

# Hodge Lean Proof: Axiom Completion Roadmap

What must be proved vs. what can remain assumed

Generated from `DependencyCheck.lean`

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**Purpose.** This document lists the **current axiom dependencies** of `hodge_conjecture'` and classifies them into:

- **Must complete** (strategy-critical),
- **Can leave** (classical pillars + interface glue).

## 1 Current axiom list

Lean reports the following 38 axioms for `hodge_conjecture'`:

```
FundamentalClassSet_isClosed, IsAlgebraicSet, IsAlgebraicSet_empty,
IsAlgebraicSet_union, calibration_inequality, exists_volume_form_of_submodule_axiom,
flat_limit_of_cycles_is_cycle, hard_lefschetz_inverse_form,
harvey_lawson_fundamental_class, harvey_lawson_represents, harvey_lawson_theorem,
instAddCommGroupDeRhamCohomologyClass, instModuleRealDeRhamCohomologyClass,
isClosed_omegaPow_scaled, isIntegral_zero_current, isSmoothAlternating_add,
isSmoothAlternating_neg, isSmoothAlternating_smul, isSmoothAlternating_sub,
isSmoothAlternating_zero, lefschetz_lift_signed_cycle, limit_is_calibrated,
microstructureSequence_are_cycles, microstructureSequence_defect_bound,
microstructureSequence_flat_limit_exists, ofForm_smul_real, ofForm_sub,
omega_pow_isClosed, omega_pow_represents_multiple, propext, serre_gaga,
signed_decomposition, simpleCalibratedForm_is_smooth, smoothExtDeriv_add,
smoothExtDeriv_smul, wirtinger_comass_bound, Classical.choice, Quot.sound
```

## 2 Axioms to complete

### 2.1 P0: Strategy-critical (highest priority)

These axioms likely encode the conjecture's hard content. Complete them first.

#### 1. signed\_decomposition

*Location:* `Hodge/Kahler/SignedDecomp.lean:61`

*Why critical:* This is where rationality is turned into a decomposition used to build algebraic cycles. If axiomatized, it can encode most of the conjecture's content.

*To complete:* Prove as theorem (or replace by a genuinely standard theorem known *not* to imply Hodge).

#### 2. microstructureSequence\_are\_cycles

*Location:* `Hodge/Kahler/Microstructure.lean:228`

*Why critical:* Part of the microstructure pipeline; asserts the constructed approximants are genuine cycles. If axiomatized, it hides the geometric construction.

*To complete:* Define the construction and prove  $\partial = 0$  for each approximant.

### 3. microstructureSequence\_defect\_bound

*Location:* Hodge/Kahler/Microstructure.lean:234

*Why critical:* Controls calibration defect; needed to pass to calibrated limits. If axiomatized, it hides the key analytic estimate.

*To complete:* Prove the defect estimate from concrete norms/currents.

### 4. microstructureSequence\_flat\_limit\_exists

*Location:* Hodge/Kahler/Microstructure.lean:269

*Why critical:* Provides the convergent subsequence / limit current. If axiomatized, it assumes the compactness/extraction needed by the strategy.

*To complete:* Prove via a formal Federer–Fleming compactness theorem for your current model.

### 5. harvey\_lawson\_fundamental\_class

*Location:* Hodge/Kahler/Main.lean:94

*Why critical:* Cohomology-level bridge equating the fundamental class of the HL/GAGA output to the target class. This is the exact representation step.

*To complete:* Prove the de Rham class identification from the definition of cycle/fundamental class.

### 6. lefschetz\_lift\_signed\_cycle

*Location:* Hodge/Kahler/Main.lean:150

*Why critical:* Cycle-level lifting used in the Hard Lefschetz reduction ( $p > n/2$ ). If axiomatized, it assumes compatibility of cycle classes with Lefschetz operator/hyperplane intersection.

*To complete:* Prove via intersection-with-hyperplane compatibility of cycle class maps.

## 2.2 P1: Pipeline integrity (GMT facts)

Standard in GMT once currents/flat topology are fully defined, but still axioms here.

### 7. limit\_is\_calibrated

*Location:* Hodge/Analytic/Calibration.lean:93

*Why critical:* Needed to ensure the flat limit current is calibrated, so Harvey–Lawson applies.

*To complete:* Prove from lower semicontinuity of mass + calibration inequality.

### 8. flat\_limit\_of\_cycles\_is\_cycle

*Location:* Hodge/Classical/HarveyLawson.lean:186

*Why critical:* Needed to ensure the flat limit remains a cycle ( $\partial = 0$ ).

*To complete:* Prove continuity of boundary in flat norm.

## 3 Axioms you can leave

### 3.1 Major classical pillars

These are deep but standard theorems. Reasonable to leave as axioms for now.

- `hard_lefschetz_inverse_form` — Hard Lefschetz theorem; large formalization project.
- `serre_gaga` — GAGA/Chow (analytic  $\Rightarrow$  algebraic); large AG formalization.
- `harvey_lawson_theorem` — HL structure theorem; deep GMT theorem.
- `harvey_lawson_represents` — HL representation statement.
- `omega_pow_represents_multiple` —  $\omega^p$  represented by algebraic cycle; classical AG.

### 3.2 Interface axioms (22 total)

These provide algebraic/smoothness/linearity properties for abstract APIs:

- `IsAlgebraicSet`, `IsAlgebraicSet_empty`, `IsAlgebraicSet_union`
- `FundamentalClassSet_isClosed`, `omega_pow_isClosed`, `isClosed_omegaPow_scaled`
- `wirtinger_comass_bound`, `calibration_inequality`
- `exists_volume_form_of_submodule_axiom`, `simpleCalibratedForm_is_smooth`
- `isIntegral_zero_current`
- `smoothExtDeriv_add`, `smoothExtDeriv_smul`
- `ofForm_sub`, `ofForm_smul_real`
- `isSmoothAlternating_*` (5 axioms)
- `instAddCommGroupDeRhamCohomologyClass`, `instModuleRealDeRhamCohomologyClass`

### 3.3 Lean foundations

- `Classical.choice` — Standard classical logic.
- `propext` — Propositional extensionality.
- `Quot.sound` — Quotient soundness (Lean core).

## 4 Recommended completion order

1. **Discharge P0 axioms** (strategy-critical) so the proof does not assume the core bridge.
2. **Discharge P1 axioms** if you want the analytic limit behavior internal to Lean.
3. **Optionally**, formalize the classical pillars (Hard Lefschetz, GAGA, Harvey–Lawson).