HW1: Using Implicit and Explicit Time-Stepping Schemes to Model the Kinematics of Falling and Simply Supported Beams*

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***Abstract*— Designing for the kinematics of soft robotics can be facilitated using efficient analytical models capable of predicting nonlinear and transient responses of non-conservative systems. One such system that is often employed for soft robotics simulations is the discrete elastic rod (DER) formulation which approximates discrete sections of robotic members as a system of point masses linked together into a chain via elastic rods. In this assignment we show that through the derivation of an objective function defined for implicit Newton Raphson time-stepping using the equation of motion and the Jacobian inversion, the transient response of a beam falling through viscous media can be predicted. This scheme has significant time-saving advantages to an explicit time-stepping scheme since the conditions for stability are less limiting to the time increment. Moreover, this assignment explores the impact of DER spatial refinement using 20-node members and the application of boundary conditions often used in conservative beam loading and deflection problems.**

I. Introduction

This assignment includes the simulated response of three problems involving discrete elastic rods. In **Problem 1** we examine the transient response of a 3-node elastic rod that falls through a viscous medium from an initially straight configuration at rest. We examine the advantages of the implicit scheme over explicit time stepping for this application and document various predictions including deformations, displacements, and terminal velocity. In **Problem 2** we extend the approach to Problem 1 to apply to a falling rod with N number of nodes. Finally, in **Problem 3** we assess the accuracy of classic beam deflection predictions using the implicit time stepping scheme and the notion of fixed, free, and loaded degrees of freedom (DOF).

II. Procedure for Paper Submission

A. *Problem 1: Rigid Spheres and Elastic Beam Falling in Viscous Flow (3 Nodes)*

For problem 1, we assume three discrete point masses concentrated at nodes between two connecting rod members. The assembly is shown in **Figure 1** with masses R1, R2, and R3 assembled left to right in blue, orange, and green, respectively.

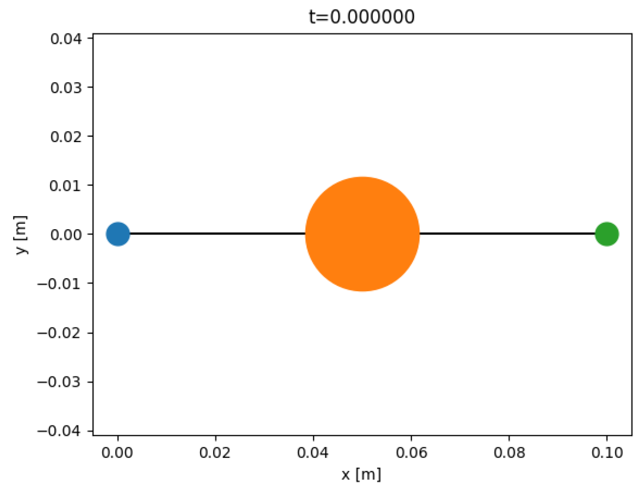

Figure 1: Initial Configuration for Falling Beam Initially at Rest

The mass radii, R1 and R3 are 5 mm and R2 is 25 mm, each have a density of 7000 kg/m^3. The elastic rods have a radius, r_0 of 1 mm and a Young's modulus of 1 GPa. In this problem the masses have two degrees of freedom each (x_n, y_n) and therefore the equation of motion includes conservative contributions for both stretching and z-bending, (y-bending and x-twisting is ignored), which are recovered by taking the negative gradient of the stretching and z-bending potentials.

The transient effect of unconservative forces (including damping within a viscous medium of 1000 N-s/m) is first resolved implicitly using a Newton Raphson time-stepping scheme. In this approach equation of motion (EOM) is rearranged to the RHS, set as the residual, and subsequently minimized to converge on the updated DOFs (q_{k+1}) within acceptable tolerances after passing an initial guess. The implicit scheme uses Jacobian inversion to evaluate q_{k+1} which (while more expensive than incremental evaluations in explicit) is much more suitable for long duration responses like steady state analysis or in this case determination of terminal velocity for the falling beam. **Figure 2** shows the transient response at discrete points in time (t = 0s, 0.01s, 0.05s, 0.1s, 1s, and 10s).

*Research supported by ABC Foundation.

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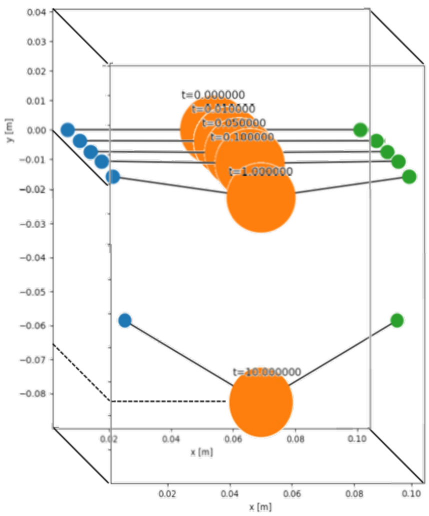

Figure 2: Transient Change in Shape and Nodal Coordinates Between 0 and 10s.

The displacement as a function of time for R2 is mostly linear over the 10s duration but the velocity approaches a steady state terminal velocity marked by a plateau in the velocity history as the viscous forces balance out the weight and elastic forces of the falling system. **Figure 3** and **Figure 4** show the displacement and velocity histories, respectively.

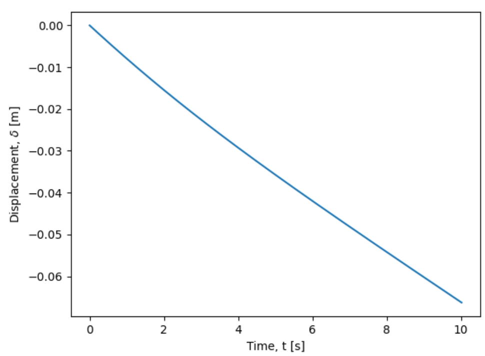

Figure 3: Displacement History of 3-Node Falling Beam

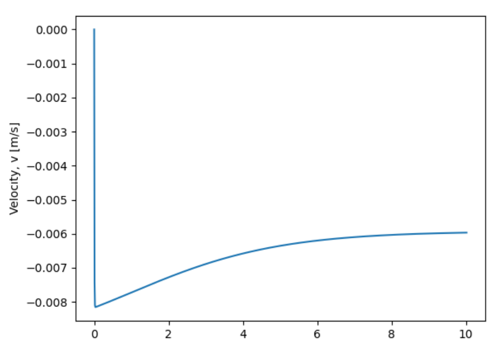

Figure 4: Velocity History of 3-Node Falling Beam

As shown in the plateau in **Figure 4,** the terminal velocity is about -0.005966 m/s.

For a system with equal radii among all three masses, the turning angle begins with some very small fluctuations but remains constant near zero for the remainder of the simulation. Physically this does not match my expectations as there should be no imbalance in a perfectly symmetric system, however based on our understanding of the Newton Raphson iterations scheme, there is some guess that predicates more accurate predictions thereafter, so it is not unusual to see such small initial fluctuations in implicit schemes such as this. **Figure 4** shows the extent of initial fluctuations in the system and the quick progression toward zero turning angle thereafter.

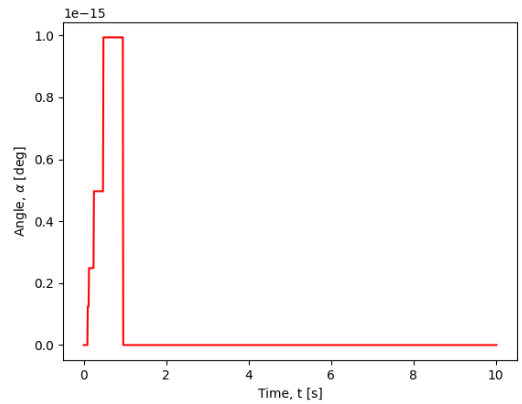

Figure 5: Velocity History of 3-Node Falling Beam

The time-stepping was then adjusted to reflect an explicit scheme, where the only unknown is the new values for the system DOFs. In such a scheme the Jacobian inversion is no longer needed and a simple rearrangement of the EOM will generate an algebraic expression capable of marching forward in time with relatively little expense per increment. However, due to stability requirements of the explicit scheme, a much smaller time step is necessary in order to avoid divergence of the transient solution so orders of magnitude greater number of increments are necessary to produce a reasonable solution. When a time-step of 1e-5s is used, the simulation produces results that match very closely with the same implicit scheme using a time step of 1e-2s, but the simulation itself takes over an hour to complete as opposed to less than a minute for the implicit scheme. For the time-scale pertinent to the steady state or terminal velocity of this system, an implicit scheme is much more efficient and practical.

Increasing the time step from 1e-5 to 1e-4s accelerates the explicit solve to finish in about 5 minutes, however it results in an instability that produces highly inaccurate results. **Figure 6-8** shows how much the predicted values for angle, velocity and displacement have blown up from the unstable time-stepping.

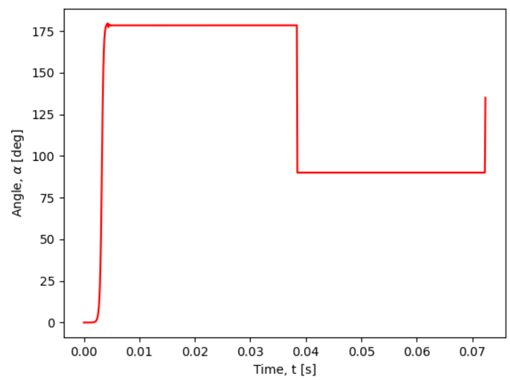

Figure 6: Angle History for Explicit Scheme Where dt = 1e-4 s.

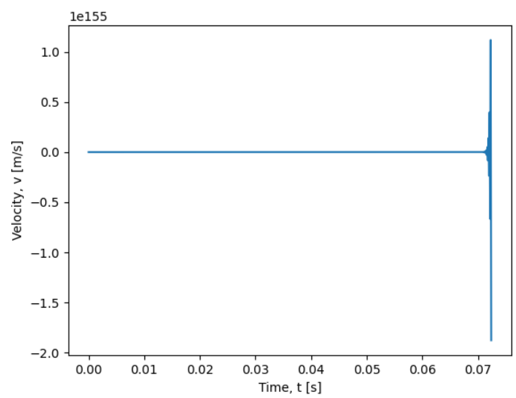

Figure 7: Velocity History for Explicit Scheme Where dt = 1e-4 s.

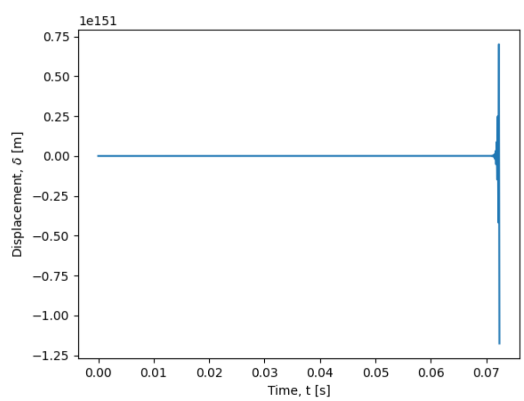

Figure 8: Displacement History for Explicit Scheme Where dt = 1e-4 s.

Next we will explore the impact of simulating a falling beam with greater spatial refinement where the number of discrete nodal masses is increased.

B. *Problem 2: Generalized Case of Elastic Beam Falling in Viscous Flow*

In this problem the implicit scheme is generalized to account for several DOF and enables the study of spatial refinement on approximations using the DER method. We start with a 5 node system (N=5) where node 3 (R_{mid}) is 25 mm in radius, while all other nodes are set to a radius (R) equal to dl/10 where dl is the length of 1 edge split evenly between the four edges along a total length of 0.1 m. Thus R = 2.5 mm, and thus the total mass remains the same as the 3 node problem discussed earlier. The 5 node model shows a slightly farther plunge after 10s but shape is still very much in family with the 3 node problem. **Figure 9 and 10** shows the overlay of the two approaches with the 3-noded model given slight transparency.

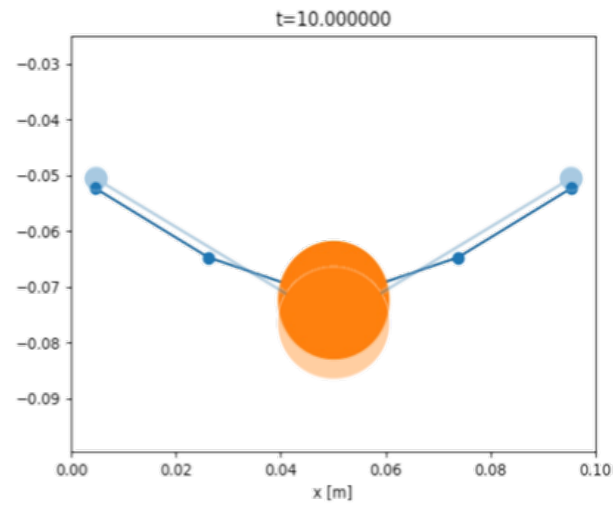

Figure 9: Generalized Implicit Deformation Shape for Falling Beam with 5 Nodes Overlaid With 3 Node Model.

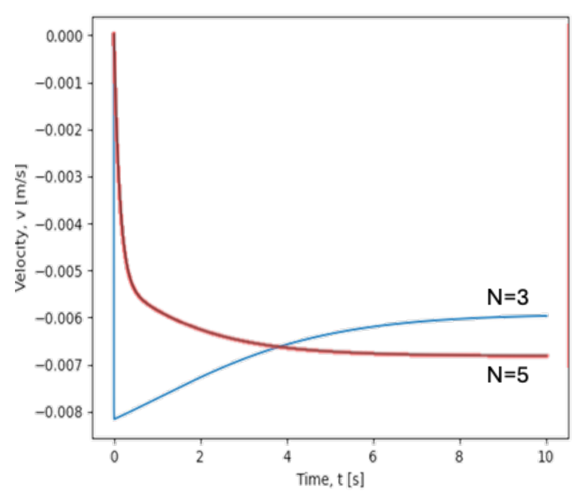

Figure 10: Generalized Implicit Velocity for Falling Beam with 5 Nodes (red) Overlaid With 3 Node Model (blue).

Next we increase the node count to 21 and evaluate the velocity history as well as the vertical position history.

First we examine the response up to 50s. In doing so we ensure a greater steady state solution over a greater temporal refinement. **Figure 11 and 12** show the displacement and velocity histories over the 50 s period.

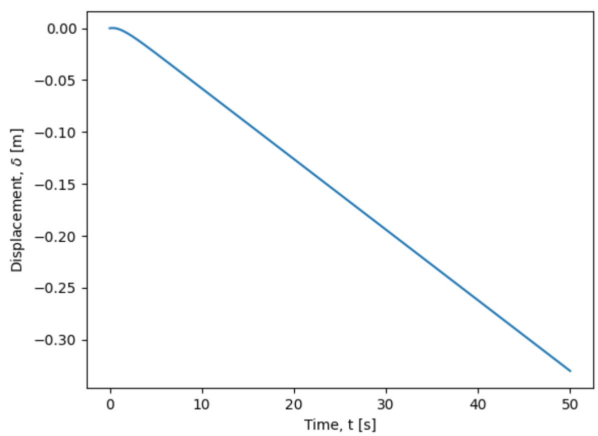

Figure 11: Generalized Implicit Displacement for Falling Beam with 21 Nodes (red) falling for 50s.

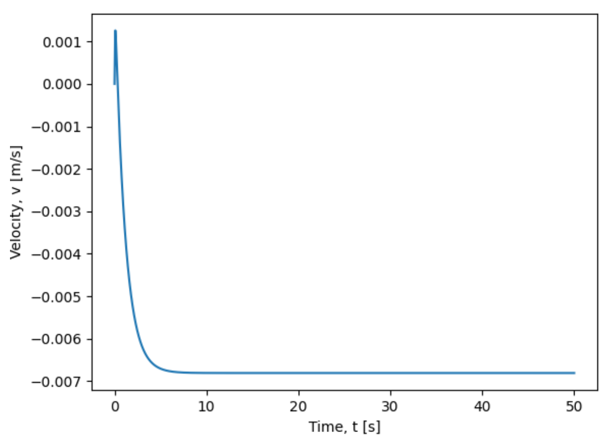

Figure 12: Generalized Implicit Velocity for Falling Beam with 21 Nodes (red) falling for 50s

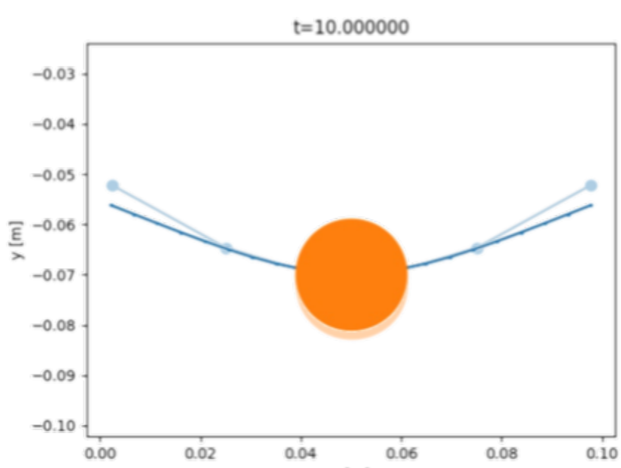

Figure 13: Generalized Implicit Deformation Shape for Falling Beam with 21 Nodes Overlaid With 3 Node Model.

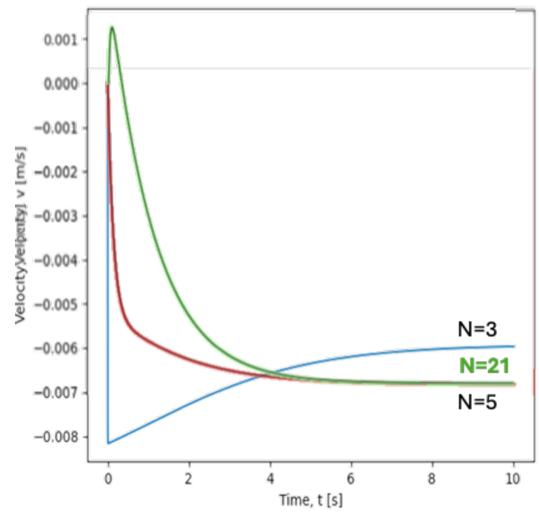

Figure 14: Generalized Implicit Velocity for Falling Beam with 21 Nodes (red) Overlaid With 3 Node Model (blue).

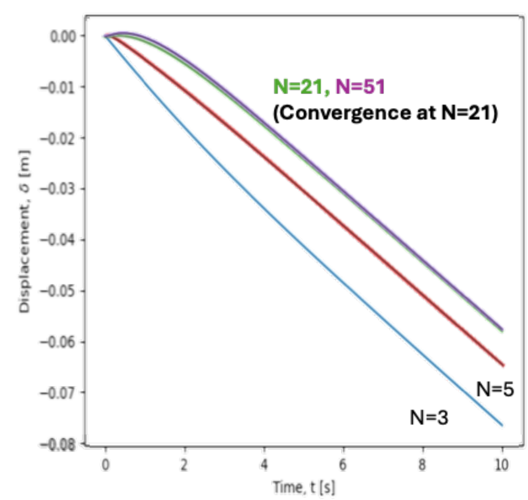

Figure 15: Generalized Implicit Velocity for Falling Beam with 21 Nodes (red) Overlaid With 3 Node Model (blue).

As shown in **Figures 12 and 13** the solution appears to converge to a single consistent result at 21 nodes. This convergence is shown in terminal velocity between the overlapping with N=5. And is further illustrated by the overlapping with displacement using 51 nodes.

Moreover terminal velocity is insensitive to temporal refinement if the total period is large enough for the solution to eventually converge. For a 50s period all dt values between 1e-2 and 1e+1 produce the same terminal velocity of 0.0068 m/s.

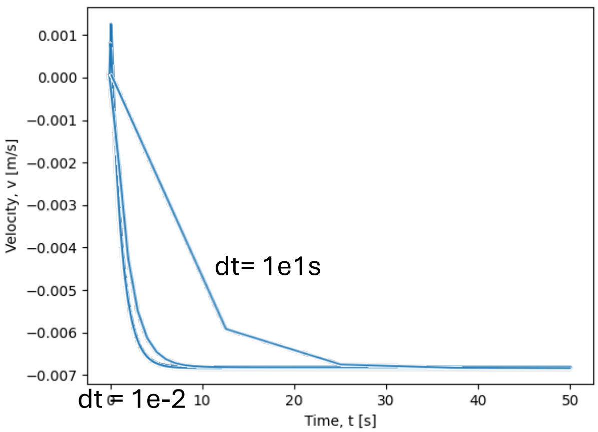

Figure 16: Terminal Velocity is Unchanged if the Total Time is Set Sufficiently High.

Setting dt will be a problem dependent decision. Users should be more careful with spatial refinement as that can lead to diverging solutions independent of how long the time period is.

B. Problem 3: Generalized Case of Elastic Beam Falling in Viscous Flow

In this problem the implicit scheme is modified to allow for boundary conditions to simulate the response of a simply supported beam under loading at a location 0.75 m from the pinned edge. The beam has an inner radius of 0.011m, an outer radius of 0.013 m and a length of 1 m. The model will use a spatial refinement of 51 nodes with x and y fixed at the pinned end and only y fixed at the opposite end. The time step used is 1e-2 s, the density of the beam is 2700 kg/m^3 and the modulus is 70 GPa.

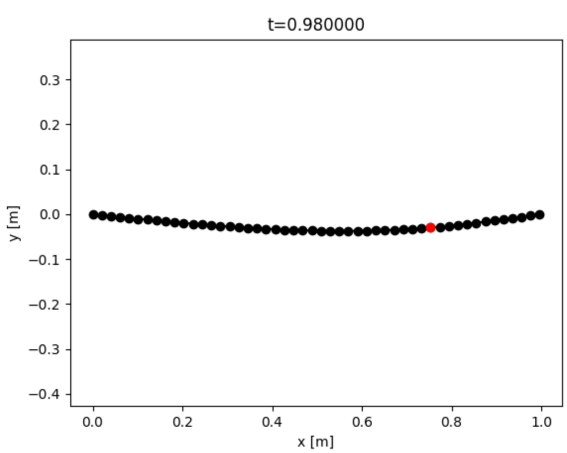

Figure 17: Simply supported beam loaded at the red node -2000 N in y.

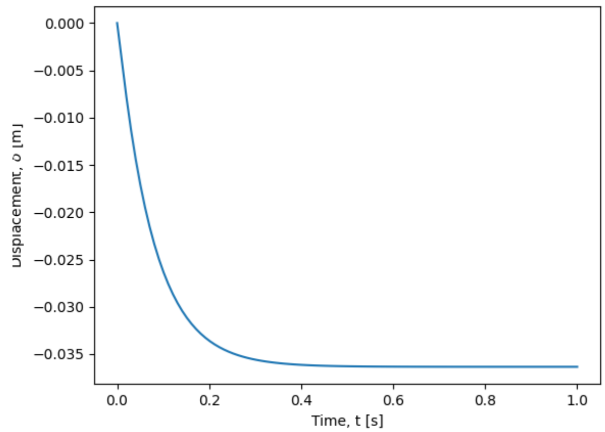

Figure 18: Simply supported beam loaded at the red node with 2000 N (Displacement history of center node).

A steady state value is reached very quickly and it matches the Euler beam theory prediction of 0.038 m. The implicit scheme predicts a max disp of about 0.037 m. This starts to deviate from Euler beam theory as the displacement becomes very large. At 20,000 N the implicit prediction is 0.23 m while the Euler prediction is about 0.38 m. Euler beam theory formula below.

$$y_{\max} = \frac{Pc\,(L^2 - c^2)^{1.5}}{9\sqrt{3}EI\,l} \qquad \text{where} \qquad c = \min(d, l - d)$$

References

[1] Helper functions and modules for code, MAE 263F.