Proof Comparison and Validity Analysis: Calibration–Coercivity Approaches to the Hodge Conjecture

Analysis of `hodg-dec-6d.tex` vs. Earlier Approaches

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Abstract

This document compares two approaches to proving the Hodge Conjecture via calibration–coercivity methods: the *original conditional approach* and the *new signed decomposition approach*. We identify the critical flaw in the original approach, explain how the new approach remedies it, and assess the validity of the resulting proof. The two files `hodg-dec-6d.tex` and `hodge-dec-6c.tex` are *textually identical*—both contain the new approach. The "original approach" is documented in auxiliary files (`hodge-blocker.tex`, `change-report.tex`).

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1 Executive Summary

1.1 File Comparison Result

A direct comparison of `hodg-dec-6d.tex` and `hodge-dec-6c.tex` reveals that **these files are identical**. Both contain the same proof using the signed decomposition approach. The version numbering (6c vs. 6d) likely reflects incremental saves of the same document.

The meaningful comparison is between:

- **Original approach:** Documented in `hodge-blocker.tex` and the "before" sections of `change-report.tex`
- **New approach:** Contained in both `hodg-dec-6d.tex` and `hodge-dec-6c.tex`

1.2 Key Finding

The original approach was **conditional** on an unproven (and often false) assumption. The new approach is designed to be **unconditional** by employing signed decomposition. However, critical analysis reveals that the proof still has unresolved dependencies that affect its validity.

2 The Original Approach (Conditional)

2.1 Core Strategy

The original proof attempted the following:

1. Let $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be a rational Hodge class.
2. Let γ_{harm} be its unique ω-harmonic representative.
3. **Assume:** $\gamma_{\text{harm}}(x) \in K_p(x)$ for all $x \in X$ (the harmonic representative lies in the calibrated cone pointwise).
4. Under this assumption, prove calibration–coercivity:
$$E(\alpha) - E(\gamma_{\text{harm}}) \;\geq\; c\,\mathrm{Def}_{\text{cone}}(\alpha).$$
5. Use energy minimization to force convergence to calibrated currents.
6. Conclude via Harvey–Lawson that the class is algebraic.

2.2 The Critical Assumption

The key step was:

> *"Since $\gamma_{\text{harm}}(x) \in K_p(x)$ for all x, the cone distance satisfies* $\mathrm{dist}_{\text{cone}}(\alpha_x) \leq \|\alpha_x - \gamma_{\text{harm},x}\|$*."*

This inequality requires the harmonic representative to lie *inside* the calibrated cone at every point, serving as a reference point for distance estimates.

2.3 Why the Assumption Fails

Proposition 2.1 (Failure of the cone-valued harmonic assumption)**.** *There exist rational Hodge classes γ on smooth projective varieties whose harmonic representatives γ_{harm} do **not** lie in the calibrated cone $K_p(x)$ at any point $x \in X$.*

Counterexample. Consider $X = S_1 \times S_2$, a product of two smooth projective surfaces, with Kähler form $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$. Let $p = 2$ and consider the $(2,2)$-class:

$$\gamma \;:=\; [\pi_1^*\omega_1^2] - [\pi_2^*\omega_2^2].$$

This is a rational Hodge class. Its harmonic representative is:

$$\gamma_{\mathrm{harm}} \;=\; \pi_1^*\omega_1^2 - \pi_2^*\omega_2^2.$$

In the Hermitian model at any point $x = (x_1, x_2) \in X$, this corresponds to a matrix with both positive and negative eigenvalues—it has *indefinite signature* everywhere. Thus:

$$\gamma_{\mathrm{harm}}(x) \;\notin\; K_2(x) \quad \text{for all } x \in X.$$

The calibrated cone $K_2(x)$ consists of positive semi-definite matrices (weakly positive forms), so a form with indefinite signature cannot belong to it. □

2.4 Consequence

The original proof was **conditional** on the assumption $\gamma_{\mathrm{harm}}(x) \in K_p(x)$, which:

- Is true for some special classes (e.g., effective classes).
- Is **false** for general Hodge classes, including difference classes.
- Was never proven in the original manuscript.

Therefore, the original approach **did not constitute a proof** of the Hodge Conjecture.

3 The New Approach (Signed Decomposition)

3.1 Core Innovation

The new approach in `hodg-dec-6d.tex` bypasses the problematic assumption entirely through the following insight:

> *We do not need the harmonic representative to be cone-valued. We only need that every Hodge class is a **difference** of two classes that admit cone-valued representatives.*

3.2 The Signed Decomposition Lemma

Lemma 3.1 (Signed Decomposition—Lemma 8.7 in the manuscript)**.** *Let $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be any rational Hodge class. Then there exist* effective *classes γ^+ and γ^- such that:*

$$\gamma \;=\; \gamma^+ - \gamma^-.$$

Moreover:

1. $\gamma^- := N[\omega^p]$ *for some rational* $N > 0$.

2. $\gamma^+ := \gamma + N[\omega^p]$ *admits a cone-valued representative.*

3. Both γ^+ *and* γ^- *are rational Hodge classes.*

Proof sketch. Let α be any smooth closed (p,p)-form representing γ. In the Hermitian model, $\alpha(x)$ corresponds to a Hermitian matrix $A(x)$ with (possibly negative) minimum eigenvalue $\lambda_{\min}(A(x))$.

The Kähler power $\omega^p(x)$ corresponds to a strictly positive definite matrix $W(x)$ with $\lambda_{\min}(W(x)) \geq c_0 > 0$ uniformly (by compactness of X).

Choose $N > \sup_x |\lambda_{\min}(A(x))|/c_0$. Then:

$$\lambda_{\min}(A(x) + N \cdot W(x)) \ \geq \ \lambda_{\min}(A(x)) + Nc_0 \ > \ 0.$$

Thus $\alpha + N\omega^p$ is strictly positive (cone-valued), and:

$$\gamma^+ := \gamma + N[\omega^p], \qquad \gamma^- := N[\omega^p].$$

□

3.3 Why γ^- is Automatically Algebraic

Lemma 3.2 (Lemma 8.8 in the manuscript)**.** *The class* $[\omega^p] = H^p$ *(where* H *is the hyperplane class) is algebraic, represented by a complete intersection of* p *generic hyperplane sections.*

Proof. By Bertini's theorem, generic hyperplanes $H_1, \ldots, H_p$ intersect X in a smooth subvariety $Z = X \cap H_1 \cap \cdots \cap H_p$ of codimension p. Its Poincaré dual is $H^p = [\omega^p]$. □

3.4 New Proof Structure

The new proof proceeds as:

1. **Signed decomposition:** Write $\gamma = \gamma^+ - \gamma^-$.
2. γ^- **is algebraic:** It equals $N[\omega^p]$, represented by complete intersections.
3. γ^+ **is effective:** It admits a cone-valued representative $\beta = \alpha + N\omega^p$ with $\beta(x) \in K_p(x)$ for all x.
4. **Effective ⇒ Algebraic:** For effective classes, the calibration–coercivity machinery produces calibrated currents via:
   - Projective tangential approximation (Lemma 8.4)
   - Automatic SYR (Theorem 8.6)
   - Harvey–Lawson structure theory
   - Chow's theorem
5. **Conclusion:** $\gamma = [Z^+] - [Z^-]$ is algebraic.

4 How the New Approach Remedies the Original Flaw

4.1 The Key Insight

Original Approach	**New Approach**
Required: $\gamma_{\text{harm}}(x) \in K_p(x)$ for all x	No assumption on γ_{harm}
Applied to: All Hodge classes directly	Applied to: Only *effective* classes
Failed for: Difference classes with indefinite signature	Handles differences by decomposition
Cone-valued form: The harmonic representative	Cone-valued form: Constructed $\alpha + N\omega^p$

4.2 Detailed Comparison

4.2.1 What Changed in the Coercivity Proof

Original Step 3:

> "Since $\gamma_{\text{harm}}(x) \in K_p(x)$ for all x, the cone distance satisfies $\text{dist}_{\text{cone}}(\alpha_x) \leq \|\alpha_x - \gamma_{\text{harm},x}\|$."

Revised Step 3:

> "By Proposition 6.6 (pointwise cone projection bound),
>
> $$\text{dist}_{\text{cone}}(\alpha_x)^2 \;\leq\; |\alpha_x^{(p+1,p-1)}|^2 + |\alpha_x^{(p-1,p+1)}|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2 + d\,\mu(x)^2.$$
>
> Integrating and using established estimates yields the bound unconditionally."

The revised version uses the Hermitian/PSD-cone structure directly, without requiring γ_{harm} to be in the cone.

4.2.2 Why the Decomposition Works

The signed decomposition sidesteps the fundamental obstruction because:

- We do **not** claim that γ has a cone-valued representative.
- We do **not** claim that γ_{harm} is cone-valued.
- We **only** claim that $\gamma^+ = \gamma + N[\omega^p]$ has a cone-valued representative (namely $\alpha + N\omega^p$).
- This is trivially true: adding a sufficiently large positive form makes any form positive.

5 Validity Assessment of the New Proof

5.1 What is Established

The following components of the proof are mathematically rigorous:

1. **Signed decomposition (Lemma 8.7): Valid.** The construction is elementary linear algebra in the Hermitian model.
2. **γ^- is algebraic (Lemma 8.8): Valid.** This follows from Bertini's theorem and is classical.
3. **Calibration–coercivity inequality (Theorem 7.1): Valid for effective classes.** The pointwise linear algebra and integration arguments are sound.
4. **Projective tangential approximation (Lemma 8.4): Valid.** Uses Bertini's theorem and projective automorphisms—standard tools.
5. **Realization from almost-calibrated sequences (Theorem 8.1): Valid.** This is a direct application of Federer–Fleming compactness and Harvey–Lawson structure theory.

5.2 Critical Dependencies

The proof's validity hinges on two key claims:

5.2.1 Automatic SYR (Theorem 8.6)

> *"Every smooth cone-valued (p, p) form β representing a rational Hodge class satisfies the Stationary Young-measure Realizability property."*

Assessment: The proof sketch appeals to:

- Carathéodory decomposition on each cube
- Dense family of calibrated submanifolds from Proposition 8.5
- "Boundary correction and varifold compactness arguments identical to Theorem 8.3"

[Warning:] This is the most delicate step. The argument requires:

1. Constructing rectifiable currents S_Q with prescribed barycentric weights
2. Controlling boundary masses across cube interfaces
3. Ensuring the homology class is exactly correct (not just approximately)
4. Varifold compactness producing the right Young-measure convergence

The manuscript provides a proof sketch for the LICD case (Theorem 8.3) but claims Theorem 8.6 follows "identically." This equivalence needs more detailed justification, as the general cone-valued case lacks the integrability structure of LICD.

5.2.2 Effective Classes are Algebraic (Theorem 8.7)

This theorem relies on Automatic SYR (Theorem 8.6), so its validity is contingent on that result.

5.3 Potential Gaps

1. **SYR construction details:** The passage from "dense family of calibrated directions" to "rectifiable currents with exact homology class" involves non-trivial geometric measure theory. The Federer–Fleming deformation theorem handles boundary filling, but ensuring exact (not approximate) cohomology requires careful attention to lattice quantization.

2. **Young-measure convergence:** Varifold compactness gives subsequential convergence, but the claim that tangent-plane Young measures converge to a field with barycenter $\beta(x)$ "almost everywhere" needs the weak-$*$ limit to be supported on calibrated planes. This follows if the approximating currents are exactly calibrated, which they are by construction.

3. **Stationarity:** The SYR definition requires the sequence $\{T_k\}$ to be "stationary," but the construction produces ψ-calibrated cycles, which are automatically stationary (zero first variation). This is fine.

5.4 Overall Verdict

Verdict: The proof is **substantially more complete** than the original conditional version. The signed decomposition strategy is sound and eliminates the false assumption about harmonic representatives.
The remaining concern is the **Automatic SYR theorem** (Theorem 8.6), which is stated with a brief proof appealing to techniques from the LICD case. A fully rigorous treatment would require more detailed verification that the lamination/filling construction works for arbitrary cone-valued forms, not just those satisfying LICD.
Conditional status: The proof is unconditional **if** Theorem 8.6 (Automatic SYR) holds. The manuscript's argument for Theorem 8.6 is plausible but abbreviated.

6 Summary: What Changed and Why

6.1 The Fundamental Shift

Original Approach	New Approach
Prove: Harmonic representative is cone-valued	Accept: Harmonic representative may not be cone-valued
Strategy: Direct attack on general γ	Strategy: Reduce to effective classes via decomposition
Assumption: $\gamma_{\text{harm}} \in K_p$ (unproven, often false)	Assumption: None (construction yields cone-valued form)
Status: Conditional	Status: Unconditional (modulo SYR details)

6.2 Why the Original Approach was Insufficient

1. **False assumption:** The claim $\gamma_{\text{harm}}(x) \in K_p(x)$ is false for difference classes.

2. **No bypass:** The original coercivity argument *required* this assumption to bound cone distance by L^2 distance.

3. **Gap not identified:** The original manuscript did not flag this as an assumption; it was stated as if proven.

6.3 How the New Approach Remedies It

1. **Signed decomposition:** Writes any γ as $\gamma^+ - \gamma^-$ with both pieces effective.
2. γ^- **handled classically:** Complete intersections are algebraic by Bertini/Lefschetz.
3. γ^+ **handled by construction:** The form $\alpha + N\omega^p$ is cone-valued by choice of N, not by any property of harmonic representatives.
4. **Coercivity reformulated:** The revised Theorem 7.1 bounds cone distance using pointwise Hermitian/PSD structure, not by assuming the harmonic form is in the cone.

7 Conclusion

The documents `hodg-dec-6d.tex` and `hodge-dec-6c.tex` are **identical** and both contain the **new signed decomposition approach**. The comparison requested is therefore between:

- The **original conditional approach** (documented in earlier files and the change report), which assumed the harmonic representative is cone-valued—an assumption that is **false in general**.
- The **new unconditional approach**, which uses signed decomposition to reduce the problem to effective classes, for which cone-valued representatives can be **constructed explicitly**.

Validity Summary

✓ Signed decomposition is **valid and elementary**.

✓ γ^- algebraic is **classical and valid**.

✓ Calibration–coercivity for effective classes is **valid**.

✓ Projective tangential approximation is **valid**.

$\sim$ Automatic SYR (Theorem 8.6) is **plausible but abbreviated**.

? Full rigor requires detailed verification of the lamination/filling construction for general cone-valued forms.

The proof represents a **major conceptual advance** over the original conditional version. Whether it constitutes a complete proof of the Hodge Conjecture depends on the detailed validity of the SYR construction, which merits careful scrutiny by experts in geometric measure theory.