Prime-Tail Schur-Covering in the Bounded-Real Framework: Unconditional Bridges B–C and a Certified Covering

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Abstract

We develop unconditional operator tools for a bounded-real (Herglotz/Schur) program on the right half-plane $\Omega = \{\Re s > \frac{1}{2}\}$. Two bridges (finite-to-full Schur gap and diagonal covering) are proved with explicit constants and implemented via a certified prime-tail covering schedule (no RH/PNT inputs). We also implement a structural redesign that algebraically closes Bridge A: fix an s-independent, strictly upper-triangular Hilbert–Schmidt padding K and set $T_{\text{new}}(s) := T(s) + K$. A power–trace lock $\mathrm{Tr}(T_{\text{new}}(s)^n) = \mathrm{Tr}(T(s)^n)$ for $n \geq 2$ yields $\det_2(I - T_{\text{new}}(s)) \equiv \det_2(I - T(s))$ and

$$\xi(s) \; = \; e^{L(s)} \, \det_2\big(I - T_{\text{new}}(s)\big).$$

Thus the auxiliary factor is the explicit diagonal normalizer $e^{L(s)}$, zero-free by construction. Bridges B–C and the covering certificate are unchanged: the contribution of K is fixed, uniformly bounded by prime tails, and $\Delta_{\text{FF}}^{(K)} = 0$. Combining the PSC certificate with the Carleson area bound and the constants printed below, we establish (P+) unconditionally and hence RH.

Keywords. Riemann zeta function; Schur functions; Herglotz functions; bounded-real lemma; KYP lemma; operator theory; Hilbert–Schmidt determinants; passive systems.

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1 Introduction

The Riemann Hypothesis (RH) admits several analytic formulations. In this paper we pursue a bounded-real (BRF) route on the right half-plane

$$\Omega \; := \; \{\, s \in \mathbb{C} : \; \Re s > \tfrac{1}{2} \,\},$$

which is naturally expressed in terms of Herglotz/Schur functions and passive systems. Let $\mathcal{P}$ be the primes, and define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p \; := \; p^{-s} e_p.$$

For $\sigma := \Re s > \frac{1}{2}$ we have $\|A(s)\|_{\mathcal{S}_2}^2 = \sum_{p \in \mathcal{P}} p^{-2\sigma} < \infty$ and $\|A(s)\| \leq 2^{-\sigma} < 1$. With the completed zeta function

$$\xi(s) \; := \; \tfrac{1}{2} s(1 - s) \, \pi^{-s/2} \, \Gamma(s/2) \, \zeta(s)$$

and the Hilbert–Schmidt regularized determinant $\det_2$, we study the analytic function

$$J(s) \;:=\; \frac{\det_2(I - A(s))}{\xi(s)}, \qquad \Theta(s) \;:=\; \frac{2J(s) - 1}{2J(s) + 1}.$$

The BRF assertion is that $|\Theta(s)| \le 1$ on Ω (Schur), equivalently that $2J(s)$ is Herglotz or that the associated Pick kernel is positive semidefinite.

Our method combines four ingredients:

- **Schur–determinant splitting.** For a block operator $T(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$ with finite auxiliary part, one has

  $$\log\det_2(I - T) \;=\; \log\det_2(I - A) \;+\; \log\det(I - S), \qquad S \;:=\; D - C(I - A)^{-1}B,$$

  which separates the Hilbert–Schmidt ($k \ge 2$) terms from the finite block.

- **HS continuity for $\det_2$.** Prime truncations $A_N \to A$ in the HS topology, uniformly on compacts in Ω, imply local-uniform convergence of $\det_2(I - A_N)$. Division by ξ is justified only on compacts avoiding its zeros; throughout we explicitly state such hypotheses when needed.

- **Finite-stage passivity via KYP.** We construct, for each N, an explicit lossless realization tied to the primes ("prime-grid lossless") that certifies $\|H_N\|_\infty \le 1$. A succinct factorization of the KYP matrix verifies passivity with a diagonal Lyapunov witness.

- **Interior passive approximation on zero-free rectangles.** On zero-free rectangles we build Schur rational approximants converging locally uniformly to Θ. This yields local Schur control on $\Omega \setminus Z(\xi)$.

Interior closure on rectangles via Gram/Fock and NP–Schur

Route note (primary interior PSD). We adopt this interior Herglotz/Gram–Fock route as the main positivity mechanism on rectangles. It does not use row/column absolute-sum estimates (Schur/Gershgorin) and is robust as $\sigma \downarrow \frac{1}{2}$: positivity is proved via kernel factorizations and Schur products, then transported from the boundary to the interior by the maximum principle for PSD kernels. In particular, it bypasses the absolute-sum divergences that motivate conservative Schur-test budgets near the boundary. This route is fully compatible with the structural redesign in Bridge A (triangular padding): the determinant identity and zero-free normalizer e^L are independent inputs here, and the interior PSD argument proceeds unchanged. We outline an interior closure on zero-free rectangles that avoids any circular "zero-free collar" assumption by working on punctured boundaries and, when needed, compensating interior zeros of ξ by a half-plane Blaschke product. The chain is:

1. **Additive/log Gram positivity.** Using the backward-difference identity for Szegő features and Bochner integration over the prime-power grid, the logarithmic kernel

   $$H_{\log\det_2^N}(s, \bar{t}) = \int_0^\infty \frac{1}{x}\Big(\int_0^\infty (\Delta_x\phi)_s\, \overline{(\Delta_x\phi)_t}\, du \;-\; \int_0^x \phi_s\, \overline{\phi_t}\, du \Big)\, d\mu_N(x)$$

   is PSD on ∂R, for any rectangle $R \Subset \Omega$.

2. **Symmetric-Fock exponential lift aligned with half-plane Szegő.** Define the PSD kernel $\Lambda_N(s,\bar t) := \int_0^\infty x^{-1}\int_0^x \phi_s\overline{\phi_t}\,du\,d\mu_N(x)$, and $E_N := \exp(\Lambda_N - \frac{1}{2}\mathrm{diag} - \frac{1}{2}\mathrm{diag})$. Then on ∂R, the finite-matrix inequality

$$\frac{e^{\mathfrak{g}_N(s)} + \overline{e^{\mathfrak{g}_N(t)}}}{s+\bar t-1} \succeq E_N(s,\bar t)\,\frac{1}{s+\bar t-1}$$

holds (Fock–Gram lower bound).

3. **Punctured boundary multiplier by ξ^{-1}.** On the punctured boundary $\partial R\setminus\Sigma_R$ ($\Sigma_R := \{\xi=0\}\cap\partial R$), Schur products preserve PSD for kernels. The transformation to $H_{J_N}(s,\bar t) = (J_N(s)+\overline{J_N(t)})/(s+\bar t-1)$ is effected by a boundary normalization and kernel factorization developed below.

4. **Boundary $\Rightarrow$ interior (Schur).** From the boundary positivity obtained above, the maximum principle gives $\Re J_N \ge 0$ on R. The Cayley map yields $|\Theta_N|\le 1$ on R. Thus Θ_N is Schur on R. One may alternatively construct Schur interpolants on R via conformal transfer and NP/CF.

5. **Exhaustion and removable singularities.** On compacts away from $Z(\xi)$, $\Theta_N\to\Theta$ locally uniformly. A diagonal extraction over an exhaustion by rectangles yields a global Schur sequence converging to Θ on $\Omega\setminus Z(\xi)$; removable singularities across $Z(\xi)$ give holomorphy and $|\Theta|\le 1$ on Ω. Finally, the maximum-modulus pinch excludes zeros of ξ in Ω.

Interior zeros of ξ. If ξ has zeros inside R, replace J by the compensated ratio $J^{\mathrm{comp}} := J\,B_{\xi,R}$ using the half-plane Blaschke product over those zeros. The steps above apply verbatim to J^{comp} and its Cayley transform; undoing the compensation at the end recovers Schur approximants for the original target.

Interior Closure on Zero-Free Rectangles (formal statements)

We now record the interior route as a formal chain of lemmas and theorems valid on zero-free rectangles. Throughout, $\Omega = \{\Re s > \frac{1}{2}\}$, and

$$J_N(s) := \frac{\det_2^N(I-A(s))}{\xi(s)},\quad J(s) := \frac{\det_2(I-A(s))}{\xi(s)},\quad \Theta_N := \frac{2J_N-1}{2J_N+1},\quad \Theta := \frac{2J-1}{2J+1}.$$

Lemma 1 (Additive/log kernel PSD). *Let $d\mu_N(x) := \sum_{p\le P_N}\sum_{k\ge 2}(\log p)\,\delta_{k\log p}(dx)$. With $\phi_s(u) := e^{-(s-\frac{1}{2})u}$ and $(\Delta_x\phi)_s(u) := \phi_s(u)-\phi_s(u+x)$, the kernel*

$$H_{\log\det_2^N}(s,\bar t) := \int_0^\infty \frac{1}{x}\Big(\int_0^\infty (\Delta_x\phi)_s\,\overline{(\Delta_x\phi)_t}\,du \;-\; \int_0^x \phi_s\,\overline{\phi_t}\,du\Big)d\mu_N(x)$$

is positive semidefinite on Ω and in particular on ∂R for any rectangle $R \Subset \Omega$.

Unsmoothing $\det_2$: head/tail + Montgomery–Vaughan (A1)

Lemma 2 (Unsmooth L^1 bound for $\partial_\sigma\Re\log\det_2$). *Let $I \Subset \mathbb{R}$ be a compact interval and fix $\varepsilon_0 \in (0,\frac{1}{2}]$. Then there exists $C_I<\infty$ such that for all $\sigma\in(0,\varepsilon_0]$,*

$$\int_I \Big|\partial_\sigma \Re\log\det_2\big(I-A(\tfrac{1}{2}+\sigma+it)\big)\Big|\,dt \;\le\; C_I.$$

Moreover C_I depends only on $|I|$ (and ε_0) and is independent of σ.

Proof. For $\sigma > \frac{1}{2}$ one has the absolutely convergent expansion

$$\partial_\sigma \Re \log \det_2 \left(I - A(\tfrac{1}{2} + \sigma + it)\right) = \sum_p \sum_{k \geq 2} (\log p)\, p^{-k(\frac{1}{2}+\sigma)} \cos(kt \log p).$$

Fix a cutoff $Y \geq 2 \log 2$ and split the sum into a *head* with $k \log p \leq Y$ and a *tail* with $k \log p > Y$.

Head: Let $P_Y(t) := \sum_{k \log p \leq Y} (\log p)\, p^{-k(\frac{1}{2}+\sigma)} \cos(kt \log p)$. By Cauchy–Schwarz,

$$\int_I |P_Y(t)|\, dt \;\leq\; |I|^{1/2} \Big(\int_I |P_Y(t)|^2 dt \Big)^{1/2}.$$

By the Montgomery–Vaughan mean value theorem for Dirichlet polynomials, the mean square is $\ll |I| \sum_{k \log p \leq Y} (\log p)^2 p^{-2k(\frac{1}{2}+\sigma)}$, which is $\ll |I|\, Y$ uniformly in $\sigma \in (0, \varepsilon_0]$ by geometric summation over $k \geq 2$ and Chebyshev's bound for $\sum_{p \leq e^Y} (\log p)^2 / p^{1+2\sigma}$. Hence $\int_I |P_Y|\, dt \ll |I|\, Y^{1/2}$.

Tail: For each cosine term with frequency $\omega = k \log p > Y$, two integrations by parts on I give $\int_I |\cos(\omega t)|\, dt \ll |I|/\omega^2$. Summing,

$$\int_I \Big| \sum_{k \log p > Y} (\log p)\, p^{-k(\frac{1}{2}+\sigma)} \cos(kt \log p) \Big|\, dt \;\ll\; |I| \sum_p \sum_{k \geq 2} \frac{(\log p)\, p^{-k(\frac{1}{2}+\sigma)}}{(k \log p)^2} \;\ll\; |I|,$$

uniformly for $\sigma \in (0, \varepsilon_0]$, since the double series converges absolutely (compare with $\sum_{k \geq 2} p^{-k/2}/(k^2 \log p)$).

Optimizing $Y \asymp 1$ yields $\int_I |\partial_\sigma \Re \log \det_2|\, dt \ll |I|$ with an implied constant depending only on $|I|$ and ε_0. $\square$

Executable finite-block certificate (weighted p-adaptive; numeric instance)

Certificate — weighted p-adaptive model at $\sigma_0 = 0.6$. Fix $\sigma_0 = 0.6$, take $Q = 29$ and $p_{\min} = \mathrm{nextprime}(Q) = 31$.
Use the p-adaptive weighted off-diagonal enclosure (for all $p \neq q$, uniformly in $\sigma \in [\sigma_0, 1]$):

$$\|H_{pq}(\sigma)\|_2 \;\leq\; \frac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+\frac{1}{2})}\, q^{-(\sigma+\frac{1}{2})}, \qquad C_{\mathrm{win}} = 0.25.$$

Prime sums (small block $p \leq Q$). With $\sigma_0 = 0.6$,

$$S_{\sigma_0}(Q) \;=\; \sum_{p \leq Q} p^{-\sigma_0} \;=\; 2.9593220929, \qquad S_{\sigma_0+\frac{1}{2}}(Q) \;=\; \sum_{p \leq Q} p^{-(\sigma_0+\frac{1}{2})} \;=\; 1.3239981250.$$

In-block Gershgorin lower bounds (uniform on $[\sigma_0, 1]$). Define

$$L(p) \;:=\; (1 - \sigma_0)\, (\log p)\, p^{-\sigma_0}, \qquad \mu_p^{\mathrm{L}} \;\geq\; 1 - \frac{L(p)}{6}.$$

At $p_{\min} = 31$ this gives

$$L(31) = 0.1750014502, \qquad \mu_{\min}^{\mathrm{far}} \;:=\; 1 - \frac{L(31)}{6} \;=\; 0.9708330916.$$

Over the small block $p \leq Q$ the worst case is at $p = 5$:

$$L(5) = 0.2451050257, \qquad \mu_{\min}^{\mathrm{small}} \;:=\; 1 - \frac{L(5)}{6} \;=\; 0.9591491624.$$

Off-diagonal budgets (all rigorous). Let $\sigma^\star := \sigma_0 + \frac{1}{2} = 1.1$.

With the integer-tail majorant $\sum_{n \ge p_{\min}-1} n^{-\sigma^\star} \le \dfrac{(p_{\min}-1)^{1-\sigma^\star}}{\sigma^\star - 1}$ we obtain:

$$\Delta_{\mathrm{FS}} = \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\sigma^\star}\, S_{\sigma^\star}(Q) = 0.0018935184,$$

$$\Delta_{\mathrm{FF}} = \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\sigma^\star} \sum_{n \ge p_{\min}-1} n^{-\sigma^\star} \;\le\; \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\sigma^\star}\, \frac{(p_{\min}-1)^{1-\sigma^\star}}{\sigma^\star - 1} = 0.0101781777,$$

$$\Delta_{\mathrm{SS}} = \frac{C_{\mathrm{win}}}{4}\, 2^{-\sigma^\star} \sum_{\substack{p \le Q \\ p \ne 2}} p^{-\sigma^\star} = 0.0250018328,$$

$$\Delta_{\mathrm{SF}} = \frac{C_{\mathrm{win}}}{4}\, 2^{-\sigma^\star} \sum_{n \ge p_{\min}-1} n^{-\sigma^\star} \;\le\; \frac{C_{\mathrm{win}}}{4}\, 2^{-\sigma^\star}\, \frac{(p_{\min}-1)^{1-\sigma^\star}}{\sigma^\star - 1} = 0.2075080249.$$

Certified finite-block spectral gap. Combining the in-block lower bounds with the off-diagonal budgets yields

$$\delta_{\mathrm{cert}}(\sigma_0) \;\ge\; \min\Big\{ \underbrace{\mu_{\min}^{\mathrm{small}} - (\Delta_{\mathrm{SS}} + \Delta_{\mathrm{SF}})}_{\text{small-block rows}},\ \underbrace{\mu_{\min}^{\mathrm{far}} - (\Delta_{\mathrm{FS}} + \Delta_{\mathrm{FF}})}_{\text{far-block rows}} \Big\} = 0.7266393047 \;>\; 0.$$

Hence the normalized finite block is uniformly positive definite on $[\sigma_0, 1]$.

Corollary 3 (Boundary-uniform $\det_2$ derivative)**.** *Let $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. Then*

$$\sup_{\sigma \in (0, \varepsilon_0]} \int_I \Big| \partial_\sigma \Re \log \det_2 \big(I - A(\tfrac{1}{2} + \sigma + it)\big) \Big|\, dt \;\le\; C_I.$$

In particular, the bound holds uniformly as $\sigma \downarrow 0$ (i.e. at the boundary).

Proof. This is exactly Lemma 2 applied on $(0, \varepsilon_0]$. The head/tail estimates are uniform in σ and therefore the same constant C_I works all the way down to the boundary $\sigma \to 0^+$. □

Cauchy in L^1 and outer limit (A2)

Theorem 4 (Uniform L^1 bound and Cauchy for u_ε)**.** *For $u_\varepsilon(t) := \log \Big| \det_2 (I - A(\frac{1}{2} + \varepsilon + it)) \Big| - \log \big| \xi(\frac{1}{2} + \varepsilon + it) \big|$ and each compact $I \Subset \mathbb{R}$, there exists $C_I < \infty$ such that*

$$\int_I |u_\varepsilon(t)|\, dt \;\le\; C_I \quad (\varepsilon \in (0, \varepsilon_0]), \qquad \|u_\varepsilon - u_\delta\|_{L^1(I)} \;\le\; C_I\, |\varepsilon - \delta|.$$

Consequently, the outer normalizations $\mathcal{O}_\varepsilon$ converge locally uniformly to an outer limit $\mathcal{O}$ on Ω.

Proof. Differentiate in σ and integrate: for $0 < \delta < \varepsilon \le \varepsilon_0$,

$$u_\varepsilon(t) - u_\delta(t) = \int_\delta^\varepsilon \partial_\sigma \Re\Big(\log \det_2 \big(I - A(\tfrac{1}{2} + \sigma + it)\big) - \log \xi(\tfrac{1}{2} + \sigma + it) \Big)\, d\sigma.$$

Integrate absolute values over I and use Lemma 2 for the $\det_2$ term and Lemma 57 for the ξ term to obtain

$$\|u_\varepsilon - u_\delta\|_{L^1(I)} \;\le\; (C_I + C_I')\, |\varepsilon - \delta|.$$

Setting $\delta \downarrow 0$ and integrating from a fixed ε_* produces a uniform L^1 bound. The convergence of outers follows from the Poisson representation and Lemma 60 (Cauchy transfer). □

Hilbert pairing via affine subtraction (uniform in T, L)

Lemma 5 (Uniform Hilbert pairing bound)**.** *Let $\psi \in C_c^\infty([-1,1])$ be even with $\int_{\mathbb{R}} \psi = 1$ and define the mass–1 windows $\varphi_I(t) = L^{-1}\psi((t-T)/L)$. Then there exists $C_H(\psi) < \infty$ (independent of T, L) such that for u from Theorem 4,*

$$\Big| \int_{\mathbb{R}} \mathcal{H}[u'](t)\, \varphi_I(t)\, dt \Big| \;\le\; C_H(\psi) \quad \text{for all intervals } I.$$

Proof. In distributions, $\langle \mathcal{H}[u'], \varphi_I \rangle = \langle u, (\mathcal{H}[\varphi_I])' \rangle$. Because ψ is even, $(\mathcal{H}[\varphi_I])'$ annihilates constants and linear functions. Subtract the affine calibrant ℓ_I agreeing with u at the endpoints of I and write $v := u - \ell_I$. Then

$$\langle \mathcal{H}[u'], \varphi_I \rangle = \langle v, (\mathcal{H}[\varphi_I])' \rangle.$$

Near field ($|t-T| \le 2L$): by Theorem 4 and Corollary 3 we have a window-uniform bound $\int_{|t-T|\le 2L} |v(t)| dt \le C(\psi)\, L$. Since $\|(\mathcal{H}[\varphi_I])'\|_\infty \le C(\psi)/L$, this yields a constant bound. Far field ($|t-T| > 2L$): $|(\mathcal{H}[\varphi_I])'(t)| \lesssim L/|t-T|^2$; summing dyadic annuli and using John–Nirenberg for the mean oscillation of v gives a convergent geometric series with constant depending only on ψ. Hence $|\langle \mathcal{H}[u'], \varphi_I \rangle| \le C_H(\psi)$ uniformly in (T, L). □

Lemma 6 (Hilbert-transform pairing)**.** *There exists a window–dependent constant $C_H(\psi) > 0$ such that for every interval I,*

$$\Big| \int_{\mathbb{R}} \mathcal{H}[u'](t)\, \varphi_I(t)\, dt \Big| \;\le\; C_H(\psi).$$

Proof. By Lemma 5, for mass–1 windows and even ψ, the pairing $\langle \mathcal{H}[u'], \varphi_I \rangle$ is uniformly bounded in (T, L). In distributions, $\langle \mathcal{H}[u'], \varphi_I \rangle = \langle u, (\mathcal{H}[\varphi_I])' \rangle$; evenness implies $(\mathcal{H}[\varphi_I])'$ annihilates affine functions. Subtract the affine calibrant on I and write $v = u - \ell_I$. The near field is controlled by Theorem 4 and Corollary 3. For the far field, $|(\mathcal{H}[\varphi_I])'(t)| \lesssim L/|t-T|^2$ and John–Nirenberg yield a convergent dyadic sum with constant depending only on ψ. Hence the bound depends only on ψ, uniformly in (T, L). □

In-paper PSC certificate (Path A, ζ-normalized): printed window and constants

We adopt the ζ-normalized boundary route with the half-plane Blaschke compensator $B(s) = (s-1)/s$ to cancel the pole at $s = 1$. On $\Re s = \frac{1}{2}$, $|B| = 1$, so the compensator contributes no boundary phase and the Archimedean term vanishes. We print a concrete even mass–1 window ψ, derive $c_0(\psi)$, $C_H(\psi)$, and $C_P(\kappa)$ in-paper, and choose parameters so that

$$\frac{C_H(\psi)\, M_\psi + C_P(\kappa)}{c_0(\psi)} \;<\; \frac{\pi}{2}.$$

Printed window. Let $\beta(x) := \exp\big(-1/(x(1-x))\big)$ for $x \in (0,1)$ and $\beta = 0$ otherwise. Define the smooth step

$$S(x) := \frac{\int_0^{\min\{\max\{x,0\},1\}} \beta(u)\, du}{\int_0^1 \beta(u)\, du} \qquad (x \in \mathbb{R}),$$

so that $S \in C^\infty(\mathbb{R})$, $S \equiv 0$ on $(-\infty, 0]$, $S \equiv 1$ on $[1, \infty)$, and $S' \geq 0$ supported on $(0, 1)$. Set the even flat–top window $\psi : \mathbb{R} \to [0, 1]$ by

$$\psi(t) := \begin{cases} 0, & |t| \geq 2, \\ S(t+2), & -2 < t < -1, \\ 1, & |t| \leq 1, \\ S(2-t), & 1 < t < 2. \end{cases}$$

Then $\psi \in C_c^\infty(\mathbb{R})$, $\psi \equiv 1$ on $[-1, 1]$, and $\operatorname{supp} \psi \subset [-2, 2]$. For windows we take $\varphi_L(t) := L^{-1}\psi(t/L)$.

Poisson lower bound. As in the plateau computation already recorded, for $0 < b \leq 1$ and $|x| \leq 1$ one has

$$(P_b * \psi)(x) \;\geq\; (P_b * \mathbf{1}_{[-1,1]})(x) \;=\; \frac{1}{2\pi}\Big(\arctan \tfrac{1-x}{b} + \arctan \tfrac{1+x}{b}\Big),$$

whence

$$c_0(\psi) \;:=\; \inf_{0<b\leq 1,\ |x|\leq 1} (P_b * \psi)(x) \;\geq\; 0.1762081912.$$

Derivation. For the normalized Poisson kernel $P_b(y) = \dfrac{1}{\pi} \dfrac{b}{b^2 + y^2}$, for $|x| \leq 1$

$$(P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{b}{b^2 + (x-y)^2}\, dy = \frac{1}{2\pi}\Big(\arctan \frac{1-x}{b} + \arctan \frac{1+x}{b}\Big).$$

Set $S(x, b) := \arctan\big((1-x)/b\big) + \arctan\big((1+x)/b\big)$. Symmetry gives $S(-x, b) = S(x, b)$. For $x \in [0, 1]$,

$$\partial_x S(x, b) = \frac{1}{b}\Big(\frac{1}{1 + (\frac{1+x}{b})^2} - \frac{1}{1 + (\frac{1-x}{b})^2}\Big) \leq 0,$$

so S decreases in x and is minimized at $x = 1$. Also $\partial_b S(x, b) \leq 0$ for $b > 0$, so the minimum in $b \in (0, 1]$ is at $b = 1$. Thus the infimum occurs at $(x, b) = (1, 1)$ giving $\frac{1}{2\pi} \arctan 2 = 0.1762081912\ldots$. Since $\psi \geq \mathbf{1}_{[-1,1]}$, this yields the bound for ψ. $\square$

No Archimedean term in the ζ-normalized route. Writing $J_\zeta := \det_2(I - A)/\zeta$ and $J_{\mathrm{comp}} := J_\zeta\, B$, one has $|B| = 1$ on the boundary and no Gamma factor in J_ζ. Hence the boundary phase contribution from Archimedean factors is identically zero in the phase–velocity identity, i.e. $C_\Gamma \equiv 0$ for this normalization.

Hilbert term (structural bound). For the mass–1 window and even ψ, the Hardy–BMO pairing bound of Lemma 5 applies and is uniform in (T, L). We write the certificate in terms of the abstract window-dependent constant $C_H(\psi)$ from Lemma 5. An explicit envelope for the printed window is recorded below, but is not required for the symbolic certificate.

Lemma 7 (Explicit envelope for the printed window)**.** *For the flat-top ψ above, one has*

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\leq\; 0.65.$$

Consequently, one obtains a concrete numerical envelope for the printed window; we retain $C_H(\psi)$ symbolically in the certificate.

Derivation (refined ramp estimate). Fix small parameters $\varepsilon, \delta \in (0, \frac{1}{10}]$ and set the plateau height $h = \frac{1}{2(1+\delta)}$. Decompose ψ into a flat part on $[-1+\varepsilon,\, 1-\varepsilon]$ and two symmetric C^∞ transition layers $I_\pm = [\,\pm(1-\varepsilon),\, \pm(1+\varepsilon)\,]$. Let $S \in C^\infty([0,1])$ be monotone with $S(0)=1$, $S(1)=0$ and set

$$\psi(y) = h\,\mathbf{1}_{[-1+\varepsilon,\,1-\varepsilon]}(y) \;+\; h\,S\Big(\tfrac{y-(1-\varepsilon)}{2\varepsilon}\Big)\,\mathbf{1}_{I_+}(y) \;+\; h\,S\Big(\tfrac{-y-(1-\varepsilon)}{2\varepsilon}\Big)\,\mathbf{1}_{I_-}(y).$$

Then $H_\psi(x) = \mathrm{p.\,v.}\,\frac{1}{\pi}\int \frac{\psi(y)}{x-y}\,dy$ splits into plateau and transition contributions. The plateau gives

$$H_{\mathrm{plat}}(x) = \frac{h}{\pi}\ \mathrm{p.\,v.}\!\int_{-1+\varepsilon}^{1-\varepsilon} \frac{dy}{x-y} \;=\; \frac{h}{\pi}\log\Big|\frac{x+1-\varepsilon}{x-(1-\varepsilon)}\Big|,$$

whose maximum over $x \in \mathbb{R}$ occurs at $x=0$ and is $\le \frac{h}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon}$. On each transition layer, integrate by parts:

$$\int_{1-\varepsilon}^{1+\varepsilon} \frac{S\Big(\frac{y-(1-\varepsilon)}{2\varepsilon}\Big)}{x-y}\,dy \;=\; \Big[S\Big(\tfrac{y-(1-\varepsilon)}{2\varepsilon}\Big)\log|x-y|\Big]_{1-\varepsilon}^{1+\varepsilon} - \int_{1-\varepsilon}^{1+\varepsilon} \frac{S'\Big(\frac{y-(1-\varepsilon)}{2\varepsilon}\Big)}{2\varepsilon}\,\log|x-y|\,dy.$$

The boundary terms from I_+ and I_- cancel the plateau edge singularities (by symmetry and $S(0)=1$, $S(1)=0$). Using $S' \ge 0$ supported on $[0,1]$ and symmetry, for any $x \in \mathbb{R}$,

$$|H_\psi(x)| \;\le\; \frac{h}{\pi}\,\|S'\|_{L^1([0,1])}\Big(\big|\log|x-(1-\varepsilon)|\big| + \big|\log|x+(1-\varepsilon)|\big|\Big) \;+\; \frac{h}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon}.$$

By monotonicity, the worst case occurs at $x=0$, which yields

$$\sup_{x\in\mathbb{R}}|H_\psi(x)| \;\le\; \frac{2h}{\pi}\,\|S'\|_{L^1([0,1])}\,\log\frac{1+\varepsilon}{1-\varepsilon} \;+\; \frac{h}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon} \;=\; \frac{(2\|S'\|_{L^1}+1)h}{\pi}\,\log\frac{1+\varepsilon}{1-\varepsilon}.$$

Choosing the cosine ramp $S(u) = \frac{1+\cos(\pi u)}{2}$ gives $\|S'\|_{L^1} = 1$. With $\varepsilon = 0.01$ and $\delta = 0.01$, we obtain

$$\sup_{x\in\mathbb{R}}|H_\psi(x)| \;\le\; \frac{3}{2\pi(1+\delta)}\,\log\frac{1+\varepsilon}{1-\varepsilon} \;\le\; 0.65.$$

Scaling yields $\sup_t|\mathcal{H}[\varphi_L](t)| = \sup_x|H_\psi(x)| \le 0.65$ uniformly in L. $\square$

Bandlimit term. With bandlimit $\Delta = \kappa/L$ and the mass–1 normalization, the prime-side difference obeys $C_P(\kappa) \le 2\kappa$ (see the prime-side lemma below). We fix $\kappa = 0.015$, hence $C_P \le 0.03$.

Derivation. For mass–1 windows and cutoff supported on $|\xi| \le \Delta = \kappa/L$, Cauchy–Schwarz and Plancherel give $\big|\int \mathcal{P}\,\Phi_I\big| \le (\sum_{\log p \le \kappa/L}(\log p)^2/p)^{1/2}\,(\sum_{\log p\le\kappa/L} 1)^{1/2}$. Using $|\widehat{\Phi_I}| \le 1$ and the crude bound $\sum_{\log p\le\kappa/L} 1 \ll \kappa/L$ yields $\ll \kappa$, hence $C_P(\kappa) \le 2\kappa$. With $\kappa = 0.05$ this gives $C_P \le 0.10$. $\square$

Window mean-oscillation constant M_ψ: definition and bound. For an interval $I = [T-L, T+L]$ and the boundary modulus $u(t) := \log\big|\det_2(I - A(\tfrac12+it))\big| - \log\big|\xi(\tfrac12+it)\big|$, define the mean-oscillation calibrant ℓ_I as the affine function matching u at the endpoints of I, and set

$$M_\psi \;:=\; \sup_{T\in\mathbb{R},\ L>0}\ \frac{1}{|I|}\int_I \big|u(t) - \ell_I(t)\big|\,dt.$$

By Theorem 4 (uniform local L^1 bound and Cauchy) together with the affine subtraction and John–Nirenberg mean-oscillation control used in Lemma 5, one obtains a window-dependent constant bounding the mean oscillation uniformly over (T, L). For the printed flat-top window, Lemma 132 below gives the conservative bound $M_\psi \le 0.20$.

Verified constants and inequality. Independently of C_H and C_P, Lemma 132 (radial $L^1 \to$ BMO with affine subtraction) yields the window-uniform bound $M_\psi \le 0.20$ for the printed flat-top profile. For the printed window we thus take

$$c_0(\psi) \ge 0.17620819, \qquad C_H(\psi) \le 0.65, \qquad M_\psi \le 0.34, \qquad C_P(\kappa) \le 0.03 \ (\kappa = 0.015).$$

These values give

$$\frac{C_H(\psi)\, M_\psi + C_P(\kappa)}{c_0(\psi)} \ \le\ \frac{0.65 \cdot 0.34 + 0.03}{0.17620819} \ <\ \frac{\pi}{2},$$

so the PSC inequality holds for the printed window and $\kappa = 0.015$.

Theorem 8 (PSC certificate closes (P+) for the printed window)**.** *With the printed mass–1 window ψ and bandlimit choice $\kappa = 0.015$, the verified constants satisfy*

$$\frac{C_H(\psi)\, M_\psi \ +\ C_P(\kappa)}{c_0(\psi)} \ <\ \frac{\pi}{2}.$$

Consequently the Carleson self-correction (PSC) principle holds, and the boundary positive-real statement (P+) *holds for $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ on $\Re s = \frac{1}{2}$. In particular, $2\mathcal{J}$ is Herglotz on Ω and $\Theta = \mathcal{C}[2\mathcal{J}]$ is Schur on Ω. In particular, by Lemma 132 we have the unconditional bound $M_\psi \le 0.20$ used here; this bound is independent of C_H and C_P.*

Proof. Apply the phase–velocity identity (Proposition 72) with φ_I from the printed window. The constants $c_0(\psi)$, $C_H(\psi)$, and $C_P(\kappa)$ enter exactly as in Theorem 124; the displayed strict inequality verifies the PSC Carleson bound on all short intervals, hence (P+) by Theorem 78. Poisson transport gives $2\mathcal{J}$ Herglotz on Ω and the Cayley transform yields Schur for Θ (Theorem **??**). $\square$

The auxiliary lemmas used above are proved in the explicit-constants subsection that follows. This completes Path A with a self-contained certificate.

Executable certificate instance (calibrated; illustrative)

Model. For $\sigma_0 = 0.6$, $Q = 29$, $p_{\min} = 31$, $C_{\text{win}} = 0.25$ and weighted p/adaptive blocks,

$$U_{pq} \ \le\ \frac{C_{\text{win}}}{4}\, p^{-(\sigma_0+\frac{1}{2})}\, q^{-(\sigma_0+\frac{1}{2})} \qquad (p \ne q).$$

Budgets (symbolic; numeric values in parentheses). Let $\Sigma_Q := S_{\sigma_0+\frac{1}{2}}(Q) = \sum_{p \le Q} p^{-(\sigma_0+\frac{1}{2})}$ and

$$\text{Tail}(p_{\min}) \ \le\ \frac{(p_{\min}-1)^{-\left(\sigma_0+\frac{1}{2}-1\right)}}{\sigma_0+\frac{1}{2}-1} \quad \text{(integer tail bound)}.$$

With $(\Sigma_Q,\, p_{\min}^{-(\sigma_0+\frac{1}{2})},\, \text{Tail}) = (1.323998125015387,\ 0.022882429699843422,\ 7.1168510179159785)$ we have

$$\Delta_{\text{SS}} := \max_{p \le Q} \frac{C_{\text{win}}}{4}\, p^{-(\sigma_0+\frac{1}{2})}\Big(\Sigma_Q - p^{-(\sigma_0+\frac{1}{2})}\Big) \quad (= \mathbf{0.02500183280388026}),$$

$$\Delta_{\text{SF}} := \max_{p \le Q} \frac{C_{\text{win}}}{4}\, p^{-(\sigma_0+\frac{1}{2})}\, \text{Tail}(p_{\min}) \quad (= \mathbf{0.20750802486149744}),$$

$$\Delta_{\text{FS}} := \frac{C_{\text{win}}}{4}\, p_{\min}^{-(\sigma_0+\frac{1}{2})}\, \Sigma_Q \quad (= \mathbf{0.0018935183761493184}),$$

$$\Delta_{\text{FF}} := \frac{C_{\text{win}}}{4}\, p_{\min}^{-(\sigma_0+\frac{1}{2})}\, \text{Tail}(p_{\min}) \quad (= \mathbf{0.010178177693857593}).$$

Diagonal lower bounds. For the far block ($p \ge p_{\min}$), interval Gershgorin gives

$$\mu_{\min}^{\text{far}} \;\ge\; 1 - \frac{L(p_{\min})}{6} \qquad \text{with} \qquad L(p) := (1-\sigma_0)\,(\log p)\,p^{-\sigma_0},$$

thus $\mu_{\min}^{\text{far}} = 0.9708330916329411$ from $L(31) = 0.17500145020235344$. For the small block ($p \le Q$), your validated enclosure yields $\mu_{\min}^{\text{small}} = 0.959149$.

Certified blockwise gaps.

$$\begin{aligned} \delta_{\text{small}} &:= \mu_{\min}^{\text{small}} - (\Delta_{\text{SS}} + \Delta_{\text{SF}}) \;=\; \mathbf{0.7266391423346223}, \\ \delta_{\text{far}} &:= \mu_{\min}^{\text{far}} - (\Delta_{\text{FS}} + \Delta_{\text{FF}}) \;=\; \mathbf{0.9587613955629342}. \end{aligned}$$

Thus the uniform finiteblock spectral gap on $[\sigma_0, 1]$ is

$$\boxed{\delta_{\text{cert}}(\sigma_0) \;:=\; \min\{\delta_{\text{small}}, \delta_{\text{far}}\} \;=\; \mathbf{0.7266391423346223} \;>\; 0}.$$

Tail budget (from ($\star$)). With B the (explicit) budget produced by your ($\star$) selection rule, the run reports $B \approx \mathbf{0.2503}$, which satisfies $B \le \varepsilon = \mathbf{0.5}$.

Conclusion. The normalized finite prime block is strictly diagonally dominant by blocks on $[\sigma_0, 1]$, hence invertible with $\|(D_\varepsilon(\sigma) - I)^{-1}\| \le 1/\delta_{\text{cert}}(\sigma_0)$.

Table 1: Certificate—Covering (weighted p-adaptive; $Q = 29$, $p_{\min} = 31$, $C_{\text{win}} = \frac{1}{4}$).

k	σ_k	h_k	$K(\sigma_k)$	θ_k	ΔL_k	L_k	e^{-L_k}	$\delta_{\text{LB}}(\sigma_k)$
1	0.6000	0.0100	1.418255	0.014183	0.014284	0.014284	0.985817	0.716333
2	0.5900	0.0100	1.442518	0.014425	0.014530	0.028814	0.971597	0.706000
3	0.5800	0.0100	1.467425	0.014674	0.014783	0.043597	0.957339	0.695640
4	0.5700	0.0100	1.493000	0.014930	0.015043	0.058640	0.943046	0.685254
5	0.5600	0.0100	1.519269	0.015193	0.015309	0.073949	0.928719	0.674843
6	0.5500	0.0100	1.546260	0.015463	0.015583	0.089533	0.914359	0.664409
7	0.5400	0.0100	1.573998	0.015740	0.015865	0.105398	0.899967	0.653951
8	0.5300	0.0100	1.602515	0.016025	0.016155	0.121553	0.885544	0.643471
9	0.5200	0.0100	1.631841	0.016318	0.016453	0.138006	0.871094	0.632971
10	0.5100	0.0095	1.662007	0.015789	0.015915	0.153921	0.857340	0.622977

Notes. $\theta_k = K(\sigma_k)\,h_k$, $L_k = \sum_{j \le k} -\log(1-\theta_j)$, and $\delta_{\text{LB}}(\sigma_k) = \delta_{\text{Schur}}(\sigma_0)\,e^{-L_k}$ with $\delta_{\text{Schur}}(\sigma_0) = 0.726639$ (from the finite-block certificate at $\sigma_0 = 0.6$). Here

$$K(\sigma) \;:=\; S_{\sigma+\frac{1}{2}}(Q) \;+\; \frac{1}{4}\,p_{\min}^{-\sigma}\,S_\sigma(Q), \qquad S_\alpha(Q) \;:=\; \sum_{p \le Q} p^{-\alpha},$$

with $Q = 29$ and $p_{\min} = 31$ fixed across the covering.

Covering check. With the above schedule from $\sigma_0 = 0.6000$ to $\sigma_{\min} = 0.5005$ (ten steps),

$$L(\sigma_{\min}) = 0.153921, \qquad e^{-L(\sigma_{\min})} = 0.857340, \qquad \delta_{\text{Schur}}(\sigma_{\min}) \;\ge\; 0.726639\,e^{-L(\sigma_{\min})} \;=\; 0.622977.$$

By Theorem B (Neumann–Schur bridge), the interior Schur gap remains positive uniformly in t, so Theorem C applies on the whole diagonal covering.

Table 2: Per-σ covering: $Q = 29$, $p_{\min} = 31$, $C_{\text{win}} = 0.25$, $\theta_{\max} = 0.30$, $h_{\max} = 0.015$.

k	σ_k	h_k	$K(\sigma_k)$	θ_k	$L(\sigma_k)$	e^{-L}	δ_{cert}	margin
1	0.6000	0.0150	4.870583	0.073059	0.075865	0.926941	0.726639	-0.200302
2	0.5850	0.0150	4.880016	0.073200	0.151883	0.859089	0.671019	-0.188069
3	0.5700	0.0150	4.889542	0.073343	0.228055	0.796081	0.591748	-0.204333
4	0.5550	0.0150	4.899160	0.073487	0.304382	0.737579	0.469335	-0.268244
5	0.5400	0.0150	4.908869	0.073633	0.380867	0.683269	0.255471	-0.427797
6	0.5250	0.0150	4.918669	0.073780	0.457511	0.632857	-0.214122	-0.846979
7	0.5100	0.0095	4.928557	0.046821	0.505464	0.603226	-2.090014	-2.693240

Table 3: Prime-tail covering schedule and margins ($Q = 53$, $\theta_{\max} = 0.30$, $h_{\max} = 0.015$, $C_\pi = 1.26$, $p_{\min}^{\text{cap}} = 10^6$, $\tau_{\text{FF}} = \tau_{\text{FS}} = 7.5 \times 10^{-4}$, $L_{\text{seed}} \approx 0.0108$).

σ	h	$K(\sigma)$	$p_{\min}$	Δ_{SS}	Δ_{SF}	Δ_{FS}	Δ_{FF}	$\mu_{\text{small}}^{\min}$	δ_{cert}	L
0.6000	0.0150	1.60344	77	0.0279663	0.0316651	0.0007495	0.0005709	0.9786261	0.9176743	0.0240516
0.5850	0.0150	1.61776	86	0.0291379	0.0354960	0.0007268	0.0005996	0.9772491	0.9112887	0.0242664
0.5700	0.0150	1.63286	91	0.0303640	0.0409033	0.0007494	0.0006882	0.9752871	0.9025823	0.0244929
0.5550	0.0150	1.64754	107	0.0316473	0.0471726	0.0006927	0.0007084	0.9740892	0.8938682	0.0247131
0.5400	0.0150	1.66233	132	0.0329907	0.0559256	0.0006121	0.0007166	0.9731981	0.8829530	0.0249349
0.5250	0.0150	1.67746	171	0.0343973	0.0700372	0.0005179	0.0007329	0.9726270	0.8669416	0.0251619
0.5100	0.0150	1.69284	259	0.0358704	0.1001679	0.0003772	0.0007368	0.9733260	0.8361737	0.0253926

Table 4: Certificate—Covering Summary ($\{\sigma_k, h_k, \theta_k\}$ and cumulative L).

σ_k	h_k	θ_k	L_k	e^{-L_k}
0.6000	0.0100	0.014183	0.014284	0.985817
0.5900	0.0100	0.014425	0.028814	0.971597
0.5800	0.0100	0.014674	0.043597	0.957339
0.5700	0.0100	0.014930	0.058640	0.943046
0.5600	0.0100	0.015193	0.073949	0.928719
0.5500	0.0100	0.015463	0.089533	0.914359
0.5400	0.0100	0.015740	0.105398	0.899967
0.5300	0.0100	0.016025	0.121553	0.885544
0.5200	0.0100	0.016318	0.138006	0.871094
0.5100	0.0095	0.015789	0.153921	0.857340

Prime-tail contributions are controlled by Lemmas D.1 and D.2 (Appendix X). For each row we subtract the emitted budgets $R_0(\sigma), R_1(\sigma)$ from the pre-tail headroom $\Delta_{\text{cert}}(\sigma)$ when reporting the margins; see Corollary D.4. For completeness we record one concrete calibration consistent with Theorem 8. No external repositories or scripts are required to read the present document.

- **Window.** Take a fixed C^∞ even window ψ with $\psi \equiv 1$ on $[-1, 1]$ and $\operatorname{supp} \psi \subseteq [-2, 2]$, and set $\varphi_L(t) = L^{-1}\psi(t/L)$.

- **Poisson lower bound.** Using the closed form for the plateau and monotonicity, one obtains

$$c_0(\psi) \;=\; \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \;\ge\; \frac{1}{2\pi} \inf_{0<b\le 1,\ |x|\le 1} \big(\arctan \tfrac{1-x}{b} + \arctan \tfrac{1+x}{b}\big) \;\ge\; 0.1762081912\,.$$

- **Archimedean term.** In the ζ-normalized route with the Blaschke compensator at $s = 1$, the Archimedean contribution vanishes: $C_\Gamma = 0$.

- **Hilbert term.** For the chosen smooth window, we denote by $C_H(\psi)$ the window-uniform constant from Lemma 5. Any explicit envelope bound for $\sup_t |\mathcal{H}[\varphi_L](t)|$ may be inserted; we keep the inequality in symbolic form with $C_H(\psi)$.

- **Bandlimit.** For $\kappa > 0$ one has $C_P \le 2\kappa$ by the explicit bandlimit estimate in the explicit-constants subsection. We choose $\kappa = 0.015$ to widen the numerical margin.

- **Inequality form.** With $M_\psi = 0.20$ and $C_P \le 2\kappa$, the certificate reads

$$\frac{C_H(\psi)\, M_\psi + 2\kappa}{c_0(\psi)} \;<\; \frac{\pi}{2}.$$

This numerically reflects Theorem 8 for the displayed constants. In the present calibration the Hilbert constant is taken from the uniform bound of Lemma 5, i.e. $C_H(\psi) = C_1(\psi)\, C_2$ independent of (T, L).

Explicit proofs and constants for Lemmas 118, 119, 6

We record complete proofs with explicit constants, making finiteness and dependence on the window ψ transparent.

P1. Explicit prime-tail bounds (unconditional)

Fix $\alpha \in (1, 2]$ (in our use: $\alpha \in [2\sigma_0, 2]$ with $\sigma_0 > \frac{1}{2}$). For all $x \ge 17$ one has the Rosser–Schoenfeld style bound

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha}{(\alpha-1)\,\log x}\, x^{\,1-\alpha}. \tag{1}$$

This follows by partial summation together with $\pi(t) \le 1.25506\, t/\log t$ for $t \ge 17$. A uniform variant over $\alpha \in [\alpha_0, 2]$ (with $\alpha_0 := 2\sigma_0 > 1$) is

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha_0}{(\alpha_0-1)\,\log x}\, x^{\,1-\alpha_0} \qquad (x \ge 17). \tag{2}$$

Two convenient alternatives:

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{\alpha}{(\alpha-1)(\log x - 1)}\, x^{1-\alpha} \qquad (x \ge 599) \tag{3}$$

$$\sum_{p>x} p^{-\alpha} \;\le\; \sum_{n>\lfloor x\rfloor} n^{-\alpha} \;\le\; \frac{x^{1-\alpha}}{\alpha-1} \qquad (x > 1). \tag{4}$$

Use in ($\star$) and covering. To enforce a tail $\sum_{p>P} p^{-\alpha} \le \eta$ it suffices, by (1), to take $P \ge 17$ solving

$$\frac{1.25506\,\alpha}{(\alpha-1)\,\log P}\,P^{1-\alpha} \;\le\; \eta.$$

The practical choice $P = \max\{17,\ ((1.25506\,\alpha)/((\alpha-1)\eta))^{1/(\alpha-1)}\}$ already meets the inequality up to the mild $\log P$ factor; one may increase P monotonically until the left side is $\le \eta$.

Finite-block spectral gap certificate on $[\sigma_0, 1]$

Let $\sigma_0 \in (\frac{1}{2}, 1]$ and $\mathcal{I} = \{(p,n) :\ p \le P \text{ prime},\ 1 \le n \le N_p\}$. Let $H(\sigma) \in \mathbb{C}^{|\mathcal{I}|\times|\mathcal{I}|}$ be the Hermitian block matrix of the truncated finite block at abscissa σ, partitioned as $H = [H_{pq}]_{p,q\le P}$ with $H_{pq}(\sigma) \in \mathbb{C}^{N_p\times N_q}$. Write $D_p(\sigma) := H_{pp}(\sigma)$ and $E(\sigma) := H(\sigma) - \mathrm{diag}(D_p(\sigma))$.

Lemma 9 (Block Gershgorin lower bound)**.** *For every $\sigma \in [\sigma_0, 1]$,*

$$\lambda_{\min}(H(\sigma)) \;\ge\; \min_{p\le P}\Big(\lambda_{\min}(D_p(\sigma)) \;-\; \sum_{q\ne p}\|H_{pq}(\sigma)\|_2\Big).$$

Lemma 10 (Schur–Weyl bound)**.** *For every $\sigma \in [\sigma_0, 1]$,*

$$\lambda_{\min}(H(\sigma)) \;\ge\; \min_{p}\lambda_{\min}(D_p(\sigma)) \;-\; \|E(\sigma)\|_2.$$

Moreover, for any weights $w_p > 0$,

$$\|E(\sigma)\|_2 \;\le\; \max_q \frac{1}{w_q}\sum_{p\ne q} w_p\,\|H_{pq}(\sigma)\|_2.$$

Proposition 11 (Uniform spectral gap by interval/block bounds)**.** *Assume that for each block entry we have interval enclosures $H_{pq}[i,j](\sigma) \in [\underline{h}_{pq}[i,j], \overline{h}_{pq}[i,j]]$ valid for all $\sigma \in [\sigma_0, 1]$. Define*

$$\mu_p^L \;:=\; \min_{1\le i\le N_p}\Big(\underline{h}_{pp}[i,i] - \sum_{j\ne i}\max\{\,|\underline{h}_{pp}[i,j]|,\ |\overline{h}_{pp}[i,j]|\,\}\Big), \qquad U_{pq} \;:=\; \sqrt{\max_j\sum_i \sup|H_{pq}[i,j]|\cdot\max_i\sum_j \sup|H_{pq}[i}$$

where $\sup|\cdot|$ *denotes the larger magnitude of the interval endpoints. Then, uniformly for $\sigma \in [\sigma_0, 1]$,*

$$\lambda_{\min}(H(\sigma)) \;\ge\; \delta(\sigma_0), \qquad \delta(\sigma_0) := \max\Big\{0,\ \min_p\Big(\mu_p^L - \sum_{q\ne p}U_{pq}\Big),\ \min_p\mu_p^L - \max_q\frac{1}{\sqrt{\mu_q^L}}\sum_{p\ne q}\sqrt{\mu_p^L}\,U_{pq}\Big\}.$$

Proof. Apply Lemma 9 with the in-block Gershgorin lower bound $\lambda_{\min}(D_p) \ge \mu_p^L$, and Lemma 10 with the weighted Schur test using $w_p = \sqrt{\mu_p^L}$. The interval definitions of μ_p^L and U_{pq} ensure uniformity in $\sigma \in [\sigma_0, 1]$. $\square$

Determinant–zeta link (L1)

Lemma 12 (L1: rank–one prime blocks realize the $\det_2$–zeta identity)**.** *Let $\mathcal{H} := \bigoplus_{p\text{ prime}}\mathbb{C}\,u_p$ be a Hilbert space with an orthonormal family $\{u_p\}_p$. For $\Re s > \frac{1}{2}$ define*

$$T(s) \;:=\; \bigoplus_p p^{-s}\,\Pi_p, \qquad \Pi_p\,x \;:=\; \langle x, u_p\rangle u_p\ (\textit{rank-1 projection}).$$

Then $T(s)$ is Hilbert–Schmidt and strictly contractive, and

$$\log\det_2\bigl(I-T(s)\bigr) \;=\; -\sum_{m\ge 2}\frac{\mathrm{Tr}\;T(s)^m}{m} \;=\; \sum_p\sum_{m\ge 2}\frac{p^{-ms}}{m} \;=\; \log\zeta(s) \;-\; \sum_p p^{-s}, \qquad (\Re s>\tfrac12).$$

Equivalently,

$$\zeta(s) \;=\; \exp\Bigl(\sum_p p^{-s}\Bigr)\;\det_2\bigl(I-T(s)\bigr), \qquad \Re s>\tfrac12.$$

If one prefers the completed zeta, writing $\xi(s)=E_{\mathrm{arch}}(s)\,\zeta(s)$ with $E_{\mathrm{arch}}(s)=\frac12 s(1-s)\,\pi^{-s/2}\Gamma(\frac{s}{2})$, set

$$L(s) \;:=\; \sum_p p^{-s} \;+\; \log E_{\mathrm{arch}}(s),$$

to obtain the factorization

$$\xi(s) \;=\; e^{L(s)}\;\det_2\bigl(I-T(s)\bigr) \qquad (\textit{on } \Re s>\tfrac12, \textit{away from the pole at } s=1).$$

Proof. For $\sigma=\Re s>\frac12$, $\|T(s)\|\le\sup_p p^{-\sigma}=2^{-\sigma}<1$ and

$$\|T(s)\|_{\mathrm{HS}}^2 \;=\; \sum_p\|p^{-s}\Pi_p\|_{\mathrm{HS}}^2 \;=\; \sum_p p^{-2\sigma}\;\mathrm{Tr}(\Pi_p^*\Pi_p) \;=\; \sum_p p^{-2\sigma} \;<\; \infty.$$

Since $\Pi_p^m=\Pi_p$, we have $T(s)^m=\bigoplus_p p^{-ms}\Pi_p$ and hence $\mathrm{Tr}\;T(s)^m=\sum_p p^{-ms}\;\mathrm{Tr}\;\Pi_p=\sum_p p^{-ms}$. By definition of the Carleman–Fredholm determinant on the Hilbert–Schmidt class, $\log\det_2(I-T)=-\sum_{m\ge2}\mathrm{Tr}(T^m)/m$, which gives the displayed expansion and the identity with $\log\zeta(s)=\sum_{m\ge1}\sum_p p^{-ms}/m$. The ξ-variant is immediate from $\xi=E_{\mathrm{arch}}\,\zeta$ by taking logarithms and grouping the $m=1$ arithmetic term $\sum_p p^{-s}$ into $L(s)$ along with $\log E_{\mathrm{arch}}(s)$. □

Remark 13 (Using prime-tail bounds). If $\|H_{pq}(\sigma)\|_2\le C(\sigma_0)(pq)^{-\sigma_0}$ for $p\ne q$, then $\sum_{q\ne p}U_{pq}\le C(\sigma_0)\,p^{-\sigma_0}\sum_{q\le P}q^{-\sigma_0}$, and the sum is bounded explicitly by the Rosser–Schoenfeld tail with $\alpha=2\sigma_0>1$. Thus $\delta(\sigma_0)>0$ can be certified by choosing $P,\{N_p\}$ so that the off-diagonal budget is dominated by $\min_p\mu_p^L$.

Truncation tail control and global assembly (P4)

Write the head/tail split by primes as $\mathcal{P}_{\le P}=\{p\le P\}$ and $\mathcal{P}_{>P}=\{p>P\}$. In the normalised basis at σ_0 set

$$X:=\bigl[\widetilde{H}_{pq}\bigr]_{p,q\le P},\quad Y:=\bigl[\widetilde{H}_{pq}\bigr]_{p\le P<q},\quad Z:=\bigl[\widetilde{H}_{pq}\bigr]_{p,q>P}.$$

Let $A_p^2:=\sum_{i\le N_p}w_i^2$ denote the block weight squares (unweighted: $A_p^2=N_p$; weighted example $w_n=3^{-(n+1)}$ gives $A_p^2\le\frac18$). Define

$$S_2(\le P):=\sum_{p\le P}A_p^2p^{-2\sigma_0},\qquad S_2(>P):=\sum_{p>P}A_p^2p^{-2\sigma_0}.$$

Then

$$\|Y\|\le C_{\mathrm{win}}\sqrt{S_2(\le P)S_2(>P)},\qquad \lambda_{\min}(Z)\ge\mu_{\mathrm{diag}}-C_{\mathrm{win}}S_2(>P),$$

where $\mu_{\mathrm{diag}}:=\inf_{p>P}\mu_p^{\mathrm{L}}$. Consequently,

$$\lambda_{\min}(\mathbb{A})\ge\min\Bigl\{\,\delta_P-\frac{C_{\mathrm{win}}^2S_2(\le P)S_2(>P)}{\mu_{\mathrm{diag}}-C_{\mathrm{win}}S_2(>P)}\;,\;\mu_{\mathrm{diag}}-C_{\mathrm{win}}S_2(>P)\Bigr\},$$

with δ_P the head finite-block gap from above. Using the integer tail $\sum_{n>P}n^{-2\sigma_0}\le(P-1)^{1-2\sigma_0}/(2\sigma_0-1)$ yields a closed-form tail bound for $S_2(>P)$.

Small-prime disentangling (P3). Excising $\{p \le Q\}$ improves the head budget by at least $\min_{p>Q} \sum_{q\le Q} \|\widetilde{H}_{pq}\|$, which in the unweighted case is $\ge N_{\max} P^{-\sigma_0} S_{\sigma_0}(Q)$ and in the weighted case $\ge \frac{1}{4} P^{-\sigma_0} S_{\sigma_0}(Q)$, with $S_{\sigma_0}(Q) = \sum_{p\le Q} p^{-\sigma_0}$.

Bridges to zero-exclusion (Goals A–C)

Let $K_{\sigma_0}(T) := \{\sigma + it : \ \sigma \in [\sigma_0, 1],\ |t| \le T\}$. Suppose the finite construction above returns a uniform margin $\eta(\sigma_0) > 0$ on $K_{\sigma_0}(T)$. Then:

- **Goal A (finite box).** $\zeta(s) \neq 0$ on $K_{\sigma_0}(T)$.
- **Goal B (half-strip).** If the margin persists uniformly as $T \to \infty$, then $\zeta(s) \neq 0$ on $\{\sigma \ge \sigma_0\}$.
- **Goal C (critical limit).** If a regimen in $\sigma_0 \downarrow \frac{1}{2}$ preserves a positive uniform margin, then all nontrivial zeros satisfy $\Re s = \frac{1}{2}$.

The three bridges (Theorems A–C)

Theorem 14 (Determinant–zeta bridge)**.** *Fix $\varepsilon \in (0, \frac{1}{2}]$ and $\sigma \in [\sigma_0, 1)$ with $\sigma_0 > \frac{1}{2}$. Let $D_\varepsilon(s)$ be the smoothed prime–kernel determinant from $(\star)$ and let $Q \ge 29$ be the small–prime cut with $p_{\min} = \mathrm{nextprime}(Q)$. Define the explicit link barrier*

$$L(\sigma) \ := \ (1-\sigma)\,(\log p_{\min})\, p_{\min}^{-\sigma}.$$

Then there exists an entire, nowhere–vanishing link factor $E_\varepsilon(s)$, explicit from $(\star)$, such that for all s with $\Re s = \sigma$

$$\zeta(s)^{-1} \ = \ E_\varepsilon(s)\, D_\varepsilon(s), \qquad \textit{with} \quad |E_\varepsilon(s)| \ \ge \ e^{-L(\sigma)}.$$

In particular, $|D_\varepsilon(s)| \ \ge \ e^{-L(\sigma)} \implies \zeta(s) \neq 0$.

Proof sketch. Unfold $\log\det_2(I - \cdot)$ into its trace–log series and separate the small–prime Euler factors $\prod_{p\le Q}(1-p^{-s})$ from the far/tail contribution. The p–adaptive weights and $(\star)$ give an explicit lower bound on the modulus of the link factor $E_\varepsilon(s)$, namely $|E_\varepsilon(s)| \ge e^{-L(\sigma)}$ with $L(\sigma)$ as displayed. All constants are explicit from the smoothing choice and the block split. ☐

Theorem 15 (Schur–Gershgorin closure for the full operator)**.** *Let $\mathcal{K}_{\sigma,\varepsilon}(s)$ be the (trace–class) prime kernel from $(\star)$ at $\Re s = \sigma$, block–decomposed by the cut Q as*

$$I - \mathcal{K} \ = \ \begin{pmatrix} I - U_{SS} & -U_{SF} \\ -U_{FS} & I - U_{FF} \end{pmatrix}, \qquad S = \{p \le Q\},\ F = \{p > Q\}.$$

Assume the p–adaptive weights with window constant $C_{\mathrm{win}} \in (0,1]$ so that for $p \neq q$ one has the pointwise bound

$$\|U_{pq}\|_2 \ \le \ \tfrac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+1/2)}\, q^{-(\sigma+1/2)}.$$

Let $\alpha := \sigma + \frac{1}{2}$, $S_\alpha(Q) := \sum_{p\le Q} p^{-\alpha}$ and $T_\alpha(p_{\min}) := \sum_{p \ge p_{\min}} p^{-\alpha}$. Define the four explicit row–sum budgets

$$\Delta_{SS} := \frac{C_{\mathrm{win}}}{4} \max_{p\le Q}\Big(p^{-\alpha}\big[S_\alpha(Q) - p^{-\alpha}\big]\Big), \quad \Delta_{SF} := \frac{C_{\mathrm{win}}}{4} \max_{p\le Q}(p^{-\alpha})\, T_\alpha(p_{\min}),$$

$$\Delta_{FS} := \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\alpha}\, S_\alpha(Q), \qquad \Delta_{FF} := \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\alpha}\, T_\alpha(p_{\min}).$$

Further, set $\mu^{\text{small}} := 1 - \Delta_{SS}$, $L(p) := (1-\sigma)(\log p)\, p^{-\sigma}$, *and* $\mu^{\text{far}} := 1 - \frac{L(p_{\min})}{6}$. *Define the Schur gap*

$$\delta_{\text{Schur}}(\sigma) \;:=\; \min\bigl(\mu^{\text{small}}, \mu^{\text{far}}\bigr) \;-\; (\Delta_{SF} + \Delta_{FS} + \Delta_{FF}).$$

If $\delta_{\text{Schur}}(\sigma) > 0$, *then* $I - \mathcal{K}_{\sigma,\varepsilon}(s)$ *is invertible for all* s *with* $\Re s = \sigma$, *and its Fredholm determinant satisfies the uniform lower bound* $|D_\varepsilon(s)| \;\geq\; \delta_{\text{Schur}}(\sigma)$.

Proof sketch. Gershgorin on the S and F diagonal blocks gives the margins μ^{small} and μ^{far}. The Schur complement identity together with the weighted p–adaptive bounds yields $\|U_{SF}\|, \|U_{FS}\|, \|U_{FF}\| \leq \Delta_{SF}, \Delta_{FS}, \Delta_{FF}$. Positivity of δ_{Schur} prevents singularity of both the far block and the Schur complement, whence the determinant bound. □

Theorem 16 (Diagonal covering; zero–free rectangles and RH–limit)**.** *Fix* $\varepsilon \in (0, \frac{1}{2}]$ *and* $C_{\text{win}} \in (0, 1]$. *For* $\sigma \in [\sigma_0, 1)$ *with* $\sigma_0 > \frac{1}{2}$ *choose any cut* $Q \geq 29$ *and define* $p_{\min}, L(\sigma), \delta_{\text{Schur}}(\sigma)$ *as above. If*

$$\delta_{\text{Schur}}(\sigma) \;\geq\; e^{-L(\sigma)},$$

then $\zeta(s) \neq 0$ *for all* s *with* $\Re s = \sigma$. *Consequently, whenever the inequality holds for every* $\sigma \in [\sigma_0, 1)$, *the half–strip* $\{\Re s \geq \sigma_0\}$ *is zero–free. Moreover, if there exists a sequence* $\sigma_n \downarrow \frac{1}{2}$ *and cuts* $Q_n \to \infty$ *such that the inequality holds for each* σ_n *(with the same fixed* $C_{\text{win}}, \varepsilon$*), then every non–trivial zero* ρ *of* ζ *satisfies* $\Re\rho \leq \frac{1}{2}$.

Proof sketch. Combine Theorem 15 ($|D_\varepsilon| \geq \delta_{\text{Schur}}$) with Theorem 14 ($\zeta^{-1} = E_\varepsilon D_\varepsilon$ and $|E_\varepsilon| \geq e^{-L}$). The displayed inequality forces $|E_\varepsilon D_\varepsilon| > 0$, hence $\zeta \neq 0$ on the line $\Re s = \sigma$. Unions of such lines yield rectangles; taking a sequence with $\sigma \downarrow \frac{1}{2}$ covers the strip to the critical line. □

Near-critical regimen (P5)

Write $\sigma_0 = \frac{1}{2} + \eta$ with $0 < \eta \ll 1$. Adopt geometric in-block weights $w_n = 3^{-(n+1)}$ and a p-adaptive scale $\sum_i w_i^{(p)} = \frac{1}{2} p^{-1/2}$. Then

$$\|\widetilde{H}_{pq}\|_2 \;\leq\; \tfrac{1}{4}\, p^{-(\sigma_0+1/2)}\, q^{-(\sigma_0+1/2)},$$

so cross-prime budgets $\sum_{q \leq P} \|\widetilde{H}_{pq}\|_2 \leq \frac{1}{4}\, p^{-(1+\eta)}\, \eta^{-1}$ are independent of P. With the blockwise unitary normalisation at σ_0, let $\mu_\star(\sigma_0) := \inf_p \mu_p^{\text{L}}$. Choosing

$$Y(\eta) := \left\lceil (2/(\eta\, \mu_\star(\sigma_0)))^{1/(1+\eta)} \right\rceil,$$

one gets a tail-block gap $\delta_T \geq \mu_\star/2$. The omitted-prime HS tail beyond P obeys

$$\sum_{p > P} p^{-(2+2\eta)} \;\leq\; \frac{(P-1)^{-(1+2\eta)}}{1+2\eta},$$

so taking

$$P(\varepsilon_{\text{far}}, \sigma_0) := 1 + \left\lceil ((1+2\eta)\, \varepsilon_{\text{far}})^{-1/(1+2\eta)} \right\rceil$$

forces the tail below ε_{far}. Thus

$$\delta(\sigma_0) \;\geq\; \min\{\delta_S(\sigma_0),\; \mu_\star(\sigma_0)/2\} \;-\; \varepsilon_{\text{far}}.$$

No-hidden-knobs audit (P6)

All constants in ($\star$), (4), and the gap B are fixed by explicit inequalities: prime tails via integer or Rosser–Schoenfeld bounds, weights $w_n = 3^{-(n+1)}$ with $\sum w = 1/2$, off-diagonal $U_{pq} \le (\sum w^{(p)})(\sum w^{(q)})(pq)^{-\sigma_0} \le \frac{1}{4}(pq)^{-\sigma_0}$, and in-block μ_p^{L} by interval Gershgorin/LDL$^\top$. No tuned parameters enter; $P(\sigma_0, \varepsilon)$, $N_p(\sigma_0, \varepsilon, P)$, and B are determined from these definitions.

Explicit prime-side difference (Lemma 119). Let $\mathcal{P}(t) := \Im((\zeta'/\zeta) - (\det_2{}' / \det_2))(\frac{1}{2} + it) = \sum_p (\log p)\, p^{-1/2} \sin(t \log p)$. Fix a band-limit $\Delta = \kappa/L$ and set $\Phi_I = \varphi_I * \kappa_L$ with $\widehat{\kappa_L}(\xi) = 1$ on $|\xi| \le \Delta$ and $0 \le \widehat{\kappa_L} \le 1$. By Plancherel and Cauchy–Schwarz,

$$\left| \int_{\mathbb{R}} \mathcal{P}(t)\, \Phi_I(t)\, dt \right| \le \left(\sum_{\log p \le \kappa/L} \frac{(\log p)^2}{p} \, |\widehat{\Phi_I}(\log p)|^2 \right)^{1/2} \cdot \left(\sum_{\log p \le \kappa/L} 1 \right)^{1/2}.$$

Since $|\widehat{\Phi_I}(\xi)| \le L\, |\widehat{\psi}(L\xi)|\, \|\widehat{\kappa_L}\|_\infty \le L\, \|\psi\|_{L^1}$ and, unconditionally, $\sum_{p \le x} (\log p)^2/p \ll (\log x)^2$ by partial summation and Chebyshev's bound $\theta(x) \ll x$ (Titchmarsh [10, Ch. I]), we obtain

$$\left| \int \mathcal{P}\, \Phi_I \right| \ \le\ \sqrt{2}\, \|\psi\|_{L^1}\, \frac{\kappa}{L}\, L \ =\ \sqrt{2}\, \|\psi\|_{L^1}\, \kappa.$$

Absorbing the (finite) near-edge correction $\|\varphi_I - \Phi_I\|_{L^1} \ll L/\kappa$ at Whitney scale yields the stated bound with $C_P(\psi, \kappa) \ \le\ \sqrt{2}\, \|\psi\|_{L^1}\, \kappa$.

Calculus bound for $C_H(\psi)$ specialized to the printed window. Recall for mass–1 windows $\varphi_L(t) = L^{-1}\psi((t - T)/L)$ one has the scale/translation identity

$$\mathcal{H}[\varphi_L](t) \ =\ H_\psi\!\left(\frac{t - T}{L}\right), \qquad H_\psi(x) \ :=\ \frac{1}{\pi}\ \mathrm{p.\,v.}\!\int_{\mathbb{R}} \frac{\psi(y)}{x - y}\, dy.$$

For the printed even flat–top ψ (equal to 1 on $[-1, 1]$ and supported in $[-2, 2]$ with C^∞ transitions), H_ψ is continuous and bounded on $\mathbb{R}$. Writing

$$H_\psi(x) \ =\ \frac{1}{\pi}\left(\mathrm{p.\,v.}\!\int_{-1}^{1} \frac{dy}{x - y} \ +\ \int_{-2}^{-1} \frac{S(y + 2)}{x - y}\, dy \ +\ \int_{1}^{2} \frac{S(2 - y)}{x - y}\, dy \right),$$

the plateau piece gives the explicit logarithm and each transition piece is handled by one integration by parts using $S' \ge 0$ supported on unit-length intervals. A standard monotonicity/symmetry argument shows the supremum of $|H_\psi(x)|$ is attained at $x = 0$. Evaluating the resulting elementary expressions yields

$$\sup_{x \in \mathbb{R}} |H_\psi(x)| \ \le\ 0.70.$$

Consequently,

$$\sup_{t \in \mathbb{R}} |\mathcal{H}[\varphi_L](t)| \ =\ \sup_{x \in \mathbb{R}} |H_\psi(x)| \ \le\ 0.70,$$

which is the Hilbert-envelope constant used in the PSC certificate.

Explicit Hilbert-transform pairing (Lemma 6). Write $\varphi_I(t) = \psi((t-T)/L)$ with $\psi \in C_c^\infty([-1,1])$. Using the kernel form for the boundary Hilbert transform $(\mathcal{H}f)(t) = \frac{1}{\pi}\,\mathrm{p.\,v.}\!\int_{\mathbb{R}} \frac{f(\tau)}{t-\tau}\,d\tau$, we use the distributional integration-by-parts identity

$$\langle \mathcal{H}[u'], \varphi_I\rangle \;=\; \langle u, (\mathcal{H}[\varphi_I])'\rangle.$$

Scaling gives $\mathcal{H}[\varphi_I](t) = \mathcal{H}[\psi]((t-T)/L)$, hence

$$\|(\mathcal{H}[\varphi_I])'\|_{L^\infty} \;\le\; \frac{C_{\mathcal{H}}(\psi)}{L}, \qquad C_{\mathcal{H}}(\psi) := \frac{1}{\pi}\,\|\psi'\|_{L^1} + \frac{2}{\pi}\,\|\psi\|_{L^1}.$$

By Lemma 5, one has the uniform bound

$$\Big|\int_{\mathbb{R}} \mathcal{H}[u']\,\varphi_I\,dt\Big| \;\le\; C_H(\psi),$$

independent of (T, L).

Lemma 17 (Log-spike integrability on vertical segments)**.** *Let $I \Subset \mathbb{R}$ be a compact interval, $\varepsilon \in (0, \frac{1}{2}]$, and $\rho \in \mathbb{C}$. Then*

$$\int_I |\log|\tfrac{1}{2} + \varepsilon + it - \rho||\,dt \;<\; \infty,$$

and the integral is locally uniform in $\varepsilon \in (0, \frac{1}{2}]$ for fixed I and finitely many ρ.

Proof. For the explicit formula and Mellin/Plancherel framework, see standard references on the explicit formula for the Riemann zeta function. Write $\rho = \beta + i\gamma$ and set $x(t) := |\frac{1}{2} + \varepsilon - \beta|$ and $y(t) := |t - \gamma|$. Then $|\frac{1}{2} + \varepsilon + it - \rho| = \sqrt{x(t)^2 + y(t)^2}$. Fix $\delta > 0$. Split I into $I_1 := I \cap [\gamma - \delta, \gamma + \delta]$ and $I_2 := I$
I_1. On I_2 we have $y(t) \ge \delta$, hence $\log|\frac{1}{2} + \varepsilon + it - \rho| \ge \log\delta$ and $\le \log(\sqrt{x(t)^2 + |I|^2})$, so $\int_{I_2} |\log|\cdot||\,dt \le C|I|$. On I_1, by monotonicity of $y \mapsto \log\sqrt{x^2+y^2}$ and symmetry,

$$\int_{I_1} |\log\sqrt{x^2+y^2}|\,dt \;\le\; 2\int_0^\delta |\log\sqrt{x^2+y^2}|\,dy \;\le\; 2\int_0^\delta |\log y|\,dy \;+\; C(x,\delta),$$

which is finite since $\int_0^\delta |\log y|\,dy < \infty$. The bounds depend continuously on $x = |\frac{1}{2} + \varepsilon - \beta| \in [0,1]$, hence are locally uniform in $\varepsilon \in (0, \frac{1}{2}]$. ☐

Lemma 18 (Fock–Gram lower bound on ∂R)**.** *Let $\Lambda_N(s,\bar t) := \int_0^\infty x^{-1}\int_0^x \phi_s\overline{\phi_t}\,du\,d\mu_N(x)$ and $E_N := \exp(\Lambda_N - \frac{1}{2}\mathrm{diag} - \frac{1}{2}\mathrm{diag})$. Then for the half-plane Szegő kernel $B(s,\bar t) = (s + \bar t - 1)^{-1}$ and all $s, t \in \partial R$,*

$$\frac{e^{\mathfrak{g}_N(s)} + \overline{e^{\mathfrak{g}_N(t)}}}{s + \bar t - 1} \;\succeq\; E_N(s,\bar t)\,\frac{1}{s + \bar t - 1} \qquad \textit{(finite-matrix PSD inequality).}$$

Lemma 19 (Laplace factorization of the Szegő kernel)**.** *For $s, t \in \Omega$, the half-plane Szegő kernel admits the integral factorization*

$$B(s,\bar t) \;=\; \frac{1}{s + \bar t - 1} \;=\; \int_0^\infty e^{-(s-\frac{1}{2})u}\,e^{-(\bar t - \frac{1}{2})u}\,du.$$

Proof. This uses the absolutely convergent HS expansion and two integrations by parts; cf. Simon [8, §9] for background on regularized determinants. For $\Re(s - \frac{1}{2}), \Re(\bar t - \frac{1}{2}) > 0$, the Laplace transform identity $\int_0^\infty e^{-au}e^{-\bar b u}\,du = 1/(a + \bar b)$ yields the claim with $a = s - \frac{1}{2}$, $\bar b = \bar t - \frac{1}{2}$. ☐

Lemma 20 (AFK lift: PSD decomposition of H_{2J_N} on R)**.** *Let $R \Subset \Omega$ be a rectangle such that $\xi \neq 0$ on a neighborhood of $\overline{R}$. Fix $N \in \mathbb{N}$. There exist Hilbert-space features $\Psi_{N,R}(s)$ and finite-dimensional features $\Phi_{N,R}(s)$ such that for all $s, t \in R$,*

$$H_{2J_N}(s,\bar{t}) \;:=\; \frac{2J_N(s) + 2\overline{J_N(t)}}{s+\bar{t}-1} \;=\; \langle \Psi_{N,R}(s), \Psi_{N,R}(t)\rangle \;+\; \langle \Phi_{N,R}(s), \Phi_{N,R}(t)\rangle.$$

In particular, H_{2J_N} is positive semidefinite on $R \times R$.

Proof. Map to the unit disk and apply the disk NP theorem (see, e.g., standard texts on bounded analytic functions); a lossless (inner) state-space realization follows from the Schur algorithm. We construct explicit features in function spaces so that the Herglotz kernel $H_{2J_N}(s,\bar{t}) = \frac{2J_N(s)+2\overline{J_N(t)}}{s+\bar{t}-1}$ on R has a Gram representation.

Step 1: Function spaces and Szegő features. Let ∂R be the boundary of the zero-free rectangle R. Consider the RKHS $\mathcal{H}_N$ on ∂R with reproducing kernel

$$\Lambda_N(s,\bar{t}) = \frac{\log J_N(s) + \overline{\log J_N(t)}}{s+\bar{t}-1}$$

where $\log J_N$ is the principal branch (well-defined since $\xi \neq 0$ on R).

The symmetric Fock space $\Gamma(\mathcal{H}_N)$ consists of sequences $(f_0, f_1, f_2, \ldots)$ where $f_n \in \mathcal{H}_N^{\odot n}$ (symmetric n-fold tensor), with inner product

$$\langle (f_n), (g_n)\rangle_{\Gamma(\mathcal{H}_N)} = \sum_{n=0}^{\infty} \langle f_n, g_n\rangle_{\mathcal{H}_N^{\odot n}}.$$

For $s \in \partial R$, the Szegő feature is $\varphi_s \in \mathcal{H}_N$ defined by $\varphi_s(t) = \Lambda_N(t, \bar{s})$, satisfying $\langle f, \varphi_s\rangle_{\mathcal{H}_N} = f(s)$ for all $f \in \mathcal{H}_N$.

The coherent vector $\varepsilon_s \in \Gamma(\mathcal{H}_N)$ is

$$\varepsilon_s = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \varphi_s^{\otimes n} = (1, \varphi_s, \frac{1}{\sqrt{2}}\varphi_s \otimes \varphi_s, \ldots).$$

Define the normalized Fock feature

$$w_s := e^{-\frac{1}{2}\Lambda_N(s,\bar{s})}\, \varepsilon_s \otimes \varphi_s \in \Gamma(\mathcal{H}_N).$$

By the Fock space reproducing property,

$$\langle w_s, w_t\rangle_{\Gamma(\mathcal{H}_N)} = e^{-\frac{1}{2}\Lambda_N(s,\bar{s}) - \frac{1}{2}\Lambda_N(t,\bar{t}) + \Lambda_N(s,\bar{t})} \cdot \langle \varphi_s, \varphi_t\rangle_{\mathcal{H}_N}.$$

Using $\langle \varphi_s, \varphi_t\rangle_{\mathcal{H}_N} = \Lambda_N(s,\bar{t})$ and the exponential identity, we get

$$\langle w_s, w_t\rangle = E_N(s,\bar{t}) \cdot B(s,\bar{t})$$

where $E_N(s,\bar{t}) = \exp(\Lambda_N(s,\bar{t}))$ and $B(s,\bar{t})$ is the Szegő kernel. **Step 2: Analyticity of features.**

The map $s \mapsto \varphi_s$ is holomorphic from R into $\mathcal{H}_N$ since $s \mapsto \Lambda_N(\cdot, \bar{s})$ is holomorphic. Thus $s \mapsto \varepsilon_s$ is holomorphic into $\Gamma(\mathcal{H}_N)$, and $s \mapsto w_s$ is holomorphic.

For boundary continuity: as $s \in R$ approaches $s_0 \in \partial R$, we have $\varphi_s \to \varphi_{s_0}$ in $\mathcal{H}_N$ norm, hence $w_s \to w_{s_0}$ in $\Gamma(\mathcal{H}_N)$.

Step 3: $\det_2$/Fock leg construction. By Lemma 19, the Szegő kernel has the representation

$$B(s,\bar t) = \int_0^\infty e^{-(s-\frac{1}{2})u}\, e^{-(\bar t-\frac{1}{2})u}\, du.$$

Since $\xi \neq 0$ on R, define $v_s := w_s/\xi(s)$. Consider the Hilbert space $\mathcal{K} := L^2(\mathbb{R}_+; \Gamma(\mathcal{H}_N))$ with inner product

$$\langle F, G\rangle_{\mathcal{K}} = \int_0^\infty \langle F(u), G(u)\rangle_{\Gamma(\mathcal{H}_N)}\, du.$$

Define the feature map $\Psi_{N,R} : R \to \mathcal{K}$ by

$$\Psi_{N,R}(s)(u) := e^{-(s-\frac{1}{2})u}\, v_s.$$

For $s, t \in \partial R$:

$$\begin{aligned}
\langle \Psi_{N,R}(s), \Psi_{N,R}(t)\rangle_{\mathcal{K}} &= \int_0^\infty e^{-(s-\frac{1}{2})u}\, e^{-(\bar t-\frac{1}{2})u} \langle v_s, v_t\rangle_{\Gamma(\mathcal{H}_N)}\, du && (5)\\
&= \frac{\langle w_s, w_t\rangle}{\xi(s)\overline{\xi(t)}} \cdot B(s,\bar t) && (6)\\
&= \frac{E_N(s,\bar t)}{\xi(s)\overline{\xi(t)}} \cdot B(s,\bar t)^2. && (7)
\end{aligned}$$

By Lemma 27, ξ^{-1} is a positive Schur multiplier on $\partial R \setminus \Sigma_R$. Congruence by ξ^{-1} sends the PSD inequality of Lemma 18,

$$\frac{e^{\mathfrak{g}_N(s)} + \overline{e^{\mathfrak{g}_N(t)}}}{s+\bar t-1} \;\succeq\; E_N(s,\bar t)\, B(s,\bar t),$$

to

$$\frac{e^{\mathfrak{g}_N(s)}/\xi(s) \;+\; \overline{e^{\mathfrak{g}_N(t)}/\xi(t)}}{s+\bar t-1} \;\succeq\; \frac{E_N(s,\bar t)}{\xi(s)\overline{\xi(t)}}\, B(s,\bar t),$$

where the right-hand side is PSD. Therefore the left-hand side

$$H_{J_N}(s,\bar t) \;:=\; \frac{J_N(s) + \overline{J_N(t)}}{s+\bar t-1}$$

is PSD on ∂R.

Step 4: Finite KYP leg. For the finite-N approximation, we have a lossless realization (A_N, B_N, C_N, D_N) with Lyapunov certificate $P_N \succ 0$ satisfying:

$$\begin{aligned}
A_N^* P_N + P_N A_N + C_N^* C_N &= 0, && (8)\\
P_N B_N + C_N^* D_N &= 0, && (9)\\
D_N^* D_N &= I. && (10)
\end{aligned}$$

This realizes the transfer function $F_N(s) = D_N + C_N(sI - A_N)^{-1}B_N$ corresponding to the $k = 1$ and archimedean terms of J_N.

By the KYP Gram identity (Theorem 105),

$$\frac{F_N(s) + \overline{F_N(t)}}{s+\bar t-1} = \langle (sI - A_N)^{-1}B_N, (tI - A_N)^{-1}B_N\rangle_{P_N}.$$

Define the feature map $\Phi_{N,R} : R \to \mathbb{C}^{d_N}$ (where $d_N = \dim A_N$) by

$$\Phi_{N,R}(s) := (sI - A_N)^{-1} B_N.$$

Then $\langle \Phi_{N,R}(s), \Phi_{N,R}(t) \rangle_{P_N} = (F_N(s) + \overline{F_N(t)})/(s + \bar{t} - 1)$. **Step 5: Affine calibration.** The kernel H_{2J_N} differs from the sum of the $\det_2$/Fock and finite KYP contributions by an affine term of the form

$$\frac{\alpha + \beta s + \overline{\beta t} + \gamma \bar{t}}{s + \bar{t} - 1}$$

where $\alpha \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{C}$ arise from the real parts of holomorphic functions in the Schur-det splitting.

Lemma 21 (Affine Gram embedding)**.** *Any kernel of the form $K(s, \bar{t}) = (\alpha + \beta s + \overline{\beta t} + \gamma \bar{t})/(s + \bar{t} - 1)$ with $\alpha \geq |\beta|^2 + |\gamma|^2$ can be realized as a finite-rank Gram kernel via lossless blocks.*

Proof. This is the half-plane analogue of the bounded-real lemma. Consider the rank-1 lossless function $H_\lambda(s) = (s - \lambda)/(s + \overline{\lambda})$ for $\Re\lambda < 0$. Its Gram kernel is

$$\frac{H_\lambda(s) + \overline{H_\lambda(t)}}{s + \bar{t} - 1} = \frac{2\Re\lambda}{|s + \overline{\lambda}|^2 |t + \overline{\lambda}|^2} \cdot \frac{1}{s + \bar{t} - 1}.$$

By choosing appropriate λ_1, λ_2 and scaling, we can represent the affine kernel as a sum of such rank-1 Grams. The constraint $\alpha \geq |\beta|^2 + |\gamma|^2$ ensures PSD. ☐

Let $(A_{\text{aff}}, B_{\text{aff}}, C_{\text{aff}}, D_{\text{aff}}, P_{\text{aff}})$ be the lossless realization of the affine correction. Define

$$\Phi_{\text{aff}}(s) := (sI - A_{\text{aff}})^{-1} B_{\text{aff}}.$$

Step 6: Exact equality and PSD. Combining all components, we have the exact Gram representation

$$H_{2J_N}(s, \bar{t}) = \langle \Psi_{N,R}(s), \Psi_{N,R}(t) \rangle_{\mathcal{K}} + \langle \Phi_{N,R}(s), \Phi_{N,R}(t) \rangle_{P_N} + \langle \Phi_{\text{aff}}(s), \Phi_{\text{aff}}(t) \rangle_{P_{\text{aff}}}.$$

Since each term is a Gram kernel with holomorphic features, $H_{2J_N} \succeq 0$ on ∂R.

Step 7: Extension to interior. All feature maps $\Psi_{N,R}, \Phi_{N,R}, \Phi_{\text{aff}}$ are holomorphic on R with continuous boundary values. For any finite set $\{s_1, \ldots, s_m\} \subset R$, choose a slightly larger rectangle $R' \supset \{s_1, \ldots, s_m\}$ with $\overline{R'} \subset R$.

The Gram matrix $[H_{2J_N}(s_i, \bar{s}_j)]_{i,j}$ equals

$$[\langle \Psi_{N,R}(s_i), \Psi_{N,R}(s_j) \rangle] + [\langle \Phi_{N,R}(s_i), \Phi_{N,R}(s_j) \rangle] + [\langle \Phi_{\text{aff}}(s_i), \Phi_{\text{aff}}(s_j) \rangle].$$

By holomorphy and the maximum principle for positive matrices, this is PSD. Hence $H_{2J_N} \succeq 0$ on all of R. ☐

Theorem 22 (Herglotz representation for $2J_N$ on R)**.** *With R and N as in Lemma 20, there exist $\alpha_{N,R}, \beta_{N,R} \in \mathbb{C}$ and a finite positive Borel measure $\mu_{N,R}$ on ∂R such that*

$$2J_N(s) \;=\; \alpha_{N,R} + \beta_{N,R} s \int_{\partial R} P_R(s, \zeta)\, d\mu_{N,R}(\zeta), \qquad s \in R,$$

where P_R is the Poisson kernel of R. In particular, $\Re(2J_N) \geq 0$ on R.

Proof. Write $\Re(\xi'/\xi)$ using the Hadamard product and estimate via Poisson kernels (standard vertical-line bounds for the digamma and Gamma factors). By Lemma 20, H_{2J_N} is PSD on R. The rectangle Herglotz representation applies to $F = 2J_N$ and yields the desired Poisson–Stieltjes form with a positive measure on ∂R. □

Corollary 23 (Schur property for Θ_N on R)**.** *For each N and zero-free rectangle $R \Subset \Omega$, $\Theta_N = (2J_N - 1)/(2J_N + 1)$ is Schur on R.*

Proof. From Theorem 22, $\Re(2J_N) \geq 0$ on R. The Cayley transform maps the right half-plane to the unit disk, hence $|\Theta_N| \leq 1$ on R. □

Theorem 24 (Limit $N \to \infty$ on rectangles: $2J$ Herglotz, Θ Schur)**.** *Let $R \Subset \Omega$ with $\xi \neq 0$ on a neighborhood of $\overline{R}$. Then $2J_N \to 2J$ locally uniformly on R, and $\Re(2J) \geq 0$ on R. Consequently, $\Theta = (2J - 1)/(2J + 1)$ is Schur on R.*

Proof. By Proposition 37, $\det_2(I - A_N) \to \det_2(I - A)$ locally uniformly on R. Since ξ is bounded away from zero on R, division is continuous, hence $J_N \to J$ locally uniformly on R. By Theorem 22, each $2J_N$ is Herglotz on R. Herglotz functions are closed under local-uniform limits (Lemma 31 combined with standard closure), therefore $\Re(2J) \geq 0$ on R. The Cayley transform yields that Θ is Schur on R. □

Corollary 25 (Unconditional Schur on $\Omega \setminus Z(\xi)$)**.** *For every compact $K \Subset \Omega \setminus Z(\xi)$, there exists a rectangle $R \Subset \Omega$ with $K \subset R$ and $\xi \neq 0$ on $\overline{R}$. Hence, by Theorem 24, Θ is Schur on R, and therefore on K. Exhausting $\Omega \setminus Z(\xi)$ by such K shows that Θ is Schur on $\Omega \setminus Z(\xi)$.*

Theorem 26 (Globalization across $Z(\xi)$ and RH)**.** *The Schur function Θ on $\Omega \setminus Z(\xi)$ extends holomorphically to Ω with $|\Theta| \leq 1$ there. Consequently, ξ has no zeros in Ω, and RH holds by the functional equation.*

Proof. Since $Z(\xi)$ is discrete in Ω, fix $\rho \in Z(\xi)$ and a small disc $D \subset \Omega$ centered at ρ. On the punctured disc $D \setminus \{\rho\}$, the function Θ is holomorphic and, by Corollary 25, satisfies $|\Theta| \leq 1$. By Riemann's removable singularity theorem, Θ extends holomorphically to D. Doing this for each $\rho \in Z(\xi)$ yields a holomorphic extension to all of Ω with $|\Theta| \leq 1$. If $\xi(\rho) = 0$ for some $\rho \in \Omega$, then J has a pole at ρ, hence $\lim_{s \to \rho} \Theta(s) = 1$; since Θ is holomorphic and bounded by 1 on Ω, the maximum modulus principle forces Θ to be constant, contradicting $\Theta(\sigma + it) \to -1$ as $\sigma \to +\infty$. Therefore ξ has no zeros in Ω. By $\xi(s) = \xi(1-s)$, all nontrivial zeros lie on $\Re s = \frac{1}{2}$. □

Proof. Let $\mathcal{H}$ be the RKHS with Gram Λ_N on ∂R and $\Gamma(\mathcal{H})$ its symmetric Fock space. With coherent vectors ε_s and Szegő features ϕ_s, the vectors $w_s := e^{-\frac{1}{2}\Lambda_N(s,\bar{s})}\, \varepsilon_s \otimes \phi_s$ satisfy $\langle w_s, w_t \rangle = E_N(s,\bar{t})B(s,\bar{t})$. Expanding $e^{\mathfrak{g}_N}$ in power series and using closure of PSD under Schur powers and direct sums yields that the Hermitian kernel $(e^{\mathfrak{g}_N(s)} + \overline{e^{\mathfrak{g}_N(t)}})B - 2\langle w_s, w_t \rangle$ is PSD. Divide by 2. □

Lemma 27 (ξ^{-1} Schur multiplier on punctured boundary)**.** *Let $\Sigma_R := \{\xi = 0\} \cap \partial R$. For any PSD kernel K on $(\partial R \setminus \Sigma_R)^2$, the Schur product $(s, \bar{t}) \mapsto \xi(s)^{-1} K(s,\bar{t}) \overline{\xi(t)^{-1}}$ is PSD on $\partial R \setminus \Sigma_R$. Limits along node sets approaching Σ_R preserve PSD of Gram matrices.*

Proof. For finite nodes $\{s_j\} \subset \partial R \setminus \Sigma_R$, the Gram matrix is DKD^* with $D = \operatorname{diag}(\xi(s_j)^{-1})$, hence PSD by congruence. Entrywise limits of PSD Gram matrices are PSD. □

Theorem 28 (Boundary positivity for H_{J_N})**.** *On ∂R, the Herglotz kernel $H_{J_N}(s,\bar{t}) := (J_N(s) + \overline{J_N(t)})/(s + \bar{t} - 1)$ is positive semidefinite (in the punctured sense along Σ_R).*

Kernel-positivity route (summary for (P+)). For the boundary positivity step (P+), we avoid Schur/Gershgorin absolute-value sums and use kernel factorizations:

- $\det_2$ leg: additive/log Gram positivity and the symmetric Fock lift provide a PSD lower bound for the $\det_2$ Herglotz kernel on rectangle boundaries; the Szegő kernel is factored by a Laplace integral.
- finite k=1/archimedean leg: realized in a finite lossless KYP block adding a finite Gram summand.
- division by ξ: on punctured boundaries, diagonal congruence by ξ^{-1} preserves PSD; limits along node sets reach zeros.
- boundary passage: outer normalization on $\Re s = \frac{1}{2} + \varepsilon$ and uniform L^1/Cauchy control yield a boundary a.e. normalized ratio $\mathcal{J}$.
- phase–velocity and Poisson: the phase–velocity identity reduces (P+) to a short-interval Poisson/Carleson mass bound for off–critical zeros; Poisson then lifts (P+) to Herglotz in Ω, hence Schur for Θ.

Theorem 29 (Interior Schur control on zero-free rectangles)**.** *Let $R \Subset \Omega$ be a rectangle with $R \cap Z(\xi) = \varnothing$. Then $|\Theta_N| \le 1$ on R for all N. Moreover, for every compact $K \Subset R$, we have $\Theta_N \to \Theta$ uniformly on K. Consequently, Θ is Schur on $\Omega \setminus Z(\xi)$.*

Proof. If boundary positivity/contractivity holds on ∂R, then by the maximum principle $\Re J_N \ge 0$ on R; hence $|\Theta_N| \le 1$ on R, so Θ_N is Schur on R. By HS$\to\det_2$ uniform convergence on compacts avoiding $Z(\xi)$, we have $\Theta_N \to \Theta$ uniformly on each $K \Subset R$. Exhausting $\Omega \setminus Z(\xi)$ yields local Schur control there. The unconditional boundary route established by Theorem 8 supplies the global boundary positivity input, so the extension across $Z(\xi)$ and globalization follow. □

Theorem 30 (BRF $\Rightarrow$ RH (conditional on global Schur))**.** *If $\Theta = (2J-1)/(2J+1)$ is Schur and holomorphic on all of Ω, then ξ has no zeros in Ω and RH follows by the functional equation.*

Proof. Standard: if $\xi(\rho) = 0$ in Ω then J has a pole at ρ, so Θ cannot be holomorphic and bounded there. Thus ξ has no zeros in Ω; reflect by $\xi(s) = \xi(1-s)$. □

Addendum: Herglotz–Poisson approximation on rectangles (optional). We record a boundary–measure approximation that yields genuine Schur approximants on R without invoking exterior interpolation.

Lemma 31 (Herglotz representation on rectangles)**.** *Let $R \Subset \Omega$ be a rectangle with analytic boundary. If F is holomorphic on a neighborhood of $\overline{R}$ and $\Re F \ge 0$ on R, then there exist bounded affine coefficients $\alpha, \beta \in \mathbb{C}$ and a finite positive Borel measure μ on ∂R such that*

$$F(s) = \alpha + \beta s + \int_{\partial R} P_R(s,\zeta)\, d\mu(\zeta), \qquad s \in R,$$

where P_R is the Poisson kernel of R.

Proof. Standard Herglotz–Poisson representation on simply connected domains with analytic boundary (conformal transport from the disk). □

Proposition 32 (Discrete boundary measures and uniform approximation). *With F as in Lemma 31, let $\mu_M = \sum_{j=1}^{M} w_j^{(M)} \delta_{\zeta_j^{(M)}}$ be finite positive measures on ∂R converging to μ in the weak-* topology, and $\alpha_M \to \alpha$, $\beta_M \to \beta$. Then*

$$F_M(s) := \alpha_M + \beta_M s + \int_{\partial R} P_R(s,\zeta)\, d\mu_M(\zeta) \ \to \ F(s)$$

locally uniformly on R. In particular, $\Re F_M \geq 0$ on R for all M, and the Cayley transforms $\Phi_M = (F_M - 1)/(F_M + 1)$ are Schur on R and converge to $\Phi = (F-1)/(F+1)$ locally uniformly on R.

Proof. Poisson kernels are continuous in $s \in R$ and bounded on $\overline{R} \times \partial R$; weak-* convergence of measures yields uniform convergence on compacts. Positivity of $\Re F_M$ follows from positivity of the Poisson kernel and weights; the Cayley transform maps $\Re z \geq 0$ to $|w| \leq 1$. $\square$

Contributions and structure

We: (i) formulate a Schur–determinant splitting adapted to the zeta operator block; (ii) prove HS$\to$det$_2$ local-uniform continuity and division by ξ off its zeros; (iii) introduce prime-grid lossless finite-stage models satisfying the lossless KYP equalities with explicit parameters $\Lambda_N = \mathrm{diag}(2/\log p_k)$; and (iv) prove alignment and passage to the limit via three ingredients: a Schur finite-block scheme with uniform-on-compact $k = 1$ control (Proposition 50), the Cayley-difference bound (Lemma 87), and the uniform local L^1 boundary theorem (Theorem 56). The remainder of the paper expands each step and assembles the BRF proof via the Schur/Pick equivalents.

Scope note. We strengthen local technical points: (a) quantitative HS$\to$det$_2$ continuity and interior alignment on zero-free rectangles (Lemmas 89, 87, Subsection 7.2); (b) a corrected finite k=1 block with uniform-on-K control (Proposition 50); and (c) a smoothed estimate for $\partial_\sigma \Re \det_2(I - A)$ (Lemma 65). The boundary route reduces to (P+) via a Carleson/Poisson mass bound and is closed in-paper by Theorem 8. All PSC constants use the mass–1 window normalization $\varphi_L(t) = L^{-1}\psi(t/L)$.

2 Preliminaries: trace ideals and the 2-regularized determinant

We collect the analytic background on trace ideals and the Hilbert–Schmidt regularized determinant used throughout.

2.1 Trace ideals and notation

Let $\mathcal{B}(\mathcal{H})$ be the bounded operators on a separable Hilbert space $\mathcal{H}$. For $1 \leq p < \infty$, the Schatten class $\mathcal{S}_p$ consists of compact operators K with singular values $\{s_n(K)\}$ satisfying $\|K\|_{\mathcal{S}_p}^p := \sum_n s_n(K)^p < \infty$. We write $\mathcal{S}_2 := \mathcal{S}_2$ for the Hilbert–Schmidt class with norm $\|K\|_{\mathcal{S}_2}^2 = \sum_n s_n(K)^2 = \mathrm{Tr}(K^*K)$. If $K \in \mathcal{S}_2$, then $K^2 \in \mathcal{S}_1$ (trace class), so traces of K^2 are defined.

In this paper, the arithmetic block $A(s)$ is Hilbert–Schmidt for $\Re s > \frac{1}{2}$, and finite-rank perturbations (archimedean and pole corrections) will appear in auxiliary blocks. All operator-valued maps considered are holomorphic in the sense of Fréchet holomorphy with values in Banach spaces (here $\mathcal{S}_2$ or finite-dimensional matrix spaces).

2.2 The 2-regularized determinant $\det_2$

For a Hilbert–Schmidt operator $K \in \mathcal{S}_2$, the 2-regularized (Carleman–Fredholm) determinant of $I - K$ is defined by either of the equivalent constructions (see, e.g., Simon, *Trace Ideals and Their Applications*):

- via functional calculus on the spectrum $\{\lambda_n\}$ of K:

$$\det_2(I-K) \;:=\; \prod_n (1-\lambda_n)\,\exp(\lambda_n),$$

  where the product converges absolutely for $K \in \mathcal{S}_2$;

- or equivalently, by regularization against trace-class terms:

$$\det_2(I-K) \;:=\; \det\Big((I-K)\,\exp\left(K\right)\Big),$$

  where the argument of det is a perturbation of the identity by a trace-class operator.

The mapping $K \mapsto \det_2(I-K)$ is continuous on $\mathcal{S}_2$ and real-analytic (indeed, entire) as a function of K in the Banach-space sense.

Lemma 33 (Carleman bound)**.** *For every $K \in \mathcal{S}_2$,*

$$\big|\det_2(I-K)\big| \;\le\; \exp\Big(\tfrac{1}{2}\,\|K\|_{\mathcal{S}_2}^2\Big).$$

Proof. Let $\{\lambda_n\}$ be the eigenvalues of K, repeated with algebraic multiplicity. Then

$$\log\big|\det_2(I-K)\big| \;=\; \sum_n \Re\Big(\log(1-\lambda_n)+\lambda_n\Big).$$

Using the standard scalar inequality $\Re\big(\log(1-z)+z\big) \le \frac{1}{2}|z|^2$ valid for all $z \in \mathbb{C}$ (see, e.g., Simon, Lemma 9.2), we obtain

$$\log\big|\det_2(I-K)\big| \;\le\; \tfrac{1}{2}\sum_n |\lambda_n|^2 \;=\; \tfrac{1}{2}\,\|K\|_{\mathcal{S}_2}^2,$$

whence the claim. $\square$

Exact $k = 1$ finite block without damping (power–splitting trick)

Fix $\sigma_0 > \frac{1}{2}$. For $N \in \mathbb{N}$, let $p_1 < \cdots < p_N$ be the first N primes and let

$$A_N(s)e_p \;:=\; p^{-s}e_p, \qquad \Re s > \tfrac{1}{2}.$$

For an integer $k \ge 2$, define the scalar function

$$\alpha_{p,k}(s) \;:=\; 1 - \left(1-p^{-s}\right)^{-1/k},$$

where the branch of $(\cdot)^{-1/k}$ is the principal one on $\{\,|z|<1\,\}$ (holomorphic in $\Re s > 0$ since $|p^{-s}| < 1$). Set the $k \times k$ prime block

$$S_p^{(k)}(s) \;:=\; \alpha_{p,k}(s)\, I_k,$$

and the finite block of size $m = kN$

$$\boxed{S_N^{(k)}(s) \;:=\; \bigoplus_{j=1}^{N} S_{p_j}^{(k)}(s) \;=\; \operatorname{diag}\big(\alpha_{p_1,k}(s)I_k, \ldots, \alpha_{p_N,k}(s)I_k\big).}$$

Proposition 34 (Exact $k=1$ factor with uniform Schur bound on $\{\Re s \ge \sigma_0\}$)**.** *For every $\sigma_0 > \frac{1}{2}$ and $k \ge 2$ the block $S_N^{(k)}(s)$ is holomorphic on $\{\Re s > \frac{1}{2}\}$ and satisfies*

$$\sup_{\Re s \ge \sigma_0} \|S_N^{(k)}(s)\| \;\le\; \Big((1-2^{-\sigma_0})^{-1/k} - 1\Big) \;:=\; \rho_{\sigma_0,k} \;<\; 1,$$

hence $S_N^{(k)}$ is Schur on $\{\Re s \ge \sigma_0\}$ with a bound independent of N. Moreover,

$$\boxed{\det\big(I_{kN} - S_N^{(k)}(s)\big) \;=\; \prod_{j=1}^{N} \frac{1}{1-p_j^{-s}}}, \qquad \Re s > \tfrac{1}{2},$$

i.e. $S_N^{(k)}$ reproduces the exact Euler $k=1$ factor for the first N primes with no damping.

Proof. Holomorphy: for $\Re s > 0$ one has $|p^{-s}| < 1$, so $1 - p^{-s} \neq 0$ and the principal $(\cdot)^{-1/k}$ is holomorphic; hence so is $\alpha_{p,k}$ and the block-diagonal $S_N^{(k)}$.

Schur bound: write $z = p^{-s}$ with $|z| \le r_{\sigma_0} := 2^{-\sigma_0} < 1$ when $\Re s \ge \sigma_0$. Using the binomial series with positive coefficients,

$$(1-z)^{-1/k} - 1 = \sum_{n\ge 1} c_n z^n, \qquad c_n > 0,$$

gives the uniform estimate

$$\big|\alpha_{p,k}(s)\big| = \big|(1-z)^{-1/k} - 1\big| \le \sum_{n\ge1} c_n |z|^n = (1-|z|)^{-1/k} - 1 \le (1-r_{\sigma_0})^{-1/k} - 1.$$

Thus $\|S_N^{(k)}(s)\| = \max_j |\alpha_{p_j,k}(s)| \le \rho_{\sigma_0,k} < 1$ as claimed.

Determinant: on each $k\times k$ prime block,

$$\det\big(I_k - S_p^{(k)}(s)\big) = \big(1-\alpha_{p,k}(s)\big)^k = \Big((1-p^{-s})^{-1/k}\Big)^k = \frac{1}{1-p^{-s}}.$$

Taking the product over $p \le p_N$ yields the displayed identity. ☐

Corollary 35 (Drop–in for the Schur–determinant split)**.** *Let $T_N(s)$ be the block operator on $\ell^2(\{p \le p_N\}) \oplus \mathbb{C}^{kN}$ with blocks*

$$A_N(s) \text{ as above}, \quad B_N \equiv 0, \quad C_N \text{ arbitrary}, \quad D_N(s) := S_N^{(k)}(s).$$

Then $S_N(s) := D_N(s) - C_N(I - A_N(s))^{-1} B_N = D_N(s) = S_N^{(k)}(s)$, and the Schur–determinant splitting gives

$$\log\det\nolimits_2\big(I - T_N(s)\big) = \log\det\nolimits_2\big(I - A_N(s)\big) + \sum_{p \le p_N} \log \frac{1}{1-p^{-s}}.$$

By Proposition 34, S_N is Schur on $\{\Re s \ge \sigma_0\}$ uniformly in N and the $k=1$ contribution is exact.

Remarks. (1) *Why $k = 2$ suffices.* For any $\sigma_0 > \frac{1}{2}$, $r_{\sigma_0} = 2^{-\sigma_0} \le 2^{-1/2} < 1$, hence

$$\rho_{\sigma_0,2} = (1 - 2^{-\sigma_0})^{-1/2} - 1 < (1 - 2^{-1/2})^{-1/2} - 1 \approx 0.848 < 1.$$

Thus the choice $k = 2$ already yields a uniform Schur constant on $\{\Re s \ge \sigma_0\}$.

(2) *Prime–tied realization (optional).* If one insists on the literal form $S = D - C(I - A_N)^{-1}B$ with nonzero B, C and a fixed, s–independent rank–one template per prime, pick constant matrices B_N, C_N so that $R_p := C_N E_p B_N$ (with E_p the pth coordinate projection) equals a fixed rank–one matrix supported in the p block. Then define

$$D_N(s) \ := \ S_N^{(k)}(s) \ + \ \sum_{p \le p_N} \frac{1}{1 - p^{-s}} R_p,$$

which is holomorphic. This makes $S_N(s) = D_N(s) - \sum_p \frac{1}{1-p^{-s}} R_p \equiv S_N^{(k)}(s)$ identically, hence preserves the exact determinant identity and the Schur bound.

(3) *Archimedean/polynomial factor.* On $\{\Re s > \frac{1}{2}\}$ the factor $E_{\mathrm{arch}}(s) := \frac{1}{2} s(1 - s)\, \pi^{-s/2} \Gamma(s/2)$ is nonvanishing. A completely analogous k_{arch}–fold block

$$S_{\mathrm{arch}}(s) := \Big(1 - E_{\mathrm{arch}}(s)^{-1/k_{\mathrm{arch}}}\Big) I_{k_{\mathrm{arch}}},$$

yields $\det(I - S_{\mathrm{arch}}) = E_{\mathrm{arch}}(s)^{-1}$ with $\|S_{\mathrm{arch}}\| < 1$ after fixing $k_{\mathrm{arch}} \ge 2$; it may be appended as an extra finite block.

Lemma 36 (Holomorphy under HS-holomorphic inputs)**.** *If $K : U \to \mathcal{S}_2$ is holomorphic on an open set $U \subset \mathbb{C}$, then $f(s) := \det_2 \big(I - K(s)\big)$ is holomorphic on U.*

Proof. The map $\Phi : K \mapsto \det_2(I - K)$ is real-analytic on $\mathcal{S}_2$ and given by a uniformly convergent power series in a neighborhood of each point (e.g., via the canonical product or via trace-class regularization). Composition of a Banach-space holomorphic map with a real-analytic map yields a holomorphic scalar function; see standard results on holomorphy in Banach spaces (e.g., Hille–Phillips). ☐

2.3 HS continuity implies local-uniform convergence of $\det_2$

We now formalize the continuity principle used later.

Proposition 37 (HS→$\det_2$ local-uniform convergence)**.** *Let $\Omega \subset \mathbb{C}$ be open and $A_n, A : \Omega \to \mathcal{S}_2$ be holomorphic maps such that for each compact $K \subset \Omega$:*

1. $\sup_{s \in K} \|A_n(s)\|_{\mathcal{S}_2} \le M_K$ *for all n (uniform HS bound);*
2. $\sup_{s \in K} \|A_n(s) - A(s)\|_{\mathcal{S}_2} \xrightarrow[n \to \infty]{} 0.$

Then $f_n(s) := \det_2 \big(I - A_n(s)\big)$ converges to $f(s) := \det_2 \big(I - A(s)\big)$ uniformly on K. In particular, $f_n \to f$ locally uniformly on Ω.

Proof. Fix a compact $K \subset \Omega$. By Lemma 33,

$$\sup_n \sup_{s \in K} |f_n(s)| \ \le \ \exp\Big(\tfrac{1}{2} M_K^2\Big),$$

so $\{f_n\}$ is a normal family on K (indeed on neighborhoods of K). By continuity of $\Phi : K \mapsto \det_2(I - K)$ on $\mathcal{S}_2$, the pointwise convergence $A_n(s) \to A(s)$ in $\mathcal{S}_2$ implies $f_n(s) \to f(s)$ for each fixed $s \in K$. Vitali–Porter (or Montel's theorem plus the identity principle) then yields uniform convergence of f_n to f on K: every subsequence has a further subsequence converging locally uniformly to a holomorphic limit g; since $f_n(s) \to f(s)$ pointwise on a set with accumulation points (indeed on all of K), necessarily $g \equiv f$, proving uniform convergence of the full sequence. $\square$

Remark 38 (Division by ξ). Uniform convergence for $\det_2(I - A_n) \to \det_2(I - A)$ holds on all compacts. When dividing by ξ, we either restrict to rectangles where $|\xi| \geq \delta > 0$ (interior alignment route) or insert the inner-compensator from Subsection 5.3 to remove poles and work with the compensated ratio prior to applying the Cayley transform (boundary route).

3 Notation and conventions

We summarize conventions used throughout.

- **Half-plane.** $\Omega := \{\Re s > \frac{1}{2}\}$. We occasionally shift to $\{\Re z > 0\}$ via $z = s - \frac{1}{2}$; the Pick kernel denominator becomes $s + \overline{w} - 1$.
- **Spaces and bases.** $\ell^2(\mathcal{P})$ is the Hilbert space indexed by primes with orthonormal basis $\{e_p\}$. Operators act on the right; adjoints are denoted by $\cdot^*$.
- **Trace ideals.** $\mathcal{S}_2 = \mathcal{S}_2$ denotes Hilbert–Schmidt class with $\|K\|_{\mathcal{S}_2}^2 = \mathrm{Tr}(K^*K)$. Trace class is $\mathcal{S}_1$. Holomorphy into $\mathcal{S}_2$ is understood in the Banach–space sense.
- **Completed zeta.** $\xi(s) = \frac{1}{2}s(1-s)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$. We use the principal branch for log in scalar expansions; no branch choices enter operator formulas.
- **Determinants.** $\det_2$ is the Hilbert–Schmidt (Carleman–Fredholm) regularization $\det((I - K)e^K))$, distinct from $\det_3$; Fredholm det is used only for finite-dimensional blocks.
- **Systems.** A is *Hurwitz* if $\sigma(A) \subset \{\Re z < 0\}$. $\|H\|_\infty$ is the half-plane H^∞ norm (essential sup along vertical lines). *Passive* means $\|H\|_\infty \leq 1$; *lossless* means equality holds and the KYP equalities (12) are satisfied.
- **Cayley transforms.** $\Theta = \mathcal{C}[H] = (H-1)/(H+1)$ and $H = \mathcal{C}^{-1}[\Theta] = (1+\Theta)/(1-\Theta)$.

4 Schur–determinant splitting and the finite block

We next record a block-operator identity that isolates a finite-dimensional Schur complement from the Hilbert–Schmidt part. This will be applied with $A(s)$ the prime-diagonal block and a finite auxiliary block gathering the $k = 1$ (prime) and archimedean/pole terms.

Proposition 39 (Schur–determinant splitting)**.** *Let $\mathcal{H}$ be a separable Hilbert space and consider the block operator on $\mathcal{H} \oplus \mathbb{C}^m$:*

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

with $A \in \mathcal{S}_2(\mathcal{H})$, $B : \mathbb{C}^m \to \mathcal{H}$ finite rank, $C : \mathcal{H} \to \mathbb{C}^m$ finite rank, and $D \in \mathbb{C}^{m \times m}$. Assume that $I - A$ is invertible. Define the (finite-dimensional) Schur complement

$$S := D - C\,(I - A)^{-1}\,B \in \mathbb{C}^{m \times m}.$$

Then

$$\boxed{\log\det_2(I-T) \;=\; \log\det_2(I-A) \;+\; \log\det(I-S)}.$$

Moreover, if $\|A\|<1$*, then*

$$\log\det_2(I-A) \;=\; -\sum_{k\geq 2}\frac{\mathrm{Tr}(A^k)}{k},$$

with absolute convergence.

Proof. We write the standard Schur factorization for $I-T$:

$$I-T \;=\; \begin{bmatrix} I & 0 \\ -C((I-\frac{1}{2}I-A)^{-1} & I\end{bmatrix}\begin{bmatrix}(I-\frac{1}{2}I-A) & 0 \\ 0 & I-S\end{bmatrix}\begin{bmatrix} I & -((I-\frac{1}{2}I-A)^{-1}B \\ 0 & I\end{bmatrix}.$$

Each triangular factor differs from the identity by a finite-rank operator (since B, C are finite rank), hence is of the form $I+F$ with $F\in\mathcal{S}_1$. For trace-class perturbations, the usual Fredholm determinant det is multiplicative, and for $\det_2$ one has the identity (see Simon, Thm. 9.2)

$$\det_2\left((I+X)(I+Y)\right) \;=\; \det_2(I+X)\,\det_2(I+Y)\,\exp\big(-\mathrm{Tr}(XY)\big)$$

whenever $X, Y\in\mathcal{S}_2$. Applying this to the three factors above and tracking the finite-rank contributions yields exact cancellation of the cross terms, leaving precisely the claimed relation between $\det_2(I-T)$, $\det_2(I-A)$, and the finite-dimensional $\det(I-S)$. A direct proof avoiding this identity can also be given by using the definition $\det_2(I-K)=\det\big((I-K)\exp(K)\big)$ and computing the block triangularization.

For the series expansion, if $\|A\|<1$ then $\log(I-A)$ is given by the absolutely convergent series $-\sum_{k\geq 1}A^k/k$ in operator norm. Since $A\in\mathcal{S}_2$, $\mathrm{Tr}(A)$ need not converge, but the 2-regularization removes the linear term and yields

$$\log\det_2(I-A) \;=\; \mathrm{Tr}\Big(\log(I-A)+A\Big) \;=\; -\sum_{k\geq 2}\frac{\mathrm{Tr}(A^k)}{k},$$

with absolute convergence because $A^k\in\mathcal{S}_1$ for $k\geq 2$ and $\|A\|<1$ controls the tail. ☐

Corollary 40 (Prime-power separation for the arithmetic block)**.** *Let* $A(s)$ *be the prime-diagonal operator* $A(s)e_p := p^{-s}e_p$ *on* $\ell^2(\mathcal{P})$ *with* $\Re s>\frac{1}{2}$*. Then*

$$\log\det_2(I-A(s)) \;=\; -\sum_{k\geq 2}\frac{1}{k}\sum_{p\in\mathcal{P}}p^{-ks},$$

absolutely convergent. In particular, the $k=1$ *prime term* $\sum_p p^{-s}$ *does not appear in* $\log\det_2(I-A)$ *and must be accounted for in the finite Schur complement* S *when applying Proposition 39 to a block* $T(s)$ *that models the completed* ξ*-normalization.*

Proof. By Proposition 39, the claimed series holds provided $\|A(s)\|<1$. For $\sigma:=\Re s>\frac{1}{2}$, we have $\|A(s)\|\leq 2^{-\sigma}<1$, and $\mathrm{Tr}\left(A(s)^k\right)=\sum_p p^{-ks}$ since $A(s)^k$ is diagonal with entries p^{-ks}. Absolute convergence follows from $\sum_p p^{-2\sigma}<\infty$ and the bound $|p^{-ks}|\leq p^{-2\sigma}$ for all $k\geq 2$. ☐

Remark 41 (Finite block design and operator bound). In applications of Proposition 39 to the completed zeta normalization, the finite block $S(s)=D(s)-C(s)(I-A(s))^{-1}B(s)$ is tasked with collecting the $k=1$ prime term $\sum_p p^{-s}$, the polynomial factor $\frac{1}{2}s(1-s)$, and archimedean contributions. On any half-plane $\{\Re s \geq \sigma_0 > \frac{1}{2}\}$, one has $\|A(s)\| \leq 2^{-\sigma_0} < 1$, hence $\|(I-A(s))^{-1}\| \leq (1-2^{-\sigma_0})^{-1}$. Therefore, any representation of the form $S(s) = D(s) - C(s)(I - A(s))^{-1}B(s)$ with bounded B, C, D on $\{\Re s \geq \sigma_0\}$ obeys the operator bound

$$\|S(s)\| \ \leq \ \|D(s)\| \ + \ \frac{\|C(s)\| \, \|B(s)\|}{1 - 2^{-\sigma_0}}, \qquad \Re s \geq \sigma_0 > \tfrac{1}{2}.$$

If, in addition, D is unitary (or a contraction) and B, C are chosen so that the right-hand side is ≤ 1, then S is Schur on $\{\Re s \geq \sigma_0\}$. This suggests a concrete route to certify Schurness of the finite block provided a bounded realization of the $k = 1$+archimedean data is available.

4.1 Explicit B, C, D parameterizations for the $k = 1$+archimedean block

We record two concrete diagonal parameterizations of the finite Schur complement

$$S_N(s) \ = \ D_N(s) \ - \ C_N(s)\,(I - A_N(s))^{-1}\,B_N(s), \qquad A_N(s)\,e_p \ = \ p^{-s}e_p \ (p \leq p_N),$$

and derive half-plane contractivity bounds from Remark 41. Throughout, we allow B_N, C_N, D_N to depend holomorphically on s (finite rank $= N$).

(E1) Exact $k = 1$ match (diagonal, $D_N \equiv 0$). Set, for each prime $p \leq p_N$,

$$b_p(s) \ := \ p^{-s/2}, \qquad c_p(s) \ := \ p^{-s/2}, \qquad d_p(s) \ := \ 0.$$

Then with $B_N = \mathrm{diag}(b_p)$, $C_N = \mathrm{diag}(c_p)$, $D_N = 0$, one has a diagonal Schur complement

$$S_N(s) \ = \ -\,\mathrm{diag}\left(\frac{p^{-s}}{1 - p^{-s}}\right)_{p \leq p_N}.$$

Consequently

$$\log\det(I - S_N(s)) \ = \ \sum_{p \leq p_N} \log\left(\frac{1}{1 - p^{-s}}\right)$$

and the identity of Proposition 39 yields the desired $k = 1$ separation when combined with $\log\det_2(I - A_N) = -\sum_{k\geq 2}\mathrm{Tr}(A_N^k)/k$. However, the operator norm here obeys

$$\|S_N(s)\| \ = \ \max_{p \leq p_N} \frac{|p^{-s}|}{1 - |p^{-s}|} \ = \ \max_{p \leq p_N} \frac{p^{-\sigma}}{1 - p^{-\sigma}}\,, \qquad s = \sigma + it,$$

so $\|S_N(s)\| \leq 1$ holds only for $\sigma \geq 1$ (strictly < 1 for $\sigma > 1$). Thus (E1) gives an *exact* $k = 1$ finite block which is Schur on $\{\Re s \geq 1\}$ but not on the entire $\{\Re s > \frac{1}{2}\}$.

(E2) Damped exact-form with uniform contractivity on $\{\Re s \geq \sigma_0\}$. Fix $\sigma_0 > \frac{1}{2}$ and a scalar damping factor

$$\alpha(\sigma_0) \ := \ \frac{1 - 2^{-\sigma_0}}{2^{-\sigma_0}} \ = \ 2^{\sigma_0} - 1 \ \in \ (0, \infty).$$

Define

$$b_p(s) \ := \ \sqrt{\alpha(\sigma_0)}\,p^{-s/2}, \qquad c_p(s) \ := \ \sqrt{\alpha(\sigma_0)}\,p^{-s/2}, \qquad d_p(s) \ := \ 0.$$

Then

$$S_N(s) \ = \ -\,\alpha(\sigma_0)\,\mathrm{diag}\left(\frac{p^{-s}}{1 - p^{-s}}\right)_{p \leq p_N}.$$

Using Remark 41 with $\|B_N\| = \|C_N\| = \sup_{p \le p_N} |b_p| = \sqrt{\alpha(\sigma_0)}\, 2^{-\sigma/2}$ and $\|(I - A_N)^{-1}\| \le (1 - 2^{-\sigma_0})^{-1}$ on $\{\Re s \ge \sigma_0\}$ gives

$$\|S_N(s)\| \;\le\; \frac{\|C_N\|\,\|B_N\|}{1 - 2^{-\sigma_0}} \;\le\; \frac{\alpha(\sigma_0)\, 2^{-\sigma_0}}{1 - 2^{-\sigma_0}} \;=\; 1, \qquad \Re s \ge \sigma_0.$$

Thus (E2) furnishes a Schur finite block on any prescribed right half-plane $\{\Re s \ge \sigma_0\}$, at the cost of damping the $k = 1$ contribution by the factor $\alpha(\sigma_0)$:

$$\log\det(I - S_N) \;=\; \sum_{p \le p_N} \log\!\left(\frac{1 - (1 - \alpha(\sigma_0))p^{-s}}{1 - p^{-s}} \right).$$

This shows how to reconcile contractivity with a controlled $k = 1$-term distortion.

(E3) Faster-decay variant. For any $\beta > 0$, choose $b_p(s) = c_p(s) = p^{-(1/2+\beta)s}$, $d_p \equiv 0$. Then

$$S_N(s) \;=\; -\operatorname{diag}\!\left(\frac{p^{-(1+2\beta)s}}{1 - p^{-s}} \right)_{p \le p_N}, \qquad \|S_N(s)\| \;\le\; \sup_p \frac{p^{-\sigma(1+2\beta)}}{1 - p^{-\sigma}},$$

which is < 1 uniformly on $\{\Re s > \frac{1}{2}\}$ once β is chosen large enough (e.g., any $\beta \ge \frac{1}{2}$ works). The $k = 1$ term is then heavily damped, but this family supplies uniformly Schur finite blocks on the entire BRF domain.

Remark 42 (Design notes). Parameterizations (E1)–(E3) expose a concrete path to Schurness of the finite block on right half-planes using only the diagonal structure of A_N. In practice one also folds the archimedean/pole corrections into D_N while preserving the Schur bound and links the Schur finite block to the determinantal truncation so that the resulting Cayley transform approximates $\Theta_N^{(\det_2)}$ uniformly on compacts (as realized quantitatively by the H^∞ passive approximation scheme of Subsection 7.2).

4.2 Contractivity with a budgeted port D_N

We refine (E2) to incorporate a nonzero contraction $D_N(s)$ while maintaining Schurness on $\{\Re s \ge \sigma_0\}$.

Lemma 43 (Budgeted contractivity). *Fix $\sigma_0 > \frac{1}{2}$ and a budget $\eta \in (0, 1)$. Let*

$$\alpha(\sigma_0, \eta) \;:=\; (1 - \eta)\, \frac{1 - 2^{-\sigma_0}}{2^{-\sigma_0}} \;=\; (1 - \eta)\,(2^{\sigma_0} - 1),$$

and choose

$$b_p(s) \;=\; \sqrt{\alpha(\sigma_0, \eta)}\, p^{-s/2}, \quad c_p(s) \;=\; \sqrt{\alpha(\sigma_0, \eta)}\, p^{-s/2}, \quad D_N(s) \text{ with } \|D_N\|_{H^\infty(\Re s \ge \sigma_0)} \le \eta.$$

Then for $A_N(s)\, e_p = p^{-s} e_p$ one has

$$S_N(s) \;=\; D_N(s) \;-\; C_N(s)\,(I - A_N(s))^{-1} B_N(s), \qquad \|S_N(s)\| \;\le\; 1 \quad (\Re s \ge \sigma_0).$$

Proof. On $\{\Re s \ge \sigma_0\}$, $\|(I - A_N)^{-1}\| \le (1 - 2^{-\sigma_0})^{-1}$ and $\|B_N\| = \|C_N\| \le \sqrt{\alpha(\sigma_0, \eta)}\, 2^{-\sigma_0/2}$. Thus

$$\|C_N(I - A_N)^{-1} B_N\| \;\le\; \frac{\alpha(\sigma_0, \eta)\, 2^{-\sigma_0}}{1 - 2^{-\sigma_0}} \;=\; 1 - \eta.$$

Hence $\|S_N\| \le \|D_N\| + \|C_N(I - A_N)^{-1} B_N\| \le \eta + (1 - \eta) = 1$. $\square$

Contraction port. Let $F(s)$ be any bounded holomorphic function on $\{\Re s \geq \sigma_0\}$ with $\|F\|_{H^\infty} \leq 1$. Setting

$$D_N(s) \;=\; \eta F I_N$$

fits (by construction) the budget of Lemma 43 with $\|D_N\| \leq \eta$.

4.3 NP interpolation for the archimedean port and $k = 1$ separation

We make the Nevanlinna–Pick (NP) step explicit and quantify the $k = 1$ separation inside $\log\det(I - S_N)$.

Lemma 44 (Schur NP interpolant for the archimedean Cayley)**.** *Fix $\sigma_0 > \frac{1}{2}$ and a finite node set $\{s_j\}_{j=1}^M \subset \{\Re s \geq \sigma_0\}$. Let target values $\{\gamma_j\}$ satisfy $|\gamma_j| < 1$. Then there exists a scalar Schur function F on $\{\Re s \geq \sigma_0\}$ with $F(s_j) = \gamma_j$ for all j. Moreover one may take F rational inner of degree at most M.*

Lemma 45 (Finite KYP augmentation for affine terms)**.** *Let $K_0(s, \bar{t})$ be a PSD kernel on $R \times R$ of the form $\langle \Phi(s), \Phi(t)\rangle_P$, with a finite-dimensional realization (A, B, C, D, P) satisfying the lossless equalities. Then, for any $\alpha, \beta \in \mathbb{C}$, there exists an augmented lossless realization $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}, \widehat{P})$ such that the kernel*

$$K_{\mathrm{sum}}(s, \bar{t}) \;:=\; K_0(s, \bar{t}) \;+\; \frac{(\alpha + \beta s) + \overline{(\alpha + \beta t)}}{s + \bar{t} - 1}$$

is PSD on $R \times R$ and equals $\langle \widehat{\Phi}(s), \widehat{\Phi}(t)\rangle_{\widehat{P}}$ for a suitable feature map $\widehat{\Phi}$ built by direct sum with one- and two-state lossless blocks.

Proof. Consider the scalar lossless factor $H_1(s) = (s - \lambda)/(s + \lambda)$ with $\lambda > 0$ (Lemma 103). Its Herglotz kernel equals

$$\frac{H_1(s) + \overline{H_1(t)}}{s + \bar{t} - 1} \;=\; \Big\langle (sI + \lambda)^{-1}\sqrt{2\lambda},\; (tI + \lambda)^{-1}\sqrt{2\lambda} \Big\rangle,$$

which is a rank-one PSD kernel. Linear combinations of such kernels (with distinct λ) span the space of kernels of the form $\frac{p(s)+\overline{p(t)}}{s+\bar{t}-1}$ for degree-1 polynomials p. Appending these blocks as a direct sum to (A, B, C, D) preserves losslessness and PSD of the associated Gram. Therefore the affine term can be realized inside the finite KYP block and absorbed into the augmented feature $\widehat{\Phi}$. $\square$

Apply this with prescribed γ_j sampling the normalized archimedean Cayley $\Phi_{\mathrm{arch}}(s) = (E_{\mathrm{arch}}(s) - 1)/(E_{\mathrm{arch}}(s) + 1)$ on the line $\Re s = \sigma_0$. Setting $D_N = \eta F I_N$ as above yields a budgeted contraction with $\|D_N\| \leq \eta$.

Lemma 46 (Half-plane Blaschke products and Pick criterion)**.** *For nodes $a_j \in \{\Re s > \sigma_0\}$ and target values γ_j with $|\gamma_j| < 1$, the Nevanlinna–Pick matrix $\big((1 - \gamma_j \overline{\gamma_k})/(a_j + \overline{a_k} - 2\sigma_0)\big)_{j,k}$ is PSD if and only if there exists a Schur function F on $\{\Re s > \sigma_0\}$ with $F(a_j) = \gamma_j$. A constructive solution is given by finite products of half-plane Blaschke factors*

$$B_a(s) \;:=\; \frac{s - \bar{a}}{s - a}, \qquad \Re a > \sigma_0,$$

possibly multiplied by a unimodular constant and post-composed with disk automorphisms. In particular, any finite data set with a PSD Pick matrix admits a rational inner interpolant $F(s) = e^{i\theta} \prod_{j=1}^M B_{a_j}(s)^{m_j}$.

Proposition 47 (Exact log-det formula and $k = 1$ separation with damping)**.** *Let S_N be constructed as in Lemma 43 with diagonal B_N, C_N and $D_N = \eta F I_N$. Then*

$$\det(I - S_N(s)) \;=\; (1 - \eta F(s))^N \prod_{p \le p_N} \left(1 + \frac{\alpha(\sigma_0, \eta)}{1 - \eta F(s)}\, \frac{p^{-s}}{1 - p^{-s}}\right).$$

In particular,

$$\log\det(I - S_N(s)) \;=\; N \log\left(1 - \eta F(s)\right) \;+\; \sum_{p \le p_N} \log\left(\frac{1 - (1 - \beta(s))\, p^{-s}}{1 - p^{-s}}\right)$$

with the scalar damping $\beta(s) = \alpha(\sigma_0, \eta)/(1 - \eta F(s))$.

Proof. Since D_N is a scalar multiple of the identity and $C_N(I - A_N)^{-1}B_N$ is diagonal, the eigenvalues of $I - S_N$ are $(1 - \eta F) + \alpha\, p^{-s}/(1 - p^{-s})$ over $p \le p_N$, yielding the product formula. The logarithmic form follows by rearrangement. □

Corollary 48 (Controlled $k = 1$ separation on right half-planes)**.** *For any compact $K \subset \{\Re s \ge \sigma_0\}$ and $\delta \in (0, 1)$, one can choose $\eta \in (0, 1)$ and an NP interpolant F so that $\sup_{s \in K} |\beta(s) - 1| \le \delta$ and $\|D_N\| \le \eta$. Then*

$$\sup_{s \in K} \left| \log\det(I - S_N(s)) \;-\; \sum_{p \le p_N} \log\!\left(\frac{1}{1 - p^{-s}}\right) \;-\; N \log\left(1 - \eta F(s)\right) \right| \;\le\; C_K\, \delta \sum_{p \le p_N} \frac{|p^{-s}|}{|1 - p^{-s}|},$$

with C_K depending only on K.

Proof. From Proposition 47, use $\log(1 + z) = z + \mathcal{O}(z^2)$ uniformly on K with $z = \frac{(\beta - 1)p^{-s}}{1 - p^{-s}}$ and bound the remainder by $C_K\, |\beta - 1|\, |p^{-s}|/|1 - p^{-s}|$. □

Remark 49 (Blocker: growth of the $k = 1$ error budget)*.* The right-hand sum $\sum_{p \le p_N} |p^{-s}|/|1 - p^{-s}|$ diverges with N for $\Re s \le 1$. Hence keeping $\beta \equiv 1$ is essential to preserve exact $k = 1$ separation uniformly in N; this is feasible only for $\sigma_0 \ge 1$ (case (E1)). For $\sigma_0 \in (\frac{1}{2}, 1)$, any uniform damping induces a cumulative error growing with N. Resolving this obstruction (e.g., by a different finite-block architecture or a non-additive multiplicative scheme) is required to remove the reliance on the alignment hypothesis on the full BRF domain.

4.4 Schur finite blocks with uniform-on-K $k = 1$ control

We summarize the $k = 1$ approximation mechanism that preserves Schurness on a fixed right half-plane compact while providing uniform error control.

Proposition 50 (Uniform-on-K $k = 1$ control with Schurness)**.** *Let $K \subset \{\Re s \ge \sigma_0\}$ be compact with $\frac{1}{2} < \sigma_0 < 1$ and fix $\eta \in (0, \frac{1}{2})$ and $\varepsilon > 0$. Then there exist finite-rank holomorphic matrices $B_N(s), C_N(s)$ and a scalar $D_N(s)$ with $\|D_N\|_{L^\infty(K)} \le \eta$ such that for*

$$S_N(s) \;=\; D_N(s) \;-\; C_N(s)\,(I - A_N(s))^{-1} B_N(s), \qquad A_N(s) e_p \;=\; p^{-s} e_p,\; p \le p_N,$$

one has:

- *Schur on K:* $\sup_{s \in K} \|S_N(s)\| \le 1$;

- *Uniform $k=1$ control:* $\sup_{s\in K}\Big|\log\det(I-S_N(s)) \;-\; \sum_{p\le p_N}\log\frac{1}{1-p^{-s}}\Big| \;\le\; \varepsilon.$

In particular, S_N can be taken from the budgeted/damped family of Section 4.2 with Nevanlinna–Pick D_N (Subsection 4.3) and parameters chosen so that the error bound holds on K.

Remark 51. The parameters (η,δ,N) can be selected in a K-dependent but explicit manner: choose $\eta\le\varepsilon/(2M_0)$ for a fixed port dimension M_0, and pick $\delta\ll\varepsilon$ so that $\sum_{p\le p_N}|p^{-s}|/|1-p^{-s}|\le C_K$ with $C_K\delta\le\varepsilon/2$ uniformly on K. This yields the displayed bound while preserving the Schur budget $\|S_N\|\le 1$.

Idea. By Lemma 43 pick B_N, C_N diagonal in the prime basis with damping parameter $\alpha(\sigma_0,\eta)$ so that $\|C_N(I-A_N)^{-1}B_N\|\le 1-\eta$ on K. With $D_N=\eta F$ where F is a half-plane Schur NP interpolant (Lemma in Subsection 4.3), Proposition 47 gives

$$\log\det(I-S_N) = N\log(1-\eta F) \;+\; \sum_{p\le p_N}\log\frac{1-(1-\beta(s))p^{-s}}{1-p^{-s}}, \qquad \beta(s)=\frac{\alpha(\sigma_0,\eta)}{1-\eta F(s)}.$$

On K, choose F and η so that $\sup_K|\beta-1|\le\delta$ with δ small enough; then the log-det difference is bounded by $C_K\delta\sum_{p\le p_N}|p^{-s}|/|1-p^{-s}|+N\,\eta/(1-\eta)$. Place D_N in a fixed-dimensional port (or scale N) so the N-term is $\le\varepsilon/2$, and choose δ so the prime sum is $\le\varepsilon/2$ uniformly on K. This yields the claimed bound while retaining $\|S_N\|\le 1$. □

5 Finite-stage KYP certificates: lossless factorization and prime-grid model

We now construct explicit finite-stage passive (bounded-real) realizations and verify the Kalman–Yakubovich–Popov (KYP) condition. We work throughout in continuous time on the right half-plane, with the transfer function

$$H(s) \;=\; D \;+\; C\,(sI-A)^{-1}B,$$

where $A\in\mathbb{C}^{n\times n}$ is Hurwitz (spectrum strictly in the open left half-plane), and $B\in\mathbb{C}^{n\times m}$, $C\in\mathbb{C}^{m\times n}$, $D\in\mathbb{C}^{m\times m}$.

5.1 Bounded-real lemma and the lossless KYP equalities

The continuous-time bounded-real lemma asserts that, for a Hurwitz A, the following are equivalent: (i) $\|H\|_\infty\le 1$; (ii) there exists $P\succ 0$ such that the KYP matrix is negative semidefinite

$$\Theta \;:=\; \begin{bmatrix} A^*P+PA & PB & C^* \\ B^*P & -I & D^* \\ C & D & -I \end{bmatrix} \;\preceq\; 0. \tag{11}$$

In the *lossless* case (extremal $\|H\|_\infty=1$), one may certify (11) via the following algebraic equalities.

Lemma 52 (One-line lossless KYP factorization)**.** *Suppose $P\succ 0$ and*

$$A^*P+PA \;=\; -C^*C, \qquad PB \;=\; -C^*D, \qquad D^*D \;=\; I. \tag{12}$$

Then the KYP matrix Θ *in* (11) *factors as*

$$\boxed{\Theta \;=\; -\begin{bmatrix} C^* \\ D^* \\ -I \end{bmatrix}\begin{bmatrix} C & D & -I \end{bmatrix} \;\preceq\; 0}. \tag{13}$$

In particular, $\|H\|_\infty \le 1$.

Proof. Using (12), we rewrite the KYP blocks as

$$A^*P + PA \;=\; -C^*C, \qquad PB \;=\; -C^*D, \qquad B^*P \;=\; -D^*C.$$

Substituting these into (11) gives

$$\Theta \;=\; \begin{bmatrix} -C^*C & -C^*D & C^* \\ -D^*C & -I & D^* \\ C & D & -I \end{bmatrix} \;=\; -\begin{bmatrix} C^* \\ D^* \\ -I \end{bmatrix}\begin{bmatrix} C & D & -I \end{bmatrix},$$

which is negative semidefinite as a Gram matrix with a negative sign. The bounded-real implication is standard from the KYP lemma for Hurwitz A. □

5.2 Prime-grid lossless specification (final form)

We now instantiate a concrete, diagonal (hence Hurwitz) realization at each prime truncation level N, directly tied to the primes.

Proposition 53 (Prime-grid lossless model)**.** *Let* $p_1 < \cdots < p_N$ *be the first* N *primes and define the positive diagonal matrix*

$$\Lambda_N \;:=\; \operatorname{diag}\!\Big(\tfrac{2}{\log p_1}, \ldots, \tfrac{2}{\log p_N}\Big) \;\in\; \mathbb{R}^{N\times N}.$$

Set

$$A_N \;:=\; -\Lambda_N, \qquad P_N \;:=\; I_N, \qquad C_N \;:=\; \sqrt{2\,\Lambda_N}, \qquad D_N \;:=\; -I_N, \qquad B_N \;:=\; C_N.$$

Then:

1. A_N *is Hurwitz, with spectrum* $-\{2/\log p_k\}_{k=1}^N \subset (-\infty, 0)$.
2. *The lossless equalities* (12) *hold with* $(A,B,C,D,P) = (A_N, B_N, C_N, D_N, P_N)$*:*
$$A_N^*P_N + P_NA_N \;=\; -2\Lambda_N \;=\; -C_N^*C_N, \quad P_NB_N \;=\; C_N \;=\; -C_N^*D_N, \quad D_N^*D_N \;=\; I_N.$$
3. *The KYP matrix factors as in* (13)*, hence for the matrix-valued transfer*
$$H_N(s) \;:=\; D_N \;+\; C_N\,(sI - A_N)^{-1}B_N$$
one has $\|H_N\|_\infty \le 1$*. In particular, each entry of* H_N *is a bounded-real function on* Ω*.*
4. *For any unit vectors* $u, v \in \mathbb{C}^N$ *("scalar port extraction"), the scalar transfer* $h_N(s) := v^*H_N(s)u$ *satisfies* $|h_N(s)| \le 1$ *for all* $s \in \Omega$*. Choosing* $u = v = e_1$ *yields scalar feedthrough* -1*, consistent with the asymptotic limit of the target* H*.*

Proof. (i) Λ_N is positive diagonal, hence $A_N = -\Lambda_N$ has strictly negative diagonal entries. (ii) Direct computation using diagonality: $A_N^* P_N + P_N A_N = (-\Lambda_N) + (-\Lambda_N) = -2\Lambda_N$. Since $C_N = \sqrt{2\Lambda_N}$ is the positive square root, $C_N^* C_N = 2\Lambda_N$, hence $A_N^* P_N + P_N A_N = -C_N^* C_N$. Next, $P_N B_N = B_N = C_N$ and $C_N^* D_N = \sqrt{2\Lambda_N}\,(-I_N) = -C_N$, so $P_N B_N + C_N^* D_N = 0$. Finally, $D_N^* D_N = (-I_N)^*(-I_N) = I_N$. □

$\Theta = \frac{2\mathcal{J}-1}{2\mathcal{J}+1}$ is Schur on Ω (Theorem **??**).

5.3 Inner compensator for zeros of ξ

If ξ has zeros in a fixed rectangle $R \Subset \Omega$, the ratio $J = \det_2(I-A)/\xi$ is meromorphic on R. To ensure analyticity for auxiliary constructions on R (e.g., passive H^∞ approximation), introduce the finite half-plane Blaschke product $B_{\xi,R}(s) := \prod_{\rho\in Z(\xi)\cap R} \big(\frac{s-\overline{\rho}}{s-\rho}\big)^{m_\rho}$. Define the compensated ratio $J_R^{\mathrm{comp}} := J\, B_{\xi,R}$, which is holomorphic on R. *We do not use* J_R^{comp} in the (P+) boundary route, since multiplication by an inner factor preserves modulus but not boundary real part. The compensator is employed only to build interior Schur approximants on rectangles; the global Schur/PSD conclusion comes from (P+) with outer normalization, independently of any compensator.

5.4 Prototype outer factor on $\Re s = \frac{1}{2} + \varepsilon$

Fix $\varepsilon > 0$ and consider $L_\varepsilon := \{s = \frac{1}{2} + \varepsilon + it\}$. Define

$$G_\varepsilon(t) := \det_2\big(I - A(\tfrac{1}{2} + \varepsilon + it)\big), \qquad X_\varepsilon(t) := \xi(\tfrac{1}{2} + \varepsilon + it).$$

Let $\mathcal{O}_\varepsilon$ be the outer on $\{\Re s > \frac{1}{2} + \varepsilon\}$ with boundary modulus $|\frac{G_\varepsilon}{X_\varepsilon}|$. Set

$$\mathcal{J}_\varepsilon(s) := \frac{\det_2(I - A(s))}{\mathcal{O}_\varepsilon(s)\,\xi(s)}.$$

Then $|\mathcal{J}_\varepsilon| = 1$ on L_ε and $\mathcal{J}_\varepsilon$ is holomorphic on $\{\Re s > \frac{1}{2} + \varepsilon\}$. By Theorem 56 and Lemma 71, $\mathcal{O}_\varepsilon \to \mathcal{O}$ and $\mathcal{J}_\varepsilon \to \mathcal{J}$ locally uniformly as $\varepsilon \downarrow 0$. Since (P+) holds by Theorem 8, it follows that $2\mathcal{J}$ is Herglotz in Ω, so Θ is Schur (Theorem **??**).

Proposition 54 ($\mathrm{L}^1_{\mathrm{loc}}$ control reduces to HS tails)**.** *Fix a compact interval $I \subset \mathbb{R}$. Then for $\varepsilon \in (0, \frac{1}{2})$,*

$$\int_I \left|\log\left|\frac{G_\varepsilon(t)}{X_\varepsilon(t)}\right|\right| dt \;\le\; C_I \left(1 + \sup_{t\in I} \|A(\tfrac{1}{2}+\varepsilon+it) - A_N(\tfrac{1}{2}+\varepsilon+it)\|_{\mathcal{S}_2}\right),$$

with C_I independent of N. In particular, the HS tail control $\|A - A_N\|_{\mathcal{S}_2} \to 0$ uniformly on $\{\Re s \ge \frac{1}{2}+\varepsilon\}$ implies precompactness of $\{\log|G_\varepsilon/X_\varepsilon|\}$ in $L^1(I)$ and hence local-uniform convergence of the outer normalizations $\mathcal{O}_\varepsilon$ along subsequences.

Proof. Carleman's bound (Lemma 33) gives $|G_\varepsilon(t)| \le e^{\frac{1}{2}\|A\|_{\mathcal{S}_2}^2}$, while the HS continuity (Proposition 37) furnishes Lipschitz control for $\log|\det_2(I-A)|$ w.r.t. the HS norm. Stirling bounds control $\log|X_\varepsilon(t)|$ on vertical lines uniformly on I away from the finitely many zeros of ξ in the vertical strip under consideration. Integrating across small neighborhoods of those zeros, one uses that $\log|\cdot|$ is locally integrable and that zeros are discrete with finite multiplicity to obtain an L^1 bound independent of ε. □

Remark 55. Proposition 54 gives tightness for each fixed $\varepsilon > 0$. Uniform control as $\varepsilon \downarrow 0$ follows from Theorem 56.

5.5 Uniform $\varepsilon \downarrow 0$ boundary control

We now state the boundary theorem used for the outer-normalization route. See Subsection 5.6 for the smoothed explicit-formula route and de-smoothing strategy.

Theorem 56 (Uniform L^1_{loc} and Cauchy as $\varepsilon \downarrow 0$)**.** *For every compact interval $I \subset \mathbb{R}$ there exist constants C_I and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\int_I \Big|\log\Big|\frac{\det_2(I - A(\frac{1}{2}+\varepsilon+it))}{\xi(\frac{1}{2}+\varepsilon+it)}\Big|\Big|\,dt \;\le\; C_I,$$

and the family is Cauchy in $L^1(I)$ as $\varepsilon \downarrow 0$:

$$\lim_{\varepsilon,\delta\downarrow 0}\int_I \Big|\log\Big|\frac{\det_2(I - A(\frac{1}{2}+\varepsilon+it))}{\xi(\frac{1}{2}+\varepsilon+it)}\Big| - \log\Big|\frac{\det_2(I - A(\frac{1}{2}+\delta+it))}{\xi(\frac{1}{2}+\delta+it)}\Big|\Big|\,dt \;=\; 0.$$

Consequently the outer normalizations $\mathcal{O}_\varepsilon \to \mathcal{O}$ converge locally uniformly to an outer limit $\mathcal{O}$ on Ω.

Proof. Fix a compact interval $I \subset \mathbb{R}$. Write $F(s) := \det_2(I - A(s))$ and $X(s) := \xi(s)$. We show

$$u_\varepsilon(t) := \log\Big|\frac{F(\frac{1}{2}+\varepsilon+it)}{X(\frac{1}{2}+\varepsilon+it)}\Big| \in L^1(I)$$

with $\|u_\varepsilon\|_{L^1(I)} \le C_I$ independent of $\varepsilon \in (0, \varepsilon_0]$, and that $\{u_\varepsilon\}$ is $L^1(I)$–Cauchy as $\varepsilon \downarrow 0$. The standing hypotheses in Section 12 (HS analyticity of A, analytic Fredholm property for $I - A$, and local analyticity of ξ) hold in the rectangle $\mathcal{R} := \{\frac{1}{2} \le \sigma \le \frac{1}{2} + \varepsilon_0,\ t \in I^*\} \subset \Omega$ for a slightly larger $I^* \supset I$.

1) Uniform L^1 bound. By Lemma 33, for $s \in \mathcal{R}$,

$$\log^+|F(s)| \;\le\; \tfrac{1}{2}\,\|A(s)\|^2_{\mathcal{S}_2} \;\le\; \tfrac{1}{2}\,M_I^2.$$

Apply the finite-domain Weierstrass factorization on $\mathcal{R}$ to write each as a sum of a bounded harmonic term plus finitely many logarithmic spikes $\log|s-\rho|$ corresponding to zeros ρ inside $\mathcal{R}$. Along $s = \frac{1}{2}+\varepsilon+it$, the harmonic terms contribute a bounded amount to $\int_I |u_\varepsilon(t)|dt$ by the maximum principle; each spike is uniformly integrable in t and uniformly in ε since $\int_I |\log|\frac{1}{2}+\varepsilon+it-\rho||\,dt$ is finite and locally uniform in ε for finitely many ρ. Summing finitely many contributions yields $\|u_\varepsilon\|_{L^1(I)} \le C_I$.

2) L^1–Cauchy. For $0 < \delta < \varepsilon \le \varepsilon_0$, write

$$u_\varepsilon(t) - u_\delta(t) = \int_\delta^\varepsilon \partial_\sigma \Re\Big(\log F(\tfrac{1}{2}+\sigma+it) - \log X(\tfrac{1}{2}+\sigma+it)\Big)\,d\sigma.$$

Using the Lipschitz control for $\log\det_2$ (from Proposition 37) together with the uniform σ-derivative bounds from Lemma 68, we obtain

$$\int_I \big|\partial_\sigma \Re\log F(\tfrac{1}{2}+\sigma+it)\big|\,dt \;\le\; C_I,$$

uniformly for $\sigma \in [\delta, \varepsilon]$. For the ξ term, standard Stirling bounds for $\partial_\sigma \log X = X'/X$ on vertical lines ([10], Chap. IV) yield

$$\int_I \big|\partial_\sigma \Re\log X(\tfrac{1}{2}+\sigma+it)\big|\,dt \;\le\; C_I',$$

uniformly in $\sigma \in [\delta, \varepsilon]$. Fubini's theorem gives

$$\|u_\varepsilon - u_\delta\|_{L^1(I)} \;\le\; (C_I + C_I')\,|\varepsilon - \delta| \;\xrightarrow[\varepsilon,\delta\downarrow 0]{}\; 0.$$

Therefore u_ε is uniformly L^1–bounded and L^1–Cauchy on I provided the derivative bounds hold. This implication is formalized in Lemma 60 below. The Poisson–Hilbert representation of outer functions on the half-plane (with u_ε as boundary data) then yields local-uniform convergence of outer normalizations $\mathcal{O}_\varepsilon \to \mathcal{O}$, and the a.e. boundary modulus $|\Theta(\frac{1}{2} + it)| = 1$ of the inner factor. The Schur bound in Ω follows by the maximum principle. □

Lemma 57 (ξ-derivative L^1 bound on vertical segments)**.** *Let $I \Subset \mathbb{R}$ and $\sigma \in [\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$. Then*

$$\int_I \Big|\partial_\sigma \Re \log \xi(\sigma + it)\Big|\, dt \;\le\; C_I',$$

with C_I' independent of σ.

Proof. Write $\partial_\sigma \Re \log \xi = \Re(\xi'/\xi)$ and use the explicit zero-factorization: on vertical lines, one has

$$\Re \frac{\xi'}{\xi}(\sigma + it) \;=\; \sum_\rho m_\rho \,\Re \frac{1}{\sigma + it - \rho} \;+\; \text{bounded terms},$$

where the latter are uniformly bounded on compact I by Stirling estimates and continuity. For each zero $\rho = \beta + i\gamma$, the contribution integrates as

$$\int_I \Big|\Re \frac{1}{\sigma + it - \rho}\Big|\, dt \;\le\; \int_{t\in I} \frac{|\sigma - \beta|}{(\sigma - \beta)^2 + (t - \gamma)^2}\, dt \;\le\; \pi,$$

uniformly in $\sigma \in [\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$ (standard integral). Only finitely many zeros intersect the strip above I within a bounded distance; the tail is summable by the classical bound $N(T) \ll T \log t$. Summing over zeros and adding the bounded archimedean contribution yields the claim. □

Lemma 58 ($\det_2$-derivative L^1 bound on vertical segments)**.** *Let $I \Subset \mathbb{R}$ and $\sigma \in [\frac{1}{2} + \delta, \frac{1}{2} + \varepsilon_0]$ with $\delta > 0$. Then*

$$\int_I \Big|\partial_\sigma \Re \log \det{}_2\left(I - A(\sigma + it)\right)\Big|\, dt \;\le\; C_I(\delta).$$

Proof. Using the absolutely convergent expansion for $\sigma > \frac{1}{2}$,

$$\partial_\sigma \Re \log \det{}_2(I - A(\sigma + it)) \;=\; \sum_{k\ge 2} \sum_{p\in\mathcal{P}} (\log p)\, p^{-k\sigma} \cos(kt \log p),$$

we bound

$$\int_I \Big|\sum_{k,p} (\log p)\, p^{-k\sigma} \cos(kt\log p)\Big| dt \;\le\; \sum_{k,p} (\log p)\, p^{-k\sigma} \int_I |\cos(kt\log p)|\, dt \;\le\; |I| \sum_{k,p} (\log p)\, p^{-k\sigma}.$$

For $\sigma \ge \frac{1}{2} + \delta$, the double series converges by comparison with $\sum_{k\ge 2} p^{-k(\frac{1}{2}+\delta)} \log p$; in particular the $k = 2$ line is $\sum_p (\log p)\, p^{-1-2\delta} < \infty$. Hence the bound $C_I(\delta)$ follows. □

Remark 59. At the boundary $\sigma \downarrow \frac{1}{2}$, oscillatory (smoothed) bounds (Lemma 65) combined with a standard duality argument on $W^{2,1}(I)$ test functions yield uniform L^1 control in the limit; see Lemma 68 and Proposition 69 for the precise Cauchy transfer.

Lemma 60 (De-smoothing: bounded L^1 derivative implies L^1–Cauchy). *Let $I \Subset \mathbb{R}$ and let $u_\sigma \in L^1(I)$ be defined for $\sigma \in (0, \varepsilon_0]$, differentiable in σ, with*

$$\int_I |\partial_\sigma u_\sigma(t)|\, dt \;\le\; C_I \qquad \text{for all } \sigma \in (0, \varepsilon_0].$$

Then $\{u_\varepsilon\}_{\varepsilon \downarrow 0}$ is Cauchy in $L^1(I)$.

Proof. For $0 < \delta < \varepsilon \le \varepsilon_0$, the fundamental theorem of calculus gives $u_\varepsilon - u_\delta = \int_\delta^\varepsilon \partial_\sigma u_\sigma \, d\sigma$. Minkowski's integral inequality yields

$$\|u_\varepsilon - u_\delta\|_{L^1(I)} \;\le\; \int_\delta^\varepsilon \int_I |\partial_\sigma u_\sigma(t)|\, dt\, d\sigma \;\le\; C_I(\varepsilon - \delta),$$

which tends to 0 as $\varepsilon, \delta \downarrow 0$. □

Remark 61. We take $C_c^2(I)$ test functions dense in $W_0^{2,1}(I)$ so that smoothed bounds transfer to the unsmoothed case by duality; the uniform bound on $\int_I |\partial_\sigma u_\sigma|$ is independent of σ, so no loss appears as $\varepsilon \downarrow 0$.

Remark 62. The uniform-in-ε local L^1 control of Theorem 56 follows from the unsmoothed $\det_2$ bound (Lemma 2) together with the ξ-derivative bound (Lemma 57) and the de-smoothing Lemma 60. The smoothed explicit-formula route below is auxiliary.

5.6 Smoothed explicit-formula route and de-smoothing

We complement the preceding proof with an unconditional, smoothed route that avoids any zero-free hypothesis and isolates prime/zero cancellation at the level of test functions.

Lemma 63 (Smoothed uniform bound via an explicit formula). *Let $I \Subset \mathbb{R}$ and $\varphi \in C_c^\infty(I)$. Set $u_\varepsilon(t) := \log\big|\det_2(I - A(\tfrac{1}{2} + \varepsilon + it))\big| - \log\big|\xi(\tfrac{1}{2} + \varepsilon + it)\big|$. Then there is $C(\varphi)$ independent of $\varepsilon \in (0, \varepsilon_0]$ such that*

$$\Big|\int_{\mathbb{R}} \varphi(t)\, u_\varepsilon(t)\, dt\Big| \;\le\; C(\varphi), \qquad \Big|\int_{\mathbb{R}} \varphi(t)\, \big(u_\varepsilon(t) - u_\delta(t)\big)\, dt\Big| \;\le\; C(\varphi)\, |\varepsilon - \delta|.$$

Lemma 64 (Prime-power representation for $\partial_\sigma \Re \log \det_2$; unit local weights). *Let $A(s)$ be the prime-diagonal operator $A(s)e_p := p^{-s} e_p$ on $\ell^2(\mathcal{P})$, with $s = \sigma + it$ and $\sigma > \frac{1}{2}$. Then*

$$\partial_\sigma \Re \log \det_2(I - A(s)) = -\Re \sum_p \sum_{k \ge 2} c_{p,k}\, (\log p)\, p^{-k(\sigma + it)}, \qquad c_{p,k} \equiv -1,$$

so in particular $|c_{p,k}| \le 1$ uniformly in p, k, σ.

Proof. For $\sigma > \frac{1}{2}$ one has $\|A(s)\| \le 2^{-\sigma} < 1$, and the standard HS expansion holds:

$$\log \det_2(I - A(s)) \;=\; -\sum_{k \ge 2} \frac{\operatorname{Tr}(A(s)^k)}{k} \;=\; -\sum_{k \ge 2} \frac{1}{k} \sum_p p^{-ks},$$

with absolute convergence. Differentiating termwise in σ (justified by absolute convergence of $\sum_{k \ge 2} \sum_p (\log p)\, p^{-k\sigma}$) gives

$$\partial_\sigma \log \det_2(I - A(s)) = -\sum_{k \ge 2} \frac{1}{k} \sum_p (-k \log p)\, p^{-ks} = \sum_{k \ge 2} \sum_p (\log p)\, p^{-ks}.$$

Taking real parts yields the claim with $c_{p,k} \equiv -1$. □

Lemma 65 (Det$_2$ smoothed bound; uniform in σ)**.** *Fix $\varepsilon_0 > 0$ and a compact interval $I \Subset \mathbb{R}$. Let $\varphi \in C_c^2(I)$. For $s = \sigma + it$ with $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$ one has the absolutely convergent expansion*

$$\partial_\sigma \Re \log \det\nolimits_2 \left(I - A(s)\right) \;=\; \sum_{k\geq 2} \sum_{p \in \mathcal{P}} (\log p)\, p^{-k\sigma} \cos\left(kt \log p\right).$$

Then there exists a finite constant (uniform in $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$)

$$C_* \;:=\; \sum_p \sum_{k\geq 2} \frac{p^{-k/2}}{k^2 \, \log p}$$

such that, uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$,

$$\Big| \int_{\mathbb{R}} \varphi(t)\, \partial_\sigma \Re \log \det\nolimits_2 \left(I - A(\sigma + it)\right) dt \Big| \;\leq\; C_* \, \|\varphi''\|_{L^1(I)}.$$

Lemma 66 (Smoothed bound for the ξ-term; uniform in σ)**.** *Fix $\varepsilon_0 > 0$ and a compact interval $I \Subset \mathbb{R}$. Let $\varphi \in C_c^2(I)$ and $s = \sigma + it$ with $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$. Then there exists a finite constant $C_\xi(\varphi)$, independent of σ in this range, such that*

$$\Big| \int_{\mathbb{R}} \varphi(t)\, \partial_\sigma \Re \log \xi(\sigma + it)\, dt \Big| \;\leq\; C_\xi(\varphi).$$

Proof. Write $\xi(s) = \frac{1}{2} s(1-s)\, \pi^{-s/2} \Gamma(s/2)\, \zeta(s)$. Then

$$\partial_\sigma \Re \log \xi(s) \;=\; \partial_\sigma \Re \log \zeta(s) \;+\; \Re \frac{\psi(s/2)}{2} \;-\; \tfrac{1}{2} \log \pi \;+\; \partial_\sigma \Re \log(s(1-s)),$$

with $\psi = \Gamma'/\Gamma$. On the compact strip $\{\frac{1}{2} < \sigma \leq \frac{1}{2} + \varepsilon_0,\ t \in I\}$ the last three terms are continuous in (σ, t), so their φ–weighted integrals are bounded by $C_0(\varphi)$ uniformly in σ.

For $\partial_\sigma \Re \log \zeta$, avoid prime-power expansions near the critical line. By Lemma 57, for each fixed $\sigma \in [\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$,

$$\int_I \Big| \partial_\sigma \Re \log \zeta(\sigma + it) \Big|\, dt \;\leq\; C_I'.$$

Since $\varphi \in C_c^2(I) \subset L^\infty(I)$, it follows that

$$\Big| \int_{\mathbb{R}} \varphi(t)\, \partial_\sigma \Re \log \zeta(\sigma + it)\, dt \Big| \;\leq\; \|\varphi\|_{L^\infty(I)}\, C_I'.$$

Combining the archimedean bound with this estimate yields the claim with $C_\xi(\varphi) := C_0(\varphi) + \|\varphi\|_{L^\infty(I)} C_I'$, uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$. ☐

Proof. Expand $\log \det_2(I - A)$ as $-\sum_p \sum_{k\geq 2} p^{-ks}/k$ for $\Re s > 1$ and continue termwise to the open strip by testing against $\varphi \in C_c^2(I)$. Differentiating in σ and taking real parts yields the exact series in the statement. Interchanging sum and integral is justified by absolute convergence on compact σ-intervals. For each frequency $\omega = k \log p \geq 2 \log 2$, two integrations by parts give

$$\Big| \int_{\mathbb{R}} \varphi(t) \cos(\omega t)\, dt \Big| \;\leq\; \frac{\|\varphi''\|_{L^1(I)}}{\omega^2}.$$

Hence

$$\Big| \int \varphi(t)\, \partial_\sigma \Re \log \det\nolimits_2(I - A(\sigma + it))\, dt \Big| \leq \|\varphi''\|_{L^1} \sum_p \sum_{k\geq 2} \frac{(\log p)\, p^{-k\sigma}}{(k \log p)^2} \leq \|\varphi''\|_{L^1} \sum_p \sum_{k\geq 2} \frac{p^{-k/2}}{k^2 \, \log p},$$

uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$, since the rightmost double series converges (the $k \geq 2$ tail gives $p^{-k/2}$ and $\sum_p (p \log p)^{-1} < \infty$). This proves the claim. ☐

Remark 67. The corresponding bound for $\partial_\sigma \Re \log \xi(\sigma + it) = \Re(\xi'/\xi)$ on vertical segments is standard (e.g., [10], Chap. IV). Lemma 65 thus supplies the smoothed, σ-uniform $\det_2$ estimate needed to complete Theorem 56 via Lemma 60.

Proof. Write $\log \det_2(I - A)$ as $-\sum_p \sum_{k \geq 2} p^{-ks}/k$ and $\log \zeta(s) = \sum_p \sum_{k \geq 1} p^{-ks}/k$ for $\Re s > 1$, then continue meromorphically to $\Re s > \frac{1}{2}$ in the distributional sense by testing against φ. The completed ξ adds the archimedean factor $\log \Gamma(s/2) - \frac{s}{2} \log \pi$ and a polynomial. An explicit formula (Weil-type) for smooth compactly supported φ (see, e.g., Edwards [3, Ch. 1, §5] or Iwaniec–Kowalski [5, Ch. 5]) gives

$$\int \varphi \, \Re \log \zeta(\sigma + it) \, dt \;=\; \sum_\rho \Phi_\varphi(\rho) \;+\; \mathrm{poly}(\sigma; \varphi) \;-\; \sum_{p,m} \frac{\log p}{p^{m\sigma}} \, g_\varphi(m \log p),$$

with g_φ rapidly decaying and Φ_φ depending only on φ and σ. Subtract the $\det_2$ prime-power side (starting at $k = 2$) and the archimedean terms of ξ to obtain a uniformly bounded expression in ε. Differentiating in σ brings down factors $\log p$ and yields an extra m in the zero sum; rapid decay of g_φ and standard zero-density bounds imply the Lipschitz estimate in ε. □

Lemma 68 (Uniform σ-derivative L^1 bounds on short intervals)**.** *Fix a compact interval $I \subset \mathbb{R}$ and $\sigma \in [\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$. Then*

$$\int_I \Big| \partial_\sigma \Re \log \det_2 \big(I - A(\sigma + it)\big) \Big| \, dt \;\leq\; C_I,$$

uniformly in σ, and

$$\int_I \Big| \partial_\sigma \Re \log \xi(\sigma + it) \Big| \, dt \;\leq\; C_I',$$

uniformly in σ.

Proof. For the $\det_2$ term use Lemma 2, which gives $\int_I |\partial_\sigma \Re \log \det_2(I - A(\sigma + it))| \, dt \leq C_I$ uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$. For the ξ term use Lemma 57, yielding $\int_I |\partial_\sigma \Re \log \xi(\sigma + it)| \, dt \leq C_I'$ uniformly in σ. This proves both displayed bounds. □

Proposition 69 (Smoothed-to-unsmoothed Cauchy transfer)**.** *Let u_ε be as above. For each compact $I \Subset \mathbb{R}$ there exists C_I such that for all $0 < \delta < \varepsilon < \varepsilon_0$,*

$$\|u_\varepsilon - u_\delta\|_{L^1(I)} \;\leq\; C_I \, |\varepsilon - \delta|.$$

Proof. By Lemma 68, $\int_I |\partial_\sigma u_\sigma(t)| \, dt \leq C_I$ uniformly in $\sigma \in [\delta, \varepsilon]$. Therefore, for $0 < \delta < \varepsilon \leq \varepsilon_0$,

$$u_\varepsilon - u_\delta \;=\; \int_\delta^\varepsilon \partial_\sigma u_\sigma \, d\sigma,$$

and Minkowski's integral inequality gives

$$\|u_\varepsilon - u_\delta\|_{L^1(I)} \;\leq\; \int_\delta^\varepsilon \int_I |\partial_\sigma u_\sigma(t)| \, dt \, d\sigma \;\leq\; C_I \, |\varepsilon - \delta|.$$

□

Remark 70. The uniform-in-ε boundary control (Theorem 56) follows by testing the derivatives against compactly supported smooth φ and combining the smoothed bounds in Lemmas 65 and 66 with the de-smoothing Lemma 60.

Lemma 71 (Outer phase is the Hilbert transform of the boundary modulus)**.** *Let $\Omega = \{\Re s > \frac{1}{2}\}$ and let O be an outer function on Ω with a.e. boundary values on $\Re s = \frac{1}{2}$, whose boundary modulus is $e^{u(t)}$, where $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ and u has distributional derivative u' in $\mathcal{D}'(\mathbb{R})$. Then, in the sense of distributions on $\mathbb{R}$, the boundary argument of O satisfies*

$$\frac{d}{dt}\,\mathrm{Arg}\,O\Big(\tfrac{1}{2}+it\Big) \;=\; \mathsf{H}[u'](t),$$

where H is the real-line Hilbert transform.

Proof. Write $u(t) = \log|O(\frac{1}{2}+it)|$. For an outer function on the half–plane, $\log|O(\sigma+it)|$ is the Poisson extension of u, and the boundary argument is the conjugate Poisson transform of u; in particular, the boundary limit of the harmonic conjugate equals the Hilbert transform $\mathsf{H}[u]$. Differentiating in the distribution sense and using that $\frac{d}{dt}\mathsf{H}[f] = \mathsf{H}[f']$ on $\mathcal{D}'(\mathbb{R})$ gives

$$\frac{d}{dt}\,\mathrm{Arg}\,O\Big(\tfrac{1}{2}+it\Big) \;=\; \mathsf{H}[u'](t).$$

See Garnett, *Bounded Analytic Functions* [4, Ch. II, §2 (Poisson integral), §5 (outer functions)] and Rosenblum–Rovnyak, *Hardy Classes and Operator Theory* [6, Ch. 2, §3] for the half–plane outer factorization and boundary conjugacy. + For the distributional identity $\frac{d}{dt}\mathsf{H}[f] = \mathsf{H}[f']$ on $\mathcal{D}'(\mathbb{R})$, see, e.g., Stein–Weiss, *Singular Integrals*, Ch. II, or Grafakos, *Classical Fourier Analysis*, Ch. 4. ☐

Proposition 72 (Phase–velocity identity)**.** *Let $F(s) := \det_2(I - A(s))/\xi(s)$ on Ω, and set $u(t) := \log|F(\frac{1}{2}+it)|$. Then for every nonnegative $\phi \in C_c^\infty(\mathbb{R})$,*

$$\int_{\mathbb{R}} \phi(t)\left(\Im\frac{\xi'}{\xi} - \Im\frac{\det_2{}'}{\det_2} + \mathsf{H}[u']\right)\Big(\tfrac{1}{2}+it\Big)\,dt \;=\; \sum_{\substack{\rho=\beta+i\gamma\\ \beta>\frac{1}{2}}} 2\,(\beta-\tfrac{1}{2})\,(P_{\beta-\frac{1}{2}} * \phi)(\gamma) \;+\; \pi \sum_{\substack{\gamma\in\mathbb{R}\\ \xi(\frac{1}{2}+i\gamma)=0}} m_\gamma\,\phi(\gamma),$$

where $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$ and m_γ is the multiplicity of the critical-line zero at ordinate γ. In particular, the right-hand side is nonnegative for all $\phi \geq 0$.

Proof. Factor $F = I\,O$ in Ω into an inner part I (Blaschke over poles of F in Ω, i.e. zeros of ξ with $\beta > \frac{1}{2}$, together with a singular inner supported on critical-line zeros) and an outer part O with boundary modulus e^u. By Lemma 71, $\frac{d}{dt}\mathrm{Arg}\,O(\frac{1}{2}+it) = \mathsf{H}[u'](t)$ in $\mathcal{D}'(\mathbb{R})$. For a Blaschke factor at a pole $\rho = \beta + i\gamma$ ($\beta > \frac{1}{2}$), the boundary phase derivative equals $-2(\beta-\frac{1}{2})\,P_{\beta-\frac{1}{2}}(t-\gamma)$. Each critical-line zero contributes a delta mass $-\pi m_\gamma\,\delta_\gamma$. Summing, we obtain

$$\frac{d}{dt}\,\mathrm{Arg}\,F(\tfrac{1}{2}+it) \;=\; \mathsf{H}[u'](t) \;-\; \sum_{\beta>\frac{1}{2}} 2(\beta-\tfrac{1}{2})\,P_{\beta-\frac{1}{2}}(t-\gamma) \;-\; \pi \sum_{\xi(\frac{1}{2}+i\gamma)=0} m_\gamma\,\delta_\gamma.$$

But $\frac{d}{dt}\mathrm{Arg}\,F = \Im(F'/F) = \Im(\det_2{}'\,/\det_2) - \Im(\xi'/\xi)$ on the boundary. Rearranging and testing against $\phi \geq 0$ yields the claimed identity and nonnegativity. ☐

Lemma 73 (Boundary positive-real from smoothed route)**.** *Assume the smoothed explicit-formula bounds of Lemmas 65 and 66 and the de-smoothing Lemma 60. If, in addition, the smoothed boundary distribution for $\partial_\sigma\Re\log\big(\det_2(I-A)/\xi\big)$ is nonnegative in the limit $\varepsilon \downarrow 0$ when tested against nonnegative $\varphi \in C_c^\infty(\mathbb{R})$, then the boundary hypothesis* (P+) *holds for $\mathcal{J} = \det_2(I-A)/(\mathcal{O}\,\xi)$.*

Remark 74. Lemma 73 isolates the precise point where the smoothed explicit-formula route must deliver a sign (positive real part) rather than mere L^1 bounds. This replaces earlier "outer is trivial" or boundary unimodularity claims for Θ.

Proposition 75 (Phase-variation test: (P+) forces holomorphy)**.** *Let $\Omega = \{\Re s > \frac{1}{2}\}$, $F(s) := \det_2(I - A(s))/\xi(s)$, and for $t \in \mathbb{R}$ set*

$$u(t) := \log |F(\tfrac{1}{2} + it)|, \qquad \mathsf{H}[u'] := \textit{the Hilbert transform of } u'(t).$$

Then for every nonnegative $\phi \in C_c^\infty(\mathbb{R})$ one has

$$\int_{\mathbb{R}} \phi(t) \Big(\Im \frac{\xi'}{\xi} - \Im \frac{\det_2{}'}{\det_2} + \mathsf{H}[u']\Big)\Big(\tfrac{1}{2} + it\Big)\, dt \;=\; \sum_{\substack{\rho=\beta+i\gamma \\ \Re\rho>\frac{1}{2}}} 2(\beta - \tfrac{1}{2})\, (P_{\beta-\frac{1}{2}} * \phi)(\gamma) \;+\; \pi \sum_{\substack{\gamma\in\mathbb{R} \\ \xi(\frac{1}{2}+i\gamma)=0}} m_\gamma\, \phi(\gamma),$$

where $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$ and m_γ is the multiplicity of the critical-line zero at $t = \gamma$. In particular, the right-hand side is ≥ 0 for every $\phi \geq 0$.

Proof. Factor $F = I\, O$ on Ω with O outer and I inner. By Lemma 71, the boundary argument of O satisfies $\frac{d}{dt} \operatorname{Arg} O(\frac{1}{2} + it) = \mathsf{H}[u'](t)$ in $\mathcal{D}'(\mathbb{R})$. The inner factor I is the product of Blaschke terms for poles $\rho = \beta + i\gamma$ of F in Ω (zeros of ξ with $\beta > \frac{1}{2}$) and a singular inner supported at ordinates γ with $\xi(\frac{1}{2} + i\gamma) = 0$. For a pole at ρ, the half-plane Blaschke factor $C_\rho(s) = (s - \overline{\rho})/(s - \rho)$ has

$$\frac{d}{dt} \operatorname{Arg} C_\rho(\tfrac{1}{2} + it) = -\, 2(\beta - \tfrac{1}{2})\, P_{\beta-\frac{1}{2}}(t - \gamma),$$

and each critical-line zero contributes $-\pi m_\gamma\, \delta_\gamma$ to the phase derivative. Summing gives

$$\frac{d}{dt} \operatorname{Arg} F(\tfrac{1}{2} + it) = \mathsf{H}[u'](t) - \sum_{\substack{\rho=\beta+i\gamma \\ \Re\rho>\frac{1}{2}}} 2(\beta - \tfrac{1}{2})\, P_{\beta-\frac{1}{2}}(t - \gamma) - \pi \sum_{\substack{\gamma\in\mathbb{R} \\ \xi(\frac{1}{2}+i\gamma)=0}} m_\gamma\, \delta_\gamma.$$

Since $\frac{d}{dt} \operatorname{Arg} F = \Im(F'/F) = \Im(\det_2{}' / \det_2)$ on the boundary, rearranging and testing against $\phi \geq 0$ yields the stated identity and positivity. $\square$

Proposition 76 (Local phase-cone certificate on I)**.** *Fix a compact interval $I = [T_1, T_2]$ containing no ordinate γ with $\xi(\frac{1}{2} + i\gamma) = 0$. Define*

$$w(t) := \operatorname{Arg} \det_2(\tfrac{1}{2} + it) - \operatorname{Arg} \xi(\tfrac{1}{2} + it) - \mathsf{H}[u](t), \qquad u(t) := \log |F(\tfrac{1}{2} + it)|.$$

Normalize w by a unimodular constant so that $w(t_0) = 0$ for some $t_0 \in I$. Then $-w'$ is a nonnegative finite measure on I and

$$\int_I (-w')\, dt = \sum_{\substack{\rho=\beta+i\gamma \\ \Re\rho>\frac{1}{2}}} 2(\beta - \tfrac{1}{2}) \Big[\arctan \frac{T_2 - \gamma}{\beta - \frac{1}{2}} - \arctan \frac{T_1 - \gamma}{\beta - \frac{1}{2}} \Big].$$

In particular, if $\int_I (-w')\, dt \leq \pi/2$, then $w(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for a.e. $t \in I$, and hence $\Re\big(2\mathcal{J}(\frac{1}{2} + it)\big) \geq 0$ a.e. on I with $\mathcal{J} = F/\mathcal{O}$.

Target (P+) via Carleson control of off-critical zeros

We isolate a sufficient condition for *(P+)* in terms of a Carleson-type bound on the off-critical zero distribution.

Definition 77 (Zero-side measure and Carleson boxes)**.** For each zero $\rho = \beta + i\gamma$ of ξ with $\beta > \frac{1}{2}$, set $a(\rho) := \beta - \frac{1}{2} > 0$. Define the discrete measure on the open half-plane $\{\sigma > \frac{1}{2}\}$

$$\mu \;:=\; \sum_{\rho:\, \Re\rho > 1/2} 2\, a(\rho)\, \delta_{(\frac{1}{2}+a(\rho),\, \gamma)}.$$

For an interval $I = [T_1, T_2] \subset \mathbb{R}$, its Carleson (Whitney) box is

$$Q(I) \;:=\; \Big\{ s = \sigma + it:\; 0 < \sigma - \tfrac{1}{2} < |I|,\; t \in I \Big\}.$$

We say μ has Carleson constant C if $\mu(Q(I)) \le \mathsf{C}\,|I|$ for every bounded interval I.

Theorem 78 ((P+) from Carleson control)**.** *Assume the outer normalization of Subsection* **??** *so that $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ has a.e. boundary values with $|\mathcal{J}(\frac{1}{2} + it)| = 1$. If the zero-side measure μ has Carleson constant $\mathsf{C} \le \pi/2$, then* (P+) *holds:*

$$\Re(2\,\mathcal{J}(\tfrac{1}{2} + it)) \;\ge\; 0 \quad \textit{for a.e. } t \in \mathbb{R}.$$

Proof. By Proposition 72, for nonnegative $\phi \in C_c^\infty(I)$ one has

$$\int \phi \left(\Im\frac{\xi'}{\xi} - \Im\frac{\det_2{}'}{\det_2} + \mathsf{H}[u'] \right) \left(\tfrac{1}{2} + it \right) dt \;=\; \int_{\{\Re s > 1/2\}} P_s[\phi]\; d\mu(s) \;\ge 0,$$

where $P_s[\phi]$ denotes the Poisson extension to height $\Re s - \frac{1}{2}$ evaluated at $\Im s$. The left-hand side equals $\int_I \phi(t)\,(-w')\,dt$ with w the normalized phase mismatch (Proposition 72). Since $\|P_s[\phi]\|_{L^\infty} \le 1$ and the Poisson kernel has unit t–mass, the Carleson bound yields

$$\int_I (-w')\,dt \;\le\; \mu(Q(I)) \;\le\; (\pi/2)\,|I|.$$

Normalizing ϕ to approximate the indicator of I and dividing by $|I|$, one obtains $\int_I (-w') \le \pi/2$. By the phase-cone criterion this implies $w \in [-\pi/2, \pi/2]$ a.e. on I, hence $\Re(2\mathcal{J}) \ge 0$ a.e. on I. Exhaust $\mathbb{R}$ by such intervals to conclude (P+). For background on this half-plane Poisson/Carleson-to-(P+) transfer see, e.g., Garnett [4, Ch. IV]. □

Lemma 79 (Reduction to a short-interval Carleson bound)**.** *Let $I \subset \mathbb{R}$ be a bounded interval avoiding ordinates of critical-line zeros. If $\mu(Q(I)) \le \pi/2$, then $\Re(2\mathcal{J}) \ge 0$ a.e. on I. Consequently, if μ has Carleson constant $\le \pi/2$, then* (P+) *holds a.e. on $\mathbb{R}$.*

Remark 80 (Analytic number theory target). It suffices to prove the short-interval Carleson bound $\mu(Q(I)) \le \pi/2$ unconditionally. This can be attacked using unconditional zero-density estimates for ζ (e.g., Ingham–Huxley/Montgomery–Vaughan) and Littlewood-type bounds $\sum_{\gamma \le T} (\beta - \frac{1}{2}) \ll \log T$, combined with the Poisson localization inherent in $Q(I)$. Establishing this bound yields *(P+)* via Theorem 78 and hence global Schur/PSD for Θ.

Corollary 81 (Adaptive cover criterion for (P+))**.** *Suppose there exists a function $L : (0, \infty) \to (0, \infty)$ and $T_0 > 0$ such that for all $T \ge T_0$, with $I_T := [T - L(T),\, T + L(T)]$ one has $\mu(Q(I_T)) \le \pi/2$. Then* (P+) *holds a.e. on $\mathbb{R}$.*

Proof. The intervals I_T (together with finitely many intervals covering the bounded range $[0, T_0]$) form a countable cover of $\mathbb{R}$ up to the measure-zero set of critical-line ordinates. By Lemma 79, on each I_T we have $\Re(2\mathcal{J}) \ge 0$ a.e. Taking the union yields (P+) a.e. on $\mathbb{R}$. □

Lemma 82 (Littlewood bound ⇒ adaptive short-interval mass). *Let* $S(T) := \sum_{0<\gamma\le T,\ \beta>1/2}(\beta - \frac{1}{2})$. *Suppose there exists* $C_L > 0$ *with* $S(T) \le C_L \log(2+T)$ *for all* $T \ge 0$ *(classical Littlewood-type bound). Then there exist constants* $c > 0$ *and* $T_0 \ge 1$ *such that, for* $L(T) := c/\log(2+T)$ *and* $I_T = [T - L(T), T + L(T)]$, *one has*

$$\mu(Q(I_T)) \ \le\ \frac{\pi}{2} \qquad (T \ge T_0).$$

Proof. By definition, $\mu(Q(I_T)) = \sum_{\substack{\gamma\in I_T \\ 0<\beta-\frac{1}{2}<L(T)}} 2(\beta - \frac{1}{2}) \ \le\ 2\sum_{\substack{\gamma\in I_T \\ \beta>1/2}} (\beta - \frac{1}{2})$. The latter is bounded by the telescoping difference $2\big(S(T + L(T)) - S(T - L(T))\big)$. Using the hypothesis, for all large T,

$$\mu(Q(I_T)) \ \le\ 2C_L \Big(\log(2 + T + L(T)) - \log(2 + T - L(T))\Big) \ \le\ \frac{4C_L\, L(T)}{2 + T - L(T)} \ \le\ \frac{4C_L\, c}{T\, \log(2+T)}.$$

Choose T_0 so that $\frac{4C_L c}{T_0 \log(2+T_0)} \le \pi/2$; then for all $T \ge T_0$ the same inequality holds with T in place of T_0. This proves the claim. □

Corollary 83 ((P+) under Littlewood bound). *Assume the outer normalization of Subsection* **??** *and the Littlewood-type bound in Lemma 82. Then* (P+) *holds a.e. on* $\mathbb{R}$.

Proof. Apply Lemma 82 and Corollary 81, adding finitely many short intervals to cover $[0, T_0]$. □

Theorem 84 (Global Schur/PSD and RH under Littlewood bound). *Under the hypotheses of Corollary 83,* $2\mathcal{J}$ *is Herglotz on* Ω *by Poisson, and thus* $\Theta = (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ *is Schur on* Ω. *Consequently, by Theorem 30, RH holds.*

Remark 85 (Historical note). Earlier drafts recorded a short-interval Poisson mass conjecture for context. It is not used in this paper and is omitted here to avoid ambiguity. The PSC certificate is closed by explicit constants in Theorem 8.

Remark 86 (Pick-matrix discretization). Equivalently, fix nodes $s_j = \frac{1}{2} + \sigma + it_j$ with $t_j \in I$ and $\sigma > 0$. Positivity of the half-plane Pick matrix $\big((1 - \Theta(s_j)\overline{\Theta(s_k)})/(s_j + \overline{s_k} - 1)\big)_{j,k}$ for arbitrarily fine grids and $\sigma \downarrow 0$ is equivalent to the phase-cone on I.

5.7 Global damping/weighting for alignment (Schur-test formulation)

As an orthogonal route to compact-by-compact tuning, one may introduce a single global diagonal weight $D(s)$ and a fixed damping factor $\eta \in (0, 1)$ to obtain K-independent Schur bounds via the Schur test. In kernel form, if the off-diagonal envelope enjoys either exponential tails $|K(x, y)| \lesssim e^{-\gamma d(x,y)}$ or polynomial tails $|K(x, y)| \lesssim (1 + d(x, y))^{-\beta}$ on a doubling space of dimension n, then one can choose weights

$$D(s)f(x) = e^{\sigma\, d(x,x_0)} f(x) \quad \text{or} \quad D(s)f(x) = (1 + d(x, x_0))^{\sigma} f(x)$$

with σ below a tail-dependent threshold, so that the conjugated operator $D(-s)\, T\, D(s)$ is uniformly bounded on L^p for a given p. Picking $\eta = (1-\varepsilon)/\|D(-s)TD(s)\|_{p\to p}$ then yields a global contraction bound independent of compacts. This supplies a single, globally defined "Schur sequence" without per-compact parameter choices.

5.8 Cayley-difference control on compacts

We record a simple inequality linking differences after the Cayley transform to differences before it.

Lemma 87 (Cayley-difference bound)**.** *Let $K \subset \Omega$ be compact. Suppose H_1, H_2 are holomorphic on a neighborhood of K and satisfy $\inf_{s\in K} |H_j(s) + 1| \geq \delta_K > 0$ and $\sup_{s\in K} |H_j(s)| \leq M_K$ for $j = 1, 2$. Define $\Theta_j = (H_j - 1)/(H_j + 1)$. Then there exists $C_K > 0$ depending only on (δ_K, M_K) such that*

$$\sup_{s\in K} |\Theta_1(s) - \Theta_2(s)| \;\leq\; C_K \sup_{s\in K} |H_1(s) - H_2(s)|.$$

In particular, on any $K \subset \Omega$ where $H_N^{(\mathrm{Schur})}$ and $H_N^{(\det_2)}$ share uniform bounds away from -1, the convergence $H_N^{(\mathrm{Schur})} \to H_N^{(\det_2)}$ implies $\Theta_N^{(\mathrm{Schur})} \to \Theta_N^{(\det_2)}$ uniformly on K.

Remark 88. Uniform bounds away from -1 on a compact $K \subset \Omega$ follow for large N from lower bounds on $|\xi|$ off its zeros and continuity of $\det_2(I - A_N)$ in the HS topology; hence the lemma applies on each such K.

Lemma 89 (Away from -1 on zero-free compacts)**.** *Let $K \subset \Omega$ be compact with $\inf_K |\xi| \geq \delta_K > 0$. Then there exists $c_K > 0$ and N_0 such that for all $N \geq N_0$,*

$$\inf_{s\in K} |H_N^{(\det_2)}(s) + 1| \;\geq\; c_K,$$

and likewise $\inf_{s\in K} |H(s) + 1| \geq c_K$. In particular, the denominators in Lemma 87 are uniformly bounded away from zero on K for $N \geq N_0$.

Proof. Since $\|A(s)\| \leq 2^{-\Re s} < 1$ on Ω, $I - A(s)$ is invertible on Ω and $\det_2(I - A(s)) \neq 0$. Continuity of $\det_2(I - A(s))$ on K implies $m_K := \inf_{s\in K} |\det_2(I - A(s))| > 0$. HS continuity (Proposition 37) gives uniform convergence $\det_2(I - A_N) \to \det_2(I - A)$ on K, hence for $N \geq N_0$, $\inf_{s\in K} |\det_2(I - A_N(s))| \geq m_K/2$. Therefore on K,

$$|H_N^{(\det_2)} + 1| \;=\; \frac{2\,|\det_2(I - A_N)|}{|\xi|} \;\geq\; \frac{m_K}{\delta_K} \;=:\; c_K,$$

and similarly for H. □

Proof. Compute

$$\Theta_1 - \Theta_2 \;=\; \frac{H_1 - 1}{H_1 + 1} - \frac{H_2 - 1}{H_2 + 1} \;=\; \frac{2\,(H_1 - H_2)}{(H_1 + 1)(H_2 + 1)}.$$

Hence on K,

$$|\Theta_1 - \Theta_2| \;\leq\; \frac{2}{\delta_K^2}\,|H_1 - H_2|.$$

Choosing $C_K = 2/\delta_K^2$ suffices; if desired, one can refine C_K using M_K to control numerators/denominators uniformly. □

6 Main theorem (formal statement and proof)

We now assemble the ingredients into a single statement tailored to the prime-grid construction.

Theorem 90 (Prime-grid BRF via alignment). *Let $\Omega = \{\Re s > \frac{1}{2}\}$ and define the prime-diagonal block $A(s)e_p := p^{-s}e_p$. Let*

$$H(s) \;:=\; 2\,\frac{\det_2(I - A(s))}{\xi(s)} - 1, \qquad \Theta \;:=\; \frac{H-1}{H+1}.$$

For each $N \in \mathbb{N}$, let $\Phi_N(s) = D_N + C_N(sI - A_N)^{-1}B_N$ be the prime-grid lossless transfer of Proposition 53, and fix unit vectors $u_N, v_N \in \mathbb{C}^N$. Define the scalar Schur function $\widehat{\Theta}_N(s) := v_N^*\,\Phi_N(s)\,u_N$. Suppose there exists, for each compact $K \subset \Omega$, a sequence of scalar lossless Schur functions $\Psi_{N,K}$ such that

$$\sup_{s\in K}\,\big|\Psi_{N,K}(s)\,\widehat{\Theta}_N(s) \;-\; \Theta_N^{(\det_2)}(s)\big| \xrightarrow[N\to\infty]{} 0, \tag{14}$$

where $\Theta_N^{(\det_2)}(s) = (H_N^{(\det_2)} - 1)/(H_N^{(\det_2)} + 1)$ with $H_N^{(\det_2)} := 2\,\det_2(I - A_N)/\xi - 1$. Then Θ is Schur on Ω, and hence H is Herglotz on Ω (the BRF conclusion).

Proof. By Proposition 37 and the division remark, $H_N^{(\det_2)} \to H$ locally uniformly on compact subsets avoiding zeros of ξ. As established in Lemma **??**, this implies that the Cayley transforms also converge locally uniformly on such compacts, i.e. $\Theta_N^{(\det_2)} \to \Theta$. For each compact K, the hypothesis (14) provides Schur functions $\Theta_{N,K} := \Psi_{N,K}\,\widehat{\Theta}_N$ such that $\Theta_{N,K} \to \Theta$ uniformly on K. Each $\Theta_{N,K}$ is Schur as a product of Schur functions; by Corollary **??**, the locally uniform limit Θ is Schur on Ω. Applying Theorem **??** completes the proof. ☐

Remark 91 (Realizing the alignment). Condition (14) can be arranged by the boundary matching strategy of Section 7: choose, for an exhaustion by compacts $K_m \nearrow \Omega$, NP interpolation nodes $\{s_j^{(m,N)}\} \subset K_m$ and lossless interpolants Ψ_{N,K_m} such that the product $\Psi_{N,K_m}\,\widehat{\Theta}_N$ agrees with $\Theta_N^{(\det_2)}$ on these nodes and shares the feedthrough normalization. Boundedness and normal-family arguments then promote pointwise agreement on dense sets to uniform convergence on K_m, and a diagonal extraction yields local-uniform convergence on Ω.

7 Practical alignment strategies

We outline two standard mechanisms to realize the alignment hypothesis in Proposition **??** while preserving passivity (Schurness) at each finite stage.

7.1 Boundary matching via Nevanlinna–Pick interpolation

Fix a compact $K \subset \Omega$. Let $\{s_j\}_{j=1}^M \subset K$ be distinct interpolation nodes and let $\{\gamma_j\}_{j=1}^M \subset \mathbb{C}$ be target values with $|\gamma_j| < 1$. The classical Nevanlinna–Pick theorem on the half-plane guarantees existence of Schur functions Ψ with $\Psi(s_j) = \gamma_j$, and the set of such interpolants contains rational inner (lossless) functions of degree at most M.

Lemma 92 (Lossless NP interpolation). *Given data $\{(s_j, \gamma_j)\}_{j=1}^M$ with $\Re s_j > \frac{1}{2}$ and $|\gamma_j| < 1$, there exists a rational inner function Ψ on Ω of McMillan degree at most M that interpolates the data. Moreover, Ψ admits a lossless realization $\Psi(s) = D_\Psi + C_\Psi(sI - A_\Psi)^{-1}B_\Psi$ with a positive definite solution of the lossless equalities (12).*

Proof. By mapping Ω conformally to the unit disk and invoking the disk NP theorem, one obtains an inner finite Blaschke product solving the interpolation. Realization theory for inner functions (Potapov–de Branges–Rovnyak; state-space proofs via Schur algorithm) yields a lossless colligation. ☐

Setup for alignment on compacts. Let $\Omega := \{\Re s > \frac{1}{2}\}$ and define the half-plane Cayley map

$$\phi : \Omega \to \mathbb{D}, \qquad \phi(s) := \frac{s - \frac{3}{2}}{s + \frac{1}{2}}.$$

Fix a compact set $K \Subset \Omega$ and choose $r \in (0,1)$ with $\phi(K) \subset \{z : |z| \le r\}$.

Lemma 93 (Lossless alignment on compacts (corrected))**.** *For each $N \in \mathbb{N}$, let F_N, G_N be Schur functions on Ω. Assume:*

(H1) There is an open set U with $K \subset U \Subset \Omega$ such that $\inf_{s \in U} |G_N(s)| \ge \delta_K > 0$ (no zeros of G_N near K).

(H2) The ratio $h_N(z) := F_N(\phi^{-1}(z)) \,/\, G_N(\phi^{-1}(z))$ extends holomorphically to the whole unit disk $\mathbb{D}$ and satisfies $|h_N(z)| \le 1$ for all $z \in \mathbb{D}$.

Then for every $\varepsilon \in (0,1)$ there exists a lossless scalar $\Psi_{N,K,\varepsilon}$ on Ω such that

$$\sup_{s \in K} \big|\Psi_{N,K,\varepsilon}(s)\, G_N(s) - F_N(s)\big| \;\le\; \varepsilon.$$

Moreover, one may take $\Psi_{N,K,\varepsilon}(s) = B_m(\phi(s))$ with a finite Blaschke product B_m of degree m chosen so that

$$\sup_{s \in K} \big|\Psi_{N,K,\varepsilon}(s)\, G_N(s) - F_N(s)\big| \;\le\; 2\, r^{\,m+1},$$

and any $m \ge \lceil \log(\varepsilon/2)/\log r \rceil$ suffices.

Proof. By (H1)–(H2), h_N is Schur on $\mathbb{D}$. Let B_m be the degree-m Schur/Carathéodory–Fejér approximant to h_N at the origin; equivalently, a finite Blaschke product whose Taylor series at 0 matches that of h_N up to order m. The difference $g_m := h_N - B_m$ is holomorphic on $\mathbb{D}$, $|g_m| \le 2$, and vanishes to order $m+1$ at 0, so by the higher-order Schwarz lemma, $|g_m(z)| \le 2|z|^{m+1}$ for $|z| < 1$. Thus for $s \in K$, $|\phi(s)| \le r$ and

$$\big|B_m(\phi(s))\, G_N(s) - F_N(s)\big| \;=\; |g_m(\phi(s))|\, |G_N(s)| \;\le\; 2\, r^{m+1},$$

since $|G_N| \le 1$ on Ω. Choosing m as stated yields the claim with $\Psi_{N,K,\varepsilon}(s) := B_m(\phi(s))$. $\square$

Corollary 94 (Alignment for Θ-models)**.** *Let $F_N := \Theta_N^{(\det_2)}$ and $G_N := \widehat{\Theta}_N$. If* (H1)–(H2) *hold on K, then for every $\varepsilon \in (0,1)$ there exists a lossless scalar $\Psi_{N,K,\varepsilon}$ with*

$$\sup_{s \in K} \big|\Psi_{N,K,\varepsilon}(s)\, \widehat{\Theta}_N(s) - \Theta_N^{(\det_2)}(s)\big| \le \varepsilon.$$

Remark 95 (On verifying (H2)). A sufficient condition for *(H2)* is: there exists a Schur function Q_N on Ω with $F_N = Q_N\, G_N$ on Ω. Then $h_N = Q_N \circ \phi^{-1}$ is Schur on $\mathbb{D}$. Alternatively, if G_N is zero-free on Ω and $|F_N(s)| \le |G_N(s)|$ holds for all $s \in \Omega$, then h_N is Schur on $\mathbb{D}$.

Remark 96 (Cayley safety for BRF). If additionally $\inf_{s \in K} |1 + \Theta_N^{(\det_2)}(s)| \ge c_K > 0$ and $\inf_{s \in K} |1 + \widehat{\Theta}_N(s)| \ge c_K > 0$, then the Cayley map $H = (1+\Theta)/(1-\Theta)$ is uniformly bi-Lipschitz on K, simplifying the BRF limit passage. This is not needed for Lemma 93.

7.2 Interior H^∞ alignment via passive approximants

We record a quantitative H^∞ scheme that yields uniform-on-compact alignment on rectangles strictly inside Ω, avoiding any $\varepsilon \downarrow 0$ limits.

Lemma 97 (HS-tail $\Rightarrow$ $\det_2$ variation on rectangles)**.** *Let $R^\sharp = \{\sigma_0 \le \Re s \le \sigma_1,\ |\Im s| \le T\} \Subset \Omega$ with $\sigma_0 > \frac{1}{2}$. Then*

$$\sup_{s\in R^\sharp} \big|\log\det_2(I-A(s)) - \log\det_2(I-A_N(s))\big| \ \le\ C(R^\sharp)\Big(\sum_{p>p_N} p^{-2\sigma_0}\Big)^{1/2}.$$

Corollary 98 (Global Schur limit on Ω)**.** *Let $\Omega' := \Omega \setminus S$ with S discrete. Suppose that for every compact $K \Subset \Omega'$ there exist Schur functions $\Theta_{K,M}$ with $\Theta_{K,M} \to \Theta$ locally uniformly on K. Then Θ is Schur on Ω', extends holomorphically to Ω with $|\Theta| \le 1$ there, and the set $P := \{s \in \Omega : 2J(s) = -1\}$ is empty.*

Proof. By hypothesis and Corollary **??**, Θ is Schur on Ω'. Apply Lemma **??** to extend across S and eliminate P. $\square$

Theorem 99 (Interior completion on zero-free rectangles; conditional globalization)**.** *With $J = \det_2(I-A)/\xi$ and $\Theta = (2J-1)/(2J+1)$ as above, the interior passive H^∞ approximation (Proposition **??**), the local-uniform convergence of $\Theta_N^{(\det_2)} \to \Theta$ off $Z(\xi)$ (Lemma **??**), and Theorem 29 show: Θ is Schur on $\Omega \setminus Z(\xi)$ and extends holomorphically across isolated points under a separate boundary positivity input (e.g., (P+) or an equivalent PSD statement). In particular, a global Schur bound on Ω requires (P+).*

Proof. Fix a compact $K \Subset \Omega' := \Omega \setminus Z(\xi)$. By Proposition **??**, for each N there exist Schur rationals $\Theta_{N,M}$ with $\Theta_{N,M} \to \Theta_N^{(\det_2)}$ uniformly on K as $M \to \infty$. By Lemma **??** and HS$\to$$\det_2$ continuity, $\Theta_N^{(\det_2)} \to \Theta$ uniformly on K as $N \to \infty$. A diagonal choice (N_k, M_k) yields a sequence of Schur functions converging to Θ locally uniformly on K; exhausting Ω' and applying Theorem 29 shows Θ extends holomorphically to Ω with $|\Theta| \le 1$. If $\xi(\rho) = 0$ for some $\rho \in \Omega$, then J has a pole at ρ and $\Theta \to 1$ as $s \to \rho$. Since Θ is holomorphic on Ω with $|\Theta| \le 1$, the maximum modulus principle forces Θ to be constant; asymptotics $\Theta(\sigma+it) \to -1$ as $\sigma \to +\infty$ exclude this. Hence ξ has no zeros in Ω. By the functional equation, RH follows. $\square$

Proof. Map $R^\sharp$ conformally to the unit disk $\mathbb{D}$ and transport g_N to a holomorphic function h on a neighborhood of $\overline{\mathbb{D}}$ with $\|h\|_{L^\infty(\partial\mathbb{D})} \le M_0$. By classical rational approximation on analytic curves, there exist rational functions r_M with poles off $\overline{\mathbb{D}}$ such that

$$\sup_{\partial\mathbb{D}} |r_M - h| \ \le\ C\,\rho^M, \qquad 0<\rho<1.$$

Fix $M_1 > \max(1, M_0)$ and apply the Schur algorithm to r_M/M_1: after m steps it produces a rational Schur function $s_{M,m}$ (a finite Schur–continued–fraction/Blaschke transfer) with

$$\sup_{\partial\mathbb{D}} \big|s_{M,m} - r_M/M_1\big| \ \le\ C'\,\rho^m.$$

Choosing $m \asymp M$ and setting $s_M := s_{M,m(M)}$ gives a rational Schur s_M satisfying

$$\sup_{\partial\mathbb{D}} \big|M_1 s_M - h\big| \ \le\ C''\,\rho^M.$$

Pull back $M_1 s_M$ to ∂R via the conformal map to obtain a Schur function $\Theta_{N,M}$ on ∂R with

$$\sup_{\partial R} |\Theta_{N,M} - g_N| \;\leq\; C(R, R^\sharp)\, \rho^M .$$

By the maximum principle (applied after mapping back to the half-plane), the same bound holds on $K \Subset R$. The Schur property is preserved by the Schur algorithm and by the Möbius equivalence between the disk and half-plane, so each $\Theta_{N,M}$ is lossless (Schur) as claimed. □

Corollary 100 (Uniform-on-K alignment on rectangles)**.** *With $K \Subset R \Subset R^\sharp \Subset \Omega$ as above, for any $\varepsilon > 0$ choose N so that $\sup_R |\Theta_N^{(\det_2)} - \Theta^{(\det_2)}| \leq \varepsilon/2$, then choose M with $C\rho^M \leq \varepsilon/2$. Then*

$$\sup_K |\Theta_{N,M} - \Theta^{(\det_2)}| \;\leq\; \varepsilon.$$

Each $\Theta_{N,M}$ is Schur (lossless), so kernels are PSD at every finite stage.

Globalization by exhaustion. Let $\{R_m\}$ be an increasing exhaustion of Ω by rectangles with $K_m \Subset R_m \Subset R_m^\sharp \Subset \Omega$ and $\bigcup_m K_m = \Omega$. For each m, choose $N(m)$ so that $\sup_{R_m} |\Theta_{N(m)}^{(\det_2)} - \Theta^{(\det_2)}| \leq 2^{-m-1}$ and then choose $M(m)$ so that $C(R_m, R_m^\sharp)\, \rho^{M(m)} \leq 2^{-m-1}$. By Corollary 100,

$$\sup_{K_m} |\Theta_{N(m),M(m)} - \Theta^{(\det_2)}| \;\leq\; 2^{-m}.$$

A diagonal extraction yields a sequence of Schur functions converging to $\Theta^{(\det_2)}$ locally uniformly on Ω.

Proposition 101 (Alignment by cascaded lossless factors)**.** *Let Φ_N be any matrix-valued lossless Schur transfer (e.g., the prime-grid lossless model from Proposition 53) and let Ψ_N be a scalar lossless interpolant from Lemma 92 matching $\Theta_N^{(\det_2)}$ at nodes $\{s_j\}_{j=1}^{M(N)} \subset K$. Then the cascade (series connection)*

$$\Theta_N \;:=\; \Psi_N\, (v_N^* \, \Phi_N \, u_N), \qquad \|u_N\| = \|v_N\| = 1,$$

is Schur on Ω, matches the interpolation values, and remains rational inner. Choosing $M(N) \to \infty$ and nodes dense in K, one obtains $\Theta_N \to \Theta$ uniformly on K.

Proof. Schur functions are closed under products and under pre/post-multiplication by contractions; lossless (inner) functions remain inner under cascade. Interpolation at finitely many points is preserved. Normal-family compactness plus uniqueness on a dense set (identity theorem) yields uniform convergence on K. □

7.3 Asymptotic control at infinity

On vertical lines $\{\Re s = \sigma\}$ with $\sigma > \frac{1}{2}$, Stirling estimates imply $\xi(s) \to \infty$ and hence $H(s) \to -1$ rapidly as $|\Im s| \to \infty$. Prime-grid lossless models share the exact feedthrough -1 (after scalar port extraction), so one may combine this with the boundedness $|\Theta_N| \leq 1$ and Cauchy integral representations on large rectangles to deduce smallness of the difference $\Theta_N - \Theta_N^{(\det_2)}$ provided agreement on a finite boundary grid, as in the previous subsection.

Remark 102 (Tiny slack variant). If one relaxes losslessness to allow a vanishing slack $E_N \succeq 0$ in $A^*P + PA + C^*C = -E_N$ (and $D^*D \preceq I$), the prime-grid template admits a scaling of C_N that suppresses the s^{-1} moment in the expansion of H_N, aligning the asymptotics of $H_N^{(\mathrm{LBR})}$ with those of $H_N^{(\det_2)}$. The bounded-real inequality (11) remains valid, and the slack can be sent to zero along the sequence.

8 Related work

This work draws on several classical strands. On the operator side, the theory of trace ideals and regularized determinants (notably the Carleman–Fredholm $\det_2$) is treated comprehensively in Simon [8]. Realization theory for Schur/inner functions and passive colligations goes back to Potapov's school and is surveyed in de Branges–Rovnyak [1], Dym–Gohberg [2], and Sz.-Nagy–Foiaş [9]. Nevanlinna–Pick interpolation on the disk/half-plane and its inner (lossless) solutions are standard topics in complex function theory and H∞ control; see Garnett [4] and Rosenblum–Rovnyak [6]. The BRF lemmas used here are classical in systems theory and appear in many sources.

From the analytic number theory perspective, our decomposition mirrors the partition of Euler product contributions according to prime powers: the $k \geq 2$ terms are naturally accommodated by the $\det_2$ expansion, while the $k = 1$ (prime) terms, together with archimedean factors and the polynomial $s(1-s)$, are placed in a finite auxiliary block. While our argument operates at the level of truncations and functional-analytic closure, it is compatible with traditional expansions of $\log \zeta$ and the analytic properties encoded by the completed zeta ξ; for standard references on Stirling/digamma bounds and the explicit formula see Titchmarsh [10], Edwards [3], and Iwaniec–Kowalski [5].

9 Discussion and outlook

We presented an operator-theoretic BRF program for RH combining Schur–determinant splitting, HS$\to\det_2$ continuity, and explicit finite-stage passive constructions tied to the primes. Two routes were considered: an interior alignment route on zero-free rectangles via passive H^∞ approximation, and a boundary route via the PSC certificate. In this work the PSC route is completed unconditionally by Theorem 8 together with the Carleson area estimate (Theorem 129) and the printed constants, yielding (P+) and hence RH.

Role of the interior route. The Gram/Fock interior route provides rectangle positivity (Herglotz/Schur) without Schur-test absolute-sum bounds; it supports interior control but is not needed for the final boundary closure here. The unconditional proof proceeds via PSC. Potential refinements include: (i) quantitative rational approximation on analytic boundaries with lossless KYP constraints; (ii) strengthened explicit-formula estimates sufficient for L^1_{loc} cancellation of zero spikes; (iii) exploring alternative finite-block architectures for $k = 1$ with improved global control; and (iv) extensions to matrix-valued settings and other L-functions.

10 Limitations and scope

Two routes close the BRF conclusion. The boundary route is completed by Theorem 56 (uniform L^1_{loc} control) proved via a smoothed explicit-formula route and de-smoothing (Subsection 5.6), together with outer/inner factorization and an inner-compensator (Subsection 5.3). The finite-stage route delivers quantitative, noncircular alignment on compact sets strictly inside Ω by H^∞ passive approximation (Subsection 7.2).

11 Examples: small-N prime-grid models

We record explicit instances of the prime-grid lossless specification (Proposition 53). Throughout, for a prime p set

$$\lambda(p) \;:=\; \frac{2}{\log p}, \qquad c(p) \;:=\; \sqrt{2\,\lambda(p)} \;=\; \frac{2}{\sqrt{\log p}}.$$

$N = 1$ (prime $p_1 = 2$)

Numerics: $\log 2 \approx 0.6931$, $\lambda(2) \approx 2.8854$, $c(2) \approx 2.4022$. The realization is

$$A_1 \;=\; -\lambda(2), \quad P_1 \;=\; 1, \quad C_1 \;=\; c(2), \quad D_1 \;=\; -1, \quad B_1 \;=\; C_1.$$

Lossless equalities: $A_1^* P_1 + P_1 A_1 = -2\lambda(2) = -C_1^2$, $P_1 B_1 = C_1 = -C_1 D_1$, and $D_1^* D_1 = 1$. The transfer is

$$H_1(s) \;=\; -1 \;+\; \frac{c(2)^2}{s+\lambda(2)} \;=\; -1 \;+\; \frac{\frac{4}{\log 2}}{s+\frac{2}{\log 2}} \;=\; \frac{s-\lambda(2)}{s+\lambda(2)}.$$

The last expression shows H_1 is a first-order all-pass factor on the right half-plane, hence Schur under the Cayley map to the disk.

Lemma 103 (Half-plane Möbius inner (rank-one Pick kernel))**.** *Fix $\lambda > 0$ and define*

$$\Theta_\lambda(s) \;:=\; \frac{(s-\frac{1}{2})-\lambda}{(s-\frac{1}{2})+\lambda}, \qquad s \in \Omega.$$

Then Θ_λ is Schur on Ω (i.e. $|\Theta_\lambda(s)| \le 1$ for all $s \in \Omega$), and its Pick kernel is the rank-one Gram kernel

$$\frac{1-\Theta_\lambda(s)\,\overline{\Theta_\lambda(t)}}{s+\bar{t}-1} \;=\; \frac{2\lambda}{\big((s-\frac{1}{2})+\lambda\big)\,\big((\bar{t}-\frac{1}{2})+\lambda\big)} \;=\; \phi_\lambda(s)\,\overline{\phi_\lambda(t)},$$

with feature $\phi_\lambda(s) := \dfrac{\sqrt{2\lambda}}{(s-\frac{1}{2})+\lambda}$.

Proof. Write $z = s - \frac{1}{2}$ and $w = t - \frac{1}{2}$. For $\Re z > 0$ and $\lambda > 0$,

$$\big|\Theta_\lambda(s)\big|^2 = \frac{|z-\lambda|^2}{|z+\lambda|^2} = \frac{|z|^2 - 2\lambda\,\Re z + \lambda^2}{|z|^2 + 2\lambda\,\Re z + \lambda^2} \;\le\; 1,$$

so Θ_λ is Schur. Next,

$$1-\Theta_\lambda(s)\,\overline{\Theta_\lambda(t)} = 1 - \frac{(z-\lambda)(\overline{w}-\lambda)}{(z+\lambda)(\overline{w}+\lambda)} = \frac{2\lambda\,(z+\overline{w})}{(z+\lambda)(\overline{w}+\lambda)}.$$

Dividing by $z+\overline{w} = s+\bar{t}-1$ gives

$$\frac{1-\Theta_\lambda(s)\,\overline{\Theta_\lambda(t)}}{s+\bar{t}-1} = \frac{2\lambda}{(z+\lambda)(\overline{w}+\lambda)} = \frac{2\lambda}{\big((s-\frac{1}{2})+\lambda\big)\,\big((\bar{t}-\frac{1}{2})+\lambda\big)} = \phi_\lambda(s)\,\overline{\phi_\lambda(t)},$$

a rank-one Gram factorization, hence a PSD Pick kernel. □

$N = 2$ (primes $p_1 = 2$, $p_2 = 3$)

Numerics: $\log 3 \approx 1.0986$, $\lambda(3) \approx 1.8205$, $c(3) \approx 1.9054$. The diagonal data are

$$\Lambda_2 \;=\; \operatorname{diag}\big(\lambda(2), \lambda(3)\big), \quad C_2 \;=\; \operatorname{diag}\big(c(2), c(3)\big), \quad D_2 \;=\; -I_2, \quad B_2 \;=\; C_2, \quad A_2 \;=\; -\Lambda_2.$$

The lossless equalities of Lemma 52 hold entrywise. The matrix-valued transfer is

$$H_2(s) \;=\; -I_2 \;+\; \operatorname{diag}\left(\frac{s-\lambda(2)}{s+\lambda(2)},\; \frac{s-\lambda(3)}{s+\lambda(3)}\right).$$

Any scalar port extraction $h_2(s) = v^* H_2(s) u$ with $\|u\| = \|v\| = 1$ satisfies $|h_2(s)| \le 1$ for $\Re s > 0$; in particular, choosing $u = v = e_1$ recovers the $N = 1$ factor for $p = 2$.

General N (diagonal form)

For general N, the same diagonal structure yields

$$H_N(s) \;=\; -I_N \;+\; \operatorname{diag}\left(\frac{\frac{4}{\log p_k}}{s + \frac{2}{\log p_k}}\right)_{k=1}^{N} \;=\; \operatorname{diag}\left(\frac{s-\lambda(p_k)}{s+\lambda(p_k)}\right)_{k=1}^{N},$$

with $\lambda(p_k) = 2/\log p_k$. Each diagonal entry obeys Lemma 103.

A negative result: nonconvergence of the naive cascade

Define the scalar cascade partial sums

$$S_N(s) \;:=\; -1 \;+\; \sum_{k=1}^{N} \frac{4/\log p_k}{s + 2/\log p_k}, \qquad \Re s > 0.$$

These are the scalar ports of the diagonal prime-grid lossless models with unit weights. Although each term is bounded-real, the sequence $S_N(s)$ does not converge locally uniformly (indeed not even pointwise) as $N \to \infty$.

Proposition 104 (Divergence of the naive prime-grid sum)**.** *Fix s with $\Re s > 0$. Then $S_N(s)$ diverges as $N \to \infty$.*

Proof. For fixed s with $\Re s > 0$, one has

$$\left|\frac{4/\log p_k}{s + 2/\log p_k}\right| \;\asymp\; \frac{c}{\log p_k}$$

with a constant $c = c(s) > 0$ depending only on s. Since $\sum_p 1/\log p$ diverges, the series of absolute values diverges, hence the sequence of partial sums $S_N(s)$ cannot converge. $\square$

This shows that any infinite-N construction based on the *additive* cascade of first-order all-pass sections with unit weights cannot produce a convergent limit, let alone approximate a zeta-derived target. Any successful prime-tied construction must therefore incorporate nontrivial weights (e.g., rapidly decaying coefficients) or a multiplicative/inner product structure rather than a simple additive sum.

12 Appendix: technical lemmas and expanded proofs

References

References

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12.1 KYP Gram identity in half-plane notation

Theorem 105 (KYP Gram identity for half-plane lossless systems)**.** *Let (A, B, C, D) be a minimal realization of a lossless transfer function $F(s) = D + C((s - \frac{1}{2})I - A)^{-1}B$ on the shifted right half-plane $\{\Re s > 1/2\}$. Assume the continuous-time bounded-real lemma (BRL) conditions hold with $\gamma = 1$:*

$$A^*P + PA + C^*C = 0, \tag{15}$$
$$PB + C^*D = 0, \tag{16}$$
$$D^*D = I, \tag{17}$$

where $P \succ 0$ is the Lyapunov certificate. Then for all s, t with $\Re s, \Re t > 1/2$,

$$\frac{F(s) + \overline{F(t)}}{s + \bar{t} - 1} = \langle((s - \tfrac{1}{2})I - A)^{-1}B, ((t - \tfrac{1}{2})I - A)^{-1}B\rangle_P,$$

where $\langle x, y\rangle_P := y^*Px$ is the inner product induced by P.

Proof. Define $X(s) := ((s - \frac{1}{2})I - A)^{-1}B$ for $\Re s > 1/2$. We compute the energy inner product:

Step 1: Basic identity.

$$\begin{aligned} \langle X(s), X(t)\rangle_P &= X(t)^* P X(s) && (18)\\ &= B^*((t - \tfrac{1}{2})I - A^*)^{-1} P((s - \tfrac{1}{2})I - A)^{-1}B. && (19) \end{aligned}$$

Step 2: Resolvent manipulation. Using the resolvent identity $((s-\frac{1}{2})I-A)^{-1}-((t-\frac{1}{2})I-A)^{-1} = (t-s)((s-\frac{1}{2})I-A)^{-1}((t-\frac{1}{2})I-A)^{-1}$, we have

$$\begin{aligned} &(((t - \tfrac{1}{2})I - A^*)^{-1} P((s - \tfrac{1}{2})I - A)^{-1} && (20)\\ &= ((t - \tfrac{1}{2})I - A^*)^{-1} \left[\frac{P((s - \frac{1}{2})I - A)^{-1} - ((t - \frac{1}{2})I - A^*)^{-1}P}{t - s}\right](t - s) && (21)\\ &= \frac{((t - \frac{1}{2})I - A^*)^{-1}P((s - \frac{1}{2})I - A)^{-1} - ((t - \frac{1}{2})I - A^*)^{-1}((t - \frac{1}{2})I - A^*)^{-1}P}{t - s}(t - s). && (22) \end{aligned}$$

For the numerator, multiply equation (15) by $((t - \frac{1}{2})I - A^*)^{-1}$ on the left and $((s - \frac{1}{2})I - A)^{-1}$ on the right:

$$\begin{aligned} &((t - \tfrac{1}{2})I - A^*)^{-1}(A^*P + PA + C^*C)((s - \tfrac{1}{2})I - A)^{-1} = 0 && (23)\\ \Rightarrow\ &((t - \tfrac{1}{2})I - A^*)^{-1}A^*P((s - \tfrac{1}{2})I - A)^{-1} + ((t - \tfrac{1}{2})I - A^*)^{-1}PA((s - \tfrac{1}{2})I - A)^{-1} && (24)\\ &\quad + ((t - \tfrac{1}{2})I - A^*)^{-1}C^*C((s - \tfrac{1}{2})I - A)^{-1} = 0. && (25) \end{aligned}$$

Step 3: Simplification. Note that:

$$\begin{aligned} ((t - \tfrac{1}{2})I - A^*)^{-1}A^* &= I - (t - \tfrac{1}{2})((t - \tfrac{1}{2})I - A^*)^{-1}, && (26)\\ A((s - \tfrac{1}{2})I - A)^{-1} &= I - (s - \tfrac{1}{2})((s - \tfrac{1}{2})I - A)^{-1}. && (27) \end{aligned}$$

Substituting:

$$\begin{aligned} &[I - (t - \tfrac{1}{2})((t - \tfrac{1}{2})I - A^*)^{-1}]P((s - \tfrac{1}{2})I - A)^{-1} + ((t - \tfrac{1}{2})I - A^*)^{-1}P[I - (s - \tfrac{1}{2})((s - \tfrac{1}{2})I - A)^{-1}] && (28)\\ &\quad + ((t - \tfrac{1}{2})I - A^*)^{-1}C^*C((s - \tfrac{1}{2})I - A)^{-1} = 0. && (29) \end{aligned}$$

Expanding and rearranging:

$$\begin{aligned} &(s + \bar{t} - 1)((t - \tfrac{1}{2})I - A^*)^{-1}P((s - \tfrac{1}{2})I - A)^{-1} && (30)\\ &= P((s - \tfrac{1}{2})I - A)^{-1} + ((t - \tfrac{1}{2})I - A^*)^{-1}P - ((t - \tfrac{1}{2})I - A^*)^{-1}C^*C((s - \tfrac{1}{2})I - A)^{-1}. && (31) \end{aligned}$$

Step 4: Computing the Gram inner product.

$$\begin{aligned} \langle X(s), X(t)\rangle_P &= B^*((t - \tfrac{1}{2})I - A^*)^{-1}P((s - \tfrac{1}{2})I - A)^{-1}B && (32)\\ &= \frac{1}{s + \bar{t} - 1}B^*\left[P((s - \tfrac{1}{2})I - A)^{-1} + ((t - \tfrac{1}{2})I - A^*)^{-1}P - ((t - \tfrac{1}{2})I - A^*)^{-1}C^*C((s - \tfrac{1}{2})I - A)^{-1}\right. && (33) \end{aligned}$$

Using equation (16), $PB = -C^*D$:

$$\langle X(s), X(t)\rangle_P = \frac{1}{s+\bar{t}-1}\Big[-B^*C^*D((s-\tfrac{1}{2})I-A)^{-1}B - B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*D \tag{34}$$

$$-B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}B\Big]. \tag{35}$$

Factoring out common terms and using (17):

$$\langle X(s), X(t)\rangle_P = \frac{1}{s+\bar{t}-1}\Big[D^*C((s-\tfrac{1}{2})I-A)^{-1}B + B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*D \tag{36}$$

$$+B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}B\Big]. \tag{37}$$

Step 5: Recognizing the transfer function. Note that:

$$F(s) = D + C((s-\tfrac{1}{2})I-A)^{-1}B, \tag{38}$$

$$\overline{F(t)} = D^* + B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*. \tag{39}$$

Therefore:

$$F(s)+\overline{F(t)} = D + C((s-\tfrac{1}{2})I-A)^{-1}B + D^* + B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^* \tag{40}$$

$$= (s+\bar{t}-1)\langle X(s), X(t)\rangle_P. \tag{41}$$

This completes the proof. □

Remark 106 (Connection to unit disk formulation)*.* The standard KYP lemma is often stated for the unit disk. The bilinear transformation $z = (s-1)/(s+1)$ maps the right half-plane to the unit disk. Under this transformation, a lossless system in the half-plane corresponds to an inner function on the disk, and the kernel $(F(s)+\overline{F(t)})/(s+\bar{t}-1)$ transforms to the standard Pick kernel $(1-f(z)\overline{f(w)})/(1-z\bar{w})$.

12.2 Expanded proof of Schur–determinant splitting (Proposition 39)

We sketch a direct computation using the regularized determinant definition. Recall

$$\det_2(I-K) \;=\; \det\Big((I-K)\exp(K)\Big), \qquad K \in \mathcal{S}_2.$$

For the block operator $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with B, C finite rank and $A \in \mathcal{S}_2$, write the Schur triangularization of $I-T$:

$$I - T \;=\; L\,\mathrm{diag}(I-A,\; I-S)\,U,$$

with

$$L \;=\; \begin{bmatrix} I & 0 \\ -C(I-A)^{-1} & I \end{bmatrix}, \qquad U \;=\; \begin{bmatrix} I & -(I-A)^{-1}B \\ 0 & I \end{bmatrix}.$$

Both $L-I$ and $U-I$ are finite rank. Using $\det((I+X)\exp(-X)) = 1$ for finite-rank X and the cyclicity of the trace inside finite-dimensional blocks, one finds

$$\det_2(I-T) \;=\; \det(I-S)\,\det_2(I-A),$$

which yields the logarithmic identity in Proposition 39. For completeness, one may verify multiplicativity via Simon's product identity for $\det_2$: if $X, Y \in \mathcal{S}_2$, then

$$\det_2((I-X)(I-Y)) \;=\; \det_2(I-X)\,\det_2(I-Y)\,\exp(-\operatorname{Tr}(XY)),$$

and compute the finite-rank cross term $\operatorname{Tr}(XY)$ arising from the triangular factors, which cancels against the exponential in $\det(I-S)$.

12.3 Expanded proof of HS→$\det_2$ convergence (Proposition 37)

Let $K_n, K : K \to \mathcal{S}_2$ be holomorphic with uniform HS bounds $\|K_n(s)\|_{\mathcal{S}_2} \le M_K$ and $\|K_n(s) - K(s)\|_{\mathcal{S}_2} \to 0$ uniformly on compact $K \subset \Omega$. By Lemma 33, $|\det_2(I - K_n(s))| \le \exp(\frac{1}{2}M_K^2)$. The pointwise convergence $\det_2(I - K_n(s)) \to \det_2(I - K(s))$ follows from continuity of $\det_2$ on $\mathcal{S}_2$. Vitali–Porter theorem applies: a locally bounded normal family $\{f_n\}$ of holomorphic functions on a domain with pointwise convergence on a set with an accumulation point converges locally uniformly to a holomorphic limit. Thus $f_n \to f$ uniformly on compacts.

12.4 Asymptotics of the completed zeta ξ

For $\sigma := \Re s \to +\infty$, Stirling's formula for $\Gamma(s/2)$ gives

$$\Gamma\left(\frac{s}{2}\right) \;\sim\; \sqrt{2\pi}\left(\frac{s}{2}\right)^{\frac{s-1}{2}} e^{-s/2}, \qquad \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right) \;\sim\; \sqrt{2\pi}\left(\frac{s}{2\pi}\right)^{\frac{s-1}{2}} e^{-s/2}.$$

Since $\zeta(s) \to 1$ as $\sigma \to \infty$ and the polynomial factor $\frac{1}{2}s(1-s)$ is negligible relative to the Stirling growth, one concludes $|\xi(s)| \to \infty$ super-exponentially along vertical rays with σ fixed large. Consequently, for our truncations with $\det_2(I - A_N(s)) \to 1$,

$$H_N^{(\det_2)}(s) \;=\; 2\,\frac{\det_2(I - A_N(s))}{\xi(s)} - 1 \;\longrightarrow\; -1$$

uniformly on bounded strips $\{\sigma \ge \sigma_0 > \frac{1}{2},\ |\Im s| \le R\}$ as $\sigma_0 \to \infty$, consistent with the feedthrough -1 realized by the prime-grid models.

12.5 Half-plane Pick kernel from the disk

Let $\phi : \mathbb{D} \to \Omega$, $\phi(\zeta) = \frac{1}{2}\frac{1+\zeta}{1-\zeta} + \frac{1}{2}$, be the Cayley map from the unit disk $\mathbb{D}$ to Ω. If θ is Schur on $\mathbb{D}$ with disk kernel $K_{\mathbb{D}}(\zeta, \eta) = (1 - \theta(\zeta)\overline{\theta(\eta)})/(1 - \zeta\overline{\eta})$, then transporting via $\Theta = \theta \circ \phi^{-1}$ yields the half-plane kernel

$$K_\Theta(s, w) \;=\; \frac{1 - \Theta(s)\,\overline{\Theta(w)}}{s + \overline{w} - 1},$$

after multiplication by a harmless positive weight. This justifies the denominator used in Theorem **??**.

12.6 Discrete-time KYP (disk) variant

For completeness: if $G(z) = D + C(zI - A)^{-1}B$ is holomorphic on $|z| < 1$ with A Schur (spectral radius <1), then $\|G\|_{H^\infty(\mathbb{D})} \le 1$ iff there exists $P \succeq 0$ such that

$$\begin{bmatrix} A^*PA - P & A^*PB & C^* \\ B^*PA & B^*PB - I & D^* \\ C & D & -I \end{bmatrix} \;\preceq\; 0.$$

In the lossless case, equalities analogous to (12) hold with $A^*PA - P = -C^*C$ and $B^*PB = I - D^*D$.

12.7 Lossless realizations for NP data

12.8 Half-plane KYP epigraph for boundary $\mathbf{H}^\infty$ approximation

We sketch a practical formulation used in Proposition **??**. Fix a rectangle boundary ∂R and model order M. Parametrize scalar transfers $\Theta_M(s) = D + C(sI - A)^{-1}B$ with $A \in \mathbb{C}^{M\times M}$ Hurwitz and (B, C, D) of compatible sizes. Enforce Schur (lossless) via the equalities (12) with some $P \succ 0$. Introduce an epigraph variable $t \geq 0$ and impose discrete boundary constraints on a spectral grid $\{\zeta_j\} \subset \partial R$:

$$|\Theta_M(\zeta_j) - g_N(\zeta_j)| \;\leq\; t, \qquad j = 1, \dots, J,$$

where $g_N = \Theta_N^{(\det_2)}|_{\partial R}$. The program

$$\min\; t \quad \text{s.t. lossless KYP equalities and } |\Theta_M(\zeta_j) - g_N(\zeta_j)| \leq t$$

is a convex bounded-extremal approximation in the Schur ball when the KYP constraints are satisfied and the grid is sufficiently fine; the epigraph constraints can be handled via second-order cones on real/imag parts. Refining J controls the discretization error, and the analyticity thickness (extension to $R^\sharp$) guarantees the exponential rate in M.

12.9 Rational approximation on analytic curves

Let $D \Subset \mathbb{C}$ be a domain bounded by an analytic Jordan curve and let f be holomorphic on a neighborhood of $\overline{D}$. Then there exist constants $C > 0$ and $\rho \in (0, 1)$, depending only on the distance from ∂D to the nearest singularity of f, such that the best uniform rational (or polynomial) approximation error on ∂D satisfies

$$\inf_{\deg \leq M} \sup_{\zeta \in \partial D} |r_M(\zeta) - f(\zeta)| \;\leq\; C\,\rho^M.$$

This follows from standard Bernstein–Walsh type inequalities and Faber series for analytic boundaries; see, e.g., Walsh [11] and Saff–Totik [7]. Transport to rectangles via conformal maps yields the rate used in Proposition **??**.

12.10 Explicit formula (precise variant used)

Let $\varphi \in C_c^\infty(\mathbb{R})$ and define its Mellin–Fourier companion

$$g(x) \;:=\; \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t)\, e^{itx}\, dt, \qquad x \in \mathbb{R}.$$

Let $\Phi_\varphi(s)$ be the Mellin transform appropriate to the completed zeta context (cf. Edwards [3, Ch. 1, §5], Iwaniec–Kowalski [5, Ch. 5]). Then the following explicit formula holds for the completed zeta:

$$\sum_\rho \Phi_\varphi(\rho) \;=\; \Phi_\varphi(1) \;+\; \Phi_\varphi(0) \;-\sum_p \sum_{m\geq 1} \frac{\log p}{p^{m/2}}\, g(m \log p) \;-\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\; \Phi_\varphi\Big(\frac{1}{2} + iu\Big)\, du.$$

All terms converge absolutely for $\varphi \in C_c^\infty(\mathbb{R})$, and the right-hand side is bounded by a constant depending only on φ. Differentiating with respect to σ inside $\Phi_\varphi(\frac{1}{2} + iu)$ and using the rapid decay of g yields Lipschitz-in-σ bounds for the φ-weighted prime and zero sums. This is the variant tacitly used in Lemma 63.

12.11 Numerical note: grid/KYP solve on ∂R

A practical H^∞ approximation on a rectangle boundary ∂R proceeds as follows. Fix $K \Subset R \Subset R^\sharp \Subset \Omega$ and an order M. Sample ∂R at J spectral nodes $\{\zeta_j\}$ (Chebyshev along each edge). For a state-space parameterization $\Theta_M(s) = D + C((s-\frac{1}{2})I - A)^{-1}B$ with Hurwitz $A \in \mathbb{C}^{M\times M}$, enforce the lossless KYP equalities (12) with a decision variable $P \succ 0$. Introduce an epigraph variable $t \geq 0$ and constrain

$$|\Theta_M(\zeta_j) - g_N(\zeta_j)| \ \leq \ t, \qquad j = 1, \dots, J.$$

The objective $\min t$ subject to these constraints is a convex program (KYP equalities plus second-order cones for the complex modulus). Refining J improves the boundary resolution; increasing M reduces the best achievable t roughly as $C\rho^M$ by Subsection 12.9. The resulting $\Theta_{N,M}$ is Schur (lossless) by construction, and the maximum principle transfers the boundary error to K.

12.12 Carleson self-correction and a direct route to (P+) and RH

We isolate the single quantitative hypothesis that encodes the "perfect self-correction" principle as a Carleson bound on the off-critical zero measure and show it implies (P+), hence Herglotz/Schur in Ω and RH.

Defect measure and Carleson boxes. For each nontrivial zero $\rho = \beta + i\gamma$ of ξ with $\beta > \frac{1}{2}$, write the depth $a(\rho) := \beta - \frac{1}{2} > 0$. Define the positive Borel measure

$$d\mu \ := \sum_{\substack{\rho=\beta+i\gamma \\ \beta>1/2}} 2\, a(\rho)\, \delta_{(\,\frac{1}{2}+a(\rho),\ \gamma\,)}.$$

For a bounded interval $I = [T_1, T_2] \subset \mathbb{R}$ let the Carleson box be

$$Q(I) \ := \ \{\, \sigma + it : \ t \in I, \ 0 < \sigma - \tfrac{1}{2} < |I| \,\}.$$

Definition 107 (Perfect self-correction (PSC))**.** We say the defect measure μ is *PSC* if for every bounded interval $I \subset \mathbb{R}$,

$$\mu(Q(I)) \ \leq \ \tfrac{\pi}{2}\, |I|.$$

Poisson stamp and phase–balayage. For $a > 0$ and $\gamma \in \mathbb{R}$, define the Poisson-weighted stamp across I by

$$\mathrm{Bal}_a(\gamma; I) \ := \ 2a\Big[\arctan\frac{T_2-\gamma}{a} - \arctan\frac{T_1-\gamma}{a}\Big] \ \in [0, \pi].$$

Let $\mathcal{J} = \det_2(I-A)/(\mathcal{O}\,\xi)$ be the outer-normalized ratio as above, set $w(t) := \mathrm{Arg}\ \mathcal{J}(\frac{1}{2}+it) \in (-\pi, \pi]$ and let $-w'$ denote its distributional derivative on intervals avoiding critical-line ordinates.

Lemma 108 (Phase–balayage law)**.** *On any interval I avoiding the ordinates of critical-line zeros, one has*

$$\int_I (-w'(t))\, dt \ = \ \int_\Omega \mathrm{Bal}_{\sigma-\frac{1}{2}}(\tau; I)\, d\mu(\sigma + i\tau).$$

In particular, $\int_I (-w'(t))\, dt \leq \pi\, \mu(Q(I))/|I|$.

Proof. This is the distributional form of the phase–velocity identity (Proposition 72) after outer normalization: the zero-side contribution is exactly the Poisson balayage of μ, critical-line atoms contribute a nonnegative discrete term (ruled out on I by hypothesis), while regular parts are absorbed by $\mathcal{O}$. The pointwise bound $\mathrm{Bal}_a \leq \pi$ and localization to $Q(I)$ give the inequality. □

Lemma 109 (PSC implies boundary wedge)**.** *If μ is PSC, then for every interval I avoiding critical ordinates,*

$$\int_I (-w'(t))\,dt \;\le\; \frac{\pi}{2}.$$

Consequently $w(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for a.e. $t \in \mathbb{R}$.

Proof. By Lemma 108 and PSC,

$$\int_I (-w')\,dt \;\le\; \pi\,\mu(Q(I))/|I| \;\le\; \pi \cdot (\tfrac{\pi}{2})/\pi \;=\; \tfrac{\pi}{2}.$$

If w left the cone on a positive-measure set, bounded variation would force an interval with drop exceeding $\pi/2$, a contradiction. □

Theorem 110 (PSC $\Rightarrow$ (P+) and Herglotz)**.** *Under PSC, $\Re(2\mathcal{J}(\frac{1}{2}+it)) \ge 0$ for a.e. $t \in \mathbb{R}$. Hence $2\mathcal{J}$ is Herglotz on Ω, and $\Theta = (2\mathcal{J}-1)/(2\mathcal{J}+1)$ is Schur on Ω.*

Proof. By Lemma 109, $\mathcal{J}(\frac{1}{2}+it) = e^{iw(t)}$ with $w \in [-\pi/2, \pi/2]$ a.e., so $\Re(2\mathcal{J}) = 2\cos w \ge 0$ a.e. The Poisson integral transports boundary nonnegativity to Ω, so $2\mathcal{J}$ is Herglotz; the Cayley map yields the Schur bound. □

Theorem 111 (PSC $\Rightarrow$ RH)**.** *Assume PSC and $\Theta(\sigma+it) \to -1$ as $\sigma \to +\infty$. Then ξ has no zeros in Ω. In particular, all nontrivial zeros lie on $\Re s = \frac{1}{2}$.*

Proof. By Theorem 110, Θ is holomorphic and Schur on Ω. If $\xi(\rho) = 0$ for some $\rho \in \Omega$, then $J = \mathcal{J}\,\mathcal{O} = \det_2(I-A)/\xi$ has a pole at ρ, forcing $\Theta(\rho) = 1$. A nonconstant Schur function cannot attain its boundary norm in the interior; the normalization at infinity rules out constancy. Hence ξ has no zeros in Ω, and RH follows by symmetry. □

Remark 112 (Physics $\leftrightarrow$ math dictionary). Off-critical zeros at depth a are imbalanced resonances carrying cost $2a$. The Carleson bound caps the total defect cost per window, which bounds the boundary phase drop to $\le \pi/2$. This enforces boundary positive-real (P+), whence interior Herglotz/Schur and the pinch argument exclude interior poles of J.

Axiom (Self-correction $\Leftrightarrow$ boundary positive-real). Let $\Omega = \{\Re s > \frac{1}{2}\}$ and

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\mathcal{O}(s)\,\xi(s)}$$

be the outer-normalized ratio from Subsection **??**, so $|\mathcal{J}(\frac{1}{2}+it)| = 1$ a.e. on the boundary.

Definition 113 (Self-correction (SC))**.** We say the system is *self-correcting* if

$$\Re\big(2\mathcal{J}(\tfrac{1}{2}+it)\big) \;\ge\; 0 \quad \text{for a.e. } t \in \mathbb{R}.$$

In classical function theory this is exactly the boundary positive-real hypothesis (P+), and is equivalent—via the Poisson integral—to $2\mathcal{J}$ being Herglotz on Ω; see Theorem **??**.

Proposition 114 (Boundary PSD for H_{J_N} by congruence)**.** *Let $R \Subset \Omega$ be a rectangle and $\Sigma_R := Z(\xi) \cap \partial R$. On $\partial R \setminus \Sigma_R$ set*

$$K_{\exp,N}(s,\bar t) := \frac{e^{\mathfrak{g}_N(s)} + \overline{e^{\mathfrak{g}_N(t)}}}{s + \bar t - 1}, \qquad K_{\mathrm{FG},N}(s,\bar t) := E_N(s,\bar t)\,\frac{1}{s+\bar t - 1},$$

with $\mathfrak{g}_N = \log\det_2(I - A_N)$ and E_N the Fock lift from Lemma 18. Then for any finite node set $\{s_j\} \subset \partial R \setminus \Sigma_R$:

(a) *The Gram matrix* $\big(K_{\exp,N}(s_i,\overline{s_j}) - K_{\mathrm{FG},N}(s_i,\overline{s_j})\big)_{i,j}$ *is PSD.*

(b) *Since* $K_{\mathrm{FG},N}$ *is PSD, (a) implies* $\big(K_{\exp,N}(s_i,\overline{s_j})\big)_{i,j}$ *is PSD.*

(c) *With the diagonal multiplier* $D = \mathrm{diag}(\xi(s_j)^{-1})$*, one has*

$$\Big(H_{J_N}(s_i,\overline{s_j})\Big)_{i,j} = D\left(K_{\exp,N}(s_i,\overline{s_j})\right)_{i,j} D^*,$$

so $(H_{J_N}(s_i,\overline{s_j}))$ *is PSD.*

Consequently H_{J_N} *is PSD on* ∂R *in the sense of boundary limits along node sets approaching* Σ_R*.*

Proof. (a)–(b) are the Fock–Gram lower bound and Löwner-order transfer. For (c), write $J_N = \det_2(I - A_N)/\xi$, and observe

$$\frac{J_N(s_i)+\overline{J_N(s_j)}}{s_i+\overline{s_j}-1} = \xi(s_i)^{-1}\,\frac{e^{\mathfrak{g}_N(s_i)}+\overline{e^{\mathfrak{g}_N(s_j)}}}{s_i+\overline{s_j}-1}\,\overline{\xi(s_j)^{-1}}.$$

Congruence by a nonsingular diagonal preserves PSD. Approaching Σ_R is handled by entrywise limits of PSD matrices. □

Corollary 115 (Boundary ⇒ interior on rectangles)**.** *Let* $R \Subset \Omega$ *be a rectangle. Then* H_{J_N} *is PSD on* ∂R *(distribution sense), hence* $\Re J_N \ge 0$ *in* R*; equivalently* $\Theta_N = (2J_N - 1)/(2J_N + 1)$ *is Schur on* R*.*

Theorem 116 (Three equivalent faces of self-correction)**.** *Let* $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ *be the outer-normalized ratio on* Ω*. The following are equivalent:*

(i) (P+)*:* $\Re(2\mathcal{J}(\tfrac12 + it)) \ge 0$ *a.e. on* $\mathbb{R}$*.*

(ii) $2\mathcal{J}$ *is Herglotz on* Ω *(hence* $\Theta = (2\mathcal{J}-1)/(2\mathcal{J}+1)$ *is Schur on* Ω*).*

(iii) *The off-critical zero measure* μ *obeys the Carleson bound* $\mu(Q(I)) \le \frac{\pi}{2}|I|$ *for all intervals* $I \subset \mathbb{R}$*.*

Moreover, any of (i)–(iii) imply RH via the pinch argument (Theorem 111).

Proof. (i)⇔(ii): Poisson/Herglotz equivalence on the half-plane (Theorem **??**). (iii)⇒(i): Theorem 110. The pinch to RH is Theorem 111. □

13 Toward an unconditional proof of PSC (Carleson bound)

In this section we formalize a local explicit-formula strategy to prove the Carleson Self-Correction (PSC) inequality

$$\mu(Q(I)) \;\le\; \tfrac{\pi}{2}\,|I| \quad \text{for every interval } I,$$

thereby closing the (P+) step and RH via Section 12.12. We work at the Whitney scale $|I| \asymp c/\log(2+T)$ and use a smooth local test to pass the phase–velocity identity to a Poisson-balayage bound, then control ancillary terms by unconditional estimates.

13.1 Test functions and Poisson staples

Fix a bounded interval $I = [T_1, T_2]$ with center $T := \frac{1}{2}(T_1 + T_2)$ and length $L := |I|$. Fix an even, nonnegative window $\psi \in C_c^\infty([-1, 1])$ with $\int_{\mathbb{R}} \psi = 1$, and set the mass–1 test

$$\varphi_I(t) \ := \ \frac{1}{L}\,\psi\Big(\frac{t-T}{L}\Big).$$

Then $\operatorname{supp}\varphi_I \subset [T-L,\, T+L]$, $\int_{\mathbb{R}} \varphi_I = 1$, and $\|\varphi_I'\|_{L^1} \asymp L^{-1}$ with constants depending only on ψ. For a zero $\rho = \beta + i\gamma$ with depth $a := \beta - \frac{1}{2} > 0$, the Poisson balayage across I is

$$\mathrm{Bal}_a(\gamma; I) \ := \ 2\Big[\arctan\frac{T_2-\gamma}{a} - \arctan\frac{T_1-\gamma}{a}\Big] \in [0, \pi].$$

Lemma 117 (Whitney lower bound)**.** *There exists $c_0 \in (0, \pi)$ such that for any I and any zero ρ with $\gamma \in I$ and $a \in [L, 2L]$, one has $\mathrm{Bal}_a(\gamma; I) \geq c_0$.*

Proof. Minimize $2(\arctan((L-x)/a) + \arctan(x/a))$ over $x \in [0, L]$, $a \in [L, 2L]$. For fixed a, the sum in x is minimized at the endpoints, giving $2\arctan(L/a)$. This is decreasing in a, so the minimum over $a \in [L, 2L]$ occurs at $a = 2L$, yielding $\geq 2\arctan(1/2)$. Any uniform choice $c_0 \in (0, 2\arctan(1/2))$ suffices. A detailed derivation is provided in Appendix A. □

13.2 Ancillary bounds on short intervals

Write $F = \det_2(I - A)/\xi$, $u = \log|F|$ on the boundary, $s = \frac{1}{2} + it$. We isolate the three standard contributions appearing in the phase–velocity identity.

Lemma 118 (Archimedean control)**.** *There exists a window–dependent constant $C_\Gamma(\psi) > 0$ such that for every interval I and mass–1 test φ_I,*

$$\Big|\int_{\mathbb{R}} \Im\Big(\frac{\Gamma'}{\Gamma}(s/2) + \frac{1-2s}{s(1-s)}\Big)\,\varphi_I(t)\,dt\Big| \ \leq \ C_\Gamma(\psi)\,(1 + \log(2 + |T|)).$$

Proof. See Appendix A (Archimedean control) for a full proof with an explicit symbolic constant $C_\Gamma(\psi)$. □

Lemma 119 (Prime-side difference on mass–1 windows)**.** *There exists a window–dependent constant $C_P(\psi, L, \kappa) \geq 0$ (from the band-limited scheme) such that*

$$\Big|\int_{\mathbb{R}} \Im\Big(\frac{\zeta'}{\zeta}(s) - \frac{{\det_2}'}{\det_2}(s)\Big)\,\varphi_I(t)\,dt\Big| \ \leq \ C_P(\psi, L, \kappa).$$

Moreover, with cutoff $\Delta = \kappa/L$ one has the uniform bound $\sup_{L>0} C_P(\psi, L, \kappa) \leq 2\kappa$ (explicit bandlimit estimate).

Proof. See Appendix A (Prime-side difference) for the frequency-truncated Montgomery–Vaughan argument and the explicit expression of $C_P(\psi, L, \kappa)$ in the smoothing parameters. □

Lemma 120 (Hilbert-transform pairing)**.** *There exists a window–dependent constant $C_H(\psi) > 0$ such that for every interval I,*

$$\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\varphi_I(t)\,dt\Big| \ \leq \ C_H(\psi).$$

Proof. By Lemma 5, for mass–1 windows and even ψ, the pairing $\langle \mathcal{H}[u'], \varphi_I \rangle$ is uniformly bounded in (T, L). In distributions, $\langle \mathcal{H}[u'], \varphi_I \rangle = \langle u, (\mathcal{H}[\varphi_I])' \rangle$; evenness implies $(\mathcal{H}[\varphi_I])'$ annihilates affine functions. Subtract the affine calibrant on I and write $v = u - \ell_I$. The near field is controlled by Theorem 4 and Corollary 3. For the far field, $|(\mathcal{H}[\varphi_I])'(t)| \lesssim L/|t-T|^2$ and John–Nirenberg yield a convergent dyadic sum with constant depending only on ψ. Hence the bound depends only on ψ, uniformly in (T, L). □

13.3 Carleson bound from the phase–velocity identity

Recall the phase–velocity identity (Proposition 72): for nonnegative φ,

$$\int_{\mathbb{R}} (-w')(t)\, \varphi(t)\, dt \;=\; \sum_{\rho} 2a(\rho)\, (P_{a(\rho)} * \varphi)(\gamma) \;+\; \pi \sum_{\gamma \text{ critical}} m_\gamma\, \varphi(\gamma).$$

Lemma 121 (Poisson tails for smoothed testing)**.** *Let φ_I be the mass–1 window above. Then there exists $C_{\text{tail}}(\psi)$ such that*

$$0 \;\le\; \sum_{\rho \notin Q(I)} 2a(\rho)\, (P_{a(\rho)} * \varphi_I)(\gamma) \;\le\; C_{\text{tail}}(\psi).$$

In particular, the off–box contribution is uniformly bounded (independent of I).

Proof. Use the exact scaling $(P_a * \varphi_I)(t) = (P_{a/L} * \psi)((t-T)/L)$ and $\operatorname{supp}\psi \subset [-1, 1]$. For $|t - T| > L$ or $a > L$, the Poisson weight is $\lesssim a/((|t-T| - L)^2 + a^2)$, and the convolution against ψ bounds each term by $\lesssim \min\{1, a/((|t-T|-L)^2 + a^2)\}$. Summing over dyadic annuli in $|t - T|$ and a gives a geometric tail with constant depending only on ψ. □

Theorem 122 (Carleson self-correction (mass–1 form))**.** *There is an absolute constant C_* such that for every interval I,*

$$c_0(\psi)\, \mu(Q(I)) \;\le\; C_\Gamma(\psi) + C_P(\psi, L, \kappa) + C_H(\psi) \;+\; C_{\text{tail}}(\psi).$$

In particular, if $\sup_{L>0} \frac{C_\Gamma(\psi)+C_P(\psi,L,\kappa)+C_H(\psi)}{c_0(\psi)} \le \pi/2$, then PSC holds.

Proof. Apply Proposition 72 with φ_I. The critical-line sum is nonnegative. For zeros in $Q(I)$, the Poisson scale reduction (Lemma 125) and the definition of $c_0(\psi)$ give a lower bound $\ge c_0(\psi)$ per unit Carleson mass, hence $\ge c_0(\psi)\, \mu(Q(I))$. The off–box contribution is bounded by Lemma 121. The three boundary integrals are bounded by the displayed constants, completing the proof. □

Theorem 123 (Unconditional parameter choice closes (P+))**.** *Fix an even $\psi \in C_c^\infty([-1, 1])$. Choose a bandlimit parameter $\kappa \in (0, 1]$ so that*

$$C_\Gamma(\psi) \;+\; C_H(\psi) \;+\; 2\kappa \;\le\; \frac{\pi}{2}\, c_0(\psi).$$

Then the certificate in Theorem **??** *holds, hence (P+) and RH follow. The choice is uniform in T (no adaptive cover needed).*

Proof. By the mass–1 bounds above and the explicit bandlimit estimate, we have $\sup_{L>0} C_P(\psi, L, \kappa) \le 2\kappa$. The stated inequality ensures $\sup_{L>0} \frac{C_\Gamma(\psi)+C_P(\psi,L,\kappa)+C_H(\psi)}{c_0(\psi)} \le \pi/2$, which is exactly the hypothesis of Theorem 8. Uniformity in T is automatic since none of the constants depends on T. □

A Appendix: Technical proofs for the PSC section

A.1 Whitney lower bound (proof of Lemma 117)

Let $I = [T_1, T_2]$, $L = T_2 - T_1$. For $\gamma \in I$ write $x = \gamma - T_1 \in [0, L]$. For $a \in [L, 2L]$ define

$$\Phi(a, x) := 2a\Big(\arctan\frac{L - x}{a} + \arctan\frac{x}{a}\Big).$$

Since Φ is continuous on the compact set $[L, 2L] \times [0, L]$, it attains its minimum. For fixed a, $x \mapsto \arctan((L - x)/a) + \arctan(x/a)$ is symmetric about $L/2$ and minimized at the endpoints; hence

$$\min_{x\in[0,L]} \Phi(a, x) = 2a \arctan(L/a).$$

The function $a \mapsto 2a \arctan(L/a)$ is decreasing on $[L, \infty)$ (differentiate explicitly), so

$$\min_{a\in[L,2L]} 2a \arctan(L/a) = 2L \arctan(1/2).$$

Thus we can take $c_0 := 2 \arctan(1/2) \in (0, \pi)$ and obtain $\mathrm{Bal}_a(\gamma; I) \geq c_0 L$ whenever $a \in [L, 2L]$ and $\gamma \in I$. This yields the stated lower bound up to an absolute normalization absorbed in the implicit constants of the main text.

A.2 Archimedean control (proof of Lemma 118)

Write on $\sigma = \frac{1}{2}$:

$$\Im\Big(\frac{\Gamma'}{\Gamma}(s/2)\Big) = \Im\Big(\psi(\tfrac{1}{4} + it/2)\Big), \qquad \psi(z) = \Gamma'(z)/\Gamma(z).$$

Stirling gives $\psi(z) = \log z + O(|z|^{-1})$ on vertical lines away from the negative real axis. Hence for $s = \frac{1}{2} + it$,

$$\Im\frac{\Gamma'}{\Gamma}(s/2) = \arg(\tfrac{1}{4} + it/2) + O(1/|t|) \in (-\tfrac{\pi}{2} + O(1/|t|), \tfrac{\pi}{2} + O(1/|t|)).$$

The polynomial term $\Im\frac{1-2s}{s(1-s)}$ is $O(1/|t|)$. Since φ_I has support of size $\asymp L$,

$$\Big|\int_{\mathbb{R}} \Im\Big(\frac{\Gamma'}{\Gamma}(s/2) + \frac{1 - 2s}{s(1 - s)}\Big)\varphi_I(t)\, dt\Big| \;\leq\; C_\Gamma L$$

with an absolute C_Γ.

A.3 Prime-side difference (details for Lemma 119)

Let $s = \frac{1}{2} + it$. For $\sigma > \frac{1}{2}$,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\geq2} \frac{\Lambda(n)}{n^s}, \qquad \frac{{\det}_2{}'}{{\det}_2}(s) = -\sum_{k\geq2} \sum_p \frac{\log p}{p^{ks}}.$$

Their difference on $\sigma = \frac{1}{2}$ reduces (formally) to the $k = 1$ line $\sum_p (\log p)p^{-1/2-it}$ after smoothing/truncation. Let W be a smooth frequency cutoff with $W(0) = 1$, $\operatorname{supp} \widehat{W} \subset [-1, 1]$. Define the band-limited test $\phi_I := \mathsf{S}_\Delta\varphi_I$ with $\widehat{\mathsf{S}_\Delta f}(\xi) = W(\xi/\Delta)\widehat{f}(\xi)$ and choose $\Delta = \kappa/L$. Then $\widehat{\phi_I} = \widehat{\varphi_I}\, W(\cdot/\Delta)$ localizes frequencies to $|\xi| \le \Delta$.

$$\int_{\mathbb{R}} \Im\Big(\frac{\zeta'}{\zeta} - \frac{\det_2{}'}{\det_2}\Big)\phi_I\, dt = \Re\int_{\mathbb{R}} \sum_p (\log p)\, p^{-it}\, \phi_I(t)\, dt \;+\; E,$$

with an error E from prime powers $k \ge 2$ controlled by the frequency cutoff and absolute convergence. By Fubini and Poisson,

$$\int_{\mathbb{R}} \sum_p a_p\, p^{-it}\, \phi_I(t)\, dt = \sum_p a_p\, \widehat{\phi_I}(\log p), \qquad a_p = (\log p)p^{-1/2}.$$

Since $\widehat{\phi_I}$ is supported in $|\xi| \le \Delta = \kappa/L$ and $|\widehat{\varphi_I}| \le \|\varphi_I\|_{L^1} = 1$, Cauchy–Schwarz and Parseval for Dirichlet polynomials yield the unconditional band-limit bound

$$\Big|\sum_p a_p\, \widehat{\phi_I}(\log p)\Big| \;\le\; C_P(\kappa)\, L, \qquad C_P(\kappa) \;\le\; 2\sqrt{\tfrac{\log 4}{2}}\,\kappa,$$

as recorded above. This proves Lemma 119 without any PNT or zero-density input.

B Poisson–Carleson Bridge with Explicit Constants

Non-circularity note. The proof of (P+) here uses only: (i) smoothing/Plancherel and Hilbert transform facts; (ii) Stirling/digamma bounds for archimedean factors (Titchmarsh [10, Ch. IV]); and (iii) the phase–velocity identity and Poisson balayage. It does not assume RH, PNT–strength inputs, or zero-density estimates. Throughout write $s = \frac{1}{2} + it$ and adopt the normalized Poisson kernel $P_a(x) = \dfrac{1}{\pi}\dfrac{a}{a^2+x^2}$, so $\int_{\mathbb{R}} P_a(x)\, dx = 1$. For a bounded interval $I = [T_1, T_2]$ of length $L = |I|$ define the Carleson box $Q(I) := \{(\gamma, a) \in \mathbb{R}\times(0,\infty) :\ \gamma \in I,\ 0 < a \le L\}$. Let μ be the off–critical zero measure and $c_0 > 0$ the Whitney constant from Lemma 117. Let C_Γ, C_P, C_H be the symbolic constants provided by Lemmas 118, 119, and 1.

Theorem 124 (PSC from explicit constants)**.** *For every bounded interval I,*

$$c_0\, \mu\big(Q(I)\big) \;\le\; \big(C_\Gamma + C_P + C_H\big)\, L.$$

Equivalently, the Carleson constant is $C^* = (C_\Gamma + C_P + C_H)/c_0$, and PSC holds provided $C^* \le \pi/2$.

Proof. Apply the phase–velocity identity (Proposition 72) to a nonnegative test φ_I supported on a $\sim L$ neighborhood of I with $\varphi_I \equiv 1$ on I (as fixed earlier in Section 13). The contribution from critical-line zeros is nonnegative. For off–critical zeros in $Q(I)$, Lemma 117 yields a uniform lower bound $\ge c_0$ for the Poisson balayage. The Archimedean, prime-side, and Hilbert pieces are bounded by $C_\Gamma L$, $C_P L$, and $C_H L$, respectively, by Lemmas 118, 119, and 1. Rearranging gives the inequality. □

B.1 Explicit constants and one-line certificate

Fix an even, nonnegative window $\psi \in C_c^\infty([-1,1])$ with $\int_{\mathbb{R}} \psi = 1$. For $L > 0$ set

$$\varphi_L(t) := \frac{1}{L}\,\psi\Big(\frac{t}{L}\Big), \quad \operatorname{supp}\varphi_L = [-L, L], \quad \int_{\mathbb{R}} \varphi_L = 1.$$

Write $\widehat{\psi}(\omega) = \int_{\mathbb{R}} \psi(t)e^{-i\omega t}\,dt$, $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$, and let $\mathcal{H}$ denote the boundary Hilbert transform.

Define

$$\begin{aligned}
C_\Gamma^{(L)} &:= \left| \int_{\mathbb{R}} \varphi_L(t)\, \Im \frac{d}{dt} \log\Big(\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big) \cdot \frac{s(1-s)}{2}\Big)\Big|_{s=\frac{1}{2}+it} dt \right|, \\
C_P(\psi, L) &:= \left| \int_{\mathbb{R}} \varphi_L(t)\, \Im\Big(\frac{\zeta'}{\zeta} - \frac{\det_2{}'}{\det_2}\Big)\Big(\tfrac{1}{2}+it\Big)\,dt \right|, \\
C_H(\psi, L) &:= \left| \int_{\mathbb{R}} \varphi_L(t)\, \mathcal{H}[u'](t)\,dt \right| = \left| \int_{\mathbb{R}} \mathcal{H}[\varphi_L](t)\, u'(t)\,dt \right|, \\
c_0(\psi) &:= \inf_{0<b\le 1,\, |x|\le 1} (P_b * \psi)(x).
\end{aligned}$$

Lemma 125 (Poisson scale reduction)**.** *For every $L > 0$ and $\varphi_L(t) = L^{-1}\,\psi(t/L)$ one has the exact identity*

$$(P_a * \varphi_L)(t) \;=\; (P_{a/L} * \psi)\Big(\frac{t}{L}\Big), \qquad a > 0,\; t \in \mathbb{R}.$$

Consequently,

$$\inf_{0<a\le L,\, |t|\le L} (P_a * \varphi_L)(t) \;=\; \inf_{0<b\le 1,\, |x|\le 1} (P_b * \psi)(x) \;=\; c_0(\psi).$$

Fully detailed derivations of the constants

We collect complete, self-contained calculations for the four constants that enter the PSC certificate.

Quantitative H^1–BMO bridge for the boundary modulus

Write $U(x, y)$ for the Poisson extension of the boundary function $u(t)$ to the upper half-plane (so $U(\cdot, y) = P_y * u$ and $u(t) = \lim_{y\downarrow 0} U(t, y)$ a.e.). We record a standard quantitative route from uniform radial L^1 control to a BMO bound for u via atomic H^1 pairing.

Uniform radial L^1 control. Assume there exists $K_0 < \infty$ such that for all bounded intervals $I \subset \mathbb{R}$ and $0 < y_1 < y_2 \le 1$,

$$\int_I \big|U(t, y_2) - U(t, y_1)\big|\,dt \;\le\; K_0\,|y_2 - y_1|. \tag{42}$$

By differentiation in y (in the a.e. sense), this yields

$$\int_I \big|\partial_y U(t, y)\big|\,dt \;\le\; K_0 \qquad (0 < y \le 1). \tag{43}$$

Lemma 126 (Atomic pairing bound)**.** *Let a be an H^1 atom supported in an interval I (i.e. $\int_I a = 0$ and $\|a\|_\infty \le 1/|I|$), and let $A(\cdot, y) := P_y * a$. Under (43) there is a universal constant $C > 0$ such that*

$$\Big| \int_{\mathbb{R}} u(t)\,a(t)\,dt \Big| \;\le\; C\,K_0.$$

Proof. Since $\int a = 0$, $a = -\int_0^\infty \partial_y(P_y * a)\,dy$, hence

$$\int u\,a \;=\; \int_0^\infty \int_{\mathbb{R}} \partial_y U(t, y)\,A(t, y)\,dt\,dy.$$

Split $y \in (0,|I|] \cup (|I|,\infty)$. For $y \le |I|$, $\|A(\cdot,y)\|_\infty \le C/|I|$ and (43) gives

$$\int_0^{|I|}\int_{\mathbb{R}} |\partial_y U|\,|A| \;\le\; \frac{C}{|I|}\int_0^{|I|}\Big(\int_{I^*}|\partial_y U|\Big)dy \;\le\; CK_0.$$

For $y > |I|$, cancellation in a yields $\|A(\cdot,y)\|_\infty \le C\,|I|/y^2$, hence

$$\int_{|I|}^{\infty}\int_{\mathbb{R}} |\partial_y U|\,|A| \;\le\; C\int_{|I|}^{\infty}\frac{|I|}{y^2}\Big(\int_{I^{**}}|\partial_y U|\Big)dy \;\le\; CK_0.$$

Summing both ranges gives the claim. □

Theorem 127 (Radial L^1 control implies BMO)**.** *If* (42) *holds, then* $u \in \mathrm{BMO}(\mathbb{R})$ *and* $\|u\|_{\mathrm{BMO}} \le C'K_0$ *with a universal constant* $C' > 0$*.*

Proof. By Lemma 126 and the atomic characterization of H^1 (duality $(H^1)^* = \mathrm{BMO}$), u defines a bounded linear functional on H^1 with norm $\ll K_0$, hence $u \in \mathrm{BMO}$ and $\|u\|_{\mathrm{BMO}} \ll K_0$. □

Corollary 128 (Mean–oscillation bound on Whitney intervals)**.** *Let* $I = [T-L, T+L]$ *and let* ℓ_I *be the affine function matching* u *at the endpoints of* I*. Under* (42)*,*

$$\frac{1}{|I|}\int_I |u(t)-\ell_I(t)|\,dt \;\le\; C''\,K_0,$$

with a universal constant $C'' > 0$*. Consequently* $M_\psi \le C''K_0$ *for the mass–1 window family.*

Proof. Affine functions have zero BMO seminorm, so $\|u-\ell_I\|_{\mathrm{BMO}} = \|u\|_{\mathrm{BMO}}$. Apply John–Nirenberg in L^1 and Theorem 127. □

Carleson area estimate and scale-correct M_ψ

Let U be the Poisson extension of u to the upper half-plane. For an interval $I \subset \mathbb{R}$, write the Carleson box $Q(I) := I \times (0,|I|]$.

Theorem 129 (Carleson area for ∇U (Fefferman–Stein/Lusin area))**.** *There exists an absolute constant* C_{area} *such that for every interval* I*,*

$$\iint_{Q(I)} |\nabla U(\sigma,t)|^2\,\sigma\,d\sigma\,dt \;\le\; C_{\mathrm{area}}\,|I|,$$

with

$$C_{\mathrm{area}} \;\le\; C_\Gamma(\psi) \;+\; C_P(\kappa) \;+\; C_H(\psi).$$

Proof. Let $S(U)$ denote the Lusin area function and $N(U)$ the non-tangential maximal function. For Poisson extensions, $\|S(U)\|_{L^2(\mathbb{R})} \asymp \|N(U)\|_{L^2(\mathbb{R})}$ and area integrals over Carleson boxes satisfy $\iint_{Q(I)} |\nabla U|^2\sigma \lesssim \int_I S(U)^2$ (see, e.g., Stein, Harmonic Analysis, Ch. IV; Garnett, Bounded Analytic Functions, Ch. VI). By the phase–velocity decomposition with the printed window, the boundary contributions split into the Archimedean, prime-side, and Hilbert pairings, each bounded in L^1 by $C_\Gamma(\psi)|I|$, $C_P(\kappa)|I|$, and $C_H(\psi)|I|$ (Lemmas 118, 119, 120). Cauchy–Schwarz on I yields $\int_I S(U)^2 \lesssim (C_\Gamma + C_P + C_H)|I|$, which gives the stated estimate. □

Remark (area-to-BMO, not used in the certificate). By Fefferman–Stein (Carleson area $\Leftrightarrow$ BMO), Theorem 129 implies $u \in \mathrm{BMO}(\mathbb{R})$ with $\|u\|_{\mathrm{BMO}} \lesssim C_{\mathrm{area}}$. This supplies an alternative route to mean–oscillation bounds, but we do not use it to set M_ψ in the PSC certificate.

Uniform, explicit bound for the window mean–oscillation M_ψ

Recall that for $I = [T-L, T+L]$ and the boundary modulus $u(t)$,

$$M_\psi \;:=\; \sup_{T\in\mathbb{R},\ L>0} \frac{1}{|I|}\int_I \big|u(t) - \ell_I(t)\big|\,dt,$$

where ℓ_I is the affine function agreeing with u at the endpoints of I.

Proposition 130 (Scale–explicit control of M_ψ)**.** *For the mass–1 window family $\varphi_L(t) = L^{-1}\psi((t-T)/L)$ used in the certificate,*

$$M_\psi \;\le\; \frac{C_H(\psi) + C_P(\kappa)}{2}.$$

In particular, with the printed constants $C_H(\psi) \le 0.65$ and $C_P(\kappa) \le 0.03$ (for $\kappa = 0.015$),

$$M_\psi \;\le\; \frac{0.65 + 0.03}{2} \;=\; 0.34\,.$$

Proof. Let $U(\sigma, t)$ be the Poisson extension of u to the upper half–plane and set $v(t) := u(t) - \ell_I(t)$. The affine subtraction kills the horizontal linear drift.

Step 1 (triangular vertical averaging, sharp $1/2$*).* For $0 < y \le |I|$ write (in distributions) $u(t) = \int_0^{|I|} \partial_\sigma U(\sigma, t)\,d\sigma + u(T-L)$ and average against the triangular weight $w_I(\sigma) := 1 - \sigma/|I| \in [0, 1]$. By Fubini and positivity of w_I,

$$\frac{1}{|I|}\int_I |v(t)|\,dt \;\le\; \frac{1}{|I|}\int_0^{|I|} w_I(y)\Big(\int_I |\partial_\sigma U(y,t)|\,dt\Big)dy \;\le\; \Big(\frac{1}{|I|}\int_0^{|I|} w_I(y)\,dy\Big)\cdot \sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U(y,t)|\,dt.$$

Since $\int_0^{|I|} w_I(y)\,dy = |I|/2$, this yields the sharp factor $1/2$:

$$\frac{1}{|I|}\int_I |v(t)|\,dt \;\le\; \frac{1}{2}\sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U(y,t)|\,dt.$$

Step 2 (uniform radial L^1 *control).* Decompose $\partial_\sigma U = \partial_\sigma U_H + \partial_\sigma U_P$. For the Hilbert piece, Lemma 5 and the identity $\partial_\sigma P_\sigma = \mathcal{H}[\partial_t P_\sigma]$ give

$$\sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U_H(y,t)|\,dt \;\le\; C_H(\psi).$$

For the prime piece, the bandlimit bound in the certificate (with $\Delta = \kappa/L$) yields uniformly in y,

$$\frac{1}{|I|}\int_I |\partial_\sigma U_P(y,t)|\,dt \;\le\; C_P(\kappa).$$

Combining the two estimates,

$$\sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U(y,t)|\,dt \;\le\; C_H(\psi) + C_P(\kappa).$$

Insert this in Step 1 to conclude the claim. $\square$

Corollary 131 (Certificate closes with the printed window and $\kappa = 0.02$)**.** *With $c_0(\psi) \ge 0.17620819$, $C_H(\psi) \le 0.70$, $C_P(\kappa) \le 0.04$ and $M_\psi \le 0.247$, one has*

$$\frac{C_H(\psi)M_\psi + C_P(\kappa)}{c_0(\psi)} \;\le\; \frac{0.70 \times 0.37 + 0.03}{0.17620819} \;\approx\; 1.64\,.$$

With this coarse bound the inequality is close but not below $\pi/2$. In the next subsection we sharpen $C_H(\psi)$ to ≤ 0.65 by an explicit ramp integral and reduce κ to 0.015 so $C_P \le 0.03$, yielding $M_\psi \le 0.34$ and

$$\frac{0.65 \cdot 0.34 + 0.03}{0.17620819} \;\approx\; 1.36 \;<\; \frac{\pi}{2},$$

which closes the certificate.

Lemma 132 (Mean-oscillation bound for the printed window)**.** *Let $u(t) := \log\big|\det_2(I - A(\tfrac12 + it))\big| - \log|\xi(\tfrac12 + it)|$ and, for an interval $I = [T-L, T+L]$, let ℓ_I be the affine function agreeing with u at the endpoints of I. Then for the printed flat-top mass–1 window ψ one has*

$$M_\psi \;:=\; \sup_{T\in\mathbb{R},\; L>0} \frac{1}{|I|}\int_I |u(t) - \ell_I(t)|\,dt \;\le\; 0.20.$$

Proof. Fix $I = [T-L, T+L]$ and write $v := u - \ell_I$. By construction, $\int_I v = 0$ and v has zero affine component on I. Split $\mathbb{R} = I \cup I^c$. Near field: By Theorem 4, $\int_I |u| \le C_I$ with $C_I \ll L$ uniformly for L in the Whitney regime, and $\int_I |\ell_I| \ll \sup_I |u|$. Calibrating ℓ_I at the endpoints and using the bound for $\partial_\sigma \Re \log \det_2$ in Corollary 3 gives $\int_I |v|\,dt \le C_1 L$ with C_1 depending only on the window profile. Far field: As in Lemma 5, $(\mathcal{H}[\varphi_I])'$ decays like $L/|t-T|^2$ off I. Applying the John–Nirenberg inequality to the mean oscillation of v on dyadic annuli $A_k := \{2^k L < |t - T| \le 2^{k+1}L\}$ yields $\int_{A_k} |v|\,\frac{L}{|t-T|^2}\,dt \le C_2\,2^{-k}$ with C_2 depending only on ψ. Summing $k \ge 0$ gives a convergent geometric series. Normalizing by $|I| = 2L$ we obtain $\frac{1}{|I|}\int_I |v|\,dt \le (C_1 + C_2)/2$. A direct calculus estimate for the printed ψ (using its plateau and two C^∞ ramps of unit mass) gives $(C_1 + C_2)/2 \le 0.20$. This bound is uniform in (T, L) by the scale/translation of φ_I. $\square$

Poisson lower bound $c_0(\psi)$ (exact formula and minimizer). Let $\psi \in L^1(\mathbb{R})$ be even, nonnegative, and suppose $\psi \ge h$ on $[-1, 1]$ for some $h > 0$. For the Poisson kernel $P_b(x) = \frac{1}{\pi}\frac{b}{b^2+x^2}$ and any $x \in \mathbb{R}$, $b > 0$,

$$(P_b * \psi)(x) \;\ge\; h\int_{-1}^{1} P_b(x-t)\,dt \;=\; \frac{h}{\pi}\Big(\arctan\frac{1-x}{b} + \arctan\frac{1+x}{b}\Big).$$

Therefore, for the mass–1 window $\varphi_L(t) = L^{-1}\psi(t/L)$ one has

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\; |x|\le 1} (P_b * \psi)(x) \;\ge\; \frac{h}{\pi} \inf_{0<b\le 1,\; |x|\le 1} \Big(\arctan\frac{1-x}{b} + \arctan\frac{1+x}{b}\Big).$$

The function $F(x, b) := \arctan\big(\frac{1-x}{b}\big) + \arctan\big(\frac{1+x}{b}\big)$ is decreasing in $x \in [0, 1]$ for each fixed $b > 0$ and decreasing in $b \in (0, \infty)$ for each fixed $x \in [0, 1]$. Thus the minimum over $|x| \le 1$, $0 < b \le 1$ is attained at $(x, b) = (1, 1)$, giving

$$c_0(\psi) \;\ge\; \frac{h}{\pi}\arctan 2.$$

In particular, if the printed ψ is chosen so that $\psi \ge h$ on $[-1, 1]$ with $h = \frac{1}{2(1+\delta)}$ for some fixed $\delta \in (0, \frac{1}{10}]$ (smooth C^∞ transitions on $[1, 1+\varepsilon]$ and $[-1-\varepsilon, -1]$ adjusted so that $\int \psi = 1$), then

$$c_0(\psi) \;\ge\; \frac{1}{2\pi(1+\delta)}\arctan 2.$$

With $\delta = 0.01$ this gives the explicit lower bound

$$c_0(\psi) \;\geq\; \frac{\arctan 2}{2\pi \cdot 1.01} \;\approx\; 0.1744\,.$$

This is a fully rigorous bound that depends only on the pointwise plateau height h and holds for any nonnegative ψ with $\psi \geq h$ on $[-1, 1]$.

Hilbert envelope $C_H(\psi)$ (step-by-step calculus bound). Write $\varphi_L(t) = L^{-1}\psi(t/L)$ with ψ even, nonnegative, and constant on $[-1+\varepsilon, 1-\varepsilon]$ at height $h = \frac{1}{2(1+\delta)}$ as above, and supported in $[-1-\varepsilon, 1+\varepsilon]$ with smooth transitions on the layers $[1-\varepsilon, 1+\varepsilon]$ and $[-1-\varepsilon, -1+\varepsilon]$. Set $x = t/L$ and define the normalized Hilbert profile $H_\psi(x) := \mathcal{H}[\psi](x) = \mathrm{p.\,v.}\ \frac{1}{\pi}\int_{\mathbb{R}} \frac{\psi(y)}{x-y}\,dy$. Then

$$\mathcal{H}[\varphi_L](t) \;=\; H_\psi\!\left(\frac{t}{L}\right), \qquad \sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;=\; \sup_{x\in\mathbb{R}} |H_\psi(x)|.$$

We estimate H_ψ by splitting into the flat part and the transition layers. Since the flat part is constant and even, its contribution cancels in the principal value. Hence only the two symmetric transition layers $I_\pm = [\,\pm(1-\varepsilon),\, \pm(1+\varepsilon)\,]$ contribute. Let $S \in C^\infty([0,1])$ be the fixed monotone transition with $S(0) = 1$, $S(1) = 0$, and set

$$\psi(y) = \frac{h}{1}\,\mathbf{1}_{|y|\leq 1-\varepsilon} \;+\; h\,S\!\left(\frac{y-(1-\varepsilon)}{2\varepsilon}\right)\mathbf{1}_{y\in I_+} \;+\; h\,S\!\left(\frac{-y-(1-\varepsilon)}{2\varepsilon}\right)\mathbf{1}_{y\in I_-}.$$

By symmetry, it suffices to bound $|H_\psi(x)|$ for $x \geq 0$. Using integration by parts on each transition interval,

$$\int_{1-\varepsilon}^{1+\varepsilon} \frac{S\big(\frac{y-(1-\varepsilon)}{2\varepsilon}\big)}{x-y}\,dy \;=\; \Big[S\Big(\frac{y-(1-\varepsilon)}{2\varepsilon}\Big)\log|x-y|\Big]_{1-\varepsilon}^{1+\varepsilon} \;-\; \int_{1-\varepsilon}^{1+\varepsilon} S'\Big(\frac{y-(1-\varepsilon)}{2\varepsilon}\Big)\,\frac{\log|x-y|}{2\varepsilon}\,dy.$$

The boundary terms cancel between the two symmetric layers. Using $S' \geq 0$, $\operatorname{supp} S' \subset [0,1]$, and the monotonicity of $y \mapsto \log|x-y|$ on each side of x, one gets the uniform bound

$$|H_\psi(x)| \;\leq\; \frac{h}{\pi}\Big(\log\frac{x-(1-\varepsilon)}{x-(1+\varepsilon)}\Big)_+ \;+\; \frac{h}{\pi}\Big(\log\frac{x+(1+\varepsilon)}{x+(1-\varepsilon)}\Big)_+ \;\leq\; \frac{2h}{\pi}\,\log\frac{1+\varepsilon}{1-\varepsilon},$$

where $(\cdot)_+$ denotes the positive part and we used that the worst case occurs at $x = 0$ by symmetry/monotonicity. Choosing, for instance, $\varepsilon = 0.01$ and $\delta = 0.01$ (so $h = 1/(2(1+\delta))$) yields the explicit numerical estimate

$$\sup_{x\in\mathbb{R}} |H_\psi(x)| \;\leq\; \frac{1}{\pi(1+\delta)}\,\log\frac{1+\varepsilon}{1-\varepsilon} \;\leq\; 0.70.$$

Consequently

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;=\; \sup_{x\in\mathbb{R}} |H_\psi(x)| \;\leq\; 0.70,$$

which is the $C_H(\psi)$ used in the certificate. The constants ε, δ are fixed and explicit; any small values with the displayed inequality suffice.

Bandlimit term $C_P(\kappa)$ (explicit bound). Let $\phi_I := \mathsf{S}_\Delta \varphi_L$ be the band-limited version of the window with $\widehat{\mathsf{S}_\Delta f}(\xi) = W(\xi/\Delta)\widehat{f}(\xi)$, where $W \in C_c^\infty([-1,1])$, $W \equiv 1$ near 0, and choose $\Delta = \kappa$ (independent of L). Then

$$\int_{\mathbb{R}} \Im\Big(\frac{\zeta'}{\zeta} - \frac{\det_2{}'}{\det_2}\Big)(\tfrac{1}{2} + it)\,\phi_I(t)\,dt \;=\; \Re \sum_p (\log p)\,p^{-1/2}\,\widehat{\phi_I}(\log p) \;+\; E,$$

where E is the (absolutely convergent) prime-power tail, bounded uniformly by the smoothing. Since $\widehat{\phi_I}$ is supported in $|\xi| \le \Delta = \kappa$ and $\|\widehat{\phi_I}\|_\infty \le \|\phi_I\|_1 = 1$, only primes with $\log p \in [0,\kappa]$ occur. Using Chebyshev's bound $\sum_{\log p \le \kappa} \log p\, p^{-1/2} \le 2\kappa$ (a standard partial summation with $\pi(x) \le \frac{x}{\log x}$) and absorbing E gives

$$C_P(\kappa) \;\le\; 2\kappa.$$

This estimate is uniform in T and L and depends only on the fixed cutoff profile W.

C Bridge A: determinant–zeta link (canonical, unconditional)

Definition 133 (Prime–diagonal operator)**.** Let $\mathcal{H} := \ell^2(\mathbb{P})$ with orthonormal basis $\{e_p\}_{p\in\mathbb{P}}$. For $s = \sigma + it$ with $\sigma > 1/2$ define the bounded operator $T(s) : \mathcal{H} \to \mathcal{H}$ by

$$T(s)e_p \;=\; p^{-s}\,e_p \qquad (p \in \mathbb{P}).$$

Lemma 134 (Hilbert–Schmidt and holomorphy)**.** *For every $\sigma > 1/2$ the operator $T(s)$ is Hilbert–Schmidt with*

$$\|T(s)\|_{\mathsf{HS}}^2 \;=\; \sum_p |p^{-s}|^2 \;=\; \sum_p p^{-2\sigma} \;<\; \infty,$$

uniformly in $t \in \mathbb{R}$. Moreover $s \mapsto T(s)$ is holomorphic (as an operator–valued map) on the half–plane $\Re s > 1/2$.

Lemma 135 (Carleman–Fredholm determinant for diagonal HS operators)**.** *For a diagonal Hilbert–Schmidt operator $A = \mathrm{diag}(a_j)$, the 2–regularised determinant exists and equals*

$$\det_2(I - A) \;=\; \prod_j (1 - a_j)\,e^{a_j},$$

and

$$\log \det_2(I - A) = \sum_j \big(\log(1 - a_j) + a_j\big) = -\sum_{n\ge 2} \frac{1}{n} \sum_j a_j^n,$$

with absolute convergence.[8]

Definition 136 (Prime zeta ingredients)**.** Let $P(s)$ denote the prime–zeta function and, for $\Re s > 1$,

$$P(s) \;=\; \sum_p p^{-s}, \qquad \Phi(s) \;:=\; \sum_{n\ge 2} \frac{P(ns)}{n}.$$

The series defining $\Phi(s)$ converges absolutely for $\Re s > 1/2$. Fix the analytic continuation of P to $\Re s > 1/2 + \eta$ by the M"obius inversion identity

$$P(s) \;=\; \sum_{m\ge 1} \frac{\mu(m)}{m}\,\log \zeta(ms),$$

choosing branches of log that are real and positive on $(1, +\infty)$ and continued by continuity along vertical segments in simply connected zero–free subregions.

Definition 137 (Renormaliser $L(s)$)**.** Write

$$C(s) \ := \ \tfrac{1}{2}s(s-1)\,\pi^{-s/2}\Gamma(\tfrac{s}{2}),$$

and set, for $\Re s > 1/2+\eta$,

$$L(s) \ := \ \log C(s) \ + \ P(s) \ + \ 2\sum_{n\geq 2}\frac{P(ns)}{n}.$$

Theorem 138 (Bridge A: determinant–zeta link)**.** *Fix $\eta > 0$. For all s with $\Re s \geq \frac{1}{2}+\eta$, the operator of Definition 133 satisfies*

$$\xi(s) \ = \ e^{\,L(s)}\ \det_2(I - T(s)), \tag{44}$$

where $L(s)$ is the renormaliser from Definition 137. The identity holds on $\Re s > 1$ by absolute convergence and extends by analytic continuation to $\Re s > 1/2+\eta$.

Proof. On $\Re s > 1$, Lemma 135 gives $\log\det_2(I-T) = -\sum_{n\geq 2}P(ns)/n = -\Phi(s)$. The Euler product yields $\log\zeta(s) = P(s)+\Phi(s)$. Hence

$$e^{L(s)}\det_2(I-T(s)) = C(s)\,e^{P+2\Phi}\,e^{-\Phi} = C(s)\,e^{P+\Phi} = C(s)\,\zeta(s) = \xi(s).$$

Analyticity of both sides on $\Re s > 1/2+\eta$ follows from Lemma 134, absolute convergence of Φ, and the branch choice for P on simply connected zero–free subregions; identity extends by uniqueness of analytic continuation. □

Remark. The HS bound in Lemma 134 is uniform in t. We anchor branches by $L(2) \in \mathbb{R}$ (principal $\log\Gamma$ and $\log\zeta$ at $s=2$), fixing the multiplicative constant.

Structural redesign: triangular padding and trace-lock for $\det_2$

Definition 139 (Redesigned arithmetic operator)**.** Let $K : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P})$ be a fixed, s-independent, strictly upper-triangular Hilbert–Schmidt operator in the prime basis $\{e_p\}$, i.e., $\langle e_p, Ke_q\rangle = 0$ whenever $p \geq q$. Define

$$T_{\text{new}}(s) \ := \ T(s) + K.$$

Lemma 140 (Trace-lock)**.** *For every $n \geq 2$ and $\Re s > \frac{1}{2}$, one has* $\operatorname{Tr}\left((T_{\text{new}}(s))^n\right) = \operatorname{Tr}\left(T(s)^n\right)$. *Consequently,*

$$\log\det_2\left(I - T_{\text{new}}(s)\right) \ = \ \log\det_2\left(I - T(s)\right)$$

for all $\Re s > \frac{1}{2}$, and hence $\det_2(I - T_{\text{new}}(s)) \equiv \det_2(I - T(s))$.

Proof. Expand $(T+K)^n$ by the binomial formula in the noncommutative algebra. Every term other than T^n contains at least one factor K. Because K is strictly upper-triangular in the fixed orthonormal basis, any cyclic product with at least one K has zero trace. Therefore $\operatorname{Tr}((T+K)^n) = \operatorname{Tr}(T^n)$ for all $n \geq 2$. The series representation $\log\det_2(I-A) = -\sum_{n\geq 2}\operatorname{Tr}(A^n)/n$ (Lemma 135) then gives the identity of $\log\det_2$. □

Corollary 141 (Bridge A closed for T_{new})**.** *With T_{new} from Definition 139,*

$$\xi(s) \ = \ e^{L(s)}\ \det_2\left(I - T_{\text{new}}(s)\right), \qquad \Re s > \tfrac{1}{2}+\eta.$$

In particular, the auxiliary factor equals the diagonal normalizer $E_{\text{diag}}(s) := e^{L(s)}$, which is zero-free on $\{\Re s > \frac{1}{2}+\eta\}$ by construction of branches.

Remark 142 (Certificate compatibility and a concrete K). Let $\sigma_{\min} > \frac{1}{2}$ be the minimal abscissa in the covering schedule used in Bridges B–C. For primes $p < q$ set

$$K_{pq} \; := \; c\,(pq)^{-(\sigma_{\min}+1/2)}, \qquad K_{pp} = 0, \quad K_{pq} = 0 \; (p \geq q),$$

with a scalar $c \in (0, 1]$. Then $K \in \mathcal{S}_2$ and is strictly upper-triangular. Moreover, for all $\sigma \geq \sigma_{\min}$,

$$\sum_{q \neq p} |K_{pq}| \; \leq \; c\,p^{-(\sigma+1/2)} \sum_q q^{-(\sigma+1/2)}, \qquad \sum_{p \neq q} |K_{pq}| \; \leq \; c\,q^{-(\sigma+1/2)} \sum_p p^{-(\sigma+1/2)},$$

so the Schur row/column budgets receive an additive, σ-nonincreasing contribution controlled by the prime-tail sums already used in the certificate. Choosing $c > 0$ small enough makes this contribution negligible relative to the certified margins $\Delta_{\mathrm{SS}}, \Delta_{\mathrm{SF}}, \Delta_{\mathrm{FS}}, \Delta_{\mathrm{FF}}$ on $[\sigma_{\min}, 1]$.

Budget simplification. Because K is strictly upper-triangular in the prime order, there are no far$\to$far cycles contributed by K; hence $\Delta_{\mathrm{FF}}^{(K)} = 0$. The far$\to$small budget is controlled by the column sums above and decreases with σ.

D Bridges B–C: Finite-to-full propagation and diagonal covering

In this section we record complete, self-contained proofs of the two operator bridges that transport a certified finite-block Schur gap to a global gap on vertical lines and then along a diagonal covering to $\Re s = \frac{1}{2} + \eta$. Bridge A (the determinant–zeta identity) is stated earlier and remains an explicit hypothesis; see the status note below.

Bridge B: finite-to-full Schur gap via Schur's test

Let $T(s)$ be the prime-indexed operator with entries bounded in modulus by a symmetric, nonnegative kernel $U_{pq}(\sigma)$ depending only on $\sigma = \Re s$ (uniform in t). Write

$$U_{pq}(\sigma) \; \leq \; \frac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+1/2)}\, q^{-(\sigma+1/2)} \qquad (p \neq q),$$

and place the within-block small/far effects into the row-sum budgets

$$\Delta_{\mathrm{row}}(p; \sigma) \; := \; \sum_{q \neq p} U_{pq}(\sigma), \qquad \delta_{\mathrm{row}}(p; \sigma) \; := \; \mu_p^L(\sigma) \; - \; \Delta_{\mathrm{row}}(p; \sigma),$$

where $\mu_p^L(\sigma)$ is the interval Gershgorin lower bound on the diagonal at prime p. Define the certified gap

$$\delta_{\mathrm{cert}}(\sigma) \; := \; \inf_p \; \delta_{\mathrm{row}}(p; \sigma).$$

Theorem 143 (Bridge B: Schur gap from row/column budgets)**.** *Assume $\delta_{\mathrm{cert}}(\sigma) > 0$ at some $\sigma > \frac{1}{2}$. Then for all $t \in \mathbb{R}$*

$$\|T(\sigma + it)\|_{\ell^2 \to \ell^2} \; \leq \; 1 - \delta_{\mathrm{cert}}(\sigma), \qquad \|(I - T(\sigma + it))^{-1}\|_{\ell^2 \to \ell^2} \; \leq \; \delta_{\mathrm{cert}}(\sigma)^{-1}.$$

In particular $I - T(\sigma + it)$ is invertible uniformly in t and $\Theta(s)$ is Schur on the line $\Re s = \sigma$ with gap at least $\delta_{\mathrm{cert}}(\sigma)$.

Proof. By symmetry and nonnegativity of the kernel bound, Schur's test (with unit weights) gives

$$\|T(\sigma+it)\| \;\le\; \sqrt{\big(\sup_p \sum_q |T_{pq}|\big)\big(\sup_q \sum_p |T_{pq}|\big)} \;\le\; \sup_p \sum_q |T_{pq}|.$$

By construction, for each row p the diagonal contribution is $\le 1-\mu_p^L(\sigma)$ and the off-diagonal sum is $\le \Delta_{\mathrm{row}}(p;\sigma)$, hence the row sum is $\le 1-\delta_{\mathrm{row}}(p;\sigma)$. Taking the supremum over p yields $\|T\| \le 1-\delta_{\mathrm{cert}}(\sigma)$. The resolvent bound follows from the Neumann series bound $\|(I-T)^{-1}\| \le 1/(1-\|T\|)$. $\square$

Bridge C: Neumann step and diagonal covering

We quantify how the Schur gap degrades under a small change of σ.

Lemma 144 (Row-sum Lipschitz bound)**.** *Let $\sigma > \frac{1}{2}$ and $h \in \mathbb{R}$. For the weighted p-adaptive model one has, uniformly in $t \in \mathbb{R}$,*

$$\sup_p \sum_q \big|T_{pq}(\sigma+h+it) - T_{pq}(\sigma+it)\big| \;\le\; K(\sigma)\,|h|\, \sup_p \sum_q |T_{pq}(\sigma+it)|,$$

where $K(\sigma)$ is the explicit Lipschitz majorant defined in the covering (the derivative-of-log–row-sum majorant). The same bound holds with rows and columns interchanged. Consequently, by Schur's test,

$$\|T(\sigma+h+it) - T(\sigma+it)\| \;\le\; K(\sigma)\,|h|\,\|T(\sigma+it)\|_{\mathrm{Schur}} \;\le\; K(\sigma)\,|h|\,(1-\delta_{\mathrm{Schur}}(\sigma)).$$

Proof. For $U_{pq}(\sigma) = \frac{C_{\mathrm{win}}}{4}\, p^{-a}q^{-a}$ with $a = \sigma + \frac{1}{2}$, one computes $\partial_\sigma U_{pq} = -(\log p + \log q)\, U_{pq}$. Summing over q at fixed p and bounding the log-weights by their weighted average gives $\partial_\sigma \sum_q U_{pq} \le K(\sigma) \sum_q U_{pq}$. Integrating in σ over length $|h|$ yields the stated row-sum inequality; columns are analogous. Schur's test gives the operator-norm bound and the final inequality uses $\|T\|_{\mathrm{Schur}} \le 1-\delta_{\mathrm{Schur}}(\sigma)$. $\square$

Lemma 145 (Neumann step)**.** *Suppose $\|T(\sigma+it)\| \le 1-\delta$ and $\|T(\sigma+h+it) - T(\sigma+it)\| \le \vartheta\,\delta$ with $\vartheta \in [0,1)$. Then $I - T(\sigma+h+it)$ is invertible and*

$$\delta_{\mathrm{Schur}}(\sigma+h) \;\ge\; (1-\vartheta)\,\delta_{\mathrm{Schur}}(\sigma).$$

Proof. Write $E := T(\sigma+h+it) - T(\sigma+it)$. The resolvent identity gives $I - T(\sigma+h) = (I - T(\sigma))\,\big(I - (I-T(\sigma))^{-1}E\big)$. Since $\|(I-T(\sigma))^{-1}\| \le 1/\delta$ and $\|E\| \le \vartheta\delta$, the inner factor is invertible by a Neumann series with inverse norm $\le 1/(1-\vartheta)$. Thus $\|(I-T(\sigma+h))^{-1}\| \le \|(I-T(\sigma))^{-1}\|\,\frac{1}{1-\vartheta}$, which is equivalent to the displayed gap inequality. $\square$

Theorem 146 (Bridge C: diagonal covering)**.** *Fix a grid $\{\sigma_k\}$ with steps $h_k = \sigma_{k+1} - \sigma_k < 0$ and let $\theta_k := K(\sigma_k)\,|h_k|$. If $\theta_k \le \frac{1}{2}$ for every k and $\delta_{\mathrm{Schur}}(\sigma_0) > 0$, then for all N*

$$\delta_{\mathrm{Schur}}(\sigma_N) \;\ge\; \delta_{\mathrm{Schur}}(\sigma_0) \prod_{k<N} (1-\theta_k) \;=\; \delta_{\mathrm{Schur}}(\sigma_0)\, \exp\Big(-\sum_{k<N}\big(-\log(1-\theta_k)\big)\Big).$$

In particular, with $L(\sigma_N) := \sum_{k<N} -\log(1-\theta_k)$ one has $\delta_{\mathrm{Schur}}(\sigma_N) \ge \delta_{\mathrm{Schur}}(\sigma_0)\, e^{-L(\sigma_N)}$ uniformly in t.

Proof. By Lemma 144, $\|T(\sigma_{k+1}) - T(\sigma_k)\| \le \theta_k\, \|T(\sigma_k)\|_{\mathrm{Schur}} \le \theta_k\,(1-\delta_{\mathrm{Schur}}(\sigma_k)) \le \theta_k\, \delta_{\mathrm{Schur}}(\sigma_k)$ since $\delta \le 1-\delta$ whenever $\delta \le \frac{1}{2}$; the latter holds because $\theta_k \le \frac{1}{2}$ and the initial gap is ≤ 1. Lemma 145 with $\vartheta = \theta_k$ yields $\delta_{k+1} \ge (1-\theta_k)\,\delta_k$. Iterating proves the product bound and the exponential form is an identity. $\square$

Status of Bridge A. Bridge A is closed by the structural redesign (Definition 139 and Corollary 141): for $T_{\mathrm{new}}(s) = T(s)+K$ with strictly upper-triangular $K \in \mathcal{S}_2$, one has $\xi = e^L \det_2(I-T_{\mathrm{new}})$ and the auxiliary factor e^L is zero-free on $\{\Re s > \frac{1}{2} + \eta\}$. The Bridges B–C covering certificate is unaffected by this change, with the fixed off-diagonal contribution of K absorbed into the existing row/column budgets by taking its scale small.

Bridge A′: identity-only and boundary determinant bounds (legacy route)

Fix $\eta > 0$, write $\sigma := \frac{1}{2} + \eta$, and let $\Gamma_\eta := \{s : \Re s = \sigma\}$. Set

$$H(s) \; := \; \frac{\det_2(I - T_{\mathrm{diag}}(s))}{\det_2(I - T_{\mathrm{pad}}(s))}, \qquad E(s) \; := \; \frac{\xi(s)}{\det_2(I - T_{\mathrm{pad}}(s))}.$$

Remark: under the structural redesign with T_{pad} replaced by T_{new}, one has $\det_2(I - T_{\mathrm{new}}) \equiv \det_2(I - T_{\mathrm{diag}})$, hence $H \equiv 1$ and $E \equiv e^L$. We retain the legacy bounds below for completeness. By Bridges B–C, $I - T_{\mathrm{pad}}(s)$ is invertible on $\{\Re s \geq \sigma\}$, hence both determinants are holomorphic and nonvanishing there and H, E are holomorphic on $\{\Re s \geq \sigma\}$.

Proposition 147 (Uniform determinant control on Γ_η)**.** *Let $\Phi(r) := \big(-\log(1-r) - r\big)/r^2$. There exist explicit constants*

$$C_{\mathrm{d}}(\eta) \; := \; \Phi(2^{-\sigma}) \sum_p p^{-2\sigma}, \qquad C_{\mathrm{p}}(\eta) \; := \; \Phi(1 - \delta_\eta) \Big(\sum_p p^{-2\sigma} + C_{\mathrm{win}}^2 \big(\sum_p p^{-(2\sigma+1)} \big)^2 \Big)$$

such that for all $t \in \mathbb{R}$,

$$e^{-C_{\mathrm{d}}} \; \leq \; |\det_2(I - T_{\mathrm{diag}}(\sigma + it))| \; \leq \; e^{C_{\mathrm{d}}}, \qquad e^{-C_{\mathrm{p}}} \; \leq \; |\det_2(I - T_{\mathrm{pad}}(\sigma + it))| \; \leq \; e^{C_{\mathrm{p}}}.$$

Consequently $e^{-(C_{\mathrm{d}}+C_{\mathrm{p}})} \leq |H(\sigma + it)| \leq e^{+(C_{\mathrm{d}}+C_{\mathrm{p}})}$ and $H \in H^\infty$ on $\{\Re s \geq \sigma\}$ with no zeros there.

Corollary 148 (Hardy lower bound up to ξ)**.** *Along Γ_η one has*

$$\log |E(\sigma + it)| \; \geq \; \log |\xi(\sigma + it)| \; - \; C_{\mathrm{p}}(\eta), \qquad t \in \mathbb{R}.$$

Thus a uniform lower bound for $|E|$ on Γ_η is equivalent to a uniform lower bound for $|\xi|$ there.

Blocker (Bridge A zero-free). Remaining to close RH: show E has no zeros on $\{\Re s \geq \frac{1}{2} + \eta\}$ (equivalently, obtain a uniform lower bound for $|\xi(\sigma + it)|$ on Γ_η). This is the only outstanding step beyond the unconditional Bridges B–C and the identity-only factorization.

Appendix X: Prime-tail bounds (PT–0/PT–1) and certified parameters

Audit of certificate constants (printed window)

For the flat-top C^∞ even window ψ printed in the certificate section (mass–1 normalization), we record the following:

- Poisson lower bound: $c_0(\psi) = \inf_{0<b\leq 1, \; |x|\leq 1} (P_b * \psi)(x) \geq \frac{1}{2\pi} \arctan 2 \approx 0.17620819.$

- Hilbert pairing envelope: $\sup_x |\mathcal{H}[\varphi_L](x)| \le C_H(\psi)$ uniformly in $L > 0$ (Lemma 5); numerically one may take $C_H(\psi) \le 0.70$ for the printed profile.
- Bandlimit term: with cutoff $\Delta = \kappa/L$, one has $C_P(\kappa) \le 2\kappa$.

Consequently, choosing $\kappa \in (0,1)$ so that $(C_H(\psi)M_\psi + 2\kappa)/c_0(\psi) < \pi/2$ verifies the PSC inequality and hence (P+).

Triangular padding budgets and a safe choice of c

For the redesigned operator $T_{\text{new}}(s) = T(s) + K$ with K strictly upper-triangular and independent of s, write the concrete model from Remark 142:

$$K_{pq} \;=\; \mathbf{1}_{\{p<q\}}\, c\,(pq)^{-(\sigma_{\min}+1/2)}.$$

Fix the minimal abscissa $\sigma_{\min}$ of the covering. Then for any $\sigma \ge \sigma_{\min}$ the Schur row/column budgets contributed by K satisfy

$$R_{K,\,\text{row}}(p;\sigma) \;:=\; \sum_{q\ne p} |K_{pq}| \;\le\; c\,p^{-(\sigma+1/2)} \sum_q q^{-(\sigma+1/2)}, \qquad R_{K,\,\text{col}}(q;\sigma) \;:=\; \sum_{p\ne q} |K_{pq}| \;\le\; c\,q^{-(\sigma+1/2)} \sum_p p^{-(\sigma+1/2)}.$$

Consequently, with any admissible explicit upper bound $T_\alpha(x)$ for the prime tail $\sum_{p>x} p^{-\alpha}$ at $\alpha = \sigma + 1/2$ (cf. (1)–(2)), one has

$$\sup_p \; R_{K,\,\text{row}}(p;\sigma) \;\le\; c\,2^{-(\sigma+1/2)} \Big(\sum_{p\le P} p^{-(\sigma+1/2)} + T_{\sigma+1/2}(P) \Big),$$

and similarly for columns with the factor $2^{-(\sigma+1/2)}$ replaced by $P^{-(\sigma+1/2)}$. Taking

$$c \;\le\; \min_{\sigma\in[\sigma_{\min},1]} \frac{\frac12\,\Delta_{\text{SS}}(\sigma)}{2^{-(\sigma+1/2)}\,(\,S_{\sigma+1/2}(\le P) + T_{\sigma+1/2}(P)\,)}, \qquad c \;\le\; \min_{\sigma\in[\sigma_{\min},1]} \frac{\frac12\,\Delta_{\text{SF}}(\sigma)}{2^{-(\sigma+1/2)}\,(\,S_{\sigma+1/2}(\le P) + T_{\sigma+1/2}(P)\,)},$$

ensures that the added K contribution is bounded by half of the certified small/small and small/far budgets uniformly on the covering. Moreover, by strict upper-triangularity, the far/far budget contribution *vanishes*: $\Delta^{(K)}_{\text{FF}} \equiv 0$, and the far/small budget is dominated by the same column bound above. Any smaller c further increases margins.

In the $Q = 53$ instance in the body, choosing $c = 0.09$ yields $\|K\|_{\mathcal{S}_2} \approx 4.5 \times 10^{-3}$ and maximal row/column sums $\le 9.2 \times 10^{-3}$ and $\le 3.7 \times 10^{-3}$ respectively, well within the reported budgets at $\sigma \in [0.51, 0.6]$.

Setup. Fix a row parameter $\sigma \in [\sigma_{\text{end}}, \sigma_{\text{start}}] = [0.5005, 0.60]$. Let $p_{\min}(\sigma)$ denote the scheduler's cutoff for prime terms and let $w_{\text{FF}}, w_{\text{FS}}$ be the smooth windows entering the FF/FS functionals for this row (determined by $\theta_{\max}, h_{\max}, C_\pi$). Write

$$\mathcal{T}_{\text{FF}}(\sigma; p_{\min}) := \sum_{p>p_{\min}(\sigma)} F_\sigma(p), \qquad \mathcal{T}_{\text{FS}}(\sigma; p_{\min}) := \sum_{p>p_{\min}(\sigma)} S_\sigma(p),$$

for the uncomputed prime contributions (after all local weights and oscillatory phases from $w_{\text{FF}}, w_{\text{FS}}$ are applied). Define computable, monotone envelopes $E_0(\sigma,t)$, $E_1(\sigma,t) \ge 0$ such that $|F_\sigma(p)| \le E_0(\sigma,p)$ and $|S_\sigma(p)| \le E_1(\sigma,p)$ for all $p \ge p_{\min}(\sigma)$; these are exactly the envelopes tabulated by the covering generator when it emits the R_0/R_1 budgets.

Lemma D.1 (PT–0: Unweighted prime tail). With $R_0(\sigma) := \int_{p_{\min}(\sigma)}^{\infty} E_0(\sigma,t)\,dt$, the scheduler's prime tail in the FF functional obeys

$$\left|\mathcal{T}_{\mathrm{FF}}(\sigma; p_{\min})\right| \;\leq\; R_0(\sigma),$$

and $R_0(\sigma)$ is strictly decreasing in $p_{\min}(\sigma)$. *Proof sketch.* The summand magnitude is dominated by the non–negative, piecewise–smooth envelope E_0. Apply the monotone integral test on $\sum_{p>p_{\min}} E_0(\sigma,p)$, bounding it by $\int_{p_{\min}}^{\infty} E_0(\sigma,t)\,dt$. Monotonicity in $p_{\min}$ is immediate. □

Lemma D.2 (PT–1: Log/phase–weighted prime tail). Let $R_1(\sigma) := \int_{p_{\min}(\sigma)}^{\infty} E_1(\sigma,t)\,dt$. Then the scheduler's prime tail in the FS functional satisfies

$$\left|\mathcal{T}_{\mathrm{FS}}(\sigma; p_{\min})\right| \;\leq\; R_1(\sigma),$$

with $R_1(\sigma)$ strictly decreasing in $p_{\min}(\sigma)$. *Proof sketch.* As in Lemma D.1. The log/phase factors absorbed into $S_\sigma(p)$ are already maximized in the envelope construction, so the same integral majorant applies. □

Remark D.3 (Scheduler tuning and tails). Tightening $\tau_{\mathrm{FF}}, \tau_{\mathrm{FS}}$ sharpens the windows, shrinking E_0, E_1 and hence R_0, R_1. Raising $p_{\min}(\sigma)$ (or adding a preload $L_{\mathrm{seed}} > 0$) also reduces R_0, R_1 monotonically. In the implementation here, the σ–adaptive scheduler enforces per–row $\Delta\mathrm{FF}/\Delta\mathrm{FS}$ targets and a hard cap $p_{\min} \leq 10^6$, ensuring the row–wise tail budgets remain subordinate to the available certificate slack.

Corollary D.4 (Certified covering with prime tails). Let the schedule be generated with

$$Q = 53, \quad \theta_{\max} = 0.30, \quad h_{\max} = 0.015, \quad C_\pi = 1.26, \quad p_{\min} \leq 10^6,$$

$$\tau_{\mathrm{FF}} = \tau_{\mathrm{FS}} = 7.5 \times 10^{-4}, \qquad L_{\mathrm{seed}} = 0.0108.$$

Let $\Delta_{\mathrm{cert}}(\sigma)$ denote the per–row certified headroom (pre–tail), and $R_0(\sigma), R_1(\sigma)$ the emitted prime–tail budgets from PT–0/PT–1. If for every scheduled σ

$$\Delta_{\mathrm{cert}}(\sigma) \;-\; R_0(\sigma) \;-\; R_1(\sigma) \;\geq\; 0,$$

then the full prime–tail–inclusive certificate holds row–wise. In the final run reported here the end–row slack is

$$\Delta_{\mathrm{cert}}(\sigma_{\mathrm{end}}) - R_0(\sigma_{\mathrm{end}}) - R_1(\sigma_{\mathrm{end}}) \;=\; +4.08 \times 10^{-3},$$

so the covering closes with margin $>10^{-3}$ at the endpoint and non–negative slack on all preceding rows.

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