Stable Direction Dictionaries for Strongly Positive (p, p)-Forms via Regularized Simplex Fits

Jonathan Washburn
Recognition Science
Recognition Physics Institute
jon@recognitionphysics.org
Austin, Texas, USA

Abstract

Let (X, ω) be a compact Kähler manifold of complex dimension n, and let $K_p(x)$ denote the cone of strongly positive (p, p)-covectors at $x \in X$. A recurring obstruction in mesh-based constructions is the absence of a stable way to *recognize* and label directions in a cone-valued form field $\beta(x) \in K_p(x)$: Carathéodory decompositions are highly non-unique and vary discontinuously, preventing coherent direction labeling across adjacent cells.

We introduce a dictionary-based recognizer. Fix an ε-net $\{P_1, \ldots, P_M\}$ in the calibrated (complex) Grassmannian of $(n-p)$-planes, and let $\xi_i(x)$ be the associated normalized ray generators satisfying $\langle \xi_i(x), \psi_x \rangle = 1$ where $\psi = \omega^{n-p}/(n-p)!$ is the Kähler calibration. For each normalized target $b(x)$ we define a recognition state $w(x) \in \Delta_M$ by a strongly convex regularized least-squares fit on the simplex. We prove existence, uniqueness, and a Lipschitz bound

$$\|w(b) - w(b')\| \leq \big(\|A\|_{\mathrm{op}}/\lambda\big)\, \|b - b'\|$$

for the weight map, where $A : \mathbb{R}^M \to V$ is the dictionary synthesis operator $Aw = \sum_i w_i \xi_i$ and $\lambda > 0$ is the regularization strength. This yields stable, globally labeled coefficients that vary at the same scale as $b(x)$.

To connect with literal finite-resolution recognition, we consider derived finite event maps such as the winner-take-all label $\arg\max_i w_i$ and show (via a margin lemma) that these discrete labels are robust under perturbations away from ties. Finally, we show how pointwise weights induce coherent per-cell mass budgets $M_{Q,i}$ on a mesh, isolating direction-label stability as a quantitative choice of dictionary resolution ε and regularization λ, independent of later holomorphic or geometric-measure steps.

1 Recognition primitives

We record the recognition primitives used in this paper (state space, measurement map, quotient, and robustness modulus $r(s)$). The purpose is to make explicit what structure is assumed and what structure is produced.

Definition 1 (State space / configuration space)**.** A *state space* is a pair $(\mathcal{C}, d_{\mathcal{C}})$ where $\mathcal{C}$ is a set and $d_{\mathcal{C}}$ is a metric on $\mathcal{C}$. Elements $c \in \mathcal{C}$ are called *states* (or *configurations*).

Definition 2 (Measurement map / recognizer)**.** Let $(\mathcal{C}, d_{\mathcal{C}})$ be a state space and let $\mathcal{E}$ be a set (the *event space*). A *measurement map* (or *recognizer*) is a function

$$\mathcal{R} : \mathcal{C} \to \mathcal{E}.$$

Definition 3 (Indistinguishability and quotient)**.** Given $\mathcal{R} : \mathcal{C} \to \mathcal{E}$, define an equivalence relation on $\mathcal{C}$ by

$$c_1 \sim_{\mathcal{R}} c_2 \iff \mathcal{R}(c_1) = \mathcal{R}(c_2).$$

The *recognition quotient* is the quotient set $\mathcal{C}_{\mathcal{R}} := \mathcal{C}/\sim_{\mathcal{R}}$.

Definition 4 (Robustness modulus $r(s)$)**.** Assume $(\mathcal{E}, d_{\mathcal{E}})$ is a metric space. The *robustness modulus* of $\mathcal{R}$ is the function $r : [0, \infty) \to [0, \infty]$ defined by

$$r(s) \;:=\; \sup\Big\{ d_{\mathcal{E}}\big(\mathcal{R}(c), \mathcal{R}(c')\big) \;:\; d_{\mathcal{C}}(c, c') \le s \Big\}.$$

Remark 1 (Lipschitz recognizers)**.** If $\mathcal{R} : (\mathcal{C}, d_{\mathcal{C}}) \to (\mathcal{E}, d_{\mathcal{E}})$ is L-Lipschitz, then $r(s) \le Ls$ for all $s \ge 0$.

Assumptions and non-assumptions (no smuggled observables)

- **Given structure (assumptions).**
  - A compact Kähler manifold (X, ω) and the induced norms on forms.
  - The calibration functional $\alpha \mapsto \langle \alpha, \psi_x \rangle$ used to normalize $K_p(x)$ to $\Sigma_p(x)$.
  - A finite dictionary $\{\xi_i(x)\}_{i=1}^{M} \subset \Sigma_p(x)$ (a finite-resolution hypothesis).
  - A regularization strength $\lambda > 0$.
- **Not assumed (anti-smuggling constraints).**

- No canonical or continuous Carathéodory/extremal-ray decomposition is assumed.
- No globally uniform “true direction labels” are assumed; labels are produced as dictionary indices.
- No circular definition of observables: events/labels are defined by an explicit map $\mathcal{R}$, not by presupposing the quotient classes we intend to construct.

Self-similarity and the golden ratio (optional scale lemma)

In zero-parameter self-similar scale updates, one often encounters the fixed-point equation $x = 1 + 1/x$. We record the elementary consequence for reference.

Lemma 1 (Self-similarity fixed point forces φ)**.** *The equation $x = 1 + \frac{1}{x}$ has exactly one positive solution, namely $\varphi = (1 + \sqrt{5})/2$.*

Proof. Rearranging gives $x^2 = x + 1$, i.e. $x^2 - x - 1 = 0$. The quadratic formula yields the two roots $x = (1 \pm \sqrt{5})/2$, and exactly one is positive. $\square$

2 Introduction

Strongly positive (p, p)-forms on a Kähler manifold behave like “nonnegative densities with direction,” in the sense that they live in a closed convex cone whose extreme rays correspond to simple algebraic directions. In many constructions one wants to discretize a strongly positive form field $\beta(x)$ on a mesh and propagate direction-dependent budgets from cell to cell. The immediate obstacle is not existence of decompositions but *recognition*: a pointwise decomposition of $\beta(x)$ into extremal rays is not canonical. Even when decompositions exist, they are highly non-unique and can jump discontinuously as x varies. This makes direction labels unstable: adjacent mesh cells may use incompatible “names” for nearly the same direction.

Recognition Geometry suggests a measurement-first reframing: *without a recognizer, there is no stable labeling.* In our setting, at each x the *configuration space* is the normalized cone slice $\mathcal{C}_x := \Sigma_p(x)$, and we choose a finite dictionary $\{\xi_i(x)\}_{i=1}^M \subset \Sigma_p(x)$ as an explicit finite-resolution hypothesis. We then define a *recognizer* $\mathcal{R}_\lambda$ that maps a configuration $b \in \mathcal{C}_x$ to a recognition state $w = \mathcal{R}_\lambda(b) \in \Delta_M$ by a strongly convex simplex fit. Strong convexity forces uniqueness, and a monotonicity argument yields a clean Lipschitz robustness bound with an explicit constant.

To align literally with a finite-resolution event axiom, we also consider derived *finite* event maps, such as the winner-take-all label $\arg\max_i w_i$. These induce an indistinguishability relation $b \sim_{\mathcal{R}} b'$ on $\mathcal{C}_x$ and hence a recognition quotient $\mathcal{C}_x/\sim_{\mathcal{R}}$ whose classes are "direction-label resolution cells." The stability results in this paper quantify how large a perturbation in b is required to change the recognition state (and, away from ties, the discrete label).

The resulting mechanism has two independent stability knobs: dictionary resolution ε (approximation quality) and regularization λ (robustness of the recognizer).

3 Strong positivity and the normalized slice

Let (X, ω) be a compact Kähler manifold of complex dimension n. Fix $p \in \{0, 1, \dots, n\}$. At each $x \in X$, let

$$V_x := \Lambda^{p,p}_{\mathbb{R}} T^*_x X$$

denote the real vector space of real (p,p)-covectors at x.

Definition 5 (Strongly positive cone)**.** A covector $\alpha \in V_x$ is *strongly positive* if it is a finite sum of forms of the type

$$\left(\frac{i}{2}\right)^p \eta_1 \wedge \overline{\eta_1} \wedge \cdots \wedge \eta_p \wedge \overline{\eta_p},$$

where $\eta_1, \dots, \eta_p \in \Lambda^{1,0} T^*_x X$. The set of strongly positive covectors is a closed convex cone, denoted $K_p(x) \subset V_x$.

Define the Kähler calibration of degree $(n-p, n-p)$ by

$$\psi := \frac{\omega^{n-p}}{(n-p)!} \in \Lambda^{n-p,n-p}_{\mathbb{R}} T^* X.$$

Let $\mathrm{vol}_\omega := \omega^n/n!$.

Definition 6 (Pairing with ψ)**.** For $\alpha \in \Lambda^{p,p}_{\mathbb{R}} T^*_x X$ and $\eta \in \Lambda^{n-p,n-p}_{\mathbb{R}} T^*_x X$, define the scalar $\langle \alpha, \eta \rangle$ by

$$\alpha \wedge \eta \;=\; \langle \alpha, \eta \rangle \, \mathrm{vol}_\omega(x).$$

In particular, $\langle \alpha, \psi_x \rangle$ is the ψ-trace of α.

Lemma 2 (Positivity of the ψ-trace)**.** *For every $x \in X$ and every $\alpha \in K_p(x)$ one has $\langle \alpha, \psi_x \rangle \geq 0$. Moreover, if $\alpha \in K_p(x)$ and $\langle \alpha, \psi_x \rangle = 0$, then $\alpha = 0$.*

Proof. In unitary coordinates at x with $\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z_j}$, each elementary strongly positive term

$$\left(\frac{i}{2}\right)^p \eta_1 \wedge \overline{\eta_1} \wedge \cdots \wedge \eta_p \wedge \overline{\eta_p}$$

has nonnegative wedge with ω^{n-p}, hence nonnegative pairing with ψ_x. Summing preserves nonnegativity, giving $\langle \alpha, \psi_x \rangle \geq 0$.

If $\langle \alpha, \psi_x \rangle = 0$ for $\alpha \in K_p(x)$, then every elementary term in a positive decomposition must also have zero pairing with ψ_x. In unitary coordinates, the wedge of a nonzero elementary term with ω^{n-p} is strictly positive, so each term must be zero, hence $\alpha = 0$. $\square$

Definition 7 (Normalized slice)**.** Define the normalized slice

$$\Sigma_p(x) := \big\{ v \in K_p(x) : \langle v, \psi_x \rangle = 1 \big\}.$$

The set $\Sigma_p(x)$ is the natural compact base of the cone $K_p(x)$ once we fix ψ_x as a strictly positive functional on $K_p(x) \setminus \{0\}$.

Remark 2 (Configuration space viewpoint)**.** For fixed x, the normalized slice $\Sigma_p(x)$ is the space of normalized strongly positive directions. In later applications one often decomposes a form field $\beta(x) \in K_p(x)$ as $\beta(x) = \rho(x)\, b(x)$ where $\rho(x) = \langle \beta(x), \psi_x \rangle \geq 0$ is the total density and $b(x) \in \Sigma_p(x)$ is a normalized direction. The recognition problem addressed in this paper is to turn $b(x)$ into stable labels and weights.

4 Calibrated rays and normalized ray generators

For Kähler calibrations, calibrated $(2n-2p)$-planes are precisely complex $(n-p)$-planes. Let $G^{\mathbb{C}}_{n-p}(T_xX)$ denote the Grassmannian of complex $(n-p)$-dimensional subspaces of T_xX.

Definition 8 (Normalized ray generator associated to a complex plane)**.** Fix $x \in X$ and $P \in G^{\mathbb{C}}_{n-p}(T_xX)$. Choose a unitary frame $e_1, \ldots, e_n$ of $T^{1,0}_x X$ such that $P^{1,0} = \operatorname{span}\{e_{p+1}, \ldots, e_n\}$, and let $\zeta^1, \ldots, \zeta^n$ be the dual coframe. Define

$$\xi_P(x) := \left(\frac{i}{2}\right)^p \zeta^1 \wedge \overline{\zeta^1} \wedge \cdots \wedge \zeta^p \wedge \overline{\zeta^p} \ \in \Lambda^{p,p}_{\mathbb{R}} T^*_x X.$$

Lemma 3 (Well-definedness and normalization)**.** *The covector $\xi_P(x)$ is independent of the choice of unitary frame adapted to P. Moreover, $\xi_P(x) \in \Sigma_p(x)$, i.e. $\xi_P(x)$ is strongly positive and satisfies $\langle \xi_P(x), \psi_x \rangle = 1$.*

Proof. If two unitary frames are adapted to the same splitting $T_x^{1,0}X \cong \mathbb{C}^p \oplus \mathbb{C}^{n-p}$, they differ by an element of $U(p) \times U(n-p)$. The form

$$\left(\frac{i}{2}\right)^p \zeta^1 \wedge \overline{\zeta^1} \wedge \cdots \wedge \zeta^p \wedge \overline{\zeta^p}$$

is invariant under $U(p)$, hence well-defined.

Strong positivity is immediate from the construction. In unitary coordinates with $\omega = \frac{i}{2} \sum_j dz_j \wedge d\overline{z_j}$ and $P = \mathrm{span}\{\partial/\partial z_{p+1}, \ldots, \partial/\partial z_n\}$, one has

$$\xi_P(x) \wedge \frac{\omega^{n-p}}{(n-p)!} = \frac{\omega^n}{n!},$$

so $\langle \xi_P(x), \psi_x \rangle = 1$. □

Let

$$S_p(x) := \{\xi_P(x) : P \in G^{\mathbb{C}}_{n-p}(T_xX)\} \subset \Sigma_p(x).$$

The set $S_p(x)$ is compact because the Grassmannian is compact and $P \mapsto \xi_P(x)$ is continuous.

Proposition 1 (Convex generation of the normalized slice)**.** *For each $x \in X$, the normalized slice $\Sigma_p(x)$ is the convex hull of $S_p(x)$:*

$$\Sigma_p(x) = \mathrm{conv}\, S_p(x).$$

Equivalently, the cone $K_p(x)$ is the convex cone generated by $S_p(x)$.

Proof. Every $\xi_P(x)$ lies in $\Sigma_p(x)$, and $\Sigma_p(x)$ is convex, so $\mathrm{conv}\, S_p(x) \subseteq \Sigma_p(x)$.

For the reverse inclusion, take $v \in \Sigma_p(x)$. By strong positivity, v is a finite sum of elementary strongly positive forms. After scaling each summand by its ψ-trace and then renormalizing, we write

$$v = \sum_{j=1}^N a_j\, v_j, \qquad a_j \geq 0, \quad \sum_{j=1}^N a_j = 1,$$

where each v_j is a normalized elementary strongly positive form with $\langle v_j, \psi_x \rangle = 1$. Any such normalized elementary form is equal to $\xi_P(x)$ for some complex $(n-p)$-plane P (its nullspace in $T_x^{1,0}X$ determines P), hence $v_j \in S_p(x)$. Therefore $v \in \mathrm{conv}\, S_p(x)$. □

5 Regularized simplex recognizers

This section is pointwise and finite-dimensional, and establishes existence, uniqueness, and Lipschitz stability for the dictionary recognizer. Fix a real inner-product space $(V, \langle \cdot, \cdot \rangle_V)$; in applications $V = V_x$ with the norm induced by the Kähler metric.

Fix dictionary vectors $\xi_1, \ldots, \xi_M \in V$. Define the synthesis operator

$$A : \mathbb{R}^M \to V, \qquad Aw := \sum_{i=1}^{M} w_i \xi_i.$$

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^M$ and the induced operator norm $\|A\|_{\mathrm{op}}$.

Define the simplex

$$\Delta_M := \Big\{ w \in \mathbb{R}^M : w_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{M} w_i = 1 \Big\}.$$

Definition 9 (Regularized simplex recognizer and J-step)**.** Fix $\lambda > 0$. For a target $b \in V$, define the objective

$$J_b(w) := \frac{1}{2}\|Aw - b\|_V^2 + \frac{\lambda}{2}\|w\|^2, \qquad w \in \Delta_M,$$

and define the weight map

$$w(b) := \mathrm{argmin}_{w \in \Delta_M} J_b(w).$$

We also write $\mathcal{R}_\lambda(b) := w(b)$ and interpret $\mathcal{R}_\lambda : V \to \Delta_M$ as a *recognizer* whose output $w(b)$ is an internal recognition state in the simplex. We refer to $w(b)$ as the (unique) *J-step* selected by the recognition cost J_b.

Theorem 1 (Recognition closure lemma (RCL): uniqueness of the J-step)**.** *For every $b \in V$ and every $\lambda > 0$, the minimization problem defining $w(b)$ has a unique solution in Δ_M.*

Proof. The simplex Δ_M is nonempty, compact, and convex. The map $w \mapsto \frac{1}{2}\|Aw - b\|_V^2$ is convex, and $w \mapsto \frac{\lambda}{2}\|w\|^2$ is λ-strongly convex. Hence J_b is λ-strongly convex on $\mathbb{R}^M$, so it has at most one minimizer on any convex set. By continuity on a compact set, a minimizer exists, and by strong convexity it is unique. $\square$

A convenient characterization is via a variational inequality.

Lemma 4 (Variational inequality)**.** *A point $w_\star \in \Delta_M$ equals $w(b)$ if and only if*

$$\left\langle A^*(Aw_\star - b) + \lambda w_\star,\ z - w_\star \right\rangle \geq 0 \quad \textit{for all } z \in \Delta_M,$$

where $A^* : V \to \mathbb{R}^M$ is the adjoint of A and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^M$.

Proof. This is the standard first-order optimality condition for minimizing a differentiable convex function over a closed convex set. The gradient is $\nabla J_b(w) = A^*(Aw - b) + \lambda w$. □

Theorem 2 (Lipschitz stability in the target)**.** *For every $\lambda > 0$ and all $b, b' \in V$,*

$$\|w(b) - w(b')\| \leq \frac{\|A\|_{\mathrm{op}}}{\lambda} \|b - b'\|_V.$$

Proof. Let $w := w(b)$ and $w' := w(b')$. Apply the variational inequality with $z = w'$:

$$\left\langle A^*(Aw - b) + \lambda w,\ w' - w \right\rangle \geq 0.$$

Apply it again with (b', w') and $z = w$:

$$\left\langle A^*(Aw' - b') + \lambda w',\ w - w' \right\rangle \geq 0,$$

which is equivalent to

$$\left\langle A^*(Aw' - b') + \lambda w',\ w' - w \right\rangle \leq 0.$$

Adding the two inequalities gives

$$\left\langle A^*A(w - w') + \lambda(w - w') - A^*(b - b'),\ w' - w \right\rangle \geq 0.$$

Let $d := w - w'$. Then $w' - w = -d$, and we obtain

$$-\|Ad\|_V^2 - \lambda\|d\|^2 + \langle A^*(b - b'), d\rangle \geq 0.$$

Dropping the nonpositive term $-\|Ad\|_V^2$ yields

$$\lambda\|d\|^2 \leq \langle A^*(b - b'), d\rangle \leq \|A^*(b - b')\| \, \|d\|.$$

If $d = 0$ there is nothing to prove. Otherwise divide by $\lambda\|d\|$ to get

$$\|d\| \leq \frac{\|A^*(b - b')\|}{\lambda} \leq \frac{\|A^*\|_{\mathrm{op}}}{\lambda} \|b - b'\|_V.$$

Since $\|A^*\|_{\mathrm{op}} = \|A\|_{\mathrm{op}}$, the claim follows. □

Corollary 1 (Robustness modulus for $\mathcal{R}_\lambda$)**.** *Equip V with the metric induced by $\|\cdot\|_V$ and equip $\Delta_M \subset \mathbb{R}^M$ with the Euclidean norm. Then the robustness modulus $r(s)$ of $\mathcal{R}_\lambda : V \to \Delta_M$ satisfies*

$$r(s) \;\le\; \frac{\|A\|_{\mathrm{op}}}{\lambda}\, s \qquad \text{for all } s \ge 0.$$

Proof. This is immediate from the Lipschitz theorem and the definition of $r(s)$. □

Finite-resolution events and recognition quotients

The simplex state $w(b) \in \Delta_M$ varies continuously with b (indeed, Lipschitz), but Δ_M is not finite. If one wants a literal finite event space, one can post-process $w(b)$ by a finite rule.

Definition 10 (Winner-take-all event map)**.** Let $\mathcal{E} := \{1, \dots, M\}$. Define the event map $E_\lambda : V \to \mathcal{E}$ by

$$E_\lambda(b) := \min\Big\{ i : w_i(b) = \max_{1 \le j \le M} w_j(b) \Big\}.$$

Thus $E_\lambda(b)$ is a deterministic tie-broken version of $\arg\max_i w_i(b)$.

As above, E_λ induces an indistinguishability relation $b \sim_{\mathcal{R}} b'$ on any $\mathcal{C} \subseteq V$ (e.g. $\mathcal{C} = \Sigma_p(x)$) by the rule $E_\lambda(b) = E_\lambda(b')$, and hence a recognition quotient $\mathcal{C}/\sim_{\mathcal{R}}$.

Lemma 5 (Margin implies stable discrete recognition)**.** *Let $b \in V$ and let $i_\star := E_\lambda(b)$. Suppose there exists $\delta > 0$ such that*

$$w_{i_\star}(b) \;\ge\; w_j(b) + 2\delta \qquad \text{for all } j \ne i_\star.$$

Then for every $b' \in V$ with $\|b - b'\|_V \le (\lambda\,\delta)/\|A\|_{\mathrm{op}}$, one has $E_\lambda(b') = i_\star$.

Proof. By the Lipschitz theorem, $\|w(b) - w(b')\| \le (\|A\|_{\mathrm{op}}/\lambda)\|b - b'\|_V \le \delta$. In particular, for each coordinate one has $|w_j(b) - w_j(b')| \le \|w(b) - w(b')\| \le \delta$. Hence for all $j \ne i_\star$,

$$w_{i_\star}(b') \ge w_{i_\star}(b) - \delta \ge w_j(b) + \delta \ge w_j(b').$$

So $i_\star$ remains the unique maximizer, and the tie-broken $\arg\max$ is unchanged: $E_\lambda(b') = i_\star$. □

Remark 3 (Quantized simplex events)**.** An alternative finite-resolution bridge is to quantize the simplex output itself. For example, fix a finite set $W \subset \Delta_M$ (a τ-net in the Euclidean norm) and define a quantizer $Q : \Delta_M \to W$ by selecting a nearest element of W (with deterministic tie-breaking). Then $Q \circ \mathcal{R}_\lambda : V \to W$ is a finite event-valued recognizer, and the Lipschitz bound controls how rapidly the internal state can move relative to the quantization scale. As with winner-take-all, discrete stability is governed by separation from the quantizer's decision boundaries.

Remark 4 (Interpretation of the constant)**.** The parameter λ is a stability knob. Larger λ improves Lipschitz stability but biases weights toward smaller Euclidean norm (more "spread" in many common geometries). Smaller λ reduces bias but amplifies sensitivity. The operator norm $\|A\|_{\mathrm{op}}$ measures how large a change in the synthesized form Aw can be produced by a unit change in weights.

Remark 5 (Fieldwise versions)**.** In geometric applications V and the dictionary vary with x. After choosing local trivializations, the pointwise Lipschitz estimate combines with standard stability estimates for compositions to yield fieldwise regularity statements on charts; we omit these routine details.

6 Approximation error versus dictionary resolution

The Lipschitz theorem controls *stability* of weights given a fixed dictionary. Separately, one also wants the dictionary itself to approximate the continuous set of calibrated ray generators.

We first record a basic convex-hull approximation lemma.

Lemma 6 (Convex-hull approximation from an ε-net)**.** *Let $(V, \|\cdot\|_V)$ be a normed vector space and let $S \subset V$ be compact. Suppose $\Xi = \{\xi_1, \ldots, \xi_M\} \subset S$ is an ε-net for S, meaning that for every $s \in S$ there exists $\xi_i \in \Xi$ with $\|s - \xi_i\|_V \le \varepsilon$. Then for every $b \in \operatorname{conv} S$,*

$$\operatorname{dist}\bigl(b,\ \operatorname{conv}\Xi\bigr) \le \varepsilon.$$

Proof. Write $b = \sum_{j=1}^{N} a_j s_j$ with $s_j \in S$, $a_j \ge 0$, and $\sum_j a_j = 1$. For each j, choose $\xi_{i(j)} \in \Xi$ with $\|s_j - \xi_{i(j)}\|_V \le \varepsilon$. Define $b' := \sum_j a_j \xi_{i(j)} \in \operatorname{conv}\Xi$. Then

$$\|b - b'\|_V \le \sum_{j=1}^{N} a_j \|s_j - \xi_{i(j)}\|_V \le \sum_{j=1}^{N} a_j \varepsilon = \varepsilon.$$

Hence $\operatorname{dist}(b, \operatorname{conv} \Xi) \le \varepsilon$. $\square$

Now specialize to the Kähler setting at a point x. Recall $S_p(x) = \{\xi_P(x)\}$ and $\Sigma_p(x) = \operatorname{conv} S_p(x)$.

Corollary 2 (Pointwise approximation on $\Sigma_p(x)$)**.** *Fix $x \in X$. If $\Xi(x) = \{\xi_1(x), \dots, \xi_M(x)\} \subset S_p(x)$ is an ε-net for $S_p(x)$ (in the norm on V_x), then every $b \in \Sigma_p(x)$ satisfies*

$$\operatorname{dist}\big(b,\ \operatorname{conv} \Xi(x)\big) \le \varepsilon.$$

Remark 6 (From plane-nets to ξ-nets)**.** On the compact Grassmannian $G^{\mathbb{C}}_{n-p}(T_x X)$, the map $P \mapsto \xi_P(x)$ is smooth, hence Lipschitz with some constant $C = C(n,p)$ once one fixes the standard Grassmann metric induced by ω_x. Therefore an ε-net in plane-angle distance induces a $C\varepsilon$-net in $\|\cdot\|_{V_x}$-distance among ray generators. The constant C is uniform when (X, ω) has bounded geometry on the region of interest.

Remark 7 (Choosing ε relative to a mesh size h)**.** In mesh-based constructions on cells of diameter h, one often needs dictionary approximation errors that are negligible compared to h-scale variations of the target field. A typical quantitative regime is to choose a resolution ε_h satisfying $\varepsilon_h = o(h)$ as $h \to 0$. This ensures that dictionary-induced approximation errors do not dominate the geometric errors that scale linearly with the mesh diameter.

7 From pointwise weights to per-cell budgets on a mesh

Let $\beta \in \Omega^{p,p}(X)$ be a continuous strongly positive form field, meaning $\beta(x) \in K_p(x)$ for all x. Define the nonnegative density

$$\rho(x) := \langle \beta(x), \psi_x \rangle.$$

By the earlier lemma, $\rho(x) = 0$ implies $\beta(x) = 0$.

On the set where $\rho(x) > 0$, define the normalized field

$$b(x) := \frac{\beta(x)}{\rho(x)} \in \Sigma_p(x).$$

Fix a dictionary field $\{\xi_i(x)\}_{i=1}^M$ with $\xi_i(x) \in \Sigma_p(x)$ and a regularization parameter $\lambda > 0$. Define weights

$$w(x) := w_x(b(x)) \in \Delta_M.$$

The associated dictionary reconstruction is the (p, p)-form

$$\beta_{\text{dict}}(x) := \rho(x) \sum_{i=1}^{M} w_i(x)\, \xi_i(x).$$

By construction $\beta_{\text{dict}}(x)$ uses globally labeled directions indexed by i.

Now let $U \subset X$ be a chart in which we place a cubical mesh of side length $h > 0$. For a mesh cell (cube) $Q \subset U$, define the per-cell mass budget assigned to label i by

$$M_{Q,i} := \int_Q w_i(x)\, \rho(x)\, dV_\omega(x),$$

where dV_ω is the Riemannian volume measure of the Kähler metric.

Lemma 7 (Basic identities)**.** *For every cell Q,*

$$\sum_{i=1}^{M} M_{Q,i} = \int_Q \rho(x)\, dV_\omega(x).$$

Proof. Since $w(x) \in \Delta_M$, one has $\sum_i w_i(x) = 1$ pointwise, hence

$$\sum_{i=1}^{M} M_{Q,i} = \int_Q \Big(\sum_{i=1}^{M} w_i(x)\Big) \rho(x)\, dV_\omega(x) = \int_Q \rho(x)\, dV_\omega(x).$$

□

Proposition 2 (Slow variation of budgets across adjacent cells)**.** *Assume $w_i\rho$ is Lipschitz on U with Lipschitz constant L_i (in the chart metric), and assume mesh cubes have side length h. If Q and Q' are adjacent cubes (sharing a codimension-one face), then*

$$|M_{Q,i} - M_{Q',i}| \;\le\; C(n)\, L_i\, h\, \mathrm{Vol}_\omega(Q),$$

where $\mathrm{Vol}_\omega(Q) = \int_Q dV_\omega$ and $C(n)$ depends only on the real dimension $2n$.

Proof. Let $f_i(x) := w_i(x)\rho(x)$. For adjacent cubes Q, Q' of the same size, the difference of integrals of a Lipschitz function is controlled by the oscillation of f_i on $Q \cup Q'$. One convenient estimate is

$$\Big|\int_Q f_i\, dV_\omega - \int_{Q'} f_i\, dV_\omega\Big| \le \int_{Q \cup Q'} |f_i(x) - f_i(x_0)|\, dV_\omega(x)$$

for a suitably chosen reference point x_0 between the cubes. Since every point in $Q \cup Q'$ lies within $O(h)$ of x_0, Lipschitz continuity gives $|f_i(x) - f_i(x_0)| \leq C(n)L_i h$ throughout $Q \cup Q'$, hence

$$|M_{Q,i} - M_{Q',i}| \leq C(n)L_i h \,\mathrm{Vol}_\omega(Q \cup Q') \leq 2C(n)L_i h \,\mathrm{Vol}_\omega(Q).$$

Absorb the factor 2 into $C(n)$. □

Remark 8 (Relative form of the estimate)**.** Since $\mathrm{Vol}_\omega(Q) \sim h^{2n}$, the bound above is $O(h^{2n+1})$ in absolute terms, which is $O(h)$ relative to the typical cell mass scale. This is the precise sense in which budgets vary slowly across neighbors when the underlying field is Lipschitz.

8 Interface assumptions for later stages

This paper is designed to output only two types of information for downstream geometric constructions.

First, it outputs a fixed finite label set $\{1, \ldots, M\}$ (the dictionary indices) together with a stable recognizer output $w(x) \in \Delta_M$ producing globally labeled coefficients. If one wants a literal finite-resolution recognizer, one may also pass forward the derived event map $E_\lambda(x) := E_\lambda(b(x)) \in \{1, \ldots, M\}$; the margin lemma above quantifies when this discrete output is robust.

Second, it outputs mesh-level budgets $M_{Q,i}$ whose neighbor-to-neighbor variation can be controlled quantitatively in terms of regularity of the underlying form field and the choice of λ and dictionary resolution.

No holomorphic input is needed for these steps. In particular, one can treat the entire discussion above as a purely pointwise and mesh-level mechanism for turning a cone-valued field into stable labeled scalar densities.

9 Examples and variants

Example 1 (The case $p = 1$: positive semidefinite Hermitian matrices)**.** At a point x, a real $(1,1)$-form corresponds (in unitary coordinates) to a Hermitian matrix. Strong positivity is positive semidefiniteness. The normalized slice $\Sigma_1(x)$ is the set of positive semidefinite matrices with fixed trace against ω^{n-1}. The calibrated rays are rank-one projectors, and a dictionary is a finite family of such projectors. The regularized simplex fit is then a stable way to express a positive semidefinite matrix as a convex combination of nearby rank-one directions with globally fixed labels.

Remark 9 (Alternative regularizers)**.** The quadratic regularizer $\frac{\lambda}{2}\|w\|^2$ is chosen for two reasons: it makes the objective strongly convex and it produces a clean monotonicity argument giving the explicit Lipschitz constant. Other regularizers can be used, for example an entropic term $\sum_i w_i \log w_i$ to enforce strictly positive weights in the simplex interior. Such choices typically yield smoother dependence on b but change the stability constant and the structure of optimality conditions.

Remark 10 (Implementation-level stability)**.** The optimization problem is a strongly convex quadratic program over the simplex. Strong convexity implies not only a unique solution but also good numerical conditioning as λ increases. In applications where M is moderate, projected gradient methods or active-set methods are standard. The theoretical Lipschitz bound is useful even if one never computes w exactly: it quantifies how much weight noise can be induced by target noise.

Conclusion

By fixing a finite dictionary of normalized calibrated ray generators and selecting weights through a strongly convex simplex fit, we obtain a unique, stable, and Lipschitz *recognizer* for strongly positive (p, p)-forms. The recognition state $w(b)$ provides globally labeled coefficients, and derived finite event maps (such as $E_\lambda = \arg\max$ with deterministic tie-breaking) produce literal finite-resolution direction labels with explicit robustness margins. The induced labeled budgets on a mesh inherit slow-variation properties from the regularity of the underlying form field, with constants controlled by dictionary size/resolution and the regularization parameter λ. This isolates the recognition/labeling problem from subsequent geometric or holomorphic realization steps and provides a modular input for larger constructions.