CS2100

Computer Organization

AY2022/23 Semester 1

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Contents

Part I

Data Representation

Data is represented as a sequence of **bits** — `0` or `1`.

8 bits form a **byte**, and a fixed multiple of bytes in a computer architecture forms a **word**.

number of bits $\iff$ number of values

n bits can represent up to 2^n values.

Conversely, we need at least $\lceil \log_2 m \rceil$ bits to represent m values.

1 Number Systems

Number systems are **weighted-positional** systems, with the **decimal** number system having a **base** (or **radix**) of 10.

number system	base	prefix
binary	2	0b
octal	8	0
decimal	10	—
hexadecimal	16	0x

The digit to the left of the decimal point has a weight of 10^0, with the exponent *incrementing* leftward and *decrementing* rightward.

decimal $\implies$ binary

For *whole numbers*, use **repeated division-by-two**:

1. divide the value by 2, and record the remainder
2. prepend the remainder to the binary representation
3. repeat until the value is 0

For *fractions*, use **repeated multiplication-by-two**:

1. multiply the value by 2, and record the carry
2. append the carry to the binary representation
3. repeat until the value is 1
4. append the 1 to the binary representation

binary $\implies$ decimal

Use **weighted summation**:

1. assign the digit to the left of the binary point a weight of 2^0
2. increment the weight exponents leftward
3. decrement the weight exponents rightward
4. multiply each digit by its weight
5. sum the products

These conversions generalize to any base!

Converting between 2 bases can be done with base-10 as an intermediary.

binary $\iff$ base-2^k

1. partition the binary representation into k-bit chunks outward from the binary point
2. convert each chunk to base-10
3. convert each base-10 value to base-2^k

Apply the reverse direction for converting from base-2^k to binary.

1.1 Negative Numbers

Signed numbers include all positive and negative values, unlike **unsigned numbers** which only include non-negative values.

We will consider n-bit values in this section.

sign-and-magnitude

The MSB is the **sign bit** — `0` for positive, `1` for negative.

Negate a value by inverting the sign bit.

- **range**: $-2^{n-1}+1$ to $2^{n-1}-1$
- **zeros**: -0_{10} and $+0_{10}$

1s complement

An n-bit number x has a negated value $-x = 2^n - x - 1$.

Negate a value by inverting *all the bits*.

- **range**: $-2^{n-1}+1$ to $2^{n-1}-1$
- **zeros**: -0_{10} (all `1`s) and $+0_{10}$ (all `0`s)

The weight of the MSB is $-2^{n-1}+1$.

2s complement

An n-bit number x has a negated value $-x = 2^n - x$.

Negate a value by inverting all the bits to the left of the *rightmost* `1`.

- **range**: -2^{n-1} to $2^{n-1}-1$
- **zeros**: 0_{10} (all `0`s)

Unlike sign-and-magnitude and 1s complement, 2s complement does not have duplicate zeros.

The weight of the MSB is -2^{n-1}.

The MSB also represents the sign — `0` for positive, `1` for negative.

Negating a fractional number in a complement representation is no different than negating a whole number!

excess representation

Distribute positive and negative values by a simple addition or subtraction by k, which is also known as the **bias**.

k is typically $2^n - 1$ for an n-bit number.

The binary value in excess-k *subtracts* k from its decimal value.

Conversely, the decimal value in excess-k *adds* k to its binary value.

1.2 Overflow

Overflow occurs when addition or subtraction goes beyond the fixed range of a signed number.

Subtraction is equivalent to addition of the negated value.

Note that in 1s addition, the carry out of the MSB is *added* to the result, but this carry out is *discarded* in 2s addition.

detecting overflow

If the MSB (sign bit) of the result is different from the MSBs of the operands, then overflow has occurred.

Note that overflow can only occur when the operands have the same sign, i.e., adding a negative number to a positive number will *never* result in overflow.

1.3 Real Numbers

Fixed-point representation fixes the number of bits for the whole number and fractional parts, which limits the range of values.

The IEEE 754 **floating-point representation** resolves this by allocating a fixed number of bits to the **sign**, **exponent**, and **mantissa**.

precision	bits	sign	exp.	mantissa	bias
single	32	1	8	23	127
double	64	1	11	52	1023

The **mantissa** is **normalized** with an implicit leading 1 bit, e.g. 110.1_2 is normalized to $1.101_2 \times 2^2$, and only 101 is stored in the mantissa.

decimal $\Longrightarrow$ IEEE 754 floating-point representation

1. convert the decimal value to normalized binary, i.e., $\pm 1.xxx \times 2^y$
2. determine the sign bit — 0 for positive, and 1 for negative
3. add the bias to the exponent y — $+127$ for single precision, and $+1023$ for double precision
4. convert the exponent to binary
5. concatenate the sign bit, exponent, and mantissa xxx

Part II

Instruction Set Architecture

2 ISA Overview

An **instruction set architecture** (ISA) is an abstraction on the interface between hardware and low-level software, which runs on a **processor**.

Processors are connected to **memory** (RAM) via a **bus**, across which code and data is transferred.

ISAs have two major design philosophies:

1. **CISC** — Complex Instruction Set Computer:
   - single instructions for complex operations
   - smaller program sizes
   - complex implementation, little room for hardware optimization
2. **RISC** — Reduced Instruction Set Computer:
   - smaller and simpler instruction set
   - software combines simpler operations to implement high-level language statements
   - room for compiler optimization

2.1 Data Storage Architecture

1. **stack architecture**:
   operands are implcitly popped from the stack
2. **accumulator architecture**:
   one operand is implcitly stored in an accumulator
3. **general-purpose register architecture**:
   operands are stored explicitly in registers, operations are register-memory or register-register
4. **memory-memory architecture**:
   operands are read from memory

2.2 Memory Architecture

Memory transfer takes place across buses:

1. **address bus**: k-bits, uni-directional from processor to memory
2. **data bus**: n-bits, bi-directional
3. **control lines**: bi-directional, e.g. read/write controls

The address bus feeds addresses from the **memory address register** to the memory.

Data is written to or read from the **memory data register**, depending on the R/W control line.

Endianness is the relative ordering of the *bytes* in a word (*not* the bits in a byte!) in memory:

- **big-endian**: MSB stored in smallest address
- **little-endian**: LSB stored in smallest address

3 Instruction Set Encoding

Instructions consist of an **opcode** — a unique code to identify the operation — as well as **operands**.

In an ISA with **fixed-length instructions**, we need to fit multiple sets of instruction types each with the same number of bits.

We use the **expanding opcode** scheme in which the opcode has variable length for different instructions.

maximizing the total number of instructions

Maximize the number of instructions in each set, starting from the *largest* set to the smallest.

minimizing the total number of instructions

Maximize the number of instructions in each set, starting from the *smallest* set to the largest.

Part III

MIPS

4 Registers

Memory access is slow. To avoid frequent memory access, temporary values are stored in the processor in **registers**, which are limited in number.

Registers *do not have data types*, unlike program variables. Instructions always assume that the data stored in a register is of the correct type.

MIPS has **32 registers**, which are referred to by *number* or *name*:

name	#	usage
`$zero`	`0`	constant value zero
`$at`	`1`	reserved for the assembler
`$v0` - `$v1`	2 - 3	values for results and expression evaluation
`$a0` - `$a3`	4 - 7	arguments
`$t0` - `$t7`	8 - 15	temporaries
`$s0` - `$s7`	16 - 23	program variables
`$t8` - `$t9`	24 - 25	temporaries
`$k0` - `$k1`	26 - 27	reserved for the OS
`$gp`	28	global pointer
`$sp`	29	stack pointer
`$fp`	30	frame pointer
`$ra`	31	return address

5 MIPS Assembly Language

The **general instruction syntax** is as follows:

`op $s0, $s1, $s2`	
	description
`op`	operation
`$s0`	destination register
`$s1`	source register 1
`$s2`	source register 2

Each instruction executes a *single* command. Each line of assembly code contains *at most* one instruction.

Almost all MIPS operations are *register-to-register*

The # hex synbol is used for comments.

5.1 Arithmetic Instructions

instruction	format	opcode/funct
`add $rd, $rs, $rt`	R	$0/20_{hex}$
`sub $rd, $rs, $rt`	R	$0/22_{hex}$
`addi $rt, $rs, imm`	I	8_{hex}
`move $s0, $s1`	psuedo	—

immediate subtraction

There is no `subi` operation as it is equivalent to `addi` with a negative constant.

assigning variables

The `move` instruction is a **psuedo-instruction** which is translated into its MIPS equivalent by the compiler:

```
add $s0, $s1, $zero
```

To assign a constant `imm` to a register:

```
addi $s0, $zero, imm
```

The immediate values in **immediate operations** such as `addi` range from -2^{15} to $2^{15} - 1$, as the 16-bit 2s complement system is used.

setting large constants

If 32-bit constants are required, use the `lui` operation to load the most-significant 16-bits first, followed by an `ori` operation to set the least-significant 16-bits:

```
lui $t0, 0xAAAA       # t0 = 0xAAAA0000
ori $t0, $t0, 0xF0F0  # t0 = 0xAAAAF0F0
```

Use the **temporary registers** `$t0` to `$t9` to store intermediate results in complex expressions.

5.2 Logical Instructions

instruction	format	opcode/funct
`sll $rd, $rt, shamt`	R	$0/00_{hex}$
`srl $rd, $rt, shamt`	R	$0/02_{hex}$
`and $rd, $rs, $rt`	R	$0/24_{hex}$
`or $rd, $rs, $rt`	R	$0/25_{hex}$
`xor $rd, $rs, $rt`	R	$0/26_{hex}$
`nor $rd, $rs, $rt`	R	$0/27_{hex}$
`andi $rt, $rs, imm`	I	C_{hex}
`ori $rt, $rs, imm`	I	D_{hex}
`xori $rt, $rs, imm`	I	E_{hex}
`lui $rt, imm`	I	F_{hex}

Bitwise `NOR` sets the result to `1` if both bits are `0`, and `0` otherwise.

Bitwise `XOR` sets the result to `1` if both bits are different, and `0` otherwise.

bitwise `NOT`

There is no `not` operation as it is equivalent to either of the following:

- `nor $rd, $rs, $zero`
- `xor $rd, $rs, $rt`, where `$rt` is all `1`s

The absence of `nori` keeps the instruction set small.

In bitshift operations (`sll` and `srl`), the empty positions are filled with zeros, and the **shift amount** is limited to **5 bits**.

multiplication and division

For multiplication/division by k, if k is a power of 2, use `sll` and `srl` respectively, setting the `shamt` to k.

Otherwise, use a loop.

5.3 Memory Instructions

Memory can be thought of as a *single-dimensional array* of memory location, with each having an **address**.

Memory addresses allow access to *bytes* of data, or **words** of data, which are usually 2^n bytes — the common unit of transfer between the processor and memory.

Word alignment occurs in memory when words begin at a *byte address* which is a multiple of the word size — 2^n bytes.

In MIPS, each word is 32 bits (4 bytes), and addresses are 32-bits long — such that 2^{30} words are addressable, each of which differing by 4.

instruction	format	opcode/funct
`lw $rt, offset($rs)`	I	23_{hex}
`sw $rt, offset($rs)`	I	$2B_{hex}$
`lb $rt, offset($rs)`	I	20_{hex}
`sb $rt, offset($rs)`	I	28_{hex}
`ulw $rt, offset($rs)`	pseudo	—
`usw $rt, offset($rs)`	pseudo	—

For the word-aligned memory instructions `lw` and `sw`, the resulting address of `$rs + offset` must be a multiple of 4.

The `offset` is a 16-bit 2s complement number.

5.4 Control Flow Instructions

instruction	format	opcode/funct
`beq $rs, $rt, label`	I	4_{hex}
`bne $rs, $rt, label`	I	5_{hex}
`j label`	J	2_{hex}
`slt $rd, $rs, $rt`	R	$0/2A_{hex}$
`slti $rt, $rs, imm`	I	A_{hex}
`blt $rs, $rt, label`	pseudo	—
`bgt $rs, $rt, label`	pseudo	—
`ble $rs, $rt, label`	pseudo	—
`bge $rs, $rt, label`	pseudo	—

`slt` and `slti` sets the result to `1` if `$rs < $rt` or `imm`, and `0` otherwise.

`beq` and `bne` are **conditional jumps** — they jump to the label if the condition is true.

Labels are written as `<label>:` to the left of a statement, but they are *not* instructions.

A `j` instruction is equivalent to `beq $s0 $s0, <label>`.

The **program counter** (PC) typically stores the address of the *next* address to be executed, and has to be modified by the branching and jump instructions.

6 Instruction Encoding

Every MIPS instruction is **32 bits** in 3 possible formats:

format	source registers	destination registers	immediate values
R	2	1	0
I	1	1	1
J	0	0	1

6.1 R-format

	`opcode`	`rs`	`rt`	`rd`	`shamt`	`funct`
bits	6	5	5	5	5	6

- `opcode := 0` for all R-format instructions,
- `funct` determines the instruction,
- `shamt := 0, rd := arith(rs, rt)` for non-shift (arithmetic) instructions, and,
- `rs := 0, rd := shift(rt, shamt)` for shift instructions.

6.2 I-format

	opcode	rs	rt	immediate
bits	6	5	5	16

- `opcode` determines the instruction since there is no `funct` field,
- `rt` determines the *destination* register since there is no `rd` field,
- `immediate` is a *signed* integer in 2s complement except for bitwise operations where it is *unsigned*,
- `rt := op(rs, immediate)` in general for all non-branching instructions, and,
- `PC := (PC + 4) + (immediate * 4)` when branching, otherwise `PC += 4`, which is the next instruction.

For branching instructions, `immediate` is the offset from the *next* instruction to the label of the *target* instruction.

PC is incremented in multiples of 4 due to word-alignment, which also means that we can now branch $2^{15} \times 4 = 2^{17}$ bytes away from PC.

6.3 J-format

	opcode	target address
bits	6	26

The last 2 bits of every instruction are always `00` due to word-alignment, so we leave them out of the target address.

This leaves with an effective range of 28 bits for the target address, and the remaining 4 bits are derived from the most significant (leftmost) bits of `PC + 4`.

The **destination address** is therefore:

	(PC+4)[0:4]	target address	00
bits	4	26	2

This creates a maximum jump range of 256 MB.

6.4 Addressing Modes

Addressing modes are used to calculate the address of an operand.

1. **register addressing** — `add, xor, etc.`:
   operand is a register
2. **immediate addresssing** — `addi, andi, etc.`:
   operand is a constant within the instruction
3. **base/displacement addressing** — `lw, sw`:
   operand is a memory location at the address the sum of a register and a constant in the instruction
4. **PC-relative addressing** — `beq, bne`:
   address is the sum of the PC and a constant in the instruction
5. **psuedo-direct addressing** — `j`:
   part of the instruction concantenated with part of the PC

7 Datapath

A collection of components that process data, and perform arithmetic, logical, and memory operations.

The **instruction execution cycle** has 5 stages:

1. **fetch**:
   get instruction from memory, address in PC
2. **decode and operand fetch**:
   determine the operation and get the operands needed
3. **execute (ALU)**:
   perform the computations to get a result
4. **execute (memory access)**:
   read from or write to memory if necessary
5. **write-back**:
   store the result of the operation

7.1 Fetch

There are 3 steps in the fetch stage:

1. Use PC to fetch the instruction from memory.
2. Increment PC by 4 to get the address of the next instruction.
3. Feed the output instruction to the decode stage.

The **instruction memory** element is a **sequential circuit** with an internal state which stores the instructions.

When supplied an instruction address *m*, it outputs the content at address *m*.

The **adder** element takes in the 32-bit PC and the 32-bit constant 4, and outputs the 32-bit sum `PC + 4`.

The **clock signal** is a *square wave* with *rising* and *falling* edges, and a period controlled by the CPU.

PC is read in the first half of the **clock period** and updated at the *next rising clock edge*.

7.2 Decode

There are 3 steps in the decode stage:

1. Read the `opcode` to determine the instruction type and field lengths.
2. Read data from all necessary registers.
3. Feed the operation and operands to the ALU stage.

The **register file** element is a collection of 32 registers which can be read from or written to.

input	bits	output	bits
read register 1	5	read data 1	32
read register 2	5	read data 2	32
write register	5		
write data	32		

The **RegWrite** control signal determines whether the instruction should read from or write to the registers:

	RegWrite	registers per instruction
read	0	≤ 2
write	1	≤ 1

7.2.1 R-format Decoding

The binary content (Inst) of R-format instructions map to the inputs as follows:

- rs ≡ Inst[25:21] ↦ rr1
- rt ≡ Inst[20:16] ↦ rr2
- rd ≡ Inst[15:11] ↦ wr

7.2.2 I-format Decoding

For I-format instructions, 2 problems arise:

1. rt, not part of imm, needs to be fed to wr
2. imm needs to be fed to the ALU, not rd2

A **multiplexer** chooses the correct Inst slice to feed to wr using the control signal **RegDst**:

	RegDst	input to wr
I-format	0	rt ≡ Inst[20:16]
R-format	1	rd ≡ Inst[15:11]

- rs ≡ Inst[25:21] ↦ rr1
- rt ≡ Inst[20:16] ↦ rr2
- rt ≡ Inst[20:16] ↦ wr

Another multiplexer chooses the correct 32-bit binary data to feed into the ALU:

	ALUSrc	input to ALU opr2
R-format or beq	0	rr2 ≡ *rt ≡ *Inst[20:16]
I-format not beq	1	imm ≡ Inst[15:0], sign extended to 32-bits

7.3 Arithmetic Logic Unit (ALU)

The ALU performs the arithmetic, shifting, logical, memory, and branching operations.

It takes the operation and operands from the decode stage, performs its computation, and feeds the result to the memory stage.

input	bits	output	bits
operand 1	32	is zero?	1
operand 2	32	result	32

The **ALUControl** signal determines the operation to perform:

ALUControl	operation
0000	and
0001	or
0010	add
0110	sub
0111	slt
1100	nor

7.3.1 Branching Instructions

A multiplexer chooses between the next instruction (PC + 4) and the branch target (BT) using the control signal **PCSrc**:

isZero	PCSrc	next instruction
false ≡ 0	0	PC + 4
true ≡ 1	1	BT ≡ PC + 4 + imm * 4

Because both register contents need to be compared, the ALUSrc control value is set to 0 for branching instructions.

BT is computed as follows:

1. left shift by 2 bits the sign-extended imm from the decode stage (multiplying the offset by 4)
2. add it to PC + 4 from the adder in the fetch stage

7.4 Memory

Only lw/lb and sw/sb instructions operate in this stage, the rest remain idle.

The **data memory** element exists in RAM:

input	bits	output	bits
address (addr)	32	read data (rdt)	32
write data (wd)	32		

addr is the effective memory address computed by the ALU, and wd is connected to rd2 from the register file from the decode stage.

The control signals **MemRead** and **MemWrite** control the memory operations:

MemRead	MemWrite	action
0	1	write wd into addr
1	0	read contents of addr into rdt
0	0	do nothing
1	1	undefined, should not occur

7.4.1 Non-memory Instructions

A multiplexer chooses between the result of the ALU and the read data from memory, using the control signal **`MemtoReg`**:

	`MemToReg`	output
non-memory instruction	`0`	ALU output
memory instruction	`1`	read data

This output is fed to the write-back stage.

7.5 Write-back

Stores, branches, and jumps remain idle in this stage as there is nothing to be written.

The result of the memory stage is fed into write data (`wd`) of the register file.

8 Control

Control signals are generated by a **control unit** based on the type of instruction.

The control unit is a **combinational circuit** which takes in the 6-bit opcode and the 5-bit function code, and outputs the 8 control signals.

`funct` is only needed to determine `ALUSrc`; every other control signal can be derived from `opcode` alone.

8.1 `ALUControl`

We start with a simplified 1-bit ALU.

1-bit MIPS ALU

4 control signals are needed:

1. **`Ainvert`**: 1 to invert A, `0` otherwise
2. **`Binvert`**: 1 to invert B, `0` otherwise
3. **`operation` (2 bits)**: select one of the 3 results

5 I/O signals:

1. **`A`**: input
2. **`B`**: input
3. **`Cin`**: carry in
4. **`Cout`**: carry out
5. **`result`**: output

These 1-bit ALUs are daisy-chained to form a 32-bit ALU.

The carries from one of the 1-bit ALUs is fed out via `Cout` and into the next one via `Cin`.

`Ainvert` and `Binvert` control the multiplexers which select between their inputs and their negated values:

`Xinvert`	output
`0`	X
`1`	NOT(X)

The `operation` control signal selects one of the 3 results from the AND gate, OR gate, and the adder ADD:

`operation`	output
`00`	AND
`01`	OR
`10`	ADD
`11`	SLT

1-bit subtraction

`Cin` is set to 1 due to 2s complement:

$A - B \equiv A + (-B) \equiv A + B' + 1$

`ALUControl`

In MSB to LSB order, `ALUControl` comprises of:

1. `Ainvert`
2. `Binvert`
3. `operation`

8.1.1 `ALUOp`

It is necessary to use the **multi-level decoding approach** to simplify the design process and size of the main controller.

`ALUOp` is an intermediate 2-bit control signal which is generated from `opcode` and used with `funct` to determine `ALUControl`.

instruction	`ALUOp`	`funct`	`ALUControl`
`lw, sw`	`00`	`XXXXXX`	`0010`
`beq`	`01`	`XXXXXX`	`0110`
`add`	`10`	`100000`	`0010`
`sub`	`10`	`100010`	`0110`
`and`	`10`	`100100`	`0000`
`or`	`10`	`100101`	`0001`
`slt`	`10`	`101010`	`0111`

In general, `ALUOp` is `10` for R-format instructions.

Part IV

Digital Logic Design

9 Boolean Algebra

order of operations

From highest precedence to lowest:

- NOT
- AND
- OR

Use parentheses to overwrite precedence.

---

1 represents **`true`** and **0** represents **`false`**.

proving boolean equations

Construct a **truth table** for the LHS and RHS of the equation, and then show that both expressions are equivalent for all inputs.

This is **proof by exhaustion**.

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duality

In boolean equations, the **dual** of an equation remains valid by interchanging:

- the operators AND and OR
- the identity elements 0 and 1

Boolean functions are parametrized by a set of variables, e.g. F1(A, B, C) = x · y · z'.

The **complement** of a boolean function is obtained by interchanging 1 and 0 of the function's output values.

9.1 Standard Forms

- **literals**: boolean variables or their complement, e.g. x or x'
- **product terms**: single literal or AND products of literals, e.g. x or x · y · z'
- **sum terms**: single literal or OR sum of literals, e.g. x or x + y + z'
- **sum-of-products (SOP) expression**: a product term or OR sum of several product terms, e.g. A · B + A' · B'
- **product-of-sums (POS) expression**: a sum term or AND product of several sum terms, e.g. (A + B + C) · D' · (D' + E')

Every boolean expression can be expressed in SOP or POS form.

9.2 Minterms and Maxterms

	...of *n* literals	example	prefix
minterm	*product term*	x' · y'	m
maxterm	*sum term*	x' + y'	M

n variables generate up to 2^n minterms and 2^n maxterms, and they are enumerated and prefixed:

x	y	minterms		maxterms	
		term	notation	term	notation
0	0	x' · y'	m0	x + y	M0
0	1	x' · y	m1	x + y'	M1
1	0	x · y'	m2	x' + y	M2
1	1	x · y	m3	x' + y'	M3

Every minterm is the complement of its maxterm and vice versa.

deriving minterm notation from expressions

1. Assign 0 to every negated variable and 1 to every non-negated variable.
2. Concatenate the result and convert to decimal.
3. Prefix the decimal number with m.

For the reverse direction, expand with · as minterms are product terms.

---

deriving maxterm notation from expressions

1. Assign 1 to every negated variable and 0 to every non-negated variable.
2. Concatenate the result and convert to decimal.
3. Prefix the decimal number with M.

For the reverse direction, expand with + as maxterms are sum terms.

9.3 Canonical Forms

canonical sum-of-products ≡ sum-of-minterms
canonical product-of-sums ≡ product-of-maxterms

deriving canonical forms

- Σm **sum-of-minterms**: OR the combinations of x, y, z, ... in F(x, y, z, ...) such that the output is 1
- ΠM **product-of-maxterms**: AND the combinations of x, y, z, ... in F(x, y, z, ...) such that the output is 0
- Σm ⟷ ΠM **conversion**: e.g. F = Σm(1, 4, 5, 6, 7) = ΠM(0, 2, 3)

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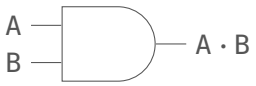
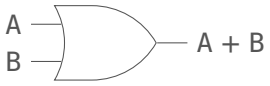
10 Logic Circuits

10.1 Logic Gates

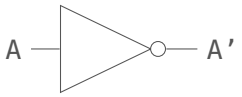
AND gate



OR gate

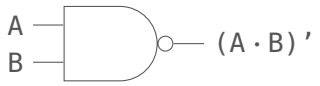


NOT gate

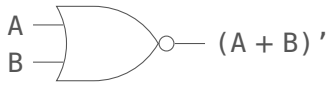


The **NAND** and **NOR** operations are the negations of the AND and OR operations respectively:

NAND gate

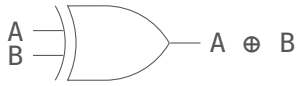


NOR gate

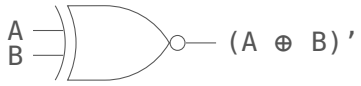


The **XOR** operation outputs 1 only if its inputs are different, and the **XNOR** operation is the complement of that:

XOR gate



XNOR gate



XNOR can also be represented by ⊙.

The **fan-in** is the number of inputs to a gate.

drawing logic circuits

Every input must be connected in a working circuit, so use 0 or 1 depending on the gate if you don't have a value to connect.

If **complemented literals** are not allowed, use a NOT gate to negate the input.

Intersecting wires are denoted by a solid circle ●.

10.2 Universal Gates

The set {AND, OR, NOT} is a **complete set of logic** — they are sufficient to build any boolean function.

{NAND} and {NOR} are also complete sets of logic.

	implementation #1	implementation #2
SOP	AND-OR circuit	NAND circuit
POS	OR-AND circuit	NOR circuit

10.3 Programmable Logic Array (PLA)

A **programmable logic array** implements sum-of-products (SOP) circuits, which allows for multiple outputs.

It has 2 stages:

1. **AND gates**: product terms
2. **OR gates**: outputs

drawing PLA diagrams

1. Draw a horizontal line for each input.
2. Draw an AND gate for each possible input combination that has at least one 1 output minterm.
3. For each input for every AND gate, add an inverter ○ if the input is 0.
4. Connect each input of every AND gate to the corresponding input line with ● at the intersection.
5. Draw a vertical line for each AND gate output.
6. Draw an OR gate for each output.
7. Connect the output line of each AND gate to the corresponding OR gate with ● at the intersection if that output is 1 .

Simplified PLA diagrams eliminate the need to draw the AND and OR gates.

interpreting simplified PLAs

- **inputs**:
  each input A, B, C, ... is paired with its negated input A', B', C', ... in horizontal lines
- **AND plane**:
  solid circles in each vertical line represent the inputs to an AND gate
- **OR plane**:
  solid circles in each horizontal line represent the inputs to an OR gate
- **ouputs**:
  each of the OR gates is (connected to) an output

10.4 Half Adder

The **half adder** adds 2 bits X and Y to produce 2 bits C and S:

- $C = X \cdot Y$
- $S = X' \cdot Y + X \cdot Y' = X \oplus Y$

inputs		outputs	
X	Y	C	S
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

10.5 Gray Code

Gray code is an unweighted binary sequence where only *one* bit changes between consecutive values, with no duplicate values.

An n-bit Gray code has 2^n values, but there are many possible Gray code sequences.

generating the standard n-bit Gray code sequence

1. Start with the n-bit binary values for `0` and `1`.
2. Mirror the first existing values, effectively doubling the number of values.
3. On the i-th doubling, replace the `0` in the $i+1$th place from the right with `1` for each of the reflected values.
4. Repeat until you have 2^n values after $n-1$ doublings.

The **standard Gray code** sequence

11 Simplification

11.1 K-maps

K-maps are a systematic way to derive simplified SOP expressions with the fewest product terms and literals.

K-maps are organized as a *matrix of squares*, where each square represents a *minterm*, and two adjacent squares differ by exactly *one literal*.

11.1.1 2-variable K-maps

a \ b	0	1
0	a' · b'	a' · b
1	a · b'	a · b

a \ b	0	1
0	m0	m1
1	m2	m3

Every row and column of `0`s and `1`s can be permuted, such that there are many alternative layouts for a K-map.

filling K-maps

Place a **`1`** in squares corresponding to a minterm of the function, and **`0`** otherwise.

a \ b	0	1
0	0	0
1	0	1

a \ b	0	1
0	0	1
1	1	0

e.g. `C = a · b` (left), `S = a · b' + a' · b` (right)

11.1.2 3-variable K-maps

a \ bc	00	01	11	10
0	m0	m1	m3	m2
1	m4	m5	m7	m6

The arrangement of the squares ensures that minterms of adjacent cells *differ by only one literal*.

This means that the labels have to be some **Gray code sequence** where each value differs from its previous by 1 bit.

K-maps **wrap-around** such that the last row is adjacent to the first row, and the last column is adjacent to the first column.

This means that `m0` is adjacent to not just `m1` and `m4`, but also `m2`, as colored above.

filling K-maps

Since K-maps are used to simplify sum-of-product expressions, we fill in each minterm (`AND` product) one-by-one with **`1`**s.

For each minterm x, we find the intersection of all the squares of x or `x'` to fill with `1`s:

- **x**: *inside* the rows/columns of x
- **`x'`**: *outside* the rows/columns of x

11.1.3 4-variable K-maps

wx / yz	00	01	11	10
00	m0	m1	m3	m2
01	m4	m5	m7	m6
11	m12	m13	m15	m14
10	m8	m9	m11	m10

The cells adjacent to `m0` have also colored above.

An *n*-variable K-map has *n* neighbors.

11.1.4 5-variable K-maps

wx / yz	00	01	11	10
00	m0	m1	m3	m2
01	m4	m5	m7	m6
11	m12	m13	m15	m14
10	m8	m9	m11	m10

v = 0

	00	01	11	10
00	m16	m17	m19	m18
01	m20	m21	m23	m22
11	m28	m29	m31	m30
10	m24	m25	m27	m26

v = 1

11.1.5 6-variable K-maps

wx / yz	00	01	11	10
00	m0	m1	m3	m2
01	m4	m5	m7	m6
11	m12	m13	m15	m14
10	m8	m9	m11	m10

ab = 00

	00	01	11	10
00	m16	m17	m19	m18
01	m20	m21	m23	m22
11	m28	m29	m31	m30
10	m24	m25	m27	m26

ab = 01

	00	01	11	10
00	m32	m33	m35	m34
01	m36	m37	m39	m38
11	m44	m45	m47	m46
10	m40	m41	m43	m42

ab = 10

	00	01	11	10
00	m48	m49	m51	m50
01	m52	m53	m55	m54
11	m60	m61	m63	m62
10	m56	m57	m59	m58

ab = 11