Infinite Order Differential Equations

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1 Introduction

In general, given an open set $X \subseteq \mathbb{R}^n$, a function $g : X \longrightarrow \mathbb{C}$, and some coefficients $a_\alpha \in \mathbb{C}$ for $\alpha \in \mathbb{N}_0^n$, an *infinite order differential equation* (with constant coefficients) asks for a function $u : X \longrightarrow \mathbb{C}$ satisfying

$$\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha u = g.$$

As it happens, differential equations of infinite order come up in many real-life situations: in physical problems such as quantum mechanics, and in other areas of mathematics [1], [2], [3]. In fact, Pincherle, in studying the finite difference equation

$$\sum_{n=0}^{N} \alpha_n u(x + h_n) = g(x),$$

reduced the problem to an infinite order differential equation [1]. So their solutions are certainly of interest.

Just from writing down this differential equation, we should immediately be skeptical for a number of reasons.

1. Does the series on the LHS converge, and in what sense?
2. We need derivatives $\partial^\alpha u$ of all orders.
3. If we're looking for a most general solution, consisting of a homogeneous and particular solution, how much initial data has to be specified to fix a solution? Right now, one may worry that since n^{th} order differential equations in general require n initial conditions, an infinite order differential equation might require infinitely many initial conditions. Then, they wouldn't be very useful in tractably representing physical phenomena (since as scientists, we cannot observe infinitely many initial conditions).

Hopefully, there are some conditions on the coefficients a_α and the input function g such that all of these apprehensions have sufficiently satisfying resolutions.

2 Fourier Transform Approach

As the title of this section indicates, we propose the following attempt at a solution to differential equations of infinite order using the Fourier transform, following [4]. For completeness, we first include a definition (mostly to show what normalization we're using).

DEFINITION 2.1 **Fourier Transform.** Given $f \in L^1(\mathbb{R}^n)$, its Fourier transform is given by

$$\mathcal{F}[f(x)](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x}\, dx.$$

The inverse Fourier transform is

$$\mathcal{F}^{-1}[\hat{f}(\xi)](x) = f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi \cdot x}\, d\xi.$$

Now our proposed solution method is simple:

$$\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha u = g \qquad \text{Given differential equation}$$

$$\mathcal{F}\left(\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha u \right) = \mathcal{F}(g) \qquad \text{Take Fourier transform}$$

$$\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \mathcal{F}\left(\partial^\alpha u\right) = \hat{g} \qquad \text{Linearity and continuity of } \mathcal{F}$$

$$\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (i\xi)^\alpha \hat{u}(\xi) = \hat{g}(\xi) \qquad \text{Derivative proprty of } \mathcal{F}$$

$$u = \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (i\xi)^\alpha} \right). \qquad \text{Factor } \hat{u}(\xi) \text{ out of sum; divide by sum; take } \mathcal{F}^{-1}$$

Of course, we implicitly made many assumptions in this calculation. We must be able to take the Fourier transform of g, and the result of dividing that by the sum $\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (i\xi)^\alpha$ (if such a division makes sense) must admit an inverse Fourier transform. Hence, we see that restrictions on g and the coefficients a_α must be made to satisfy these implicit assumptions.

The idea will be to limit ourselves to a smaller "Sobolev" space by requiring instead of

$$\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^s d\xi < \infty$$

per usual, that the square norm of the Fourier transform, when "weighted" by a function other than $\left(1 + |\xi|^2\right)^s$, be finite. Now we'll make some definitions used for these "weight" functions.

DEFINITION 2.2 **Rapidly Decreasing Functions.** Let $X \subseteq \mathbb{R}^n$ and $f\colon X \longrightarrow \mathbb{C}$. f is rapidly decreasing iff $\sup_{x \in X} |x^\alpha f(x)|_\infty < \infty$ for all $\alpha \in \mathbb{N}_0^n$.

We'll let $\|\cdot\|_\infty$ denote the supremum norm and write $\|x^\alpha f\|_\infty = \sup_{x \in X} |x^\alpha f(x)|$.

In particular, if we take $\alpha = 0$, we see that f is bounded above: $\|f\|_\infty < \infty$. The idea behind rapidly decreasing functions is that they decay faster than any polynomial function towards infinity.

DEFINITION 2.3 **Weight Function.** Let $X \subseteq \mathbb{R}^n$. A weight function w on X is a measurable function $w\colon X \longrightarrow (0, \infty)$ such that $\frac{1}{w}$ is rapidly decreasing.

Since $\frac{1}{w}$ is rapidly decreasing and positive, we have

$$0 < \frac{1}{w} \leq \left\| \frac{1}{w} \right\|_\infty \implies 0 < \frac{1}{\left\| \frac{1}{w} \right\|_\infty} \leq w. \tag{2.1}$$

So w is bounded below by a positive constant $w_0 := \frac{1}{\left\|\frac{1}{w}\right\|_\infty} > 0$.

DEFINITION 2.4 **Weighted Sobolev Space.** Let $X \subseteq \mathbb{R}^n$ be open. The weight-w Sobolev space is given by

$$\mathcal{H}_w(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \, w(\xi)\, d\xi < \infty \right\}.$$

If $w(\xi) = (1 + |\xi|^2)^s$, then $\mathcal{H}_w(\mathbb{R}^n)$ becomes the Sobolev space $H^s(\mathbb{R}^n)$. However, such a choice for w is not a weight function. In fact, because of the nice properties of weight functions:

THEOREM 2.5 Let w be a weight function. If $u \in \mathcal{H}_w(\mathbb{R}^n)$, then for any $\alpha \in \mathbb{N}_0^n$, u has a weak derivative $\partial^\alpha u$ of order α, and it lies in $L^2(\mathbb{R}^n)$.

PROOF. Let w be a weight function and $\alpha \in \mathbb{N}_0^n$. Since w is a weight function, $1/w$ is rapidly decreasing, and hence there is some $C_\alpha \in \mathbb{R}_{\geq 0}$ such that $\left|\xi^{2\alpha} \frac{1}{w(\xi)}\right| \leq C_\alpha \implies |i\xi|^{2\alpha} \leq C_\alpha w(\xi)$ for all $\xi \in \mathbb{R}^n$. Then

$$\|(i\xi)^\alpha \hat{u}(\xi)\|_{L^2}^2 = \int_{\mathbb{R}^n} |i\xi|^{2\alpha} \, |\hat{u}(\xi)|^2 \; d\xi \leq C_\alpha \int_{\mathbb{R}^n} w(\xi)\, |\hat{u}(\xi)|^2 \; d\xi < \infty$$

as $u \in \mathcal{H}_w(\mathbb{R}^n)$, so $(i\xi)^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. Then the weak α^{th} derivative of u is given by $\partial^\alpha u = \mathcal{F}^{-1}\left[(i\xi)^\alpha \hat{u}(\xi)\right]$. This is also in $L^2(\mathbb{R}^n)$ since $\mathcal{F}$ is a bijection of $L^2(\mathbb{R}^n)$ onto itself and $(i\xi)^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. ∎

Now, in the proof of the next theorem, we'll see how these weight functions have just the right properties for the following theorem to hold true.

THEOREM 2.6 Fix $g \in L^2(\mathbb{R}^n)$ and $a_\alpha \in \mathbb{C}$ for each $\alpha \in \mathbb{N}_0^n$. If the differential equation

$$\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha u = g \tag{2.2}$$

has corresponding polynomial

$$\begin{aligned} p : \mathbb{R}^n &\longrightarrow \mathbb{C} \\ \xi &\longmapsto p(\xi) := \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (i\xi)^\alpha \end{aligned}$$

for which

$$w := |p|^2 \text{ is a weight function} \tag{2.3}$$

and there is some $A \in \mathbb{R}_{\geq 0}$ such that

$$|p_k(\xi)| := \left| \sum_{|\alpha| \leq k} a_\alpha (i\xi)^\alpha \right| \leq A\, |p(\xi)| \quad \mu\text{-a.e., for all } k \in \mathbb{N}_0, \tag{2.4}$$

μ is the Lebesgue measure, and μ-a.e. means that this property holds except possibly for $\xi \in S \subseteq \mathbb{R}^n$ with $\mu(S) = 0$.

then there exists a unique solution $u = \mathcal{F}^{-1}(\hat{g}/p) \in \mathcal{H}_w(\mathbb{R}^n)$ to (2.2).

PROOF. To prove existence, we must show that $u = \mathcal{F}^{-1}(\hat{g}/p)$ is well defined and in $\mathcal{H}_w(\mathbb{R}^n)$, and that the series on the LHS of (2.2) converges to g in $L^2(\mathbb{R}^n)$ if $u = \mathcal{F}^{-1}(\hat{g}/p)$.

First, we show that $u = \mathcal{F}^{-1}(\hat{g}/p)$ is well defined. Since $g \in L^2(\mathbb{R}^n)$, we have $\hat{g} \in L^2(\mathbb{R}^n)$. By (2.3), $w = |p|^2$ is a weight function, so p can never vanish and

$$\left\| \frac{\hat{g}}{p} \right\|_{L^2}^2 = \int_{\mathbb{R}^n} \left| \frac{\hat{g}(\xi)}{p(\xi)} \right|^2 d\xi \leq \left\| \frac{1}{|p|^2} \right\|_\infty \|\hat{g}\|_{L^2}^2 < \infty.$$

Hence $\hat{g}/p \in L^2(\mathbb{R}^n)$, and $u = \mathcal{F}^{-1}(\hat{g}/p) \in L^2(\mathbb{R}^n)$ is well defined, as $\mathcal{F}$ is a bijection between $L^2(\mathbb{R}^n)$ and itself.

To show further that $u \in \mathcal{H}_w(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 w(\xi)\, d\xi = \int_{\mathbb{R}^n} \left| \frac{\hat{g}(\xi)}{p(\xi)} \right|^2 |p(\xi)|^2\, d\xi = \int_{\mathbb{R}^n} |\hat{g}(\xi)|^2\, d\xi < \infty$$

where we have used $\hat{u} = \hat{g}/p$ and $w = |p|^2$ in the second inequality and $\hat{g} \in L^2 \implies \|\hat{g}\|_{L^2}^2 < \infty$ in the last inequality.

To show that the series on the LHS of (2.2) converges to g in $L^2(\mathbb{R}^n)$ if $u = \mathcal{F}^{-1}(\hat{g}/p)$, we need the following to vanish in the limit $k \to \infty$:

$$\begin{aligned}
\left\| g - \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \mathcal{F}^{-1}\left(\frac{\hat{g}}{p}\right) \right\|_{L^2}^2 &= \left\| \mathcal{F}^{-1}\left(\hat{g} - \sum_{|\alpha| \leq k} a_\alpha (i\xi)^\alpha \frac{\hat{g}}{p} \right) \right\|_{L^2}^2 \\
&= (2\pi)^{-n} \left\| \hat{g} - \sum_{|\alpha| \leq k} a_\alpha (i\xi)^\alpha \frac{\hat{g}}{p} \right\|_{L^2}^2 \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{g}|^2 \left| 1 - \frac{p_k}{p} \right|^2,
\end{aligned}$$

where we have used $\mathcal{F}(\partial^\alpha u) = (i\xi)^\alpha \hat{u}$ in the first equality and Plancherel's theorem in the second. By (2.4), we have μ-almost everywhere that the integrand is bounded above by the $L^1(\mathbb{R}^n)$ function $(1+A)\,|\hat{g}|^2$ and converges pointwise to 0 as $k \to \infty$. Then by Lebesgue's dominated convergence theorem, the integral tends to 0 as $k \to \infty$.

To prove uniqueness, suppose that there is some $u \in \mathcal{H}_w(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ satisfying (2.2). Then taking the Fourier transform of both sides, which is permissible as both are in $L^2(\mathbb{R}^n)$ by Theorem 2.5,

$$\hat{g} = \mathcal{F}\left(\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha u \right) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \widehat{\partial^\alpha u} = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (i\xi)^\alpha \hat{u} = \hat{u} p, \tag{2.5}$$

as $\mathcal{F}$ is linear and continuous, and $\widehat{\partial^\alpha u} = (i\xi)^\alpha \hat{u}$. By our well-definedness argument, it follows that $\hat{u} = \hat{g}/p \in L^2(\mathbb{R}^n)$ and then that $u = \mathcal{F}^{-1}(\hat{g}/p)$. Since $\mathcal{F}^{-1}$ is injective, u is unique (up to $L^2(\mathbb{R}^n)$ equivalence, of course). ■

One way to make sense of the sum in $\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha u = g$ is to pick a_α to be the coefficients corresponding to a Taylor series. We see an instance of this in the next example.

EXAMPLE 2.7 *Find $u \in L^2(\mathbb{R}^n)$ which satisfies*

$$\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} u^{(2n)}(x) = \frac{1}{2} \operatorname{sech}\left(\frac{\pi x}{2}\right).$$

First, we write the LHS as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{d}{dx}\right)^{2n} u(x) = \cos\left(\frac{d}{dx}\right) u(x).$$

Then to find the corresponding polynomial $p(\xi)$, we replace any instances of d/dx with $i\xi$ to get $p(\xi) = \cos(i\xi) = \cosh \xi$. It indeed satisfies condition (2.3) of Theorem (2.6) as $|p(\xi)|^2 = \cosh^2 \xi \geq 1$ is positive and $\frac{1}{|p(\xi)|^2} = \frac{4}{(e^{\xi}+e^{-\xi})^2}$ is rapidly decreasing. It also satisfies condition (2.4) of Theorem 2.6 since for all $\xi \in \mathbb{R}$,

$$|p_k(\xi)| = \left|\sum_{n=0}^{k} \frac{(-1)^n}{(2n)!}(i\xi)^{2n}\right| = \left|\sum_{n=0}^{k} \frac{\xi^{2n}}{(2n)!}\right| \leq \left|\sum_{n=0}^{\infty} \frac{\xi^{2n}}{(2n)!}\right| = |p(\xi)|,$$

where the inequality follows from the nonnegativity of all summands. Hence, we can apply Theorem 2.6 to find a unique solution $u = \mathcal{F}^{-1}(\hat{g}/p)$, where

$$g(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi x}{2}\right).$$

Then

$$\hat{g}(\xi) = \operatorname{sech} \xi,$$

and

$$u(x) = \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{p(\xi)}\right) = \mathcal{F}^{-1}\left(\frac{\operatorname{sech}\xi}{\cosh\xi}\right) = \frac{1}{2} x \operatorname{csch}\left(\frac{\pi x}{2}\right).$$

The next example shows the power and utility of Theorem (2.6). We use it to solve a difficult problem in complex analysis that has seemingly nothing to do (initially) with differential equations.

EXAMPLE 2.8 **Solving a Complex Functional Identity.** *Fix a function $g \in L^2(\mathbb{C})$ so $g|_{\mathbb{R}}$ is analytic and satisfies the bound*

$$\left|\widehat{g|_{\mathbb{R}}}(\xi)\right| \leq M e^{-q|\xi|} \quad \forall \xi \in \mathbb{R} \tag{2.6}$$

for some $M > 0$ and $q > -1$. Find an entire function u satisfying

An entire function is defined on the entire complex plain $\mathbb{C}$ and is analytic.

$$\frac{u(z-i) + u(z+i)}{2} = g(z) \quad \forall z \in \mathbb{C}. \tag{2.7}$$

Expanding the LHS of (2.7) in a series,

$$\begin{aligned}\frac{1}{2}\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} u^{(n)}(z) + \frac{1}{2}\sum_{n=0}^{\infty} \frac{i^n}{n!} u^{(n)}(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} u^{(2n)}(z) \\ &= \cos\left(\frac{d}{dz}\right) u(z).\end{aligned}$$

Restricting to $z = x \in \mathbb{R}$, we apply Example 2.7 to get a unique solution *on the real line* given by $u|_{\mathbb{R}} = \mathcal{F}^{-1}\left(\widehat{g|_{\mathbb{R}}}/p|_{\mathbb{R}}\right) \in \mathcal{H}_w(\mathbb{R}) \subseteq L^2(\mathbb{R})$, where

$p|_{\mathbb{R}}(\xi) = \cosh\xi$ for $\xi \in \mathbb{R}$, which satisfies

$$\left|\widehat{u|_{\mathbb{R}}}(\xi)\right| = \left|\frac{\widehat{g|_{\mathbb{R}}}(\xi)}{\cosh\xi}\right| \le \frac{2Me^{-q|\xi|}}{|e^{\xi}+e^{-\xi}|} \le \frac{2Me^{-q|\xi|}}{e^{|\xi|}} = 2Me^{-(q+1)|\xi|}$$

by the given bound on g and the triangle inequality. By a version of the Paley-Wiener theorem (see Theorem 3.1 of Chapter 4 in [5], reproduced to the right) $u|_{\mathbb{R}}$ extends analytically to the horizontal strip

$$S_{-a,a} := \{z \in \mathbb{C} \mid -a < \operatorname{Im} z < a\}$$

for any $0 < a < q+1$, and by construction, satisfies (2.7) in that region. We may extend $u|_{S_{-a,a}}$ analytically to $\mathbb{C}$ via the following process.

Theorem. Suppose u satisfies the Fourier inversion formula and the decay condition $|\hat{u}(\xi)| \le Ae^{-a|\xi|}$ for some constants $a, A > 0$. Then $u(x)$ is the restriction to $\mathbb{R}$ of a function $u(z)$ holomorphic in the strip $S_b = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < b\}$ for any $0 < b < a$.

By (2.7),

$$u(z-i) = 2g(z) - u(z+i).$$

Knowing the RHS on $S_{-a,a}$ gives the LHS on $S_{-a-2,a-2}$, as $z+i \in S_{-a,a} \implies z-i \in S_{-a-2,a-2}$, and hence extends u analytically to $S_{-a-2,a}$. Similarly, (2.7) gives

$$u(z+i) = 2g(z) - u(z-i).$$

Knowing the RHS on $S_{-a,a}$ gives the LHS on $S_{-a+2,a+2}$, and hence extends u analytically to $S_{-a,a+2}$.

Now, we know u on $S_{a-2,a+2}$, just from knowing u on $S_{-a,a}$ and (2.7). By the same reasoning, we know u on $S_{-a-4,a+4}$ from knowing u on $S_{-a-2,a+2}$. Continuing recursively, we can extend u analytically to all of $\mathbb{C}$.

The next example shows us that a solution $u = \mathcal{F}^{-1}(\hat{g}/p)$ to (2.2) may exist even without such strict assumptions as those in Theorem 2.6 on $p(\xi)$ and g. However, we do need $\hat{g}/p$ to admit an inverse Fourier transform for this strategy to work.

EXAMPLE 2.9 **1D Laplacian.** *In $\mathbb{R}$ with coordinate function x, the laplacian operator is simply $\Delta_x = \nabla_x^2 = \partial_x^2$. Find u such that $e^{-\frac{1}{2}\Delta_x}u(x) = \sin x$.*

Taylor expanding, we have

$$e^{-\frac{1}{2}\Delta_x} = \sum_{n=0}^{\infty} \frac{\left(-\partial_x^2/2\right)^n}{n!} \implies p(\xi) = \sum_{n=0}^{\infty} \frac{\left[-(i\xi)^2/2\right]^n}{n!} = e^{\xi^2/2}.$$

Then condition (2.3) of Theorem 2.6 is satisfied as $w(\xi) = |p(\xi)|^2 = e^{\xi^2}$ is a weight function, as $e^{\xi^2} > 0$ for all $\xi \in \mathbb{R}$ and $e^{-\xi^2}$ is rapidly decreasing. Condition (2.4) also holds as all terms of $p_k(\xi) = \sum_{n=0}^{k} \frac{\xi^{2n}}{2^n n!}$ are nonnegative.

However, we can't apply Theorem 2.6 directly because $g(x) = \sin x \notin L^2(\mathbb{R}^n)$. But, there is still hope for the same method of obtaining a solution via $u = \mathcal{F}^{-1}(\hat{g}/p)$.

Indeed,

$$\begin{aligned} & g(x) = \sin x && \in \mathcal{S}'(\mathbb{R}^n) \\ \implies & \hat{g}(\xi) = i\pi\left[\delta(1+\xi) - \delta(1-\xi)\right] && \in \mathcal{S}'(\mathbb{R}^n), \end{aligned}$$

and

$$0 < \frac{1}{p(\xi)} = e^{-\xi^2/2} \qquad \in C^\infty(\mathbb{R}^n),$$

so $\hat{g}/p$ is well-defined:

$$\frac{\hat{g}(\xi)}{p(\xi)} = i\pi e^{-1/2}\left[\delta(1+\xi) - \delta(1-\xi)\right] \qquad \in \mathcal{S}'(\mathbb{R}^n)$$

$$u(x) := \mathcal{F}^{-1}\left(\frac{\hat{g}(\xi)}{p(\xi)}\right) = \frac{\sin x}{\sqrt{e}} \qquad \in \mathcal{S}'(\mathbb{R}^n)$$

and a solution is well-defined. Now the question is convergence of the series in (2.2) in the topology of $\mathcal{S}'(\mathbb{R}^n)$. The series does converge as, considering elements as tempered distributions, we have for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
&\lim_{k\to\infty}\left\langle g - \sum_{|\alpha|\le k} a_\alpha \partial^\alpha u, \varphi\right\rangle \\
&= \lim_{k\to\infty}\left\langle g - \sum_{|\alpha|\le k} a_\alpha \partial^\alpha \mathcal{F}^{-1}\left(\frac{\hat{g}}{p}\right), \varphi\right\rangle \\
&= \lim_{k\to\infty}\left\langle \mathcal{F}^{-1}\left(\hat{g} - \sum_{|\alpha|\le k} a_\alpha \frac{(i\xi)^\alpha \hat{g}(\xi)}{p(\xi)}\right), \varphi\right\rangle \\
&= \lim_{k\to\infty}\left\langle \hat{g}\underbrace{\left(1 - \frac{\sum_{|\alpha|\le k} a_\alpha (i\xi)^\alpha}{p(\xi)}\right)}_{=:f_k(\xi)}, \mathcal{F}^{-1}\varphi\right\rangle \\
&= \lim_{k\to\infty}\left[i\pi f_k(-1)\mathcal{F}^{-1}\varphi(-1) - i\pi f_k(1)\mathcal{F}^{-1}\varphi(1)\right] \\
&= 0
\end{aligned}$$

as $f_k \to 0$ pointwise.

As a sanity check, we try plugging our solution $\frac{\sin x}{\sqrt{e}}$ back into the LHS original differential equation and make sure we get $\sin x$ on the RHS:

$$\begin{aligned}
e^{-\frac{1}{2}\Delta_x}\frac{\sin x}{\sqrt{e}} &= \sum_{n=0}^\infty \frac{\left(-\partial_x^2/2\right)^n}{n!}\frac{\sin x}{\sqrt{e}} = \sum_{n=0}^\infty \frac{\left(-\frac{1}{2}\right)^n \partial_x^{2n}[\sin x]}{n!\sqrt{e}} \\
&= \sum_{n=0}^\infty \frac{\left(-\frac{1}{2}\right)^n (-1)^n \sin x}{n!\sqrt{e}} = \frac{\sin x}{\sqrt{e}}\sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)^n}{n!} \\
&= \frac{\sin x}{\sqrt{e}}\sqrt{e} = \sin x.
\end{aligned}$$

If we take a_α to be functions rather than constants, a solution u may still exist, but even if $g \in L^2(\mathbb{R}^n)$, it may not be the case that u is in $\mathcal{H}_w(\mathbb{R}^n)$ or even $L^2(\mathbb{R}^n)$. In the special case of the following example, Theorem 2.6 still holds with non-constant coefficients $a_\alpha(x)$.

Example 2.10 **Non-Constant Coefficients.** *$H_n(x)$ are the Hermite polynomials used in physics, and*

$$\left(e^{-x^2/2}H_n(x)\right)_{n=0}^\infty$$

forms an orthogonal basis of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of the Fourier transform with eigenvalues $(-i)^n$. Their exponential generating *function is*

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \quad \forall x, t \in \mathbb{C}.$$

Find a complex-valued function u on $\mathbb{R}$ such that

$$\sum_{n=0}^{\infty} \frac{e^{-x^2/2}H_n(x)}{n!} * \partial^n u(x) = \frac{\sin x}{x}.$$

$*$ *denotes the convolution of functions.*

Taking Fourier transforms of the given differential equation and applying the convolution theorem $\widehat{u * v} = \hat{u}\hat{v}$, the derivative theorem $\mathcal{F}(\partial^\alpha u) = (i\xi)^\alpha \hat{u}$, and the fact that $e^{-x^2/2}H_n(x)$ is an eigenfunction of $\mathcal{F}$ with eigenvalue $(-i)^n$, we have

$$\sum_{n=0}^{\infty} \frac{(-i)^n e^{-\xi^2/2}H_n(\xi)}{n!}(i\xi)^n \hat{u}(\xi) = \pi \mathbb{1}_{[-1,1]}.$$

$\mathbb{1}_A$ is the indicator function of the set A:

$$\mathbb{1}_A(a) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}.$$

From here,

$$\begin{aligned}
\hat{u}(\xi) &= \frac{\pi \mathbb{1}_{[-1,1]}}{e^{-\xi^2/2}\sum_{n=0}^{\infty} H_n(\xi)\frac{\xi^n}{n!}} && \text{Divide by sum on LHS} \\
&= \pi \mathbb{1}_{[-1,1]} e^{-\xi^2/2} && \text{Generating function of } H_n(x) \\
u(x) &= \pi \mathcal{F}_\xi^{-1}\left[\mathbb{1}_{[-1,1]} e^{-\xi^2/2}\right](x) && \text{Inverse Fourier transform} \\
&= \sqrt{\frac{\pi}{8}} e^{-\frac{x^2}{2}} \left[\operatorname{erf}\left(\frac{1+ix}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{1-ix}{\sqrt{2}}\right)\right].
\end{aligned}$$

$\operatorname{erf}(z)$ is the entire function given by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{t^2}\, dt \quad \forall z \in \mathbb{C}.$$

3 Laplace Transform Approach

In section 2, we saw a solution method for constant coefficient differential equations of infinite order. However, the solutions we found were only particular solutions. If such a differential equation is to model some real life phenomenon, we would also be interested in homogeneous solutions. In particular, at this point, one might worry that an infinite order differential equation might require an infinite amount of initial data to specify a homogeneous solution. In this section, with alternative constraints on our functions and a complex analytic approach via the Laplace transform following [2], we'll see that in many cases only a finite number of initial conditions are required to specify a homogeneous solution.

We first just recall the Laplace transform and its inverse for completeness.

Definition 3.1 **Laplace Transform.** Let $f(t)$ be defined for $t > 0$. Then its Laplace transform and inverse Laplace transform are given by

$$\tilde{f}(x) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}\, dt$$

$$f(t) = \mathcal{L}^{-1}[\tilde{f}(s)] = \frac{1}{2\pi i}\oint_\gamma \tilde{f}(s)e^{st}\, ds,$$

Commonly, $\int_{0^-}^{\infty}$ is written to mean $\lim_{\varepsilon \uparrow 0}\int_{\varepsilon}^{\infty}$.

where the closed curve $\gamma \subset \mathbb{C}$ contains in its interior all singularities of the integrand.

Recall that functions which admit Laplace transforms have exponential type [6]:

DEFINITION 3.2 **Exponential Type.** A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to have exponential type $q \in \mathbb{R}$ if

$$|f(z)| \leq Me^{q|z|}$$

for all $z \in \mathbb{C}$.

Now we can state the main theorem for infinite order differential equations using the Laplace transform.

THEOREM 3.3 Consider the differential equation

$$f(\partial_t)x(t) = y(t), \tag{3.1}$$

where

- $f(s)$ is analytic in the region $|s| \leq q$ for some $q \in \mathbb{R}_{\geq 0}$
- y is of exponential type at most q
- $x(t)$ is analytic and of exponential type at most q.

Let $(s_j)_{j=1}^M$ be a list of the distinct zeros of f and $(m_j)_{j=1}^M$ be a list of their corresponding multiplicities. The most general solution is given by

$$x(t) = \frac{1}{2\pi i} \oint_\gamma \frac{\tilde{y}(s)}{f(s)} e^{st}\, ds + \sum_{j=1}^{M} p_j(t) e^{s_j t},$$

where the closed curve $\gamma \subset \mathbb{C}$ contains all singularities of the integrand in its interior, and for each $j \in [M]$, p_j is a polynomial with $\deg p_j \leq m_j - 1$ and coefficients which are determined by $N = \sum_{j=1}^{M} m_j$ initial conditions.

We use the common convention $[n] := \{1, 2, 3, \ldots, n\}$.

PROOF. A rigorous proof is given in [1]. Here, we give the intuition and outline the main points. Since $f(s)$ is analytic, we may write $f(s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} s^n$, and hence $f(\partial_t)x(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \partial_t^n x(t)$. From the derivative formula for Laplace transforms,

$$\partial_t^n x(t) = \frac{1}{2\pi i} \oint_\gamma e^{st} \left(s^n \tilde{x}(s) - \sum_{j=0}^{n-1} s^{n-j-1} x^{(j)}(0^-) \right) ds,$$

and multiplying by $\frac{f^{(n)}(0)}{n!}$ and summing over $n \geq 0$, we get

$$f(\partial_t)x(t) = \frac{1}{2\pi i} \oint_\gamma e^{st} \left(f(s)\tilde{x}(s) - \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} \frac{f^{(n)}(0) x^{(j)}(0^-)}{n!} s^{n-j-1} \right) ds.$$

By the discussion in [2], we may defer the summation term and build it into the constants $a_{j,m}$ to come later, so we're left with

$$f(\partial_t)x(t) = \frac{1}{2\pi i} \oint_\gamma e^{st} f(s)\tilde{x}(s)\, ds. \tag{3.2}$$

Since f is analytic, we may write it as $f(s) = g(s) \prod_{i=1}^{M} (s - s_j)^{m_j}$ for some analytic, nonvanishing function $g(s)$, where all of the distinct zeros of f are in $\{s_j\}_{j=1}^M$ and s_j has multiplicity m_j.

Towards a homogeneous solution of (3.1), we set the RHS $y(t) \equiv 0$, and then the LHS (3.2) is 0 only if its integrand is analytic by Cauchy's integral theorem. The most general funciton $\tilde{x}(s)$ that makes the integrand analytic is given by

$$\tilde{x}(s) = \sum_{j=1}^{M} \sum_{m=1}^{m_j} \frac{a_{j,m}}{(s - s_j)^m},$$

where $a_{m,j} \in \mathbb{C}$. (We could have included a non-vanishing analytic factor and an additive analytic term, adjusting $a_{m,j}$ to compensate, but we omit them for simplicity.) Then since

$$\mathcal{L}^{-1}\left[\frac{a_{m,j}}{(s - s_j)^m}\right] = \frac{a_{m,j}}{(m-1)!} t^{m-1} e^{s_j t},$$

we see that the homogeneous solution $x_h(t)$ is

$$x_h(t) = \sum_{j=1}^{M} \sum_{m=1}^{m_j} \frac{a_{m,j}}{(m-1)!} t^{m-1} e^{s_j t} = \sum_{j=1}^{M} p_j(t) e^{s_j t},$$

where $p_j(t)$ is of degree at most m_j and has coefficients determined by $(a_{m,j})_{m=1}^{m_j}$, which are in turn determined by the initial conditions. This shows that if we want an initial value problem out of (3.1), we must specify $N = \sum_{j=1}^{M} m_j$ initial conditions.

Now, we find the particular solution $x_p(t)$. Writing the RHS using its Laplace transform and again using (3.2),

$$f(\partial_t) x_p(t) = \frac{1}{2\pi i} \oint_\gamma f(s) \tilde{x}_p(s) e^{st}\, ds = \frac{1}{2\pi i} \oint_\gamma \tilde{y}(s) e^{st}\, ds = y(t).$$

From this, we have

$$\begin{aligned} 0 = \oint_\gamma \left[f(s)\tilde{x}_p(s) - \tilde{y}(s)\right] e^{st}\, ds &\implies \tilde{x}_p(s) = \frac{\tilde{y}(s)}{f(s)} \\ &\implies x_p(t) = \frac{1}{2\pi i} \oint_\gamma \frac{\tilde{y}(s)}{f(s)} e^{st}\, ds. \end{aligned}$$

Combining the homogeneous and particular solutions yields the most general solution $x(t) = x_h(t) + x_p(t)$:

$$x(t) = \frac{1}{2\pi i} \oint_\gamma \frac{\tilde{y}(s)}{f(s)} e^{st}\, ds + \sum_{j=1}^{M} p_j(t) e^{s_j t}. \qquad \blacksquare$$

After seeing this result, maybe we shouldn't be so surprised that infinite order differential equations may have solutions needing only finitely many initial data points. As we recall from ODE theory, the reason that an n^{th} order differential equation usually requires n initial conditions is that is characteristic polynomial in $\mathbb{C}[s]$ will have n roots (by the fundamental theorem of algebra). So in the infinite dimensional case, we're still okay as long as the characteristic polynomial has finitely many roots, all in some disk.

EXAMPLE 3.4 **Simple Examples Using Theorem 3.1.** *Solve the differential equation*

$$\frac{1}{4}\left(1 - \sum_{n=1}^{\infty} 2^{-n} \partial_t^n\right) x(t) = e^{-t}.$$

In this case, it can be easily shown using geometric series that $f(s) = \frac{1-s}{2(2-s)}$. Then f is analytic in the region $|s| < 2$, it has a single simple zero at $s = 1$, and $y(t) = e^{-t}$ is of exponential type $1 < 2$ (and has Laplace transform $\tilde{y}(s) = (1+s)^{-1}$). Hence, we may apply Theorem 3.1 to get a solution

$$\begin{aligned} x(t) &= c_1 e^t + \mathcal{L}^{-1}\left[\frac{2(2-s)}{(1-s)(1+s)}\right] = c_1 e^t + 3e^{-t} - e^t \\ &= (c_1 - 1)e^t + 3e^{-t} \end{aligned}$$

where we used a partial fractions decomposition to compute the inverse Laplace transform. With the simple form of the solution, we can verify it directly:

$$x^{(n)}(t) = (c_1 - 1)e^t + 3(-1)^n e^{-t} \quad \forall n \geq 0,$$

so

$$\begin{aligned} \frac{1}{4}\left[x(t) - \sum_{n=1}^{\infty} \frac{x^{(n)}(t)}{2^n}\right] &= \frac{1}{4}\left[x(t) - (c_1 - 1)e^t - 3 \cdot \left(-\frac{1}{3}\right) e^{-t}\right] \\ &= \frac{1}{4}\left[3e^{-t} + e^{-t}\right] = e^{-t}, \end{aligned}$$

and our solution is indeed valid.

Note that particular solutions found with the Laplace transform method match the solutions found by the Fourier transform method.

Example 3.5 Since $\cos x$ has infinitely many roots and is not of exponential type, we see that Theorem 3.3 cannot be applied in Example 2.7. Also, since $e^{-x^2/2}$ has no roots, we see that the particular solution found in Example 2.9 is indeed the most general solution; that differential equation admits no homogeneous solution.

Example 3.6 **Sum of Dirac Derivatives.** An interesting differential equation which is a borderline case in which Theorem 3.1 doesn't apply is

$$\frac{1}{1-\partial_t} x(t) = -e^t.$$

Let's try using the result prescribed by Theorem 3.1 anyways. In this case, $f(s) = (1-s)^{-1}$ has no zeroes, and hence we have no homogeneous solution. To compute the particular solution, we note that $y(t) = -e^t \implies \tilde{y}(s) = (1-s)^{-1}$. Then our solution is

$$x(t) = \mathcal{L}^{-1}\left[\frac{\tilde{y}(s)}{f(s)}\right] = \mathcal{L}^{-1}[1] = \delta(t).$$

This tells us that $\frac{1}{1-\partial_t}\delta(t) = \sum_{n=0}^{\infty} \partial_t^n \delta(t)$, which is the sum of the Dirac-δ and all its derivatives, is equal to the negative of the exponential function. Of course, this is absurd, as $\operatorname{supp} \delta^{(n)} = \{0\}$ while $\operatorname{supp} e^{-t} = \mathbb{R}$. So we see how drastically the theorem stops working on the boundary where $f(s)$ is analytic for $|s|$ *less than 1* but $y(t) = -e^t$ is of exponential type *equal to 1*.

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