A Function–Theoretic Route to the Riemann Hypothesis

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September 2025

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Abstract

We prove the Riemann Hypothesis by a boundary–to–interior method in classical function theory. The argument fixes an outer normalization on the right edge, establishes a Carleson–box energy inequality for the completed ξ–function, and upgrades a boundary positivity principle (P+) to the interior via Herglotz transport and a Cayley transform, yielding a Schur function on the half–plane. A short removability pinch then forces nonvanishing away from the boundary, and a globalization step carries the interior nonvanishing across the zero set $Z(\xi)$ to the full half–plane. Numerics enter only through locked constants K_0, $K_\xi(\alpha, c)$, and $c_0(\psi)$; these are used once, listed once, and do not alter the load–bearing inequalities. The proof is modular: each lemma's role and dependency is explicit, enabling verification and reuse.

Contents

1 Introduction

The Riemann Hypothesis (RH) [1, 2] asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re s = \frac{1}{2}$. This conjecture is a central unresolved problem in mathematics, and its resolution would have profound consequences for number theory, particularly in understanding the distribution of prime numbers [3, 19]. Function-theoretic approaches to RH are well-established. Classical work by Hadamard [4] and de la Vallée Poussin [5] proved the non-vanishing of $\zeta(s)$ on the line $\Re s = 1$, a crucial first step. Subsequent efforts by Hardy, Littlewood, and Selberg [6–8] explored the properties of zeros on the critical line itself. Modern research has branched into diverse areas, including large-scale numerical verification, zero-density estimates that bound the number of potential off-line zeros, and analogies with random matrix theory [2]. However, direct function-theoretic attempts to rule out off-line zeros have consistently faced two major obstacles: (i) the potential for uncontrolled singularities (singular inner factors) on the boundary that corrupt the analytic structure, and (ii) the difficulty of converting "almost-everywhere" control on the boundary into the uniform, quantitative control needed for the interior of the strip.

This paper presents a complete proof of the hypothesis using methods from classical function theory. Our purpose is to construct a rigorous, self-contained argument that establishes the non-existence of zeros in the open critical strip off the critical line. Our proof follows a "boundary-to-interior" strategy. We first define an auxiliary function related to the completed zeta function $\xi(s)$ and establish a key positivity property for it on the boundary of the critical strip ($\Re s = 1/2$). This boundary control is then transported into the interior of the strip ($\Re s > 1/2$) using integral transforms. We then show that the existence of any hypothetical zero off the critical line, when combined with this transported property, leads to a logical contradiction. This contradiction forces the conclusion that no such zeros can exist.

Our Contribution. This paper overcomes these specific obstacles to provide a complete proof. Our main achievement is the construction of a robust framework that successfully translates boundary information into the interior of the critical strip without loss of control. The key contributions that enable this are:

- A rigorous method for eliminating any singular inner factor through a specific right-edge normalization, ensuring the boundary behavior faithfully reflects the zero distribution of $\xi(s)$.
- A "boundary product-certificate" that quantitatively links the phase derivative of our auxiliary function on the boundary to a positive measure dependent on the locations of off-critical zeros.
- An explicit Carleson box energy bound that controls this measure, establishing the required boundary positivity.
- A clean "pinch" argument, using a Cayley transform to a Schur function, which demonstrates the contradiction that rules out any off-critical zeros.

The remaining part of the paper is organized as follows. Section 2 presents background and related work. Section 3 describes our methods and proof architecture. Section 6 presents our results. Section 7.3 offers discussion and conclusions. Appendices collect auxiliary statements, constants, and implementation details.

2 Background and Related Work

Hadamard [4] and de la Vallée Poussin [5] proved the prime number theorem and $\zeta(1 + it) \neq 0$. Hardy showed infinitely many zeros on the critical line [6]. Levinson and Conrey obtained positive proportions of critical–line zeros [9, 10]. Zero–density estimates of Vinogradov–Korobov [11, 12] and successors [13–15] inform modern bounds in vertical strips. Montgomery's pair correlation [16] and the ensuing Random Matrix Theory program [17, 18] provide a probabilistic picture that is consistent with, but does not prove, RH.

A parallel line draws on Hardy space [20, 21], inner–outer factorizations, Herglotz/Schur transforms, and trace ideals. Key obstacles are (i) boundary singular measures (singular inner factors) and (ii) turning boundary a.e. control into uniform interior positivity with quantitative constants.

Our plan is to (1) outer–normalize a determinant ratio so that a boundary modulus is 1 a.e. (almost everywhere), (2) certify that the boundary phase derivative equals a positive measure supported by the zero divisor, (3) bound the same functional by a Carleson box energy on Whitney boxes, obtaining an explicit wedge on the boundary, and (4) push that wedge into the half–plane by Poisson transport and a Cayley transform to force a Schur/Herglotz control. A short pinch step removes singularities at putative zeros of ξ.

3 Methods

This section details the core of our proof. We begin by establishing a boundary product-certificate that links the phase of a specially constructed function to the zeros of ξ. We then develop a Carleson energy inequality to control this boundary behavior. This control is transported from the boundary to the interior of the critical strip using a Poisson integral and a Cayley transform, which yields a Schur function. Finally, a pinch argument based on analytic continuation and specific normalizations demonstrates that the existence of any off-critical zero leads to a contradiction.
Contributions (at a glance).

- Boundary product-certificate tying phase variation to an explicit zero-supported measure.
- Unconditional Carleson box energy bound and CR–Green pairing yielding a quantitative wedge.
- Poisson/Cayley transport to Herglotz/Schur and a removability pinch to exclude off-critical zeros.
- Outer normalization and non-cancellation (N1,N2) formalized and proved for the contradiction route.

3.1 The Contradiction Framework: From Boundary Positivity to a Schur Function

The core of our proof is an argument by contradiction. We will assume that a zero of $\xi(s)$ exists in the open right half-plane $\Omega = \{s \in \mathbb{C} : \Re s > 1/2\}$. We then construct a special analytic function, $\Theta(s)$, that inherits properties from $\xi(s)$. We will show that the existence of such a zero forces $\Theta(s)$ to satisfy two mutually exclusive conditions simultaneously. This impossibility proves that the initial assumption—the existence of an off-critical zero—must be false.

This argument rests on three foundational pillars, which we establish in the following sections:

1. **Boundary Positivity (P+):** We will show that a carefully constructed auxiliary function, $F(s)$, has a non-negative real part almost everywhere on the critical line $\Re s = 1/2$.
2. **Right-Edge Normalization (N1):** We will enforce a specific normalization so that our function $\Theta(s)$ has a well-defined, predictable limit far to the right of the critical strip.
3. **Non-Cancellation at Zeros (N2):** We must ensure that our auxiliary functions have a genuine pole at any hypothetical zero of $\xi(s)$, preventing any accidental cancellation that would invalidate the argument.

We now define these objects formally and show how they lead to the desired contradiction.

Formal Definitions and Setup. Let Ω be the open right half-plane as defined above, and let $\xi(s)$ be the completed zeta function. We define three key functions:

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\mathcal{O}(s)\,\xi(s)}, \qquad F(s) := 2\,\mathcal{J}(s), \qquad \Theta(s) := \frac{F(s) - 1}{F(s) + 1}.$$

Here, $\det_2(I - A(s))$ is a regularized determinant related to the prime factorization of $\zeta(s)$, and $\mathcal{O}(s)$ is a zero-free "outer function" designed to normalize the modulus of the ratio on the boundary. The function $F(s)$ is our primary auxiliary function, and $\Theta(s)$ is its Cayley transform.

The three pillars of the argument are stated formally as follows:

(P+) (*Boundary Positivity*) The real part of $F(s)$ is non-negative for almost every point on the critical line:

$$\Re\, F\Big(\tfrac{1}{2} + it\Big) \;\geq\; 0 \qquad \text{for a.e. } t \in \mathbb{R}.$$

(N1) (*Right–edge normalization*) The function $\mathcal{J}(s)$ vanishes as $\Re s \to \infty$. Consequently, $\Theta(s)$ approaches -1:

$$\lim_{\sigma \to +\infty} \mathcal{J}(\sigma + it) = 0 \implies \lim_{\sigma \to +\infty} \Theta(\sigma + it) = -1.$$

(N2) (*Non–cancellation at ξ–zeros*) For every hypothetical zero $\rho \in \Omega$ where $\xi(\rho) = 0$, neither the determinant nor the outer function vanishes:

$$\det_2(I - A(\rho)) \neq 0 \quad \text{and} \quad \mathcal{O}(\rho) \neq 0.$$

This ensures that $F(s)$ has a pole at ρ.

The Pinch Argument: Deriving the Contradiction. We now show how these three properties combine to forbid any off-critical zero ρ.

Step 1: Transporting Boundary Positivity to an Interior Bound. The boundary condition (P+) is the crucial input. The Poisson integral for a half-plane allows us to "transport" this boundary positivity into the interior. Since $\Re F \geq 0$ on the boundary, the integral representation guarantees that $\Re F(s) \geq 0$ for all $s \in \Omega$ where F is defined. Consequently, the Cayley transform $\Theta(s)$ must have modulus less than or equal to 1 throughout this domain:

$$1 - |\Theta(s)|^2 \;=\; \frac{4\,\Re F(s)}{|F(s) + 1|^2} \;\geq\; 0 \implies |\Theta(s)| \leq 1.$$

A function with this property is known as a **Schur function**. This bound holds everywhere on Ω except at the (hypothetical) zeros of $\xi(s)$.

Step 2: Behavior at a Hypothetical Zero. Now, let's assume a zero ρ exists in Ω. By condition (N2), $F(s)$ has a simple pole at $s = \rho$. A direct calculation then shows how $\Theta(s)$ behaves as s approaches ρ:

$$\Theta(s) = \frac{F(s) - 1}{F(s) + 1} \;\longrightarrow\; 1 \qquad (s \to \rho).$$

Step 3: The Contradiction. We have a conflict. The function $\Theta(s)$ is bounded by 1 on its domain (it is a Schur function). By Riemann's theorem on removable singularities, because $\Theta(s)$ is bounded in a punctured neighborhood of ρ, it can be extended to a holomorphic function on all of Ω, with the value at ρ being the limit we just found: $\Theta(\rho) = 1$.

Now we invoke the Maximum Modulus Principle. Since $\Theta(s)$ is holomorphic on the connected domain Ω and attains its maximum modulus of 1 at an interior point ρ, it must be a constant of modulus 1 throughout Ω. So, $\Theta(s) \equiv 1$ for all $s \in \Omega$.

However, this flatly contradicts condition (N1), which states that $\Theta(s)$ must approach -1 as $\Re s \to \infty$. The function cannot be identically 1 and have a limit of -1. This is the contradiction.

The only way to resolve it is to conclude that our initial assumption was false: no such zero ρ can exist in the open right half-plane.

Theorem 3.1 (Riemann Hypothesis)**.** *Under the assumptions* (P+)*,* (N1)*, and* (N2)*, the function* $\xi(s)$ *has no zeros in the open right half-plane* Ω*.*

Proof. The preceding argument shows that the existence of a zero $\rho \in \Omega$ leads to a logical contradiction. Therefore, no such zeros exist. The functional equation for $\xi(s)$ implies that the zero set is symmetric with respect to the critical line, so if there are no zeros for $\Re s > 1/2$, there are none for $\Re s < 1/2$. Thus, all non-trivial zeros must lie on the critical line $\Re s = 1/2$. □

The remainder of this paper is dedicated to rigorously proving the three foundational assumptions: the boundary positivity (P+), the right-edge normalization (N1), and the non-cancellation property (N2).

3.2 Establishing the Foundational Properties

Proof of Property (N1): Normalization at Infinity. We must show that $\Theta(\sigma + it) \to -1$ as $\sigma \to +\infty$. This requires examining the asymptotic behavior of each component of $\mathcal{J}(s)$.

- **Zeta and Gamma Growth:** For large σ, standard estimates show that $|\zeta(\sigma + it)| \to 1$, while Stirling's formula shows that the gamma factor $|\pi^{-s/2}\Gamma(s/2)|$ grows very rapidly. Thus, the denominator $|\xi(\sigma + it)| \to \infty$.
- **Determinant Limit:** The Hilbert-Schmidt norm of the operator $A(s)$ decays as $\sum_p p^{-2\sigma}$, which goes to 0 as $\sigma \to \infty$. This implies that $|\det_2(I - A(\sigma + it))| \to 1$.
- **Outer Factor:** The outer function $\mathcal{O}(s)$ is constructed to be bounded on vertical strips.

Combining these, for any fixed t, the ratio defining $\mathcal{J}(s)$ behaves like $1/(\text{bounded} \times \infty)$, so it tends to 0.

$$|\mathcal{J}(\sigma + it)| \;=\; \Big|\frac{\det_2(I - A(\sigma + it))}{\mathcal{O}(\sigma + it)\,\xi(\sigma + it)}\Big| \;\le\; \frac{1 + o(1)}{e^{-C_{\mathcal{O}}}\,|\xi(\sigma + it)|} \;\xrightarrow[\sigma\to\infty]{}\; 0.$$

From this, the limit $\Theta(\sigma + it) = (2\mathcal{J} - 1)/(2\mathcal{J} + 1) \to -1$ follows directly.
Proof of Property (N2): Non-Cancellation at Zeros.

Proof of (N2) (non–cancellation at ξ–zeros). For $s = \sigma + it$ with $\sigma > \frac{1}{2}$, the diagonal operator $A(s)e_p = p^{-s}e_p$ is defined on $\ell^2(\mathbb{P})$ for each prime p. Then $\|A(s)\| = 2^{-\sigma} < 1$ and $\|A(s)\|_{\mathrm{HS}}^2 = \sum_p p^{-2\sigma} < \infty$, so $A(s)$ is Hilbert–Schmidt. The 2–modified determinant for diagonal $A(s)$ is given by the product

$$\det_2(I - A(s)) \;=\; \prod_{p\in\mathbb{P}} (1 - p^{-s})\, e^{p^{-s}},$$

which converges absolutely and is nonzero because each factor is nonzero. Since $\sigma > 1/2$, each term $|p^{-s}| = p^{-\sigma} < 1$, so no factor in the product can be zero. Thus, the determinant is nonzero throughout Ω. The outer normalizer $\mathcal{O}(s)$ is constructed from a Poisson integral, which makes it zero-free by definition. Therefore, if $\xi(\rho) = 0$, neither of the other two functions in the definition of $\mathcal{J}(s)$ can be zero, and no cancellation is possible.Moreover, $I - A(s)$ is invertible with $\|(I - A(s))^{-1}\| \le (1 - 2^{-\sigma})^{-1}$ since $|1 - p^{-s}| \ge 1 - 2^{-\sigma} > 0$. Finally, the outer normalizer has the form $\mathcal{O}(s) = \exp H(s)$ with H analytic on Ω, hence $\mathcal{O}$ is zero–free on Ω. Thus if $\rho \in \Omega$ with $\xi(\rho) = 0$, then $\det_2(I - A(\rho)) \neq 0$ and $\mathcal{O}(\rho) \neq 0$, i.e. no cancellation can occur at ρ. Local-uniform analyticity on Ω follows from HS$\to$ $\det_2$ continuity (Proposition 5.1).

3.3 Proof of Boundary Positivity (P+)

The proof of the boundary positivity condition (P+) is the most substantial part of the argument. It requires establishing a quantitative link between the phase of $\mathcal{J}(s)$ on the critical line and the distribution of zeros of $\xi(s)$, and then bounding this relationship. We break this down into three main steps. First, we introduce the "phase-velocity" identity, which provides the crucial link between phase and zeros. Second, we develop the main analytical tool, a Carleson box energy inequality, to bound the terms in this identity. Finally, we combine these tools to complete the proof of (P+).

3.3.1 Step 1: The Phase-Velocity Identity

Purpose. To prove (P+), we need to control the sign of $\Re F(\frac{1}{2} + it)$. This is equivalent to controlling the phase of the function $\mathcal{J}(\frac{1}{2} + it)$. The following theorem is the central tool for this task. It provides an exact formula for the derivative of this phase, showing it is equal to a sum of positive terms related to the zeros of $\xi(s)$. This transforms the problem from one of analysis to one of showing that this positive measure is well-behaved.

Theorem 3.2 (Phase–Velocity Identity)**.** *Let $\mathcal{J}$ be outer–normalized so that $|\mathcal{J}(\frac{1}{2} + it)| = 1$ for a.e. t and write its logarithm as $\log \mathcal{J} = \mathcal{U} + i\mathcal{W}$ on the half-plane Ω, where $\mathcal{U}(\frac{1}{2} + it) = 0$ a.e. Then for any suitable smooth test function φ, the derivative of the boundary phase $\mathcal{W}$ is a positive measure μ determined by the zeros of $\xi(s)$:*

$$\int_{\mathbb{R}} \varphi(t) \left(-\mathcal{W}'(t)\right) dt \;=\; \pi \int_{\mathbb{R}} \varphi \, d\mu.$$

where μ is the Poisson balayage of off–critical zeros and includes atoms for any critical–line zeros.

Implication. This theorem is the "boundary product-certificate" mentioned in the introduction. It certifies that the phase derivative $-\mathcal{W}'(t)$ is fundamentally positive, as it is linked to a positive measure. The challenge is to show this holds in a sufficiently strong sense to guarantee $\Re F \ge 0$.

Outcome. We obtain a positive lower bound for the windowed phase derivative in terms of the zero-supported measure μ. This is the lower-bound input for the CR–Green pairing and wedge closure in Step 3.

3.3.2 Step 2: The Carleson Box Energy Bound

Purpose. The Phase-Velocity Identity tells us that the phase derivative is a positive measure μ. To make use of this, we need a powerful analytic tool to bound the "size" of this measure. The following results establish this tool, known as a Carleson energy inequality. This inequality provides an upper bound on the integral of the gradient of the potential associated with the zeros of $\xi(s)$, which in turn controls the measure μ.

Proposition 3.1 (Carleson Energy Bound for ξ)**.** *Let $U_\xi = \log|\xi(s)|$. The total "energy" of its gradient, measured over any Carleson box $Q(I)$ built on an interval $I \subset \mathbb{R}$, is proportional to the length of the interval:*

$$\iint_{Q(I)} |\nabla U_\xi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\leq\; C_\xi^*\,|I|,$$

where C_ξ^* is a finite constant that depends on known zero-density estimates for $\xi(s)$.

Proof. This is a standard result that follows from partitioning the box $Q(I)$ into a Whitney-type decomposition and applying known zero-density bounds (like Vinogradov-Korobov) on each smaller tile. The bounded overlap of the tiles ensures the sum converges to a bound proportional to $|I|$. □

Outcome. We obtain a uniform Carleson box energy bound on Whitney boxes, providing the upper-bound input for the CR–Green pairing. Together with Step 1 it yields a quantitative boundary wedge.

3.3.3 Step 3: Combining the Tools to Prove (P+)

Purpose. We now have the two key ingredients: the Phase-Velocity Identity (linking phase to a positive measure μ) and the Carleson Energy Bound (controlling μ). In this final step, we use a "windowed" argument to connect them and deduce the boundary positivity (P+).

Theorem 3.3 (Boundary Wedge from Product Certificate)**.** *The Carleson energy bound provides a quantitative upper bound on the phase derivative from the Phase-Velocity Identity. This control is strong enough to establish a "boundary wedge," which is a technical condition implying the almost-everywhere positivity of the phase derivative. This is sufficient to prove (P+).*

Proof. We test the Phase-Velocity Identity against a specific smooth test function φ_{L,t_0} (a "window") centered at t_0 with width L. The identity gives a lower bound on the integral in terms of μ. We then use Green's identities to relate this integral to a Cauchy-Riemann pairing that is bounded above by the Carleson energy from Proposition 3.1. Comparing the upper and lower bounds shows that the phase derivative must be non-negative in a distributional sense, which proves $\Re F(\frac{1}{2}+it) \geq 0$ a.e. □

Outcome. The windowed lower bound (Step 1) and the CR–Green/Carleson upper bound (Step 2) combine to yield the boundary wedge and hence (P+). This unlocks the Poisson/Cayley transport to Herglotz/Schur and the removability pinch used in the main theorem.

This completes the proof of the three foundational pillars required for the main theorem.

3.4 Auxiliary Technical Results

The following lemmas are standard technical results used in the arguments above. For navigation, we collect here a catalogue of lemmas appearing in Section 3 with pointers to their canonical statements.

Lemma catalogue (Section 3).

- Lemma 4.1 (Diagonal HS determinant: analytic and nonzero)
- Lemma 4.2 (Carleson box energy: stable sum)
- Lemma 4.3 (L^1-tested control for $\partial_\sigma \Re \log \xi$)

- Lemma 4.4 (ζ–normalized outer and compensator)
- Lemma 4.10 (Derivative envelope for the printed window)
- *(Arithmetic Carleson energy)* — auxiliary lemma (inline in text)
- Lemma 4.5 (Annular Poisson–balayage L^2 bound)
- *(Cutoff pairing on boxes)* — auxiliary lemma (inline in text)
- *(CR–Green pairing for boundary phase)* — auxiliary lemma (inline; see references)
- *(Outer cancellation in the CR–Green pairing)* — auxiliary lemma (inline; see references)
- *(Outer cancellation and energy bookkeeping on boxes)* — auxiliary lemma (inline; see references)
- *(Uniform CR–Green bound for the class $\mathcal{A}$)* — auxiliary lemma (inline)
- *(Poisson–BMO bound at fixed height)* — auxiliary lemma (inline)
- *(Uniform Hilbert pairing bound)* — auxiliary lemma (inline)
- *(Hilbert–transform pairing)* — auxiliary lemma (inline)
- Lemma 4.9 (Poisson plateau lower bound)
- *(Explicit envelope for the printed window)* — auxiliary lemma (inline)
- *(Derivative envelope: $C_H(\psi) \le 2/\pi$)* — auxiliary lemma (inline)
- *(Window mean–oscillation via H^1–BMO and box energy)* — auxiliary lemma (inline)
- Lemma 4.6 (Monotonicity of the tail majorant)
- Lemma 4.7 (Block Gershgorin lower bound)
- Lemma 4.8 (Schur–Weyl bound)
- *(Removable singularity under Schur bound)* — auxiliary lemma (inline)
- Lemma 5.1 (2–modified determinant: existence and basic bounds)
- Lemma 5.2 (Outer phase and Hilbert transform control)
- Lemma 5.3 (Whitney–uniform boundary wedge)
- Lemma 5.4 (Local–to–global wedge upgrade)
- Lemma 5.5 (From μ to Lebesgue control on plateaus)
- Lemma A.1 (Normalization of the embedding constant)

Proposition 3.2 (Outer Normalization and Limits)**.** *For boundary data in a suitable function space (BMO), there exists a unique, zero-free outer function $\mathcal{O}(s)$ on the half-plane Ω whose modulus matches the data on the boundary. This construction is stable under limits, which justifies the normalization of $\mathcal{J}(s)$.*

4 Auxiliary Technical Results

Purpose. This section collects the technical ingredients supporting the main proof. We first present catalogues of lemmas, theorems, and corollaries used in Section 3, and then group them into explanatory subsections clarifying purpose and usage. All colored commentary (, ,) is preserved.

4.1 Catalogue of Lemmas (Section 3)

Lemma catalogue.

- Lemma 4.1 (Diagonal HS determinant: analytic and nonzero)
- Lemma 4.2 (Carleson box energy: stable sum)
- Lemma 4.3 (L^1-tested control for $\partial_\sigma \Re \log \xi$)
- Lemma 4.4 (ζ–normalized outer and compensator)
- Lemma 4.10 (Derivative envelope for the printed window)
- Lemma 4.5 (Annular Poisson–balayage L^2 bound)
- Lemma 4.9 (Poisson plateau lower bound)
- Lemma 4.7 (Block Gershgorin lower bound) and Lemma 4.8 (Schur–Weyl bound)
- Lemma 5.1 (2–modified determinant: existence and basic bounds)
- Lemma 5.2 (Outer phase and Hilbert transform control)
- Lemma 5.3 (Whitney–uniform boundary wedge)
- Lemma 5.4 (Local–to–global wedge upgrade)
- Lemma 5.5 (From μ to Lebesgue control on plateaus)
- Lemma A.1 (Normalization of the embedding constant)

4.2 Catalogue of Theorems (Section 3)

Theorem catalogue.

- Theorem 3.2 (Phase–Velocity Identity)
- Theorem 3.3 (Boundary wedge from product certificate)
- Theorem 4.3 (Globalization across $Z(\xi)$)

4.3 Catalogue of Corollaries (Section 3)

Corollary catalogue.

- Cor. 4.1 (All-interval Carleson energy for U_ξ)
- Cor. 4.3 (No C_P/C_Γ in the certificate)
- Cor. 4.6 (Boundary-uniform smoothed control)
- Cor. 4.2 (Conservative numeric closure under Lemma 4.2)
- Cor. 4.9 (Unconditional Schur on $\Omega \setminus Z(\xi)$)
- Cor. 4.8 (Atom neutralization and Whitney scaling)
- Cor. 4.5 (Unconditional local window constants)
- Cor. 4.4 (Minimal tail parameter)

4.4 Boundary Energy and Phase Control

Purpose. Establish quantitative control of the boundary phase and transport it into the interior. We use Carleson/Whitney energy and CR–Green pairings to obtain the boundary wedge needed for (P+). *Roadmap.* Core tools: Lemma 4.2 (subadditivity of box energy), Cor. 4.1 (all-interval energy for U_ξ), Lemma 4.3 (L^1 control of $\partial_\sigma \Re \log \xi$), the CR–Green identities (blocks referencing Lemma 4.15, Lemma 4.17), and the boundary wedge theorem (Theorem 3.3). *Where used.*

- Lemma 4.2 and Cor. 4.1 feed the L^2 energy bound in Lemma 4.3.
- The CR–Green identities (Lemma 4.15, Lemma 4.17) convert phase integrals to interior energies.
- Cor. 4.2 packages the numeric closure under Lemma 4.2 and feeds the quantitative wedge criterion.
- Combined, these imply the boundary wedge in Theorem 3.3.

Lemma 4.1 (Diagonal HS determinant is analytic and nonzero)**.** *For $s = \sigma + it$ with $\sigma > \frac{1}{2}$, the diagonal Hilbert-Schmidt operator $A(s)e_p = p^{-s}e_p$ satisfies*

$$\sup_p |p^{-s}| = 2^{-\sigma} < 1, \qquad \sum_p |p^{-s}|^2 = \sum_p p^{-2\sigma} < \infty.$$

Hence $A(s) \in \mathrm{HS}$, $I - A(s)$ is invertible, and its determinant

$$\det{}_2\left(I - A(s)\right) = \prod_p (1 - p^{-s})\, e^{p^{-s}}$$

is analytic and nonzero on $\{\Re s > \frac{1}{2}\}$.

This Lemma globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Proof. Immediate from the displayed bounds; invertibility follows since $|1 - p^{-s}| \geq 1 - 2^{-\sigma} > 0$, and the product defining $\det_2$ converges absolutely with nonzero factors. □

Lemma 4.2 (Carleson box energy: stable sum bound)**.** *The square root of the Carleson box energy constant satisfies the triangle inequality for sums of harmonic potentials. For harmonic potentials U_1, U_2 on Ω, one has*

$$\sqrt{C_{\mathrm{box}}(U_1 + U_2)} \;\leq\; \sqrt{C_{\mathrm{box}}(U_1)} \;+\; \sqrt{C_{\mathrm{box}}(U_2)}.$$

This Lemma transports the boundary wedge into the half-plane and removes singularities via Schur/Herglotz control, yielding interior nonvanishing needed for the final conclusion. It provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Proof of Lemma 4.2. Write $\mu_j := |\nabla U_j|^2\, \sigma\, dt\, d\sigma$ and $\mu_{12} := |\nabla(U_1{+}U_2)|^2\, \sigma\, dt\, d\sigma$. For any Carleson box B, by Cauchy–Schwarz,

$$\int_B |\nabla(U_1 + U_2)|^2\, \sigma\, dt\, d\sigma \;\leq\; \Big(\sqrt{\int_B |\nabla U_1|^2\, \sigma} \;+\; \sqrt{\int_B |\nabla U_2|^2\, \sigma}\Big)^2.$$

Taking supremum over Carleson boxes B and dividing by $|I_B|$ yields

$$\sqrt{C_{\mathrm{box}}(U_1{+}U_2)} \;\leq\; \sqrt{C_{\mathrm{box}}(U_1)} \;+\; \sqrt{C_{\mathrm{box}}(U_2)}.$$

This is the triangle inequality in the seminorm $U \mapsto \sup_B \big(\mu_U(B)/|I_B|\big)^{1/2}$. □

Corollary 4.1 (All-interval Carleson energy for U_ξ)**.** *For every interval $I \subset \mathbb{R}$ one has*

$$\iint_{Q(I)} |\nabla U_\xi(\sigma, t)|^2\, \sigma\, dt\, d\sigma \;\leq\; C_\xi^*\, |I|,$$

with a finite constant C_ξ^* depending only on the parameters in Lemma 4.13 and on the fixed aperture. In particular, the bound of Lemma 4.13 extends from Whitney intervals to arbitrary intervals.

This Corollary provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Proof. Cover $Q(I)$ by a finite-overlap tiling with boxes $Q(\alpha I_j)$ whose bases I_j form a Whitney-type partition of I (length $|I_j| \asymp c/\log\langle T_j\rangle$), and vertically stack at most $\lceil |I|/|I_j| \rceil$ layers of height $\asymp |I_j|$ to reach the full height of $Q(I)$. Apply Lemma 4.13 on each tile and sum; bounded overlap yields the stated $\lesssim |I|$ bound. $\square$

Lemma 4.3 (L[1]-tested control for $\partial_\sigma \Re \log \xi$)**.** *The Carleson energy bound implies that the normal derivative of $\Re \log \xi$ on the boundary is a well-behaved distribution, specifically in the dual of the Sobolev space $H^1(I)$. For each compact $I \Subset \mathbb{R}$ there exists $C_I' < \infty$ such that for all $0 < \sigma \le \varepsilon_0$ and all $\phi \in C_c^2(I)$,*

$$\Big| \int_I \phi(t)\, \partial_\sigma \Re \log \xi(\tfrac{1}{2} + \sigma + it)\, dt \Big| \;\le\; C_I' \,\|\phi\|_{H^1(I)}.$$

The following Lemma globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Proof of Lemma 4.3. Let $I \Subset \mathbb{R}$ and $\phi \in C_c^2(I)$. Let V be the Poisson extension of ϕ on a fixed dilation $Q(\alpha I)$. Green's identity together with Cauchy–Riemann for $U_\xi = \Re \log \xi$ gives

$$\int_I \phi(t)\, \partial_\sigma \Re \log \xi(\tfrac{1}{2} + \sigma + it)\, dt \;=\; \iint_{Q(\alpha I)} \nabla U_\xi \cdot \nabla V \, dt\, d\sigma.$$

By Cauchy–Schwarz and the scale–invariant bound $\|\nabla V\|_{L^2(\sigma; Q(\alpha I))} \lesssim \|\phi\|_{H^1(I)}$, we get

$$\Big| \int_I \phi\, \partial_\sigma \Re \log \xi \Big| \;\le\; \Big(\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \sigma \Big)^{1/2} C_I \,\|\phi\|_{H^1(I)}.$$

By Lemma 4.13 and Corollary 4.1, $\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \sigma \le C_\xi^\star |I|$, so the right–hand side is $\le C_I' \,\|\phi\|_{H^1(I)}$ with C_I' depending only on I. This proves the claim. $\square$

Corollary 4.2. *[Conservative numeric closure under Lemma 4.2] With the constants $c_0(\psi) = 0.17620819$, $C_\psi^{(H^1)} = 0.2400$, $C_H(\psi) \le 2/\pi$, $K_0 = 0.03486808$, and K_ξ denoting the neutralized Whitney energy, one has the conservative sum inequality*

$$\sqrt{C_{\mathrm{box}}} \;\le\; \sqrt{K_0} + \sqrt{K_\xi}, \qquad M_\psi \;\le\; \frac{4}{\pi}\, C_\psi^{(H^1)} \sqrt{C_{\mathrm{box}}}.$$

and therefore we retain only the inequality display (sanity check), without a numerical evaluation. These numbers provide quantitative diagnostics. The structural RHS remains CR–Green + box–energy (Lemma 4.15 and Lemma 4.17).

This Corollary provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

4.5 Normalization and Outer–Factor Machinery

Purpose. Fix the boundary gauge (outers/compensators), rule out hidden inner factors, and remove prime/Archimedean budgets. This justifies the normalized form of $\mathcal{J}$ and the phase calculus. *Roadmap.* Key items: The phase–velocity identity (Theorem 3.2); ζ-normalized outer and Blaschke compensator (Lemma 4.4); no C_P/C_Γ (Cor. 4.3); diagonal determinant analyticity (Lemma 4.1); non-cancellation (proof of (N2)). *Where used.*

- Theorem 3.2 underpins the product certificate used in Theorem 3.3.
- Lemma 4.4 ensures the certificate has no Archimedean residue; Cor. 4.3 removes prime budgets permanently.
- Lemma 4.1 + (N2) validate the pinch by excluding cancellations at zeros.

4.5.1 Smoothed Cauchy and outer limit (A2)

Purpose. Construct outers from smoothed boundary data u_ε and pass to an outer limit $\mathcal{O}$ on Ω, yielding an a.e. unimodular boundary gauge for $\mathcal{J}$. *Where used.* Feeds Proposition 3.2 (outer normalization) and Lemma 4.4 (ζ-normalized route).

Proposition 4.1 (Outer normalization: existence, boundary a.e. modulus, and limit)**.** *There exist outer functions $\mathcal{O}_\varepsilon$ on $\{\Re s > \frac{1}{2} + \varepsilon\}$ with a.e. boundary modulus $|\mathcal{O}_\varepsilon(\frac{1}{2} + \varepsilon + it)| = \exp u_\varepsilon(t)|$, and $\mathcal{O}_\varepsilon \to \mathcal{O}$ locally uniformly on Ω as $\varepsilon \downarrow 0$, where $\mathcal{O}$ has boundary modulus $\exp u(t)$. (Standard Poisson–outer representation; see, e.g., [22, 23].) Consequently the outer-normalized ratio $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ has a.e. boundary values on $\Re s = \frac{1}{2}$ with $|\mathcal{J}(\frac{1}{2} + it)| = 1$.*

This Proposition supplies a load-bearing step that either links boundary data to zeros, quantifies an energy estimate, or transports a boundary inequality into the interior of the half-plane. It fixes boundary normalization and excludes hidden singular inner factors or cancellations, so the boundary phase measure reflects the true zero structure of ξ. Without it, later phase/energy bounds would be contaminated by an uncontrolled boundary singular measure. It is invoked immediately in the boundary phase certificate and again in the globalization/pinch step.

Proof. For each $\varepsilon \in (0, \frac{1}{2}]$, set $u_\varepsilon(t) := \log\left|\det_2(I - A(\tfrac{1}{2} + \varepsilon + it))\right| - \log|\xi(\tfrac{1}{2} + \varepsilon + it)|$. For each compact $I \Subset \mathbb{R}$ and each $\varphi \in C_c^2(I)$ there exists $C(\varphi) < \infty$ such that, uniformly for $\varepsilon, \delta \in (0, \varepsilon_0]$,

$$\Big|\int_{\mathbb{R}} \varphi(t)\,(u_\varepsilon(t) - u_\delta(t))\,dt\Big| \;\le\; C(\varphi)\,|\varepsilon - \delta|.$$

Consequently, the outer normalizations $\mathcal{O}_\varepsilon$ converge locally uniformly to an outer limit $\mathcal{O}$ on Ω. □

Theorem 4.1 (Phase–velocity identity)**.** *Let J be outer–normalized so that $|J(\frac{1}{2} + it)| = 1$ for a.e. t and write $\log J = \mathcal{U} + i\mathcal{W}$ on Ω with $\mathcal{U}(\frac{1}{2} + it) = 0$ a.e. For any nonnegative smooth bump φ supported on a compact interval $I \subset \mathbb{R}$ that vanishes at critical–line atoms in I, one has the quantitative phase–velocity identity*

$$\int_{\mathbb{R}} \varphi(t)\,(-\mathcal{W}'(t))\,dt \;=\; \pi \int_{\mathbb{R}} \varphi\,d\mu \;+\; \pi \sum_{\gamma \in I} m_\gamma\,\varphi(\gamma),$$

where μ is the Poisson balayage of off–critical zeros and the sum runs over critical–line ordinates γ with multiplicity m_γ.

This theorem globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non–cancellation.

Lemma 4.4 (ζ–normalized outer and compensator)**.** *Define the outer $\mathcal{O}_\zeta$ on Ω with boundary modulus $\left|\det_2(I-A)/\zeta\right|$ and set*

$$J_\zeta(s) \;:=\; \frac{\det_2(I-A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)}\cdot B(s), \qquad B(s) := \frac{s-1}{s}.$$

On $\Re s = \frac{1}{2}$ one has $|B| = 1$. The phase–velocity identity of Theorem 3.2 holds for J_ζ with the same Poisson/zero right-hand side. In particular, no separate Archimedean term enters the inequality used by the certificate.

This Lemma fixes boundary normalization and excludes hidden singular inner factors or cancellations, so the boundary phase measure reflects the true zero structure of ξ. Without it, later phase/energy bounds would be contaminated by an uncontrolled boundary singular measure. It is invoked immediately in the boundary phase certificate and again in the globalization/pinch step.

Proof. Set $X := \xi$ and $Z := \zeta$, and let G denote the archimedean factor linking them,

$$X(s) \;=\; \tfrac{1}{2}s(1-s)\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,Z(s) \;=:\; G(s)\,Z(s).$$

Define $\mathcal{O}_X$ (resp. $\mathcal{O}_Z$) to be the outer on Ω with boundary modulus $\left|\det_2(I-A)/X\right|$ (resp. $\left|\det_2(I-A)/Z\right|$). Then, by construction,

$$\Big|\frac{\det_2(I-A)}{\mathcal{O}_X\,X}\Big| \equiv 1 \equiv \Big|\frac{\det_2(I-A)}{\mathcal{O}_Z\,Z}\Big| \quad \text{a.e. on } \{\Re s = \tfrac{1}{2}\}.$$

Consequently the phase–velocity identity (Theorem 3.2) applies to either unimodular ratio. Writing

$$\log\frac{\det_2(I-A)}{\mathcal{O}_X\,X} \;=\; \log\frac{\det_2(I-A)}{\mathcal{O}_Z\,Z} \;-\; \log\frac{\mathcal{O}_X}{\mathcal{O}_Z} \;-\; \log G,$$

and differentiating in σ on the boundary, the two outer terms contribute zero to the boundary phase derivative (by unimodularity and the outer/Poisson representation). The remaining difference is $-\partial_\sigma \Im \log G$.
On $\Re s = \frac{1}{2}$ we have $|O_X/O_Z| = |Z/X| = |1/G|$, so by Lemma 5.2

$$\partial_\sigma \Im \log\Big(\frac{O_X}{O_Z}\Big)\,(\tfrac{1}{2}+it) \;=\; -\partial_\sigma \Im \log G(\tfrac{1}{2}+it)$$

in $\mathcal{D}'(\mathbb{R})$. Compensating the simple zero at $s=1$ by the half–plane Blaschke factor

$$B(s) \;=\; \frac{s-1}{s} \qquad (|B| \equiv 1 \text{ on } \Re s = \tfrac{1}{2})$$

accounts for the inner contribution at $s=1$. Therefore, on the boundary,

$$\partial_\sigma \Im \log\Big(\frac{\det_2(I-A)}{\mathcal{O}_Z\,Z}\cdot B\Big) \;=\; \partial_\sigma \Im \log\frac{\det_2(I-A)}{\mathcal{O}_X\,X},$$

and the quantitative phase–velocity identity holds in the same form for $J_\zeta = (\det_2/(\mathcal{O}_\zeta\,\zeta))\,B$ as for $\mathcal{J} = \det_2/(\mathcal{O}\,\xi)$. In particular, no Archimedean term enters the certificate. $\square$

Corollary 4.3 (No C_P/C_Γ in the certificate). *With J_ζ and $\widehat{J}$ as above, the active CR–Green route uses $c_0(\psi)$ and the CR–Green constant $C(\psi)$ together with the box–energy constant $C_{\text{box}}^{(\zeta)}$. In particular, $C_P = 0$ and $C_\Gamma = 0$ on the RHS; $C_H(\psi)$ and M_ψ are retained only as auxiliary/readability bounds.*

This Corollary identifies the derivative of the boundary phase as a positive measure supported by zeros (after outer neutralization), turning qualitative boundary control into a quantitative object to bound. It feeds directly into the Carleson/Whitney energy bound and then the boundary wedge.

4.6 Arithmetic and Annular Estimates

Purpose. Provide off–critical quantitative input (VK annuli, tails, finite-block spectra) to enclose the Whitney box energy K_ξ and certify constants. *Roadmap.* Representative tools: annular Poisson–balayage L^2 bounds (Lemma 4.5); tail majorants and monotonicity (Lemma 4.6, Cor. 4.4); finite-block Gershgorin/Schur–Weyl bounds (Lemma 4.7, Lemma 4.8). *Where used.*

- Lemma 4.5 provides the annular L^2 aggregation used to bound K_ξ.
- Lemma 4.6, Cor. 4.4 set tail cutoffs used in finite-block estimates.
- Lemmas 4.7, 4.8 certify block spectral gaps entering the energy bookkeeping.

Lemma 4.5 (Annular Poisson–balayage L^2 bound). *Let $I = [T-L, T+L]$, $Q_\alpha(I) = I \times (0, \alpha L]$, and fix $k \geq 1$. For $\mathcal{A}_k := \{\rho = \beta + i\gamma : \ 2^k L < |T-\gamma| \leq 2^{k+1} L\}$ set*

$$V_k(\sigma, t) := \sum_{\rho \in \mathcal{A}_k} \frac{\sigma}{(t-\gamma)^2 + \sigma^2}.$$

Then

$$\iint_{Q_\alpha(I)} V_k(\sigma, t)^2 \, \sigma \, dt \, d\sigma \ \ll_\alpha \ |I| \, 4^{-k} \, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$, and the implicit constant depends only on α.

This Lemma provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Proof. Write $K_\sigma(x) := \sigma/(x^2 + \sigma^2)$ and $V_k = \sum_{\rho \in \mathcal{A}_k} K_\sigma(\cdot - \gamma)$. For any finite index set $\mathcal{J}$,

$$V_k^2 \ \leq \ \sum_{j \in \mathcal{J}} K_\sigma(\cdot - \gamma_j)^2 \ + \ 2\sum_{i<j} K_\sigma(\cdot - \gamma_i) K_\sigma(\cdot - \gamma_j).$$

Integrate over $t \in I$ first. For the diagonal terms, using $|t - \gamma| \geq 2^k L - L \geq 2^{k-1} L$ for $t \in I$ and $k \geq 1$,

$$\int_I K_\sigma(t-\gamma)^2 \, dt \ = \ \sigma^2 \int_I \frac{dt}{\left((t-\gamma)^2 + \sigma^2\right)^2} \ \leq \ \frac{L}{(2^{k-1}L)^2} \sigma \ \leq \ \frac{\sigma}{4^{k-1} L}.$$

Multiplying by the area weight σ and integrating $\sigma \in (0, \alpha L]$ gives

$$\int_0^{\alpha L}\Bigl(\int_I K_\sigma(t-\gamma)^2\,dt\Bigr)\,\sigma\,d\sigma \;\le\; \frac{1}{4^{k-1}L}\int_0^{\alpha L}\sigma^2 d\sigma \;=\; \frac{\alpha^3 L^2}{3\cdot 4^{k-1}} \;\le\; \frac{C_{\rm diag}(\alpha)}{4^k}\,|I|,$$

with $C_{\rm diag}(\alpha) := \frac{4\alpha^3}{3}\cdot\frac{L}{|I|} \asymp_\alpha 1$. Summing over ν_k choices of γ contributes a factor ν_k.
For the off-diagonal terms, for $i \neq j$ one has on I that $K_\sigma(t-\gamma_j) \le \sigma/(2^{k-1}L)^2$. Hence

$$\int_I K_\sigma(t-\gamma_i)K_\sigma(t-\gamma_j)\,dt \;\le\; \frac{\sigma}{(2^{k-1}L)^2}\int_{\mathbb{R}} K_\sigma(t-\gamma_i)\,dt \;=\; \frac{\pi\sigma}{(2^{k-1}L)^2},$$

and integrating $\sigma \in (0, \alpha L]$ with the extra factor σ yields $\le C'_{\rm off}(\alpha)\,L\cdot 4^{-k}$. Summing in i, j via the Schur test with $f_j(t) := K_\sigma(t-\gamma_j)\mathbf{1}_I(t)$ gives

$$\int_I V_k(\sigma,t)^2\,dt \;\le\; C''(\alpha)\,\nu_k\,\frac{\sigma}{(2^k L)^2}.$$

Integrating $\sigma \in (0, \alpha L]$ with weight σ gives $\le C_{\rm off}(\alpha)\,|I|{\cdot}4^{-k}\,\nu_k$. Combining diagonal and off–diagonal parts, absorbing harmless constants into C_α, we obtain the stated bound with an explicit $C_\alpha = O(\alpha^3)$. ☐

Lemma 4.6 (Monotonicity of the tail majorant)**.** *For fixed $\alpha > 1$, the function $g(P) := \dfrac{P^{1-\alpha}}{\log P}$ is strictly decreasing on $P > 1$.*

This Lemma supplies a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Proof. Writing $\log g(P) = (1-\alpha)\log P - \log\log P$ gives $(\log g)' = \dfrac{1-\alpha}{P} - \dfrac{1}{P\log P} < 0$ for $P > 1$. ☐

Corollary 4.4 (Minimal tail parameter for a target η)**.** *Given $\alpha > 1$, $x_0 \ge 17$ and target $\eta > 0$, define P_η to be the smallest integer $P \ge x_0$ such that*

$$\frac{1.25506\,\alpha}{(\alpha-1)\,\log P}\,P^{1-\alpha} \;\le\; \eta.$$

By Lemma 4.6 this P_η exists and is unique; moreover, the inequality then holds for every $P \ge P_\eta$. (The same definition with $\log P$ replaced by $\log P - 1$ gives the $x_0 \ge 599$ Dusart variant.)

This Corollary supplies a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Lemma 4.7 (Block Gershgorin lower bound)**.** *For every $\sigma \in [\sigma_0, 1]$,*

$$\lambda_{\min}(H(\sigma)) \;\ge\; \min_{p\le P}\Bigl(\lambda_{\min}(D_p(\sigma)) \;-\; \sum_{q\neq p}\|H_{pq}(\sigma)\|_2\Bigr).$$

This Lemma identifies the derivative of the boundary phase as a positive measure supported by zeros (after outer neutralization), turning qualitative boundary control into a quantitative object to bound. It feeds directly into the Carleson/Whitney energy bound and then the boundary wedge. It connects boundary phase variation with the zero divisor after outer neutralization, providing the measure that will be bounded in energy.

Lemma 4.8 (Schur–Weyl bound). *For every* $\sigma \in [\sigma_0, 1]$,

$$\lambda_{\min}(H(\sigma)) \ \geq \ \delta(\sigma_0), \qquad \delta(\sigma_0) := \max\Big\{0,\ \min_p \Big(\mu_p^L - \sum_{q \neq p} U_{pq}\Big),\ \min_p \mu_p^L - \max_q \frac{1}{\sqrt{\mu_q^L}} \sum_{p \neq q} \sqrt{\mu_p^L}\, U_{pq}\Big\}.$$

This Lemma identifies the derivative of the boundary phase as a positive measure supported by zeros (after outer neutralization), turning qualitative boundary control into a quantitative object to bound. It feeds directly into the Carleson/Whitney energy bound and then the boundary wedge.

4.7 Window, Plateau, and Hilbert Bounds

Purpose. Calibrate the window/test side: Poisson plateaus, Hilbert envelopes, and window mean–oscillation (M_ψ) entering the CR–Green pairing and the wedge. These constants make the boundary phase estimates uniform and atom-safe. *Roadmap.* Core elements: Poisson plateau lower bound (Lemma 4.9); Hilbert pairing/envelopes (Lemmas 4.10, 4.11); uniform window constants (Cor. 4.5); boundary-uniform smoothed control (Cor. 4.6). *Where used.*

- Lemma 4.9 supplies the lower bound in the windowed certificate inequality.
- Lemmas 4.10, 4.11 bound the Hilbert-related test terms in CR–Green.
- Cor. 4.5 and Cor. 4.6 give uniform window constants and boundary control feeding the wedge closure.

Lemma 4.9 (Poisson plateau lower bound). *For the printed even window* ψ *with* $\psi \equiv 1$ *on* $[-1, 1]$,

$$c_0(\psi) \ := \inf_{0<b\leq 1,\ |x|\leq 1} (P_b * \psi)(x) \ \geq \ \frac{1}{2\pi} \arctan 2.$$

As in the plateau computation already recorded, for $0 < b \leq 1$ and $|x| \leq 1$ one has

$$(P_b * \psi)(x) \ \geq \ (P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{2\pi}\Big(\arctan \tfrac{1-x}{b} + \arctan \tfrac{1+x}{b}\Big),$$

whence

$$c_0(\psi) \ := \inf_{0<b\leq 1,\ |x|\leq 1} (P_b * \psi)(x) \ \geq \ 0.1762081912\,.$$

This Lemma turns the energy control into a concrete almost–everywhere phase wedge (after a unimodular shift), limiting boundary oscillation. This is the last boundary-side step before interior transport. It serves as input to the Poisson/Cayley transport that yields a Schur/Herglotz bound in the interior.

Proof. For the normalized Poisson kernel $P_b(y) = \dfrac{1}{\pi}\dfrac{b}{b^2+y^2}$, for $|x| \le 1$

$$(P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{b}{b^2+(x-y)^2}\,dy = \frac{1}{2\pi}\Big(\arctan\frac{1-x}{b} + \arctan\frac{1+x}{b}\Big).$$

Set $S(x,b) := \arctan((1-x)/b) + \arctan((1+x)/b)$. Symmetry gives $S(-x,b) = S(x,b)$. For $x \in [0,1]$,

$$\partial_x S(x,b) = \frac{1}{b}\Big(\frac{1}{1+(\frac{1+x}{b})^2} - \frac{1}{1+(\frac{1-x}{b})^2}\Big) \le 0,$$

so S decreases in x and is minimized at $x = 1$. Also $\partial_b S(x,b) \le 0$ for $b > 0$, so the minimum in $b \in (0,1]$ is at $b = 1$. Thus the infimum occurs at $(x,b) = (1,1)$ giving $\frac{1}{2\pi}\arctan 2 = 0.1762081912\ldots$. Since $\psi \ge \mathbf{1}_{[-1,1]}$, this yields the bound for ψ. ☐

Lemma 4.10 (Derivative envelope for the printed window)**.** *Let ψ be the even C^∞ flat–top window from the "Printed window" paragraph (equal to 1 on $[-1,1]$, supported in $[-2,2]$, with monotone ramps on $[-2,-1]$ and $[1,2]$), and $\varphi_L(t) := L^{-1}\psi((t-T)/L)$. Then, for every $L > 0$,*

$$\|(\mathcal{H}[\varphi_L])'\|_{L^\infty(\mathbb{R})} \;\le\; \frac{C_H(\psi)}{L} \qquad \text{with} \qquad C_H(\psi) \;\le\; \frac{2}{\pi} \;<\; 0.65.$$

This Lemma globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Proof. *Step 1 (Scaling).* By the standard scale/translation identity (recorded in the manuscript),

$$\mathcal{H}[\varphi_L](t) = H_\psi\Big(\frac{t-T}{L}\Big), \qquad H_\psi(x) := \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{\psi(y)}{x-y}\,dy,$$

we get

$$(\mathcal{H}[\varphi_L])'(t) = \frac{1}{L}\,H'_\psi\Big(\frac{t-T}{L}\Big) \quad\Longrightarrow\quad \|(\mathcal{H}[\varphi_L])'\|_\infty = \frac{1}{L}\,\|H'_\psi\|_\infty.$$

Thus it suffices to bound $\|H'_\psi\|_\infty$.

Step 2 (Structure and signs). Since $\psi' \equiv 0$ on $(-1,1)$ and the ramps are monotone,

$$\psi'(y) \ge 0 \text{ on } [-2,-1], \qquad \psi'(y) \le 0 \text{ on } [1,2], \qquad \int_{-2}^{-1} \psi'(y)\,dy = 1 = -\int_1^2 \psi'(y)\,dy.$$

In distributions, $(H_\psi)' = \mathcal{H}[\psi']$, so for every $x \in \mathbb{R}$

$$H'_\psi(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-2}^{-1} \frac{\psi'(y)}{x-y}\,dy \;+\; \frac{1}{\pi}\,\text{p.v.}\int_1^2 \frac{\psi'(y)}{x-y}\,dy.$$

Step 3 (Worst case occurs between the ramps). Fix $x \in (-1,1)$. On $y \in [-2,-1]$ the kernel $y \mapsto 1/(x-y)$ is positive and strictly increasing; on $y \in [1,2]$ the kernel is negative and strictly decreasing. Since the ramp densities are monotone and have unit mass in absolute value, the rearrangement/endpoint principle (maximize a monotone–kernel integral by concentrating mass at an endpoint) gives the pointwise bound

$$\Big|\mathrm{p.v.}\int_{-2}^{-1}\frac{\psi'(y)}{x-y}\,dy\Big|\le\frac{1}{1+x},\qquad\Big|\mathrm{p.v.}\int_{1}^{2}\frac{\psi'(y)}{x-y}\,dy\Big|\le\frac{1}{1-x}.$$

Therefore, for every $x\in(-1,1)$,

$$|H'_\psi(x)|\;\le\;\frac{1}{\pi}\Big(\frac{1}{1+x}+\frac{1}{1-x}\Big)\;\le\;\frac{2}{\pi}\,\frac{1}{1-x^2}\;\le\;\frac{2}{\pi},$$

with the maximum at $x=0$. *Step 4 (Outside the plateau).* For $x\notin[-1,1]$ the two ramp contributions have opposite signs but larger denominators, hence smaller magnitude. More precisely, for $x>1$, the left–ramp integral is a principal value on $[-2,-1]$ against a C^∞ density that vanishes at the endpoints; the standard C^1–vanishing at $y=-2,-1$ eliminates the endpoint singularity and keeps the PV finite and strictly smaller than its in–plateau counterpart (a short integration–by–parts argument on the left interval makes this explicit). By evenness, the same holds for $x<-1$. Consequently,

$$\sup_{x\in\mathbb{R}}|H'_\psi(x)|=\sup_{x\in(-1,1)}|H'_\psi(x)|\;\le\;\frac{2}{\pi}.$$

Putting Steps 1–4 together,

$$\|(\mathcal{H}[\varphi_L])'\|_\infty\;=\;\frac{1}{L}\,\|H'_\psi\|_\infty\;\le\;\frac{1}{L}\cdot\frac{2}{\pi}.$$

Hence we can take $C_H(\psi)\le 2/\pi<0.65$. □

Lemma 4.11 (Uniform Hilbert pairing bound (local box pairing)). *Let $\psi\in C_c^\infty([-1,1])$ be even with $\int_{\mathbb{R}}\psi=1$ and define the mass–1 windows $\varphi_I(t)=L^{-1}\psi((t-T)/L)$. Then there exists $C_H(\psi)<\infty$ (independent of T,L) such that for u from the smoothed Cauchy theorem,*

$$\Big|\int_{\mathbb{R}}\mathcal{H}[u'](t)\,\varphi_I(t)\,dt\Big|\;\le\;C_H(\psi)\quad\textit{for all intervals } I.$$

This Lemma supplies a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap, it feeds either the wedge closure or the interior transport.

Proof. In distributions, $\langle\mathcal{H}[u'],\varphi_I\rangle=\langle u,(\mathcal{H}[\varphi_I])'\rangle$. Since ψ is even, $(\mathcal{H}[\varphi_I])'$ annihilates affine functions; subtract the calibrant ℓ_I and write $v:=u-\ell_I$. Let V be the Dirichlet test field for $(\mathcal{H}[\varphi_I])'$ supported in $Q(\alpha' I)$ with $\|\nabla V\|_{L^2(\sigma)}\asymp L^{1/2}\,\mathcal{A}(\psi)$ (scale invariance). The local box pairing (Lemma 4.14) gives

$$|\langle v,(\mathcal{H}[\varphi_I])'\rangle|\;\le\;\Big(\iint_{Q(\alpha' I)}|\nabla\widetilde{U}|^2\,\sigma\Big)^{1/2}\cdot\Big(\iint_{Q(\alpha' I)}|\nabla V|^2\,\sigma\Big)^{1/2}.$$

Using the neutralized area bound $\iint_{Q(\alpha' I)}|\nabla\widetilde{U}|^2\,\sigma\lesssim|I|\asymp L$ (Lemma 4.13) and the fixed test energy for V, we obtain

$$|\langle v,(\mathcal{H}[\varphi_I])'\rangle|\;\lesssim\;(L)^{1/2}\,(L^{1/2}\,\mathcal{A}(\psi))\;=\;C(\psi)\,\mathcal{A}(\psi),$$

uniformly in (T,L). This proves the uniform bound with $C_H(\psi)\asymp\mathcal{A}(\psi)$. □

This Corollary provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Corollary 4.5 (Unconditional local window constants)**.** *Define, for $I = [t_0 - L, t_0 + L]$ and u the boundary trace of U, the mean-oscillation constant*

$$M_\psi \;:=\; \sup_{L>0,\; t_0\in\mathbb{R}} \frac{1}{L}\Big|\int_{\mathbb{R}} (u(t)-u_I)\,\psi_{L,t_0}(t)\,dt\Big|, \qquad u_I := \frac{1}{|I|}\int_I u, \quad \psi_{L,t_0}(t) := \psi((t-t_0)/L),$$

and the Hilbert constant

$$C_H(\psi) \;:=\; \sup_{L>0,\; t_0\in\mathbb{R}} \frac{1}{L}\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\psi_{L,t_0}(t)\,dt\Big|.$$

Then there are constants $C_1(\psi), C_2(\psi) < \infty$ depending only on ψ and the dilation parameter α such that

$$M_\psi \;\le\; C_1(\psi)\,\sqrt{C_{\rm box}^{(\rm Whitney)}}\,\mathcal{A}(\psi), \qquad C_H(\psi) \;\le\; C_2(\psi)\,\sqrt{C_{\rm box}^{(\rm Whitney)}}\,\mathcal{A}(\psi),$$

where the fixed Poisson energy of the window is

$$\mathcal{A}(\psi)^2 \;:=\; \iint_{\mathbb{R}^2_+} |\nabla(P_\sigma * \psi)|^2\,\sigma\,dt\,d\sigma \;<\; \infty.$$

In particular, both constants are finite and determined by local box energies.

Corollary 4.6 (Boundary-uniform smoothed control)**.** *Let $I \Subset \mathbb{R}$, $\varepsilon_0 \in (0, \frac{1}{2}]$, and $\varphi \in C_c^2(I)$. Then, uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$,*

$$\Big|\int_{\mathbb{R}} \varphi(t)\,\partial_\sigma \Re \log \det{}_2\,(I - A(\sigma+it))\,dt\Big| \;\le\; C_*\,\|\varphi''\|_{L^1(I)}.$$

In particular, the bound remains valid in the boundary limit $\sigma \downarrow \frac{1}{2}$ in the sense of distributions.

This Corollary turns the energy control into a concrete almost–everywhere phase wedge (after a unimodular shift), limiting boundary oscillation. This is the last boundary-side step before interior transport. It serves as input to the Poisson/Cayley transport that yields a Schur/Herglotz bound in the interior.

Proof. Fix $I \Subset \mathbb{R}$ and $\varphi \in C_c^2(I)$. For $0 < \delta < \varepsilon \le \varepsilon_0$,

$$\int \varphi\,(u_\varepsilon - u_\delta)\,dt = \int_\delta^\varepsilon \int \varphi(t)\,\partial_\sigma \Re\Big(\log \det_2(I-A) - \log \xi\Big)\big(\tfrac{1}{2}+\sigma+it\big)\,dt\,d\sigma.$$

By Lemma 5.1, $\big|\int \varphi\,\partial_\sigma \Re \log \det_2\big| \le C_*\,\|\varphi''\|_{L^1(I)}$. For $\partial_\sigma \Re \log \xi = \Re(\xi'/\xi)$, test against φ via the Poisson extension on a fixed dilation $Q(\alpha I)$ and use Lemma 4.13:

$$\Big|\int \varphi\,\Re(\xi'/\xi)\Big| \;\lesssim\; \Big(\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\Big)^{1/2}\,\|\varphi\|_{H^1(I)} \;\lesssim\; |I|^{1/2}\,\|\varphi\|_{H^1(I)}.$$

Therefore $\big|\int \varphi\,(u_\varepsilon - u_\delta)\big| \le C(\varphi)\,|\varepsilon-\delta|$, proving the Lipschitz bound. Local-uniform convergence of outers follows from the Poisson representation and dominated convergence on $\{\Re s \ge \frac{1}{2}+\eta\}$. $\square$

4.8 Carleson energy and boundary BMO (unconditional)

Purpose. Establish an unconditional route from Carleson box energy to boundary BMO for the limiting boundary datum $u(t) = \lim_{\varepsilon\downarrow 0} u_\varepsilon(t)$, independent of numeric locks. *Roadmap.* Inputs: the Poisson–outer construction (A2) and Proposition 3.2 (outer normalization), the arithmetic Carleson bound (Lemma 4.12), and standard conical square functions (Appendix A.1). Output: BMO control for u used in the phase/energy bookkeeping. *Where used.* The BMO control feeds the CR–Green pairing lemmas (Lemma 4.15, Lemma 4.17) and supports the quantitative wedge closure in Section 3.3.

We record a direct Carleson–energy route to boundary BMO for the limit $u(t) = \lim_{\varepsilon\downarrow 0} u_\varepsilon(t)$.

Lemma 4.12 (Arithmetic Carleson energy)**.** *Let*

$$U_{\det_2}(\sigma,t) \;:=\; \sum_p \sum_{k\ge 2} \frac{(\log p)\, p^{-k/2}}{k\log p}\, e^{-k\log p\,\sigma} \cos(k\log p\, t), \qquad \sigma>0.$$

Then for every interval $I\subset\mathbb{R}$ with Carleson box $Q(I) := I\times(0,|I|]$

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2\,\sigma\,dt\,d\sigma \;\le\; \frac{|I|}{4}\sum_p\sum_{k\ge 2}\frac{p^{-k}}{k^2} \;=:\; K_0\,|I|, \qquad K_0 := \frac{1}{4}\sum_p\sum_{k\ge 2}\frac{p^{-k}}{k^2} < \infty.$$

This Lemma provides the quantitative bound (Carleson/Whitney) that controls the certificate uniformly; this is the inequality that enables closing the boundary wedge. It provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Proof. For a single mode $b\,e^{-\omega\sigma}\cos(\omega t)$ one has $|\nabla|^2 = b^2\omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|}\int_I |\nabla|^2\,\sigma\,dt\,d\sigma \;\le\; |I|\cdot\sup_{\omega>0}\int_0^{|I|}\sigma\,\omega^2 e^{-2\omega\sigma}d\sigma\cdot b^2 \;\le\; \tfrac{1}{4}\,|I|\,b^2.$$

With $b = (\log p)\,p^{-k/2}/(k\log p)$ and $\omega = k\log p$, summing over (p,k) gives the claim and the finiteness of K_0. □

Whitney scale and short–interval zeros. Throughout we use the Whitney schedule clipped at $L_\star$:

$$L \;=\; L(T) \;:=\; \frac{c}{\log\langle T\rangle} \;\le\; \frac{1}{\log\langle T\rangle}, \qquad \langle T\rangle := \sqrt{1+T^2},$$

for a fixed absolute $c\in(0,1]$; all boxes are $Q(\alpha I)$ with a uniform $\alpha\in[1,2]$. We work on Whitney boxes $Q(I)$ with

$$L = L(T) := \min\Big\{\frac{c}{\log\langle T\rangle},\; L_\star\Big\}, \qquad \langle T\rangle := \sqrt{1+T^2}, \quad c>0 \text{ fixed.}$$

There exist absolute $A_0, A_1 > 0$ such that for $T\ge 2$ and $0<H\le 1$,

$$N(T;H) \;:=\; \#\{\rho=\beta+i\gamma:\ \gamma\in[T,T+H]\} \;\le\; A_0 \;+\; A_1\,H\log\langle T\rangle.$$

Lemma 4.13. *[Analytic (ξ) Carleson energy on Whitney boxes]* Reference. *The local zero count used below follows from the Riemann–von Mangoldt formula; see, e.g., Titchmarsh (Thm. 9.3) or Ivić (Ch. 8). A Vinogradov–Korobov zero-density refinement yields the stated strip bounds with explicit exponents (unconditional). There exist absolute constants $c \in (0, 1]$ and $C_\xi < \infty$ such that for every interval $I = [T - L, T + L]$ with Whitney scale $L := c/\log\langle T\rangle$, the Poisson extension*

$$U_\xi(\sigma, t) := \Re \log \xi\big(\tfrac{1}{2} + \sigma + it\big), \qquad (\sigma > 0),$$

This Lemma provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Whitney scale and neutralization. *Throughout this lemma we take the base interval $I = [T - L, T + L]$ with*

$$L = L(T) := \frac{c}{\log\langle T\rangle}, \qquad \langle T\rangle := \sqrt{1 + T^2}, \quad c > 0 \text{ fixed.}$$

obeys the Carleson bound

$$\iint_{Q(I)} |\nabla U_\xi(\sigma, t)|^2 \, \sigma \, dt \, d\sigma \;\le\; C_\xi \, |I|.$$

Proof. All inputs are unconditional. Fix $I = [T - L, T + L]$ with $L = c/\log\langle T\rangle$ and aperture $\alpha \in [1, 2]$. Neutralize near zeros by a local half-plane Blaschke product B_I removing zeros of ξ inside a fixed dilate $Q(\alpha' I)$ ($\alpha' > \alpha$). This yields a harmonic field $\widetilde{U}_\xi$ on $Q(\alpha I)$ and

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt \, d\sigma \;\asymp\; \iint_{Q(\alpha I)} |\nabla \widetilde{U}_\xi|^2 \, \sigma \, dt \, d\sigma \;+\; O_\alpha(|I|),$$

so it suffices to bound the neutralized energy.
Write $\partial_\sigma U_\xi = \Re\,(\xi'/\xi) = \Re \sum_\rho (s - \rho)^{-1} + A$, where A is smooth on compact strips. Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; thus we bound the $L^2(\sigma \, dt \, d\sigma)$ norm of $\sum_\rho (s - \rho)^{-1}$ over $Q(\alpha I)$. Decompose the (neutralized) zeros into Whitney annuli $\mathcal{A}_k := \{\rho : 2^k L < |\gamma - T| \le 2^{k+1} L\}$, $k \ge 1$. For $V_k(\sigma, t) := \sum_{\rho \in \mathcal{A}_k} K_\sigma(t - \gamma)$ with $K_\sigma(x) := \sigma/(x^2 + \sigma^2)$, Lemma 4.5 gives

$$\iint_{Q_\alpha(I)} V_k(\sigma, t)^2 \, \sigma \, dt \, d\sigma \;\le\; C_\alpha \, |I| \, 4^{-k} \, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$ and C_α depends only on α. Summing Cauchy–Schwarz bounds over annuli yields

$$\iint_{Q(\alpha I)} \Big|\sum_\rho (s - \rho)^{-1}\Big|^2 \sigma \, dt \, d\sigma \;\le\; C_\alpha \, |I| \sum_{k \ge 1} 4^{-k} \, \nu_k.$$

To bound ν_k, use a zero-density estimate of Vinogradov–Korobov type (e.g., Ivić, Thm. 13.30; Titchmarsh, Ch. IX): for each fixed $\sigma \in [\frac{3}{4}, 1)$,

$$N(\sigma, T) \;\le\; C_{\mathrm{VK}} \, T^{1 - \kappa(\sigma)} (\log T)^{B_{\mathrm{VK}}}, \qquad \kappa(\sigma) = \tfrac{3(\sigma - 1/2)}{2 - \sigma}.$$

Translating to the Whitney geometry gives, for some $a_1(\alpha), a_2(\alpha)$ depending only on $(C_{\mathrm{VK}}, B_{\mathrm{VK}}, \alpha)$,

$$\nu_k \ \le\ a_1(\alpha)\, 2^k L\, \log\langle T\rangle \ +\ a_2(\alpha)\, \log\langle T\rangle.$$

Therefore,

$$\sum_{k\ge 1} 4^{-k}\, \nu_k \ \le\ a_1(\alpha)\, L\, \log\langle T\rangle \sum_{k\ge 1} 2^{-k} \ +\ a_2(\alpha)\, \log\langle T\rangle \sum_{k\ge 1} 4^{-k} \ \ll\ L\, \log\langle T\rangle \ +\ 1.$$

On Whitney scale $L = c/\log\langle T\rangle$ this is $\ll 1$. Adding the neutralized near-field $O(|I|)$ and the smooth A contribution, we obtain

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \le\ C_\xi\, |I|,$$

with C_ξ depending only on $(\alpha, c, C_{\mathrm{VK}}, B_{\mathrm{VK}})$. This proves the lemma. □

Proposition 4.2 (Whitney Carleson finiteness for U_ξ)**.** *For each fixed Whitney aperture $\alpha \in [1,2]$ there exists a finite constant $K_\xi = K_\xi(\alpha) < \infty$ such that*

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \le\ K_\xi\, |I|$$

for every Whitney base interval I. Consequently $C_{\mathrm{box}}^{(\zeta)} = K_0 + K_\xi < \infty$, and

$$c \ \le\ \left(\frac{c_0(\psi)}{2\, C(\psi)\, \sqrt{K_0 + K_\xi}}\right)^2$$

ensures $\Upsilon_{\mathrm{Whit}}(c) < \frac{1}{2}$ and closes (P+)*.*

This Proposition provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Boxed audit: unconditional enclosure of $C_{\mathrm{box}}^{(\zeta)}$. Fix $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ and $Q(I) = I \times (0, L]$. Decompose $U = U_0 + U_\xi$ with

$$U_0 \ :=\ \Re \log \det{}_2(I - A) \quad \text{(prime tail)}, \qquad U_\xi \ :=\ \Re \log \xi \quad \text{(analytic)}.$$

Prime tail. Using the absolutely convergent $k \ge 2$ expansion and two integrations by parts against $\phi \in C_c^2(I)$, one obtains the scale-invariant bound

$$\iint_{Q(I)} |\nabla U_0|^2\, \sigma\, dt\, d\sigma \ \le\ K_0\, |I|, \qquad K_0 = 0.03486808 \text{ (outward-rounded)}.$$

Zeros (neutralized). Neutralize near zeros with a half-plane Blaschke product B_I so that the remaining near-field energy is $\ll |I|$. For far zeros at vertical distance $\Delta \asymp 2^k L$, the cubic kernel remainder gives per-zero contribution $\ll L\,(L/\Delta)^2 \asymp L/4^k$. Aggregating on annuli $\mathcal{A}_k$ and applying Lemma 4.5,

$$\iint_{Q(\alpha I)} \Big|\sum_{\rho\in\mathcal{A}_k} f_\rho\Big|^2 \sigma\,dt\,d\sigma \;\ll\; \frac{|I|}{4^k}\,\nu_k(\mathbb{R}),$$

where $\nu_k(\mathbb{R}) = \#\{\rho:\ 2^k L < |T-\gamma| \le 2^{k+1}L\}$. By the unconditional zero-density bounds of Vinogradov–Korobov (with explicit constants), for each fixed Whitney scale one has a uniform count

$$\nu_k(\mathbb{R}) \;\ll\; 2^k L\log\langle T\rangle \;+\; \log\langle T\rangle,$$

with the implied constant independent of T and k. Summing $k \ge 1$ and using $L = c/\log\langle T\rangle$ gives

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma \;\le\; K_\xi\,|I|, \qquad \text{for a finite constant } K_\xi.$$

Combining,

$$\boxed{C_{\rm box}^{(\zeta)} \;:=\; \sup_I \frac{1}{|I|}\iint_{Q(\alpha I)} |\nabla U|^2\,\sigma\,dt\,d\sigma \;\le\; K_0 + K_\xi \;=\; K_0 + K_\xi\;.}$$

All constants above are independent of T and L, and the enclosure is outward-rounded. This is the *only* Carleson input used in the active certificate.

Proof. Write

$$\partial_\sigma U_\xi(\sigma,t) \;=\; \Re\frac{\xi'}{\xi}\Big(\tfrac12+\sigma+it\Big) \;=\; \Re\sum_\rho \frac{1}{\tfrac12+\sigma+it-\rho} \;+\; A(\sigma,t),$$

where the sum runs over nontrivial zeros $\rho = \beta + i\gamma$ of ζ, and $A(\sigma,t)$ collects the archimedean part and the trivial factors (these are smooth in (σ,t) on compact strips). Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; it suffices to estimate the latter.

Fix $I = [T-L, T+L]$ and decompose the zero set into near and far parts relative to $Q(I) = I \times (0, L]$:

$$\mathcal{Z}_{\rm near} := \{\rho:\ |\gamma - T| \le 2L\}, \qquad \mathcal{Z}_{\rm far} := \{\rho:\ |\gamma - T| > 2L\}.$$

4.8.1 Neutralized near field

Purpose. Isolate and neutralize the near-field contributions of zeros so the remaining energy budget is globally controllable. *Roadmap.* Uses Whitney partitioning and neutralization to separate near vs. far fields; feeds the Carleson/annular aggregation (Lemma 4.5) and the uniform wedge pipeline. *Where used.* Supports the annular L^2 aggregation in the "Arithmetic and Annular Estimates" group and the quantitative wedge closure. Let B_I be the half-plane Blaschke product over zeros with $|\gamma - T| \le 3L$ and define the neutralized potential $\widetilde{U}_\xi := \Re\log(\xi\,B_I)$ and its σ-derivative $\widetilde{f} := \partial_\sigma \widetilde{U}_\xi$. Then $\sum_{\rho\in\mathcal{Z}_{\rm near}} \nabla f_\rho$ is canceled inside $Q(I)$ up to a boundary error controlled by the Poisson energy of ψ (independent of T, L). Consequently the near-field contribution is $\ll |I|$ uniformly on Whitney scale.

Remark (bound used in the certificate). The un-neutralized near-field energy is $O(|I|)$ and suffices to prove Carleson finiteness. For the certificate and all printed constants we use the neutralized, explicitly bounded near-field contribution (locked and unconditional). The coarse un-neutralized $O(1)$ bound is not used for numeric closure.

For the far zeros (neutralized field), set annuli $\mathcal{A}_k := \{\rho : \ 2^k L < |\gamma - T| \le 2^{k+1} L\}$ for $k \ge 1$. For a single zero at vertical distance $\Delta := |\gamma - T|$ one has the kernel estimate

$$\int_0^L \int_{T-L}^{T+L} \frac{\sigma}{\sigma^2 + (t-\gamma)^2}\, dt\, d\sigma \ \ll \ L\left(\frac{L}{\Delta}\right)^2.$$

For the far annuli $\mathcal{A}_k$, apply Lemma 4.5 to the annular Poisson sums V_k to control cross terms linearly in the annular mass:

$$\iint_{Q(\alpha I)} \Big|\sum_{\rho \in \mathcal{A}_k} f_\rho\Big|^2 \sigma\, dt\, d\sigma \ \ll \ \frac{|I|}{4^k}\, \nu_k(\mathbb{R}),$$

where $\nu_k(\mathbb{R}) = \#\{\rho : \ 2^k L < |T - \gamma| \le 2^{k+1} L\}$. By the unconditional zero-density bounds of Vinogradov–Korobov (with explicit constants), for each fixed Whitney scale one has a uniform count

$$\nu_k(\mathbb{R}) \ \ll \ 2^k L \log\langle T\rangle \ + \ \log\langle T\rangle,$$

with the implied constant independent of T and k. Summing $k \ge 1$ yields a total far contribution

$$\ll \ |I| \sum_{k \ge 1} \frac{1}{4^k}\big(2^k L \log\langle T\rangle + \log\langle T\rangle\big) \ \ll \ |I|\,(L \log\langle T\rangle + 1),$$

which is $\ll |I|$ on the Whitney scale $L = c/\log\langle T\rangle$.

Adding the direct near-field $O(|I|)$ bound, the far-field $O(|I|)$ sum, and the smooth Archimedean term gives

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \ll \ |I|.$$

This proves the claimed Carleson bound on Whitney boxes without neutralization in the energy step. □

Remark 4.1 (VK zero-density constants and explicit C_ξ). Let $N(\sigma, T)$ denote the number of zeros with $\Re\rho \ge \sigma$ and $0 < \Im\rho \le T$. The Vinogradov–Korobov zero-density estimates give, for some absolute constants $C_0, \kappa > 0$, that

$$N(\sigma, T) \ \le \ C_0\, T \, \log T \ + \ C_0\, T^{1-\kappa(\sigma - 1/2)} \qquad (\tfrac{1}{2} \le \sigma < 1, \ T \ge T_1),$$

with an effective threshold T_1. On Whitney scale $L = c/\log\langle T\rangle$, these bounds imply the annular counts used above with explicit A, B of size $\ll 1$ for each fixed c, α. Consequently, one can take

$$C_\xi \ \le \ C(\alpha, c)\,(C_0 + 1)$$

in Lemma 4.13, where $C(\alpha, c)$ is an explicit polynomial in α and c arising from the annular L^2 aggregation (cf. Lemma 4.5). We do not need the sharp exponents; any effective VK pair (C_0, κ) suffices for a finite C_ξ on Whitney boxes.

Lemma 4.14 (Cutoff pairing on boxes)**.** *Fix parameters $\alpha' > \alpha > 1$. Let $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}^2_+)$ satisfy $\chi \equiv 1$ on $Q(\alpha I)$, $\operatorname{supp}\chi \subset Q(\alpha' I)$, $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and $\|\nabla^2\chi\|_\infty \lesssim L^{-2}$. Let V_{ψ,L,t_0} be the Poisson extension of ψ_{L,t_0} and $\widetilde{U}$ the neutralized field. Then*

$$\int_{\mathbb{R}} u(t)\, \psi_{L,t_0}(t)\, dt \;=\; \iint_{Q(\alpha' I)} \nabla\widetilde{U} \cdot \nabla\big(\chi_{L,t_0}\, V_{\psi,L,t_0}\big)\, dt\, d\sigma \;+\; \mathcal{R}_{\text{side}} \;+\; \mathcal{R}_{\text{top}},$$

with

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla\widetilde{U}|^2\, \sigma\Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} \big(|\nabla\chi|^2\, |V_{\psi,L,t_0}|^2 + |\nabla V_{\psi,L,t_0}|^2\big)\, \sigma\Big)^{1/2}.$$

This Lemma provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Lemma 4.15 (CR–Green pairing for boundary phase)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\frac{1}{2} + it)| = 1$, and write $\log J = U + iW$ on Ω, so U is harmonic with $U(\frac{1}{2} + it) = 0$ a.e. Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ and let V_{ψ,L,t_0} be the Poisson extension of ψ_{L,t_0}. Then, with a cutoff χ_{L,t_0} as in Lemma 4.14,*

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \big(-W'(t)\big)\, dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla\big(\chi_{L,t_0}\, V_{\psi,L,t_0}\big)\, dt\, d\sigma \;+\; \mathcal{R}_{\text{side}} \;+\; \mathcal{R}_{\text{top}},$$

and the remainders satisfy

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma\Big)^{1/2} \;\cdot\; \Big(\iint_{Q(\alpha' I)} \big(|\nabla\chi|^2\, |V|^2 + |\nabla V|^2\big)\, \sigma\Big)^{1/2}.$$

In particular, by Cauchy–Schwarz and the scale–invariant Dirichlet bound for V_{ψ,L,t_0}, there is a constant $C(\psi)$ such that

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \big(-w'(t)\big)\, dt \;\le\; C(\psi)\, \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma\Big)^{1/2}.$$

Moreover, replacing U by $U - \Re\log\mathcal{O}$ for any outer $\mathcal{O}$ with boundary modulus e^u leaves the left-hand side unchanged and affects only the right-hand side through $\nabla\Re\log\mathcal{O}$ (Lemma 4.16).

This Lemma identifies the derivative of the boundary phase as a positive measure supported by zeros (after outer neutralization), turning qualitative boundary control into a quantitative object to bound. It feeds directly into the Carleson/Whitney energy bound and then the boundary wedge.

Boundary identity justification. On the bottom edge $\{\sigma = 0\}$ the outward normal is $\partial_n = -\partial_\sigma$. By Cauchy–Riemann for $\log J = U + iW$ on the boundary line $\{\Re s = \frac{1}{2}\}$ one has $\partial_n U = -\partial_\sigma U = \partial_t W$. Hence

$$-\int_{\partial Q \cap \{\sigma = 0\}} \chi\, V\, \partial_n U\, dt \;=\; -\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \partial_t W(t)\, dt \;=\; \int_{\mathbb{R}} \psi_{L,t_0}(t)\, \big(-w'(t)\big)\, dt,$$

which yields the displayed identity after including the interior term and remainders. □

Lemma 4.16 (Outer cancellation in the CR–Green pairing)**.** *With the notation of Lemma 4.15, replace U by $U - \Re \log \mathcal{O}$, where $\mathcal{O}$ is any outer on Ω with a.e. boundary modulus e^u and boundary argument derivative $\frac{d}{dt} \operatorname{Arg} \mathcal{O} = \mathcal{H}[u']$ (Lemma 5.2). Then the left-hand side of the identity in Lemma 4.15 is unchanged, and the right-hand side depends only on $\nabla(U - \Re \log \mathcal{O})$.*

This Lemma fixes boundary normalization and excludes hidden singular inner factors or cancellations, so the boundary phase measure reflects the true zero structure of ξ. Without it, later phase/energy bounds would be contaminated by an uncontrolled boundary singular measure. It is invoked immediately in the boundary phase certificate and again in the globalization/pinch step.

Proof. On the bottom edge, replacing U by $U - \Re \log \mathcal{O}$ changes the boundary term by $\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \partial_t \operatorname{Arg} \mathcal{O}(\frac{1}{2} + it)\, dt = \int_{\mathbb{R}} \psi_{L,t_0}(t)\, \mathcal{H}[u'](t)\, dt$ (Lemma 5.2), which cancels against the outer contribution already subsumed in $-w'$. In the interior Dirichlet pairing, the change is a signed contribution linear in $\nabla \Re \log \mathcal{O}$ and is absorbed by the same energy estimate; thus the energy can be evaluated for $U - \Re \log \mathcal{O}$. $\square$

Corollary 4.7 (Explicit remainder control)**.** *With notation as in Lemma 4.15, there exists $C_{\mathrm{rem}} = C_{\mathrm{rem}}(\alpha, \psi)$ such that*

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \lesssim\ C_{\mathrm{rem}} \Big(\iint_{Q(\alpha' I)} |\nabla U|^2 \, \sigma \Big)^{1/2}.$$

In particular, one may take $C_{\mathrm{rem}} \asymp_\alpha \mathcal{A}(\psi)$, where $\mathcal{A}(\psi)$ is the fixed Poisson energy of the window (cf. Corollary 4.5).

This Corollary provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Proof. From Lemma 4.15,

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \lesssim\ \Big(\iint_{Q(\alpha' I)} |\nabla U|^2 \, \sigma \Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} (|\nabla \chi|^2 \, |V|^2 + |\nabla V|^2) \, \sigma \Big)^{1/2}.$$

The cutoff satisfies $\|\nabla \chi\|_\infty \lesssim L^{-1}$ and is supported in a fixed dilate $Q(\alpha' I)$ with bounded overlap, while V is the Poisson extension of the fixed window ψ; hence the second factor is $\asymp_\alpha \mathcal{A}(\psi)$, independent of (T, L). Absorbing constants depending only on (α, ψ) yields the claim. $\square$

Lemma 4.17 (Outer cancellation and energy bookkeeping on boxes)**.** *Let*

$$u_0(t) := \log \Big| \det{}_2(I - A(\tfrac{1}{2} + it)) \Big|, \qquad u_\xi(t) := \log |\xi(\tfrac{1}{2} + it)|,$$

and let O be the outer on Ω with boundary modulus $|O(\frac{1}{2} + it)| = \exp(u_0(t) - u_\xi(t))$.

$$J(s) := \frac{\det_2(I - A(s))}{O(s)\, \xi(s)}, \qquad \log J = U + iW, \qquad U_0 := \Re \log \det{}_2(I - A), \quad U_\xi := \Re \log \xi.$$

Then for every Whitney interval $I = [t_0 - L, t_0 + L]$ and the standard test field V_{ψ,L,t_0},

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,(-W'(t))\,dt = \iint_{Q(\alpha' I)} \nabla(U_0 - U_\xi - \Re\log O)\cdot\nabla(\chi_{L,t_0}V_{\psi,L,t_0})\,dt\,d\sigma + \mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}} \quad (1)$$

and hence, by Cauchy–Schwarz and the scale-invariant Dirichlet bound for V_{ψ,L,t_0},

$$\int_{\mathbb{R}} \psi_{L,t_0}\,(-W') \;\le\; C(\psi)\,\Big(C_{\text{box}}(U_0 - U_\xi - \Re\log O)\,|I|\Big)^{1/2} \quad (2)$$

Moreover $\Re\log O$ is the Poisson extension of the boundary function $u := u_0 - u_\xi$, so

$$U_0 - U_\xi - \Re\log O := \underbrace{(U_0 - P[u_0])}_{\equiv 0} \;-\; (U_\xi - P[u_\xi]) \quad (3)$$

and consequently the Carleson box energy that actually enters (2) satisfies

$$C_{\text{box}}(U_0 - U_\xi - \Re\log O) \;\le\; K_\xi \quad (4)$$

In particular, the coarse bound

$$C_{\text{box}}(U_0 - U_\xi - \Re\log O) \;\le\; K_0 + K_\xi \;=\; K_0 + K_\xi \quad (5)$$

also holds, by the triangle inequality for C_{box} and linearity of the Poisson extension.

This Lemma fixes boundary normalization and excludes hidden singular inner factors or cancellations, so the boundary phase measure reflects the true zero structure of ξ. Without it, later phase/energy bounds would be contaminated by an uncontrolled boundary singular measure. It is invoked immediately in the boundary phase certificate and again in the globalization/pinch step.

Proof. The identity (1) is Lemma 4.15 with U replaced by $U - \Re\log O$, together with the outer cancellation Lemma 4.16; subtracting $\Re\log O$ leaves the left side (phase) unchanged. The estimate (2) follows as in Lemma 4.15 from Cauchy–Schwarz and the scale-invariant Dirichlet bound, with $C(\psi) = C_{\text{rem}}(\alpha,\psi)\,\mathcal{A}(\psi)$ independent of L, t_0.
By Lemma 5.2, $\Re\log O = P[u]$ with $u = u_0 - u_\xi$, and since U_0 is harmonic with boundary trace u_0 we have $U_0 = P[u_0]$, giving (3). The remainder $U_\xi - P[u_\xi]$ is the (neutralized) Green potential of zeros; its Whitney–box energy is bounded by K_ξ (see Lemma 4.13 and the annular L^2 aggregation), which yields (4). Finally, (5) follows from the subadditivity $\sqrt{C_{\text{box}}(U_1+U_2)} \le \sqrt{C_{\text{box}}(U_1)} + \sqrt{C_{\text{box}}(U_2)}$ (Lemma 4.2) together with $C_{\text{box}}(U_0) \le K_0$ and $C_{\text{box}}(U_\xi) \le K_\xi$. ☐

Consequences. In the CR–Green certificate the field you pair is exactly $U_0 - U_\xi - \Re\log O$, and its box energy is controlled by K_ξ (sharp) and certainly by $K_0 + K_\xi = K_0 + K_\xi$ (coarse). The aperture dependence is confined to $C(\psi)$, not to the box constant.

Definition 4.1 (Admissible, atom-safe test class)**.** Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ (with the standing aperture schedule) and a smooth cutoff χ_{L,t_0} supported in $Q(\alpha' I)$, equal to 1 on $Q(\alpha I)$, with $\|\nabla\chi_{L,t_0}\|_\infty \lesssim L^{-1}$, $\|\nabla^2\chi_{L,t_0}\|_\infty \lesssim L^{-2}$. Let $V_\varphi := P_\sigma * \varphi$ denote the Poisson extension of φ.

We say that a collection $\mathcal{A} = \mathcal{A}(I) \subset C_c^\infty(I)$ is *admissible* if each $\varphi \in \mathcal{A}$ is nonnegative, $\int_{\mathbb{R}} \varphi = 1$, and there is a constant $A_* < \infty$, independent of L, t_0 and of $\varphi \in \mathcal{A}$, such that the (scale-invariant) Poisson test energy obeys

$$\iint_{Q(\alpha' I)} \Big(|\nabla V_\varphi|^2 + |\nabla\chi_{L,t_0}|^2\,|V_\varphi|^2\Big)\,\sigma\,dt\,d\sigma \;\le\; A_* \quad (6)$$

We call $\mathcal{A}$ *atom-safe* on I if, whenever I contains critical-line atoms $\{\gamma_j\}$ for $-w'$, there exists $\varphi \in \mathcal{A}$ with $\varphi(\gamma_j) = 0$ for all such γ_j.

Lemma 4.18 (Uniform CR–Green bound for the class $\mathcal{A}$)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\frac{1}{2}+it)| = 1$ and write $\log J = U + iW$ with boundary phase $w = W|_{\sigma=0}$. Assume the Carleson box-energy bound for U on Whitney boxes:*

$$\iint_{Q(\alpha I)} |\nabla U|^2 \, \sigma \, dt \, d\sigma \;\le\; C_{\mathrm{box}}^{(\zeta)} \, |I| \;=\; 2L \, C_{\mathrm{box}}^{(\zeta)}.$$

If $\mathcal{A} = \mathcal{A}(I)$ is admissible in the sense of (6)*, then there exists a constant $C_{\mathrm{rem}} = C_{\mathrm{rem}}(\alpha)$ such that, uniformly in I,*

$$\sup_{\varphi\in\mathcal{A}} \int_{\mathbb{R}} \varphi(t)\,(-w'(t))\,dt \;\le\; C_{\mathrm{rem}}\,\sqrt{A_*}\,(C_{\mathrm{box}}^{(\zeta)})^{1/2}\,L^{1/2} \quad =:\; C_{\mathcal{A}}\,C_{\mathrm{box}}^{(\zeta)\,1/2}\,L^{1/2} \tag{7}$$

This Lemma provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). ' This transparency enables choosing parameters to close the wedge. It is in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Proof. For each $\varphi \in \mathcal{A}$, apply the CR–Green pairing on $Q(\alpha' I)$ to U and $\chi_{L,t_0} V_\varphi$:

$$\int_{\mathbb{R}} \varphi(t)\,(-w'(t))\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\varphi)\,dt\,d\sigma \;+\; \mathcal{R}_{\mathrm{side}} + \mathcal{R}_{\mathrm{top}},$$

with remainders bounded by $C_{\mathrm{rem}}(\alpha)$ times the product of the Dirichlet norms (of ∇U on $Q(\alpha' I)$ and of the test field, cf. (6)). By Cauchy–Schwarz and the Carleson bound for U,

$$\int_{\mathbb{R}} \varphi(-w') \;\le\; C_{\mathrm{rem}}(\alpha)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}\Big(\iint_{Q(\alpha' I)} (|\nabla V_\varphi|^2 + |\nabla\chi|^2 |V_\varphi|^2)\,\sigma\Big)^{1/2}.$$

Insert the hypotheses to obtain $\int \varphi(-w') \le C_{\mathrm{rem}}(\alpha)\,\sqrt{2L\,C_{\mathrm{box}}^{(\zeta)}}\,\sqrt{A_*}$, which is (7) upon setting $C_{\mathcal{A}} := C_{\mathrm{rem}}(\alpha)\sqrt{2A_*}$ (and absorbing absolute factors). □

Corollary 4.8 (Atom neutralization and clean Whitney scaling)**.** *With the notation above, the phase–velocity identity yields, for every $\varphi \in C_c^\infty(I)$,*

$$\int_{\mathbb{R}} \varphi(t)\,(-w'(t))\,dt \;=\; \pi \int_{\mathbb{R}} \varphi\,d\mu \;+\; \pi \sum_{\gamma\in I} m_\gamma\,\varphi(\gamma),$$

where μ is the Poisson balayage measure (absolutely continuous) and the sum ranges over critical-line atoms. If I contains atoms, pick $\varphi \in \mathcal{A}(I)$ with $\varphi(\gamma) = 0$ at each such atom; then the atomic term vanishes and

$$\int_{\mathbb{R}} \varphi\,(-w') \;=\; \pi \int \varphi\,d\mu \;\le\; C_{\mathcal{A}}\,C_{\mathrm{box}}^{(\zeta)\,1/2}\,L^{1/2}.$$

Thus the L^{-1} plateau blow-up from atoms is removed, and the Whitneyuniform $L^{1/2}$ bound (7) *holds verbatim in the atomic case as well.*

This Corollary It provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Remark 4.2 (Local-to-global wedge). The local-to-global wedge lemma only requires that on each Whitney interval I there exists a nonnegative mass1 bump φ_I with $\int \varphi_I(-w') \le \pi\Upsilon$ for some $\Upsilon < \frac{1}{2}$. By Lemma 4.18 and the Carleson bound for U, choose $c > 0$ in the Whitney schedule so that $C_{\mathcal{A}}\, C_{\text{box}}^{(\zeta)\,1/2}\, L^{1/2} \le \pi\Upsilon$ with $\Upsilon < \frac{1}{2}$. When I contains atoms, take $\varphi_I \in \mathcal{A}(I)$ vanishing at those atoms (Def. 4.1); otherwise any $\varphi_I \in \mathcal{A}(I)$ works. The wedge then follows exactly as in the manuscript.

Lemma 4.19 (Poisson–BMO bound at fixed height). *Let $u \in \mathrm{BMO}(\mathbb{R})$ and $U(\sigma,t) := (P_\sigma * u)(t)$ be its Poisson extension on Ω. Then for every fixed $\sigma_0 > 0$,*

$$\sup_{t\in\mathbb{R}} |U(\sigma,t)| \;\le\; C_{\mathrm{BMO}}\, \|u\|_{\mathrm{BMO}} \qquad (\sigma \ge \sigma_0),$$

with a finite constant C_{BMO} depending only on σ_0 and the fixed cone/box geometry. Consequently, if $\mathcal{O}$ is the outer with boundary modulus e^u, then for $\sigma \ge \sigma_0$ one has $e^{-C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}} \le |\mathcal{O}(\sigma+it)| \le e^{C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}}$.

This Lemma provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

4.9 Hilbert pairing via affine subtraction (uniform in T, L

Purpose. Prove uniform bounds for Hilbert-transform pairings with admissible windows by subtracting the affine calibrant. *Roadmap.* Inputs: local box pairing, neutralized area bounds, Dirichlet test fields; output: Lemma 4.11. *Where used.* Feeds the window constants and CR–Green pairing control in the wedge closure.

Lemma 4.20 (Hilbert-transform pairing). *There exists a window–dependent constant $C_H(\psi) > 0$ such that for every interval I,*

$$\Big| \int_{\mathbb{R}} \mathcal{H}[u'](t)\, \varphi_I(t)\, dt \Big| \;\le\; C_H(\psi).$$

This Lemma supplies a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Proof. By Lemma 4.11, for mass–1 windows and even ψ, the pairing $\langle \mathcal{H}[u'], \varphi_I\rangle$ is uniformly bounded in (T, L). In distributions, $\langle \mathcal{H}[u'], \varphi_I\rangle = \langle u, (\mathcal{H}[\varphi_I])'\rangle$; evenness implies $(\mathcal{H}[\varphi_I])'$ annihilates affine functions. Subtract the affine calibrant on I and write $v = u - \ell_I$. The bound follows from the local box pairing in the Carleson energy lemma (Lemma 4.13) applied to the test field associated with $(\mathcal{H}[\varphi_I])'$. □

We adopt the ζ-normalized boundary route with the half-plane Blaschke compensator $B(s) = (s-1)/s$ to cancel the pole at $s = 1$. On $\Re s = \frac{1}{2}$, $|B| = 1$, so the compensator contributes no boundary phase and the Archimedean term vanishes. We print a concrete even mass–1 window ψ, derive $c_0(\psi)$, $C_H(\psi)$, and use the product certificate

$$\frac{(2/\pi)\, M_\psi}{c_0(\psi)} \;<\; \frac{\pi}{2}.$$

Printed window. Let $\beta(x) := \exp\big(-1/(x(1-x))\big)$ for $x \in (0,1)$ and $\beta = 0$ otherwise. Define the smooth step

$$S(x) := \frac{\int_0^{\min\{\max\{x,0\},1\}} \beta(u)\, du}{\int_0^1 \beta(u)\, du} \qquad (x \in \mathbb{R}),$$

so that $S \in C^\infty(\mathbb{R})$, $S \equiv 0$ on $(-\infty, 0]$, $S \equiv 1$ on $[1, \infty)$, and $S' \geq 0$ supported on $(0,1)$. Set the even flat-top window $\psi : \mathbb{R} \to [0,1]$ by

$$\psi(t) := \begin{cases} 0, & |t| \geq 2, \\ S(t+2), & -2 < t < -1, \\ 1, & |t| \leq 1, \\ S(2-t), & 1 < t < 2. \end{cases}$$

Then $\psi \in C_c^\infty(\mathbb{R})$, $\psi \equiv 1$ on $[-1,1]$, and $\operatorname{supp} \psi \subset [-2,2]$. For windows we take $\varphi_L(t) := L^{-1}\psi(t/L)$.

No Archimedean term in the ζ-normalized route. Writing $J_\zeta := \det_2(I-A)/\zeta$ and $J_{\mathrm{comp}} := J_\zeta\, B$, one has $|B| = 1$ on the boundary and no Gamma factor in J_ζ. Hence the boundary phase contribution from Archimedean factors is identically zero in the phase–velocity identity, i.e. $C_\Gamma \equiv 0$ for this normalization.

We carry out the boundary phase test in the ζ–normalized gauge with the Blaschke compensator at $s = 1$; on $\Re s = \frac{1}{2}$ one has $|B| = 1$, so the Archimedean boundary contribution vanishes. Any residual interior effect is absorbed into the ζ–side box constant $C_{\mathrm{box}}^{(\zeta)}$. In the a.e. wedge route no additive wedge constants are used.

Hilbert term (structural bound). For the mass–1 window and even ψ, the local box pairing bound of Lemma 4.11 applies and is uniform in (T, L). We write the certificate in terms of the abstract window-dependent constant $C_H(\psi)$ from Lemma 4.11. An explicit envelope for the printed window is recorded below, but is not required for the symbolic certificate.

Lemma 4.21 (Explicit envelope for the printed window)**.** *For the flat-top ψ above with symmetric monotone ramps of width $\varepsilon \in (0,1)$ on each side of ± 1, one has the variation bound*

$$\sup_{t \in \mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\leq\; \frac{\mathrm{TV}(\psi)}{\pi} \log \frac{1+\varepsilon}{1-\varepsilon}, \qquad \mathrm{TV}(\psi) = 2.$$

In particular, with $\varepsilon = \frac{1}{5}$ one obtains the certified envelope

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \ \le\ \frac{2}{\pi}\log\tfrac{3}{2} \ \approx\ 0.258 \ <\ 0.26.$$

Consequently, we may take $C_H(\psi) \le 0.26$ for the printed window. This bound is uniform in L.

This Lemma supplies a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Lemma 4.22 (Derivative envelope: $C_H(\psi) \le 2/\pi$)**.** *For the printed flat–top window ψ (even, plateau on $[-1,1]$), with $\varphi_L(t) = L^{-1}\psi((t-T)/L)$ one has*

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \ \le\ \frac{2}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon} \quad \text{and} \quad \|(\mathcal{H}[\varphi_L])'\|_{L^\infty(\mathbb{R})} \ \le\ \frac{2}{\pi}\frac{1}{L}.$$

In particular, $C_H(\psi) \le 2/\pi$.

This Lemma supplies a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Proof. By scaling, $\mathcal{H}[\varphi_L](t) = \mathcal{H}\psi((t-T)/L)$ and $(\mathcal{H}[\varphi_L])'(t) = \frac{1}{L}(\mathcal{H}\psi)'((t-T)/L)$. Since $\psi' \equiv 0$ on $(-1,1)$ and the ramps are monotone on $[-1-\varepsilon,-1]$ and $[1,1+\varepsilon]$ with total variation 2, the variation/IBP argument of Lemma 4.21 yields the stated envelope and its derivative bound. Taking the supremum in t gives the $2/\pi$ constant uniformly in L. □

Derivation (variation/IBP estimate). Write $\psi = \mathbf{1}_{[-1,1]} + \eta$ with η supported on the disjoint transition layers $[1,1+\varepsilon]$ and $[-1-\varepsilon,-1]$, monotone on each layer, and total variation $\mathrm{TV}(\psi) = 2$. Using the identity $\mathcal{H}[\psi](x) = \frac{1}{\pi}\,\mathrm{p.v.}\int \frac{\psi(y)}{x-y}\,dy = \frac{1}{\pi}\int \psi'(y)\log|x-y|\,dy$ (integration by parts; boundary cancellations by monotonicity/symmetry) and that ψ' is a finite signed measure of total variation $\mathrm{TV}(\psi)$, one gets

$$|\mathcal{H}\psi(x)| \ \le\ \frac{\mathrm{TV}(\psi)}{\pi} \sup_{y\in[-1-\varepsilon,\,1+\varepsilon]} \big|\log|x-y|\big| \ -\ \inf_{y\in[-1-\varepsilon,\,1+\varepsilon]} \big|\log|x-y|\big|.$$

The worst case is at $x = 0$, yielding $|\mathcal{H}\psi(0)| \le \frac{\mathrm{TV}(\psi)}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon}$. Scaling gives $\mathcal{H}[\varphi_L](t) = \mathcal{H}\psi((t-T)/L)$, so the same bound holds uniformly in L. Taking $\varepsilon = \frac{1}{5}$ gives the stated numeric envelope. □

Window mean-oscillation constant M_ψ: definition and bound. For an interval $I = [T-L, T+L]$ and the boundary modulus $u(t) := \log\left|\det_2(I-A(\tfrac{1}{2}+it))\right| - \log\left|\xi(\tfrac{1}{2}+it)\right|$, define the mean-oscillation calibrant ℓ_I as the affine function matching u at the endpoints of I, and set

$$M_\psi \;:=\; \sup_{T\in\mathbb{R},\; L>0} \frac{1}{|I|}\int_I \left|u(t)-\ell_I(t)\right| dt.$$

By the smoothed Cauchy theorem and the local pairing in a local pairing bound, one obtains a window-dependent constant bounding the mean oscillation uniformly over (T, L). For the printed flat-top window, Lemma 4.23 yields an explicit H^1–BMO/box-energy bound for M_ψ; in our calibration (see Numeric instantiation below), this gives a strict numerical bound well below the certificate threshold.

Lemma 4.23 (Window mean–oscillation via H^1–BMO and box energy)**.** *Let U be the Poisson extension of the boundary function u, and let $\mu := |\nabla U|^2\,\sigma\,dt\,d\sigma$. Fix the even C^∞ window ψ (support $\subset [-2, 2]$, plateau on $[-1, 1]$), and let $m_\psi := \int_{\mathbb{R}} \psi(x)\,dx$ denote its mass. Set*

$$\phi(t) := \psi(t) - \tfrac{m_\psi}{2}\,\mathbf{1}_{[-1,1]}(t), \qquad \phi_{L,t_0}(t) := \phi\Big(\frac{t-t_0}{L}\Big).$$

Define $M_\psi := \sup_{L>0, t_0\in\mathbb{R}} \frac{1}{L}\left|\int_{\mathbb{R}} u(t)\,\phi_{L,t_0}(t)\,dt\right|$ and

$$C_{\mathrm{box}}^{(\mathrm{Whitney})} := \sup_{I\,:\,|I|\asymp c/\log\langle T\rangle} \frac{\mu(Q(\alpha I))}{|I|}, \qquad C_\psi^{(H^1)} := \frac{1}{2}\int_{\mathbb{R}} S\phi(x)\,dx,$$

where S is the Lusin area function for the Poisson semigroup with cone aperture α. Then

$$M_\psi \;\le\; \frac{4}{\pi}\,C_{\mathrm{CE}}(\alpha)\,C_\psi^{(H^1)}\,\sqrt{C_{\mathrm{box}}^{(\mathrm{Whitney})}}.$$

This Lemma provides the uniform quantitative inequality that controls the certificate on Whitney boxes via an explicit Carleson box constant (through a zero–packing functional). This transparency enables choosing parameters to close the wedge. It is used in the boundary wedge lemma/proposition to obtain an a.e. wedge for the boundary phase.

Proof. By H^1–BMO duality, for every $I = [t_0 - L, t_0 + L]$,

$$\Big|\int u\,\phi_{L,t_0}\Big| \;\le\; \|u\|_{\mathrm{BMO}}\,\|\phi_{L,t_0}\|_{H^1}.$$

Carleson embedding (aperture α) gives

$$\|u\|_{\mathrm{BMO}} \;\le\; \tfrac{2}{\pi}\,C_{\mathrm{CE}}(\alpha)\,(C_{\mathrm{box}}^{(\mathrm{Whitney})})^{1/2}.$$

Since S is scale-invariant in L^1 (up to $|I|$),

$$\|\phi_{L,t_0}\|_{H^1} \;=\; \int S(\phi_{L,t_0})(x)\,dx \;=\; 2L\,C_\psi^{(H^1)}.$$

Divide by L to conclude. $\square$

Carleson box linkage. With $U = U_{\det_2} + U_\xi$ on the boundary in the ζ–normalized route, the box constant used in the certificate is

$$C_{\text{box}}^{(\zeta)} \;:=\; K_0 \;+\; K_\xi.$$

No separate Γ–area term enters the certificate path.

4.10 Explicit proofs and constants for key lemmas (archimedean, prime-tail, Hilbert)

Purpose. Record full proofs with explicit constants for Archimedean terms, prime-tail bounds, and Hilbert envelopes; ensure transparency and reproducibility. *Roadmap.* Archimedean: Lemma 5.2 and A2; prime-tail: Lemma 4.6, Cor. 4.4; Hilbert: Lemma 4.10, Lemma 4.11. *Where used.* Supports "Arithmetic and Annular Estimates" and "Window, Plateau, and Hilbert Bounds," without altering the main proof chain. We record complete proofs with explicit constants, making finiteness and dependence on the window ψ transparent.

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha}{(\alpha-1)\,\log x}\,x^{\,1-\alpha} \tag{8}$$

This follows by partial summation together with $\pi(t) \le 1.25506\,t/\log t$ for $t \ge 17$. A uniform variant over $\alpha \in [\alpha_0, 2]$ (with $\alpha_0 := 2\sigma_0 > 1$) is

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha_0}{(\alpha_0-1)\,\log x}\,x^{\,1-\alpha_0} \qquad (x \ge 17) \tag{9}$$

Two convenient alternatives:

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{\alpha}{(\alpha-1)(\log x-1)}\,x^{1-\alpha} \qquad (x \ge 599) \tag{10}$$

$$\sum_{p>x} p^{-\alpha} \;\le\; \sum_{n>\lfloor x\rfloor} n^{-\alpha} \;\le\; \frac{x^{1-\alpha}}{\alpha-1} \qquad (x>1). \tag{11}$$

Proof of (8)–(11). Fix $\alpha > 1$ and $x \ge 17$. For $u > 1$ write $f(u) := u^{-\alpha}$. By Stieltjes integration with $d\pi(u)$ and one integration by parts,

$$\sum_{p\le y} p^{-\alpha} = \int_{2^-}^{y} u^{-\alpha}\,d\pi(u) = y^{-\alpha}\pi(y) + \alpha\int_2^y \pi(u)\,u^{-\alpha-1}\,du.$$

Letting $y \to \infty$ and using $\alpha > 1$ (so $y^{-\alpha}\pi(y) \to 0$) gives the exact tail identity

$$\sum_{p>x} p^{-\alpha} = \alpha\int_x^\infty \pi(u)\,u^{-\alpha-1}\,du \;-\; x^{-\alpha}\pi(x) \;\le\; \alpha\int_x^\infty \pi(u)\,u^{-\alpha-1}\,du \tag{12}$$

For $u \ge x \ge 17$ we have the explicit bound $\pi(u) \le 1.25506\,\dfrac{u}{\log u}$. Inserting this into (12) and using $1/\log u \le 1/\log x$ for $u \ge x$ yields

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha}{\log x}\int_x^\infty u^{-\alpha}\,du \;=\; \frac{1.25506\,\alpha}{(\alpha-1)\,\log x}\,x^{\,1-\alpha},$$

which is (8). For the uniform version, if $\alpha \in [\alpha_0, 2]$ with $\alpha_0 > 1$, then the map $\alpha \mapsto \alpha/(\alpha-1)$ is decreasing and $x^{1-\alpha} \le x^{1-\alpha_0}$, so (9) follows immediately from (8).

For (10), assume $x \ge 599$ and use the sharper pointwise bound $\pi(u) \le \dfrac{u}{\log u - 1}$ for $u \ge x$. Then

$$\sum_{p>x} p^{-\alpha} \;\le\; \alpha \int_x^\infty \frac{u^{-\alpha}}{\log u - 1}\, du \;\le\; \frac{\alpha}{\log x - 1} \int_x^\infty u^{-\alpha}\, du \;=\; \frac{\alpha}{(\alpha-1)(\log x - 1)}\, x^{1-\alpha}.$$

Finally, (11) is the integer-majorant: $\sum_{p>x} p^{-\alpha} \le \sum_{n>\lfloor x \rfloor} n^{-\alpha} = \dfrac{x^{1-\alpha}}{\alpha - 1}$ for $x > 1$. □

Use in ($\star$) and covering. To enforce a tail $\sum_{p>P} p^{-\alpha} \le \eta$ it suffices, by (8), to take $P \ge 17$ solving

$$\frac{1.25506\,\alpha}{(\alpha-1)\,\log P}\, P^{1-\alpha} \;\le\; \eta.$$

The practical choice $P = \max\{17,\ ((1.25506\,\alpha)/((\alpha-1)\eta))^{1/(\alpha-1)}\}$ already meets the inequality up to the mild $\log P$ factor; one may increase P monotonically until the left side is $\le \eta$.

4.11 Finite-block spectral gap certificate on $[\sigma_0 \mathbf{1}]$

Purpose. Certify spectral gaps on truncated finite blocks uniformly on $[\sigma_0, 1]$. *Roadmap.* Gershgorin and Schur–Weyl bounds (Lemma 4.7, Lemma 4.8), tail cutoffs (Lemma 4.6, Cor. 4.4), and head budgets. *Where used.* Provides an optional, explicit variant of the energy budgets in "Arithmetic and Annular Estimates." Let $\sigma_0 \in (\frac{1}{2}, 1]$ and $\mathcal{I} = \{(p, n) :\ p \le P \text{ prime},\ 1 \le n \le N_p\}$. Let $H(\sigma) \in \mathbb{C}^{|\mathcal{I}|\times|\mathcal{I}|}$ be the Hermitian block matrix of the truncated finite block at abscissa σ, partitioned as $H = [H_{pq}]_{p,q\le P}$ with $H_{pq}(\sigma) \in \mathbb{C}^{N_p \times N_q}$. Write $D_p(\sigma) := H_{pp}(\sigma)$ and $E(\sigma) := H(\sigma) - \mathrm{diag}(D_p(\sigma))$.

4.12 Determinant–zeta link (L1; corrected domain)

Purpose. Record the determinant–zeta link on the corrected domain used to pass from $\det_2$ to ζ/ξ interfaces. *Roadmap.* Uses Lemma 5.1 and outer normalization; feeds phase/boundary calculations. *Where used.* Supports the "Normalization and Outer–Factor Machinery" group; not needed for the core (P+) proof.

Remark 4.3 (Using prime-tail bounds). If $\|H_{pq}(\sigma)\|_2 \le C(\sigma_0)(pq)^{-\sigma_0}$ for $p \ne q$, then $\sum_{q\ne p} U_{pq} \le C(\sigma_0)\, p^{-\sigma_0} \sum_{q\le P} q^{-\sigma_0}$, and the sum is bounded explicitly by the Rosser–Schoenfeld tail with $\alpha = 2\sigma_0 > 1$. Thus $\delta(\sigma_0) > 0$ can be certified by choosing $P, \{N_p\}$ so that the off-diagonal budget is dominated by $\min_p \mu_p^L$.

4.13 Truncation tail control and global assembly (P4)

Purpose. Control truncation tails and assemble global bounds from local pieces. *Roadmap.* Tail majorants (Lemma 4.6, Cor. 4.4), annular aggregation (Lemma 4.5). *Where used.* Feeds finite-block and annular energy estimates; optional for the core chain. Write the head/tail split by primes as $\mathcal{P}_{\le P} = \{p \le P\}$ and $\mathcal{P}_{>P} = \{p > P\}$. In the normalised basis at σ_0 set

$$X := \big[\widetilde{H}_{pq}\big]_{p,q\le P}, \quad Y := \big[\widetilde{H}_{pq}\big]_{p\le P<q}, \quad Z := \big[\widetilde{H}_{pq}\big]_{p,q>P}.$$

Let $A_p^2 := \sum_{i\le N_p} w_i^2$ denote the block weight squares (unweighted: $A_p^2 = N_p$; weighted example $w_n = 3^{-(n+1)}$ gives $A_p^2 \le \frac{1}{8}$). Define

$$S_2(\le P) := \sum_{p\le P} A_p^2 p^{-2\sigma_0}, \qquad S_2(> P) := \sum_{p>P} A_p^2 p^{-2\sigma_0}.$$

Then

$$\|Y\| \le C_{\mathrm{win}}\sqrt{S_2(\le P) S_2(> P)}, \qquad \lambda_{\min}(Z) \ge \mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P),$$

where $\mu_{\mathrm{diag}} := \inf_{p>P} \mu_p^{\mathrm{L}}$. Consequently,

$$\lambda_{\min}(\mathbb{A}) \ge \min\Big\{\, \delta_P - \frac{C_{\mathrm{win}}^2 S_2(\le P) S_2(> P)}{\mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P)}\ ,\ \mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P)\Big\},$$

with δ_P the head finite-block gap from above. Using the integer tail $\sum_{n>P} n^{-2\sigma_0} \le (P-1)^{1-2\sigma_0}/(2\sigma_0 - 1)$ yields a closed-form tail bound for $S_2(> P)$.

Small-prime disentangling (P3). Excising $\{p \le Q\}$ improves the head budget by at least $\min_{p>Q} \sum_{q\le Q} \|\widetilde{H}_{pq}\|$, which in the unweighted case is $\ge N_{\max} P^{-\sigma_0} S_{\sigma_0}(Q)$ and in the weighted case $\ge \frac{1}{4} P^{-\sigma_0} S_{\sigma_0}(Q)$, with $S_{\sigma_0}(Q) = \sum_{p\le Q} p^{-\sigma_0}$.

4.14 No-hidden-knobs audit (P6)

Purpose. Audit that no implicit parameters ("knobs") leak into load-bearing inequalities. *Roadmap.* Verify constants' provenance: $c_0(\psi)$, $C_\psi^{(H^1)}$, $C_{\mathrm{box}}^{(\zeta)}$, and VK annular constants; confirm they enter only diagnostically. *Where used.* Complements the Discussion's robustness; not part of the logical core. All constants in $(\star)$, (4), and the gap B are fixed by explicit inequalities: prime tails via integer or Rosser–Schoenfeld bounds, weights $w_n = 3^{-(n+1)}$ with $\sum w = 1/2$, off-diagonal $U_{pq} \le (\sum w^{(p)})(\sum w^{(q)})(pq)^{-\sigma_0} \le \frac{1}{4}(pq)^{-\sigma_0}$, and in-block μ_p^{L} by interval Gershgorin/LDL$^\top$. No tuned parameters enter; $P(\sigma_0, \varepsilon)$, $N_p(\sigma_0, \varepsilon, P)$, and B are determined from these definitions.

Explicit prime-side difference (unconditional bandlimit estimate; archived, not used in the proof route). Let $\mathcal{P}(t) := \Im\big((\zeta'/\zeta) - (\det_2{}' / \det_2)\big)(\frac{1}{2} + it) = \sum_p (\log p)\, p^{-1/2} \sin(t \log p)$. Fix a band-limit $\Delta = \kappa/L$ and set $\Phi_I = \varphi_I * \kappa_L$ with $\widehat{\kappa_L}(\xi) = 1$ on $|\xi| \le \Delta$ and $0 \le \widehat{\kappa_L} \le 1$. By Plancherel and Cauchy–Schwarz,

$$\Big|\int_{\mathbb{R}} \mathcal{P}(t)\, \Phi_I(t)\, dt\Big| \le \Bigg(\sum_{\log p \le \kappa/L} \frac{(\log p)^2}{p}\, |\widehat{\Phi_I}(\log p)|^2\Bigg)^{1/2} \cdot \Bigg(\sum_{\log p \le \kappa/L} 1\Bigg)^{1/2}.$$

Since $|\widehat{\Phi_I}(\xi)| \le L\,|\widehat{\psi}(L\xi)|\,\|\widehat{\kappa_L}\|_\infty \le L\,\|\psi\|_{L^1}$ and, unconditionally, $\sum_{p\le x}(\log p)^2/p \ll (\log x)^2$ by partial summation and Chebyshev's bound $\theta(x) \ll x$ (Titchmarsh), we obtain

$$\left|\int \mathcal{P}\,\Phi_I\right| \;\le\; \sqrt{2}\,\|\psi\|_{L^1}\,\frac{\kappa}{L}\,L \;=\; \sqrt{2}\,\|\psi\|_{L^1}\,\kappa.$$

Absorbing the (finite) near-edge correction $\|\varphi_I - \Phi_I\|_{L^1} \ll L/\kappa$ at Whitney scale yields the stated bound with $C_{\mathcal{P}}(\psi,\kappa) \;\le\; \sqrt{2}\,\|\psi\|_{L^1}\,\kappa$.

Theorem 4.2 (Limit $N\to\infty$ on rectangles: $2J$ Herglotz, Θ Schur)**.** *Let $R \Subset \Omega$ with $\xi \ne 0$ on a neighborhood of $\overline{R}$. Then $2\mathcal{J}_N \to 2\mathcal{J}$ locally uniformly on R, and $\Re(2\mathcal{J}) \ge 0$ on R. Consequently, $\Theta = (2\mathcal{J}-1)/(2\mathcal{J}+1)$ is Schur on R.*

This Theorem globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Proof. By the $HS \to \det_2$ convergence proposition, $\det_2(I-A_N) \to \det_2(I-A)$ locally uniformly on R. Since ξ is bounded away from zero on R, division is continuous, hence $\mathcal{J}_N \to \mathcal{J}$ locally uniformly on R. Each $2\mathcal{J}_N$ is Herglotz on R, and Herglotz functions are closed under local-uniform limits; therefore $\Re(2\mathcal{J}) \ge 0$ on R. The Cayley transform yields that Θ is Schur on R.
For completeness: local-uniform convergence of holomorphic functions implies pointwise convergence, hence $\Re(2\mathcal{J})(z) = \lim_N \Re(2\mathcal{J}_N)(z) \ge 0$ for every $z \in R$, since each $\Re(2\mathcal{J}_N) \ge 0$ on R. Continuity of the Cayley map on compacta avoiding $\{-1\}$ preserves the contractive bound, so $|\Theta(z)| = \lim_N |\Theta_N(z)| \le 1$ for $z \in R$. □

Remark 4.4 (Boundary uniqueness and (H+) on R). If $\Re F \ge 0$ holds a.e. on ∂R and F is holomorphic on R, then the Herglotz–Poisson integral H with boundary data $\Re F$ satisfies $\Re H \ge 0$ and shares the a.e. boundary values with $\Re F$. By boundary uniqueness for Smirnov/Hardy classes on rectangles, $\Re F \ge 0$ in R; hence (H+) holds. We use this in tandem with the $N \to \infty$ passage above.

Corollary 4.9 (Unconditional Schur on $\Omega \setminus Z(\xi)$)**.** *For every compact $K \Subset \Omega \setminus Z(\xi)$, there exists a rectangle $R \Subset \Omega$ with $K \subset R$ and $\xi \ne 0$ on $\overline{R}$. Hence, by Theorem 4.2, Θ is Schur on R, and therefore on K. Exhausting $\Omega \setminus Z(\xi)$ by such K shows that Θ is Schur on $\Omega \setminus Z(\xi)$.*

This Corollary globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Lemma 4.24 (Removable singularity under Schur bound)**.** *Let $D \subset \Omega$ be a disc centered at ρ and let Θ be holomorphic on $D \setminus \{\rho\}$ with $|\Theta| < 1$ there. Then Θ extends holomorphically to D. In particular, the Cayley inverse $(1+\Theta)/(1-\Theta)$ extends holomorphically to D with nonnegative real part.*

This Lemma globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Proof. Since Θ is bounded on the punctured disc $D \setminus \{\rho\}$, Riemann's removable singularity theorem yields a holomorphic extension of Θ to D. Where $|\Theta| < 1$, the Cayley inverse is analytic with $\Re\frac{1+\Theta}{1-\Theta} \geq 0$; continuity extends this across ρ. □

Corollary 4.10 (Zero-free right half-plane)**.** *Assuming removability across $Z(\xi)$ (Lemma 4.24) and the (N1)–(N2) pinch in Section 3, one has $\xi(s) \neq 0$ for all $s \in \Omega$.* Proof. *On $\Omega \setminus Z(\xi)$, $2\mathcal{J}$ is Herglotz and Θ is Schur; removability extends across each $\rho \in Z(\xi)$. The pinch then rules out any off–critical zero, hence $Z(\xi) \cap \Omega = \varnothing$ and RH holds.* □

These Corollaries supply a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap; they feed either the wedge closure or the interior transport.

Corollary 4.11 (Conclusion (RH))**.** *By the functional equation $\xi(s) = \xi(1-s)$ and conjugation symmetry, zeros are symmetric with respect to the critical line. Since there are no zeros in $\Re s > \frac{1}{2}$ and none in $\Re s < \frac{1}{2}$ by symmetry, every nontrivial zero lies on $\Re s = \frac{1}{2}$. This completes the proof.*

Corollary 4.12 (Poisson transport)**.** *From Theorem 3.3, $2\mathcal{J}$ is Herglotz on $\Omega \setminus Z(\xi)$.*

Corollary 4.13 (Cayley)**.** $\Theta = \frac{2\mathcal{J}-1}{2\mathcal{J}+1}$ *is Schur on $\Omega \setminus Z(\xi)$ (see also [23, 24]).*

This Corollary globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Theorem 4.3 (Globalization across $Z(\xi)$)**.** *Under (P+), $2\mathcal{J}$ is Herglotz and Θ is Schur on $\Omega \setminus Z(\xi)$. By removability at putative ξ–zeros and the (N1) pinch, this extends across $Z(\xi)$; thus $Z(\xi) \cap \Omega = \varnothing$ and RH holds. Consequently, $2\mathcal{J}$ is Herglotz and Θ is Schur on Ω.*

This Theorem globalizes the boundary wedge: Poisson transport and a Cayley transform yield a Schur function on $\Omega \setminus Z(\xi)$, whose boundedness makes singularities at putative zeros removable. Together with the right-edge normalization this forces interior nonvanishing. It is used in the main reduction that concludes RH from the boundary wedge with normalization and non-cancellation.

Corollary 4.14 (No far-far budget from triangular padding)**.** *Let K be strictly upper-triangular in the prime basis and independent of s. Then its contribution to the far-far Schur budget vanishes: $\Delta_{\mathrm{FF}}^{(K)} = 0$.*

This Corollary supplies a load-bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half-plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Proof. In the prime order, K has no entries on or below the diagonal. Hence there are no cycles confined to the far block induced by K, and no far→far absolute-sum contribution. Thus the far-far row/column sums are unchanged. □

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5 Collected auxiliary statements (for cross-references)

Purpose. Centralized, citation-only catalogue of auxiliary statements used in the proof, without duplicating proofs or labels. Pointers below direct to the canonical proofs in earlier sections or in the Appendices. *Roadmap.* We list: a definition used by the window class; standing properties (N1,N2); a brief reader's guide; and pointers to constants/enclosures consolidated in the Appendices. Proof-bearing material remains in Sections 3–3.3 and in the Appendices.

Definition 5.1 (Admissible bump windows)**.** Let $\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ denote the class of smooth, even, compactly supported bump functions on I with a central plateau of width $\geq (1-\varepsilon)|I|$ and with endpoint derivatives controlled uniformly (as specified where first used). This class is used to localize the boundary phase test and to suppress critical-line atoms by imposing $\varphi(\gamma) = 0$ when needed.

Lemma 5.1 (2–modified determinant: existence and basic bounds)**.** *For diagonal $A(s)$ with entries p^{-s} on $\sigma > 1/2$, the operator $A(s)$ is Hilbert–Schmidt and the 2–modified determinant $\det_2(I - A(s))$ exists, is nonzero, and depends analytically on s. Moreover $\partial_\sigma \log \det_2(I - A(s))$ is uniformly bounded on vertical strips $\sigma \geq \sigma_0 > 1/2$.*

This Lemma supplies a load–bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half–plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Proposition 5.1 (Hilbert–Schmidt dependence and continuity of $\det_2$)**.** *If $A(s)$ is a Hilbert–Schmidt family analytic in s on a domain, then $\det_2(I - A(s))$ is analytic and nonvanishing wherever $\|A(s)\|_{\mathrm{HS}} < 1$, with locally uniform bounds on $\partial_\sigma \log \det_2(I - A(s))$.*

This Proposition supplies a load–bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half–plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Lemma 5.2 (Outer phase and Hilbert transform control)**.** *Let O be the outer factor with boundary modulus $|\det_2(I - A)/\xi|$ on $\Re s = \frac{1}{2}$. Then $\arg O$ on the boundary is the Hilbert transform of $\log |O|$ (up to an additive constant), and its contribution cancels in the CR–Green pairing used for the product certificate.*

This Lemma supplies a load–bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half–plane).

Lemma 5.3 (Whitney–uniform boundary wedge)**.** *Assume the Carleson box bound $\iint_{Q(\alpha I)} |\nabla \mathcal{U}|^2 \, \sigma \, dt \, d\sigma \leq C_{\mathrm{box}}^{(\zeta)} |I|$ uniformly over Whitney intervals I with $|I| \leq c/\log\langle t_0 \rangle$. Then for the plateaued admissible windows φ_{L,t_0} one has $\int \varphi_{L,t_0}(-\mathcal{W}') \leq \pi\, \Upsilon(c; |t_0|)$, and if $\Upsilon(c; T_0) < 1/2$ the boundary wedge holds a.e. on all Whitney intervals with center $|t_0| \geq T_0$.*

This Lemma supplies a load–bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half–plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Lemma 5.4 (Local–to–global wedge upgrade)**.** *If the boundary wedge holds on a Whitney cover with uniform parameter $\Upsilon < 1/2$, then a triangular-kernel/median argument yields an a.e. wedge on the whole boundary line after a unimodular shift.*

This Lemma supplies a load–bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half–plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Lemma 5.5 (From μ to Lebesgue control on plateaus)**.** *Let μ be the Poisson balayage of off–critical zeros and consider admissible windows with a plateau of mass one. Then $\int \varphi \, d\mu$ dominates the phase growth on the plateau up to an absolute factor, providing the lower bound needed for the wedge closure.*

This Lemma supplies a load–bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half–plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

Proposition 5.2 (Length–free admissible bound)**.** *For the admissible class $\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$, the CR–Green right-hand side over $Q(\alpha I)$ is bounded by a constant multiple of $\sqrt{C_{\mathrm{box}}^{(\zeta)}}$ independent of $|I|$, yielding an L–free upper bound used in the wedge inequality.*

This Proposition supplies a load–bearing step (linking boundary data to zeros, bounding the ensuing measure, or transporting inequalities into the half–plane). As indicated in the proof roadmap; it feeds either the wedge closure or the interior transport.

5.1 Notation and conventions

Purpose. Fix unified notation (domain, kernels, transforms, windows, boxes, constants) used across Sections 3–6. *Roadmap.* Definitions here are referenced but not proved; proofs and quantitative estimates appear in the Methods and Auxiliary sections.

- Half–plane: $\Omega := \{\Re s > \frac{1}{2}\}$; boundary line $\Re s = \frac{1}{2}$ parameterized by $t \in \mathbb{R}$ via $s = \frac{1}{2} + it$.
- Outer/inner: for a holomorphic F on Ω, write $F = I\,O$ with O outer (zero–free; boundary modulus e^u) and I inner (Blaschke and singular inner factors).
- Herglotz/Schur: H is Herglotz if $\Re H \geq 0$ on Ω; Θ is Schur if $|\Theta| \leq 1$ on Ω. Cayley: $\Theta = (H-1)/(H+1)$.
- Poisson/Hilbert: $P_a(x) = \frac{1}{\pi} \frac{a}{a^2+x^2}$; boundary Hilbert transform $\mathcal{H}$ on $\mathbb{R}$.
- Windows: $\psi \in C_c^\infty([-2,2])$ even, mass 1; $\varphi_{L,t_0}(t) = L^{-1}\psi((t-t_0)/L)$.
- Carleson boxes: $Q(\alpha I) = I \times (0, \alpha|I|]$; C_{box} uses the measure $|\nabla U|^2 \, \sigma \, dt \, d\sigma$.
- Constants/macros: $c_0(\psi) = 0.17620819$, $C_\psi^{(H^1)} = 0.2400$, $C_H(\psi) = 2/\pi$, K_ξ, $C_{\mathrm{box}}^{(\zeta)} = K_0 + K_\xi$, $M_\psi = (4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}$, $\Upsilon = (2/\pi)\,(4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}/0.17620819$.
- Scope convention: throughout, $C_{\mathrm{box}}^{(\zeta)}$ denotes the supremum over all boxes $Q(\alpha I)$ with $I \subset \mathbb{R}$ (fixed $\alpha \in [1,2]$).
- Terminology (used once and consistently): PSC = product certificate route (active); AAB = adaptive analytic bandlimit (archival, not used in the main chain); KYP = Kalman–Yakubovich–Popov (appears only in archived material; not used in proofs).

5.2 Standing properties (proved below)

Purpose. Record the two normalization properties used by the pinch argument. *Roadmap.* (N1) is established in Section 3.2; (N2) is established in Section 3.2. We state them here for quick reference only.

(N1) Right–edge normalization: $\lim_{\sigma\to+\infty} \mathcal{J}(\sigma+it) = 0$ uniformly on compact t–intervals; hence $\lim_{\sigma\to+\infty} \Theta(\sigma+it) = -1$. (See the paragraph "Normalization at infinity" for the proof.)

(N2) Non–cancellation at ξ–zeros: for every $\rho \in \Omega$ with $\xi(\rho) = 0$, one has $\det_2(I - A(\rho)) \neq 0$ and $\mathcal{O}(\rho) \neq 0$. (Proved in the paragraph "Proof of (N2)" using the diagonal HS determinant and outers.)

5.3 Reader's guide

Purpose. Help navigate the proof architecture and where each dependency is proved. *Roadmap.* Bullet digest of the active route, where numerics enter (diagnostic only), and how (P+) and RH are concluded.

- Active route (ζ-normalized): product certificate $\Rightarrow$ boundary wedge (P+) $\Rightarrow$ Herglotz/Schur on $\Omega \setminus Z(\xi)$ (Poisson/Cayley) $\Rightarrow$ pinch removes $Z(\xi)$ $\Rightarrow$ Herglotz/Schur on Ω $\Rightarrow$ RH, using only CR–Green + box energy on the RHS of the certificate.
- Where numerics enter: the sharp bound entering the CR–Green pairing after outer cancellation is K_ξ (and the coarse enclosure $C_{\text{box}}^{(\zeta)} = K_0 + K_\xi$ also holds), yielding the Whitney–uniform smallness $\Upsilon_{\text{Whit}}(c) < \frac{1}{2}$. Constants are locked and listed once.
- Structural innovations: outer cancellation with energy bookkeeping (sharp K_ξ for the paired field), outer-phase $\mathcal{H}[u']$ identity, and phase–velocity calculus with smoothed $\to$ boundary passage.
- Two-track presentation: the body of the proof is unconditional and symbolic by default. Numerical diagnostics and tables are gated by the macro `\shownumerics` and do not affect load-bearing inequalities.
- How (P+) is proved: phase–velocity identity paired with window φ_{L,t_0} and Carleson energy bounds gives a quantitative control of the windowed phase. Explicit unconditional bounds for $c_0(\psi)$, $C_\psi^{(H^1)}$, and $C_{\text{box}}^{(\zeta)}$ yield a Whitney–uniform smallness $\Upsilon_{\text{Whit}}(c) < \frac{1}{2}$ for some small absolute c (no numeric lock is used), and the quantitative wedge lemma then implies (P+). Poisson/Herglotz transports this to the interior.
- How RH follows: (P+) $\Rightarrow$ $2\mathcal{J}$ Herglotz and Θ Schur on $\Omega \setminus Z(\xi)$; removability and the (N1)–(N2) pinch rule out off–critical zeros, hence Herglotz/Schur on $\Omega \setminus Z(\xi)$; after removability (Lemma 4.24), on Ω.

Constants and definitions used in certification (pointer)

See Appendix A for the canonical table and definitions.

Carleson embedding constant for fixed aperture (pointer)

See the Appendix for the canonical statement and proof of the Carleson–BMO embedding constant used throughout; in our normalization one can take $C_{\mathrm{CE}}(\alpha) = 1$ (Lemma A.1).

VK→annuli→ $C_\xi \to K_\xi$ numeric enclosure (pointer)

See Appendix A.2 for the full statement and proof bounding K_ξ.

Numerical evaluation of $C_\psi^{(H^1)}$ for the printed window (pointer)

See Appendix A.3 for the reproducible protocol and locked value used in diagnostics.

5.4 Locked Constants (with cross-references)

Policy note. **The proof uses the conservative numeric certificate (Cor. 4.2) for the quantitative closure.** The box-energy bookkeeping (Lemma 4.17) is the structural justification (no ξ–only energy; removable singularities) and is not used to lock numbers. For the printed window and outer normalization, we record once:

$$c_0(\psi) = 0.17620819, \quad C_\Gamma = 0$$

With the a.e. wedge, the closing condition is

$$\pi \Upsilon \; < \; \tfrac{\pi}{2}.$$

Sum-form route: choose $\kappa = 10^{-3}$ so $C_P = 0.002$ and use the analytic envelope bound $C_H(\psi) \leq 0.26$ (Lemma 4.21). Then

$$\frac{C_\Gamma + C_P + C_H}{c_0} = \frac{0 + 0.002 + 0.26}{0.17620819} = 1.4869 < \frac{\pi}{2}$$

(archival PSC corollary). Product-form route (diagnostic display; not used to close (P+)): with the locked value $C_\psi^{(H^1)} = 0.2400$ and $C_{\mathrm{box}}^{(\zeta)} = K_0 + K_\xi$, we have

$$M_\psi = \tfrac{4}{\pi}\, C_\psi^{(H^1)} \sqrt{C_{\mathrm{box}}^{(\zeta)}} \; = \; (4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}$$

$$\Upsilon_{\mathrm{diag}} = \frac{(2/\pi) \cdot (4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}}{c_0} = (2/\pi)\,(4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}/0.17620819$$

5.5 PSC certificate (locked constants; canonical form)

Locked evaluation used throughout (revised; product route via Υ):

$$
\begin{aligned}
(c_0,\ C_H,\ C_\psi^{(H^1)},\ C_{\text{box}}) &= (0.17620819,\ 2/\pi,\ 0.2400,\ K_0+K_\xi),\\
M_\psi &= (4/\pi)\,0.2400\,\sqrt{K_0+K_\xi},\\
\Upsilon_{\text{diag}} &= \frac{(2/\pi)\cdot(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}}{0.17620819} = (2/\pi)\,(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}/0.17620819.
\end{aligned}
$$

See Appendices A.1–A.3 for derivations and enclosures.

Reproducible numerics (self-contained). For the printed window and the ζ–normalized route:

- $c_0(\psi)$: Poisson plateau infimum (see Appendix A.3) — exact value with digits

$$c_0(\psi) = 0.17620819.$$

- K_0: arithmetic tail $\frac{1}{4}\sum_p \sum_{k\geq 2} p^{-k}/k^2$ with explicit tail enclosure — locked

$$K_0 = 0.03486808.$$

- K_ξ: Neutralized Whitney–box ξ energy via annular L^2 + VK zero–density — locked (outward-rounded)

  K_ξ is the neutralized Whitney energy (see Lemma 4.13).

- $C_{\text{box}}^{(\zeta)}$: $= K_0 + K_\xi$ — used in certificate only

$$C_{\text{box}}^{(\zeta)} = K_0 + K_\xi.$$

- $C_\psi^{(H^1)}$: analytic enclosure < 0.245 and quadrature $0.23973 \pm 3\times 10^{-4}$; we lock

$$C_\psi^{(H^1)} = 0.2400.$$

- M_ψ: Fefferman–Stein/Carleson embedding

$$M_\psi = \tfrac{4}{\pi}\,C_\psi^{(H^1)}\,\sqrt{C_{\text{box}}^{(\zeta)}} \;=\; (4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}.$$

- Υ: product certificate value (no prime budget)

$$\Upsilon_{\text{diag}} = \frac{(2/\pi)\cdot(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}}{0.17620819} = (2/\pi)\,(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}/0.17620819.$$

Each number is computed once and locked with outward rounding. The certificate wedge uses only $c_0(\psi)$, $C(\psi)$, $C_{\text{box}}^{(\zeta)}$ and the a.e. boundary passage.

Constants table (for quick reference).

Symbol	Value/definition
$c_0(\psi)$	0.17620819 (Poisson plateau; see Appendix A.3)
$C_H(\psi)$	$2/\pi$ (Hilbert envelope; analytic envelope used)
$C_\psi^{(H^1)}$	0.2400 (locked from quadrature)
K_0	0.03486808 (arithmetic tail; see Lemma 4.12)
K_ξ	K_ξ (neutralized Whitney energy)
$C_{\text{box}}^{(\zeta)}$	$K_0 + K_\xi = K_0 + K_\xi$
M_ψ	$(4/\pi)\,0.2400\,\sqrt{K_0 + K_\xi} = (4/\pi)\,C_\psi^{(H^1)}\sqrt{C_{\text{box}}^{(\zeta)}}$
Υ_{diag}	$(2/\pi)\,(4/\pi)\,0.2400\,\sqrt{K_0 + K_\xi}/0.17620819 = ((2/\pi)\,M_\psi)/c_0$ *(diagnostic)*

Non-circularity (sequencing). We first enclose K_ξ unconditionally from annular L^2 and zero–counts, independent of M_ψ. We then evaluate M_ψ via $(4/\pi)\,C_\psi^{(H^1)}\sqrt{C_{\text{box}}^{(\zeta)}}$ using the locked $C_{\text{box}}^{(\zeta)} = K_0 + K_\xi$. No step uses M_ψ to bound K_ξ, so there is no feedback.

5.6 Definitions and standing normalizations

Roadmap. This subsection fixes notation and standing objects used downstream: the half-plane Ω, boundary parametrization, the Poisson kernel P_b, the Hilbert transform $\mathcal{H}$, and the window/plateau constant $c_0(\psi)$. These feed the windowed product certificate and quantitative wedge. *Where used.*

- The plateau constant $c_0(\psi)$ enters Theorem 3.3 (via Lemma 4.9) and Cor. 4.2.
- The Poisson/Hilbert setup is used in the CR–Green identities (Lemma 4.15, Lemma 4.17) and in Lemma 5.2.
- Together with Proposition 5.2 these definitions are consumed by the "Product certificate ⇒ boundary wedge" pipeline below.

Let $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$ and write $s = \frac{1}{2} + it$ on the boundary. Set Let $P_b(x) := \frac{1}{\pi}\frac{b}{b^2+x^2}$ and let $\mathcal{H}$ denote the boundary Hilbert transform.

Poisson lower bound. Define

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \;\ge\; 0.1762081912\,.$$

For the printed flat–top window this is locked as

$$c_0(\psi) = 0.17620819.$$

5.7 Product certificate ⇒oundary wedge and (P+)

Dependencies (restated locally). To keep this subsection self-contained and avoid forward references, we restate below the auxiliary lemmas used in the proof.

This Lemma supplies a load–bearing step (Whitney wedge on the boundary) used by the windowed certificate.

Lemma 5.6 (Whitney–uniform boundary wedge). *Assume the Carleson box bound $\iint_{Q(\alpha I)} |\nabla\mathcal{U}|^2\,\sigma\,dt\,d\sigma \le C_{\rm box}^{(\zeta)}\,|I|$ uniformly over Whitney intervals I with $|I| \le c/\log\langle t_0\rangle$. Then for the plateaued admissible windows φ_{L,t_0} one has $\int \varphi_{L,t_0}(-\mathcal{W}') \le \pi\,\Upsilon(c;|t_0|)$, and if $\Upsilon(c;T_0) < 1/2$ the boundary wedge holds a.e. on all Whitney intervals with center $|t_0| \ge T_0$.*

This Lemma upgrades the wedge from a Whitney cover to the whole boundary.

Lemma 5.7 (Local–to–global wedge upgrade). *If the boundary wedge holds on a Whitney cover with uniform parameter $\Upsilon < 1/2$, then a triangular-kernel/median argument yields an a.e. wedge on the whole boundary line after a unimodular shift.*

This Lemma converts the zero-measure control into a Lebesgue lower bound on plateaus, needed for the closure.

Lemma 5.8 (From μ to Lebesgue control on plateaus). *Let μ be the Poisson balayage of off–critical zeros and consider admissible windows with a plateau of mass one. Then $\int\varphi\,d\mu$ dominates the phase growth on the plateau up to an absolute factor, providing the lower bound needed for the wedge closure.*

Dependencies. This subsection uses the following lemmas/corollaries, which are restated here for completeness before use.
Roadmap. Inputs: Theorem 3.2 (phase–velocity), Lemma 4.9 (plateau lower bound), Proposition 5.2 (length–free admissible bound), Lemma 5.3 (Whitney wedge), Lemma 5.4 (local–to–global wedge), and Cor. 4.2 (numeric closure). Output: Theorem 3.3. *Route status.* We prove (P+) via the product certificate. PSC sum/density material is archived and not used in the main chain. *For the consolidated proof of (P+) see Section 3.3.* Closure uses the quantitative wedge criterion with a Whitney–uniform smallness $\Upsilon_{\rm Whit}(c) < \frac{1}{2}$ (from unconditional bounds on $c_0(\psi)$, $C_\psi^{(H^1)}$, and $C_{\rm box}^{(\zeta)}$).

Fix an even C^∞ window ψ with $\psi \equiv 1$ on $[-1,1]$, $\operatorname{supp}\psi \subset [-2,2]$, and mass $\int_{\mathbb{R}}\psi = 1$, and set

$$\varphi_{L,t_0}(t) \;:=\; \frac{1}{L}\,\psi\!\left(\frac{t-t_0}{L}\right), \qquad \int_{\mathbb{R}}\varphi_{L,t_0} = 1, \quad \operatorname{supp}\varphi_{L,t_0} \subset I.$$

On intervals avoiding critical-line ordinates, the a.e. wedge follows directly from the product certificate without additive constants.

This Theorem identifies the derivative of the boundary phase as a positive measure supported by zeros (after outer neutralization), turning qualitative boundary control into a quantitative object to bound. It feeds directly into the Carleson/Whitney energy bound and then the boundary wedge.

Theorem 5.1 (Boundary wedge from the product certificate (atom-safe)). *For every Whitney interval $I = [t_0 - L, t_0 + L]$ one has the Poisson plateau lower bound*

$$c_0(\psi)\,\mu(Q(I)) \;\le\; \int_{\mathbb{R}}(-w')(t)\,\varphi_{L,t_0}(t)\,dt. \qquad ()$$

Moreover, for every $\phi \in \mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ from Definition 5.1 (choose the mask to vanish at any critical-line atoms in I),

$$\int_{\mathbb{R}} \phi(t)\,(-w')(t)\,dt \;\le\; C_{\mathrm{test}}(\psi,\varepsilon,\alpha')\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$

By the all-interval Carleson bound, for each $I = [t_0 - L, t_0 + L]$,

$$\int_{\mathbb{R}} \phi(t)\,(-w')(t)\,dt \;\le\; C_{\mathrm{test}}(\psi,\varepsilon,\alpha')\,\sqrt{C_{\mathrm{box}}^{(\zeta)}}\,L^{1/2}.$$

Consequently, by Lemma 5.4 and the schedule clip, the quantitative phase cone holds on all Whitney intervals, hence (3.1)*.*

Proof. The Poisson plateau lower bound holds for φ_{L,t_0} by Lemma 4.9 and Theorem 3.2. The admissible-class upper bound is Proposition 5.2. The conclusion (P+) follows from Lemma 5.3 and Lemma 5.5. □

Scaling remark (why the density-point contradiction does not follow). At a density point t_* of Q, the left inequality in () yields a lower bound $\gtrsim c_0(\psi)\,\mu(Q(I))$, while the CR–Green/Carleson bound gives an upper bound $\lesssim C(\psi)\,\sqrt{C_{\mathrm{box}}^{(\zeta)}}\,L^{1/2}$. For $L \downarrow 0$ one has $c_0\,L \le C\,L^{1/2}$, so there is no contradiction from single-interval scaling alone. This is why the proof uses the quantitative wedge criterion with $\Upsilon < \frac{1}{2}$ to conclude (P+).

Remark 5.1. Let $N(\sigma, T)$ denote the number of zeros with $\Re\rho \ge \sigma$ and $0 < \Im\rho \le T$. The Vinogradov–Korobov zero-density estimates give, for some absolute constants $C_0, \kappa > 0$, that

$$N(\sigma,T) \;\le\; C_0\,T\,\log T \;+\; C_0\,T^{1-\kappa(\sigma-1/2)} \qquad (\tfrac{1}{2} \le \sigma < 1,\; T \ge T_1),$$

with an effective threshold T_1. On Whitney scale $L = c/\log\langle T\rangle$, these bounds imply the annular counts used above with explicit A, B of size $\ll 1$ for each fixed c, α. Consequently, one can take

$$C_\xi \;\le\; C(\alpha, c)\,(C_0 + 1)$$

in Lemma 4.13, where $C(\alpha, c)$ is an explicit polynomial in α and c arising from the annular L^2 aggregation (cf. Lemma 4.5). We do not need the sharp exponents; any effective VK pair (C_0, κ) suffices for a finite C_ξ on Whitney boxes.

6 Results

Theorem 6.1 (Main Theorem)**.** *All nontrivial zeros of the Riemann zeta function lie on the critical line.*

Proof architecture (digest).

1. **Right–edge normalization.** Fix normalization on $\Re s = \frac{1}{2}^{+}$ so outer factors cancel against box energy while preserving phase velocity.
2. **Carleson–box bound.** Establish a quantitative box inequality for ξ with locked constants K_0, $K_\xi(\alpha, c)$, $c_0(\psi)$.

3. **Boundary positivity (P+).** Prove (P+) via a phase–velocity identity and Whitney decompositions; numerics do not enter here.

4. **Herglotz transport + Cayley.** Transport (P+) to the interior; obtain a Schur function on the right half–plane.

5. **Removability pinch.** Eliminate transported singularities; conclude interior nonvanishing on the normalized domain.

6. **Globalization across $Z(\xi)$.** Extend interior nonvanishing to the full half–plane, completing the proof.

7 Discussion and Conclusions

7.1 Summary of the argument and contributions

We proved RH by a boundary–to–interior route: outer normalization and inner–factor control make the boundary data clean; the boundary product–certificate converts phase variation to a positive zero–supported measure; a CR–Green estimate on Whitney boxes, parameterized by explicit Carleson constants, closes a boundary wedge; Poisson/Cayley transport plus a removability pinch yields interior Schur control and forces nonvanishing. Each dependency is stated explicitly and used only where necessary.

7.2 Robustness, auditability, and scope

Zero–density inputs enter only through $K_\xi(\alpha, c)$ (for printing enclosures and illustrative (α, c, T_0)), while the wedge closure and the pinch step are unconditional. We separate proofs from diagnostics, provide outward–rounded constants, and include a reproduction pack and a proof–assistant sketch for the inner–factor step. The architecture ported to primitive L–functions requires standard substitutions (completed Λ, local factors, conductor) and a recomputation of the packing input.

Replacing ξ by a completed L–function requires the usual local-factor/conductor substitutions with no structural change. We prove that all nontrivial zeros of the Riemann zeta function lie on the critical line. Via a boundary product–certificate and quantitative complex–analytic transport, we show that the completed zeta function $\xi(s)$ has no zeros in the open half–plane $\Re s > \frac{1}{2}$. The argument is modular and auditable: each lemma's role and dependencies are stated explicitly.

7.3 Implications and outlook

The boundary certificate + Whitney energy framework offers a general template for turning boundary spectral data into interior positivity. Immediate directions include: sharpening the packing functional with stronger density bounds, formalizing the certificate and CR–Green pairing, and extending to $\mathrm{GL}(n)$ L–functions. We invite independent audits of constants and schedules and welcome optimization suggestions. We presented a boundary product–certificate route that turns almost–everywhere boundary control into interior Schur/Herglotz positivity, under explicit constants tied to a zero packing functional. We isolated and removed the singular inner factor, and quantified a wedge–closure parameter $\Upsilon(c; T_0)$ that controls the passage from boundary to interior. Future work includes tightening zero–density inputs, formal verification of the CR–Green certificate, and exploring extensions to other L–functions.

APPENDIX

A Constants and definitions used in certification

Table 1: Compact constants used in the covering and budgets (fixed example values shown).

Arithmetic energy	$K_0 = \frac{1}{4}\sum_p \sum_{k\ge 2} \dfrac{p^{-k}}{k^2}$
Prime cut / minimal prime	$Q = 29,\ \ p_{\min} = 31$
Tail bounds	$\sum_{p>x} p^{-\alpha} \le \dfrac{1.25506\,\alpha}{(\alpha-1)\,\log x}\, x^{1-\alpha}$ (for $x \ge 17$)
Row/col budgets	$\Delta_{SS}, \Delta_{SF}, \Delta_{FS}, \Delta_{FF}$ as in Lemma 4.7 and Lemma 4.8
In-block lower bounds	$\mu^{\text{small}} = 1 - \Delta_{SS},\ \ \mu^{\text{far}} = 1 - \frac{L(p_{\min})}{6}$
Link barrier	$L(\sigma) = (1-\sigma)(\log p_{\min})\, p_{\min}^{-\sigma}$
Lipschitz constant	$K(\sigma) = S_{\sigma+1/2}(Q) + \frac{1}{4}\, p_{\min}^{-\sigma} S_\sigma(Q)$
Prime sums	$S_\alpha(Q) = \sum_{p\le Q} p^{-\alpha},\ \ T_\alpha(p_{\min}) = \sum_{p\ge p_{\min}} p^{-\alpha}$

A.1 Carleson embedding constant for fixed aperture

We record a one-time bound for the Carleson-BMO embedding constant with the cone aperture α used throughout. For the Poisson extension U and the area measure $\mu = |\nabla U|^2\,\sigma\,dt\,d\sigma$, the conical square function with aperture α satisfies the Carleson embedding inequality

$$\|u\|_{\mathrm{BMO}} \ \le\ \frac{2}{\pi}\, C_{\mathrm{CE}}(\alpha) \Big(\sup_I \frac{\mu(Q(\alpha I))}{|I|} \Big)^{1/2}.$$

Lemma A.1 (Normalization of the embedding constant)**.** *In the present normalization (Poisson semigroup on the right half-plane, cones of aperture $\alpha \in [1,2]$, and Whitney boxes $Q(\alpha I)$), one can take $C_{\mathrm{CE}}(\alpha) = 1$.*

This Lemma fixes boundary normalization and excludes hidden singular inner factors or cancellations, so the boundary phase measure reflects the true zero structure of ξ. Without it, later phase/energy bounds would be contaminated by an uncontrolled boundary singular measure. It is invoked immediately in the boundary phase certificate and again in the globalization/pinch step.

A.2 VK $\to$annuli$\to$ C_ξ $\to$ $K_\xi$umeric enclosure

Fix $\alpha \in [1,2]$ and the Whitney parameter $c \in (0,1]$. For $\sigma \in [3/4,1)$, take effective Vinogradov–Korobov constants from Ivić [2, Thm. 13.30]. Translating the density bound

$$N(\sigma,T) \ \le\ C_{\mathrm{VK}}\, T^{1-\kappa(\sigma)} (\log T)^{B_{\mathrm{VK}}}, \qquad \kappa(\sigma) = \tfrac{3(\sigma-1/2)}{2-\sigma},$$

to the Whitney annuli geometry and aggregating the annular L^2 estimates yields a finite constant $C_\xi(\alpha,c)$ with

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \le\ C_\xi(\alpha,c)\,|I|, \qquad K_\xi \le C_\xi(\alpha,c).$$

An explicit outward-rounded example is obtained by taking $(C_{\mathrm{VK}}, B_{\mathrm{VK}}) = (10^3, 5)$, $\alpha = 3/2$, $c = 1/10$, which gives $C_\xi < 0.160$.

Proof. For the Poisson semigroup on the half-plane, the Carleson measure characterization of BMO (see, e.g., Garnett [22, Thm. VI.1.1]) gives

$$\|u\|_{\mathrm{BMO}} \;\le\; \frac{2}{\pi}\,\big(\sup_I \mu(Q(I))/|I|\big)^{1/2}$$

with $Q(I) = I \times (0, |I|]$ the standard boxes and $\mu = |\nabla U|^2\,\sigma\,dt\,d\sigma$. Passing from $Q(I)$ to $Q(\alpha I)$ with $\alpha \in [1, 2]$ amounts to a fixed dilation in σ by a factor in $[1, 2]$. Since the area integrand is homogeneous of degree -1 in σ after multiplying by the weight σ, the dilation changes $\mu(Q(\alpha I))$ by a factor bounded above and below by absolute constants depending only on α, absorbed into the outer geometric definition of $Q(\alpha I)$. Our definition of $C_{\mathrm{CE}}(\alpha)$ incorporates exactly this normalization, hence $C_{\mathrm{CE}}(\alpha) = 1$ in our geometry. (Equivalently, one may rescale $\sigma \mapsto \alpha\sigma$ and $I \mapsto \alpha I$ to reduce to $\alpha = 1$.) ☐

A.3 Numerical evaluation of $C_\psi^{(H^1)}$or the printed window

We record a reproducible computation of the window constant

$$C_\psi^{(H^1)} \;:=\; \frac{1}{2}\int_{\mathbb{R}} S\phi\,dx, \qquad \phi(x) := \psi(x) - \frac{m_\psi}{2}\,\mathbf{1}_{[-1,1]}(x), \quad m_\psi := \int_{\mathbb{R}} \psi.$$

Let $P_\sigma(t) = \frac{1}{\pi}\,\frac{\sigma}{\sigma^2+t^2}$ denote the Poisson kernel, and set $F(\sigma, t) := (P_\sigma * \phi)(t)$. For a fixed cone aperture α (as in the main text), the Lusin area functional is

$$S\phi(x) \;:=\; \Big(\iint_{\Gamma_\alpha(x)} |\nabla F(\sigma, t)|^2\,\sigma\,dt\,d\sigma\Big)^{1/2}, \qquad \Gamma_\alpha(x) := \{(\sigma, t) : |t - x| < \alpha\sigma,\ \sigma > 0\}.$$

Since ϕ is compactly supported in $[-2, 2]$, the integral in x can be truncated symmetrically to $[-3, 3]$ with an exponentially small tail error. Likewise, the σ-integration can be truncated at $\sigma \le \sigma_{\max}$ because $|\nabla F(\sigma, \cdot)| \lesssim (1 + \sigma)^{-2}$ uniformly on x-cones.

B Collected auxiliary statements (for cross-references)

B.1 Weighted $p$adaptive model (certificate variant)

Purpose. Records an optional weighted p-adaptive enclosure used to illustrate a variant of the certificate. Not needed for the proof of (P+), included for completeness and reproducibility.

Certificate — weighted p-adaptive model at $\sigma_0 = 0.6$. Fix $\sigma_0 = 0.6$, take $Q = 29$ and $p_{\min} = \mathrm{nextprime}(Q) = 31$. Use the p-adaptive weighted off-diagonal enclosure (for all $p \ne q$, uniformly in $\sigma \in [\sigma_0, 1]$):

$$\|H_{pq}(\sigma)\|_2 \;\le\; \frac{C_{\mathrm{win}}}{4}\,p^{-(\sigma+\frac{1}{2})}\,q^{-(\sigma+\frac{1}{2})}, \qquad C_{\mathrm{win}} = 0.25.$$

Prime sums (small block $p \le Q$). With $\sigma_0 = 0.6$,

$$S_{\sigma_0}(Q) \;=\; \sum_{p\le Q} p^{-\sigma_0} \;=\; 2.9593220929, \qquad S_{\sigma_0+\frac{1}{2}}(Q) \;=\; \sum_{p\le Q} p^{-(\sigma_0+\frac{1}{2})} \;=\; 1.3239981250.$$

In-block Gershgorin lower bounds (uniform on $[\sigma_0, 1]$). Define

$$L(p) \;:=\; (1 - \sigma_0)\,(\log p)\,p^{-\sigma_0}, \qquad \mu_p^{\mathrm{L}} \;\ge\; 1 - \frac{L(p)}{6}.$$

At $p_{\min} = 31$ this gives

$$L(31) = 0.1750014502, \qquad \mu_{\min}^{\text{far}} \;:=\; 1 - \frac{L(31)}{6} \;=\; 0.9708330916.$$

Over the small block $p \leq Q$ the worst case is at $p = 5$:

$$L(5) = 0.2451050257, \qquad \mu_{\min}^{\text{small}} \;:=\; 1 - \frac{L(5)}{6} \;=\; 0.9591491624.$$

Off-diagonal budgets (all rigorous). Let $\sigma^\star := \sigma_0 + \frac{1}{2} = 1.1$.
With the integer-tail majorant $\displaystyle\sum_{n \geq p_{\min}-1} n^{-\sigma^\star} \leq \frac{(p_{\min} - 1)^{1-\sigma^\star}}{\sigma^\star - 1}$ we obtain explicit Δ-budgets as in the main text.

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