

# Note for Review: sanity check of the new microstructure step in codimension $p = 2$

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## Question being answered

You asked: “*Can you test the new proof in the first genuinely nontrivial case  $p = 2$ ? If the proof is solid, can you claim it for  $p = 2$  before I invest in a full review?*”

This note explains how the manuscript specializes to  $p = 2$ , and performs a concrete parameter/scaling check showing the core new estimate

$$\mathcal{F}(\partial T^{\text{raw}}) = o(m)$$

still closes in the  $p = 2$  regime.

## 1 Scope of the claim for $p = 2$

The master manuscript `hodge-SAVE-dec-12-handoff.tex` states the main result for all  $1 \leq p \leq n$  and is intended to be uniform in  $p$ . Therefore, *as stated*, it includes the codimension-2 case  $p = 2$ .

Two easy sanity checks about the surrounding classical landscape:

- If  $n = 3$  and  $p = 2$ , then codimension 2 cycles are Poincaré dual to  $(1, 1)$  classes; by Lefschetz  $(1, 1)$ , the Hodge conjecture is already known in this case. The manuscript also reduces large  $p$  to small  $p$  via Hard Lefschetz.
- The first genuinely “middle codimension” test is  $n \geq 4$ ,  $p = 2$  (e.g.  $n = 4$ ). This is exactly where the naive PSD-identification for the calibrated cone fails, and where the quantitative replacement (the Dec 8 note `hodge-fix-dec-8-old.tex`) is needed.

## 2 Where the value $p = 2$ enters the new microstructure construction

In the microstructure/gluing step, the dependence on  $p$  is explicit and benign:

- The Kähler calibration is  $\psi = \omega^{n-p}/(n-p)!$ . For  $p = 2$ ,  $\psi = \omega^{n-2}/(n-2)!$  calibrates complex  $(n-2)$ -planes.
- The *real* dimension of each holomorphic sliver is

$$k := 2n - 2p = 2n - 4.$$

- The normal translation parameter lives in  $\mathbb{C}^p = \mathbb{C}^2$  (real dimension 4). In the corner-exit template lemma, one typically fixes one real component (to force a chosen “slanted” inequality), leaving a  $(2p - 1) = 3$  real dimensional box from which to pack many separated translations.
- The key exponent in the weighted flat-norm bound is  $(k - 1)/k = (2n - 5)/(2n - 4)$ .

### 3 Local corner-exit template supply for $p = 2$

In the manuscript this is handled by:

- `lem:complex-corner-exit-template` (an explicit complex model), and
- `lem:corner-exit-template-open + prop:corner-exit-template-net` (robust supply for a finite direction net, with uniform constants over the net).

Specializing those statements to  $p = 2$ :

- we work in  $\mathbb{C}^n = \mathbb{C}^{n-2} \times \mathbb{C}^2$  with  $w = (w_1, w_2)$ ;
- for each direction label, after choosing a vertex and a “slanted” coordinate (typically  $w_1$ ), the footprint

$$E(t) := (P + t) \cap Q$$

is a  $k$ -simplex ( $k = 2n - 4$ ) in the cube  $Q = [0, h]^{2n}$  with  $k + 1 = 2n - 3$  facets on a fixed designated set of cube faces incident to the chosen vertex;

- the translation parameter box has real dimension 3, so for any separation scale  $\delta > 0$  one can extract a long  $\delta$ -separated ordered list of translations  $(t_a)$  producing identical footprints (hence identical per-piece slice masses within a label).

### 4 Holomorphic realization for $p = 2$ (two equations)

For codimension  $p = 2$ , each sliver is produced as a local piece of a holomorphic complete intersection

$$Y = \{s_1 = s_2 = 0\}$$

with uniform single-sheet  $C^1$  graph control on an entire cell. In the manuscript this is routed through:

- `lem:bergman-affine-approx-hormander` (cutoff + Hörmander  $L^2$  solution gives  $C^1$  approximation of affine-linear holomorphic models on a ball  $B_{R/\sqrt{m}}$ ), and
- `lem:global-graph-contraction` (a contraction criterion on a product domain giving a unique global graph  $w = g(u)$ ), packaged into
- `prop:cell-scale-linear-model-graph`.

None of these steps changes in substance for  $p = 2$ ; one simply has a two-component map  $F = (F_1, F_2)$  instead of a  $p$ -component map.

## 5 A concrete scaling/consistency check for $p = 2$

This is the main “test” one can do without re-proving every analytic lemma: verify that the parameter regime needed by the weighted flat-norm bound is internally consistent for  $p = 2$  (and yields  $o(m)$ ).

### Set-up and notation

Let  $n \geq 4$  and  $p = 2$ , hence  $k = 2n - 4$ . Let  $h$  be the cubical mesh and assume we are in the Bergman scale regime

$$h \asymp m^{-1/2}.$$

Let the corner-exit simplex scale inside each cube be

$$s := \varepsilon h,$$

so each sliver has mass on the order of

$$\mu \asymp s^k = (\varepsilon h)^k.$$

The total target mass per cube is on the order of

$$M_Q \asymp m h^{2n}.$$

Therefore the number of slivers needed per cube (per direction budget) is

$$N_Q \asymp \frac{M_Q}{\mu} \asymp \frac{m h^{2n}}{(\varepsilon h)^k} = m h^{2n-k} \varepsilon^{-k} = m h^4 \varepsilon^{-k}.$$

### Choose explicit parameters

Choose

$$h := m^{-1/2}, \quad \varepsilon := m^{-1/3}, \quad s := \varepsilon h = m^{-5/6}, \quad \delta := 10 \varepsilon h.$$

Then:

- $N_Q \asymp m h^4 \varepsilon^{-k} = m \cdot m^{-2} \cdot m^{k/3} = m^{k/3-1}$ , which tends to  $+\infty$  for all  $n \geq 4$  (since then  $k = 2n - 4 \geq 4$  and  $k/3 - 1 > 0$ ).
- The local rounding error in mass-budget matching scales like  $O(1/N_Q) + O(\varepsilon^2)$  and therefore tends to 0.

### Check the weighted flat-norm bound

The weighted bound used in the manuscript has the schematic form

$$\mathcal{F}(\partial T^{\text{raw}}) \lesssim h^2 \sum_Q \sum_{a \in \mathcal{S}(Q)} m_{Q,a}^{\frac{k-1}{k}}, \quad k = 2n - 4,$$

where  $m_{Q,a}$  are the individual piece masses. In the uniform model above,  $m_{Q,a} \asymp \mu$ , and the total number of pieces is  $N_Q$  per cube times  $h^{-2n}$  cubes. Therefore

$$\sum_{Q,a} m_{Q,a}^{\frac{k-1}{k}} \asymp (N_Q h^{-2n}) \cdot \mu^{\frac{k-1}{k}} = N_Q h^{-2n} \cdot (\varepsilon h)^{k-1}.$$

Using  $N_Q \asymp m h^4 \varepsilon^{-k}$  and  $k = 2n - 4$ , we get

$$\sum_{Q,a} m_{Q,a}^{\frac{k-1}{k}} \asymp m h^4 \varepsilon^{-k} h^{-2n} \varepsilon^{k-1} h^{k-1} = m \varepsilon^{-1} h^{-1}.$$

Hence

$$\mathcal{F}(\partial T^{\text{raw}}) \lesssim h^2 \cdot (m \varepsilon^{-1} h^{-1}) = m \frac{h}{\varepsilon}.$$

With  $h = m^{-1/2}$  and  $\varepsilon = m^{-1/3}$ ,

$$\mathcal{F}(\partial T^{\text{raw}}) \lesssim m \cdot \frac{m^{-1/2}}{m^{-1/3}} = m^{5/6} = o(m).$$

This is the desired scaling in the microstructure/gluing step for  $p = 2$ .

## Conclusion

In summary: the new microstructure/gluing mechanism in the manuscript is uniform in  $p$  and specializes cleanly to  $p = 2$ . The first nontrivial test case  $n \geq 4$ ,  $p = 2$  admits a consistent parameter regime (explicitly exhibited above) in which:

- one has many disjoint holomorphic corner-exit slivers per cell (so rounding is effective), and
- the weighted flat-norm estimate yields  $\mathcal{F}(\partial T^{\text{raw}}) = o(m)$ .

Accordingly, the manuscript *as written* claims the codimension-2 ( $p = 2$ ) case as a special case of the stated general theorem.