic graph).
- Each edge is chosen only once.

Tree:
V nodes
V – 1 edges
No cycles

- Every node has to have at least one incoming edge in the MST, so this is the minimum spanning tree.

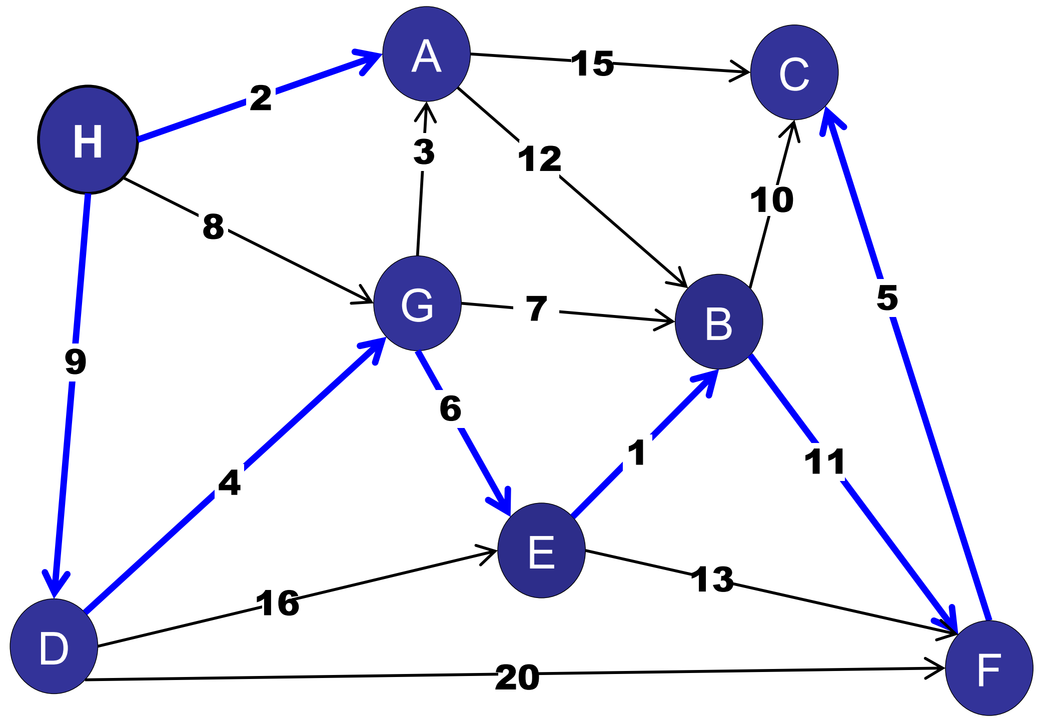
Directed Minimum Spanning Tree

For a directed acyclic graph with one "root":

For every node except the root: add minimum weight incoming edge.

Conclusion: Minimum Spanning Tree

O(E) time



Maximum Spanning Tree

A MaxST is a spanning tree of maximum weight.

How do you find a MaxST?

Maximum Spanning Tree

Reweighting a spanning tree:

– What happens if you add a constant k to the weight of every edge?

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Kruskal's Algorithm

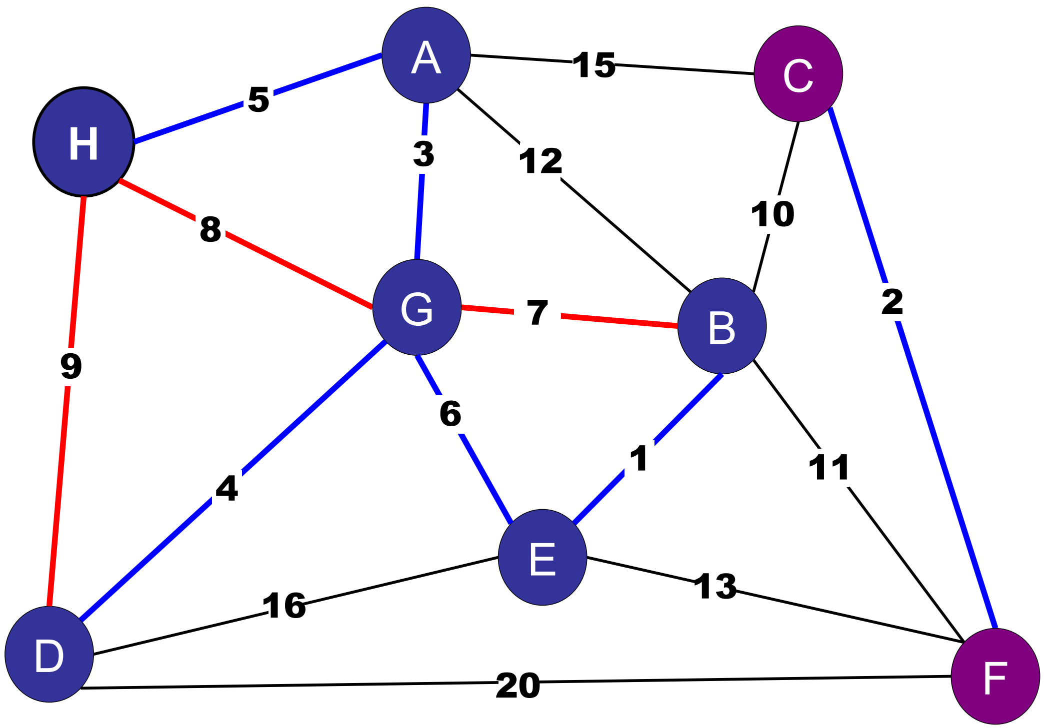
Kruskal's Algorithm. (Kruskal 1956)

Basic idea:

- Sort edges by weight.
- Consider edges in ascending order:
  - If both endpoints are in the **same** blue tree, then color the edge red.
  - Otherwise, color the edge blue.

What matters?

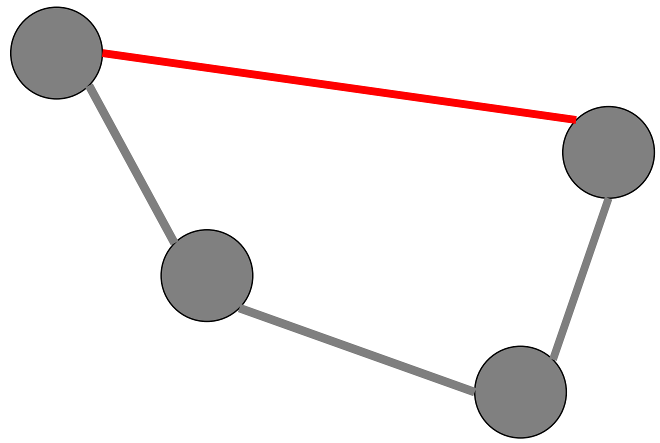
- Relative edge weights.
- Absolute edge weights have no impact.



Generic MST Algorithm

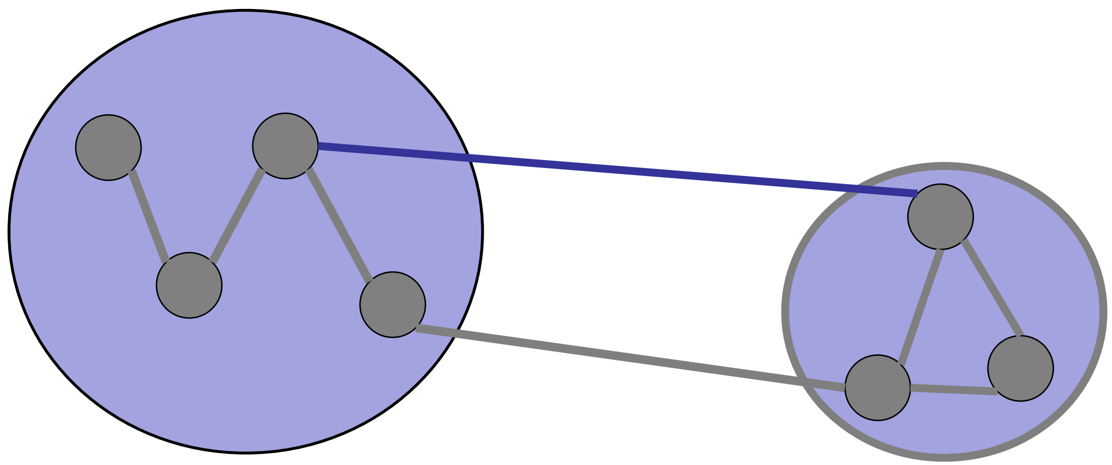
Red rule:

If C is a cycle with no red arcs, then color the max-weight edge in C red.



Blue rule:

If D is a cut with no blue arcs, then color the min-weight edge in D blue.



Maximum Spanning Tree

Reweighting a spanning tree:

- What happens if you add a constant k to the weight of every edge?

No change!

We can add or subtract weights without effecting the MST.

(Very *different* from shortest paths…)

Maximum Spanning Tree

MST with negative weights?

Maximum Spanning Tree

MST with negative weights?

No problem!

1. Reweight MST by adding a big enough value to each edge so that it is positive.

2. Actually, no need to reweight. Only relative edge weights matter, so negative weights have no bad impact.

Maximum Spanning Tree

A MaxST is a spanning tree of maximum weight.

How do you find a MaxST?

Easy!

1. Multiply each edge weight by -1.
2. Run MST algorithm.
3. MST that is “most negative” is the maximum.

Maximum Spanning Tree

A MaxST is a spanning tree of maximum weight.

How do you find a MaxST?

Or… run Kruskal's in reverse.

Roadmap

Minimum Spanning Trees

- The MST Problem
- Basic Properties of an MST
- Generic MST Algorithm
- Prim's Algorithm
- Kruskal's Algorithm
- Variations
- **Boruvka's Algorithm**

MST Algorithms

Classic:

- Prim’s Algorithm
- Kruskal’s Algorithm

Modern requirements:

- Parallelizable
- Faster in “good” graphs (e.g., planar graphs)
- Flexible

Boruvka’s Algorithm

Origin: 1926

- Otakar Boruvka
- Improve the electrical network of Moravia

Based on generic algorithm:

- Repeat: add all “obvious” blue edges.
- Very simple, very flexible.

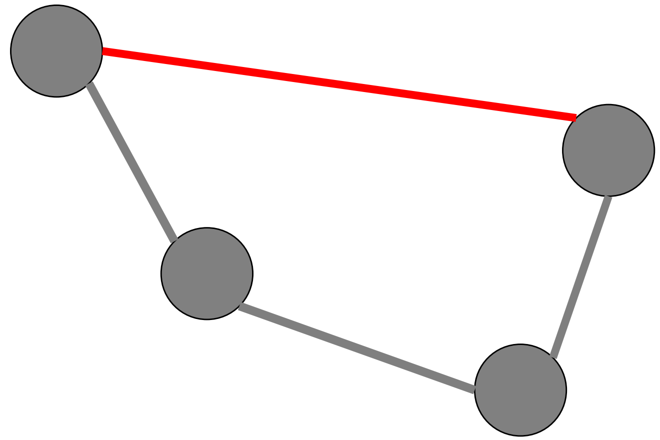
Generic MST Algorithm

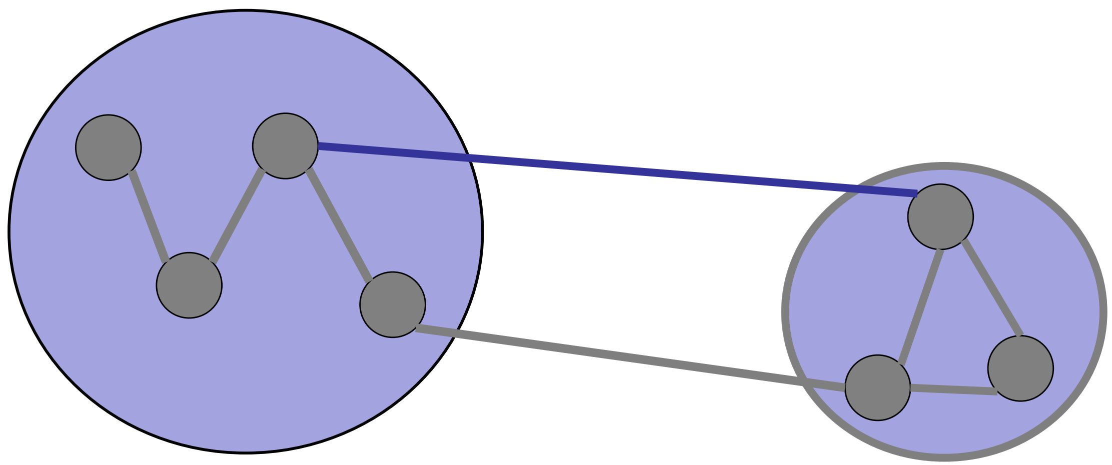
Greedy Algorithm:

Repeat:

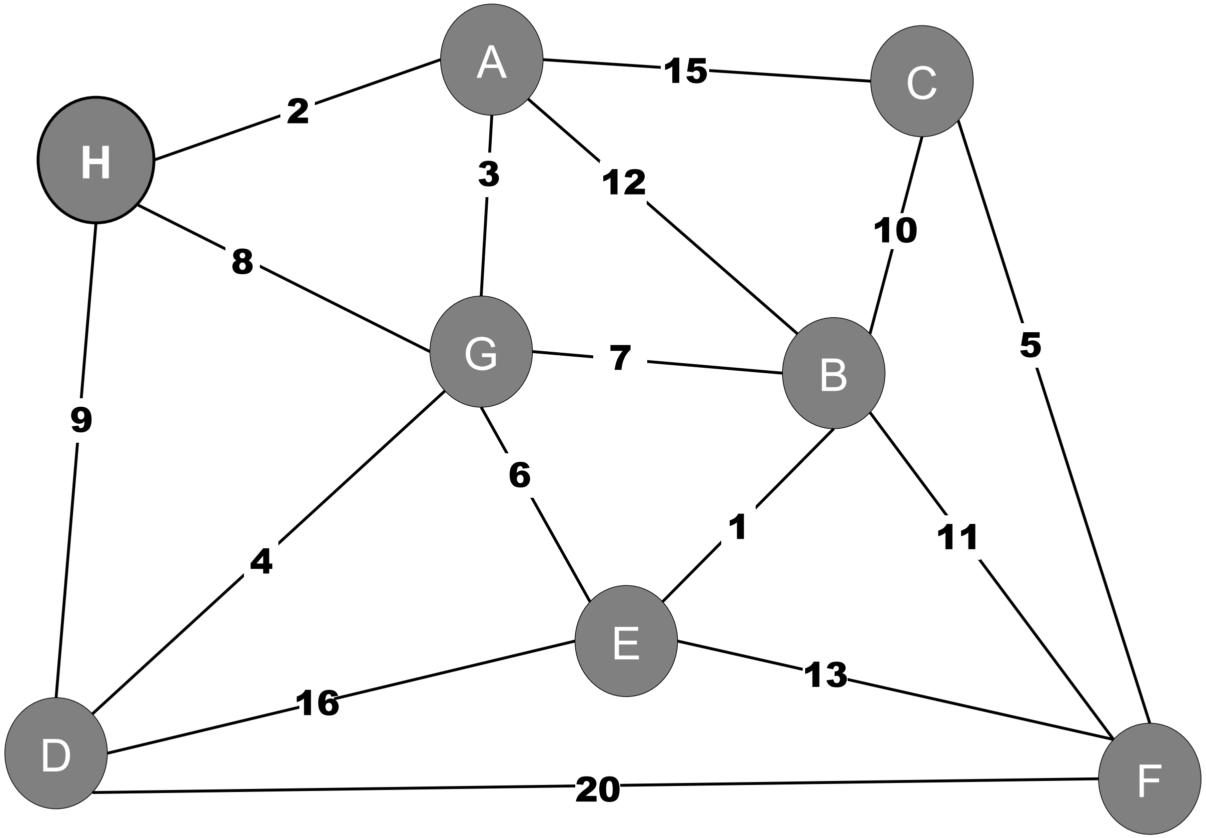
Apply red rule or blue rule to an arbitrary edge.

until no more edges can be colored.





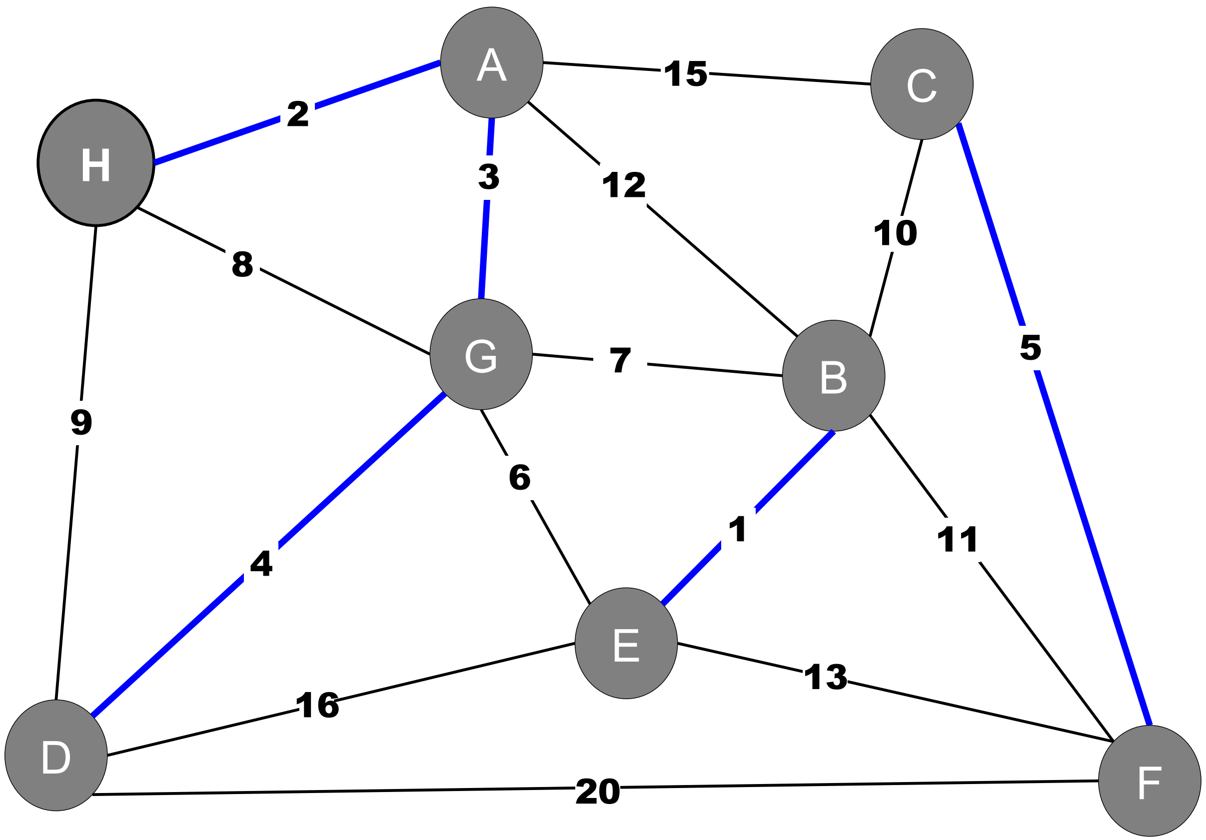
Boruvka's Example



Which edges are "obviously" in the MST?

Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Boruvka's Example



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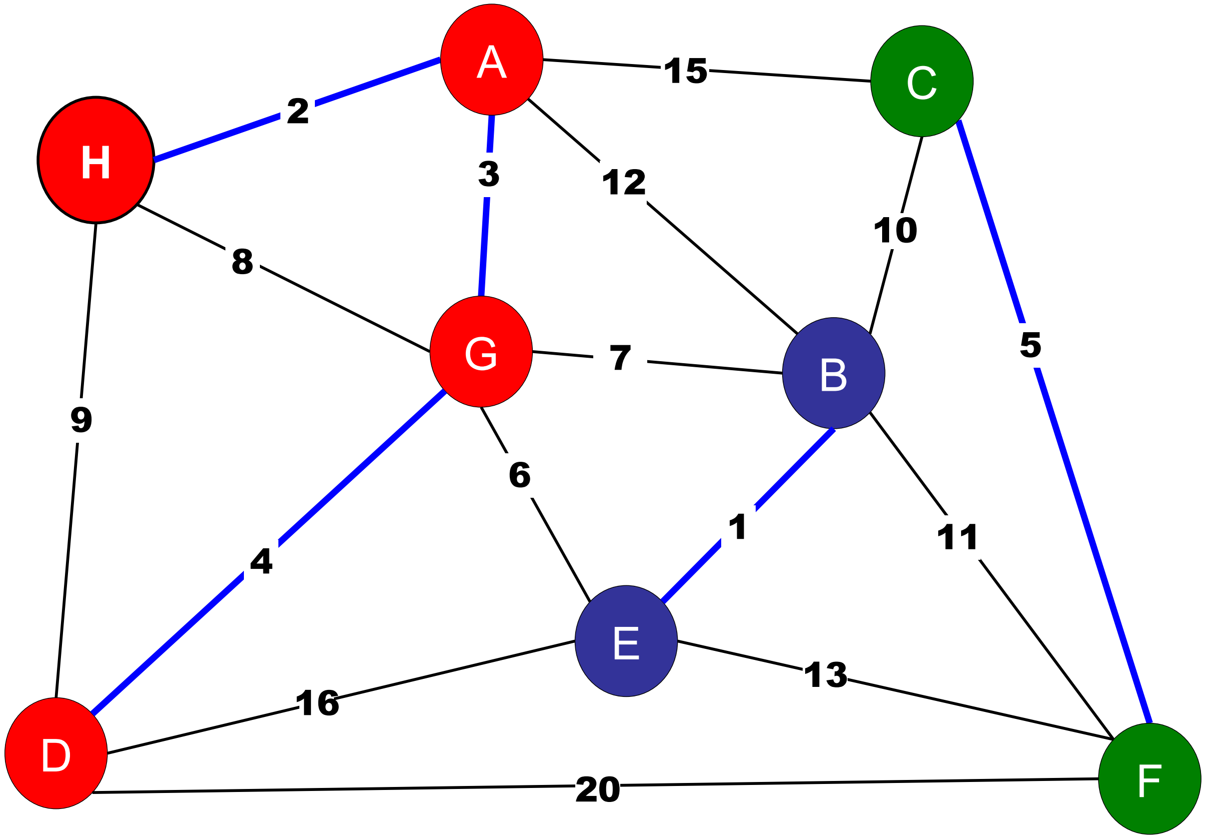
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

For every node: add minimum adjacent edge.

Add at least n/2 edges.

Boruvka's Example



Look at connected components…

At most n/2 connected components.

Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
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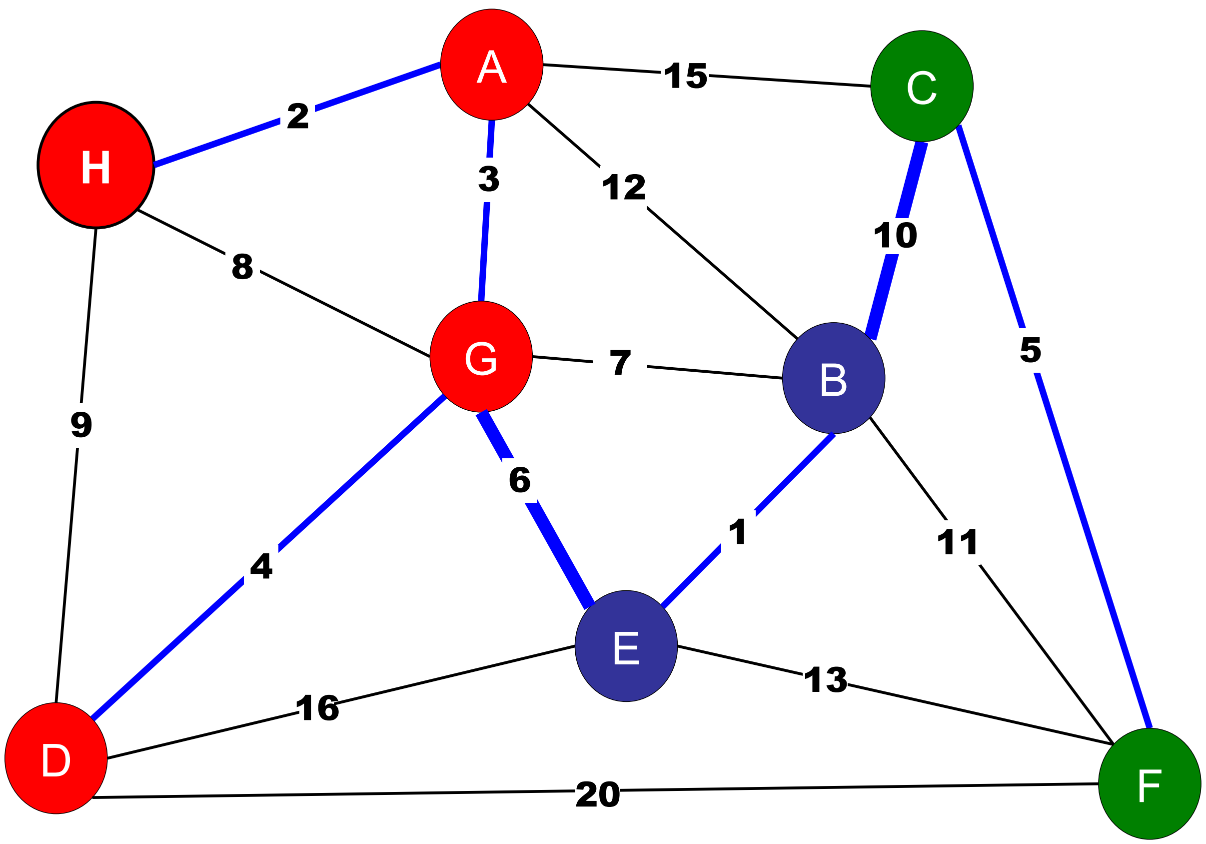
Boruvka's Example



Repeat: for every connected components, add minimum outgoing edge.

Weight	Edge
1	(E,B)
2	(C,F)
3	(A,G)
4	(D,G)
5	(C,F)
6	(E,G)
7	(B,G)
8	(G,H)
9	(D,G)
10	(B,C)
11	(B,F)
12	(A,B)
13	(E,F)
15	(A,C)
16	(D,E)
20	(D,F)

Boruvka's Algorithm

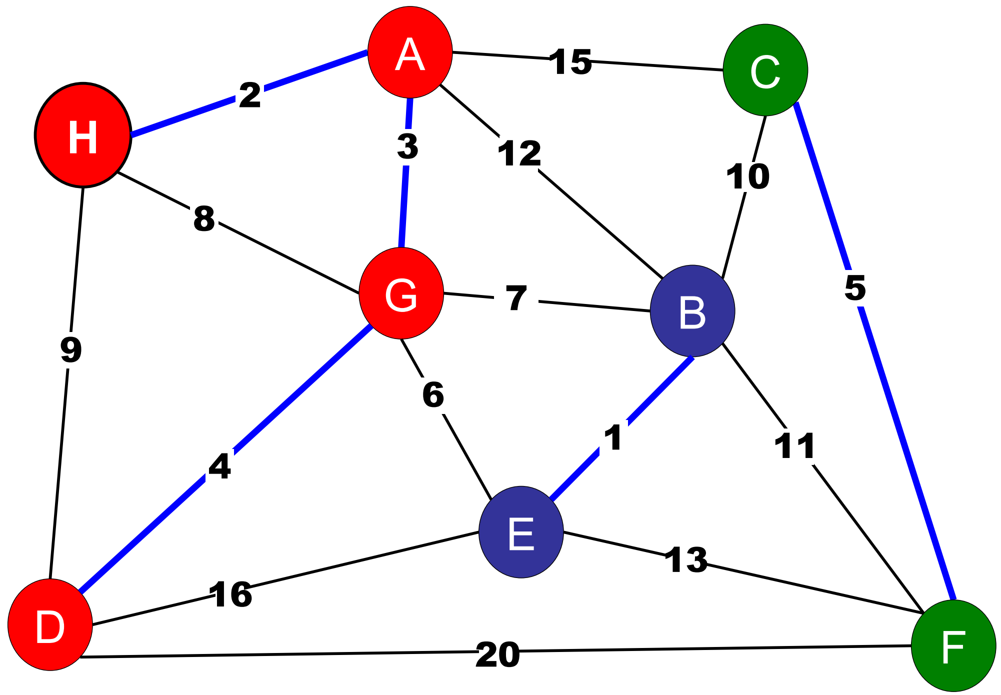
Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

One "Boruvka" Step:

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.



Boruvka's Algorithm

Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

For each node: store a component identifier.

One "Boruvka" Step:

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.

Boruvka's Algorithm

For each node: store a component identifier.

Boruvka's Algorithm

Initially:

– Create n connected components, one for each node in the graph.

DFS or BFS:

Check if edge connects two components.

Remember minimum cost edge connected to each component.

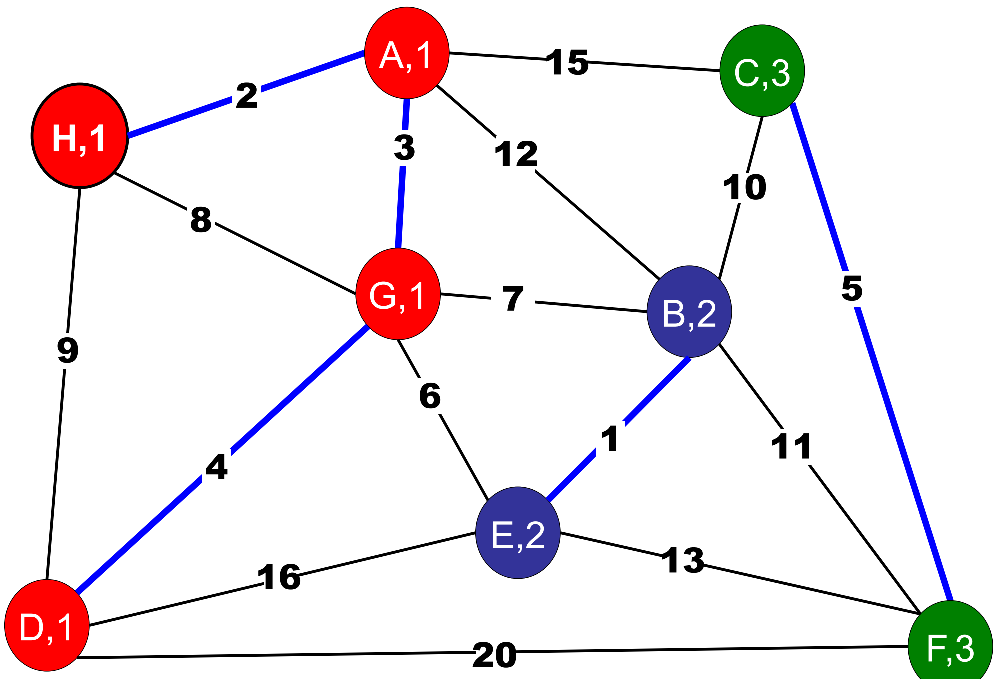
One "Boruvka" Step:

– For each connected component, search for the minimum weight outgoing edge.

– Add selected edges.

– Merge connected components.

Component	1	2	3
Min cost edge	(G,E), 6	(G,E), 6	(B,C), 10
To be merged	1 and 2	1 and 2	2 and 3



Boruvka's Algorithm

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- Merge connected components.

Component	1	2	3
Min cost edge	(G,E), 6	(G,E), 6	(B,C), 10
To be merged	1 and 2	1 and 2	2 and 3
New ID:	1	1	1

For each node: store a component identifier.

DFS or BFS:

Check if edge connects two components.

Remember minimum cost edge connected to each component.

Scan every node:

Computer new component ids.

Update component ids.

Mark added edges.

Boruvka’s Algorithm

Boruvka’s Algorithm

Initially:

- Create n connected components, one for each node in the graph.

One “Boruvka” Step: O(V+E)

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.

For each node: O(V)
store a component identifier.

DFS or BFS: O(V + E)
Check if edge connects two components.

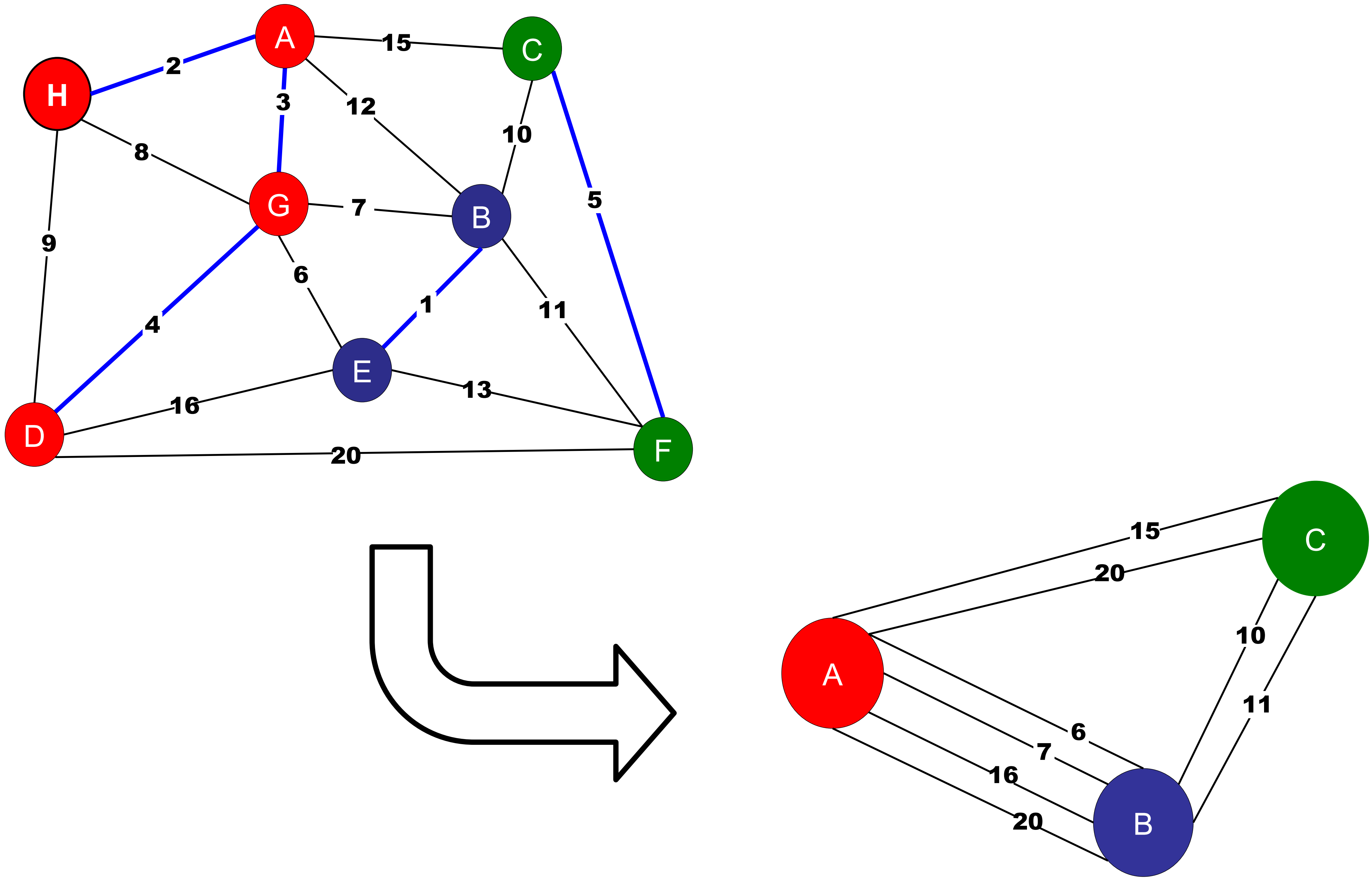
Remember minimum cost edge connected to each component.

Scan every node: O(V)
Computer new component ids.

Update component ids.

Mark added edges.

Boruvka's Example: Contraction



Boruvka's Algorithm

Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

In each "Boruvka" Step: O(V+E)

- Assume k components, initially.
- At least k/2 edges added.

Count edges:

Each component adds one edge.

Some choose same edge.

Each edge is chosen by at most two different components.

Boruvka’s Algorithm

Boruvka’s Algorithm

Initially:

- Create n connected components, one for each node in the graph.

In each “Boruvka” Step: O(V+E)

- Assume k components, initially.
- At least k/2 edges added.
- At least k/2 components merge.

Merging:
Each edge merges two components

Boruvka’s Algorithm

Boruvka’s Algorithm

Initially:

- Create n connected components, one for each node in the graph.

In each “Boruvka” Step: O(V+E)

- Assume k components, initially.
- At least k/2 edges added.
- At least k/2 components merge.
- At end, at most k/2 components remain.

Boruvka's Algorithm

Boruvka's Algorithm

Initially:

n components

At each step:

k components ➔ k/2 components.

Termination:

1 component

Conclusion:

At most O(log V) Boruvka steps.

Boruvka's Algorithm

Boruvka's Algorithm

Initially:

n components

At each step:

k components ➔ k/2 components.

Termination:

1 component

Conclusion:

At most O(log V) Boruvka steps.

Total time:

$O((E+V)\log V) = O(E \log V)$

Boruvka's Algorithm

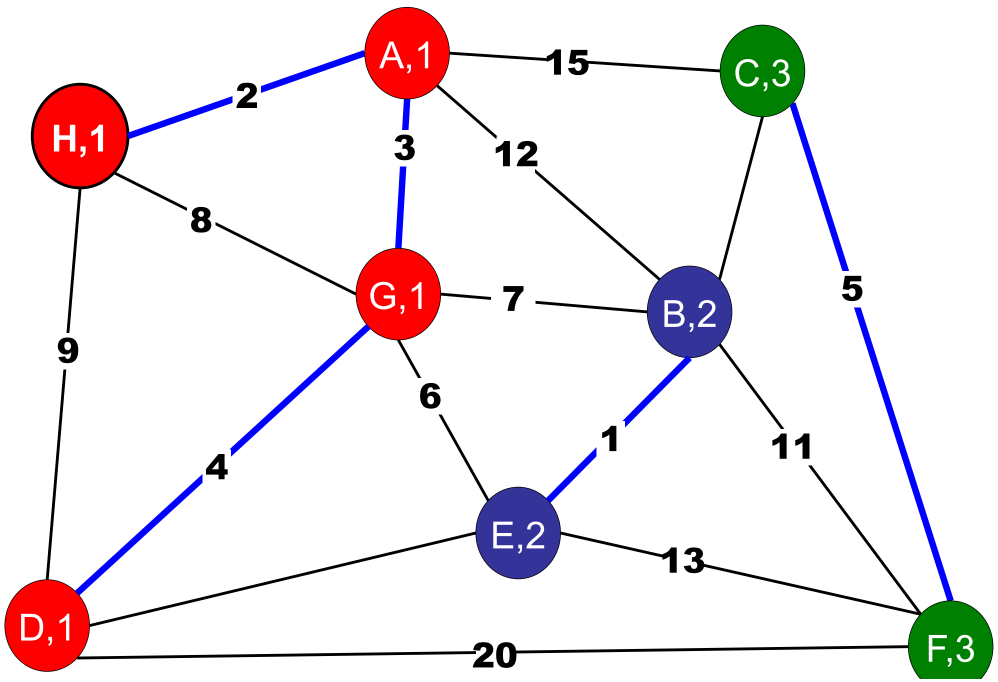
Boruvka's Algorithm

Initially:

- Create n connected components, one for each node in the graph.

One "Boruvka" Step: O(V+E)

- For each connected component, search for the minimum weight outgoing edge.
- Add selected edges.
- Merge connected components.



Roadmap

So far:

Minimum Spanning Trees

- Prim’s Algorith
- Kruskal’s Algorithm
- Boruvka’s Algorithm

Minimum Spanning Tree Summary

Classic greedy algorithms: O(E log V)

- Prim's (Priority Queue)
- Kruskal's (Union-Find)
- Boruvka's

Best known: $O(m\ \alpha(m, n))$

- Chazelle (2000)

Holy grail and major open problem: $O(m)$

Minimum Spanning Tree Summary

Classic greedy algorithms: O(E log V)

- Prim's (Priority Queue)
- Kruskal's (Union-Find)
- Boruvka's

Best known: $O(m\ \alpha(m, n))$

- Chazelle (2000)

Holy grail and major open problem: O(m)

- Randomized: Karger-Klein-Tarjan (1995)
- Verification: Dixon-Rauch-Tarjan (1992)

CS2040S
Data Structures and Algorithms

All about minimum spanning trees…