

# 6-Degree-of-Freedom Ball Balancing Robot

Inverse Kinematics (IK)

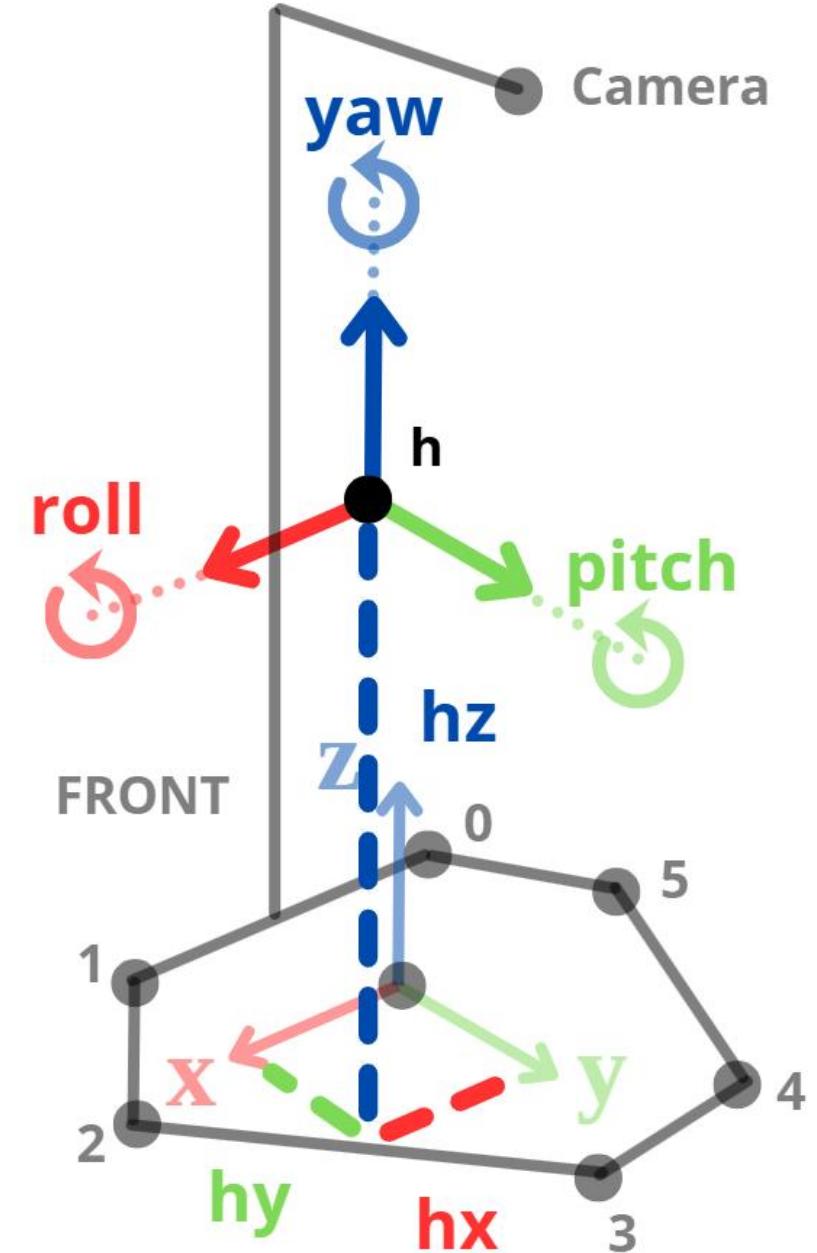
# IK main objective

Given a desired h point:

$$h = \{hx, hy, hz, roll, pitch, yaw\}$$

Find x6 motor angles:

$$\text{angles} = \{0, 1, 2, 3, 4, 5\}$$



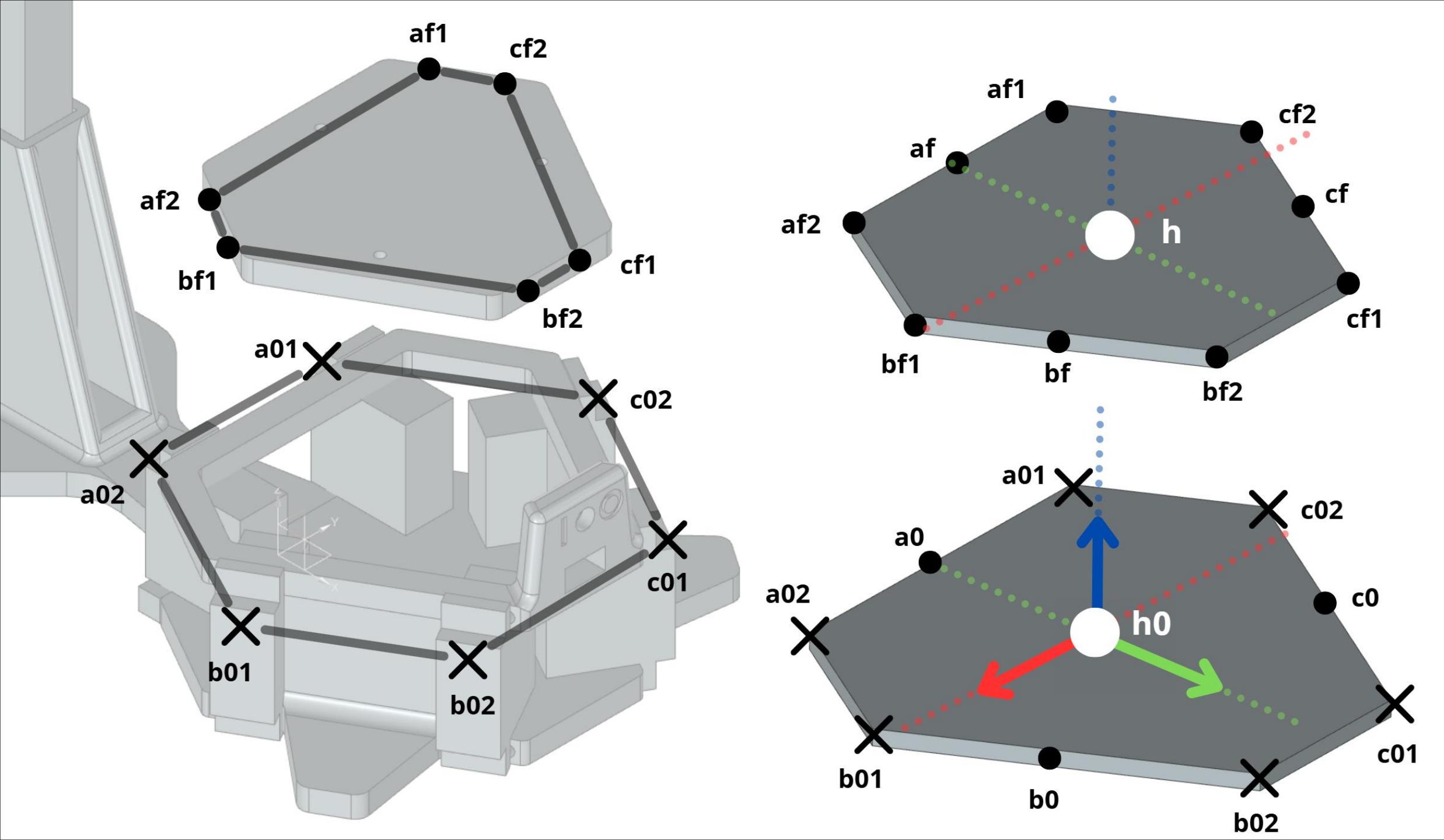
**x6 Platform  
Coordinates**

hx      →  
hy      →  
hz      →  
roll    →  
pitch   →  
yaw     →

**Inverse  
Kinematics**

**x6 Motor  
Angles**

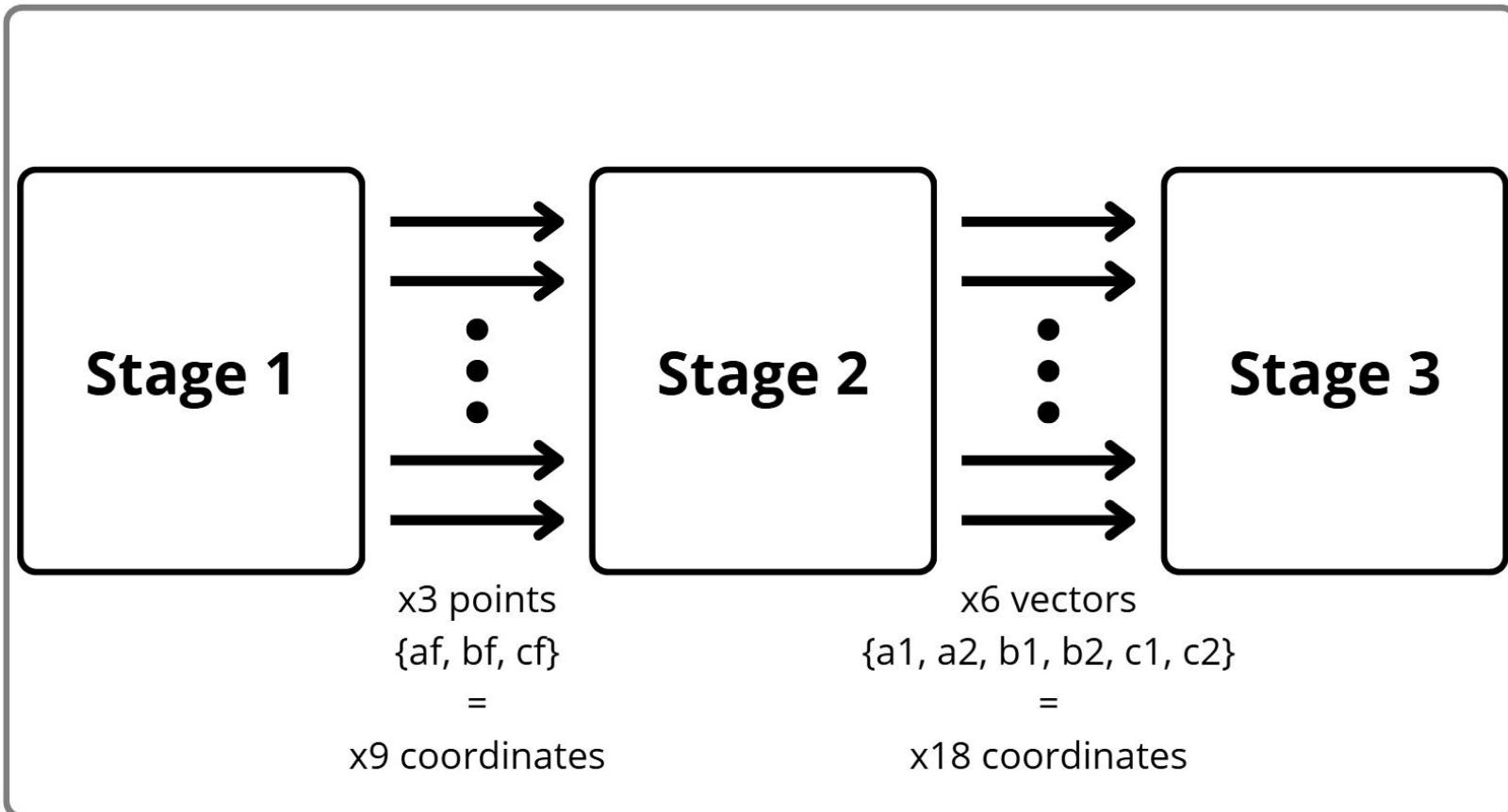
→ angle0  
→ angle1  
→ angle2  
→ angle3  
→ angle4  
→ angle5



## Inverse Kinematics

x6 Platform  
Coordinates

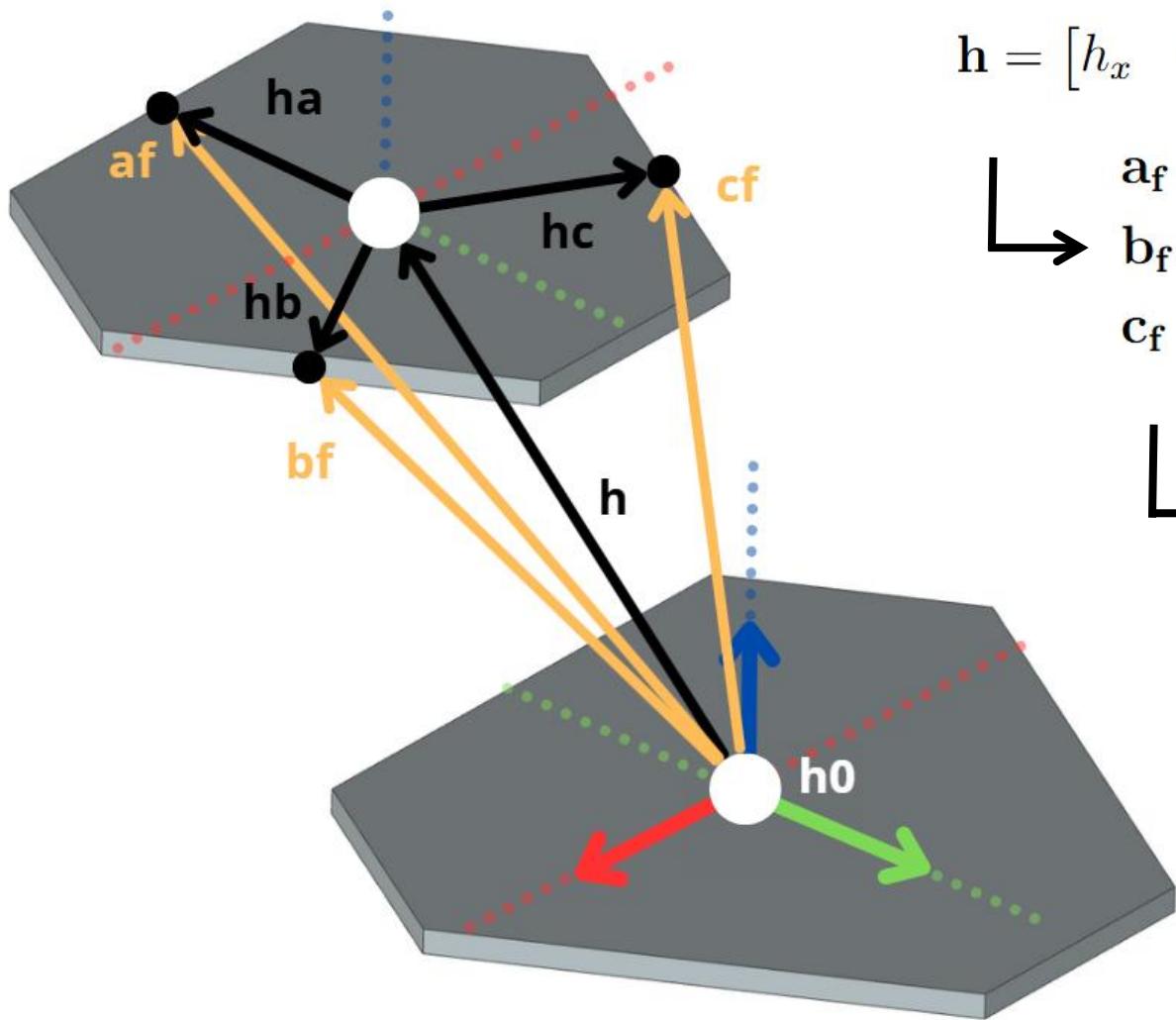
hx       $\rightarrow$   
hy       $\rightarrow$   
hz       $\rightarrow$   
roll     $\rightarrow$   
pitch    $\rightarrow$   
yaw     $\rightarrow$



x6 Motor  
Angles

$\rightarrow$  angle0  
 $\rightarrow$  angle1  
 $\rightarrow$  angle2  
 $\rightarrow$  angle3  
 $\rightarrow$  angle4  
 $\rightarrow$  angle5

# Stage 1: from $\mathbf{h}$ to $\{\mathbf{af}, \mathbf{bf}, \mathbf{cf}\}$



$$\mathbf{h} = [h_x \ h_y \ h_z]^T$$

$$\mathbf{a}_f = \mathbf{h} + \mathbf{ha}$$

$$\mathbf{b}_f = \mathbf{h} + \mathbf{hb}$$

$$\mathbf{c}_f = \mathbf{h} + \mathbf{hc}$$

$$\mathbf{ha} = \text{Rot}(\alpha, \beta, \gamma) \mathbf{ha}_{\text{initial}}$$

$$\mathbf{hb} = \text{Rot}(\alpha, \beta, \gamma) \mathbf{hb}_{\text{initial}}$$

$$\mathbf{hc} = \text{Rot}(\alpha, \beta, \gamma) \mathbf{hc}_{\text{initial}}$$

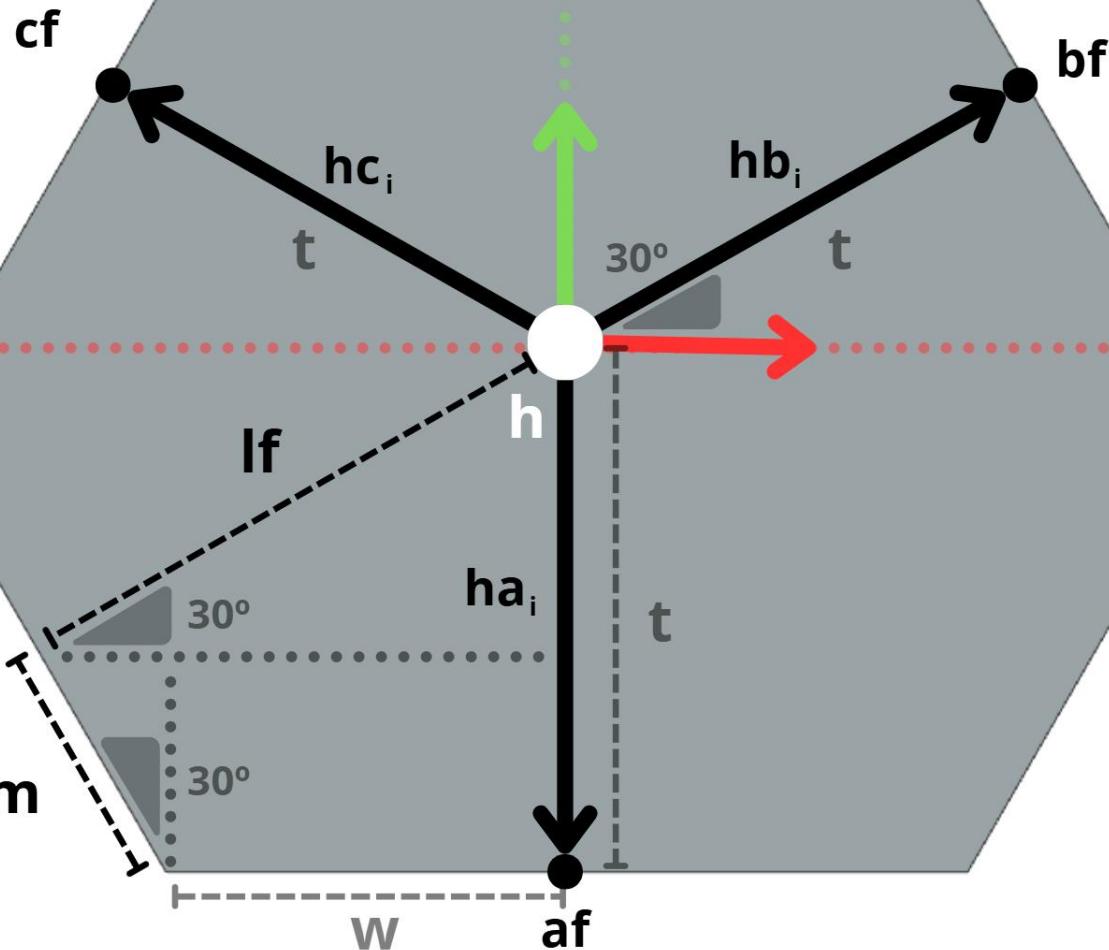
Define  
using  
known  
distances

$$\text{Rot}(yaw, pitch, roll) = \text{Rot}(\alpha, \beta, \gamma) =$$

$$= \text{Rot}(\hat{z}, \alpha) \text{Rot}(\hat{y}, \beta) \text{Rot}(\hat{x}, \gamma) ?$$

$$t = l_f \sin(30^\circ) + m \cos(30^\circ) = \frac{l_f}{2} + \frac{m\sqrt{3}}{2}$$

$$w = l_f \cos(30^\circ) - m \sin(30^\circ) = \frac{l_f\sqrt{3}}{2} - \frac{m}{2}$$

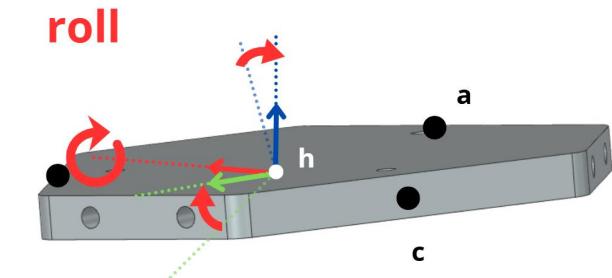
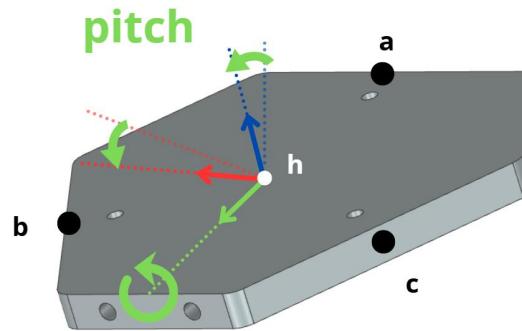
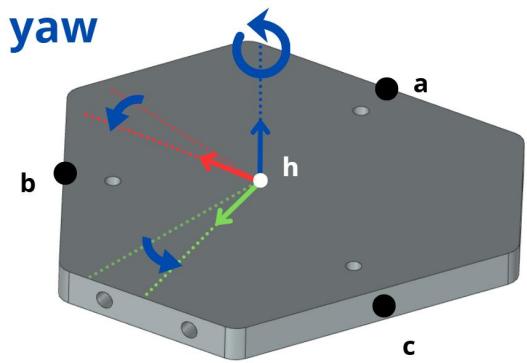
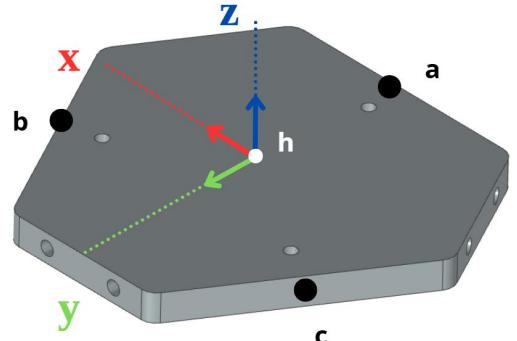


$$\mathbf{ha}_i = [0 \quad -t \quad 0]^T$$

$$\begin{aligned}\mathbf{hb}_i &= [t \cos(30^\circ) \quad t \sin(30^\circ) \quad 0]^T \\ &= \left[ \frac{\sqrt{3}t}{2} \quad \frac{t}{2} \quad 0 \right]^T\end{aligned}$$

$$\begin{aligned}\mathbf{hc}_i &= [-t \cos(30^\circ) \quad t \sin(30^\circ) \quad 0]^T \\ &= \left[ -\frac{\sqrt{3}t}{2} \quad \frac{t}{2} \quad 0 \right]^T\end{aligned}$$

# \*Rotation Roll-Pitch-Yaw (XYZ) = Rotation Euler ZYX



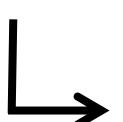
$ha_{initial}$   
 $hb_{initial}$   
 $hc_{initial}$

$$\mathbf{Rot}(\hat{z}, \alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Rot}(\hat{y}, \beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$\mathbf{Rot}(\hat{x}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$

✓  $\mathbf{Rot}(yaw, pitch, roll) = \mathbf{Rot}(\alpha, \beta, \gamma) = \mathbf{Rot}(\hat{z}, \alpha)\mathbf{Rot}(\hat{y}, \beta)\mathbf{Rot}(\hat{x}, \gamma)$



$$\left| \begin{array}{l} \mathbf{ha} = \mathbf{Rot}(\alpha, \beta, \gamma)ha_{initial} \\ \mathbf{hb} = \mathbf{Rot}(\alpha, \beta, \gamma)hb_{initial} \\ \mathbf{hc} = \mathbf{Rot}(\alpha, \beta, \gamma)hc_{initial} \end{array} \right.$$

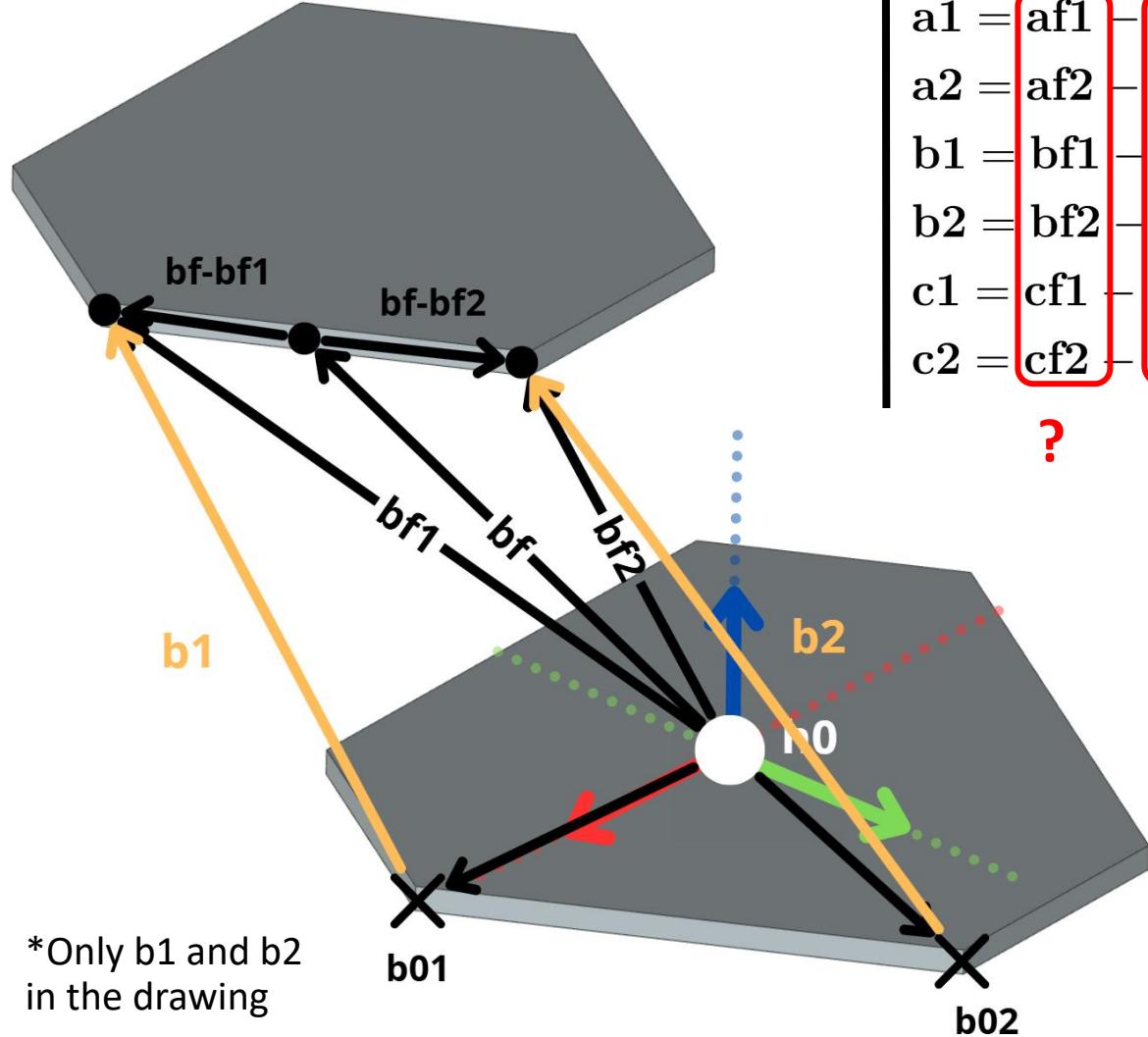


$$\left| \begin{array}{l} \mathbf{a_f} = \mathbf{h} + \mathbf{ha} \\ \mathbf{b_f} = \mathbf{h} + \mathbf{hb} \\ \mathbf{c_f} = \mathbf{h} + \mathbf{hc} \end{array} \right.$$

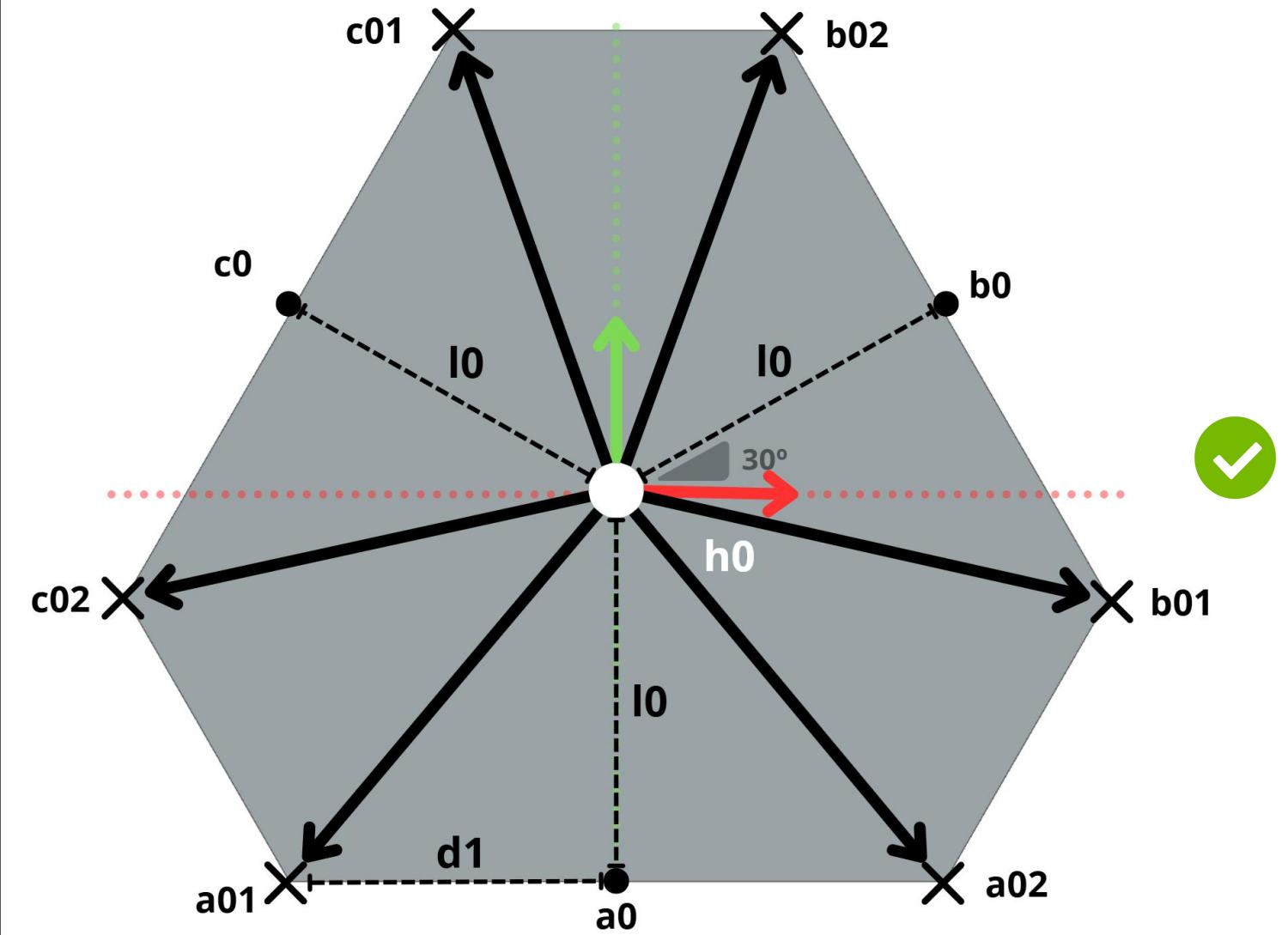


**Stage 1 solved** ✓

# Stage 2: from $\{af, bf, cf\}$ to $\{af1, af2, bf1, bf2, cf1, cf2\}$



$\begin{array}{l} a1 = af1 \\ a2 = af2 \\ b1 = bf1 \\ b2 = bf2 \\ c1 = cf1 \\ c2 = cf2 \end{array}$	<span style="color: red;">Define using known distances</span>	$\rightarrow$	$\begin{array}{l} af1 = af + af_{\rightarrow}af1 \\ af2 = af + af_{\rightarrow}af2 \\ bf1 = bf + bf_{\rightarrow}bf1 \\ bf2 = bf + bf_{\rightarrow}bf2 \\ cf1 = cf + cf_{\rightarrow}cf1 \\ cf2 = cf + cf_{\rightarrow}cf2 \end{array}$
?	?	<span style="color: green;">From Stage 1</span>	?
$\begin{array}{l} af_{\rightarrow}af1 = Rot(\alpha, \beta, \gamma) \\ af_{\rightarrow}af2 = Rot(\alpha, \beta, \gamma) \\ bf_{\rightarrow}bf1 = Rot(\alpha, \beta, \gamma) \\ bf_{\rightarrow}bf2 = Rot(\alpha, \beta, \gamma) \\ cf_{\rightarrow}cf1 = Rot(\alpha, \beta, \gamma) \\ cf_{\rightarrow}cf2 = Rot(\alpha, \beta, \gamma) \end{array}$	<span style="color: red;">Define using known distances</span>	$\begin{array}{l} af_{\rightarrow}af1_{initial} \\ af_{\rightarrow}af2_{initial} \\ bf_{\rightarrow}bf1_{initial} \\ bf_{\rightarrow}bf2_{initial} \\ cf_{\rightarrow}cf1_{initial} \\ cf_{\rightarrow}cf2_{initial} \end{array}$	



$$\mathbf{a}_{01} = [-d_1 \quad -l_0 \quad 0]^T$$

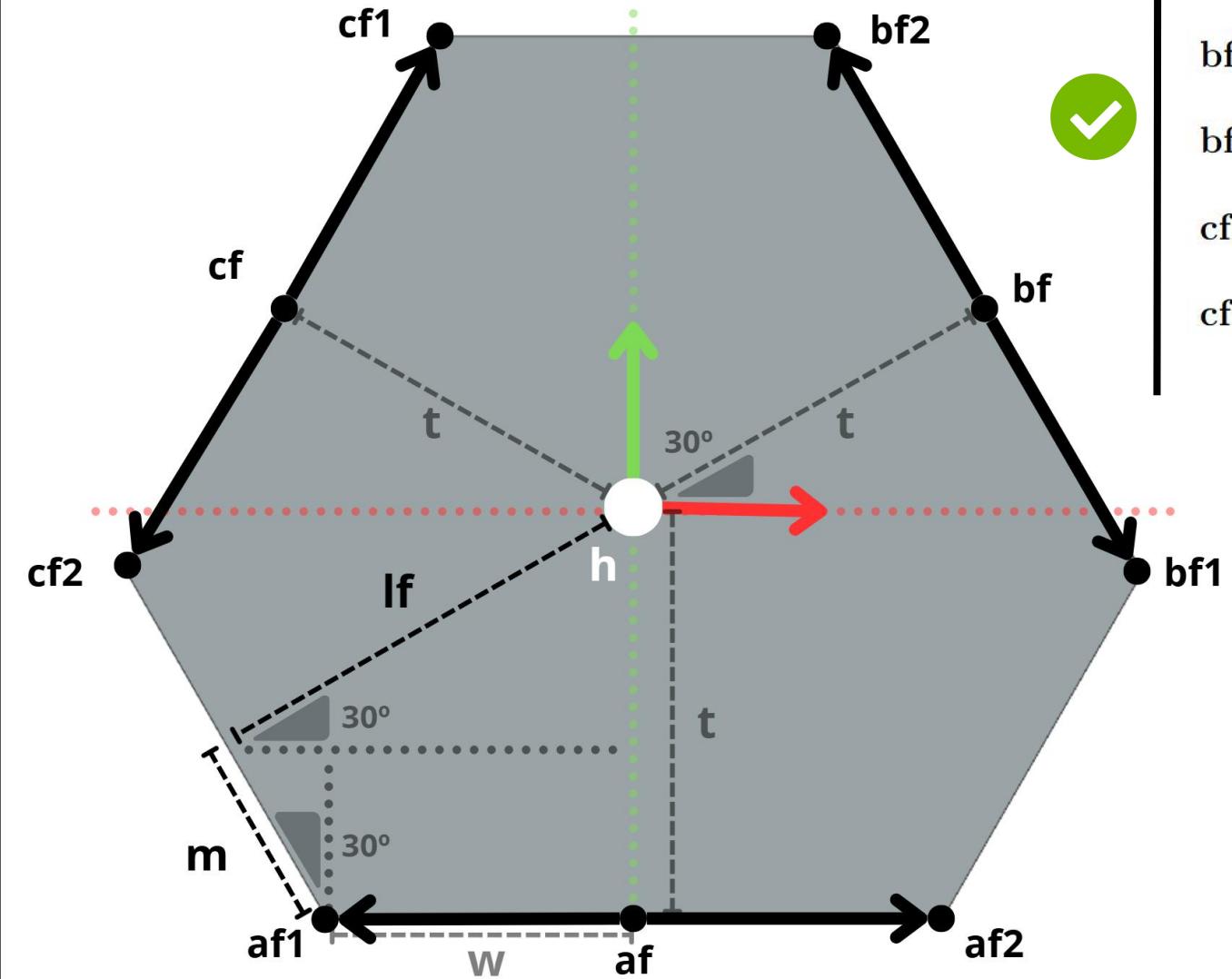
$$\mathbf{a}_{02} = [d_1 \quad -l_0 \quad 0]^T$$

$$\mathbf{b}_{01} = \left[ \frac{l_0\sqrt{3}}{2} + \frac{d_1}{2} \quad \frac{l_0}{2} - \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$

$$\mathbf{b}_{02} = \left[ \frac{l_0\sqrt{3}}{2} - \frac{d_1}{2} \quad \frac{l_0}{2} + \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$

$$\mathbf{c}_{01} = \left[ -\frac{l_0\sqrt{3}}{2} + \frac{d_1}{2} \quad \frac{l_0}{2} + \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$

$$\mathbf{c}_{02} = \left[ -\frac{l_0\sqrt{3}}{2} - \frac{d_1}{2} \quad \frac{l_0}{2} - \frac{d_1\sqrt{3}}{2} \quad 0 \right]^T$$



$$af \rightarrow af1_{initial} = [-w \ 0 \ 0]^T$$

$$af \rightarrow af2_{initial} = [w \ 0 \ 0]^T$$

$$bf \rightarrow bf1_{initial} = [w \sin(30^\circ) \ -w \cos(30^\circ) \ 0]^T = \left[ \frac{w}{2} \ -\frac{w\sqrt{3}}{2} \ 0 \right]^T$$

$$bf \rightarrow bf2_{initial} = [-w \sin(30^\circ) \ w \cos(30^\circ) \ 0]^T = \left[ -\frac{w}{2} \ \frac{w\sqrt{3}}{2} \ 0 \right]^T$$

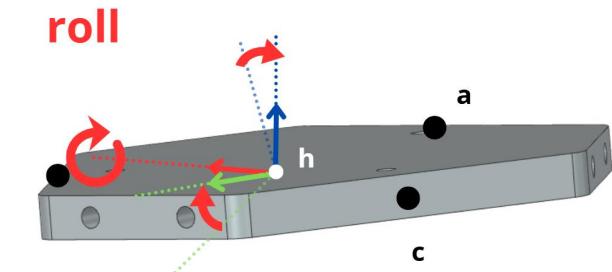
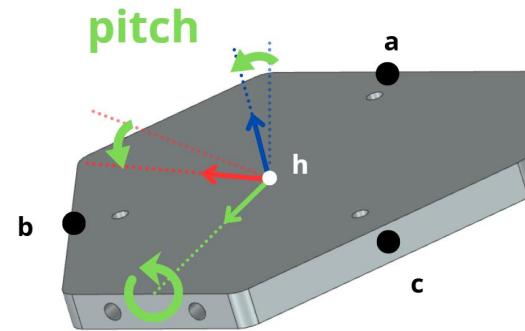
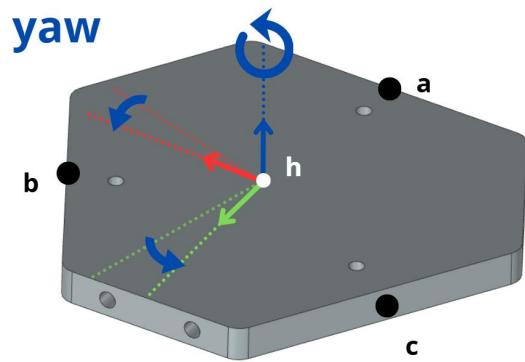
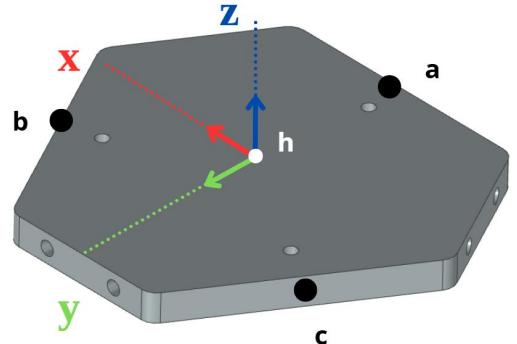
$$cf \rightarrow cf1_{initial} = [w \sin(30^\circ) \ w \cos(30^\circ) \ 0]^T = \left[ \frac{w}{2} \ \frac{w\sqrt{3}}{2} \ 0 \right]^T$$

$$cf \rightarrow cf2_{initial} = [-w \sin(30^\circ) \ -w \cos(30^\circ) \ 0]^T = \left[ -\frac{w}{2} \ -\frac{w\sqrt{3}}{2} \ 0 \right]^T$$

$$t = l_f \sin(30^\circ) + m \cos(30^\circ) = \frac{l_f}{2} + \frac{m\sqrt{3}}{2}$$

$$w = l_f \cos(30^\circ) - m \sin(30^\circ) = \frac{l_f\sqrt{3}}{2} - \frac{m}{2}$$

# \*Rotation Roll-Pitch-Yaw (XYZ) = Rotation Euler ZYX



$af \rightarrow af_{initial}$

$af \rightarrow af_{2initial}$

$bf \rightarrow bf_{1initial}$

$bf \rightarrow bf_{2initial}$

$cf \rightarrow cf_{1initial}$

$cf \rightarrow cf_{2initial}$

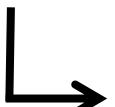
$$\text{Rot}(\hat{z}, \alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rot}(\hat{y}, \beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$\text{Rot}(\hat{x}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$



$$Rot(yaw, pitch, roll) = Rot(\alpha, \beta, \gamma) = Rot(\hat{z}, \alpha)Rot(\hat{y}, \beta)Rot(\hat{x}, \gamma)$$



$$af \rightarrow af_1 = Rot(\alpha, \beta, \gamma) af \rightarrow af_{initial}$$

$$af \rightarrow af_2 = Rot(\alpha, \beta, \gamma) af \rightarrow af_{2initial}$$

$$bf \rightarrow bf_1 = Rot(\alpha, \beta, \gamma) bf \rightarrow bf_{1initial}$$

$$bf \rightarrow bf_2 = Rot(\alpha, \beta, \gamma) bf \rightarrow bf_{2initial}$$

$$cf \rightarrow cf_1 = Rot(\alpha, \beta, \gamma) cf \rightarrow cf_{1initial}$$

$$cf \rightarrow cf_2 = Rot(\alpha, \beta, \gamma) cf \rightarrow cf_{2initial}$$



$$af_1 = af + af \rightarrow af_1$$

$$af_2 = af + af \rightarrow af_2$$

$$bf_1 = bf + bf \rightarrow bf_1$$

$$bf_2 = bf + bf \rightarrow bf_2$$

$$cf_1 = cf + cf \rightarrow cf_1$$

$$cf_2 = cf + cf \rightarrow cf_2$$



$$a1 = af_1 - a01$$

$$a2 = af_2 - a02$$

$$b1 = bf_1 - b01$$

$$b2 = bf_2 - b02$$

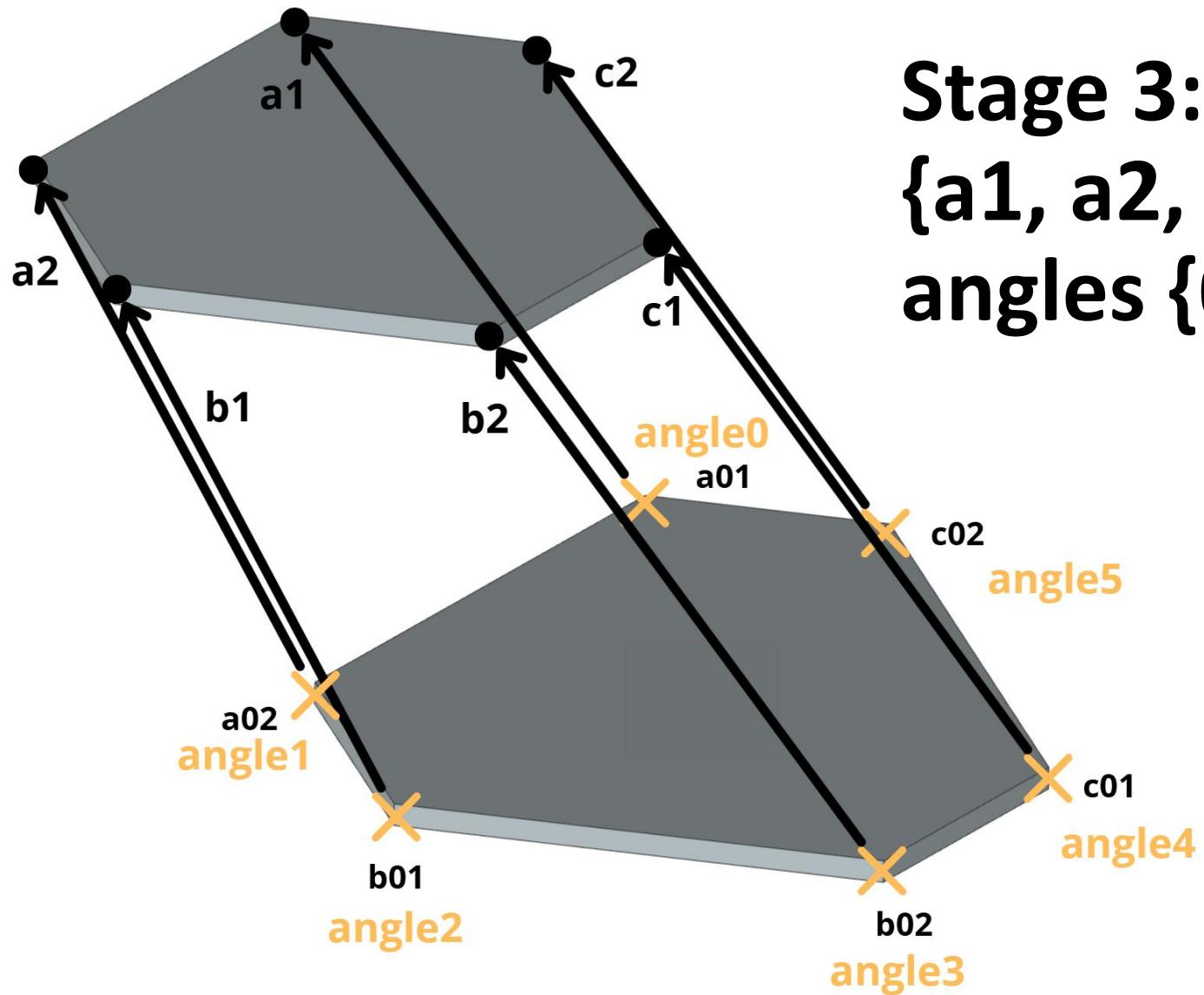
$$c1 = cf_1 - c01$$

$$c2 = cf_2 - c02$$

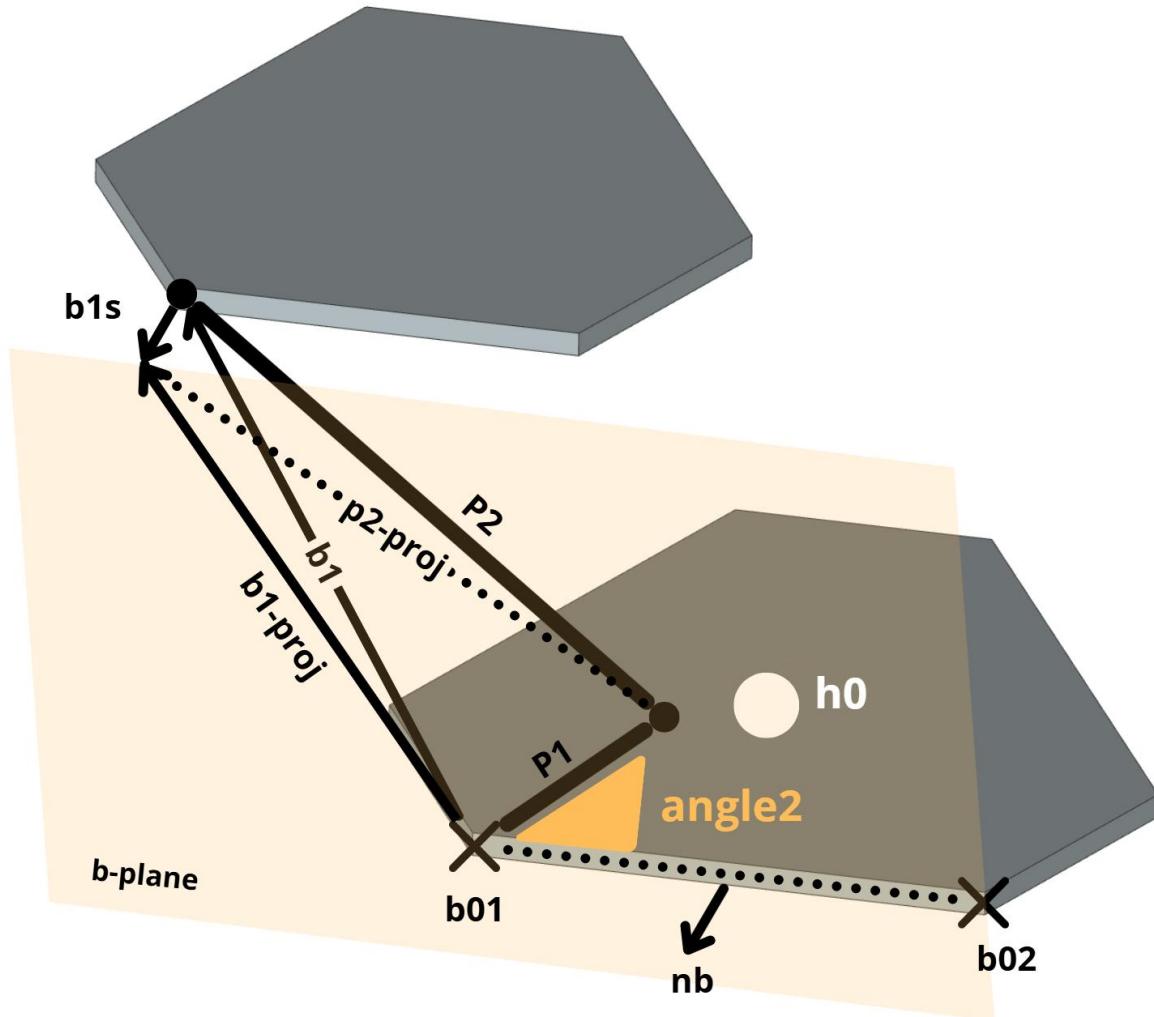
**Stage 2 solved**



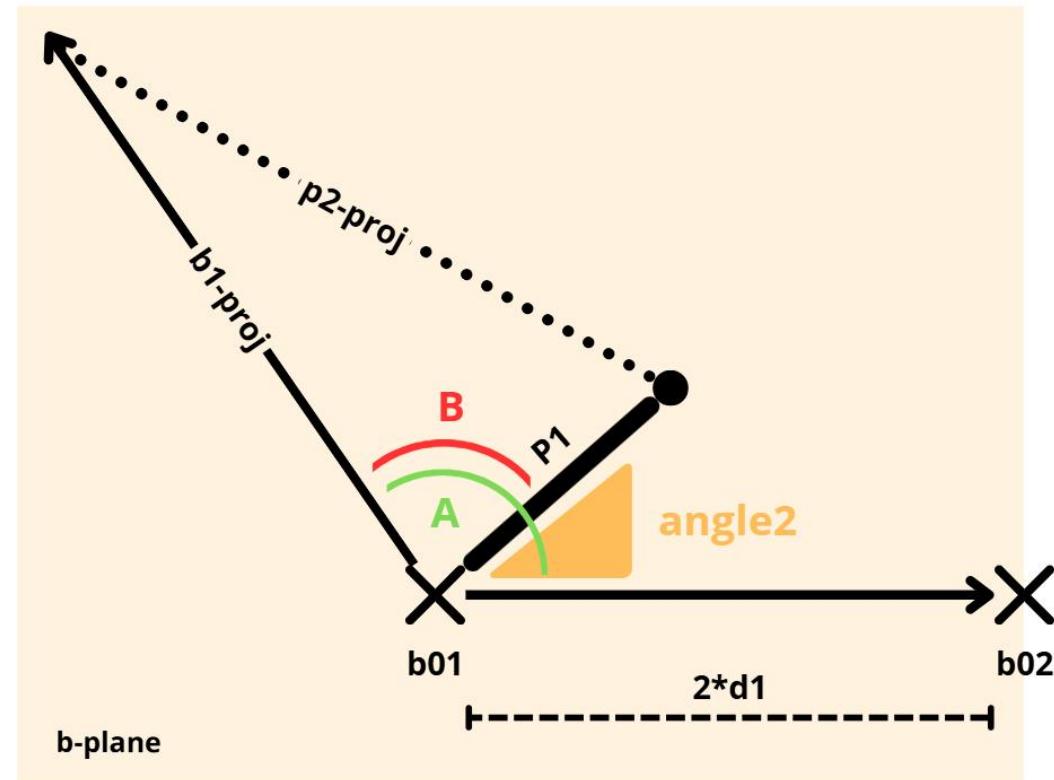
**Stage 3: from vectors**  
 $\{a_1, a_2, b_1, b_2, c_1, c_2\}$  to  
angles  $\{0, 1, 2, 3, 4, 5\}$



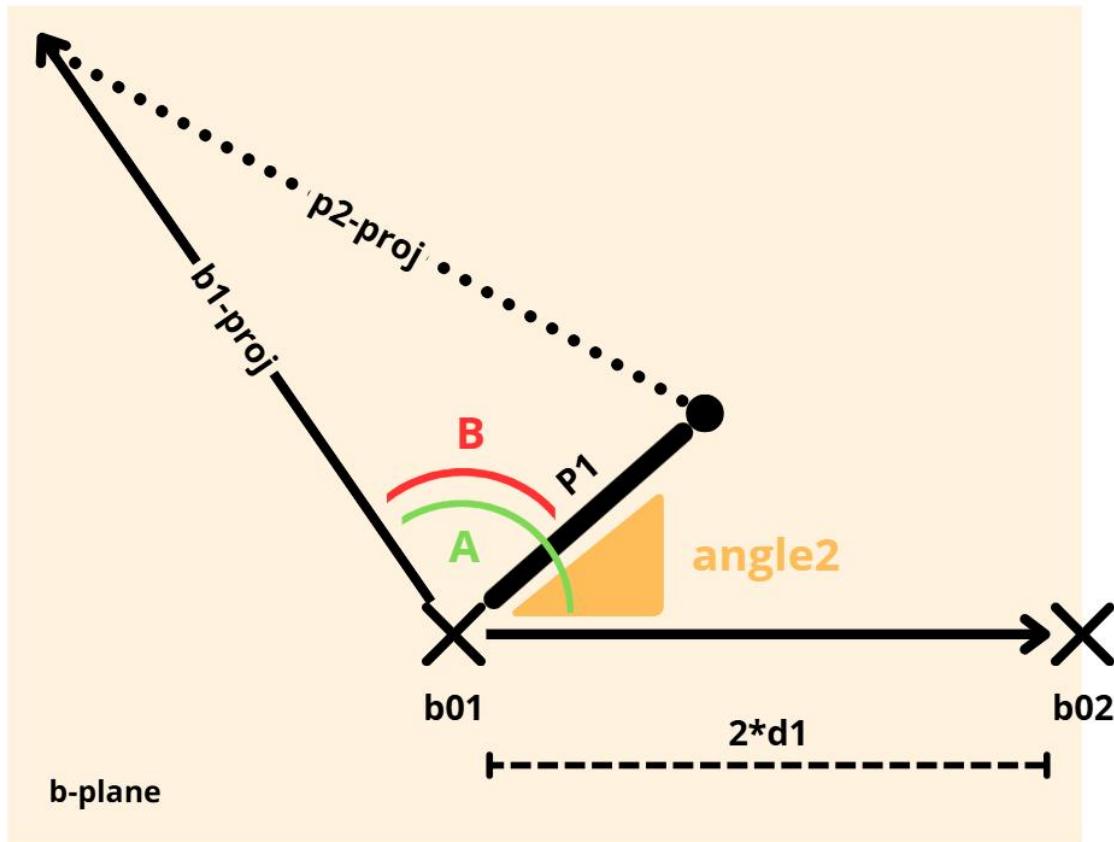
# Stage 3: from b1 to angle2 isolated, same for the rest



$$\text{angle}_2 = \frac{(A + B)}{\pi}^{180^\circ}$$



# Stage 3: from $b_1$ to angle2 isolated, same for the rest



$$\text{angle}_2 = (A + B) \frac{180^\circ}{\pi}$$

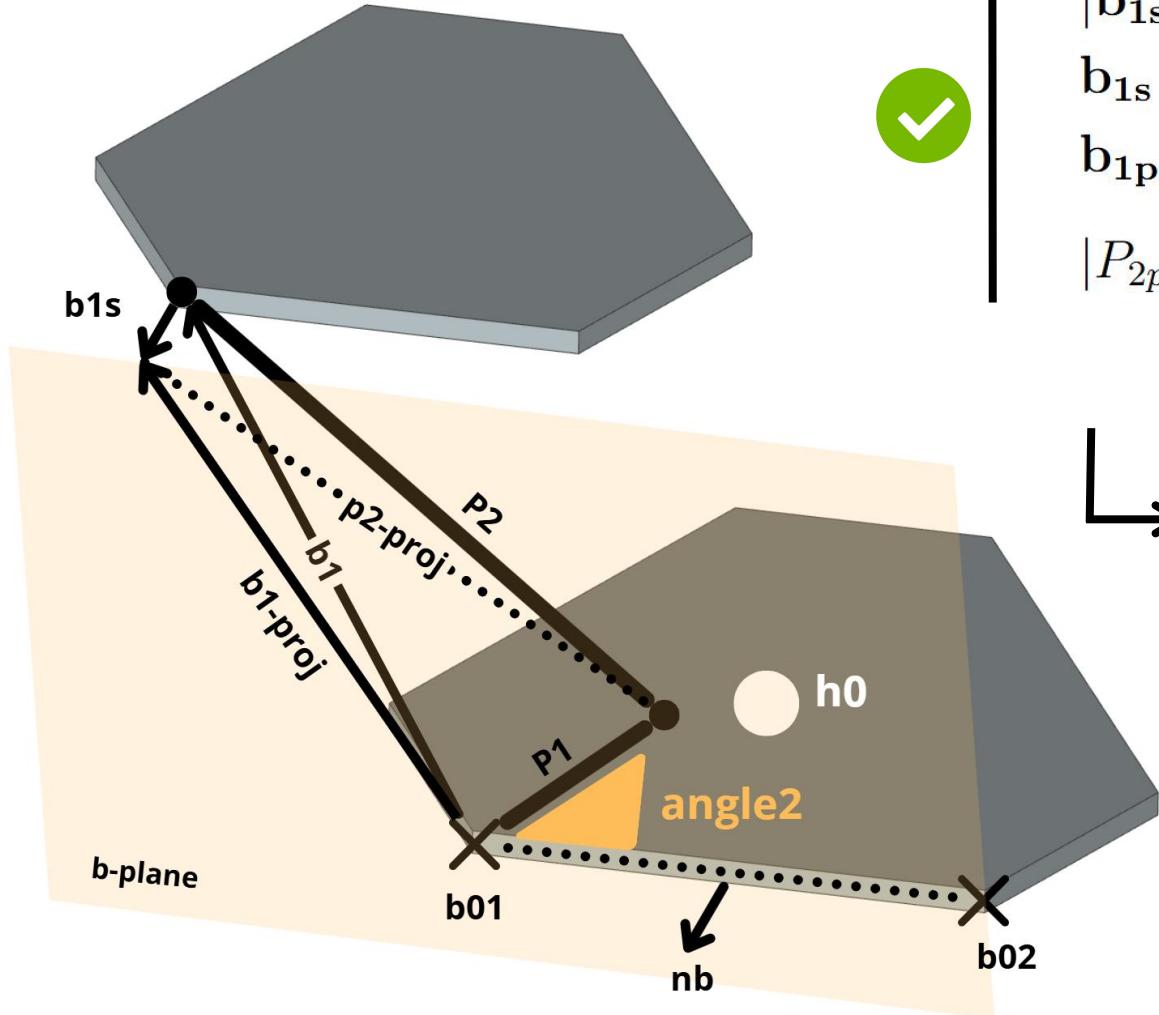
$$A = \arccos \left( \frac{\mathbf{b}_{1\text{proj}} \cdot \mathbf{b}_{01 \rightarrow 02}}{2d_1 |\mathbf{b}_{1\text{proj}}|} \right)$$

Law of Cosines

$$P_{2\text{proj}}^2 = |\mathbf{b}_{1\text{proj}}|^2 + P_1^2 - 2|\mathbf{b}_{1\text{proj}}|P_1 \cos(B)$$

$$B = \arccos \left( \frac{|\mathbf{b}_{1\text{proj}}|^2 + P_1^2 - P_{2\text{proj}}^2}{2|\mathbf{b}_{1\text{proj}}|P_1} \right)$$

# Stage 3: from $b_1$ to angle2 isolated, same for the rest



$$|b_{1s}| = |b_1 \cdot n_b| \quad \text{Defined}$$

$$b_{1s} = n_b |b_{1s}|$$

$$b_{1\text{proj}} = b_1 + b_{1s}$$

$$|P_{2\text{proj}}| = \sqrt{P_2^2 - |b_{1s}|^2}$$

$$A = \arccos \left( \frac{b_{1\text{proj}} \cdot b_{01 \rightarrow 02}}{2d_1 |b_{1\text{proj}}|} \right)$$

$$B = \arccos \left( \frac{|b_{1\text{proj}}|^2 + P_1^2 - P_{2\text{proj}}^2}{2|b_{1\text{proj}}| P_1} \right)$$

$$\text{angle}_2 = (A + B) \frac{180^\circ}{\pi} \rightarrow$$

Stage 3  
solved



**x6 Platform  
Coordinates**

hx      →  
hy      →  
hz      →  
roll    →  
pitch   →  
yaw     →

**Inverse  
Kinematics**

**x6 Motor  
Angles**

→ angle0  
→ angle1  
→ angle2  
→ angle3  
→ angle4  
→ angle5

