Strategy and Progress (Unconditional Hodge Closure)

Dec 2025

This file is a living lab notebook for pushing the manuscript toward a **valid, unconditional** proof.

1 Current status (Dec 2025)

1.1 The core dependency

- The manuscript's claimed "unconditional Hodge" conclusion flows through **`thm:automatic-syr`** ("Automatic SYR for cone-valued forms").
- **Unconditional closure requires a fully rigorous proof of the realization/microstructure step** that produces a ψ**-calibrated integral cycle** in the target class (after clearing denominators).

1.2 What is standard (once the key existence is granted)

- **Federer–Fleming compactness + calibration inequality** (already packaged as `thm:realization-from-almost`).
- **Harvey–Lawson** (calibrated integral current $\Rightarrow$ analytic cycle) and **Chow** (projective analytic $\Rightarrow$ algebraic).
- Signed decomposition reduction is algebraic bookkeeping (not the core obstacle).

1.3 What is not yet proven (the "life-or-death" blockers)

1. **Prompt 6 / Prompt 8 (core missing theorem):**
   - From a smooth closed strongly positive (p, p) representative β of a rational class, prove existence of a multiple $m[\gamma^+]$ represented by a ψ**-calibrated integral cycle** (equivalently stable mass equals the calibration pairing).
2. **Microstructure / gluing control in Step 4.2:**
   - The manuscript explicitly flags the missing piece: a quantitative estimate making $\mathcal{F}(\partial T^{\mathrm{raw}})$ small, robustly under cancellation.
3. **A scheme that avoids the "diffuse density vs codimension-p mass" trap,** or proves a new rigidity principle that collapses smooth strong positivity into algebraicity.

2 Work completed so far (high-signal)

2.1 1) Flat-norm gluing route made explicit in the manuscript

- In `hodge-dec-6-handoff.tex` Substep 4.2:
  - Added `prop:transport-flat-glue`: **small-angle sheets + transverse W_1 matching across faces $\Rightarrow$ $\mathcal{F}(\partial T^{\mathrm{raw}})$ small** (quantitative).
  - Added `prop:integer-transport` + `rem:integer-transport-algo`: reduced "produce W_1 matching" to **grid quantization + integer rounding + integer network flow** (discrete combinatorics), isolating the remaining *geometric* difficulty.

2.2 2) Prompt docs upgraded

- In `ai-gap-prompts-and-proofs.tex`:
  - Added a flat-norm criterion (`lem:prompt8-flat-glue`) and transport control (`lem:prompt8-w1`) + W_1 quantization (`lem:w1-quantize`).
  - Added a closedness-cancellation remark explaining why the flat-norm dual is the right target.

2.3 3) Obstruction knowledge improved

- We explicitly identified the need to **separate**:
  - what is purely discrete/rounding/transport (solvable), vs
  - what is geometric: realizing the discrete transverse measures by actual calibrated holomorphic sheets with controlled angles and controlled face-slice geometry.

3 The plan (what we try next, inch by inch)

3.1 A) Make the transport hypotheses provable (not assumed)

Target: prove a lemma of the form:

> For the specific holomorphic sheet families produced by `thm:local-sheets`, the restriction of ∂S_Q to a face admits a transverse-parameter measure model (with uniform tubular charts), and the induced measures on adjacent cubes can be chosen/adjusted to satisfy the required W_1 face matching.

Concrete subtasks:

- Prove uniform tubular-coordinate control for each sheet family on a face (geometry estimate).
- Define explicitly the transverse measures $\mu_{Q\to F}$ in the Kähler chart.
- Prove the Lipschitz slice-functional estimate needed for Kantorovich–Rubinstein.

3.2 B) Resolve the scaling tension in the current cube-local paradigm

We must reconcile three competing needs in any cube-by-cube sampling approach:

- **integer availability** (need many sheets per cube to match weights),
- **small variation** (need mesh small so the target data changes slowly),
- **small boundary mismatch** (need face mismatches small in flat norm).

If this cannot be reconciled, we must switch paradigms (see C).

3.2.1 Scaling sanity check (important)

Let h be the cube side length and m the denominator-clearing multiplier. The "target cube mass" is $M_Q \sim m\,h^{2n}$, while a single calibrated sheet crossing Q has mass $A_Q \sim h^{2(n-p)}$. Thus the expected number of sheets per cube is

$$N_Q \sim \frac{M_Q}{A_Q} \sim m\,h^{2p}.$$

- To have **enough integer degrees of freedom** locally (and make rounding errors small), we want $N_Q \gg 1$, i.e. $m\,h^{2p} \gg 1$.
- To make β nearly constant on cubes, we want $h \to 0$.
- In naive estimates, **total face mismatch** scales like "(variation across a face) $\times$ (total boundary mass)", which heuristically leads to a global mismatch of order $\sim m\,h$ (variation $\sim h$, total flux $\sim m$), so driving mismatch to 0 at fixed m suggests $h \ll 1/m$.

For $p > 1$, the requirements $m\,h^{2p} \gg 1$ and $h \ll 1/m$ are incompatible at large m:

$$h \ll m^{-1} \quad \Rightarrow \quad m\,h^{2p} \ll m \cdot m^{-2p} = m^{1-2p} \to 0.$$

This indicates that any successful construction must use **cancellation/closedness in an essential way** (flat-norm route), not naive mass-of-boundary estimates.

3.3 C) Search for a replacement mechanism (avoid cube-local diffuse matching)

Possible directions to explore (each must end in a classical proof, not just RS motivation):

- **Rigidity/Bochner-type**: show a rational class admitting a smooth strongly positive representative must be algebraic (or severely restricted).
- **Global probabilistic/algebraic averaging**: represent smooth positive forms as averages of algebraic cycles and attempt a rational/integral extraction argument.
- **Stationarity constraints on Young measures**: find a PDE-type necessary condition that forces realizable barycenters to come from genuine analytic strata.

3.4 D) RS/Recognition-guided hypotheses (kept separate from classical proof)

We will record RS-inspired "finite resolution" hypotheses (polyhedrality / finite-mode decompositions) as optional bridges, but every step needed for unconditional closure must be proven classically or reduced to known theorems.

4 Next concrete actions

1. Strengthen `prop:transport-flat-glue` into a fully cited lemma package: tubular neighborhood existence + slice Lipschitz estimate + KR duality.

2. Audit `thm:local-sheets` to see whether it can actually realize prescribed transverse placements across faces (not just inside one cube).

3. Decide whether Step 4's cube-local scheme is fundamentally incompatible with fixed-class realization for $1 < p < n-1$, and if so, pivot to a global replacement mechanism.

4.1 Latest incremental progress (today)

- Added `rem:transport-hypotheses` in `hodge-dec-6-handoff.tex` clarifying that hypotheses (a)–(b) of `prop:transport-flat-glue` really do hold for the flat-sheet model and persist (up to $O(\varepsilon)$) after the holomorphic upgrade.

- This isolates the remaining unknown in the transport route to a single hard requirement: **construct translation parameters so that adjacent cubes satisfy face-by-face W_1 matching (and do so consistently across all faces of each cube).**

- Added a new subsection in `hodge-blocker.tex` formalizing the "cube-consistency" issue as a finite-dimensional **marginal realization / coupling problem** for discrete translation parameters, and proved a basic existence lemma (`lem:discrete-marginals`) for the coordinate-projection case (via flow/matching + induction). This is not the full geometric problem, but it cleanly separates "combinatorics is not the blocker" from "geometry/plane-slice maps are."

- Added `lem:w1-linear-stability` in `hodge-blocker.tex`: a clean estimate $W_1(L_{\#}\mu, L'_{\#}\mu) \leq \|L - L'\| \int \|y\| \, d\mu$, useful for bounding face-measure mismatches when adjacent cubes use the *same* underlying transverse translation measure but have slightly different face-slice maps (due to small angle changes).

- Added `cor:angle-to-face` in `hodge-blocker.tex`: combines transport control + linear stability to give an explicit "small angle change $\Rightarrow$ small flat-norm face mismatch" scaling bound in the flat model.

- Added `lem:slice-angle-lip` in `hodge-blocker.tex`: a clean (sketched) estimate that the face-slice functional $\Sigma_F^P(t)(\eta)$ varies Lipschitzly with the plane angle, at scale $\alpha\, h^{k-1}$.

- Added a "cube-consistency" formulation to Prompt 8 in `ai-gap-prompts-and-proofs.tex` (`conj:cube-consistent`): the remaining transport route can be seen as a **simultaneous pushforward approximation problem** (one discrete measure on translation space must approximate many face measures at once).

- Updated `hodge-dec-6-handoff.tex rem:transport-hypotheses` to explicitly state this "same translation multiset pushes forward to all faces" viewpoint, so the reader sees that the remaining obstruction is truly a *simultaneous* matching constraint, not independent per-face tuning.

- Added `lem:w1-auto` + `rem:w1-auto` in `hodge-dec-6-handoff.tex`: if adjacent cubes use the **same translation template** and their face-slice maps differ by $O(h)$ (smooth geometry), then the induced transverse measures are automatically W_1-close at scale $O(h^2 \cdot N)$. This is a key simplification: it suggests we may not need to *solve* the full cube-consistent matching problem, only keep the translation template coherent and handle cohomology constraints globally.

- **Correction:** the resulting global flat-norm bound from this mechanism scales like $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim m\,h + O(\varepsilon m)$, not mh^2. (Counting faces in a $2n$-dimensional cubulation gives $O(h^{-2n})$ faces.)
- Added `lem:w1-template-edit` + `rem:w1-multiplicity` in `hodge-dec-6-handoff.tex`: if adjacent cubes use the same template but integer sheet counts vary slowly, then the extra W_1 error from insertions/deletions is $O(rh)$ and is absorbed into the same h^2N scaling provided $r \lesssim hN$. This reduces the remaining matching analysis to a **rounding/Diophantine "slow variation of counts"** lemma.
- Added `lem:slow-variation-rounding` in `hodge-dec-6-handoff.tex`: a concrete bound showing that rounding a Lipschitz target $n_Q = mh^{2p}f(x_Q)$ automatically yields neighbor differences $|N_Q - N_{Q'}| \le Lmh^{2p+1} + 1$, and hence $|N_Q - N_{Q'}| \lesssim hN_Q$ once f is bounded below and mh^{2p+1} is large.
- Added `lem:barany-grinberg` in `hodge-dec-6-handoff.tex` and used it to justify the Substep 4.3 claim that one can meet all cohomology pairing constraints to within $< 1/2$ simultaneously by rounding in fixed dimension (discrepancy bound depends only on $b = \dim H^{2n-2p}$).
- Rewrote the proof of `prop:cohomology-match` in `hodge-dec-6-handoff.tex` to explicitly implement the Bárány–Grinberg rounding (replace the old "LLL/continued fractions" handwave). The proof now: (i) encodes rounding as $N = \lfloor n \rfloor + \varepsilon$, (ii) bounds each sheet-piece pairing vector by $O(h^{2(n-p)})$, (iii) applies discrepancy in dimension b to get $< 1/2$ simultaneous error.
- Added `rem:param-tension` in `hodge-dec-6-handoff.tex`: makes explicit the fixed-m tension between needing $h \to 0$ for small gluing error (template route gives $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim mh$) and the fact that the naive constant-mass sheet model has expected local counts $N_Q \sim mh^{2p} \to 0$ as $h \to 0$ for $p > 1$.
- Added `lem:mass-tunable` + `rem:sliver` in `hodge-dec-6-handoff.tex`: in the flat model the mass of a translated plane slice through a cell is continuous down to 0, suggesting a fixed-m escape route via **many tiny "sliver" sheet pieces** (large N but small per-piece mass).
- Added `conj:sliver-local` + `rem:sliver-close` in `hodge-dec-6-handoff.tex`: a precise quantitative local target stating what "sliver microstructure" would need in the *holomorphic upgrade* (many calibrated pieces with tiny mass, whose weighted transverse measure approximates a smooth density in W_1).
- Added `lem:sliver-stability` in `hodge-dec-6-handoff.tex`: shows that once a holomorphic sheet is a small-C^1 graph over an affine calibrated plane slice, (i) its mass in the cell is comparable up to a $1+O(\varepsilon^2)$ factor, and (ii) disjointness inside the cell persists under separation $\gg \varepsilon h$. This reduces "sliver masses" in `conj:sliver-local` to the flat-model tunability plus sufficiently strong C^1 approximation.

- Added `rem:jet-separation` in `hodge-dec-6-handoff.tex`: notes that high powers L^k can separate jets at finitely many points, supporting the feasibility (at least at the "local existence of many pieces" level) of realizing many prescribed small-mass local plane pieces in the projective holomorphic upgrade.
- Added `lem:plane-section-continuity` in `hodge-blocker.tex`: in the flat model, the intersection mass of a translated plane with a cell varies continuously from 0 to a maximum. This highlights a possible "sliver microstructure" mechanism: split fixed total mass into many tiny pieces to keep sheet counts large even at small h.
- Added `lem:sliver-ball-scaling` in `hodge-blocker.tex`: an explicit formula for the mass of a plane section of a ball, giving a clean toy scaling for how "sliver" masses decay as the plane approaches the boundary.
- Added `conj:sliver-micro` in `ai-gap-prompts-and-proofs.tex`: a sharpened formulation of what fixed-m realization would need in a local chart—many tiny calibrated "sliver pieces" per cell whose transverse measures approximate a smooth density in W_1 without mass blow-up.