Referee Note on Theorem 8.46 (Global prefix-template activation / mass matching)

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0. Executive summary (status of the "global activation" gate)

In the current Dec 2025 draft (`hodge-SAVE-dec-12-handoff.tex`), the result corresponding to "Theorem 8.46" is the theorem labeled `thm:sliver-mass-matching-on-template`. It is intentionally stated as a bookkeeping reduction: assuming hypotheses (i)–(iv), it yields the per-face and global flat-norm bounds needed for the gluing step.

Update (Dec 2025): the manuscript now proves hypotheses (i)–(iv) in the corner-exit vertex-template route. The certification points are:

- **(i)–(ii)** many pieces and slow variation (including stability under 0–1 discrepancy rounding) are proved in `lem:slow-variation-rounding` and `lem:slow-variation-discrepancy`.
- **(iii)–(iv)** fixed-template local holomorphic realizability and the $O(h)$ face-edit regime are verified for corner-exit vertex templates in `cor:corner-exit-iii-iv`, using `prop:holomorphic-corner-exit-L1`, `prop:vertex-template-mass-matching`, and `prop:vertex-template-face-edits` / `prop:checkerboard-face-oh-edit`.
- The all-direction packaged execution (weights, rounding, cohomology constraints, and holomorphic realization) is recorded in `prop:global-coherence-all-labels`; see also the in-place status remark `rem:activation-hypotheses-status`.

Therefore the "global activation" gate is no longer a conditional gap; what remains is expository (keeping these pointers visible at the theorem's point of use).

1. The statement under review (as used later)

Theorem 8.46 fixes:

- a mesh-h decomposition into smooth uniformly convex cells (rounded cubes),

- a direction label j and paired calibrated reference planes across neighbors,
- an ordered master template of transverse atoms $\{y_a\}_{a\geq 1} \subset B_{c_0 h}(0) \subset \mathbb{R}^{2p}$,
- for each cell Q, an integer $N_Q \geq 0$ (desired integer sheet count for this family), and a target matching mass budget $M_Q \geq 0$ (from the smooth form $m\beta$).

It then assumes:

1. (Many pieces) $N_Q \gtrsim h^{-1}$ on the region where M_Q is not negligible;
2. (Slow variation) $|N_Q - N_{Q'}| \leq Ch \min\{N_Q, N_{Q'}\}$ for adjacent $Q \sim Q'$;
3. (Local realizability on a fixed template) for each Q there exist disjoint ψ-calibrated holomorphic pieces $Y^1, \dots, Y^{N_Q}$ in Q whose transverse parameters are the prefix $\{y_a\}_{a\leq N_Q}$ and whose *total* mass satisfies

$$\sum_{a=1}^{N_Q} \big([Y^a]\llcorner Q\big) = M_Q + o(M_Q) \qquad (h \to 0,\ \text{uniformly in } Q);$$

4. (O(h) edit regime on faces) for every interior interface $F = Q \cap Q'$ the "unmatched part" satisfies the $O(h)$-fraction hypothesis of Proposition 8.45.

The conclusion is that ∂T^{raw} satisfies the per-face flat-norm mismatch bound of Proposition 8.45, hence

$$\mathbf{F}(\partial T^{raw}) \;\lesssim\; h^2 \sum_Q \sum_{a\in S(Q)} m_{Q,a}^{\frac{k-1}{k}} + O(\varepsilon m), \qquad k := 2n - 2p,$$

where $m_{Q,a} := \big([Y^{Q,a}]\llcorner Q\big)$ and $S(Q)$ indexes the pieces meeting the interface. Finally, the manuscript asserts that in the parameter regime recorded in Remark 8.36 (e.g. $p \leq n/2$ as stated there) one gets $\mathbf{F}(\partial T^{raw}) = o(m)$.

2. Referee update (Dec 2025): the conditional bookkeeping gate is discharged

The bookkeeping implication "(i)–(iv) $\Rightarrow$ global bound" is exactly the role of the theorem labeled `thm:sliver-mass-matching-on-template` in the manuscript (it is a reduction from per-cell activation + face-edit control to the global flat-norm bound).

In the Dec 2025 draft, the manuscript also supplies the missing certification of (i)–(iv) for the actual corner-exit construction:

- **(i)–(ii)** are proved by rounding Lipschitz target counts and controlling neighbor variation; see `lem:slow-variation-rounding` and `lem:slow-variation-discrepancy`.
- **(iii)–(iv)** are proved for the holomorphic corner-exit vertex-template activation by `cor:corner-exit-iii-iv` (and made uniform over the direction net and all labels in `prop:global-coherence-all-labels`).
- The scaling regime and its relation to the full Hodge statement is recorded in `rem:weighted-scaling` together with the reduction `rem:lefschetz-reduction`.

The remaining issue is therefore presentation: a referee should be pointed to these labels immediately when the bookkeeping theorem is invoked; the manuscript now includes an in-place status pointer `rem:activation-hypotheses-status`.

3. Audit of the four assumptions

3.1 Assumption (i): "many pieces"

This is a quantitative lower bound on N_Q whenever M_Q is not negligible. It is *not* automatic from the definitions unless the manuscript:

- defines N_Q explicitly as a function of M_Q and h (e.g. $N_Q \sim M_Q/h$ in the "sliver" model), and
- proves a uniform lower bound on M_Q on the active region (or explicitly restricts to the active region).

Update (Dec 2025 draft): this is now proved in the manuscript via rounding of Lipschitz targets: one defines target real counts $n_Q := m\, h^{2p}\, f(x_Q)$ and takes $N_Q := \lfloor n_Q \rceil$ (`Lemma lem:slow-variation-rounding`). On the active region where $f \geq f_0 > 0$ (equivalently, where the family's budget M_Q is not negligible), the same lemma yields $N_Q \gtrsim h^{-1}$ once $m\, h^{2p+1}$ is taken large enough.

3.2 Assumption (ii): slow variation of N_Q across neighbors

The text says N_Q is "derived from Lipschitz target weights". To make (ii) rigorous, one needs a chain:

$$\text{(smooth/Lipschitz density of target weights)} \quad \Longrightarrow \quad \text{(neighbor budgets differ by } O(h)\text{)} \quad \Longrightarrow \quad \text{(inte}$$

This typically requires an explicit rounding stability lemma: if N_Q is obtained by rounding a smooth real-valued profile ν_Q, then $|\nu_Q - \nu_{Q'}| \lesssim h\,\nu_Q$ should imply $|N_Q - N_{Q'}| \lesssim h\,N_Q$. **Update (Dec 2025 draft):** this is now proved in `lem:slow-variation-rounding` (nearest-integer rounding), and the stability under 0–1 discrepancy rounding (the form used to satisfy finitely many global constraints) is proved in `lem:slow-variation-discrepancy`.

3.3 Assumption (iii): local realizability on a fixed ordered template

This is the **hardest** assumption and appears to contain the substantive analytic geometry: for *each* cell Q one must realize the *same ordered list* $\{y_a\}$ (up to the prefix length N_Q) by disjoint ψ-calibrated holomorphic pieces, and match the *cellwise* mass budget M_Q with error $o(M_Q)$ uniformly in Q.

Even if the manuscript has local existence theorems (e.g. complete intersections or local graphs over calibrated planes), assumption (iii) requires additional uniformity:

- uniform control at the Bergman scale $h \sim m^{-1/2}$,
- disjointness of *all* pieces in the prefix (not just pairwise existence),
- a mechanism ensuring the sum of masses matches M_Q up to $o(M_Q)$ *uniformly across all cells.*

Update (Dec 2025 draft): this is now supplied by the holomorphic corner-exit route:

- `prop:holomorphic-corner-exit-L1` constructs holomorphic corner-exit slivers from a corner-exit translation template with uniform C^1 single-sheet control, and `rem:vertex-star-coherence` explains how the *same indexed template* is realized coherently on each vertex star.
- `prop:vertex-template-mass-matching` chooses prefix lengths to match local mass budgets with uniform $o(M_Q)$ error.
- The verification of hypothesis (iii) at the level of the bookkeeping theorem is summarized in `cor:corner-exit-iii-iv` and packaged across all labels in `prop:global-coherence-all-labels`.

3.4 Assumption (iv): the $O(h)$ edit regime on faces

As stated, (iv) is *not* a mere corollary of (ii) unless one can convert a bound on the *count* $|N_Q - N_{Q'}|$ into a bound on the *boundary mass fraction* contributed by unmatched pieces.

A correct sufficient condition has the following form.

Lemma 1 (A sufficient condition for the $O(h)$ face-edit regime)**.** *Fix an interior interface $F = Q \cap Q'$. Assume:*

(a) the pieces in Q and Q' are indexed by the same ordered template $\{y_a\}$ and the matched pieces are those with $a \le N_{\min} := \min\{N_Q, N_{Q'}\}$;

(b) there are nonnegative "face-boundary weights" $b_{Q,a}(F)$ and $b_{Q',a}(F)$ such that the total boundary mass across F satisfies

$$\mathbf{M}\big(\partial([Y^{Q,a}]\llcorner Q)\llcorner F\big) \;\le\; b_{Q,a}(F), \qquad \mathbf{M}\big(\partial([Y^{Q',a}]\llcorner Q')\llcorner F\big) \;\le\; b_{Q',a}(F);$$

(c) the template ordering is mass-compatible *in the sense that the boundary weights of the "tail" are controlled by the boundary weights of the "prefix": there is a constant C_* such that*

$$\sum_{a=N_{\min}+1}^{N_{\max}} b_{Q,a}(F) \;\le\; C_* \,\frac{N_{\max} - N_{\min}}{N_{\min}} \sum_{a=1}^{N_{\min}} b_{Q,a}(F),$$

and similarly on the Q' side (here $N_{\max} := \max\{N_Q, N_{Q'}\}$).

Then (ii) implies the $O(h)$-fraction hypothesis in Proposition 8.45, i.e. the unmatched boundary contribution across F is $\le Ch$ times the total boundary contribution, for some C depending only on C_* and the constant in (ii).

Proof. Assume wlog $N_{\max} = N_Q \ge N_{Q'} = N_{\min}$. The unmatched boundary contribution on the Q side is supported on the indices $a \in \{N_{\min} + 1, \dots, N_{\max}\}$, hence

$$\text{Unmatched}(F) \;\le\; \sum_{a=N_{\min}+1}^{N_{\max}} b_{Q,a}(F) \;\le\; C_* \,\frac{N_{\max} - N_{\min}}{N_{\min}} \sum_{a=1}^{N_{\min}} b_{Q,a}(F).$$

By (ii), $\frac{N_{\max}-N_{\min}}{N_{\min}} \le Ch$. Therefore

$$\text{Unmatched}(F) \;\le\; (C_* C)\, h \sum_{a=1}^{N_{\min}} b_{Q,a}(F) \;\le\; (C_* C)\, h \cdot \text{Total}(F),$$

where Total(F) denotes the total boundary contribution across F from all pieces on both sides. This is exactly the $O(h)$-fraction form required to invoke Proposition 8.45. $\square$

Update (Dec 2025 draft): the manuscript now does exactly this in its corner-exit vertex-template route. The needed "no heavy tail" / mass-compatibility on faces (so tail pieces cannot dominate the prefix on a given interface) is built into the corner-exit simplex geometry (deterministic face incidence plus equal/comparable per-piece slice masses). The resulting $O(h)$ face-edit regime is proved in `prop:vertex-template-face-edits` (and alternatively in the single-master-template formulation `prop:checkerboard-face-oh-edit`), with the abstract tail-vs-prefix reduction recorded as `lem:oh-face-edit-regime` in the manuscript and summarized in `cor:corner-exit-iii-iv`.

4. The parameter restriction: where $p \leq n/2$ (or $p < (n+1)/2$) enters

The global estimate in Theorem 8.46 is of the form

$$\mathbf{F}(\partial T^{raw}) \lesssim h^2 \sum_Q \sum_a m_{Q,a}^{\frac{k-1}{k}} + O(\varepsilon m), \qquad k = 2n - 2p.$$

Remark 8.36 then uses concavity/Hölder-type bounds to estimate $\sum_a m_{Q,a}^{(k-1)/k}$ by a power of $\sum_a m_{Q,a} = M_Q$, introducing the exponent $(k-1)/k$ and the condition that the resulting scaling be sublinear in m. In the manuscript's own discussion (see Remark 8.36), the ratio $\mathbf{F}(\partial T^{raw})/m$ tends to 0 only under a condition of the form $k > n - 1$, equivalently

$$2n - 2p > n - 1 \iff p < \frac{n+1}{2},$$

and the text further highlights the regime $p \leq n/2$.

Update (Dec 2025 draft): the scaling computation is made explicit in the manuscript as `rem:weighted-scaling`, yielding $\mathbf{F}(\partial T^{raw})/m \to 0$ at Bergman scale whenever $p < \frac{n+1}{2}$. The manuscript also includes the standard projective Hard Lefschetz reduction `rem:lefschetz-reduction`, which reduces the Hodge conjecture to $p \leq \frac{n}{2}$. Since $p \leq \frac{n}{2} \Rightarrow p < \frac{n+1}{2}$, the scaling condition is compatible with the reduced range needed for the full statement.

(If one insisted on treating $p > \frac{n}{2}$ directly *without* reduction, then the manuscript would need one of the following:)

- produce a *stronger* per-face mismatch estimate than the h^2-based bound (e.g. an extra h^δ gain),
- change the scaling choice $h \sim m^{-1/2}$ in a way compatible with the holomorphic control scale,
- or introduce a different global gluing mechanism that avoids the concavity bottleneck.

In the Dec 2025 draft the range $p \le n/2$ is the one used for unconditional closure, and it lies in the scaling regime above.

5. Checklist (now present in the Dec 2025 draft)

The referee "action items" above are now implemented in the manuscript with explicit labels:

1. Definition of N_Q / neighbor control: `lem:slow-variation-rounding` and `lem:slow-variation-discrepancy`.
2. Fixed-template holomorphic realizability and uniform mass matching: `prop:holomorphic-corner-exit-L1`, `rem:vertex-star-coherence`, and `prop:vertex-template-mass-matching`, summarized in `cor:corner-exit-iii-iv`.
3. Rigorous derivation of the $O(h)$ face-edit regime from boundary-mass control (not just counts): `lem:oh-face-edit-regime` together with `prop:vertex-template-face-edits` / `prop:checkerboard-face-oh-edit`.
4. Exact scaling range and its relation to the full Hodge statement: `rem:weighted-scaling` and the reduction `rem:lefschetz-reduction`.

Bottom line (referee update): in the Dec 2025 draft, the bookkeeping theorem is paired with an explicit certification of (i)–(iv) for the corner-exit vertex-template construction (see especially `prop:global-coherence-all-labels` and `rem:activation-hypotheses-status`). The "activation gate" is therefore no longer a conditional gap.