

6-Degree-of-Freedom Ball Balancing Robot

Inverse Kinematics (IK)

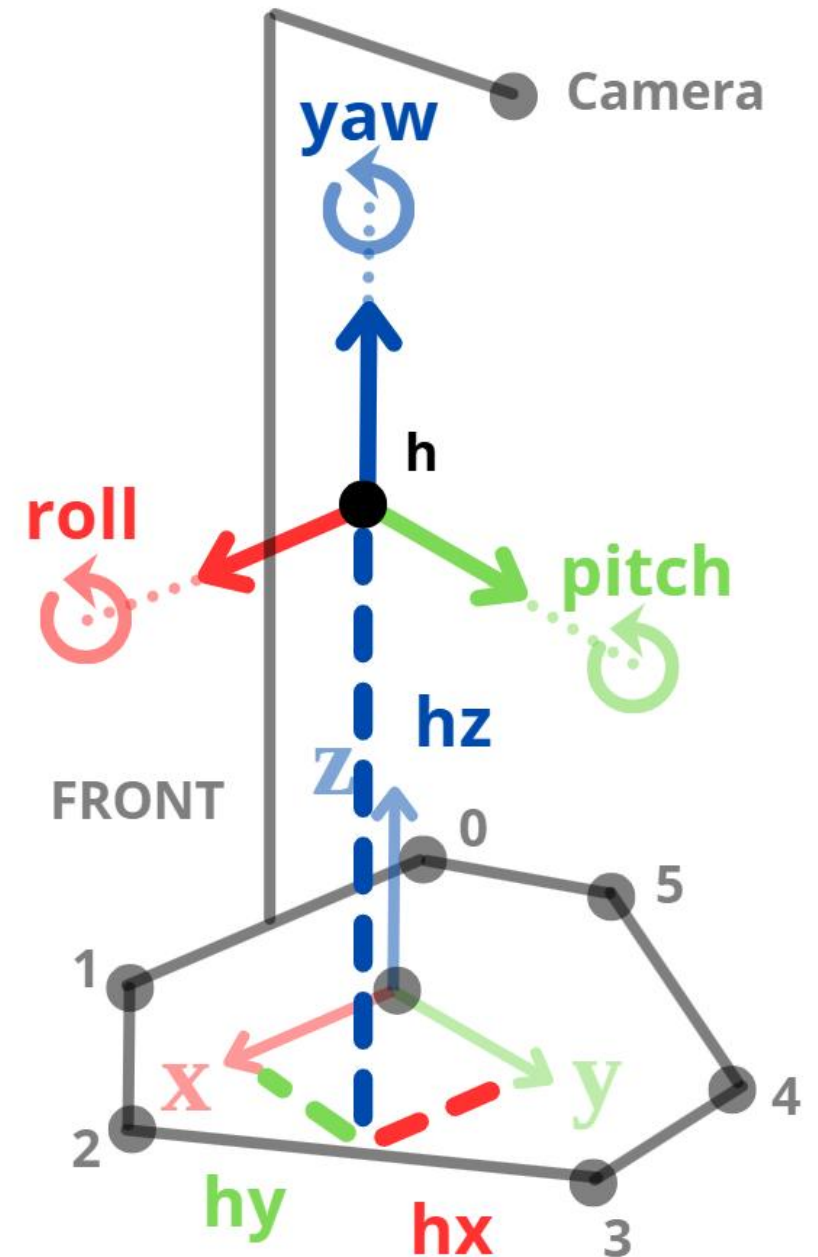
IK main objective

Given a desired h point:

$h = \{h_x, h_y, h_z, \text{roll}, \text{pitch}, \text{yaw}\}$

Find x6 motor angles:

$\text{angles} = \{0, 1, 2, 3, 4, 5\}$



**x6 Platform
Coordinates**

hx



hy



hz



roll



pitch



yaw



**Inverse
Kinematics**

**x6 Motor
Angles**



angle0



angle1



angle2



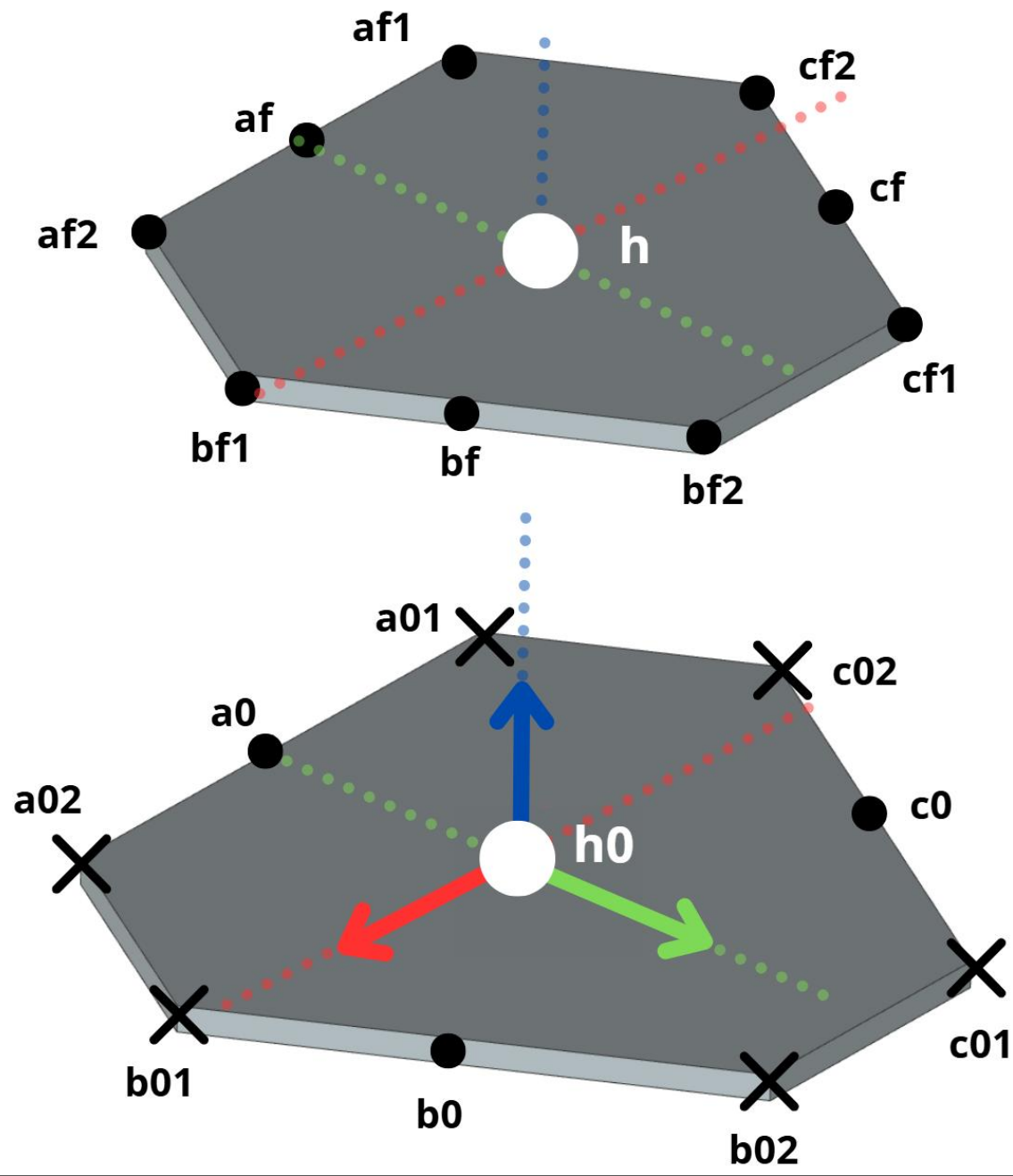
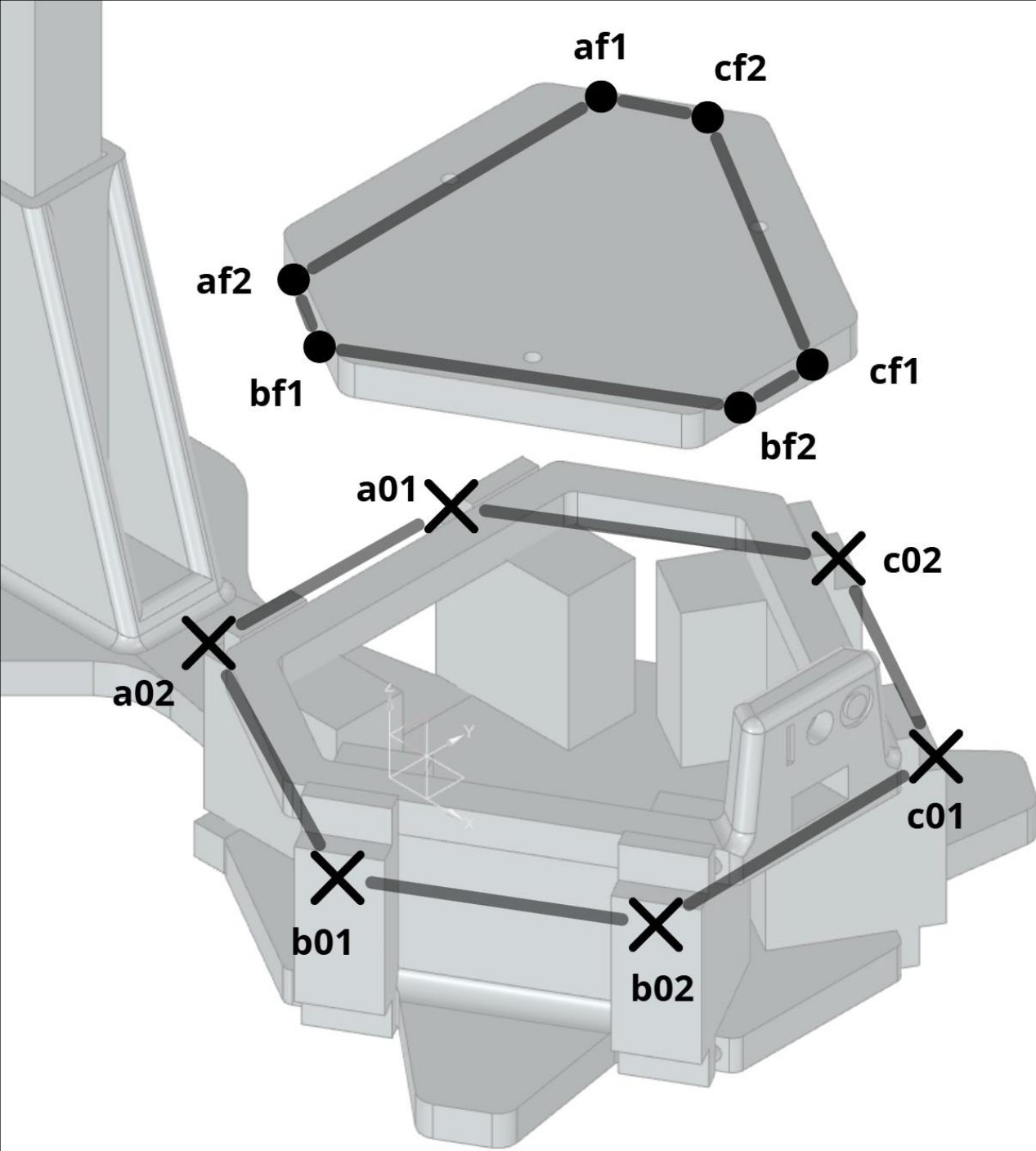
angle3



angle4



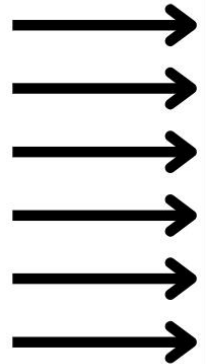
angle5



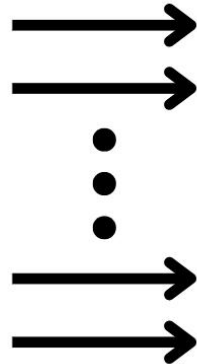
Inverse Kinematics

**x6 Platform
Coordinates**

hx
hy
hz
roll
pitch
yaw



Stage 1

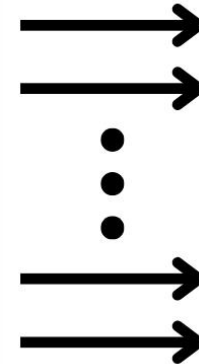


x3 points
{af, bf, cf}

=

x9 coordinates

Stage 2



x6 vectors
{a1, a2, b1, b2, c1, c2}

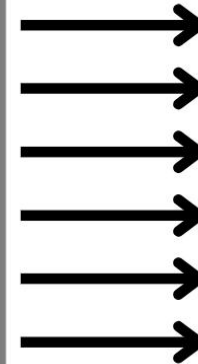
=

x18 coordinates

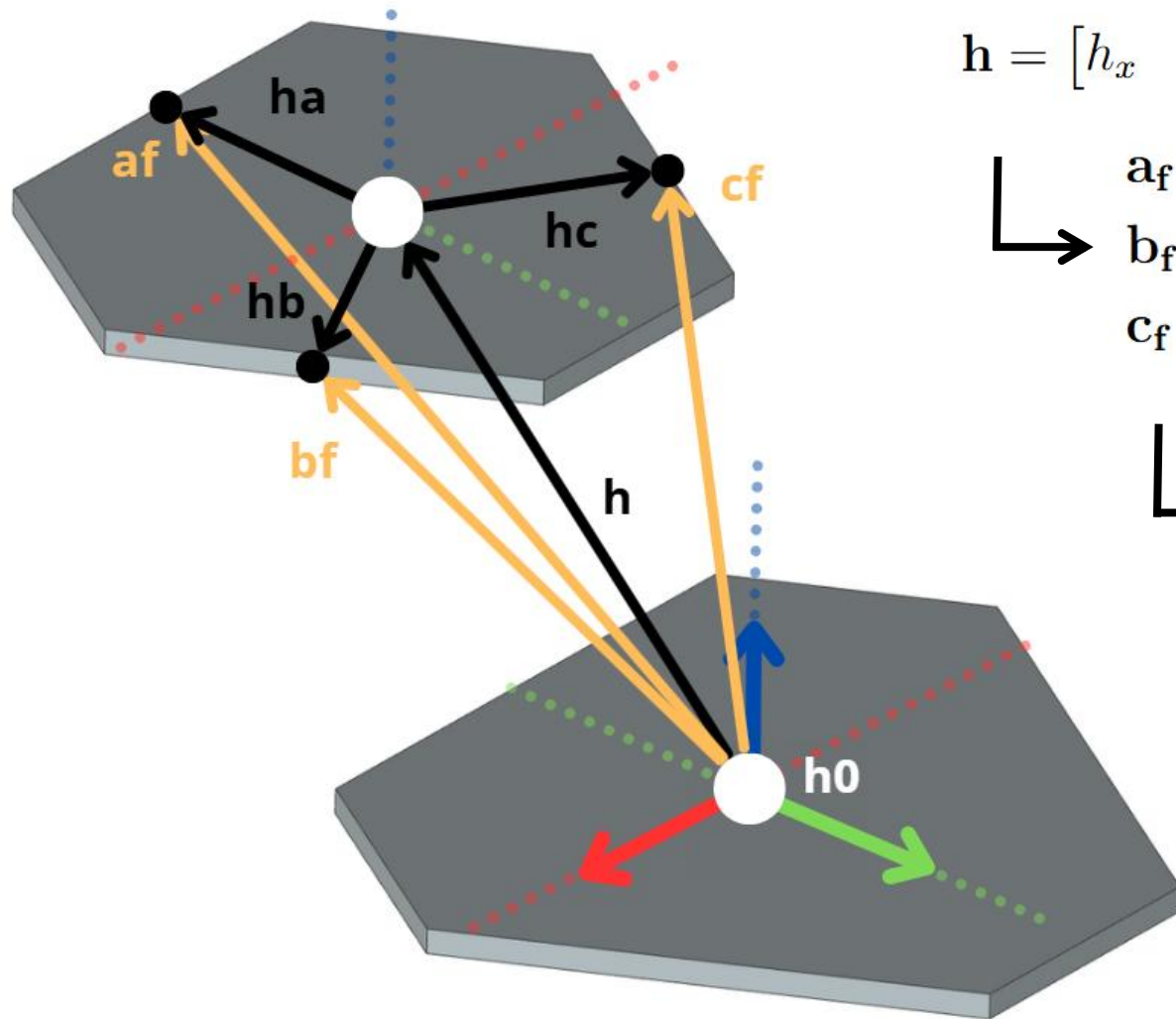
Stage 3

**x6 Motor
Angles**

angle0
angle1
angle2
angle3
angle4
angle5



Stage 1: from h to $\{af, bf, cf\}$



$$\mathbf{h} = [h_x \ h_y \ h_z]^T$$

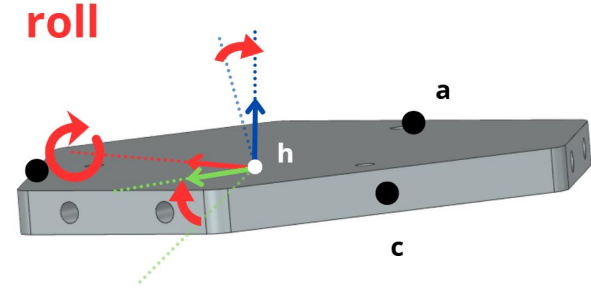
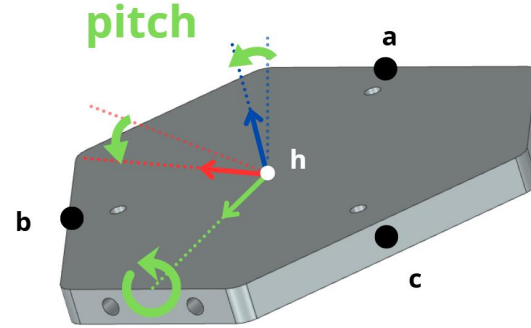
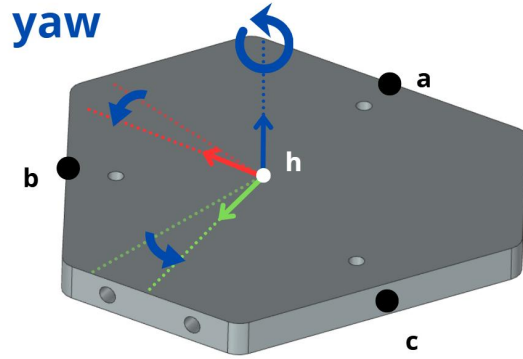
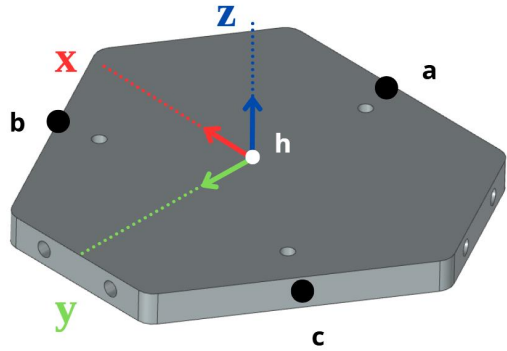
$$\begin{aligned} \mathbf{a}_f &= \mathbf{h} + \mathbf{h}_a \\ \mathbf{b}_f &= \mathbf{h} + \mathbf{h}_b \\ \mathbf{c}_f &= \mathbf{h} + \mathbf{h}_c \end{aligned}$$

$$\begin{aligned} \mathbf{h}_a &= \text{Rot}(\alpha, \beta, \gamma) \mathbf{h}_{a_{initial}} \\ \mathbf{h}_b &= \text{Rot}(\alpha, \beta, \gamma) \mathbf{h}_{b_{initial}} \\ \mathbf{h}_c &= \text{Rot}(\alpha, \beta, \gamma) \mathbf{h}_{c_{initial}} \end{aligned}$$

Define using known distances

$$\begin{aligned} \text{Rot}(\text{yaw}, \text{pitch}, \text{roll}) &= \text{Rot}(\alpha, \beta, \gamma) = \\ &= \boxed{\text{Rot}(\hat{z}, \alpha)} \boxed{\text{Rot}(\hat{y}, \beta)} \boxed{\text{Rot}(\hat{x}, \gamma)} \end{aligned} \quad ?$$

***Rotation Roll-Pitch-Yaw (XYZ) = Rotation Euler ZYX**



$$\begin{bmatrix} ha_{initial} \\ hb_{initial} \\ hc_{initial} \end{bmatrix}$$

$$\mathbf{Rot}(\hat{z}, \alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Rot}(\hat{y}, \beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$\mathbf{Rot}(\hat{x}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$



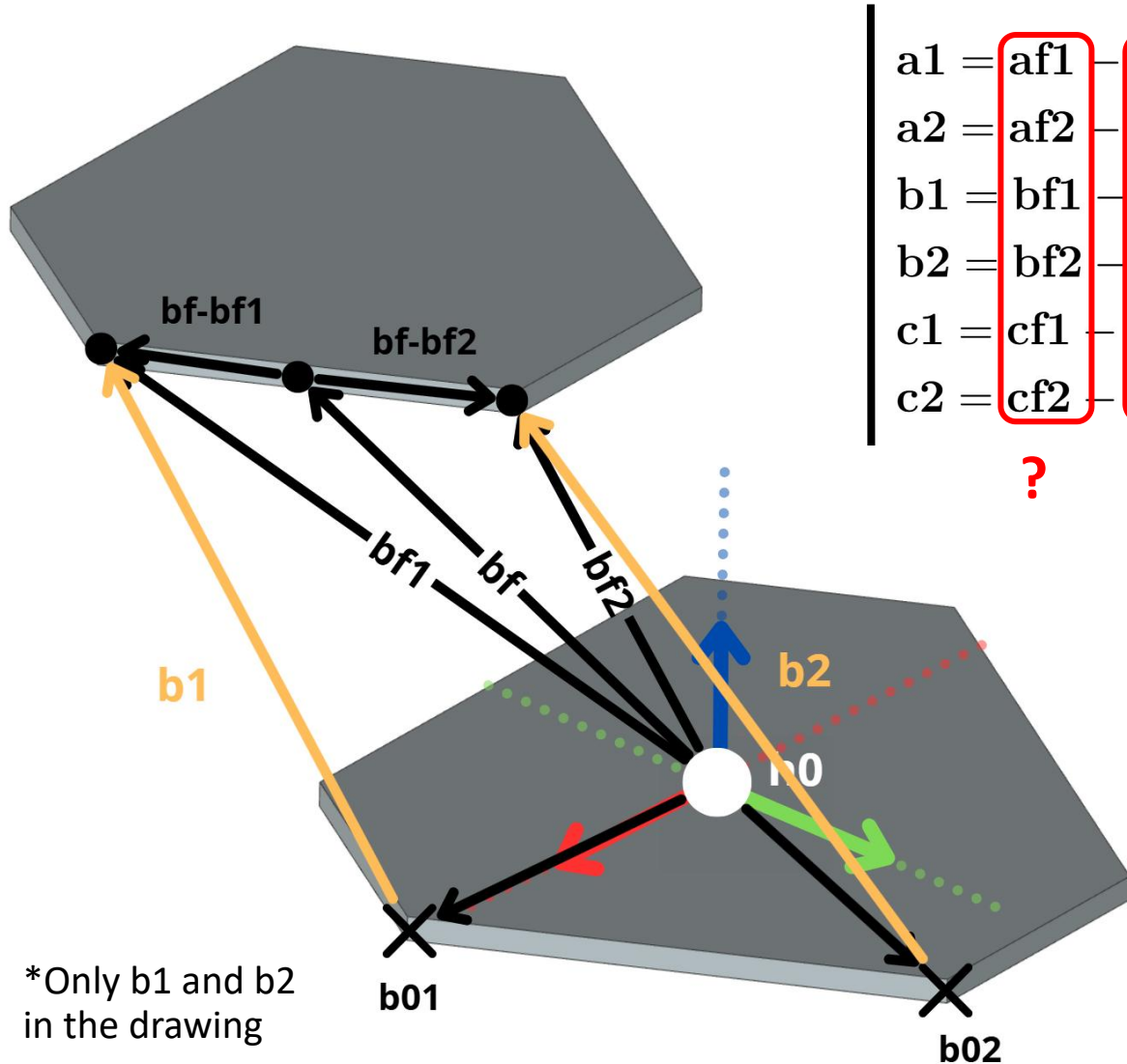
$$Rot(yaw, pitch, roll) = Rot(\alpha, \beta, \gamma) = Rot(\hat{z}, \alpha)Rot(\hat{y}, \beta)Rot(\hat{x}, \gamma)$$

$$\rightarrow \begin{cases} ha = Rot(\alpha, \beta, \gamma)ha_{initial} \\ hb = Rot(\alpha, \beta, \gamma)hb_{initial} \\ hc = Rot(\alpha, \beta, \gamma)hc_{initial} \end{cases}$$

$$\rightarrow \begin{cases} a_f = h + ha \\ b_f = h + hb \\ c_f = h + hc \end{cases}$$

Stage 1 solved 

Stage 2: from {af, bf, cf} to {af1, af2, bf1, bf2, cf1, cf2}

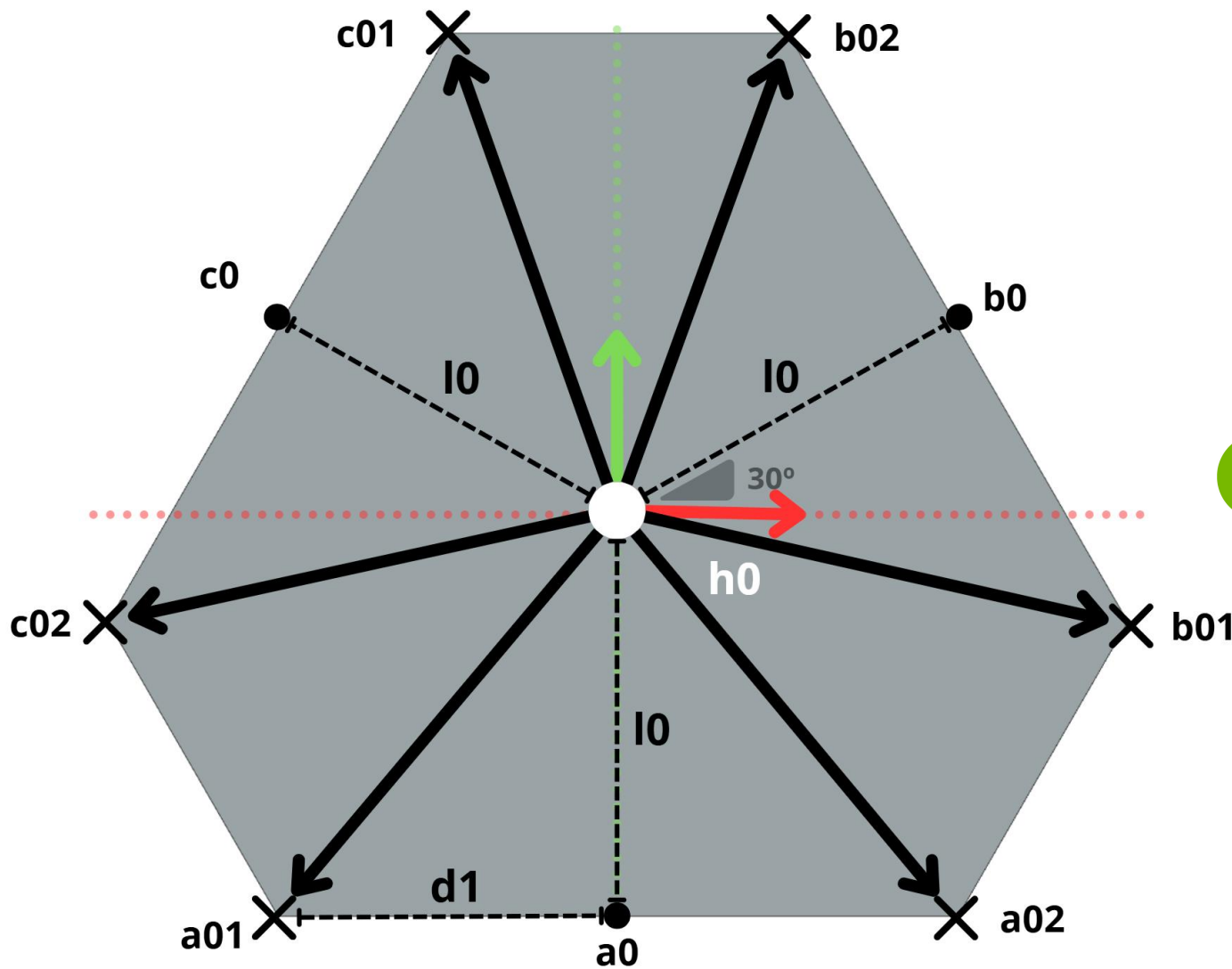


$$\begin{array}{l} a1 = \boxed{af1} - \boxed{a01} \\ a2 = \boxed{af2} - \boxed{a02} \\ b1 = \boxed{bf1} - \boxed{b01} \\ b2 = \boxed{bf2} - \boxed{b02} \\ c1 = \boxed{cf1} - \boxed{c01} \\ c2 = \boxed{cf2} - \boxed{c02} \end{array} \quad \begin{array}{l} \text{Define} \\ \text{using} \\ \text{known} \\ \text{distances} \end{array} \quad ?$$

$$\begin{array}{l} af1 = \boxed{af} + \boxed{af \rightarrow af1} \\ af2 = \boxed{af} + \boxed{af \rightarrow af2} \\ bf1 = \boxed{bf} + \boxed{bf \rightarrow bf1} \\ bf2 = \boxed{bf} + \boxed{bf \rightarrow bf2} \\ cf1 = \boxed{cf} + \boxed{cf \rightarrow cf1} \\ cf2 = \boxed{cf} + \boxed{cf \rightarrow cf2} \end{array} \quad ?$$

From Stage 1

$$\begin{array}{l} af \rightarrow af1 = \boxed{Rot(\alpha, \beta, \gamma)} \boxed{af \rightarrow af1_{initial}} \\ af \rightarrow af2 = \boxed{Rot(\alpha, \beta, \gamma)} \boxed{af \rightarrow af2_{initial}} \\ bf \rightarrow bf1 = \boxed{Rot(\alpha, \beta, \gamma)} \boxed{bf \rightarrow bf1_{initial}} \\ bf \rightarrow bf2 = \boxed{Rot(\alpha, \beta, \gamma)} \boxed{bf \rightarrow bf2_{initial}} \\ cf \rightarrow cf1 = \boxed{Rot(\alpha, \beta, \gamma)} \boxed{cf \rightarrow cf1_{initial}} \\ cf \rightarrow cf2 = \boxed{Rot(\alpha, \beta, \gamma)} \boxed{cf \rightarrow cf2_{initial}} \end{array} \quad \begin{array}{l} \text{Define} \\ \text{using} \\ \text{known} \\ \text{distances} \end{array}$$



$$\mathbf{a}_{01} = [-d_1 \quad -l_0 \quad 0]^T$$

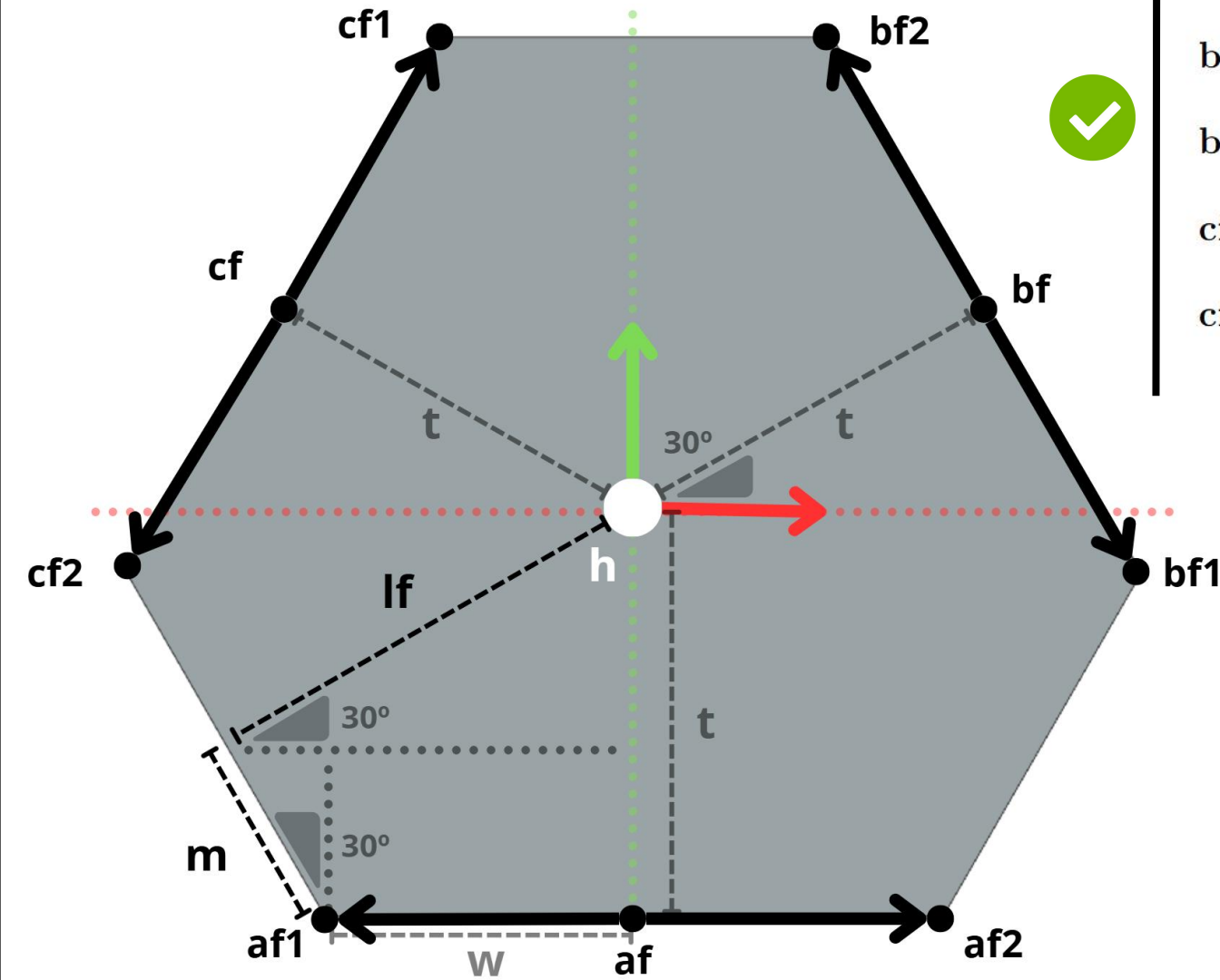
$$\mathbf{a}_{02} = [d_1 \quad -l_0 \quad 0]^T$$

$$\mathbf{b}_{01} = \left[\frac{l_0\sqrt{3}}{2} + \frac{d_1}{2} \quad \frac{l_0}{2} - \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$

$$\mathbf{b}_{02} = \left[\frac{l_0\sqrt{3}}{2} - \frac{d_1}{2} \quad \frac{l_0}{2} + \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$

$$\mathbf{c}_{01} = \left[-\frac{l_0\sqrt{3}}{2} + \frac{d_1}{2} \quad \frac{l_0}{2} + \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$

$$\mathbf{c}_{02} = \left[-\frac{l_0\sqrt{3}}{2} - \frac{d_1}{2} \quad \frac{l_0}{2} - \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$



$$\mathbf{af} \rightarrow \mathbf{af1}_{initial} = [-w \ 0 \ 0]^T$$

$$\mathbf{af} \rightarrow \mathbf{af2}_{initial} = [w \ 0 \ 0]^T$$

$$\mathbf{bf} \rightarrow \mathbf{bf1}_{initial} = [w \sin(30^\circ) \ -w \cos(30^\circ) \ 0]^T = \left[\frac{w}{2} \ -\frac{w\sqrt{3}}{2} \ 0 \right]^T$$

$$\mathbf{bf} \rightarrow \mathbf{bf2}_{initial} = [-w \sin(30^\circ) \ w \cos(30^\circ) \ 0]^T = \left[-\frac{w}{2} \ \frac{w\sqrt{3}}{2} \ 0 \right]^T$$

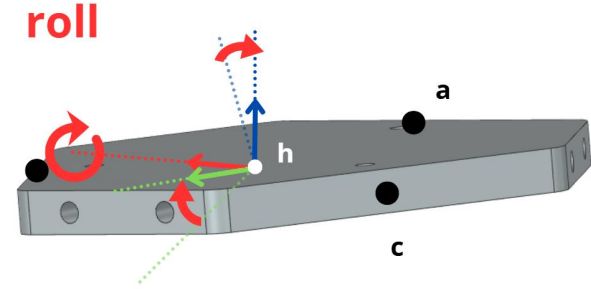
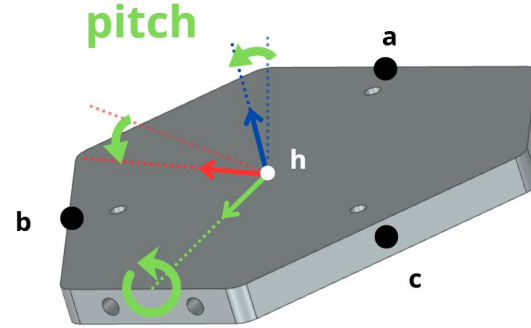
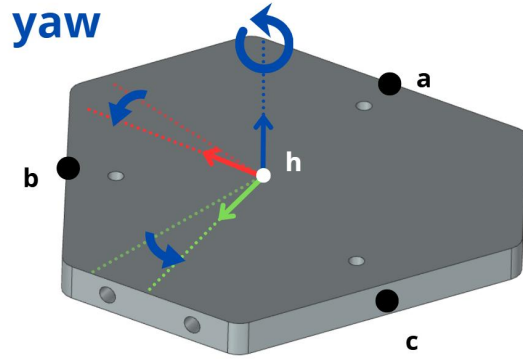
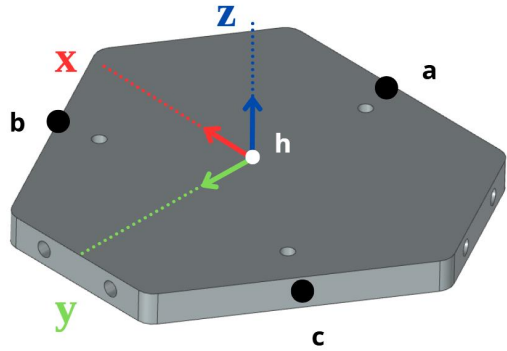
$$\mathbf{cf} \rightarrow \mathbf{cf1}_{initial} = [w \sin(30^\circ) \ w \cos(30^\circ) \ 0]^T = \left[\frac{w}{2} \ \frac{w\sqrt{3}}{2} \ 0 \right]^T$$

$$\mathbf{cf} \rightarrow \mathbf{cf2}_{initial} = [-w \sin(30^\circ) \ -w \cos(30^\circ) \ 0]^T = \left[-\frac{w}{2} \ -\frac{w\sqrt{3}}{2} \ 0 \right]^T$$

$$t = l_f \sin(30^\circ) + m \cos(30^\circ) = \frac{l_f}{2} + \frac{m\sqrt{3}}{2}$$

$$w = l_f \cos(30^\circ) - m \sin(30^\circ) = \frac{l_f \sqrt{3}}{2} - \frac{m}{2}$$

*Rotation Roll-Pitch-Yaw (XYZ) = Rotation Euler ZYX



$af \rightarrow af1_{initial}$

$af \rightarrow af2_{initial}$

$bf \rightarrow bf1_{initial}$

$bf \rightarrow bf2_{initial}$

$cf \rightarrow cf1_{initial}$

$cf \rightarrow cf2_{initial}$

$$\text{Rot}(\hat{z}, \alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rot}(\hat{y}, \beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$\text{Rot}(\hat{x}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$



$$\text{Rot}(\text{yaw}, \text{pitch}, \text{roll}) = \text{Rot}(\alpha, \beta, \gamma) = \text{Rot}(\hat{z}, \alpha) \text{Rot}(\hat{y}, \beta) \text{Rot}(\hat{x}, \gamma)$$



$$\begin{aligned} af \rightarrow af1 &= \text{Rot}(\alpha, \beta, \gamma) af \rightarrow af1_{initial} \\ af \rightarrow af2 &= \text{Rot}(\alpha, \beta, \gamma) af \rightarrow af2_{initial} \\ bf \rightarrow bf1 &= \text{Rot}(\alpha, \beta, \gamma) bf \rightarrow bf1_{initial} \\ bf \rightarrow bf2 &= \text{Rot}(\alpha, \beta, \gamma) bf \rightarrow bf2_{initial} \\ cf \rightarrow cf1 &= \text{Rot}(\alpha, \beta, \gamma) cf \rightarrow cf1_{initial} \\ cf \rightarrow cf2 &= \text{Rot}(\alpha, \beta, \gamma) cf \rightarrow cf2_{initial} \end{aligned}$$



$$\begin{aligned} af1 &= af + af \rightarrow af1 \\ af2 &= af + af \rightarrow af2 \\ bf1 &= bf + bf \rightarrow bf1 \\ bf2 &= bf + bf \rightarrow bf2 \\ cf1 &= cf + cf \rightarrow cf1 \\ cf2 &= cf + cf \rightarrow cf2 \end{aligned}$$



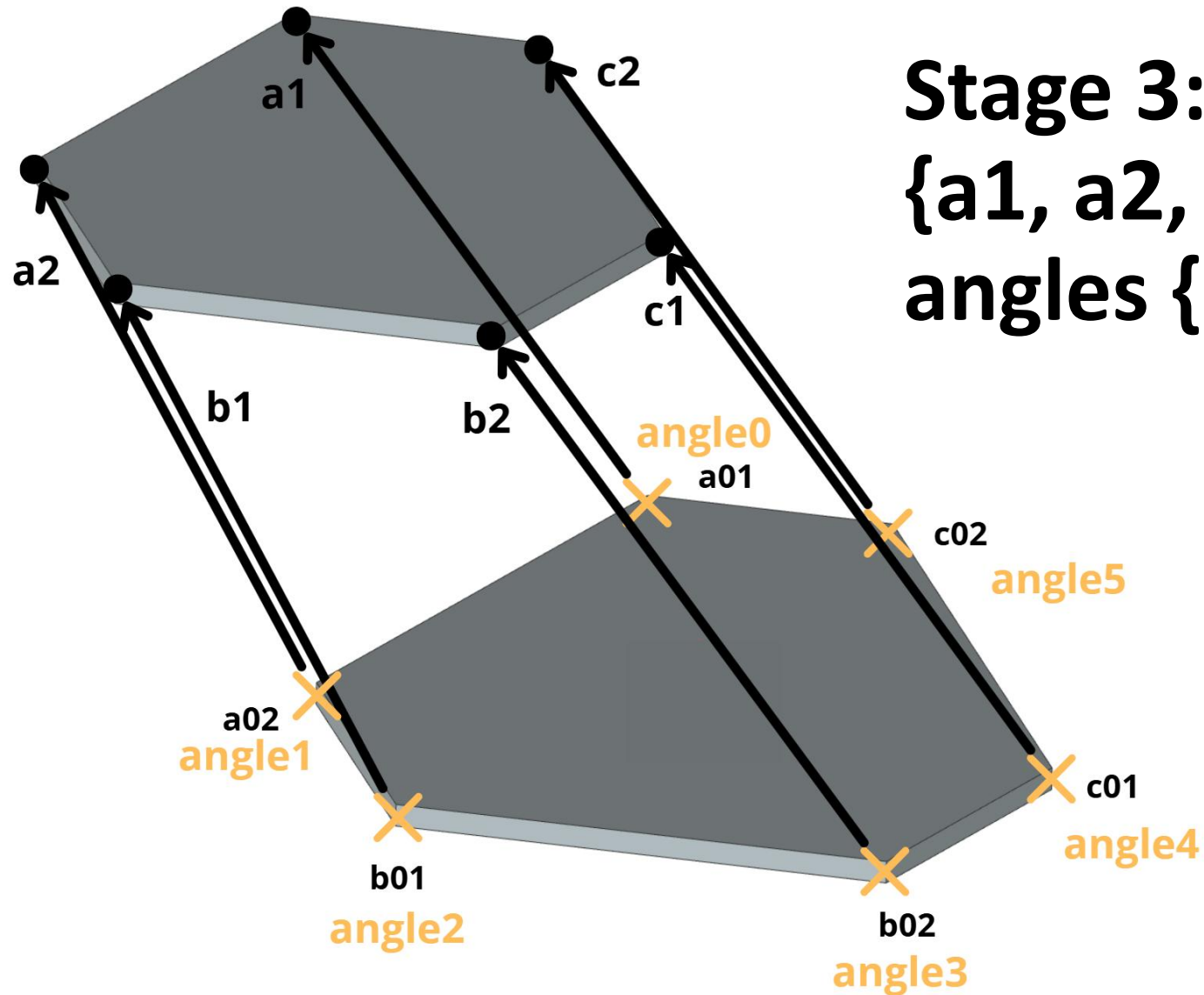
$$\begin{aligned} a1 &= af1 - a01 \\ a2 &= af2 - a02 \\ b1 &= bf1 - b01 \\ b2 &= bf2 - b02 \\ c1 &= cf1 - c01 \\ c2 &= cf2 - c02 \end{aligned}$$



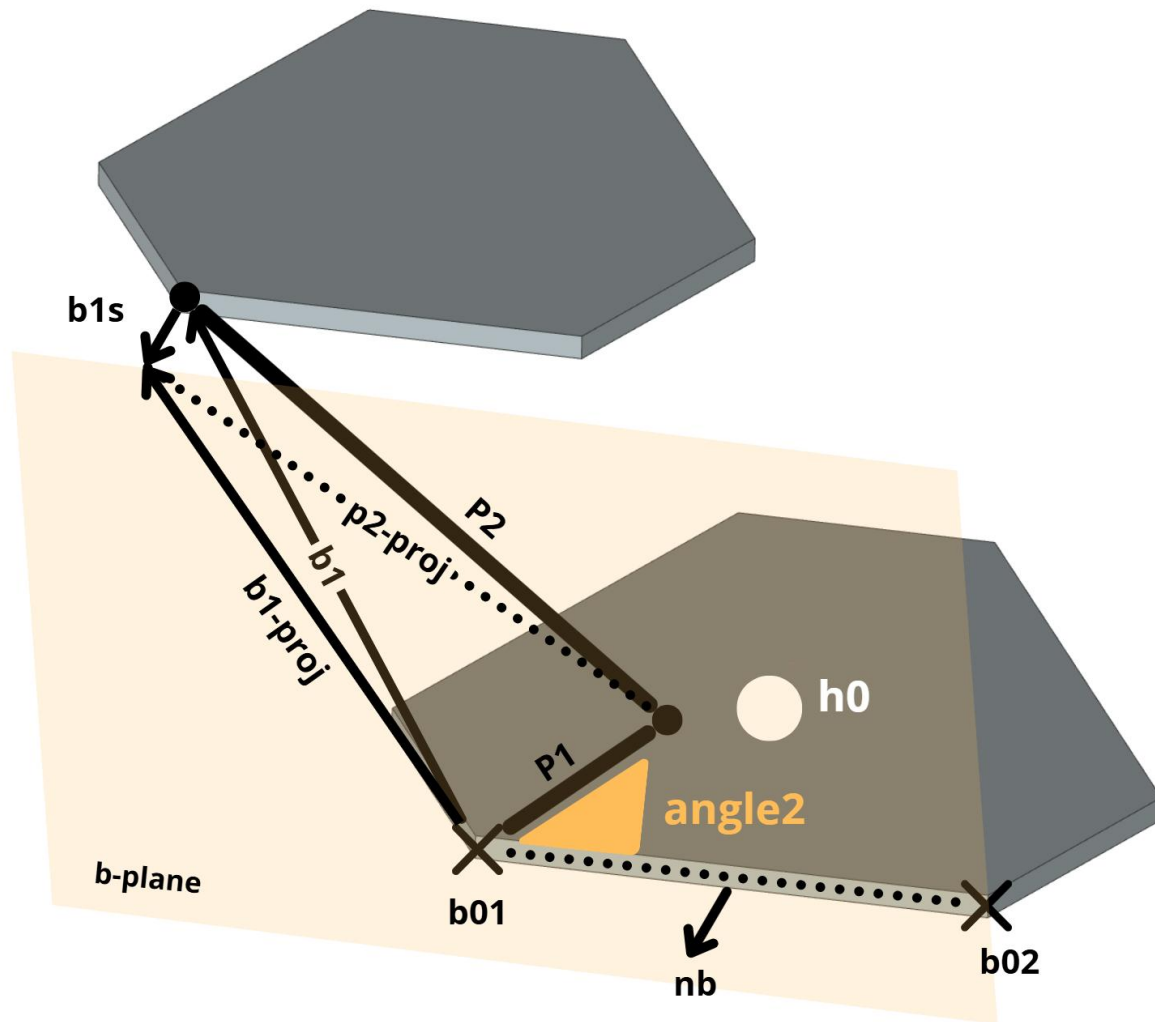
**Stage 2
solved**



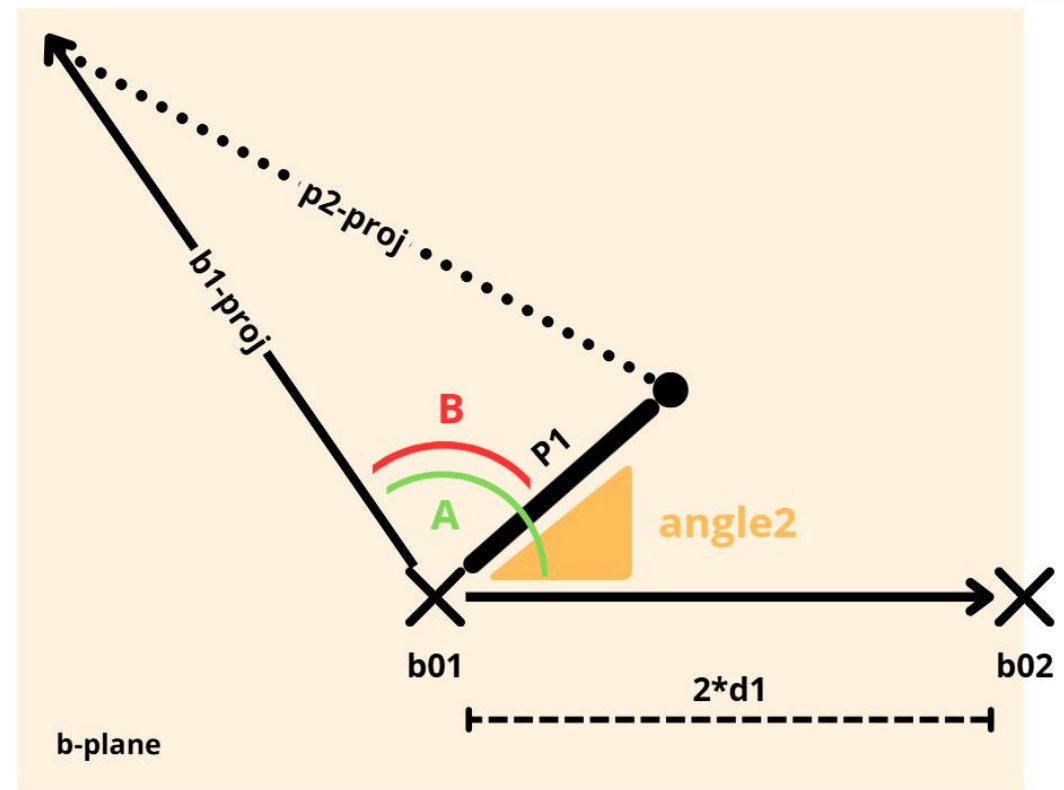
**Stage 3: from vectors
 $\{a1, a2, b1, b2, c1, c2\}$ to
angles $\{0, 1, 2, 3, 4, 5\}$**



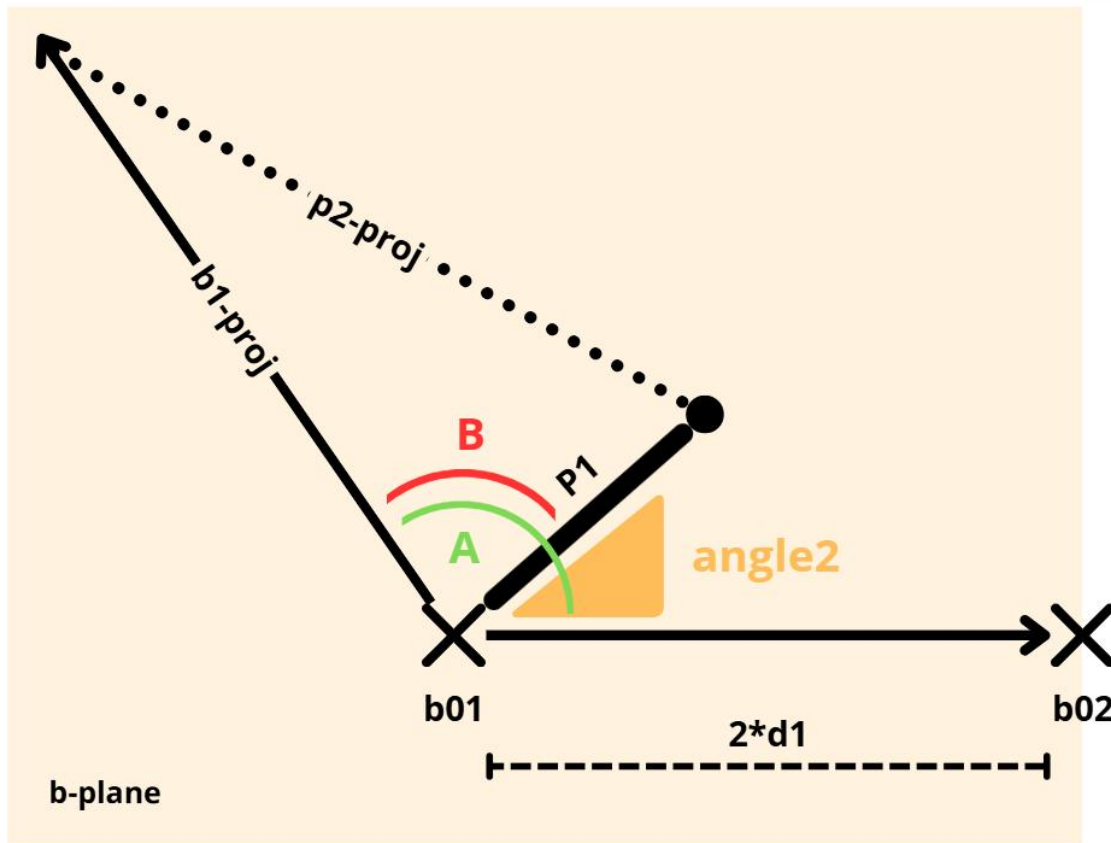
Stage 3: from b1 to angle2 isolated, same for the rest



$$angle_2 = \boxed{(A + B)} \frac{180^\circ}{\pi}$$



Stage 3: from b1 to angle2 isolated, same for the rest



$$\text{angle}_2 = (A + B) \frac{180^\circ}{\pi}$$

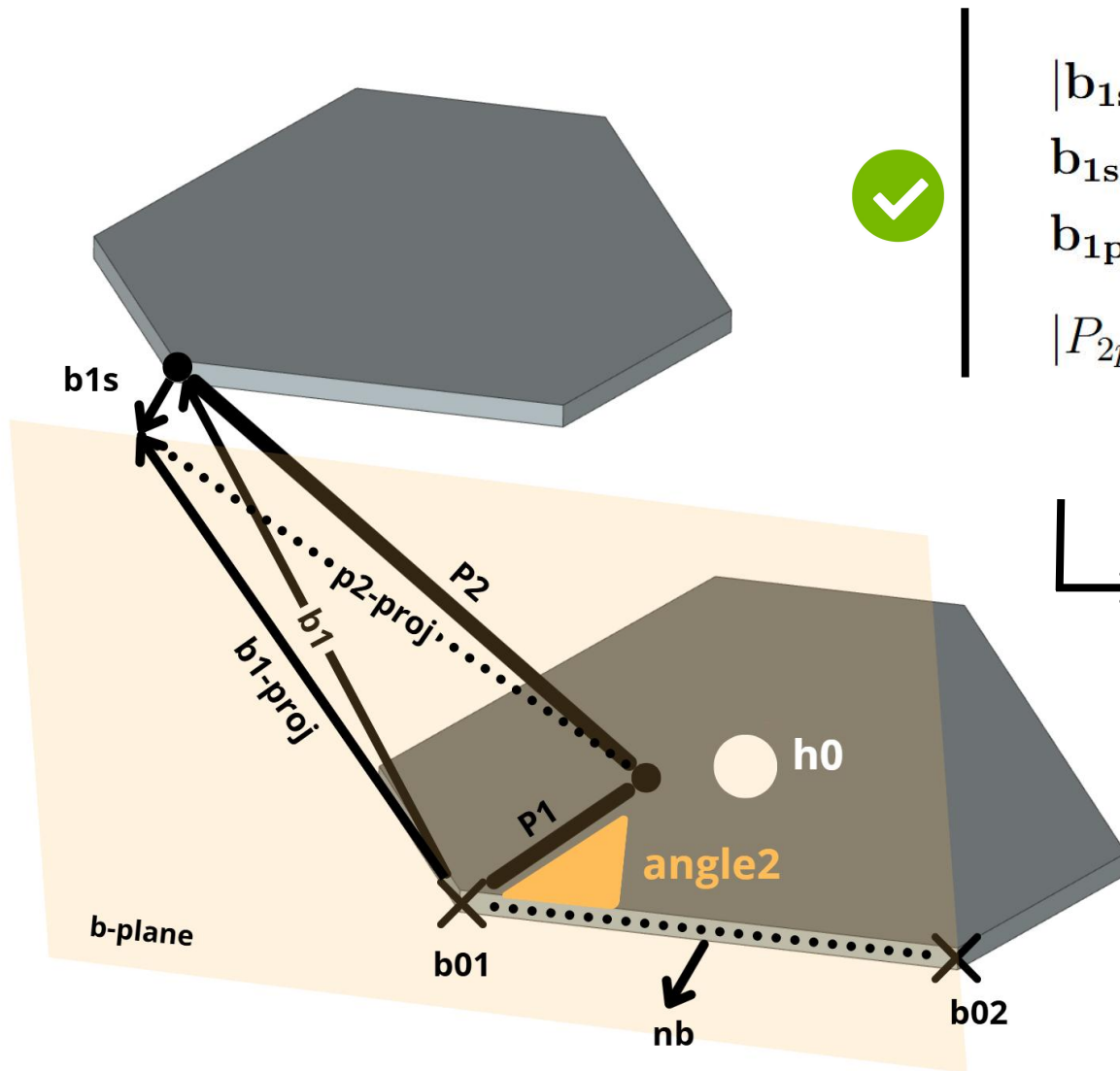
$$A = \arccos \left(\frac{b_{1\text{proj}} \cdot b_{01 \rightarrow 02}}{2d_1 |b_{1\text{proj}}|} \right) \text{ Defined}$$

Law of Cosines

$$P_{2\text{proj}}^2 = |b_{1\text{proj}}|^2 + P_1^2 - 2|b_{1\text{proj}}|P_1 \cos(B)$$

$$B = \arccos \left(\frac{|b_{1\text{proj}}|^2 + P_1^2 - P_{2\text{proj}}^2}{2|b_{1\text{proj}}|P_1} \right)$$

Stage 3: from b1 to angle2 isolated, same for the rest

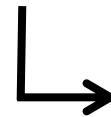


$$|b_{1s}| = |b_1 \cdot \mathbf{n}_b| \quad \text{Defined}$$

$$b_{1s} = \mathbf{n}_b |b_{1s}|$$

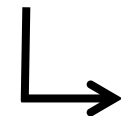
$$b_{1proj} = b_1 + b_{1s}$$

$$|P_{2proj}| = \sqrt{P_2^2 - |b_{1s}|^2}$$



$$A = \arccos \left(\frac{b_{1proj} \cdot b_{01 \rightarrow 02}}{2d_1 |b_{1proj}|} \right)$$

$$B = \arccos \left(\frac{|b_{1proj}|^2 + P_1^2 - P_{2proj}^2}{2|b_{1proj}|P_1} \right)$$



$$\text{angle}_2 = (A + B) \frac{180^\circ}{\pi}$$



**Stage 3
solved**



**x6 Platform
Coordinates**

hx

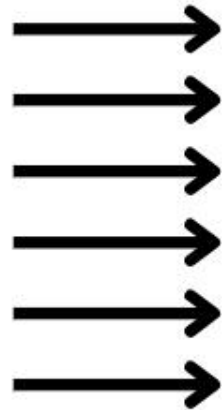
hy

hz

roll

pitch

yaw



**Inverse
Kinematics**

**x6 Motor
Angles**

angle0

angle1

angle2

angle3

angle4

angle5

