Technical Memo:
Does the Proof Imply the (False) Positive Hodge Conjecture?

Internal Review

December 27, 2025

Executive Summary

Short answer: The proof does *not* imply the positive Hodge conjecture in the sense of Demailly–Voisin. The construction is sound for its stated hypotheses, but the manuscript would benefit from an explicit remark clarifying this distinction.

Risk level: [redacted] (requires clarifying exposition, not a gap in logic).

---

1 The Concern

A referee raised the following objection regarding Proposition 8.108 (the Microstructure/Gluing Estimate, `prop:glue-gap`) and its application to Harvey–Lawson:

Concern (Paraphrased). "Approximating a non-integrable plane field with integrable holomorphic sheets incurs a systematic 'filling mass' penalty proportional to the field's curvature. If this mass does not vanish in the limit, the resulting current is not calibrated, which invalidates the Harvey–Lawson theorem. Check whether this implies the (known-false) positive Hodge conjecture."

The "positive Hodge conjecture" referred to is the statement (known to be **false** in intermediate codimension by work of Voisin and others) that every *limit of positive classes* is algebraic. The gap between differential positivity and algebraic positivity is well-documented.

2 Key Propositions Under Scrutiny

The chain of logic is:

1. **Proposition 8.108** (`prop:glue-gap`): Given the raw current $T^{\text{raw}} = \sum_Q S_Q$, if face mismatches admit a translation model with displacement $\Delta_F \lesssim h^2$, then

$$\mathcal{F}(\partial T^{\text{raw}}) \leq \text{Mass}(U_h) = o(m).$$

2. **Proposition 8.114** (`prop:almost-calibration`): If $\mathrm{Mass}(U_\epsilon) \to 0$, then the corrected cycles $T_\epsilon = S - U_\epsilon$ satisfy

$$\mathrm{Def}_{\mathrm{cal}}(T_\epsilon) := \mathrm{Mass}(T_\epsilon) - \int_{T_\epsilon} \psi \longrightarrow 0.$$

3. **Theorem 8.116** (`thm:syr-realization` / `thm:realization-from-almost`): A sequence with vanishing calibration defect converges to a ψ-calibrated integral current, which Harvey–Lawson identifies as a positive sum of complex analytic subvarieties.

The concern is: *Can the filling mass* $\mathrm{Mass}(U_h)$ *actually be forced to zero, or is there a geometric obstruction from non-integrability?*

3 Analysis of the Argument

3.1 What the manuscript claims

The manuscript's parameter schedule (Section 8, lines 2144–2162) specifies:

1. Fix m first (to clear denominators in the integral lattice).
2. Send mesh size $h_j \downarrow 0$.
3. The local parameters $\varepsilon_j, \delta_j \to 0$ as functions of h_j.

The key scaling estimate (Remark 8.110, `rem:weighted-scaling`) gives, at mesh scale h:

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim m^{\frac{k-1}{k}} h^{2-\frac{2n}{k}} \varepsilon^{-\frac{2p}{k}}, \qquad k := 2n - 2p.$$

For **fixed** m and ε, as $h \to 0$:

- When $p < n/2$: exponent $2 - 2n/k > 0$, so estimate $\to 0$. ✓
- When $p = n/2$: exponent $= 0$ (borderline). Handled by Lemma 8.164 via slow-variation. ✓
- When $p > n/2$: reduced to $p \le n/2$ by Hard Lefschetz (Remark 8.170). ✓

3.2 Why non-integrability does not obstruct

Remark 8.116 (`rem:gluing`) directly addresses this:

> **Objection:** "The plane field $x \mapsto \beta(x)$ is generically non-integrable. Local sheets cannot be glued without accumulating mass."
>
> **Response:** This conflates (a) integrating a plane field into a foliation, with (b) building many separate calibrated sheets. The construction does (b), not (a).

The crucial points are:

1. We **never integrate** the cone-valued form β. It is only a "design target" for local Carathéodory decompositions.

2. Each cell Q uses a **finite dictionary** of calibrated directions (from a ε-net on the calibrated Grassmannian). The holomorphic sheets are algebraic complete intersections—their existence is guaranteed by Bertini, independent of β's integrability.

3. The **gluing mismatch** on faces $F = Q \cap Q'$ is bounded by displacement $\times$ slice mass. The displacement is $O(h^2)$ when adjacent cells use **the same translation template** (corner-exit coherence).

4. The non-integrability of β affects *which directions* appear in the Carathéodory decomposition, but **not the approximation error**, which depends only on mesh size.

3.3 Why this is NOT the positive Hodge conjecture

The positive Hodge conjecture (false) would assert:

> Every class that is a *limit* of positive (1,1)-classes (or more generally, is represented by a positive current) is algebraic.

The manuscript's claim is different:

> Every **rational Hodge class** $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ is algebraic.

The distinction:

- The manuscript starts with a class that *becomes cone-positive* after adding $N[\omega^p]$ for $N \gg 1$. This is guaranteed for any Hodge class by the signed decomposition lemma.
- A cone-positive class **by definition** admits a smooth closed cone-valued representative β. This is *stronger* than being a limit of positive classes.
- The Demailly–Voisin counterexamples concern classes that are limits of positive classes but do **not** themselves admit smooth positive representatives. Such classes are not cone-positive in the manuscript's sense.

4 Potential Vulnerabilities

Observation 1 (Where the argument could fail—but doesn't)**.**

1. ***Hidden m-dependence:*** *If the estimate required $m \to \infty$ along with $h \to 0$, the fixed-class hypothesis would fail.*

   Status: The parameter schedule explicitly fixes m first. The scaling exponents are negative in h for $p \leq n/2$, independent of m.

2. ***Curvature-dependent lower bound:*** *If there were a geometric lower bound* $\mathrm{Mass}(U_h) \geq C(\kappa_\beta) > 0$ *depending on the Frobenius curvature κ_β of the plane field, the argument would fail.*

Status: No such bound exists because the construction does not integrate the plane field. The filling is produced by Federer–Fleming isoperimetry, which depends only on the flat norm of $\partial T^{\mathrm{raw}}$, not on β's integrability.

3. ***Cohomology mismatch:*** *The discrepancy rounding (Proposition 8.113) requires approximation errors $< 1/2$.*

   Status: The filling mass $\mathrm{Mass}(U_h) \to 0$ makes the pairing error with integral classes arbitrarily small.

5 Recommendations

1. **Add an explicit remark** distinguishing the manuscript's claim from the positive Hodge conjecture. Suggested location: after Theorem **??** or in the introduction.

2. **Clarify in Remark 8.116** (`rem:gluing`) that non-integrability affects direction *selection* but not the *approximation error bounds*, which are purely metric/mesh-dependent.

3. **Consider adding a subsection** titled "Relation to the Positive Hodge Conjecture" that:

   - States the (false) positive Hodge conjecture precisely.
   - Explains why cone-positivity (smooth closed cone-valued representative) is strictly stronger than differential positivity (positive current representative).
   - Notes that the signed decomposition guarantees cone-positivity for $\gamma + N[\omega^p]$.

6 Conclusion

Risk Assessment. The reviewer's concern identifies a place where the exposition could be clearer, not a logical gap. The construction does not imply the positive Hodge conjecture because:

1. It applies to **cone-positive** classes (smooth closed cone-valued representatives), not arbitrary limits of positive classes.

2. The filling mass estimate depends on **mesh geometry**, not on the curvature/integrability of the target form.

3. The parameter schedule fixes m first and sends $h \to 0$, so no secret blow-up occurs.

Recommended action: Add 1–2 clarifying remarks; no structural changes needed.