Correction: Proof of the Microstructure/Gluing Estimate (`prop:glue-gap`)

This note provides a self-contained corrected proof of the "Microstructure/gluing estimate" as it appears in `Hodge-v6-w-Jon-Update-MERGED.tex` (Proposition `\ref{prop:glue-gap}`). The issue in the manuscript version is purely in the *written proof* of the proposition: the statement already contains the correct bound (displacement × slice mass), but the short proof sketch incorrectly drops the slice-mass factor and references boundary faces rather than interior interfaces.

Minimal background

We use only the standard "flat norm" inequality: if T is a current and Q is any current with $\partial Q = T$, then

$$\mathcal{F}(T) \leq \mathrm{Mass}(Q),$$

since one may take $R = 0$ in the definition $\mathcal{F}(T) = \inf\{\mathrm{Mass}(R) + \mathrm{Mass}(Q) : T = R + \partial Q\}$.

Translation filling lemma

Lemma 1 (Flat-norm stability under translation). *Let S be an integral ℓ-cycle in $\mathbb{R}^d$ (so $\partial S = 0$) with finite mass. For any translation vector $v \in \mathbb{R}^d$, let $\tau_v(x) = x+v$ and $(\tau_v)_\# S$ be the pushforward. Then there exists an integral $(\ell+1)$-current Q such that*

$$\partial Q = (\tau_v)_\# S - S, \qquad \mathrm{Mass}(Q) \leq \|v\| \, \mathrm{Mass}(S).$$

Proof. Let $H : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ be the straight-line homotopy $H(t,x) = x + tv$. Define $Q := H_\#([0,1] \times S)$. Since $\partial([0,1] \times S) = \{1\} \times S - \{0\} \times S - [0,1] \times \partial S$ and $\partial S = 0$, we have $\partial Q = (\tau_v)_\# S - S$. The map H has Jacobian bounded by $\|v\|$ in the t-direction, hence $\mathrm{Mass}(Q) \leq \|v\| \mathrm{Mass}(S)$. $\square$

Corrected gluing estimate

Proposition 1 (Microstructure/gluing estimate (corrected proof)). *Let $k \geq 1$ and let*

$$T^{\mathrm{raw}} = \sum_Q S_Q$$

be an integral k-current obtained by summing integral k-currents S_Q supported on cells Q of a cubulation. Assume the boundary admits an oriented face decomposition

$$\partial T^{\mathrm{raw}} = \sum_F B_F$$

as a sum over interior interfaces $F = Q \cap Q'$ where each face mismatch is

$$B_F = \big(\partial S_Q\big)\llcorner F - \big(\partial S_{Q'}\big)\llcorner F.$$

Assume moreover that for each interface F there exist cycle slices $\Sigma_F(u)$ on F depending on $u \in \Omega_F \subset \mathbb{R}^{2p}$ such that:

- *for some multisets $\{u_{F,a}\}_{a=1}^{N_F}$ and $\{u'_{F,a}\}_{a=1}^{N_F}$,*

$$B_F = \sum_{a=1}^{N_F} \Sigma_F(u_{F,a}) - \sum_{a=1}^{N_F} \Sigma_F(u'_{F,a});$$

- *in face coordinates, each $\Sigma_F(u)$ is obtained from $\Sigma_F(0)$ by translation by u;*
- *there exists a matching $\sigma_F \in S_{N_F}$ with a uniform displacement bound*

$$\|u_{F,a} - u'_{F,\sigma_F(a)}\| \le \Delta_F \qquad \text{for all } a.$$

Define, for each F and a, the translation vector $v_{F,a} := u'_{F,\sigma_F(a)} - u_{F,a}$ and let $Q_{F,a}$ be the integral k-current provided by Lemma 1 applied (in the face chart) to the $(k-1)$-cycle $\Sigma_F(u_{F,a})$, so that

$$\partial Q_{F,a} = \Sigma_F(u_{F,a}) - \Sigma_F(u'_{F,\sigma_F(a)}), \qquad \operatorname{Mass}(Q_{F,a}) \le \|v_{F,a}\| \operatorname{Mass}(\Sigma_F(u_{F,a})).$$

Set

$$Q_F := \sum_{a=1}^{N_F} Q_{F,a}, \qquad U := \sum_F Q_F, \qquad T := T^{\mathrm{raw}} - U.$$

Then:

(i) *U is an integral k-current and $\partial U = \partial T^{\mathrm{raw}}$, hence T is an integral k-cycle.*

(ii) *The mass of U satisfies*

$$\operatorname{Mass}(U) \le \sum_F \operatorname{Mass}(Q_F) \le \sum_F \Delta_F \sum_{a=1}^{N_F} \operatorname{Mass}(\Sigma_F(u_{F,a})).$$

(iii) *In particular,*

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \le \operatorname{Mass}(U).$$

Proof. Fix an interface F and its matching σ_F. Summing the boundary identities for $Q_{F,a}$ gives

$$\partial Q_F = \sum_{a=1}^{N_F} \big(\Sigma_F(u_{F,a}) - \Sigma_F(u'_{F,\sigma_F(a)})\big) = B_F.$$

Summing over all interfaces yields

$$\partial U = \sum_F \partial Q_F = \sum_F B_F = \partial T^{\mathrm{raw}},$$

so $T = T^{\mathrm{raw}} - U$ is a cycle.

For the mass bound, by the triangle inequality and the estimate from Lemma 1,

$$\operatorname{Mass}(Q_F) \le \sum_{a=1}^{N_F} \operatorname{Mass}(Q_{F,a}) \le \sum_{a=1}^{N_F} \|u_{F,a} - u'_{F,\sigma_F(a)}\| \operatorname{Mass}(\Sigma_F(u_{F,a})) \le \Delta_F \sum_{a=1}^{N_F} \operatorname{Mass}(\Sigma_F(u_{F,a})).$$

Summing over F gives (ii). Finally, since $\partial U = \partial T^{\mathrm{raw}}$, taking $Q := U$ and $R := 0$ in the flat norm definition yields $\mathcal{F}(\partial T^{\mathrm{raw}}) \le \operatorname{Mass}(U)$, proving (iii). ☐