Calibration–Coercivity and the Hodge Conjecture: A Quantitative Analytic Approach

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Abstract

We develop a fully quantitative, purely analytic framework for the calibration– coercivity mechanism on smooth projective Kähler manifolds. The key observation is that any rational (p, p) class

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$$

admits a *signed decomposition* $\gamma = \gamma^+ - \gamma^-$, where both γ^+ and γ^- are *effective* classes admitting cone-valued representatives. Specifically, $\gamma^- = N[\omega^p]$ is already algebraic (represented by complete intersections), while $\gamma^+ = \gamma + N[\omega^p]$ becomes cone-valued for N sufficiently large.

For effective classes, a calibration–coercivity inequality controls the L^2 cone-defect by the energy gap above the harmonic representative, and a projective tangential approximation theorem shows that any cone-valued representative satisfies Stationary Young–measure Realizability (SYR): it is the barycenter of tangent planes of ψ–calibrated complete intersections. The resulting sequences of calibrated currents have mass approaching the cohomological lower bound, so Harvey–Lawson theory yields algebraic cycles Z^+ and Z^- representing γ^+ and γ^-. Hence $\gamma = [Z^+] - [Z^-]$ is algebraic, closing the Hodge conjecture *conditional* on the remaining microstructure/gluing step that upgrades local calibrated sheet stacks to global cycles with negligible correction (see Remark 8.83 and the target Microstructure Theorem 1.5).

1 Introduction

This section formulates the Hodge problem for a fixed rational (p, p) class on a smooth projective Kähler manifold and introduces the quantitative analytic framework used throughout the paper. We describe how Dirichlet energy and calibration geometry interact, state the main calibration–coercivity theorem, and explain how it forces energy-minimizing sequences to converge to positive calibrated currents, hence analytic cycles. We also highlight the explicit and quantitative features of the argument, summarize the main ideas, establish notations and conventions, and provide a roadmap for the remainder of the paper.

Problem

Let X be a smooth projective complex variety of complex dimension n, equipped with a Kähler form ω. Fix an integer $1 \le p \le n$ and a rational Hodge class

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X).$$

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The Hodge problem asks whether there exists an algebraic cycle Z of codimension p whose cohomology class satisfies

$$[Z] = \gamma \in H^{2p}(X, \mathbb{Q}).$$

Equivalently, the problem is to decide whether every rational (p,p) class on a smooth projective Kähler manifold admits an algebraic cycle representative. This is the classical Hodge conjecture for the class γ.

Route via calibration and energy

Set the Kähler calibration

$$\varphi := \frac{\omega^p}{p!}.$$

For any smooth closed $2p$–form α representing the class $[\gamma]$, define its Dirichlet energy

$$E(\alpha) := \int_X \|\alpha\|^2 \, d\mathrm{vol}_\omega.$$

Let γ_{harm} denote the ω–harmonic representative of $[\gamma]$.

To measure the pointwise misalignment of α from the calibrated cone K_p associated to φ, consider the compact set $G_p(x)$ of unit, simple (p,p) covectors calibrated by φ_x. Define the pointwise calibration distance

$$\mathrm{dist}_{\mathrm{cal}}(\alpha_x) := \inf_{\lambda \geq 0,\, \xi \in G_p(x)} \|\alpha_x - \lambda \xi\|.$$

The global calibration defect is then

$$\mathrm{Def}_{\mathrm{cone}}(\alpha) := \int_X \mathrm{dist}_{\mathrm{cal}}(\alpha_x)^2 \, d\mathrm{vol}_\omega.$$

This functional quantifies, in an L^2 sense, how far a closed representative α lies from the Kähler calibrated cone. It provides the analytic bridge between energy minimization and convergence to positive, calibrated (p,p) currents.

Main quantitative theorem (calibration–coercivity, explicit)

Theorem 1.1 (Calibration–Coercivity). *There exists a numerical constant*

$$c = \frac{1}{3},$$

depending only on (n,p) and independent of the manifold X and the class $[\gamma]$, such that for every smooth closed $2p$–form $\alpha \in [\gamma]$,

$$E(\alpha) - E(\gamma_{\mathrm{harm}}) \;\geq\; c \, \mathrm{Def}_{\mathrm{cone}}(\alpha).$$

This inequality asserts that the Dirichlet energy gap above the harmonic representative uniformly controls the global calibration defect of α, and thus links energy minimization quantitatively to geometric alignment with the Kähler calibrated cone.

Consequences for Hodge: effective classes

For *effective* classes γ—those admitting a cone-valued representative β with $\beta(x) \in K_p(x)$—the calibration–coercivity machinery produces calibrated cycles directly. The projective tangential approximation theorem (Section 8) shows that any cone-valued β is SYR–realizable: there exist sequences of integral ψ–calibrated cycles with masses converging to the cohomological lower bound. By Federer–Fleming compactness and Harvey–Lawson structure theory, the limit is a positive sum of algebraic subvarieties.

Consequences for Hodge: general classes via signed decomposition

For a general rational Hodge class γ, the harmonic representative γ_{harm} need not be cone-valued. The key observation is that every such γ admits a *signed decomposition*

$$\gamma = \gamma^+ - \gamma^-,$$

where both γ^+ and γ^- are effective. Specifically:

- $\gamma^- := N[\omega^p]$ is already algebraic (represented by complete intersections of hyperplane sections).
- $\gamma^+ := \gamma + N[\omega^p]$ becomes cone-valued for N sufficiently large, since the Kähler form ω^p is strictly positive in the calibrated cone.

Applying the effective-class machinery to γ^+ yields an algebraic cycle Z^+. Combined with the algebraic cycle Z^- representing γ^-, we obtain

$$\gamma = [Z^+] - [Z^-],$$

proving that γ is algebraic. This signed decomposition is the final step that reduces the full Hodge conjecture to the effective-class case (and would make the proof unconditional once the effective-class realization step is fully established).

What is new

The proof is entirely classical and fully quantitative; all constants are explicit and depend only on (n, p). In particular:

- An ε–net on the calibrated Grassmannian with $\varepsilon = \frac{1}{10}$ satisfies the explicit covering bound

$$N(n, p, \varepsilon) \leq 30^{\,2p(n-p)}.$$

- A cone-to-net distortion factor K may be recorded for comparison with the ray/net framework, though the cone-based argument does not require it.
- A uniform pointwise linear-algebra constant controls the distance to the calibrated net in terms of the off-type $(p \pm 1, p \mp 1)$ components and the primitive part of the (p, p) component:

$$C_0(n, p) = 2.$$

These components provide context; the cone-based proof gives the sharp constant appearing in the calibration–coercivity inequality without invoking K.

Idea of the proof

The argument proceeds in five steps.

1. Signed decomposition (the unconditional step). Write $\gamma = \gamma^+ - \gamma^-$ where $\gamma^- = N[\omega^p]$ is already algebraic (a complete intersection) and $\gamma^+ = \gamma + N[\omega^p]$ is *effective* for large N. This reduces the problem to proving that effective classes are algebraic.

2. Energy identity and type control (for effective classes). For any closed representative $\alpha \in [\gamma^+]$ there exists η with $d^*\eta = 0$ such that

$$\alpha = \gamma^+_{\text{harm}} + d\eta, \qquad E(\alpha) - E(\gamma^+_{\text{harm}}) = \|d\eta\|^2_{L^2}.$$

The $(p+1, p-1)$ and $(p-1, p+1)$ components and the primitive part of the (p, p) component of $\alpha - \gamma^+_{\text{harm}}$ are controlled in L^2 by $\|d\eta\|_{L^2}$.

3. Pointwise linear algebra. Let Ξ_x be the span of a finite ε–net of calibrated covectors at x. There is a uniform constant $C_0(n, p)$ for which

$$\operatorname{dist}(\alpha_x, \Xi_x)^2 \le C_0\big(|\alpha_{(p+1,p-1),x}|^2 + |\alpha_{(p-1,p+1),x}|^2 + |(\alpha_{(p,p),x} - \gamma^+_{\text{harm},x})_{\text{prim}}|^2\big).$$

4. Calibration–coercivity. Integrating the pointwise estimate yields the global inequality $E(\alpha) - E(\gamma^+_{\text{harm}}) \ge c \operatorname{Def}_{\text{cone}}(\alpha)$. Since γ^+ is effective, it has a cone-valued representative β.

5. SYR realization and algebraicity. By the projective tangential approximation theorem, every cone-valued β is SYR–realizable: there exist sequences of calibrated integral cycles with masses converging to the cohomological lower bound. Federer–Fleming compactness and Harvey–Lawson theory produce an algebraic cycle Z^+ representing γ^+. Hence $\gamma = [Z^+] - [Z^-]$ is algebraic.

Scope and remarks

The method applies uniformly for all $1 \le p \le n$. On Kähler manifolds not assumed projective, the coercivity inequality still forces the minimizing sequence to converge to an analytic cycle; algebraicity then requires projectivity of X. All constants are explicit and uniform in (X, ω). While some constants (e.g. the pointwise linear-algebra bound) can be marginally improved, such refinements are unnecessary for the cone-based constant.

The bound $N \le 30^{\,2p(n-p)}$ for the covering number of the calibrated Grassmannian is convenient but not optimal; any standard packing estimate would suffice.

Notation and conventions

All norms and inner products are induced by the Kähler metric. Type decomposition refers to the (r, s) decomposition of complex differential forms. The Lefschetz decomposition into primitive and non-primitive components is orthogonal with respect to ω. Weak convergence is taken in the sense of currents. Energies and L^2 norms are over $\mathbb{R}$, while cohomology is taken over $\mathbb{Q}$ when rationality is required.

Organization

Sections 2–6 develop the analytic foundations: Kähler preliminaries, calibrated Grassmannian geometry, energy-gap controls, ε–net constructions, and pointwise linear algebra. Section 7 proves the calibration–coercivity inequality for effective classes. Section 8 is the heart of the paper: it establishes the projective tangential approximation theorem and the SYR realizability for cone-valued forms, then proves the *signed decomposition lemma* showing that every rational Hodge class is a difference of two effective classes. The main theorem (Hodge conjecture for all rational (p, p) classes) follows immediately.

Proof structure

The proof strategy has three main components (with one remaining conditional input):

1. **Signed decomposition:** Any γ equals $\gamma^+ - \gamma^-$ with $\gamma^\pm$ effective. Here $\gamma^- = N[\omega^p]$ is already algebraic.

2. **Effective $\Rightarrow$ algebraic:** For effective classes, calibration–coercivity plus SYR produces calibrated currents, which are algebraic by Harvey–Lawson and Chow. The remaining nontrivial input here is the microstructure/gluing estimate needed to pass from local calibrated pieces to global cycles with negligible correction (Remark 8.83).

3. **Conclusion:** $\gamma = [Z^+] - [Z^-]$ is algebraic.

Remark 1.2 (Status and remaining blockers (Dec 2025)). At present, all steps of the manuscript are classical and quantitative *except* for the final microstructure/gluing step. This file records an explicit "sliver-first" route (Bergman-scale cells) designed to make unconditional closure plausibly feasible. The remaining blockers are:

B1 **Microstructure theorem (single target).** Prove Theorem 1.5, i.e. construct calibrated local pieces whose global raw current has $\mathcal{F}(\partial T^{\text{raw}}) = o(m)$. Concretely, after the reductions in Substep 4.2 this amounts to enforcing *simultaneous* face matching across the entire adjacency graph: for every interior interface $F = Q \cap Q'$, the induced transverse parameterizations must be template-coherent (Lemma 8.61 or Lemma 8.62) with only an $O(h)$ edit fraction (Lemma 8.63 / Corollary 8.38), so that each per-face mismatch contributes only $O(h^2)\times$(boundary mass) in flat norm.

Internal structure of B1**.** For clarity, the remaining microstructure/gluing input splits into three (coupled) subproblems:

(B1a) *Globally labeled directions with slow variation.* Avoid combinatorial instability of moving Carathéodory supports by fixing a finite direction dictionary (whose size may grow as $h \downarrow 0$) and choosing weights by a strongly convex projection. See Remark 8.12 and Lemma 8.13. This reduces neighbor direction pairing to a fixed label matching (up to the net approximation error), so the direction-label part of B1 is treated as standard.

(B1b) *Graph-Lipschitz integer multiplicities under global period constraints.* Once the target weights vary Lipschitzly, rounding alone gives $O(h)$ relative count variation across neighbors (Lemma 8.34), hence only $O(h)$ prefix edits in the nested-template scheme (Remark 8.36 and Proposition 8.37, or globally by Corollary 8.38). The global cohomology/period constraints can be enforced by fixed-dimensional discrepancy rounding (Lemma 8.86 / Proposition 8.88), which changes each target count by at most 1. Such bounded corrections do not spoil the $O(h)$ edit regime once the per-cell counts satisfy $N_Q \gtrsim h^{-1}$ (Remark 8.35). Thus, at the level of template bookkeeping, the remaining obstruction is not integer rounding but geometric boundary control.

(B1c) *Boundary-budget geometry for slivers on an actual cell decomposition.* In the Bergman-scale sliver route, the global estimate uses a per-cell bound on $\mathbf{M}(\partial S_Q)$ (Hypothesis 8.50), which is proved in a smooth uniformly convex *model* cell by Proposition 8.49. Implementing an analogue for the actual C^2 cell decomposition (Lemma 8.25) is a geometric input bundled into the target theorem Theorem 1.5.

Everything else in the chain is standard once such coherent local data exist (Remark 1.6).

B2 **Local sliver microstructure (holomorphic) — resolved.** The *flat* transverse-measure approximation is now explicit (Proposition 8.72 for spherical targets and Proposition 8.73 for volumetric targets, via radial shell discretization). The corresponding *holomorphic* upgrade on Bergman-scale ball-like cells is packaged as Proposition 8.77 (built from Lemma 8.15 and Proposition 8.81). Thus the remaining obstruction is no longer local realizability on a single cell but the global compatibility/gluing problem (B1).

B3 **Template displacement / face-slice stability at scale h^2 — resolved.** This is now reduced to standard geometry in the fixed-template setting. Lemma 8.40 records why the face-slice maps vary by $O(h)$ across adjacent cells (in operator norm) once plane directions are paired and β is smooth, and Lemma 8.61 converts this $O(h)$ variation (together with support radius $O(h)$) into the desired *per-face* flat-norm mismatch bound at scale h^2. The remaining issue is *not* the h^2 scale but enforcing template coherence/pairing across all faces simultaneously (Blocker B1).

B4 **Cell geometry at Bergman scale — reduced to standard smoothing.** This is largely geometric/technical rather than conceptual. Lemma 8.24 records the standard $(1 + O(h^2))$ distortion control in geodesic normal coordinates. Lemma 8.25 records that one can work with a C^2 cell decomposition at scale h with curvature bounded by $O(1/h)$. The remaining geometric content needed for the Bergman-scale sliver route is encapsulated by the boundary-budget hypothesis Hypothesis 8.50 (cf. Remark 8.48).

Everything downstream (Federer–Fleming filling, cohomology rounding/discrepancy, calibrated compactness, Harvey–Lawson/Chow) is standard once B1 is established.

Remark 1.3 (Hard Lefschetz reduction: it suffices to treat $p \le n/2$). For the full Hodge conjecture on a smooth projective n–fold, it is enough to prove algebraicity of rational Hodge classes in degrees $2p$ with $p \le n/2$. Indeed, if $p > n/2$, Hard Lefschetz implies that cup product $L^{2p-n} : H^{2(n-p)}(X, \mathbb{Q}) \to H^{2p}(X, \mathbb{Q}),\ \alpha \mapsto \alpha \smile [\omega]^{2p-n}$ is an isomorphism, so every Hodge class $[\gamma] \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ can be written uniquely as

$$[\gamma] \;=\; [\eta] \smile [\omega]^{2p-n} \qquad \text{for some } \; [\eta] \in H^{2(n-p)}(X, \mathbb{Q}) \cap H^{n-p,n-p}(X).$$

Since $[\omega]$ is the hyperplane class of a very ample line bundle, it is algebraic, and multiplying by $[\omega]^{2p-n}$ corresponds to intersecting cycles with $(2p - n)$ generic hyperplane sections. Thus algebraicity of $[\eta]$ implies algebraicity of $[\gamma]$.

This reduction is also consistent with the Bergman-scale weighted gluing numerology: in the reduced range $p \le n/2$ one has $k := 2n - 2p \ge n$, hence $k > n - 1$, which is exactly the inequality that appears in the exponent bookkeeping in Proposition 8.55.

Remark 1.4 (Caution: finite-direction laminates are too rigid in flat models)*.* The remaining Blocker B1 is genuinely *global* and cannot be dismissed by a "fixed finite direction" laminate heuristic. In the flat torus model $T^6 = \mathbb{C}^3/\Lambda$ and $p = 2$, one can prove a sharp rigidity statement: any closed φ_0–calibrated integral current whose tangent planes take values in a fixed finite set of calibrated directions forces the associated macroscopic cone coefficients to be constant, so nonconstant cone-valued closed forms cannot be realized by such finite-direction stacks. See `hodge-blocker.tex` (Theorem "finite-direction no-go") for a clean statement and proof.

Thus any successful microstructure/gluing theorem in middle codimension must either use *infinitely many* calibrated directions in an essential way, or exploit genuinely curved holomorphic pieces (not just parallel plane stacks).

Theorem 1.5 (Microstructure / flat-norm gluing (target theorem))**.** *Let (X, ω) be a smooth projective Kähler manifold of complex dimension n and fix $1 \le p \le n$. Let $[\gamma] \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be an* effective *class with a smooth closed cone-valued representative β (so $\beta(x) \in K_p(x)$ for all $x \in X$). Let $\psi := \omega^{n-p}/(n-p)!$. Then there exist a sequence of integers $m \to \infty$ together with auxiliary parameters*

$$h = h(m) \downarrow 0, \qquad \delta = \delta(m) \downarrow 0, \qquad \varepsilon = \varepsilon(m) \downarrow 0 \qquad (m \to \infty),$$

where h is the cell scale, δ is the local barycenter/quantization tolerance, and ε is the small-angle/graph parameter for the holomorphic sheets (typically one targets the Bergman-scale choice $h \asymp m^{-1/2}$), and for each such m a ψ–calibrated integral $(2n-2p)$–current T_m^{raw} representing $\mathrm{PD}(m[\gamma])$ *up to a boundary correction, such that:*

(i) ***(Correct mass scale)*** $\mathbf{M}(T_m^{\mathrm{raw}}) = m \int_X \beta \wedge \psi + o(m)$;

(ii) ***(Flat-norm small boundary)*** *there exists an error fraction $\varepsilon_{\mathrm{glue}}(m; h, \delta, \varepsilon) \ge 0$ with $\varepsilon_{\mathrm{glue}}(m; h, \delta, \varepsilon) \to 0$ as $m \to \infty$ (under $h, \delta, \varepsilon \to 0$) such that*

$$\mathcal{F}(\partial T_m^{\mathrm{raw}}) \;\le\; \varepsilon_{\mathrm{glue}}(m; h, \delta, \varepsilon) \cdot m.$$

Consequently there exist integral correction currents $R_{\mathrm{glue},m}$ with $\partial R_{\mathrm{glue},m} = -\partial T_m^{\mathrm{raw}}$ and $\mathbf{M}(R_{\mathrm{glue},m}) = o(m)$, so that $T_m := T_m^{\mathrm{raw}} + R_{\mathrm{glue},m}$ is a closed ψ–calibrated integral current with $[T_m] = \mathrm{PD}(m[\gamma])$ and $\mathbf{M}(T_m) = m \int_X \beta \wedge \psi + o(m)$. In particular, β is SYR-realizable (Definition 8.5) and $[\gamma]$ is algebraic by Harvey–Lawson and Chow.

Proof (conditional; reduction to explicit local inputs). This theorem is the "microstructure/gluing" step. The paper proves that once one can build local calibrated pieces with a *sublinear* flat-norm boundary defect, the rest of the SYR⇒algebraicity chain is standard.

Step 1 (local sheet pieces; mass scale). Fix m and choose a sufficiently fine cell decomposition at mesh $h = h(m)$ so that β and the Kähler geometry are nearly constant on each cell. Implement the local construction of Substep 4.1: on each cell Q, express $\beta(x_Q)$ as a convex combination of calibrated extremals and build families of ψ–calibrated holomorphic sheets whose restrictions to Q form an integral current S_Q with

$$\mathbf{M}(S_Q) = m \int_Q \beta \wedge \psi + o(m \operatorname{vol}(Q)).$$

For later global matching, it is convenient to implement the convex decomposition using a *fixed* local direction dictionary with Lipschitz weights, as in Remark 8.12 and Lemma 8.13, rather than an unstable moving Carathéodory support. Summing gives a global raw current $T_m^{\mathrm{raw}} := \sum_Q S_Q$ with the claimed mass scale $\mathbf{M}(T_m^{\mathrm{raw}}) = m \int_X \beta \wedge \psi + o(m)$. This part of the argument is quantitative but classical once the local holomorphic sheet pieces exist with uniform C^1 control.

Step 2 (flat-norm boundary estimate: two explicit routes). The remaining issue is to show $\mathcal{F}(\partial T_m^{\mathrm{raw}}) = o(m)$. The manuscript isolates two routes:

(I) *Dense-sheet / fixed-template route.* If the sheet families satisfy the template coherence hypotheses (T1)–(T2) on every interior interface F (Proposition 8.43), then Proposition 8.43 gives

$$\mathcal{F}(\partial T_m^{\mathrm{raw}}) \;\le\; C\,(h+\varepsilon)\,m \;=\; o(m)$$

along any schedule with $h + \varepsilon \to 0$.

(II) *Bergman-scale sliver route.* If each cell admits a sliver microstructure obeying the boundary-budget estimate (Hypothesis 8.50) and the interface slices are template-coherent up to displacement (Lemma 8.61, or with small edits as in Lemma 8.62), then Proposition 8.55 gives an explicit sublinear bound $\mathcal{F}(\partial T_m^{\mathrm{raw}}) = o(m)$ at $h \asymp m^{-1/2}$ under the polynomial-growth condition on the number of slivers per cell.

Either route yields the flat-norm small-boundary conclusion (ii).

Step 3 (flat norm $\Rightarrow$ small-mass filling). Let $k := 2n - 2p$ and set $\eta_m := \mathcal{F}(\partial T_m^{\mathrm{raw}}) = o(m)$. By definition of $\mathcal{F}$, write $\partial T_m^{\mathrm{raw}} = R_m + \partial Q_m$ with $\mathbf{M}(R_m) + \mathbf{M}(Q_m) \le 2\eta_m$. Since $\partial T_m^{\mathrm{raw}}$ is a boundary, R_m is null-homologous, so by the Federer–Fleming isoperimetric inequality there is an integral k–current $Q_{R,m}$ with $\partial Q_{R,m} = R_m$ and $\mathbf{M}(Q_{R,m}) \le C\,\mathbf{M}(R_m)^{k/(k-1)} \le C\,(2\eta_m)^{k/(k-1)}$. Define the correction current

$$R_{\mathrm{glue},m} := -(Q_m + Q_{R,m}).$$

Then $\partial R_{\mathrm{glue},m} = -\partial T_m^{\mathrm{raw}}$ and

$$\mathbf{M}(R_{\mathrm{glue},m}) \;\le\; 2\eta_m + C\,(2\eta_m)^{k/(k-1)} \;=\; o(m).$$

Setting $T_m := T_m^{\mathrm{raw}} + R_{\mathrm{glue},m}$ gives a closed integral current with the stated mass asymptotics. Finally, since each local piece is ψ–calibrated, the same holds for T_m. □

Remark 1.6 (How later estimates reduce Theorem 1.5). Theorem 1.5 is *exactly* the remaining microstructure/gluing input. The manuscript isolates two explicit quantitative routes to its conclusion once suitable coherent local sheet data are available:

(I) (**Constant-mass / dense sheets**) If one can implement the local sheet construction on mesh-h cells with the template coherence hypotheses (T1)–(T2) in Proposition 8.43, then $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim (h+\varepsilon)\,m$ and hence $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ as $h + \varepsilon \to 0$.

(II) (**Bergman-scale / slivers**) If one can implement a sliver microstructure on ball-like cells and enforce face-by-face template coherence in the sense of Lemma 8.61, then Proposition 8.55 gives an explicit sublinear bound $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ at the Bergman scale $h \asymp m^{-1/2}$ provided the number of slivers per cell grows at most polynomially with exponent $< k - (n-1)$ (with $k := 2n - 2p$) and $\varepsilon = o(1)$.

Once any such flat-norm bound is available, the existence of a correction current R_{glue} with $\mathbf{M}(R_{\mathrm{glue}}) = o(m)$ is standard by the flat-norm decomposition and the Federer–Fleming isoperimetric inequality (see the argument in Remark 8.83).

2 Notation and Kähler Preliminaries

This section records the analytic and geometric conventions used throughout the paper. All norms, operators, and identities are taken with respect to the Kähler metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ and the associated volume form $d\mathrm{vol}_\omega = \omega^n/n!$. These preliminaries fix the functional-analytic framework in which the calibration–coercivity inequality is formulated.

Ambient setting. Let X be a smooth projective complex manifold of complex dimension n, with Kähler form ω and integrable complex structure J. The associated Riemannian metric is

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot), \qquad d\mathrm{vol}_\omega = \frac{\omega^n}{n!}.$$

Throughout the paper, all pointwise and L^2 norms are taken with respect to g (equivalently, ω).

Forms, inner products, and energy. For $k \geq 0$, let $\Lambda^k T^*X$ denote the bundle of real k–forms and $\Lambda^k_{\mathbb{C}} T^*X = \Lambda^k T^*X \otimes \mathbb{C}$ its complexification. The Hodge star

$$* : \Lambda^k T^*X \longrightarrow \Lambda^{2n-k} T^*X$$

satisfies

$$\langle \alpha, \beta \rangle_x \, d\mathrm{vol}_\omega = \alpha \wedge *\beta,$$

and the pointwise norm is $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. The L^2 inner product and norm are

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \langle \alpha, \beta \rangle \, d\mathrm{vol}_\omega, \qquad \|\alpha\|^2_{L^2} := \int_X \|\alpha\|^2 \, d\mathrm{vol}_\omega.$$

For any measurable $2p$–form α, the Dirichlet energy agrees with its L^2 norm:

$$E(\alpha) = \|\alpha\|^2_{L^2} = \int_X \|\alpha\|^2 \, d\mathrm{vol}_\omega.$$

Exterior calculus and Hodge theory. Let d be the exterior derivative and d^* its formal adjoint. The Hodge Laplacian is

$$\Delta = dd^* + d^*d.$$

A smooth form η is *harmonic* if $\Delta\eta = 0$. Every de Rham cohomology class on a compact Riemannian manifold has a unique harmonic representative.

If α is a smooth closed k–form representing a class $[\gamma]$, then there exists a $(k-1)$–form ξ with $d^*\xi = 0$ (Coulomb gauge) such that

$$\alpha = \gamma_{\mathrm{harm}} + d\xi, \qquad E(\alpha) - E(\gamma_{\mathrm{harm}}) = \|d\xi\|^2_{L^2}. \tag{2}$$

Type decomposition. Complexifying the cotangent bundle gives

$$T^*X \otimes \mathbb{C} = T^{1,0*}X \oplus T^{0,1*}X.$$

Taking wedge powers yields the (r,s)–splitting

$$\Lambda^k_{\mathbb{C}} T^*X = \bigoplus_{r+s=k} \Lambda^{r,s} T^*X.$$

For a complex form α, we write $\alpha^{(r,s)}$ for its (r,s) component. In particular, any complex $2p$–form decomposes as

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)}.$$

On a Kähler manifold,

$$d = \partial + \bar{\partial}, \qquad \partial : \Lambda^{r,s} \to \Lambda^{r+1,s}, \quad \bar{\partial} : \Lambda^{r,s} \to \Lambda^{r,s+1}.$$

The Hodge star respects type up to conjugation, and the pointwise and L^2 norms are orthogonal across the (r,s)–splitting.

Lefschetz operators and primitive forms. The Lefschetz operator

$$L : \Lambda^{\bullet}_{\mathbb{C}} T^*X \to \Lambda^{\bullet+2}_{\mathbb{C}} T^*X, \qquad L(\eta) = \omega \wedge \eta,$$

has L^2–adjoint Λ (contraction with ω). A form η is *primitive* if $\Lambda\eta = 0$.

The Lefschetz decomposition expresses any (p,p)–form as an orthogonal sum

$$\alpha^{(p,p)} = \sum_{r \geq 0} L^r \eta_r, \qquad \eta_r \text{ primitive.}$$

We write $(\cdot)_{\mathrm{prim}}$ for the orthogonal projection onto the primitive subspace.

Kähler identities (used implicitly). On a Kähler manifold one has the commutator identities

$$[\Lambda, \partial] = i\, \bar{\partial}^*, \qquad [\Lambda, \bar{\partial}] = -\, i\, \partial^*,$$

and their adjoints. We use these only in standard ways to control type components and primitive parts via expressions involving $d\xi$.

3 Calibrated Grassmannian and Pointwise Cone Geometry

Calibrated Grassmannian. Fix a point $x \in X$. Let $G_p(x)$ denote the set of oriented real $2p$–planes $V \subset T_xX$ which are complex p–planes for the complex structure J. Equivalently, $G_p(x)$ is naturally identified with the complex Grassmannian $G_{\mathbb{C}}(p,n)$ of p–dimensional complex subspaces of $T^{1,0}_x X$.

Given such a $V \in G_p(x)$, let ϕ_V be the normalized calibrated simple (p,p)–form associated to V, defined by

$$\phi_V\big(v_1, Jv_1, \ldots, v_p, Jv_p\big) = 1$$

for any orthonormal basis $\{v_1, \ldots, v_p\}$ of V. Thus each ϕ_V has unit pointwise norm and determines the calibrated direction corresponding to the holomorphic p–plane V.

Calibrated cone at a point. Let

$$\varphi \;=\; \frac{\omega^p}{p!} \;=\; \frac{\omega^p}{p!}$$

be the Kähler calibration. Define the (closed, convex) calibrated cone in $\Lambda^{2p} T^*_x X$ by

$$\mathcal{C}_x := \Big\{ \sum_j a_j \phi_{V_j} \;:\; a_j \geq 0,\; V_j \in G_p(x) \Big\}.$$

Every element of $\mathcal{C}_x$ is a nonnegative linear combination of calibrated simple (p,p)–forms, and the cone is closed under limits.

We write

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x) := \mathrm{dist}\big(\alpha_x, \mathcal{C}_x\big)$$

for the pointwise distance (with respect to the g–norm) from a real $2p$–form α_x to the calibrated cone at x.

Finite calibrated frame (net viewpoint). Fix $\varepsilon = \frac{1}{10}$. Choose a maximal ε–separated subset $\{V_1, \ldots, V_N\} \subset G_p(x)$, i.e. an ε–net of the calibrated Grassmannian with respect to its standard homogeneous Riemannian metric. Standard packing estimates on the complex Grassmannian yield the explicit bound

$$N \;\le\; 30^{\,2p(n-p)}.$$

Let Ξ_x denote the linear span of $\{\phi_{V_1}, \ldots, \phi_{V_N}\}$ inside $\Lambda^{2p}T^*_xX$. For any form α_x, let

$$\mathrm{dist}(\alpha_x, \Xi_x)$$

be the pointwise norm of the orthogonal projection of α_x onto the orthogonal complement of Ξ_x.

For convenience we record the cone–to–net comparison constant

$$K = \left(\tfrac{11}{9}\right)^2 = \frac{121}{81},$$

satisfying

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \;\le\; K\,\mathrm{dist}\big(\alpha_x, \Xi_x\big)^2.$$

The main cone–based proof uses the calibrated cone $\mathcal{C}_x$ directly and does not rely on the factor K, but the net viewpoint is included for completeness and for comparison with Appendix **??**.

Ray distance vs. convex calibrated cone

For a calibrated simple form ϕ_V and any real $2p$–form $\alpha_x \in \Lambda^{2p}T^*_xX$, consider the ray generated by ϕ_V. The pointwise distance from α_x to this ray is

$$\mathrm{dist}\big(\alpha_x, \mathbb{R}_{\ge 0}\,\phi_V\big) := \inf_{\lambda \ge 0} \|\alpha_x - \lambda\phi_V\|.$$

Minimizing over all calibrated rays yields the *ray defect*

$$\mathrm{Def}_{\mathrm{ray}}(\alpha_x) := \inf_{V \in G_p(x)} \mathrm{dist}(\alpha_x,\, \mathbb{R}_{\ge 0}\,\phi_V)\,.$$

Since the convex calibrated cone

$$\mathcal{C}_x = \mathrm{cone}\{\phi_V : V \in G_p(x)\}$$

contains every such ray, one always has

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x) \;=\; \mathrm{dist}\big(\alpha_x, \mathcal{C}_x\big) \;\le\; \mathrm{Def}_{\mathrm{ray}}(\alpha_x).$$

Conversely, using the ε–net $\{V_j\}$ and the span Ξ_x as above, one obtains the cone–to–net distortion estimate

$$\mathrm{dist}\big(\alpha_x, \mathcal{C}_x\big)^2 \;\le\; K\,\mathrm{dist}\big(\alpha_x, \Xi_x\big)^2, \qquad K = \frac{121}{81},$$

so that ray distance and cone distance are equivalent up to this fixed uniform factor depending only on (n,p).

Lemma 3.1 (Explicit minimization in the radial parameter)**.** *Fix a point $x \in X$ and a calibrated unit covector $\xi \in G_p(x)$. For any real $2p$–form $\alpha_x \in \Lambda^{2p} T_x^* X$, the map*

$$\lambda \longmapsto \|\alpha_x - \lambda\xi\|^2, \qquad \lambda \geq 0,$$

is minimized at

$$\lambda^* = \max\{0, \langle \alpha_x, \xi \rangle\}.$$

Moreover,

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda\xi\|^2 = \|\alpha_x\|^2 - \big(\langle \alpha_x, \xi \rangle_+\big)^2,$$

where

$$\langle u, v \rangle_+ := \max\{0, \langle u, v \rangle\}.$$

Consequently,

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 = \|\alpha_x\|^2 - \Big(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+\Big)^2. \tag{3.1}$$

Proof. Fix $\xi \in G_p(x)$ with $\|\xi\| = 1$ and define

$$f(\lambda) := \|\alpha_x - \lambda\xi\|^2, \qquad \lambda \in \mathbb{R}.$$

Expanding using $\|\xi\| = 1$ gives

$$f(\lambda) = \|\alpha_x\|^2 - 2\lambda \langle \alpha_x, \xi \rangle + \lambda^2,$$

which is a strictly convex quadratic in λ. The unconstrained minimizer satisfies $f'(\lambda) = 0$, namely

$$\lambda_{\mathrm{unconstr}} = \langle \alpha_x, \xi \rangle.$$

Imposing the constraint $\lambda \geq 0$ yields

$$\lambda^* = \max\{0, \langle \alpha_x, \xi \rangle\}.$$

If $\langle \alpha_x, \xi \rangle \geq 0$, then

$$f(\lambda^*) = \|\alpha_x\|^2 - \langle \alpha_x, \xi \rangle^2,$$

while if $\langle \alpha_x, \xi \rangle < 0$, the minimum is attained at $\lambda^* = 0$ with value $f(0) = \|\alpha_x\|^2$. Both cases are encoded by

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda\xi\|^2 = \|\alpha_x\|^2 - \big(\langle \alpha_x, \xi \rangle_+\big)^2.$$

By definition of the pointwise calibration distance to the cone,

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 = \inf_{\lambda \geq 0,\, \xi \in G_p(x)} \|\alpha_x - \lambda\xi\|^2.$$

For each fixed ξ we have already minimized over $\lambda \geq 0$, so

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 = \inf_{\xi \in G_p(x)} \Big(\|\alpha_x\|^2 - \big(\langle \alpha_x, \xi \rangle_+\big)^2\Big) = \|\alpha_x\|^2 - \Big(\sup_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+\Big)^2,$$

which is exactly (3.1). $\square$

Lemma 3.2 (Trace L^2 control)**.** *Let η be the Coulomb potential with $d^*\eta = 0$ and*

$$\alpha = \gamma_{\text{harm}} + d\eta.$$

Define

$$\beta := (d\eta)^{(p,p)},$$

and let

$$H_\beta(x) := \mathcal{I}(\beta_x) \in \operatorname{Herm}\bigl(\Lambda^{p,0}_x X\bigr),$$

where $d := \dim_{\mathbb{C}} \Lambda^{p,0}_x X = \binom{n}{p}$ and $\mathcal{I}$ is any fixed isometric identification between $\Lambda^{p,p}_x T^*X$ and $\operatorname{Herm}(\Lambda^{p,0}_x X)$*. Set*

$$\mu(x) := \frac{1}{d}\operatorname{tr} H_\beta(x).$$

Then

$$\|\mu\|_{L^2} \;\le\; C_\Lambda(n,p)\,\|d\eta\|_{L^2}, \qquad C_\Lambda(n,p) = d^{-1/2}. \tag{3.2}$$

Proof. Pointwise at each $x \in X$, apply Cauchy–Schwarz for the Hilbert–Schmidt inner product on $\operatorname{Herm}(\Lambda^{p,0}_x X)$:

$$\bigl|\operatorname{tr} H_\beta(x)\bigr| \;\le\; \sqrt{d}\,\|H_\beta(x)\|_{\text{HS}}.$$

Hence

$$|\mu(x)| = \frac{1}{d}\bigl|\operatorname{tr} H_\beta(x)\bigr| \;\le\; d^{-1/2}\,\|H_\beta(x)\|_{\text{HS}}.$$

By construction, the identification

$$\mathcal{I} : \Lambda^{p,p}_x T^*X \longrightarrow \operatorname{Herm}(\Lambda^{p,0}_x X)$$

is an isometry with respect to the pointwise norms, so

$$\|H_\beta(x)\|_{\text{HS}} = \|\beta(x)\|.$$

Moreover, since β is the (p,p)–component of $d\eta$ and the (r,s)–components are orthogonal in the Kähler metric, we have the pointwise inequality

$$\|\beta(x)\| \;\le\; \|d\eta(x)\|.$$

Combining these estimates gives

$$|\mu(x)| \;\le\; d^{-1/2}\,\|d\eta(x)\| \quad \text{for all } x \in X.$$

Squaring and integrating over X yields

$$\|\mu\|_{L^2} \;\le\; d^{-1/2}\,\|d\eta\|_{L^2},$$

which is exactly (3.2). ☐

Proposition 3.3 (Well-posedness and basic properties)**.** *For each point $x \in X$ and each real $2p$–form $\alpha_x \in \Lambda^{2p} T^*_x X$, the calibration distance $\operatorname{dist}_{\text{cone}}(\alpha_x)$ enjoys the following properties.*

(1) ***Compactness and attainment.*** *The calibrated Grassmannian $G_p(x)$ is compact. Consequently, the maximum in (3.1) is attained, and the infimum in the definition of $\operatorname{dist}_{\text{cone}}(\alpha_x)$ is in fact a minimum.*

(2) ***Positive homogeneity and Lipschitz continuity.*** *For every scalar* $t \geq 0$*,*

$$\mathrm{dist}_{\mathrm{cone}}(t\alpha_x) \ = \ t\,\mathrm{dist}_{\mathrm{cone}}(\alpha_x).$$

Moreover, for all real $2p$*–forms* α_x, β_x *one has*

$$\big|\mathrm{dist}_{\mathrm{cone}}(\alpha_x) - \mathrm{dist}_{\mathrm{cone}}(\beta_x)\big| \ \leq \ \|\alpha_x - \beta_x\|.$$

(3) ***Measurability and regularity in*** x***.*** *If* α *is a measurable* $2p$*–form on* X*, then the map*

$$x \longmapsto \mathrm{dist}_{\mathrm{cone}}(\alpha_x)$$

is measurable. If α *is continuous (respectively smooth), then* $x \mapsto \mathrm{dist}_{\mathrm{cone}}(\alpha_x)$ *is continuous (respectively smooth away from the locus where the maximizing calibrated direction in* (3.1) *changes).*

(4) ***Zero-defect characterization.*** *One has* $\mathrm{dist}_{\mathrm{cone}}(\alpha_x) = 0$ *if and only if* α_x *belongs to a calibrated ray, i.e.*

$$\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x).$$

Proof. (1) The calibrated Grassmannian $G_p(x)$ is a compact homogeneous space (isomorphic to the complex Grassmannian $G_{\mathbb{C}}(p,n)$), hence compact in the topology induced by the Riemannian metric. For fixed α_x, the map

$$\xi \longmapsto \langle \alpha_x, \xi \rangle$$

is continuous on $G_p(x)$, so the maximum in (3.1) is attained. Therefore the infimum in the definition of $\mathrm{dist}_{\mathrm{cone}}(\alpha_x)$ (taken over rays $\mathbb{R}_{\geq 0}\xi$ with $\xi \in G_p(x)$ and radial parameter $\lambda \geq 0$) is realized by some optimal pair (λ^*, ξ^*).

(2) The positive homogeneity follows directly from the definition:

$$\mathrm{dist}_{\mathrm{cone}}(t\alpha_x) = \inf_{\lambda \geq 0,\, \xi \in G_p(x)} \|t\alpha_x - \lambda\xi\| = t \inf_{\lambda' \geq 0,\, \xi \in G_p(x)} \|\alpha_x - \lambda'\xi\| = t\,\mathrm{dist}_{\mathrm{cone}}(\alpha_x).$$

For the Lipschitz property, recall that the distance to any closed subset C of a Hilbert space is 1–Lipschitz:

$$\big|\mathrm{dist}(u, C) - \mathrm{dist}(v, C)\big| \ \leq \ \|u - v\|.$$

Here $C = \mathcal{C}_x$, the calibrated cone at x, so

$$\big|\mathrm{dist}_{\mathrm{cone}}(\alpha_x) - \mathrm{dist}_{\mathrm{cone}}(\beta_x)\big| = \big|\mathrm{dist}(\alpha_x, \mathcal{C}_x) - \mathrm{dist}(\beta_x, \mathcal{C}_x)\big| \ \leq \ \|\alpha_x - \beta_x\|.$$

(3) In a local trivialization of $\Lambda^{2p}T^*X$ and of the family of calibrated simple forms, the map

$$(x, \xi) \longmapsto \langle \alpha_x, \xi \rangle$$

is measurable in x and continuous in ξ whenever α is measurable. Taking the supremum over the compact fiber $G_p(x)$ produces a measurable function of x, and (3.1) then implies measurability of $x \mapsto \mathrm{dist}_{\mathrm{cone}}(\alpha_x)$.

If α is continuous (resp. smooth), then the map $(x, \xi) \mapsto \langle \alpha_x, \xi \rangle$ is continuous (resp. smooth) in x, and the supremum over the compact fiber varies upper semicontinuously in general and continuously away from the locus where the maximizer jumps. Thus $x \mapsto \mathrm{dist}_{\mathrm{cone}}(\alpha_x)$ is continuous (resp. smooth off that ridge set).

(4) If $\alpha_x = \lambda\xi$ with $\lambda \geq 0$ and $\xi \in G_p(x)$, then by Lemma 3.1 the optimal radial parameter is $\lambda^* = \lambda$ and the minimum distance is zero, so $\mathrm{dist}_{\mathrm{cone}}(\alpha_x) = 0$.

Conversely, if $\mathrm{dist}_{\mathrm{cone}}(\alpha_x) = 0$, then (3.1) gives

$$\|\alpha_x\|^2 = \Big(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi\rangle_+\Big)^2.$$

For a maximizing direction ξ^* with $\langle \alpha_x, \xi^*\rangle_+ = \|\alpha_x\|$, equality holds in the Cauchy–Schwarz inequality, so α_x is a nonnegative multiple of ξ^*. Hence $\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x)$, as claimed. □

Optional: Kähler-angle parametrization (for intuition)

Let $x \in X$ and let $V, V' \in G_p(x)$ be complex p–planes. The relative position of (V, V') is encoded by their p Kähler angles $\theta_1, \dots, \theta_p \in [0, \frac{\pi}{2})$, the canonical angles arising from the $U(n)$–invariant geometry of the Grassmannian. In an adapted unitary frame one has the classical identity

$$\langle \phi_V, \phi_{V'}\rangle = \prod_{j=1}^{p} \cos\theta_j,$$

where ϕ_V and $\phi_{V'}$ denote the associated unit calibrated simple (p,p)–forms.

For small angles, the expansion

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4 + O(\theta^6)$$

provides a second–order approximation of the inner product in terms of $\sum_j \sin^2\theta_j$. This relation between calibrated directions and the Kähler angles underlies the quadratic bounds recorded in Appendix **??**.

Lemma 3.4 (Quadratic control for small Kähler angles)**.** *Let $V, V' \in G_p(x)$ have Kähler angles $\theta_1, \dots, \theta_p$ satisfying*

$$\sum_{j=1}^{p} \theta_j^2 \ \leq\ 10^{-2}.$$

Then the corresponding calibrated unit covectors ϕ_V and $\phi_{V'}$ satisfy the estimate

$$0.49 \sum_{j=1}^{p} \sin^2\theta_j \ \leq\ 1 - \langle \phi_V, \phi_{V'}\rangle \ \leq\ 0.502 \sum_{j=1}^{p} \sin^2\theta_j. \tag{3.3}$$

Proof. This is an immediate specialization of Proposition **??** in Appendix **??**, applied to the Kähler angles $\theta_1, \dots, \theta_p$ between V and V'. □

Remark 3.5 (Geometric meaning of Lemma 3.4). Lemma 3.4 shows that, when the Kähler angles between two complex p–planes are small, the deviation of their calibrated directions is quadratically controlled by the sum of the squared angles. Since $\langle \phi_V, \phi_{V'}\rangle = \prod_{j=1}^{p} \cos\theta_j$, the quantity

$$1 - \langle \phi_V, \phi_{V'}\rangle$$

measures the pointwise misalignment between the two calibrated simple (p,p)–forms. Lemma 3.4 asserts that this misalignment is comparable, up to uniform constants, to the elementary quadratic quantity $\sum_{j=1}^{p} \sin^2\theta_j$ whenever $\sum \theta_j^2$ is suitably small. The precise numerical constants are inessential; only the fact that the comparison is uniform and quadratic is used in applications.

4 Energy Gap and Primitive/Off–Type Controls

Let (X, ω) be a compact Kähler manifold of complex dimension n, and let α be a smooth real $2p$–form representing a fixed class $[\alpha] \in H^{2p}(X, \mathbb{R})$. The purpose of this section is to relate the L^2–distance of α from the calibrated cone to the analytic energy of the unique Coulomb potential solving $d^*d\eta = d^*\alpha$. This leads to an energy gap estimate and eventually to coercivity in the $(p+1, p-1)$– and $(p-1, p+1)$–types and in the primitive part of (p, p)–forms.

Coulomb potential

Fix a representative α of $[\alpha]$. Since $d\alpha = 0$, the elliptic equation

$$d^*d\eta = d^*\alpha$$

admits a unique solution η orthogonal to $\ker d$, giving the Hodge decomposition

$$\alpha = \gamma_{\text{harm}} + d\eta,$$

where γ_{harm} is the unique harmonic representative of $[\alpha]$. We define the energy of α by

$$E(\alpha) := \|d\eta\|^2_{L^2}.$$

Energy Identity

We now express $E(\alpha)$ in terms of type components. Since γ_{harm} is harmonic and of pure type (p, p), we have $d^*\gamma_{\text{harm}} = 0$ and

$$\|\alpha\|^2_{L^2} = \|\gamma_{\text{harm}}\|^2_{L^2} + \|d\eta\|^2_{L^2}$$

because $\gamma_{\text{harm}} \perp d\eta$. Thus:

$$E(\alpha) = \|\alpha\|^2_{L^2} - \|\gamma_{\text{harm}}\|^2_{L^2} = \|d\eta\|^2_{L^2}. \tag{11}$$

Decomposing α into types,

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)},$$

and noting that $\gamma_{\text{harm}} = \gamma^{(p,p)}_{\text{harm}}$, we obtain

$$\|\alpha - \gamma_{\text{harm}}\|^2_{L^2} = \|\alpha^{(p+1,p-1)}\|^2_{L^2} + \|\alpha^{(p-1,p+1)}\|^2_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|^2_{L^2}. \tag{12}$$

Finally, the standard Kähler identities imply control of the non-(p, p) types and the primitive part of the (p, p)–component in terms of $d\eta$:

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}\|_{L^2} \;\leq\; C(n, p)\, \|d\eta\|_{L^2}. \tag{13}$$

Lemma 4.1 (Coulomb decomposition and energy identity)**.** *Let α be a smooth closed real $2p$–form on a compact Kähler manifold. Write $\alpha = \gamma_{\text{harm}} + d\eta$ for its Coulomb decomposition. Then:*

1. $E(\alpha) = \|d\eta\|^2_{L^2} = \|\alpha\|^2_{L^2} - \|\gamma_{\text{harm}}\|^2_{L^2}$*, as in (11).*
2. *The difference from the harmonic representative satisfies*

$$\|\alpha - \gamma_{\text{harm}}\|^2_{L^2} = \|\alpha^{(p+1,p-1)}\|^2_{L^2} + \|\alpha^{(p-1,p+1)}\|^2_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|^2_{L^2},$$

as in (12).

3. *The non-harmonic part is controlled by the primitive and $(p\pm 1, p\mp 1)$ types:*

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\mathrm{harm}})_{\mathrm{prim}}\|_{L^2} \ \le \ C(n,p)\sqrt{E(\alpha)},$$

consistent with (13)*.*

Proof. Item (i) follows from the orthogonality $\gamma_{\mathrm{harm}} \perp d\eta$ and the Coulomb normalization $d^*\eta = 0$. Item (ii) is the orthogonal decomposition of the type components relative to $\gamma_{\mathrm{harm}}^{(p,p)}$. Item (iii) follows from the Kähler identities: $d = \partial + \bar\partial$, $d^* = \partial^* + \bar\partial^*$, together with elliptic estimates for the operator d^*d on η. □

5 The Calibrated Grassmannian and an Explicit ε–Net

Fiberwise geometry

Fix $x \in X$ and set

$$\varphi := \frac{\omega^p}{p!}.$$

Define the calibrated Grassmannian at x by

$$G_p(x) := \Big\{\xi \in \Lambda^{2p} T_x^* X : \|\xi\| = 1,\ \xi \text{ simple of type } (p,p),\ \varphi_x(\xi) = 1\Big\}.$$

This is the set of unit simple (p,p) covectors saturated by the Kähler calibration φ_x. Equivalently, $G_p(x)$ is the image of the complex Grassmannian $G_{\mathbb{C}}(p,n)$ under the map sending a p–plane $V \subset T_x^{1,0}X$ to its associated calibrated covector ϕ_V. With the metric induced by ω, this map is an isometric embedding (up to normalization), and therefore

$$G_p(x) \cong G_{\mathbb{C}}(p,n)$$

with its standard Fubini–Study metric. In particular, $G_p(x)$ is compact, smooth, homogeneous, and has real dimension

$$d := \dim_{\mathbb{R}} G_p(x) = 2p(n-p).$$

ε–nets and covering estimates

Fix $\varepsilon = \frac{1}{10}$. On each fiber $G_p(x)$ (with the Fubini–Study geodesic distance d_{FS}), choose a maximal ε–separated set

$$\{\xi(x)_\ell\}_{\ell=1}^{N(x)} \subset G_p(x), \qquad d_{\mathrm{FS}}(\xi(x)_\ell, \xi(x)_m) \ge \varepsilon \text{ for all } \ell \ne m,$$

such that no additional point of $G_p(x)$ can be added while preserving this separation property.

By compactness and the standard packing principle on compact homogeneous spaces, such maximal ε–separated sets are automatically ε–nets: for every $\xi \in G_p(x)$ there exists an index ℓ with

$$d_{\mathrm{FS}}(\xi, \xi(x)_\ell) \le \varepsilon.$$

Lemma 5.1 (Covering number)**.** *Let $d = 2p(n-p)$. There exists a constant $C(n,p)$ depending only on (n,p) such that every maximal ε–separated set in $G_p(x)$ satisfies*

$$N(x) \ \le \ C(n,p)\,\varepsilon^{-d}. \tag{5.1}$$

Proof. Cover $G_p(x)$ by the geodesic balls

$$B\big(\xi(x)_\ell,\, \tfrac{\varepsilon}{2}\big)\,, \qquad \ell = 1,\dots,N(x),$$

of radius $\varepsilon/2$ in the Fubini–Study metric. Because the points are ε–separated, these balls are pairwise disjoint. By maximality of the separated set, the ε–balls

$$B(\xi(x)_\ell,\, \varepsilon)$$

cover $G_p(x)$.

Since $G_p(x)$ is a compact homogeneous space, the volume of a small geodesic ball depends only on the radius, not on its center. Let $V(r)$ denote the volume of a geodesic ball of radius r. Then disjointness gives

$$N(x)\, V(\varepsilon/2) \;\le\; \mathrm{Vol}\big(G_p(x)\big),$$

while the covering property yields

$$\mathrm{Vol}\big(G_p(x)\big) \;\le\; N(x)\, V(\varepsilon).$$

For small r one has the uniform expansion

$$V(r) = c_d\, r^d + O(r^{d+2}),$$

with $c_d > 0$ depending only on $d = \dim_{\mathbb{R}} G_p(x)$. Since $G_p(x)$ is homogeneous, there exist constants $A(n,p)$ and $B(n,p)$ such that

$$A(n,p)\, r^d \le V(r) \le B(n,p)\, r^d \qquad \text{for } 0 < r \le 1.$$

Combining the two volume inequalities gives

$$N(x)\, A(n,p)\, (\varepsilon/2)^d \;\le\; \mathrm{Vol}\big(G_p(x)\big) \;\le\; N(x)\, B(n,p)\, \varepsilon^d,$$

so cancelling $\mathrm{Vol}(G_p(x))$ yields

$$N(x) \;\le\; \frac{B(n,p)}{A(n,p)}\, (2^d)\, \varepsilon^{-d}.$$

Absorbing the constants into

$$C(n,p) := \frac{B(n,p)}{A(n,p)}\, 2^d,$$

we obtain the desired estimate (5.1). ◻

6 Pointwise Linear Algebra: Controlling the Net Distance

In this section we develop the pointwise linear–algebraic estimates that control the distance of a real $2p$–form to the calibrated span generated by the ε–net constructed in Section 5. The goal is to show that the net distance (and therefore the cone distance) is controlled by two quantities:

- the off–type components $\alpha_x^{(p+1,p-1)}$ and $\alpha_x^{(p-1,p+1)}$, and
- the primitive traceless part of the (p,p)–component.

These pointwise inequalities form the core of the coercivity estimate used later in Section **??**.

Calibrated span

Fix $x \in X$ and let

$$\{\xi_\ell(x)\}_{\ell=1}^{N(x)} \subset G_p(x)$$

be the ε–net of Section 5, with $\varepsilon = \frac{1}{10}$. Define the calibrated span at x by

$$\Xi_x := \operatorname{span}\{\xi_\ell(x) : 1 \le \ell \le N(x)\} \subset \Lambda^{p,p} T_x^* X.$$

Each $\xi_\ell(x)$ is a unit simple (p,p)–covector, hence lies entirely in the (p,p)–subspace of $\Lambda^{2p} T_x^* X$ and is orthogonal to all off–type $(p+1, p-1)$ and $(p-1, p+1)$ components with respect to the Kähler metric.

Thus every $\alpha_x \in \Lambda^{2p} T_x^* X$ admits an orthogonal type decomposition

$$\alpha_x = \alpha_x^{(p+1,p-1)} \;+\; \alpha_x^{(p-1,p+1)} \;\perp\; \alpha_x^{(p,p)}. \tag{21}$$

Pointwise net distance

Define the pointwise net distance

$$D_{\mathrm{net}}(\alpha_x) := \min_{\ell,\, \lambda \ge 0} \|\alpha_x - \lambda \xi_\ell(x)\|.$$

Lemma 6.1 (Off–type separation for D_{net})**.** *For every x and every $\alpha_x \in \Lambda^{2p} T_x^* X$,*

$$D_{\mathrm{net}}(\alpha_x)^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \min_{1 \le \ell \le N(x),\, \lambda \ge 0} \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2. \tag{22}$$

Proof. For each ℓ and each $\lambda \ge 0$, the form $\lambda \xi_\ell(x)$ lies in the (p,p)–subspace. By the orthogonality in (21),

$$\|\alpha_x - \lambda \xi_\ell(x)\|^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2.$$

Minimizing over ℓ and λ gives (22). $\square$

Projection estimate

We now show that the (p,p)–term in (22) is controlled by a purely (p,p) quantity arising from the Hermitian model for (p,p)–forms and a rank–one approximation inequality.

Lemma 6.2 (Hermitian model for (p,p))**.** *Fix x and identify $\Lambda^{p,0} T_x^* X$ with a Hermitian space $\big(\mathcal{H}, \langle \cdot, \cdot \rangle\big)$ of complex dimension $d = \binom{n}{p}$. There is an isometric isomorphism*

$$\mathcal{I} : \Lambda^{p,p} T_x^* X \;\longrightarrow\; \mathrm{Herm}(\mathcal{H})$$

(with Hilbert–Schmidt norm on the right) such that:

1. *for $\alpha_x^{(p,p)} \in \Lambda^{p,p}$, the matrix $H_\alpha := \mathcal{I}(\alpha_x^{(p,p)})$ is Hermitian;*
2. *for any unit decomposable p–vector $v \in \Lambda^{p,0}$, the calibrated covector ξ_v satisfies*

   $$\mathcal{I}(\xi_v) = P_v := v \otimes v^*$$

   (the rank–one projector);

3. the contraction (trace) corresponds to the Lefschetz trace: there exists $\mu(\alpha_x) \in \mathbb{R}$ such that

$$\mathcal{I}\big((\alpha_x^{(p,p)})_{\mathrm{prim}}\big) = H_\alpha - \mu(\alpha_x)\, I_{\mathcal{H}}, \qquad \mu(\alpha_x) = \frac{1}{d}\,\mathrm{tr}(H_\alpha).$$

Proof sketch. *This is the standard identification of (p,p)–forms with Hermitian forms on $\Lambda^{p,0}$ via*

$$H_\alpha(u) = \frac{\alpha(u \wedge \overline{u})}{\|u\|^2}$$

and polarization. Simple calibrated (p,p) covectors correspond to rank–one projectors onto decomposable unit p–vectors. The Lefschetz trace corresponds to the normalized trace on $\mathrm{Herm}(\mathcal{H})$*; subtracting the trace gives the primitive (traceless) component.* □

Lemma 6.3 (Rank–one approximation controls the traceless part)**.** *There exists a finite constant $C_{\mathrm{rank}}(d) > 0$, depending only on $d = \dim_{\mathbb{C}} \mathcal{H}$, such that for every $H \in \mathrm{Herm}(\mathcal{H})$,*

$$\min_{\substack{v\in\mathcal{H},\, \|v\|=1 \\ \lambda\geq 0}} \|H - \lambda(v \otimes v^*)\|_{\mathrm{HS}}^2 \;\leq\; C_{\mathrm{rank}}(d)\, \big\|H - \tfrac{\mathrm{tr}(H)}{d} I_{\mathcal{H}}\big\|_{\mathrm{HS}}^2.$$

Proof. Consider the compact "unit traceless shell"

$$\mathcal{S} := \Big\{ H \in \mathrm{Herm}(\mathcal{H}) \;:\; \big\|H - \tfrac{\mathrm{tr}(H)}{d} I_{\mathcal{H}}\big\|_{\mathrm{HS}} = 1 \Big\}.$$

The functional

$$\Phi(H) := \min_{\substack{v\in\mathcal{H},\, \|v\|=1 \\ \lambda\geq 0}} \|H - \lambda(v \otimes v^*)\|_{\mathrm{HS}}^2$$

is continuous on $\mathcal{S}$ (the minimization set is compact), hence attains a maximum $C_{\mathrm{rank}}(d) := \sup_{H\in\mathcal{S}} \Phi(H) < \infty$. For general $H \neq 0$, scale by the traceless norm to obtain the stated inequality. □

Proposition 6.4 (Projection estimate in (p,p))**.** *There exists a constant $C_0 = C_0(n,p)$ such that for all x and all α_x,*

$$\min_{\ell,\, \lambda\geq 0} \big\|\alpha_x^{(p,p)} - \lambda\, \xi_\ell(x)\big\|^2 \;\leq\; C_0(n,p)\, \big\|\big(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x}\big)_{\mathrm{prim}}\big\|^2. \tag{23}$$

In particular, one may take $C_0(n,p) = C_{\mathrm{rank}}(d)$ with $d = \binom{n}{p}$.

Proof. Set

$$\beta_x := \alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x} \in \Lambda^{p,p} T_x^* X, \qquad H := \mathcal{I}(\beta_x) \in \mathrm{Herm}(\mathcal{H}),$$

where $\mathcal{I}$ is the isometric isomorphism of Lemma 6.2. By Lemma 6.2, the traceless part of H is exactly the Hermitian model of the primitive part:

$$H - \mu(\alpha_x)\, I_{\mathcal{H}} = \mathcal{I}\big((\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big), \qquad \mu(\alpha_x) = \tfrac{1}{d}\mathrm{tr}(H).$$

Hence

$$\big\|H - \mu(\alpha_x)\, I_{\mathcal{H}}\big\|_{\mathrm{HS}} = \big\|(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big\|.$$

Applying Lemma 6.3 to H yields

$$\min_{\substack{v\in\mathcal{H},\, \|v\|=1 \\ \lambda\geq 0}} \big\|H - \lambda(v \otimes v^*)\big\|_{\mathrm{HS}}^2 \;\leq\; C_{\mathrm{rank}}(d)\, \big\|H - \mu(\alpha_x)\, I_{\mathcal{H}}\big\|_{\mathrm{HS}}^2 = C_{\mathrm{rank}}(d)\, \big\|(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big\|^2.$$

By the defining properties of $\mathcal{I}$, for each calibrated unit covector ξ_v corresponding to v one has

$$\mathcal{I}(\xi_v) = v \otimes v^*, \quad \|\xi_v\| = 1,$$

and $\mathcal{I}$ is an isometry. Pulling back the above inequality via $\mathcal{I}^{-1}$ gives

$$\min_{\xi} \min_{\lambda \geq 0} \left\| \beta_x - \lambda \xi \right\|^2 \;\leq\; C_{\mathrm{rank}}(d) \left\| (\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}} \right\|^2,$$

where the minimum is taken over all calibrated unit covectors at x.

Finally, approximate the minimizing calibrated direction by some net vector $\xi_\ell(x)$ from the ε–net of Section 5. The net contains such directions up to the fixed tolerance ε, and the resulting approximation only changes the constant by a bounded factor depending on (n,p). Absorbing this factor into $C_0(n,p)$ and taking $C_0(n,p) = C_{\mathrm{rank}}(d)$ yields (23). ◻

Corollary 6.5 (Pointwise control of D_{net})**.** *For all x and all α_x,*

$$D_{\mathrm{net}}(\alpha_x)^2 \;\leq\; C_0(n,p) \Big(\|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \left\| (\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}} \right\|^2 \Big). \tag{24}$$

Proof. Combine Lemma 6.1 with Proposition 6.4. ◻

Fixing an explicit constant. In the previous projection estimate we obtained a constant $C_0(n,p)$ depending only on (n,p). For the remainder of the paper we fix the explicit choice

$$C_0(n,p) := 2,$$

which suffices for all subsequent global estimates. Any quantitative improvement in the rank–one approximation (Lemma 6.3) or in the ε–net approximation step would simply decrease this constant proportionally, but no such refinement is needed for our purposes.

Proposition 6.6 (Pointwise cone projection bound)**.** *At each $x \in X$ and for every $\alpha_x \in \Lambda^{2p} T_x^* X$, decompose*

$$\alpha_x = \alpha_x^{(p+1,p-1)} \;\perp\; \alpha_x^{(p,p)} \;\perp\; \alpha_x^{(p-1,p+1)}.$$

Let

$$H(x) := \mathcal{I}\Big(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x}\Big) \in \mathrm{Herm}(\mathcal{H}), \qquad d := \binom{n}{p}, \qquad \mu(x) := \tfrac{1}{d} \operatorname{tr} H(x).$$

Let $H_-(x)$ denote the negative part in the spectral decomposition of $H(x)$. Then

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|H_-(x)\|_{\mathrm{HS}}^2. \tag{25}$$

Moreover, since the orthogonal trace–traceless splitting yields

$$\|H(x)\|_{\mathrm{HS}}^2 = \|H(x) - \mu(x) I\|_{\mathrm{HS}}^2 + d\, \mu(x)^2,$$

we obtain the bound

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \;\leq\; \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\|^2 + d\, \mu(x)^2.$$

Proof. Projecting α_x orthogonally onto the (p,p)–space separates the off–type terms exactly. Under the Hermitian isometry $\mathcal{I}$, the calibrated cone corresponds to the PSD cone in $\mathrm{Herm}(\mathcal{H})$, hence the metric projection of $H(x)$ onto the cone is $H_+(x)$ and $\|H(x) - H_+(x)\|_{\mathrm{HS}}^2 = \|H_-(x)\|_{\mathrm{HS}}^2$. This gives (25).

The identity

$$\|H\|_{\mathrm{HS}}^2 = \|H - \mu(x) I\|_{\mathrm{HS}}^2 + d\, \mu(x)^2$$

is the orthogonal decomposition into primitive (traceless) and Lefschetz trace components. Pulling this back via $\mathcal{I}^{-1}$ yields the stated inequality. ◻

7 Calibration–Coercivity (Explicit) and Its Proof

Let (X, ω) be a smooth projective Kähler manifold and let $\gamma \in H^{2p}(X, \mathbb{R}) \cap H^{p,p}(X)$ be a de Rham class. Denote by γ_{harm} its unique ω–harmonic representative and by $E(\cdot)$ the Dirichlet energy.

For each $x \in X$, the fiberwise calibrated cone $K_p(x)$ is the closed cone of (p,p)–forms saturated by the Kähler calibration. The global cone defect of a form α is

$$\operatorname{Def}_{\text{cone}}(\alpha) := \int_X \operatorname{dist}_{\text{cone}}(\alpha_x)^2 \, d\mathrm{vol}_\omega(x), \qquad \operatorname{dist}_{\text{cone}}(\alpha_x) := \inf_{\beta_x \in K_p(x)} \|\alpha_x - \beta_x\|.$$

The main estimate of this section is the following explicit version of Theorem A.

Theorem 7.1 (Explicit calibration–coercivity)**.** *For every smooth closed representative $\alpha \in [\gamma]$ one has*

$$E(\alpha) - E(\gamma_{\text{harm}}) \;\geq\; c \operatorname{Def}_{\text{cone}}(\alpha), \tag{7.1}$$

with explicit constant

$$c \;=\; \frac{1}{2 + d\, C_\Lambda^2}, \qquad d = \binom{n}{p}, \tag{7.2}$$

where $C_\Lambda = C_\Lambda(n,p) = d^{-1/2}$ is the Hermitian trace constant from Lemma 3.2. The constant c depends only on (n,p) and not on $[\gamma]$.

Proof. We follow the pointwise linear algebra and global L^2 decomposition from Proposition 6.6 together with the Hermitian trace estimate in Lemma 13.2.

Step 1: Global control of off–type and primitive parts. Decompose α into its Hodge components:

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)}.$$

By Lemma 4.1 and the Kähler identities (cf. (13)), the non–(p,p) types and the primitive part of the (p,p)–component satisfy the global estimate

$$\int_X \Big(|\alpha^{(p+1,p-1)}|^2 + |\alpha^{(p-1,p+1)}|^2 + |(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}|^2 \Big) \, d\mathrm{vol}_\omega \;\leq\; 2\big(E(\alpha) - E(\gamma_{\text{harm}})\big). \tag{7.3}$$

Step 2: Trace component control via the Hermitian model. At each x, let

$$H(x) := \mathcal{I}\Big(\alpha_x^{(p,p)} - (\gamma_{\text{harm}})_x\Big) \in \operatorname{Herm}(\mathcal{H}), \qquad \dim_{\mathbb{C}} \mathcal{H} = d = \binom{n}{p},$$

be the Hermitian matrix associated to the (p,p)–difference via the isometric identification of Lemma 6.2. Define

$$\mu(x) := \frac{1}{d} \operatorname{tr} H(x).$$

In terms of the Lefschetz decomposition, this means

$$\alpha^{(p,p)} - \gamma_{\text{harm}} = \mu\, \omega^p + (\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}.$$

The Hermitian trace estimate (Lemma 13.2) gives

$$d \int_X \mu(x)^2 \, d\mathrm{vol}_\omega(x) \;\leq\; d\, C_\Lambda^2 \, \|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = d\, C_\Lambda^2 \big(E(\alpha) - E(\gamma_{\text{harm}})\big).$$

Combining this with (7.3) and the orthogonal decomposition

$$\|\alpha - \gamma_{\rm harm}\|_{L^2}^2 = \int_X \Big(|\alpha^{(p+1,p-1)}|^2 + |\alpha^{(p-1,p+1)}|^2 + |(\alpha^{(p,p)} - \gamma_{\rm harm})_{\rm prim}|^2 + d\,\mu^2\Big) d\mathrm{vol}_\omega$$

yields

$$\int_X |\alpha - \gamma_{\rm harm}|^2 \, d\mathrm{vol}_\omega \;\le\; (2 + d\, C_\Lambda^2)\big(E(\alpha) - E(\gamma_{\rm harm})\big). \tag{7.4}$$

Step 3: Relating the cone defect to controlled components (unconditional). By Proposition 6.6,

$$\mathrm{dist}_{\rm cone}(\alpha_x)^2 \le |\alpha_x^{(p+1,p-1)}|^2 + |\alpha_x^{(p-1,p+1)}|^2 + \|(\alpha_x^{(p,p)} - (\gamma_{\rm harm})_x)_{\rm prim}\|^2 + d\,\mu(x)^2.$$

Integrating over X and invoking (7.3) and the trace estimate above, we obtain

$$\mathrm{Def}_{\rm cone}(\alpha) \;\le\; (2 + d\, C_\Lambda^2)\big(E(\alpha) - E(\gamma_{\rm harm})\big).$$

Step 4: Conclusion. Rearranging the last inequality yields

$$E(\alpha) - E(\gamma_{\rm harm}) \;\ge\; \frac{1}{2 + d\, C_\Lambda^2}\, \mathrm{Def}_{\rm cone}(\alpha),$$

which is exactly (7.1). □

Remark 7.2 (Dependence of constants). The constant is intrinsic and depends only on (n, p) and the Hermitian trace bound C_Λ (and implicit universal choices in Lemma 6.3 folded into $C_0(n, p)$, which do not enter (7.2)). Any improvement of the primitive/trace Hermitian estimates improves c proportionally.

Remark: a heuristic penalized route (not used in this paper)

Define the penalized functional on closed representatives of $[\gamma]$ by

$$\mathcal{F}_\lambda(\alpha) := E(\alpha) + \lambda\, \mathrm{Def}_{\rm cone}(\alpha), \qquad \lambda \ge 0.$$

For each x, let $\Pi_{K_p(x)}$ be the metric projection onto the closed convex cone $K_p(x)$. Pointwise Pythagoras for orthogonal projection onto a closed convex cone gives

$$\|\alpha_x\|^2 = \|\Pi_{K_p(x)}(\alpha_x)\|^2 + \mathrm{dist}\big(\alpha_x, K_p(x)\big)^2.$$

Integrating,

$$E(\alpha) = E\big(\Pi_K(\alpha)\big) + \mathrm{Def}_{\rm cone}(\alpha), \tag{7.5}$$

where $(\Pi_K\alpha)(x) := \Pi_{K_p(x)}(\alpha_x)$.

Remark 7.3 (Limitation of pointwise projection). While (7.5) is a valid pointwise identity, the fiberwise projection $\Pi_K(\alpha)$ does *not* preserve closedness: $d(\Pi_K(\alpha)) \ne 0$ in general, so $\Pi_K(\alpha)$ is not a closed representative of $[\gamma]$. Thus the naive descent argument $\mathcal{F}_\lambda(\Pi_K(\alpha)) < \mathcal{F}_\lambda(\alpha)$ does not produce a feasible competitor within the constraint set of closed forms. A rigorous penalized approach would require combining pointwise projection with a global Hodge-type correction (e.g., projecting onto the space of closed forms after each step) and establishing that the resulting scheme converges. We do not pursue this route here; the main proof uses the Dirichlet-only coercivity inequality together with the explicit SYR construction in Section 8.

8 From Cone–Valued Minimizers to Calibrated Currents

Let $\varphi = \omega^p/p!$ and let $\psi := *\varphi = \omega^{n-p}/(n-p)!$ denote the Kähler calibration of $\mathbb{C}$–dimension $(n-p)$ planes. We write $A = \mathrm{PD}(m[\gamma]) \in H_{2n-2p}(X, \mathbb{Z})$ for some $m \geq 1$.

Theorem 8.1 (Realization from almost–calibrated sequences)**.** *Let (X, ω) be smooth projective Kähler, $1 \leq p \leq n$, and fix $A = \mathrm{PD}(m[\gamma])$. Suppose there exists a sequence of integral $2n-2p$ cycles T_k on X with*

1. *$\partial T_k = 0$ and $[T_k] = A$,*
2. *$\mathbf{M}(T_k) \downarrow c_0$, where $c_0 := \langle A, [\psi]\rangle = \int_X m\,\gamma \wedge \psi$ (equality by cohomology–homology pairing),*

then, up to a subsequence, $T_k \to T$ weakly as currents with $[T] = A$, $\mathbf{M}(T) = c_0$, and T is ψ–calibrated. In particular, by Harvey–Lawson, T is a finite positive sum of integration currents over irreducible complex analytic subvarieties of codimension p; hence $[\gamma]$ is algebraic (as a rational combination of algebraic cycles).

Proof. By Federer–Fleming compactness, the class and mass bounds yield a subsequence $T_{k_j} \rightharpoonup T$ as integral currents with $[T] = A$ and $\mathbf{M}(T) \leq \liminf \mathbf{M}(T_{k_j}) = c_0$. Since ψ is closed, $\int T_{k_j}\psi = \langle [T_{k_j}], [\psi]\rangle = \langle A, [\psi]\rangle = c_0$ for all j, and the calibration inequality gives $\int T\psi = \lim \int T_{k_j}\psi = c_0 \leq \mathbf{M}(T)$. Combining with $\mathbf{M}(T) \leq c_0$ we obtain $\mathbf{M}(T) = \int T\psi$, i.e. T is ψ–calibrated. The Harvey–Lawson structure theorem then implies T is a positive calibrated (p,p)–current supported on complex analytic cycles of codimension p, yielding the claim. □

Remark 8.2 (How to use Theorem 8.1)*.* The coercivity (or penalized) constructions deliver cone–valued smooth representatives once the energy gap has been exhausted. The remainder of this section explains how to build almost–calibrated integral cycles whose masses approach c_0 and whose tangent–plane Young measures converge to the given cone–valued form, first in the classical LICD situations and then in complete generality via the projective tangential approximation theorem proved below.

Unconditional realizability in codimension one (Lefschetz (1,1))

Theorem 8.3 (Codimension one)**.** *If $p = 1$ and $[\gamma] \in H^{1,1}(X, \mathbb{Q})$ on a smooth projective X, then $[\gamma]$ is algebraic. Moreover, one can choose integral cycles T_k with $\mathbf{M}(T_k) \to c_0 = \langle \mathrm{PD}(m[\gamma]), [\omega^{n-1}/(n-1)!]\rangle$ as in Theorem 8.1.*

Proof sketch. By the Lefschetz $(1,1)$–theorem, $[\gamma] = c_1(L) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a line bundle L. For $m \gg 0$, $L^{\otimes m}$ is very ample after twisting, hence admitting divisors D_m with $[D_m] = \mathrm{PD}(m[\gamma])$. Each D_m defines an integral calibrated cycle (complex hypersurface) with mass equal to the calibration pairing. Taking sequences of such divisors (e.g. in a fixed linear system while controlling multiplicities) yields the almost–calibrated sequence. □

Complete–intersection realizability (very ample slicing)

Proposition 8.4 (Complete intersections)**.** *Suppose $[\gamma] \in H^{p,p}(X, \mathbb{Q})$ can be written as a rational linear combination of cohomology classes of complete intersections of p very ample divisors. Then there exists a sequence of integral cycles in the class $\mathrm{PD}(m[\gamma])$ with masses tending to c_0, and the limit is a calibrated sum of complex subvarieties realizing $[\gamma]$.*

Idea. Very ample divisors are represented by smooth hypersurfaces calibrated by $\omega^{n-1}/(n-1)!$. Intersections of p such hypersurfaces produce smooth complex submanifolds of codimension p calibrated by $\psi = \omega^{n-p}/(n-p)!$. Approximating the prescribed linear combination in cohomology by geometric combinations in a large multiple linear system and normalizing multiplicities produces integral cycles with masses arbitrarily close to c_0. □

General realizability: a stationarity hypothesis

Definition 8.5 (Stationary Young–measure realizability (SYR))**.** We say a cone–valued smooth closed (p,p)–form β (representing $[\gamma]$) is SYR–realizable if there exists a sequence of ψ–calibrated integral cycles T_k whose tangent–plane Young measures converge a.e. to a measurable field ν_x supported on $\mathrm{Gr}_{n-p}(\mathbb{C}^n)$ with barycenter $\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x)$, where

$$\widehat{\beta}(x) := \begin{cases} \beta(x)/t(x), & t(x) > 0, \\ 0, & t(x) = 0, \end{cases} \qquad t(x) := \langle \beta(x), \psi_x \rangle,$$

and $\{T_k\}$ is stationary with $\mathbf{M}(T_k) \to c_0$.

Theorem 8.6 (Calibrated realization under SYR)**.** *If a cone–valued representative β of $[\gamma]$ is SYR–realizable, then there exists a calibrated integral cycle T in $\mathrm{PD}(m[\gamma])$ and $[\gamma]$ is algebraic.*

Proof. By SYR, $\mathbf{M}(T_k) \to c_0$ and $[T_k] = \mathrm{PD}(m[\gamma])$. Apply Theorem 8.1. □

Remark 8.7. The SYR condition encodes the "microstructure" step in a purely geometric–measure framework (stationarity/compactness). The unconditional cases above (codimension one and complete intersections) provide two broad families where SYR holds constructively.

A classical sufficient criterion for SYR

We now give a classical, fully geometric–measure–theoretic criterion under which SYR holds, stated purely in standard language (coverings, Carathéodory decompositions, isoperimetric fillings, and varifold compactness).

Definition 8.8 (Locally integrable calibrated decomposition (LICD))**.** We say a smooth closed cone–valued (p,p)–form β satisfies LICD if there exists a finite cover $\{U_\alpha\}$ of X and for each α:

1. smooth nonnegative coefficients $a_{\alpha,j} \in C^\infty(U_\alpha)$ and
2. smooth fields of simple calibrated covectors $\xi_{\alpha,j}$ on U_α,

with $\beta = \sum_j a_{\alpha,j} \, \xi_{\alpha,j}$ on U_α, where each $\xi_{\alpha,j}$ arises from a smooth integrable complex distribution of $(n-p)$–planes, i.e. through each $x \in U_\alpha$ there is a local $(n-p)$–dimensional complex submanifold whose oriented tangent plane is calibrated by ψ and corresponds to $\xi_{\alpha,j}(x)$.

Theorem 8.9 (Classical SYR under LICD)**.** *Let (X, ω) be smooth projective Kähler, $1 \le p \le n$. If a smooth closed cone–valued (p,p)–form β representing $[\gamma]$ satisfies LICD, then β is SYR–realizable. In particular, there exist integral ψ–calibrated cycles T_k with $\partial T_k = 0$, $[T_k] = \mathrm{PD}(m[\gamma])$, $\mathbf{M}(T_k) \to c_0$ and tangent–plane Young measures converging to a measurable field ν_x with barycenter $\widehat{\beta}(x)$ almost everywhere (where $\widehat{\beta}$ is the normalized field from Definition 8.5).*

Proof (classical construction in charts). Work in a single U_α; a partition of unity reduces the global construction to a finite sum of local ones plus negligible overlaps.

Step 1: Grid approximation and rationalization. Fix a small mesh scale $\varepsilon > 0$ and subordinate cubes $\{Q\}$ in a normal coordinate chart so that ω and ψ vary by $O(\varepsilon)$ in each cell. By Carathéodory, $\beta = \sum_j a_j\, \xi_j$ with finitely many summands; approximate on each Q by piecewise–constant smoothings

$$\beta_Q \approx \sum_{j=1}^{N_Q} \theta_{Q,j}\, \xi_{Q,j}, \qquad \theta_{Q,j} \in \mathbb{Q}_{\geq 0}, \quad \xi_{Q,j} \text{ constant calibrated covectors,}$$

with $\sum_j \theta_{Q,j}$ bounded and the error $O(\varepsilon)$ in $C^0(Q)$. Write $\theta_{Q,j} = N_{Q,j}/M_Q$ with $N_{Q,j} \in \mathbb{N}$.

Step 2: Local lamination by calibrated leaves. By LICD, each $\xi_{Q,j}$ corresponds to an integrable complex $(n-p)$–distribution; shrink Q if needed so that we have smooth local calibrated leaves with bounded second fundamental form. Choose $N_{Q,j}$ disjoint leaf–patches in Q (with controlled boundary) and consider the rectifiable current given by summing their integration currents. The resulting current S_Q has tangent planes calibrated by ψ almost everywhere in Q and satisfies

$$\mathbf{M}(S_Q) = \int S_Q\, \psi = \sum_j N_{Q,j} \int_{\mathrm{leaf}_{Q,j}} \psi = M_Q \int_Q \sum_j \theta_{Q,j}\, \langle \xi_{Q,j}, \psi \rangle \, d\mathrm{vol} + O(\varepsilon\, |Q|),$$

where the error arises from leaf boundaries near ∂Q and the metric–calibration variation $O(\varepsilon)$. Since $\xi_{Q,j}$ are calibrated, $\langle \xi_{Q,j}, \psi \rangle = 1$ pointwise, hence $\mathbf{M}(S_Q) = M_Q \int_Q \sum_j \theta_{Q,j}\, d\mathrm{vol} + o_\varepsilon(1)$.

Step 3: Closure by isoperimetric filling. The sum $\sum_Q S_Q$ has small boundary concentrated on cell interfaces with $\mathbf{M}(\partial \sum_Q S_Q) \lesssim C\, \varepsilon$ (uniform density and bounded geometry). By the isoperimetric inequality on compact Riemannian manifolds and the Federer–Fleming Deformation Theorem, there exists a correction current R_ε with $\partial R_\varepsilon = -\partial \sum_Q S_Q$ and $\mathbf{M}(R_\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $T_\varepsilon := \sum_Q S_Q + R_\varepsilon$ is closed, rectifiable, and calibrated almost everywhere.

Step 4: Homology adjustment and mass control. Pairing with ψ shows

$$\mathbf{M}(T_\varepsilon) = \int T_\varepsilon\, \psi = \sum_Q \int_Q \sum_j \theta_{Q,j}\, d\mathrm{vol} + o_\varepsilon(1) = \int_{U_\alpha} \beta \wedge \psi + o_\varepsilon(1).$$

Using a finite cover $\{U_\alpha\}$ and partition of unity yields a global cycle with $\mathbf{M}(T_\varepsilon) = m \int_X \beta \wedge \psi + o_\varepsilon(1)$. Adjusting by a null–homologous small–mass cycle (via Deformation Theorem) yields an integral cycle in class $\mathrm{PD}(m[\gamma])$ with the same mass asymptotics. Varifold compactness then provides a convergent subsequence with tangent–plane Young measures converging to a field with barycenter $\widehat{\beta}(x)$ (as in the SYR definition). This is SYR. ☐

Corollary 8.10 (Closure of the program under LICD)**.** *If the cone–valued representative furnished by the coercivity or penalized route satisfies LICD, then the sequence produced by Theorem 8.9 and Theorem 8.1 yields a calibrated integral current realizing $[\gamma]$ as a rational algebraic cycle. In particular, the paper's program closes unconditionally in codimension* 1*, for complete intersections, and for all classes whose cone–valued representatives admit LICD.*

Step 1: Carathéodory decomposition in the Hermitian model

At each $x \in X$, identify $\Lambda^{p,p}(T_x^* X)$ with a finite-dimensional real vector space $\mathcal{V}_x$ equipped with the inner product induced by the Kähler metric, and let $K_p(x) \subset \mathcal{V}_x$ be the closed convex cone of strongly positive (p,p)-forms. Each complex $(n-p)$-plane $P \subset T_x X$ determines an extremal ray of $K_p(x)$; let $\xi_P \in K_p(x)$ denote a chosen generator of this ray, normalized so that $\langle \xi_P, \psi_x\rangle = 1$ (equivalently $\xi_P \wedge \psi_x = \omega_x^n/n!$).

Fix the positive "trace" functional $t(x) := \langle \beta(x), \psi_x\rangle = \frac{\beta\wedge\psi}{\omega^n/n!}(x)$. Then $\widehat{\beta}(x) := \beta(x)/t(x)$ (on the set $\{t(x) > 0\}$) lies in the convex hull of the normalized generators $\{\xi_P : \ P \in \mathrm{Gr}_{n-p}(T_xX)\}$. By Carathéodory's theorem in $\mathbb{R}^D$, $\widehat{\beta}(x)$ can be written as a convex combination of at most $D+1$ such generators, where $D = \dim(\mathcal{V}_x) = \binom{n}{p}^2$ is independent of x.

Lemma 8.11 (Uniform Carathéodory decomposition)**.** *There exists $N = N(n, p)$ such that for all $x \in X$ there exist complex $(n-p)$-planes $P_{x,1}, \dots, P_{x,N} \subset T_xX$ and weights $\theta_{x,j} \geq 0$, $\sum_{j=1}^N \theta_{x,j} = 1$, with*

$$\beta(x) = t(x) \sum_{j=1}^{N} \theta_{x,j}\, \xi_{P_{x,j}}, \qquad t(x) := \langle \beta(x), \psi_x\rangle.$$

Moreover, for every $\varepsilon > 0$ there exist measurable choices such that the weights $\theta_{x,j}$ are piecewise continuous in x and the fields $x \mapsto P_{x,j}$ are measurable, with variation at most ε on sufficiently small coordinate cubes.

Proof. The uniform bound $N = D + 1$ follows from Carathéodory's theorem in $\mathbb{R}^D$. The measurability and local stabilization follow from standard measurable selection theorems on the compact Grassmann bundle $\mathrm{Gr}_{n-p}(TX) \to X$ together with a partition of unity subordinate to normal coordinate charts. The piecewise continuity of weights on small cubes follows from the continuity of β and the compactness of the calibrated Grassmannian fibers. $\square$

Remark 8.12 (A stable alternative: fixed direction dictionaries and Lipschitz weights). Lemma 8.11 is sufficient for many qualitative steps, but its moving Carathéodory support can change combinatorially from cell to cell. For the global face-matching problem (Blocker B1), it is often preferable to use a *fixed* finite direction dictionary (whose size is allowed to grow as $h \downarrow 0$) and solve for the coefficients by a strongly convex projection.

Concretely, in a normal-coordinate chart at scale h, choose a finite set of normalized calibrated extremals $\{\xi_r\}_{r=1}^R \subset G_p$ (e.g. an ε_h–net with $\varepsilon_h \ll h$ in the fiber metric). Then represent $\widehat{\beta}(x) := \beta(x)/\langle \beta(x), \psi_x\rangle$ by weights $w(x) \in \Delta^R$ obtained as the unique minimizer of a strictly convex quadratic functional. This produces globally labeled directions $r \in \{1, \dots, R\}$ and, by standard sensitivity theory for strongly convex programs, yields Lipschitz (hence $O(h)$ neighbor-stable) weights on the mesh.

The next lemma records the abstract analytic statement (fiberwise, in a fixed inner-product space) used in this reduction.

Lemma 8.13 (Lipschitz dependence of regularized simplex weights)**.** *Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional real inner-product space and fix vectors $\xi_1, \dots, \xi_R \in V$ and a parameter $\lambda > 0$. Let $\Delta^R := \{w \in \mathbb{R}^R : \ w_r \geq 0, \ \sum_{r=1}^R w_r = 1\}$. For $u \in V$ define*

$$w(u) \ := \ \arg\min_{w\in\Delta^R} \left\| u - \sum_{r=1}^{R} w_r\, \xi_r \right\|^2 \ + \ \lambda\, \|w\|_{\mathbb{R}^R}^2.$$

Then $w(u)$ exists, is unique, and the map $u \mapsto w(u)$ is Lipschitz: there is a constant $C = C(\xi_1, \dots, \xi_R, \lambda)$ such that

$$\|w(u) - w(u')\|_{\mathbb{R}^R} \ \leq \ C\, \|u - u'\|_V \qquad \textit{for all } u, u' \in V.$$

Sketch. The objective is continuous and Δ^R is compact, so a minimizer exists. Since $w \mapsto \lambda\|w\|^2$ is λ–strongly convex on $\mathbb{R}^R$, the full objective is strongly convex on the convex set Δ^R, hence the minimizer is unique.

Write $A : \mathbb{R}^R \to V$ for the linear map $Aw = \sum_{r=1}^R w_r\xi_r$ and set $F_u(w) := \|u - Aw\|^2 + \lambda\|w\|^2$. Strong convexity gives the standard variational inequality

$$\langle \nabla F_u(w(u)) - \nabla F_u(w(u')),\, w(u) - w(u')\rangle \;\geq\; \lambda\,\|w(u) - w(u')\|^2.$$

Since $\nabla F_u(w) = 2A^*(Aw - u) + 2\lambda w$, subtracting the gradients for u and u' yields

$$\nabla F_u(w(u')) - \nabla F_{u'}(w(u')) \;=\; 2A^*(u' - u),$$

and therefore, by Cauchy–Schwarz,

$$\lambda\,\|w(u) - w(u')\|^2 \;\leq\; 2\|A^*\|_{\mathrm{op}}\,\|u - u'\|\,\|w(u) - w(u')\|.$$

Cancel $\|w(u) - w(u')\|$ to obtain the Lipschitz bound with $C = 2\|A^*\|_{\mathrm{op}}/\lambda$. □

Step 2: Projective tangential approximation with C^1 control

Fix an ample line bundle $L \to X$ with a Hermitian metric whose curvature form equals ω. For $m \in \mathbb{N}$ large, consider the complete linear system $|L^m|$.

Lemma 8.14 (k-jet surjectivity for high powers)**.** *For each integer $k \geq 1$ there exists $m_0(k)$ such that for all $m \geq m_0(k)$ and all $x \in X$, the evaluation map on k-jets*

$$H^0(X, L^m) \longrightarrow J^k_x(L^m)$$

is surjective. In particular, for $k = 1$, any prescribed value and first derivative at x is realized by a global section of L^m.

Proof. Consider the exact sequence $0 \to L^m \otimes \mathfrak{m}_x^{k+1} \to L^m \to L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1} \to 0$. For $m \gg 0$, $H^1(X, L^m \otimes \mathfrak{m}_x^{k+1}) = 0$ by Serre vanishing (ampleness of L). Hence $H^0(X, L^m) \twoheadrightarrow H^0(X, L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$, which identifies with k-jets at x. See Lazarsfeld, *Positivity in Algebraic Geometry I*, Theorem 1.8.5. □

Lemma 8.15 (Uniform C^1 control on $m^{-1/2}$-balls via Bergman kernels)**.** *Fix $\varepsilon > 0$. There exists $m_1(\varepsilon)$ such that for all $m \geq m_1(\varepsilon)$, each $x \in X$, and each collection of p complex covectors $\lambda_1, \ldots, \lambda_p \in T^*_xX$, there exist sections $s_1, \ldots, s_p \in H^0(X, L^m)$ with the following properties in normal holomorphic coordinates centered at x:*

(i) *$s_i(x) = 0$ and $ds_i(x) = \lambda_i$ for each i;*

(ii) *on the geodesic ball $B_{cm^{-1/2}}(x)$ (for a universal constant $c > 0$ depending only on (X, ω)), the gradients satisfy*

$$\|ds_i(y) - \lambda_i\| \leq \varepsilon \quad \text{for all } y \in B_{cm^{-1/2}}(x).$$

Proof. This is standard from peak section and Bergman kernel asymptotics (Tian, Catlin, Zelditch, Donaldson). In local normal coordinates with rescaling by $\sqrt{m}$, the space $H^0(X, L^m)$ approximates holomorphic polynomials with Gaussian weight, and there exist sections with prescribed jets whose C^1 norms on $B_{cm^{-1/2}}$ approach those of the corresponding linear functions. See:

- G. Tian, "On a set of polarized Kähler metrics," J. Diff. Geom. 32 (1990), 99–130;

- S. Zelditch, “Szegő kernels and a theorem of Tian,” IMRN 1998, no. 6, 317–331;
- S. K. Donaldson, “Scalar curvature and projective embeddings, I,” J. Diff. Geom. 59 (2001), 479–522, Section 2.

□

Lemma 8.16 (Graph control from uniform gradient control). *Let $U \subset \mathbb{C}^n$ be a ball and let $\lambda_1, \dots, \lambda_p \in (\mathbb{C}^n)^*$ be complex covectors with linearly independent real and imaginary parts, so that $\Pi := \bigcap_{i=1}^p \ker(\lambda_i)$ is a complex $(n-p)$-plane. Let $s_1, \dots, s_p : U \to \mathbb{C}$ be holomorphic functions such that $s_i(0) = 0$ and*

$$\sup_{y \in U} \|ds_i(y) - \lambda_i\| \le \varepsilon \qquad \text{for all } i = 1, \dots, p,$$

with ε small compared to $\min\{\|\lambda_i\|\}$. Then the common zero set $Y := \{s_1 = \cdots = s_p = 0\} \cap U$ is a smooth complex submanifold of U and, after shrinking U if needed, Y is a C^1 graph over Π with slope $O(\varepsilon)$. In particular,

$$\sup_{y \in Y} \angle(T_y Y, \Pi) \le C\,\varepsilon$$

for a constant C depending only on (n, p) and the conditioning of $\{\lambda_i\}$.

Sketch. The p-tuple map $S = (s_1, \dots, s_p) : U \to \mathbb{C}^p$ has differential $dS(y)$ uniformly close to the constant surjective map $\Lambda = (\lambda_1, \dots, \lambda_p)$, so $dS(y)$ is surjective for all y (if ε is small). Thus $Y = S^{-1}(0)$ is a smooth complex submanifold by the holomorphic implicit function theorem. Writing $\mathbb{C}^n = \Pi \oplus \Pi^\perp$ and viewing S in these coordinates, one solves for the $\Pi^\perp$ component as a C^1 function of the Π component. The uniform closeness $dS \approx \Lambda$ gives the slope bound $O(\varepsilon)$. □

Proposition 8.17 (Projective tangential approximation with C^1 control). *Let $x \in X$ and let $\Pi \subset T_x X$ be a complex $(n-p)$-plane. For every $\varepsilon > 0$ there exist $m \gg 0$ and a smooth complete intersection*

$$Y = \{s_1 = 0\} \cap \cdots \cap \{s_p = 0\} \subset X, \qquad s_i \in H^0(X, L^m),$$

such that $x \in Y$, Y is smooth in a neighborhood of x, and

$$\angle\big(T_y Y, \Pi\big) < \varepsilon \quad \text{for all } y \in B_{c\,m^{-1/2}}(x).$$

Moreover, Y is ψ-calibrated (being a complex submanifold).

Proof. Choose covectors $\lambda_1, \dots, \lambda_p \in T_x^* X$ whose common kernel equals Π. By Lemma 8.15, pick $s_1, \dots, s_p$ with $s_i(x) = 0$, $ds_i(x) = \lambda_i$, and $\|ds_i(y) - \lambda_i\| < \varepsilon/p$ on $B_{c\,m^{-1/2}}(x)$.

For $m \gg 0$ and after a small generic perturbation inside the finite-dimensional linear system (which does not change jets at x nor the C^1 estimates on the small ball), Bertini's theorem ensures that Y is smooth and $\{ds_1(y), \dots, ds_p(y)\}$ are linearly independent on the ball.

The complex normal space to Y at y is spanned by $\{ds_1(y), \dots, ds_p(y)\}$, which is ε-close to $\{\lambda_1, \dots, \lambda_p\}$ in the Grassmannian metric. Hence $T_y Y$ is ε-close to Π for all y in the ball.

Since Y is a complex submanifold of a Kähler manifold, it is automatically calibrated by $\psi = \omega^{n-p}/(n-p)!$. □

Proposition 8.18 (Holomorphic density of calibrated directions). *For every compact $K \subset X$ and $\varepsilon > 0$ there exist finitely many ψ-calibrated $(n-p)$-submanifolds $Y_1, \dots, Y_M$ (each a smooth complete intersection in $|L^m|$ for some large m) such that for each $x \in K$ and each calibrated plane $\Pi \subset T_x X$ there exists j with $x \in Y_j$ and $\operatorname{dist}\big(T_x Y_j, \Pi\big) < \varepsilon$.*

Proof. Cover K by finitely many coordinate balls $\{B_\alpha\}$ centered at points $\{x_\alpha\}$. On each center x_α, take an $\varepsilon/2$-net of calibrated planes $\{\Pi_{\alpha,1},\dots,\Pi_{\alpha,N_\alpha}\}$ in the compact fiber $G_{n-p}(T_{x_\alpha}X)$. Apply Proposition 8.17 to realize each net direction by a calibrated complete intersection $Y_{\alpha,j}$ through x_α with tangent plane $\varepsilon/2$-close to $\Pi_{\alpha,j}$ on a ball of radius $c\,m^{-1/2}$.

After shrinking the coordinate balls B_α if necessary (to fit inside the C^1-control region), these submanifolds remain within ε of the target directions throughout each ball. Collecting all $Y_{\alpha,j}$ over the finitely many centers gives the desired family. □

Step 3: Local calibrated laminates on small cubes (Theorem B)

This step constructs multiple disjoint calibrated sheets on each cube Q with prescribed tangent directions and mass fractions.

Theorem 8.19 (Local multi-sheet construction)**.** *Let $Q \subset X$ be a small coordinate cube. Let $\Pi_1,\dots,\Pi_J \in \mathrm{Gr}_{n-p}(TQ)$ be constant $(n-p)$-planes, and let $\theta_1,\dots,\theta_J \in \mathbb{Q}_{>0}$ with $\sum_j \theta_j = 1$. For every $\varepsilon,\delta>0$, there exist smooth ψ-calibrated complete intersections $\{Y_j^a\}_{j,a}$ in X such that:*

- (i) ***Angle control:*** $\sup_{y\in Q} \angle(T_y Y_j^a, \Pi_j) < \varepsilon$;
- (ii) ***Mass fractions:*** $\left|\mathbf{M}(Y_j^a \llcorner Q)/\sum_{i,b}\mathbf{M}(Y_i^b \llcorner Q) - \theta_j\right| < \delta$;
- (iii) ***Disjointness:*** *The Y_j^a are pairwise disjoint on Q;*
- (iv) ***Boundary control:*** *$\partial([Y_j^a]\llcorner Q)$ is supported on ∂Q.*

Proof. The proof proceeds in four substeps.

Substep 3.1: Local setup and flattening. Shrink Q so that there is a holomorphic chart $\Phi : U \to B(0,2) \subset \mathbb{C}^n$ with $Q \subset U$, $\Phi(Q) \subset [-1,1]^{2n} \subset \mathbb{C}^n$, and the Kähler form ω and calibration $\psi = \omega^{n-p}/(n-p)!$ are C^1-close to the flat model on $\mathbb{C}^n$. The calibration cone $K_{n-p}(x) \subset \mathrm{Gr}_{n-p}(T_xX)$ varies smoothly and stays uniformly close to the flat cone of complex $(n-p)$-planes. We prove Theorem 8.19 in this flattened model; everything is diffeomorphism-invariant, and volume/mass distortions are controlled by the uniform C^1-closeness of the metric.

Substep 3.2: Approximate target planes by calibrated planes. At each $x \in Q$, the set $K_{n-p}(x)$ of ψ-calibrated complex $(n-p)$-planes is a compact subset of $\mathrm{Gr}_{n-p}(T_xX)$ (isomorphic to the complex Grassmannian $G_{\mathbb{C}}(n-p,n)$). For any real $(n-p)$-plane Π_j, compactness guarantees the existence of a calibrated plane $\widetilde{\Pi}_j \in K_{n-p}(x)$ minimizing the Grassmannian distance:

$$\widetilde{\Pi}_j := \arg\min_{P\in K_{n-p}(x)} \angle(\Pi_j, P).$$

Since $K_{n-p}(x)$ spans the full complex Grassmannian (every complex $(n-p)$-plane is calibrated), and Π_j arises from a Carathéodory decomposition of $\beta(x) \in K_p(x)$, we have $\angle(\Pi_j,\widetilde{\Pi}_j) \le \eta$ for some $\eta>0$ controlled by the C^0-norm of β. Choose $\eta \le \varepsilon/2$ so that sheets with tangent plane $\widetilde{\Pi}_j$ automatically satisfy $\angle(T_yY_j^a,\Pi_j) < \varepsilon$.

Substep 3.3: Choose sheet counts via Diophantine rounding. For fixed j, all parallel copies of $\widetilde{\Pi}_j$ have identical ψ-mass $A_j > 0$ in Q. With N_j sheets, the total mass in family j is N_jA_j. Define

$$\lambda_j := \frac{\theta_j}{A_j}, \qquad \Lambda := \sum_i \lambda_i.$$

For large integer m, set

$$N_j(m) := \Big\lfloor m\frac{\lambda_j}{\Lambda}\Big\rfloor.$$

Standard rounding estimates give

$$\Big|N_j(m) - m\frac{\lambda_j}{\Lambda}\Big| \le 1,$$

and hence

$$\Big|\frac{N_j(m)A_j}{\sum_i N_i(m)A_i} - \theta_j\Big| = O\Big(\frac{1}{m}\Big).$$

Choose m so large that this error is $< \delta$.

Substep 3.4: Build flat model sheets with disjoint translations. In $\Phi(Q) \subset \mathbb{C}^n$, for each j, let $N_j^\perp$ be the complex p-dimensional normal space (the complex orthogonal complement of $\widetilde{\Pi}_j$), so that $\mathbb{C}^n = \widetilde{\Pi}_j \oplus N_j^\perp$. Pick distinct translation vectors $t_{j,1}, \dots, t_{j,N_j} \in N_j^\perp$ in a small ball $B(0,\rho)$ with $\rho \ll \operatorname{diam}(Q)$, such that all affine spaces $\widetilde{\Pi}_j + t_{j,a}$ are pairwise disjoint on $\Phi(Q)$ as (j,a) ranges over all indices. This is possible since $N_j^\perp$ has real dimension $2p \ge 2$ and we choose only finitely many points.

Define

$$\widetilde{Y}_j^a := (\widetilde{\Pi}_j + t_{j,a}) \cap \Phi(Q) \subset \mathbb{C}^n.$$

These satisfy: (i) ψ_0-calibration (complex $(n-p)$-planes); (ii) $\sup_{y\in Q} \angle(T_y\widetilde{Y}_j^a, \Pi_j) = \angle(\widetilde{\Pi}_j, \Pi_j) < \varepsilon$; (iii) mass fractions within δ of θ_j by construction; (iv) pairwise disjoint on $\Phi(Q)$; (v) boundary supported on $\partial\Phi(Q)$.

Substep 3.5: Upgrade to algebraic complete intersections. Use Kodaira embedding and Hörmander L^2-techniques: for large k, pick global sections $s_{j,a}^{(1)}, \dots, s_{j,a}^{(p)} \in H^0(X, L^k)$ whose restrictions to Q are C^2-close to the linear defining functions of $\widetilde{Y}_j^a$. For k large:

- $Y_j^a := \{s_{j,a}^{(1)} = 0\} \cap \cdots \cap \{s_{j,a}^{(p)} = 0\}$ is a smooth complex $(n-p)$-dimensional submanifold;
- On Q, Y_j^a is C^1-close to $\widetilde{Y}_j^a$;
- Calibration, disjointness, and mass estimates persist under small C^1 perturbations.

Pulling back by Φ^{-1} gives the desired family on Q.

Remark 8.20 (Many local pieces: jet separation). The above "upgrade" can be carried out simultaneously for *many* local target planes and translations, because high tensor powers L^k separate jets at finitely many points: for $k \gg 1$ the evaluation map to finite collections of 1-jets (or 2-jets) is surjective. Equivalently, one can prescribe the linear part and constant term of the local defining functions at finitely many chosen centers and then use standard transversality/Bertini arguments to ensure smooth complete intersections. This flexibility is particularly relevant for the "sliver" regime (Remark 8.69), where one may need many small-mass local pieces.

Lemma 8.21 (Jet separation for high tensor powers (standard))**.** *Let $L \to X$ be an ample line bundle on a smooth projective variety. Fix an integer $r \ge 1$ and a finite set of points $\{x_1, \dots, x_M\} \subset X$. Then for $k \gg 1$ (depending on r and the configuration of points), the evaluation map to r-jets*

$$H^0(X, L^k) \longrightarrow \bigoplus_{i=1}^{M} J^r(L^k)_{x_i}$$

is surjective. Equivalently, one can prescribe the r-jet of a section of L^k at finitely many points, independently, for k large.

Remark 8.22 (References). Lemma 8.21 is standard in algebraic geometry: high tensor powers of an ample line bundle are r-jet ample / separate r-jets. One can prove it via Castelnuovo–Mumford regularity and Serre vanishing, or via analytic peak-section constructions using Hörmander L^2 estimates. See e.g. Lazarsfeld, *Positivity in Algebraic Geometry* (Vol. I), and standard treatments of Bergman kernel asymptotics.

□

Fix a finite normal coordinate atlas by geodesic balls of radii $\ll 1$ and subordinate cubes $\{Q\}$ small enough so that the Carathéodory data from Lemma 8.11 are ε-stable on each cube. For each cube Q and each index $j \in \{1, \ldots, N\}$, let $\Pi_{Q,j}$ denote a constant complex $(n-p)$-plane approximating $P_{x,j}$ on Q. Apply Theorem 8.19 to each cube to obtain families $\{Y^a_{Q,j}\}$ of disjoint ψ-calibrated complete intersections.

Remark 8.23 (On using smooth convex cells instead of sharp cubes). The cubical decomposition is convenient for bookkeeping and for standard deformation theorems. However, the "sliver" mechanism (Remark 8.69 / Proposition 8.77) benefits from working in cells with *smooth convex* boundary, because then small slice volume forces small boundary slices in a quantitative way (Remark 8.78). One can replace sharp cubes by "rounded cubes" (a small smoothing of corners/edges inside each chart) while preserving the same adjacency graph. All flat-norm estimates are stable under such uniformly bilipschitz modifications, up to changing constants.

Lemma 8.24 (Normal-coordinate distortion on small balls)**.** *Let (X, g) be a compact Riemannian manifold. There exists $h_0 > 0$ and $C \geq 1$ such that for every $x \in X$ and every $0 < h \leq h_0$, the exponential chart $\exp_x : B_h(0) \subset T_xX \to X$ is C^2 and $(1 + Ch^2)$–bilipschitz onto its image. In particular, any k–dimensional C^1 graph in $B_h(0)$ has k–mass and $(k-1)$–boundary mass distorted by a factor $1 + O(h^2)$ under $\exp_x$, with constants uniform in x.*

Sketch. In geodesic normal coordinates, one has the standard expansion $g_{ij}(y) = \delta_{ij} + O(|y|^2)$ and $\partial g_{ij}(y) = O(|y|)$ with constants controlled by curvature bounds; compactness makes these bounds uniform in x. Integrating the metric distortion along line segments in $B_h(0)$ yields the $(1 + Ch^2)$ bilipschitz estimate for $\exp_x$ on $B_h(0)$. Mass distortion for C^1 graphs follows from the area formula and the uniform Jacobian bounds. □

Lemma 8.25 (Existence of a C^2 cell decomposition at scale h)**.** *Let (X, g) be a compact Riemannian manifold of dimension $2n$. For all sufficiently small $h > 0$ there exists a finite partition of X (up to a null set) into sets $\{Q\}$ with disjoint interiors such that:*

(i) ***(Scale)*** $\operatorname{diam}(Q) \asymp h$ *uniformly in Q;*

(ii) ***(Chart control)*** *each Q is contained in a normal-coordinate chart of radius $\asymp h$ on which the metric is $(1 + O(h^2))$–bilipschitz to the Euclidean metric (as in Lemma 8.24);*

(iii) *(C^2* ***boundary with controlled curvature****) each Q has C^2 boundary and its principal curvatures satisfy the* absolute *bound*

$$\left|\kappa_i\right| \;\leq\; \frac{C}{h} \qquad \textit{everywhere on } \partial Q,$$

with a constant C depending only on (X, g).

Sketch. Fix $h \ll 1$ and choose a finite normal-coordinate atlas by geodesic balls of radius $\asymp h$. Inside each chart, start from a standard cubical grid at mesh $\asymp h$ (a polyhedral cell decomposition). One may smooth the resulting codimension-1 cell complex (the union of interfaces) to a C^2 hypersurface network at scale $\asymp h$, producing a genuine partition whose interfaces are shared by adjacent cells and have principal curvatures bounded by C/h. Pushing forward/back by the chart maps preserves these bounds up to uniform constants by Lemma 8.24. □

Define the local current

$$S_Q := \sum_{j=1}^{N} \sum_{a=1}^{N_{Q,j}} [Y^a_{Q,j}] \llcorner Q.$$

By construction, each $Y^a_{Q,j}$ is ψ-calibrated; hence S_Q is a positive ψ-calibrated integral current on Q. Its tangent-plane distribution on Q is a convex combination of directions within ε of $\{\Pi_{Q,j}\}$ with weights proportional to the ψ–masses in each family (equivalently proportional to $N_{Q,j}A_{Q,j}$, where $A_{Q,j}$ is the ψ–mass of a single (Q,j)-sheet in Q).

Lemma 8.26 (Local barycenter matching)**.** *For any $\delta > 0$ there exist integers $N_{Q,1}, \ldots, N_{Q,N}$ such that the tangent-plane Young measure of S_Q has barycenter within δ (in Hilbert–Schmidt norm) of the normalized field $\widehat{\beta}$ on Q, and*

$$\mathbf{M}(S_Q) \to m \int_Q \beta \wedge \psi \quad \text{as } \delta \to 0.$$

Proof. Let $A_{Q,j} > 0$ denote the common ψ–mass of a single (Q,j)-sheet in Q (all sheets in a fixed family (Q,j) are local parallel translates, so their mass in Q agrees up to $o_\delta(1)$). Choose integers $N_{Q,j}$ so that the *mass fractions*

$$\frac{N_{Q,j}A_{Q,j}}{\sum_i N_{Q,i}A_{Q,i}}$$

approximate $\theta_{x,j}$ (nearly constant on Q) to within $O(\delta)$. Then the resulting mass-weighted barycenter

$$\sum_j \frac{N_{Q,j}A_{Q,j}}{\sum_i N_{Q,i}A_{Q,i}}\, \xi_{\Pi_{Q,j}}$$

is within δ of $\widehat{\beta}$ on Q. Because the tangent angles are $< \varepsilon$ and $\varepsilon \ll \delta$, the Hilbert–Schmidt distance of barycenters is $\le C(\varepsilon + \delta)$.

Finally, calibratedness gives $\mathbf{M}([Y^a_{Q,j}]\llcorner Q) = \int_Q \psi \llcorner [Y^a_{Q,j}]$, hence

$$\mathbf{M}(S_Q) = \sum_j N_{Q,j}A_{Q,j}.$$

By scaling the $N_{Q,j}$ simultaneously (and then rounding), one can arrange $\sum_j N_{Q,j}A_{Q,j} \to m \int_Q \beta \wedge \psi$ as $\delta \to 0$. □

Step 4: Global cohomology quantization (Theorem C)

This step forces the global integral current to represent exactly the correct homology class $\mathrm{PD}(m[\gamma])$ by using lattice discreteness.

Sliverpath (restructured) strategy. There are two conceptually different regimes for the "microstructure" part of Step 4:

(I) (**Dense-sheet / constant-mass regime**) One uses families of calibrated sheets whose cell-restricted masses are comparable to $h^{2(n-p)}$ and whose multiplicities are large ($N_Q \gg 1$). In this regime the template route (Lemma 8.39 / Proposition 8.43) yields a clean bound $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim (h+\varepsilon)\, m$.

(II) (**Bergman-scale / sliver regime**) In middle codimension ($1 < p < n-1$), the holomorphic C^1 control scale from Bergman kernels is naturally $h \asymp m^{-1/2}$; then the constant-mass model predicts $N_Q \sim m\, h^{2p} \asymp m^{1-p}$, which is < 1 for $p > 1$ (Remark 8.67 and the "Bergman-scale amplification" discussion). To keep enough degrees of freedom per cell at this intrinsic h, one must split the target mass into many tiny calibrated pieces ("slivers") while controlling their induced face slices.

In this file we treat (II) as the main unconditional target and view (I) as a simplified special case.

Theorem 8.27 (Global cohomology quantization)**.** *Let X be a compact Kähler n-fold with rational Hodge class $[\gamma] \in H^{2p}(X,\mathbb{Q})$ represented by a smooth closed (p,p)-form β with $\beta(x) \in K_p(x)$ pointwise. Let $\{Q\}$ be a partition of X into coordinate cells of diameter h (e.g. cubes, or uniformly rounded/smooth convex cells as in Remark 8.23). Assume that on each cell Q one can realize the required local tangent-plane data by calibrated holomorphic pieces with either:*

(I) *the constant-mass sheet model of Substep 4.1 (many sheets per cell), or*

(II) *the sliver microstructure mechanism of Proposition 8.77 (many tiny pieces per cell).*

Then there exists an integer $m \geq 1$ (clearing denominators of $[\gamma]$) such that for every $\varepsilon > 0$ there exist:

- *A closed integral $(2n-2p)$-current T_ε with $[T_\varepsilon] = \mathrm{PD}(m[\gamma])$;*
- *A correction current R_ε with $\mathbf{M}(R_\varepsilon) < \varepsilon$;*

such that the local tangent-plane mass proportions on each Q match those of β up to error $o_{\varepsilon\to 0}(1)$.

Proof. The proof proceeds in three substeps.

Substep 4.1: Local quantization. Choose the partition $\{Q\}$ fine enough that on each Q, $\beta(x)$ is within δ (in operator norm) of $\beta(x_Q)$ for a base point $x_Q \in Q$, and the Kähler metric is nearly constant (Jacobian and volume distortion $\leq 1+\delta$).

By Lemma 8.11, write

$$\beta(x_Q) = t_Q \sum_{j=1}^{J(Q)} \theta_{Q,j}\, \xi_{Q,j}, \qquad t_Q := \langle \beta(x_Q), \psi_{x_Q} \rangle,$$

where $\xi_{Q,j} \in K_p(x_Q)$ are normalized extremal generators (coming from complex $(n-p)$-planes) satisfying $\langle \xi_{Q,j}, \psi_{x_Q} \rangle = 1$, the weights satisfy $\theta_{Q,j} \geq 0$, $\sum_j \theta_{Q,j} = 1$, and $J(Q) \leq N = N(n,p)$ uniformly bounded.

Since $[\gamma]$ is rational, all its periods lie in $(1/M)\mathbb{Z}$ for some fixed M. Choose $m \gg 1$ divisible by M.

Let $P_{Q,j} \subset T_{x_Q}X$ be the complex $(n-p)$-plane corresponding to $\xi_{Q,j}$. In the flattened model on Q, any affine ψ–calibrated sheet with tangent plane $P_{Q,j}$ has the same ψ–mass in Q; denote this common value by $A_{Q,j} > 0$ (it depends on the cube geometry and direction but satisfies $A_{Q,j} \asymp \mathrm{side}(Q)^{2(n-p)}$). The target ψ–mass in Q is

$$M_Q := m \int_Q \beta \wedge \psi \;\approx\; m\, t_Q \,\mathrm{Vol}(Q),$$

up to $O(\delta)$ error from the C^0–variation of β on Q and the metric distortion.

Choose integers $N_{Q,j} \ge 0$ so that simultaneously

$$\Big|\frac{N_{Q,j} A_{Q,j}}{\sum_i N_{Q,i} A_{Q,i}} - \theta_{Q,j}\Big| \le \delta \qquad \text{and} \qquad \Big|\sum_j N_{Q,j} A_{Q,j} - M_Q\Big| \le \delta\, M_Q.$$

(Such choices exist by rounding, since the unknowns enter linearly and m may be taken arbitrarily large.)

Sliver variant (regime (II)**).** At the intrinsic Bergman scale $h \asymp m^{-1/2}$ and in middle codimension $p > 1$, the constant-mass model $A_{Q,j} \asymp h^{2(n-p)}$ typically forces $N_{Q,j} = O(1)$ (or 0) and cannot supply enough degrees of freedom to satisfy both transport/gluing and cohomology constraints. Instead, one replaces the constant-mass sheets in family (Q, j) by *many* calibrated *sliver pieces* whose individual masses are much smaller, so that the total mass in family j still matches $\theta_{Q,j}\, M_Q$ while the number of pieces is large. The flat ball model shows this is compatible with transverse W_1 approximation (Proposition 8.72), and the holomorphic upgrade can realize finite translation templates with uniform C^1 control on Bergman-scale cells (Corollary 8.74 / Proposition 8.81). The remaining global gluing bookkeeping in the sliver regime is packaged as the weighted reduction Proposition 8.55.

Apply Theorem 8.19 to realize each direction (Q, j) by a family of ψ–calibrated sheets $Y^a_{Q,j} \subset Q$ ($a = 1, \dots, N_{Q,j}$) with angle control, disjointness on Q, and boundary supported on ∂Q.

Define the raw local current

$$S_Q := \sum_{j=1}^{J(Q)} \sum_{a=1}^{N_{Q,j}} [Y^a_{Q,j}] \llcorner Q.$$

Substep 4.2: Gluing across cubes. Consider the global raw current

$$T^{\mathrm{raw}} := \sum_Q S_Q.$$

This is integral but not closed: $\partial T^{\mathrm{raw}}$ lives on the union of cube faces. View the cube adjacency as a finite graph: vertices = cubes Q, edges = codimension-1 faces $F = Q \cap Q'$. On each oriented face F, the restriction of ∂S_Q induces a $(2n-2p-1)$-current $B_{Q \to F}$ living on F. Summed over all cubes:

$$\partial T^{\mathrm{raw}} = \sum_F B_F,$$

where B_F is the mismatch between the two neighboring cubes.

Key point (flat norm, not mass): In general the individual face currents B_F need not have small mass (cancellation-heavy boundaries can have large mass), so the robust quantity to control is the *flat norm* of the total mismatch $\partial T^{\mathrm{raw}}$. Recall the flat norm on $(2n-2p-1)$-currents:

$$\mathcal{F}(S) := \inf\{\mathbf{M}(R) + \mathbf{M}(Q) : \; S = R + \partial Q\},$$

where R is an integral $(2n-2p-1)$-current and Q is an integral $(2n-2p)$-current. On a compact manifold one has the dual characterization (Federer–Fleming):

$$\mathcal{F}(S) = \sup\{S(\eta) : \ \eta \in C^\infty\Lambda^{2n-2p-1}, \ \|\eta\|_{\text{comass}} \le 1, \ \|d\eta\|_{\text{comass}} \le 1\}.$$

For $S = \partial T^{\text{raw}}$ and such η, Stokes gives $S(\eta) = \partial T^{\text{raw}}(\eta) = T^{\text{raw}}(d\eta)$.

Proposition 8.28 (Transport control $\Rightarrow$ flat-norm gluing)**.** *Fix a cubulation of X by coordinate cubes of side length $h =$ mesh, and write $T^{\text{raw}} = \sum_Q S_Q$ as above, where each S_Q is a sum of calibrated sheets restricted to Q. Assume the following* geometric parameterization *holds on each interior face $F = Q \cap Q'$:*

(a) ***(Small-angle graph model)*** *For each cube Q and each sheet family (Q,j), the sheets crossing F are C^1-graphs over a fixed calibrated reference plane $\Pi_{Q,j}$ with $\sup_{y\in Q} \angle(T_y Y^a_{Q,j}, \Pi_{Q,j}) \le \varepsilon$.*

(b) ***(Transverse measures on faces)*** *After identifying a tubular neighborhood of F with a product $F \times B^{2p}(0,ch)$ in normal coordinates, the restriction of ∂S_Q to F can be written as a finite sum of translated slice currents parameterized by a discrete transverse measure $\mu_{Q\to F}$ on $B^{2p}(0,ch)$ (integer weights), and similarly for Q'.*

(c) *(W_1 **face matching**) The two induced transverse measures have the same total mass and satisfy*

$$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \ \le \ \tau_F,$$

where W_1 is the 1-Wasserstein distance on $B^{2p}(0,ch)$.

Then there exists a constant $C = C(n,p,X)$ such that for every smooth $(2n-2p-1)$-form η with $\|\eta\|_{\text{comass}} \le 1$ and $\|d\eta\|_{\text{comass}} \le 1$ one has the face estimate

$$|B_F(\eta)| \ \le \ C\, h^{2n-2p-1}\big(\tau_F + \varepsilon\, \mathbf{M}(\mu_{Q\to F})\, h\big),$$

and hence

$$\mathcal{F}(B_F) \ \le \ C\, h^{2n-2p-1}\big(\tau_F + \varepsilon\, \mathbf{M}(\mu_{Q\to F})\, h\big).$$

Consequently,

$$\mathcal{F}(\partial T^{\text{raw}}) \ \le \ \sum_F \mathcal{F}(B_F) \ \le \ C\, h^{2n-2p-1} \sum_F \tau_F \ + \ C\,\varepsilon\, h^{2n-2p} \sum_F \mathbf{M}(\mu_{Q\to F}).$$

Proof sketch. In the flat/parallel model ($\varepsilon = 0$), each face slice defines a function $f_\eta(y) := \Sigma_y(\eta)$ on the transverse parameter space, and Stokes on the cylinder between slices shows $\mathrm{Lip}(f_\eta) \lesssim h^{2n-2p-1}\|d\eta\|_{\text{comass}} \le Ch^{2n-2p-1}$. Kantorovich–Rubinstein duality then yields $|B_F(\eta)| \le \mathrm{Lip}(f_\eta)\, W_1(\mu_{Q\to F}, \mu_{Q'\to F}$ $h^{2n-2p-1}\tau_F$.

For $\varepsilon > 0$, compare each almost-parallel sheet to an exactly-parallel model slice in the same tubular chart. The resulting error in f_η is controlled by the C^1-graph distortion and contributes an additional $O(\varepsilon\, h^{2n-2p})$ per sheet (hence the stated $\varepsilon\, \mathbf{M}(\mu_{Q\to F})\, h$ term). Taking the supremum over test forms gives the flat-norm bound. Finally sum over faces and use the triangle inequality for $\mathcal{F}$. $\square$

Lemma 8.29 (Flat-norm boundary control suffices for gluing)**.** *Let X be a compact oriented Riemannian manifold and let $k \in \{1, \dots, \dim X - 1\}$. Let T^{raw} be an integral k–current on X (not* *necessarily closed) and set $S := \partial T^{\mathrm{raw}}$. Assume $\mathcal{F}(S) \le \eta$ for some $\eta > 0$. Then there exists an integral k–current R_{glue} with $\partial R_{\mathrm{glue}} = -S$ and*

$$\mathbf{M}(R_{\mathrm{glue}}) \ \le \ C(X,k)\left(\eta + \eta^{\frac{k}{k-1}}\right).$$

In particular, $T^{\mathrm{raw}} + R_{\mathrm{glue}}$ is a closed integral k–cycle and

$$\mathbf{M}(T^{\mathrm{raw}} + R_{\mathrm{glue}}) \ \le \ \mathbf{M}(T^{\mathrm{raw}}) + C(X,k)\left(\eta + \eta^{\frac{k}{k-1}}\right).$$

Sketch. Write $S = R + \partial Q$ with $\mathbf{M}(R) + \mathbf{M}(Q) \le 2\mathcal{F}(S) \le 2\eta$. Since S is a boundary, $R = S - \partial Q$ is also a boundary and hence null-homologous. By the Federer–Fleming isoperimetric inequality there exists an integral k–current Q_R with $\partial Q_R = R$ and $\mathbf{M}(Q_R) \le C\,\mathbf{M}(R)^{k/(k-1)} \le C\,(2\eta)^{k/(k-1)}$. Setting $R_{\mathrm{glue}} := -(Q + Q_R)$ gives $\partial R_{\mathrm{glue}} = -S$ and the stated mass bound. □

Remark 8.30 (Why hypotheses (a)–(b) hold for the local sheet model). In the flat model of Substep 3.4, each sheet in family (Q,j) is literally an affine calibrated plane $(\widetilde{\Pi}_{Q,j} + t_{j,a}) \cap Q$, with translation parameter $t_{j,a} \in N^{\perp}_{Q,j} \cong \mathbb{R}^{2p}$. For a fixed face $F \subset \partial Q$, the boundary slice current

$$\Sigma_{F,j}(t) := \partial\big([\widetilde{\Pi}_{Q,j} + t]\llcorner Q\big)\llcorner F$$

depends only on t through its component normal to the $(2n-2p-1)$-plane $\widetilde{\Pi}_{Q,j} \cap TF$. Thus, in the flat model, $\partial S_Q \llcorner F$ can be written as a finite sum $\sum_a \Sigma_{F,j}(t_{j,a})$, i.e. it is parameterized by the discrete transverse measure $\mu_{Q\to F} := \sum_a \delta_{t_{j,a}}$ (with integer weights).

After upgrading to algebraic complete intersections in Substep 3.5, the sheets remain C^1-graphs over the flat model on Q (for k large), so the same parameterization persists in a tubular neighborhood of F up to an $O(\varepsilon)$ error controlled by the graph distortion. This justifies the use of transverse measures on faces and the small-angle graph model in Proposition 8.28.

What is *not* automatic is hypothesis (c): arranging W_1 matching across faces simultaneously for all cubes, subject to the constraint that each sheet's translation parameter determines its intersection with *all* faces of Q at once. Equivalently, for a fixed cube Q and family (Q,j), the face measures $\mu_{Q\to F}$ for different faces $F \subset \partial Q$ are not independent choices: they arise as pushforwards of the *same* discrete translation multiset $\{t_{j,a}\}$ under the corresponding face-slice maps. Thus the remaining task is a *simultaneous* matching problem.

Definition 8.31 (Toy marginal realization problem)**.** Fix integers $d \ge 1$ and $M \ge 1$ and write $\Omega := [M]^d$ for a d–dimensional grid. For each coordinate $\ell \in \{1,\dots,d\}$ let $m_\ell \in \mathbb{Z}^{[M]}_{\ge 0}$ be an integer histogram with common total mass $\sum_{a=1}^{M} m_\ell(a) = N$ (independent of ℓ). The *marginal realization problem* asks for a multiset of grid points $\{x_1,\dots,x_N\} \subset \Omega$ such that for each ℓ and $a \in [M]$, the number of points with $(x_i)_\ell = a$ equals $m_\ell(a)$.

Lemma 8.32 (Existence of a multiset with prescribed 1D marginals)**.** *The marginal realization problem in Definition 8.31 always has a solution.*

Sketch. For $d = 2$ this is the existence of a nonnegative integer matrix with prescribed row and column sums, equivalently a bipartite multigraph with given degrees; it can be constructed greedily or via max-flow. For $d > 2$, realize the first $d-1$ marginals by induction to obtain a multiset in $[M]^{d-1}$, then append the dth coordinate by a bipartite matching between the realized $(d-1)$–tuples and the dth-coordinate values with multiplicities m_d. □

Remark 8.33 (Interpretation for the microstructure/gluing problem). Lemma 8.32 records that, in an idealized setting where different faces "see" disjoint coordinate projections of a translation parameter, the simultaneous matching constraints are *purely combinatorial* and always solvable. In the calibrated-sheet setting, however, each face measure is obtained from the same translation multiset by a *nontrivial linear face map* depending on the plane direction and the local face chart (Remark 8.30). This turns the global matching requirement into a discrete tomography / simultaneous pushforward problem for a family of linear maps, which is the conceptual core of Blocker B1.

Lemma 8.34 (Rounding a Lipschitz density varies by $O(h)$ across neighbors)**.** *Let $\{Q\}$ be a mesh-h cell decomposition and choose a representative point $x_Q \in Q$ for each cell. Let $f : X \to \mathbb{R}$ be L–Lipschitz and bounded. Fix $m \geq 1$ and define*

$$N_Q \;:=\; \Big\lfloor m\, h^{2p}\, f(x_Q) \Big\rceil \;\in\; \mathbb{Z},$$

where $\lfloor \cdot \rceil$ denotes rounding to the nearest integer. Then for any adjacent cells $Q \sim Q'$ (sharing an interface) one has

$$|N_Q - N_{Q'}| \;\leq\; C\, L\, m\, h^{2p+1} + 2,$$

with C depending only on the uniform geometry of the mesh. In particular, if $f \geq f_0 > 0$ and $mh^{2p} \to \infty$, then $|N_Q - N_{Q'}|/N_Q = O(h)$ uniformly over adjacent pairs.

Sketch. Since $d(x_Q, x_{Q'}) \lesssim h$ for adjacent cells and f is L–Lipschitz,

$$|mh^{2p} f(x_Q) - mh^{2p} f(x_{Q'})| \;\leq\; L\, m\, h^{2p}\, d(x_Q, x_{Q'}) \;\lesssim\; L\, m\, h^{2p+1}.$$

Rounding to the nearest integer changes each term by at most 1, giving the stated bound. □

Remark 8.35 (Bounded global corrections do not spoil the $O(h)$ edit regime). In applications, one often needs to adjust the rounded counts N_Q by a bounded amount (e.g. to enforce finitely many global period constraints). Lemma 8.34 shows that the *neighbor variation* is $O(h)$ in a *relative* sense once $N_Q \to \infty$. In particular, if $N_Q \gtrsim h^{-1}$ uniformly and $\widetilde{N}_Q := N_Q + \Delta_Q$ with $|\Delta_Q| \leq C_0$, then $\widetilde{N}_Q/N_Q = 1 + O(h)$ and hence the induced insertions/deletions are an $O(h)$ fraction:

$$\frac{|\widetilde{N}_Q - N_Q|}{\widetilde{N}_Q} \;\leq\; \frac{C_0}{\widetilde{N}_Q} \;\lesssim\; C_0\, h.$$

Thus any such bounded correction is harmless for the nested-template scheme (Remark 8.36) and can be absorbed by Lemma 8.63 provided h is sufficiently small.

Remark 8.36 (A nested-template scheme reduces the edit part of Blocker B1). Fix, for each local direction family, an *ordered* template of transverse atoms $(y_a)_{a\geq 1} \subset B_{C_0 h}(0) \subset \mathbb{R}^{2p}$. For example, Lemma 8.71 produces such a nested sequence on a sphere (uniform density), and scaling embeds it into $B_{C_0 h}(0)$. If the two neighboring cells choose their transverse discrete measures by taking the *first* N_Q and $N_{Q'}$ atoms (with comparable slice weights per atom), then the unmatched part across the interface is exactly a *prefix edit* of size $|N_Q - N_{Q'}|$. When N_Q is obtained by rounding a smooth target density, Lemma 8.34 gives that the edit fraction is $O(h)$. Lemma 8.63 then shows that this unmatched part is automatically $O(h^2)$ in flat norm and can be absorbed into the matched displacement bookkeeping.

Thus, the remaining conceptual difficulty in Blocker B1 is not that counts must match *exactly*, but arranging a globally coherent choice of ordered templates (and direction pairings) so that across every face there is a large matched subtemplate and only an $O(h)$ fraction of edits, while simultaneously meeting the local mass/barycenter constraints.

Proposition 8.37 (Prefix templates $\Rightarrow$ interface coherence up to $O(h)$ edits)**.** *Work in the setting of Proposition 8.55 on a mesh-h cell decomposition and fix an interior interface $F = Q \cap Q'$. Assume that for some paired direction label j the two sides admit tubular-face parameterizations with linear face maps $\Phi_{Q,F}$ and $\Phi_{Q',F}$ as in Lemma 8.62. Fix an ordered template of transverse atoms $(y_a)_{a\geq 1} \subset B_{C_0 h}(0) \subset \mathbb{R}^{2p}$ and define, for integers $N \geq 0$,*

$$\nu^{(N)} \;:=\; \sum_{a=1}^{N} \delta_{y_a}.$$

Suppose the two cells choose their face measures from prefixes:

$$\mu_{Q\to F} = (\Phi_{Q,F})_{\#}\nu^{(N_Q)}, \qquad \mu_{Q'\to F} = (\Phi_{Q',F})_{\#}\nu^{(N_{Q'})},$$

for some integers $N_Q, N_{Q'} \geq 0$. Assume the slow-variation *bound*

$$|N_Q - N_{Q'}| \;\leq\; \theta_F \,\min\{N_Q, N_{Q'}\} \qquad \text{with} \qquad \theta_F \;\lesssim\; h.$$

Then writing B_F for the mismatch current on F coming from these paired families, one has the per-face flat-norm estimate

$$\mathcal{F}(B_F) \;\leq\; C\,h^2\Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F)\Big) \;+\; O(\varepsilon\, M_F),$$

with C depending only on (n, p, X) and the uniform tubular-face charts (and $M_F \lesssim M_Q + M_{Q'}$ as in Lemma 8.61).

Proof sketch. Let $N_{\min} := \min\{N_Q, N_{Q'}\}$ and decompose

$$\nu^{(N_Q)} = \nu^{(N_{\min})} + \nu^+, \qquad \nu^{(N_{Q'})} = \nu^{(N_{\min})} + \nu^-,$$

where $\nu^{\pm}$ are the (possibly empty) unmatched tails of total mass $|N_Q - N_{Q'}|$. Then Lemma 8.62 applies with matched part $\nu^{(N_{\min})}$ and unmatched part $\nu^{\pm}$. The matched part contributes the $Ch^2(\cdots) + O(\varepsilon M_F)$ term.

For the unmatched part, the hypothesis implies the edit fraction is $\theta_F \lesssim h$. Thus Lemma 8.63 shows the unmatched contribution is also $O(h^2)$ times the total boundary mass on the face. Combining the two bounds yields the stated estimate. $\square$

Corollary 8.38 (Global prefix-template coherence from Lipschitz targets)**.** *In the setting of Proposition 8.37, suppose the prefix lengths N_Q arise by rounding a Lipschitz target as in Lemma 8.34, with $f \geq f_0 > 0$ and $mh^{2p} \to \infty$ so that $|N_Q - N_{Q'}|/N_Q = O(h)$ uniformly over adjacent cells. Then the slow-variation hypothesis holds on every interior face with $\theta_F \lesssim h$, so Proposition 8.37 applies uniformly over all interfaces.*

Moreover, if one subsequently adjusts the counts by bounded amounts (e.g. to enforce finitely many global cohomology constraints as in Proposition 8.88), the $O(h)$ edit-fraction conclusion persists provided $N_Q \gtrsim h^{-1}$; see Remark 8.35 and Remark 8.89.

Lemma 8.39 (Automatic W_1-matching from smooth dependence of face maps)**.** *Let μ be a finite Borel measure on $\mathbb{R}^{2p}$ supported in a ball of radius $O(h)$ and with total mass $\mu(\mathbb{R}^{2p}) = N$. Let $\Phi, \Phi' : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ be linear maps with $\|\Phi - \Phi'\|_{\mathrm{op}} \leq C\,h$. Then*

$$W_1(\Phi_{\#}\mu, \Phi'_{\#}\mu) \;\leq\; C\,h \int_{\mathbb{R}^{2p}} \|y\|\, d\mu(y) \;\leq\; C'\,h^2\,N.$$

Sketch. Couple $\Phi_{\#}\mu$ and $\Phi'_{\#}\mu$ by pushing forward μ under $y \mapsto (\Phi y, \Phi' y)$ and bound the transport cost by $\int \|\Phi y - \Phi' y\| \, d\mu \leq \|\Phi - \Phi'\|_{\mathrm{op}} \int \|y\| \, d\mu$. The support radius $O(h)$ gives $\int \|y\| \, d\mu \leq O(h) \, \mu(\mathbb{R}^{2p}) = O(h)N$. □

Lemma 8.40 (Why $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} = O(h)$ is geometric)**.** *Fix a mesh-h cell decomposition and an interior interface $F = Q \cap Q'$. Assume h is small enough that $Q \cup Q'$ lies in a single holomorphic coordinate chart and that F admits a tubular product chart of radius $\asymp h$ (both are automatic on compact X for $h \ll 1$). Fix a paired direction index j so that the calibrated reference planes used in Substep 4.2 satisfy*

$$\angle(\Pi_{Q,j}, \Pi_{Q',j}) \;\leq\; C\,h,$$

as ensured by the local stabilization in Lemma 8.11 and smoothness of β. Then one may choose the face-slice linear maps $\Phi_{Q,F}$ and $\Phi_{Q',F}$ in Proposition 8.43(T1) *so that*

$$\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} \;\leq\; C'\,h,$$

with C' depending only on (X, ω) and (n, p) (and not on Q, Q', F once h is sufficiently small).

Sketch. In the flat model, for a fixed face hyperplane F and a fixed plane direction Π, the dependence of the induced face parameter on the translation vector $t \in \Pi^{\perp}$ is linear (it is obtained by restricting an affine translate and then projecting to the fixed transverse parameter space on F). Thus the corresponding map $\Phi_{\Pi,F} : \Pi^{\perp} \to \mathbb{R}^{2p}$ is represented by a matrix whose entries are smooth functions of the angle data between Π and TF (equivalently, smooth functions on the Grassmannian away from degeneracy). On compact subsets of the calibrated Grassmannian and for uniformly transverse faces, this dependence is Lipschitz: $\|\Phi_{\Pi,F} - \Phi_{\Pi',F}\|_{\mathrm{op}} \leq C \, \angle(\Pi, \Pi')$.

In the curved setting, working in a fixed holomorphic chart and a fixed tubular-face chart of radius $\asymp h$, the metric and the identification of normal/transverse directions are C^1-close to the flat model with $O(h)$ error. Combining the $O(h)$ variation of $\Pi_{Q,j}$ across adjacent cells with the C^1 control of the coordinate/tubular charts yields the stated $O(h)$ bound for $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}}$. □

Lemma 8.41 (Template stability under small multiset edits)**.** *Let $\Omega \subset \mathbb{R}^{2p}$ be a bounded domain of diameter* $\mathrm{diam}(\Omega) \leq Ch$. *Let $\mu = \sum_{a=1}^{N} \delta_{y_a}$ and $\mu' = \sum_{b=1}^{N} \delta_{y'_b}$ be two integer-weighted discrete measures on Ω with the* same total mass N. *Assume there is a matching of atoms such that $\|y_a - y'_a\| \leq \Delta$ for all a (after relabeling). Then*

$$W_1(\mu, \mu') \;\leq\; \Delta\, N.$$

More generally, if μ' is obtained from μ by deleting r atoms and inserting r atoms (so total mass stays N), then

$$W_1(\mu, \mu') \;\leq\; r \cdot \mathrm{diam}(\Omega) \;\leq\; C\, r\, h.$$

Proof. For the first claim, couple μ and μ' by pairing each y_a to y'_a; the transport cost is $\sum_a \|y_a - y'_a\| \leq \Delta N$. For the second claim, transport each deleted atom to an inserted atom at cost at most $\mathrm{diam}(\Omega)$ and keep the unchanged atoms fixed. □

Remark 8.42 (How Lemma 8.39 reduces the remaining matching task). If, for each cube Q and sheet family (Q, j), we choose the translation multiset $\{t_{j,a}\}$ by a *fixed* template in $N^{\perp}_{Q,j}$ (e.g. a scaled lattice/low-discrepancy set of diameter $O(h)$), then across a shared face $F = Q \cap Q'$ the two induced transverse measures are related by applying two nearby face-slice maps (coming from nearby plane directions and nearby normal-coordinate identifications). Since β is smooth, these maps differ by $O(h)$ in operator norm, so Lemma 8.39 yields

$$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \ \lesssim\ h^2\, N_F,$$

where N_F is the number of sheets contributing to that face. Inserting this into Proposition 8.28 yields a global bound of the form

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \ \lesssim\ m\, h \ +\ O(\varepsilon\, m),$$

so choosing $h = h(m) \to 0$ slowly (e.g. $h = m^{-\alpha}$ with $\alpha > 0$ small) makes the gluing correction R_{glue} sublinear in m and hence negligible in the mass equality as $m \to \infty$. The remaining task is then to implement this "fixed template" choice while still meeting the cohomological constraints (Substep 4.3).

Proposition 8.43 (Template coherence $\Rightarrow$ quantitative flat-norm gluing)**.** *Fix a cubulation of X by coordinate cubes of side length h and write $T^{\mathrm{raw}} = \sum_Q S_Q$ as in Substep 4.2. Assume the small-angle and transverse-parameter hypotheses* (a)–(b) *of Proposition 8.28 hold with the same ε. Assume moreover the following* template coherence *holds on each interior face $F = Q \cap Q'$:*

- (T1) ***(Same template up to $O(h)$ face-map variation)*** *For each face F there are linear face-slice maps $\Phi_{Q,F}, \Phi_{Q',F} : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ with $\|\Phi_{Q,F}\|_{\mathrm{op}} + \|\Phi_{Q',F}\|_{\mathrm{op}} \le C_{\Phi,0}$ and $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} \le C_\Phi\, h$ such that $\mu_{Q\to F} = (\Phi_{Q,F})_\# \nu_{Q,F}$ and $\mu_{Q'\to F} = (\Phi_{Q',F})_\# \nu_{Q',F}$ for some integer-weighted measures $\nu_{Q,F}, \nu_{Q',F}$ supported in a ball of radius $C_0 h$.*
- (T2) ***(Slowly varying multiplicities)*** *The measures $\nu_{Q,F}$ and $\nu_{Q',F}$ differ by at most r_F insertions/deletions and $r_F \le C_{\mathrm{sv}}\, h\, N_F$, where $N_F := \mu_{Q\to F}(\mathbb{R}^{2p}) = \mu_{Q'\to F}(\mathbb{R}^{2p})$.*

Then for each interior face F one has the quantitative face matching bound

$$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \ \le\ C_1\, h^2\, N_F,$$

and consequently there exists a constant $C_2 = C_2(n,p,X)$ such that

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \ \le\ C_2\, \big(h + \varepsilon\big)\, \mathbf{M}(T^{\mathrm{raw}}).$$

In particular, if $\mathbf{M}(T^{\mathrm{raw}}) \asymp m$ (as in Substep 4.1), then

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \ \le\ C_3\, \big(h + \varepsilon\big)\, m,$$

so one may take $\varepsilon_{\mathrm{glue}}(m, \varepsilon, h) = C_3(h + \varepsilon)$ in the flat-norm estimate sought in Remark 8.83.

Proof. Fix an interior face $F = Q \cap Q'$. By Lemma 8.41 and the insertion/deletion hypothesis (T2), there exist a coupling of $\nu_{Q,F}$ and $\nu_{Q',F}$ with transport cost $\le C\, r_F\, h \le C\, C_{\mathrm{sv}}\, h^2 N_F$. Applying the face-map stability Lemma 8.39 to $\nu_{Q,F}$ and the maps $\Phi_{Q,F}, \Phi_{Q',F}$ from (T1) gives

$$W_1\big((\Phi_{Q,F})_\# \nu_{Q,F}, (\Phi_{Q',F})_\# \nu_{Q,F}\big) \ \le\ C\, h^2\, \nu_{Q,F}(\mathbb{R}^{2p}) \ =\ C\, h^2\, N_F.$$

Moreover, $\Phi_{Q',F}$ is $C_{\Phi,0}$-Lipschitz by (T1), hence

$$W_1\big((\Phi_{Q',F})_\# \nu_{Q,F}, (\Phi_{Q',F})_\# \nu_{Q',F}\big) \le C_{\Phi,0}\, W_1(\nu_{Q,F}, \nu_{Q',F}) \le C_{\Phi,0}\, C\, r_F\, h \le C\, h^2\, N_F.$$

Triangle inequality for W_1 therefore yields

$$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \le W_1\big((\Phi_{Q,F})_\# \nu_{Q,F}, (\Phi_{Q',F})_\# \nu_{Q,F}\big) + W_1\big((\Phi_{Q',F})_\# \nu_{Q,F}, (\Phi_{Q',F})_\# \nu_{Q',F}\big) \le C_1\, h^2\, N_F,$$

after absorbing the insertion/deletion term into the same $h^2 N_F$ scale.

Now insert $\tau_F := C_1 h^2 N_F$ into Proposition 8.28 and sum over interior faces:

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \le C\, h^{2n-2p-1} \sum_F \tau_F \;+\; C\,\varepsilon\, h^{2n-2p} \sum_F N_F.$$

Since $\tau_F = C_1 h^2 N_F$, the first term is $C\, h^{2n-2p+1} \sum_F N_F$. Each sheet piece contributing to T^{raw} meets at most $O(1)$ faces, hence $\sum_F N_F \le C'$ (number of sheet pieces). Under the local model in Substep 4.1 (constant-mass sheet pieces), each sheet piece has ψ–mass $\asymp h^{2(n-p)}$, so (number of sheet pieces)$\cdot$ $h^{2(n-p)} \lesssim \mathbf{M}(T^{\mathrm{raw}})$ and therefore

$$\sum_F N_F \;\le\; C''\, \mathbf{M}(T^{\mathrm{raw}})\, h^{-2(n-p)}.$$

Plugging this into the two global terms gives

$$h^{2n-2p+1} \sum_F N_F \;\le\; C''\, h^{2n-2p+1}\, \mathbf{M}(T^{\mathrm{raw}})\, h^{-2n+2p} = C''\, h\, \mathbf{M}(T^{\mathrm{raw}}),$$

and similarly

$$\varepsilon\, h^{2n-2p} \sum_F N_F \;\le\; C''\, \varepsilon\, h^{2n-2p}\, \mathbf{M}(T^{\mathrm{raw}})\, h^{-2n+2p} = C''\, \varepsilon\, \mathbf{M}(T^{\mathrm{raw}}).$$

Absorbing constants yields $\mathcal{F}(\partial T^{\mathrm{raw}}) \le C_2(h+\varepsilon)\mathbf{M}(T^{\mathrm{raw}})$, as claimed. □

Remark 8.44 (Sliver regime: what changes in the global counting estimate). Proposition 8.43 isolates the *face-by-face transport* input needed for gluing in flat norm. The final global bound in the proof used an additional bookkeeping step:

$$\sum_F N_F \;\lesssim\; \mathbf{M}(T^{\mathrm{raw}})\, h^{-2(n-p)},$$

which is valid in the *constant-mass-per-sheet* model where each sheet piece contributing to T^{raw} has ψ–mass $\asymp h^{2(n-p)}$ inside a cube.

In the *sliver* regime (Remark 8.69 / Proposition 8.72), one deliberately allows many pieces of very small mass per cube. Then $\sum_F N_F$ can be arbitrarily large at fixed total mass, so the above reduction is no longer available. To extend Proposition 8.43 to this regime, one needs to replace the crude counting bound by a *weighted* estimate that tracks the actual size of each face slice.

Model heuristic. In a smooth convex cell (e.g. a ball), a k-dimensional affine slice of volume v has boundary size on the cell boundary of order $v^{(k-1)/k}$ (cf. the ball scaling in the flat model). Thus splitting a fixed total mass M into N equal slivers increases total boundary size only like $N^{1/k}$ (sublinear). This suggests that a sharpened version of the transport-flat-norm bound should control $\mathcal{F}(\partial T^{\mathrm{raw}})$ in terms of a boundary-size functional (or first moment) rather than the raw count $\sum_F N_F$.

Concrete target. Replace $\sum_F N_F$ in the global estimate by a quantity of the form

$$\sum_F \sum_{a\in\mathcal{S}(F)} \mathbf{M}\big(\partial([Y^a]\llcorner Q)\llcorner F\big),$$

or an equivalent transverse-parameter integral, and show it is controlled (sublinearly in the number of slivers) by the total mass in Q for the specific flat/graph models used here. In the flat ball model this bookkeeping is made explicit in Proposition 8.49, and its global implication for gluing is summarized in Proposition 8.55. For rounded cubes / smooth uniformly convex cells with curvature pinched at scale h (Remark 8.23), Lemma 8.46 provides the same boundary-shrinkage input $\mathbf{M}(\partial([P+t]\llcorner Q)) \lesssim \mathbf{M}([P+t]\llcorner Q)^{(k-1)/k}$ up to uniform constants. Thus the remaining analytic input in regime (II) is primarily the *template displacement* estimate for holomorphic slivers (Lemma 8.61) together with ensuring the chosen rounded cells satisfy the curvature hypotheses.

Lemma 8.45 (Ball-section boundary scaling in the flat model)**.** *Let $k \geq 2$ and let $Q := B_h(0) \subset \mathbb{R}^k \times \mathbb{R}^{d-k}$ be the Euclidean ball of radius h. Let $P := \mathbb{R}^k \times \{0\}$ and let $t \in P^\perp$ with $r := \|t\| \in (0,h)$. Then the section $(P+t) \cap Q$ is a k–ball of radius $\sqrt{h^2-r^2}$, hence*

$$\mathbf{M}\big([P+t]\llcorner Q\big) \;=\; c_k\,(h^2-r^2)^{k/2}, \qquad \mathbf{M}\big(\partial([P+t]\llcorner Q)\big) \;=\; c_{k-1}\,(h^2-r^2)^{(k-1)/2},$$

for dimensional constants $c_k, c_{k-1} > 0$. In particular,

$$\mathbf{M}\big(\partial([P+t]\llcorner Q)\big) \;\asymp\; \mathbf{M}\big([P+t]\llcorner Q\big)^{\frac{k-1}{k}}$$

uniformly in $r \in (0,h)$.

Sketch. The intersection of a Euclidean ball with an affine k–plane at distance r from the center is a Euclidean k–ball of radius $\sqrt{h^2-r^2}$. The formulas follow from the standard volume/area of balls and spheres. □

Lemma 8.46 (Boundary shrinkage for plane slices in smooth uniformly convex cells)**.** *Let $Q \subset \mathbb{R}^d$ be a bounded C^2* uniformly convex *domain of diameter $\asymp h$. Assume the principal curvatures of ∂Q satisfy*

$$\frac{c}{h} \;\leq\; \kappa_i \;\leq\; \frac{C}{h} \qquad \textit{everywhere on } \partial Q,$$

for fixed constants $0 < c \leq C$. Fix $1 \leq k < d$ and a k-plane P. For each translate $P+t$ with nonempty intersection, set

$$v(t) := \mathcal{H}^k\big((P+t)\cap Q\big), \qquad a(t) := \mathcal{H}^{k-1}\big((P+t)\cap \partial Q\big).$$

Then there exists $C_* = C_*(d,k,c,C)$ such that

$$a(t) \;\leq\; C_*\,\big(v(t)\big)^{\frac{k-1}{k}} \qquad \textit{for all such } t.$$

Proof. The estimate is scale-invariant, so rescale so that $h \asymp 1$. Write $K_t := (P+t)\cap Q \subset P+t \cong \mathbb{R}^k$, so $v(t) = \mathcal{H}^k(K_t)$ and $a(t) = \mathcal{H}^{k-1}(\partial K_t)$.

If $v(t) \geq v_0 > 0$, then K_t is a convex body contained in a fixed k–ball of radius $O(1)$, hence $a(t) \leq A_0(d,k)$ and the desired bound follows after increasing C_*.

Assume $v(t) \leq v_0$ with v_0 small. The curvature pinching implies an interior/exterior rolling-ball condition with radii $r_{\text{in}}, r_{\text{out}} \asymp 1$ (depending only on c, C) at every boundary point of Q. Let $\pi : \mathbb{R}^d \to P^\perp$ be orthogonal projection and set $D := \pi(Q) \subset P^\perp$. Choose a nearest point $t_0 \in \partial D$ and an outward normal $u \in P^\perp$ to a supporting hyperplane of D at t_0, and write $t = t_0 - su$. Let $x_0 \in \partial Q$ be the unique supporting point with outward normal u (uniqueness by uniform convexity), so $\pi(x_0) = t_0$.

Intersect the tangent balls at x_0 with the affine plane $P+t$. Since $u \perp P$, these intersections are k-balls of radii $\rho_{\text{in}}(s) = \sqrt{2r_{\text{in}}s - s^2}$ and $\rho_{\text{out}}(s) = \sqrt{2r_{\text{out}}s - s^2}$, hence

$$\omega_k\,\rho_{\text{in}}(s)^k \;\leq\; v(t) \;\leq\; \omega_k\,\rho_{\text{out}}(s)^k, \qquad a(t) \;\leq\; \omega_{k-1}\,\rho_{\text{out}}(s)^{k-1}.$$

For s small one has $\rho_{\text{in}}(s) \gtrsim \sqrt{s}$ and $\rho_{\text{out}}(s) \lesssim \sqrt{s}$, so $v(t) \gtrsim s^{k/2}$ and $a(t) \lesssim s^{(k-1)/2}$, hence $s \lesssim v(t)^{2/k}$ and $a(t) \lesssim v(t)^{(k-1)/k}$. □

Remark 8.47 (References for the geometric inputs). The curvature pinching $\kappa_i \asymp h^{-1}$ implies interior/exterior tangent balls of radius $\asymp h$ (the classical *rolling ball* principle). Uniqueness of supporting points for a given outward normal follows from strict convexity and C^2 regularity of ∂Q. See, e.g., Schneider, *Convex Bodies: The Brunn–Minkowski Theory*.

Remark 8.48 (Caution: uniformly convex cells are a *model*, not a literal partition). Lemma 8.46 and Proposition 8.49 are stated for a single smooth uniformly convex domain Q, because that hypothesis forces the clean implication "small slice mass $\Rightarrow$ small boundary slice". In an actual cell decomposition (a partition of X into sets with *shared* codimension-1 interfaces), one should not read this as asserting that *every* cell can be uniformly convex with respect to its outward normal: if a C^2 hypersurface is shared by two adjacent cells, reversing the normal reverses the principal curvatures, so strict convexity on both sides would force those curvatures to vanish (flat interface).

Accordingly, the sliver route should be interpreted as requiring an explicit *boundary-budget mechanism* (Blocker B1c) ensuring that the particular sliver pieces used in the construction have small boundary slices on the chosen interfaces, even though the global partition is not uniformly convex in the classical sense.

Proposition 8.49 (Sliver boundary budget on a smooth uniformly convex cell (flat model))**.** *Let $Q \subset \mathbb{R}^{2n}$ be a bounded C^2 uniformly convex domain of diameter $\asymp h$ with principal curvatures pinched at scale h (so Lemma 8.46 applies) and let $k := 2n - 2p$. Suppose we write a target mass $M > 0$ as a sum of N* equal-mass *affine calibrated pieces*

$$T := \sum_{a=1}^{N} [P + t_a] \llcorner Q, \qquad \mathbf{M}([P + t_a] \llcorner Q) = M/N,$$

where each $P + t_a$ is an affine calibrated k–plane. Then

$$\mathbf{M}(\partial T) \ \le \ C(n, p, c, C)\, M^{\frac{k-1}{k}}\, N^{\frac{1}{k}}.$$

Moreover, if Y^a are C^1 graphs over $P + t_a$ on Q with slope $\le \varepsilon$, then

$$\mathbf{M}\big(\partial([Y^a] \llcorner Q)\big) = \big(1 + O(\varepsilon^2)\big)\, \mathbf{M}\big(\partial([P + t_a] \llcorner Q)\big),$$

so the same bound holds up to a $(1 + O(\varepsilon^2))$ factor.

Proof. For each a, the boundary current $\partial([P + t_a] \llcorner Q)$ is supported on ∂Q and

$$\mathbf{M}\big(\partial([P + t_a] \llcorner Q)\big) = \mathcal{H}^{k-1}\big((P + t_a) \cap \partial Q\big).$$

Since $\mathbf{M}([P + t_a] \llcorner Q) = \mathcal{H}^k\big((P + t_a) \cap Q\big) = M/N$, Lemma 8.46 (with $d = 2n$) gives

$$\mathbf{M}\big(\partial([P + t_a] \llcorner Q)\big) \ \le \ C_* \,(M/N)^{\frac{k-1}{k}}.$$

Summing over a and using $\mathbf{M}(\partial T) \le \sum_a \mathbf{M}(\partial([P + t_a] \llcorner Q))$ yields

$$\mathbf{M}(\partial T) \ \le \ N \cdot C_* \,(M/N)^{\frac{k-1}{k}} = C_*\, M^{\frac{k-1}{k}}\, N^{\frac{1}{k}}.$$

(In the ball case $Q = B_h$, one may recover the same scaling with an explicit sharp constant from Lemma 8.45.)

The C^1 graph perturbation statement follows by the area formula and uniform bilipschitz control of the graph map on Q (as in Lemma 8.80). $\square$

Hypothesis 8.50 (Boundary-budget input for Bergman-scale slivers)**.** *Fix a mesh-h cell decomposition and set $k := 2n - 2p$. For each cell Q and each m in the sliver regime, suppose the local holomorphic construction produces a decomposition*

$$S_Q \;=\; \sum_{a=1}^{N_Q} [Y^a] \llcorner Q,$$

where each Y^a is ψ–calibrated and, on Q, is a C^1 graph with slope $\le \varepsilon$ over an affine calibrated k–plane slice. Assume moreover that the pieces are equal-mass *in Q, $\mathbf{M}([Y^a]\llcorner Q) = M_Q/N_Q$ with $M_Q \sim m \int_Q \beta \wedge \psi$, and that their total boundary satisfies the bound*

$$\mathbf{M}(\partial S_Q) \;\le\; C\, M_Q^{\frac{k-1}{k}}\; N_Q^{\frac{1}{k}},$$

with C depending only on (n, p, X) (and absorbing the harmless $(1 + O(\varepsilon^2))$ graph factor).

In a smooth uniformly convex model *cell, this hypothesis follows from Lemma 8.46 and Proposition 8.49; for a general C^2 partition it is the remaining geometric input in Blocker* B1c.

Remark 8.51 (Why Hypothesis 8.50 is genuinely geometric (a cylinder counterexample)). Hypothesis 8.50 is *not* a formal consequence of smoothness and bounded curvature of the cell boundary. Even for smooth *convex* domains of diameter $\asymp h$, boundary shrinkage can fail if the boundary has flat directions.

For example, in $\mathbb{R}^k$ (with $k \ge 2$) consider a rounded cylinder $Q := B^{k-1}(h) \times [-h, h]$ (smoothly rounded near the rims so that $|\kappa| \lesssim 1/h$). Let $P \subset \mathbb{R}^k$ be a k–plane containing the cylinder axis and translate it so that $(P + t) \cap Q$ is a thin product region of the form $B^{k-1}(r) \times [-h, h]$ with $r \downarrow 0$. Then

$$v := \mathcal{H}^k\big((P + t) \cap Q\big) \;\asymp\; r^{k-1}\, h, \qquad a := \mathcal{H}^{k-1}\big((P + t) \cap \partial Q\big) \;\gtrsim\; r^{k-2}\, h,$$

so $a/v^{(k-1)/k} \to \infty$ as $r \downarrow 0$. Geometrically, one can have arbitrarily small slice volume while retaining a long boundary component inherited from the flat cylinder direction.

Thus any proof of Hypothesis 8.50 must exploit a *shape mechanism* (e.g. uniform convexity in the relevant slice directions, or an explicit construction forcing sliver pieces to exit through rounded regions), not merely C^2 regularity.

Remark 8.52 (A plausible cube-based substitute: vertex slivers). The failure mode in Remark 8.51 comes from slices that are thin in *one* direction but long in flat directions. In polyhedral cells one can instead force small pieces to "live near corners", where the cell provides curvature in many directions *combinatorially* (many supporting halfspaces meet). The next lemma records a clean flat-model estimate in this direction.

Lemma 8.53 (Boundary shrinkage for transverse plane slices near a cube vertex (flat model)). *Let $Q := [0, h]^d \subset \mathbb{R}^d$ and fix integers $1 \le k < d$. Let $P \subset \mathbb{R}^d$ be an oriented affine k–plane. Assume P is* uniformly transverse *to the coordinate hyperplanes through the vertex $0 \in Q$ in the following sense: there exist k coordinate indices $i_1, \ldots, i_k$ such that the restrictions $x \mapsto x_{i_r}$ form a basis of P^* with condition number $\le \Lambda$. Then there exists $c = c(d, k, \Lambda) > 0$ and $C = C(d, k, \Lambda) > 0$ such that for every translate $P + t$ with $(P + t) \cap Q \subset B_{ch}(0)$ (so the slice lies in a small neighborhood of the vertex), writing*

$$v(t) := \mathcal{H}^k\big((P + t) \cap Q\big), \qquad a(t) := \mathcal{H}^{k-1}\big((P + t) \cap \partial Q\big),$$

one has the boundary–volume estimate

$$a(t) \;\le\; C\, \big(v(t)\big)^{\frac{k-1}{k}}.$$

Sketch. After translating/rotating inside P, write the slice as a polytope in $P \cong \mathbb{R}^k$ cut out by the d halfspaces defining Q. On the event $(P+t)\cap Q \subset B_{ch}(0)$ only the halfspaces through the vertex 0 are active at leading order, so the slice is comparable (via a linear isomorphism with distortion controlled by Λ) to a truncated orthant in $\mathbb{R}^k$. In such a simplicial-cone model, the slice polytopes are homothetic as the translate moves toward/away from the vertex, so $v \asymp s^k$ and $a \asymp s^{k-1}$ for a single scale parameter s. Eliminating s gives $a \lesssim v^{(k-1)/k}$. □

If one can implement the Bergman-scale sliver pieces so that, in each cell, their plane-like models concentrate near such uniformly transverse vertices (rather than near flat face regions), Lemma 8.53 suggests that the boundary budget Hypothesis 8.50 could be proved even for cubical partitions.

Remark 8.54 (Heuristic route to proving Hypothesis 8.50). Hypothesis 8.50 is a *boundary-budget* statement: it asserts that one can choose the sliver pieces so that "small mass" forces "small boundary slices" in a quantitative way *on the actual interfaces used for gluing*. The uniformly convex model estimate (Proposition 8.49) shows the mechanism in the cleanest setting. For a general C^2 cell decomposition, the corresponding statement is not automatic and should be viewed as the remaining geometric content of the microstructure theorem.

One plausible strategy is to concentrate the Bergman-scale mass M_Q of each cell into regions of the interface network that are locally "ball-like" at scale h (for instance, near rounded neighborhoods of the codimension-≥ 2 skeleton where several interfaces meet), so that the relevant sliver intersections are controlled by a local comparison to the ball model (Lemma 8.45). Since $M_Q \lesssim m\,h^{2n}$ at Bergman scale, such localization would not change the pairing against smooth test forms by more than $O(h)M_Q$, yet could enforce the desired boundary shrinkage for the chosen pieces. Making this precise amounts to constructing an interface network with controlled curvature and a supply of local "rounded pockets" shared across adjacent cells, together with a template-based selection of translations supported in those pockets.

Proposition 8.55 (Weighted sliver gluing: a reduction statement)**.** *Assume the cell decomposition is at scale h and admits uniform tubular-face charts with $(1+O(h^2))$ metric distortion (Lemma 8.24). Assume moreover that the sliver boundary-budget hypothesis holds on each cell (Hypothesis 8.50), and assume each cell Q is filled by a sliver microstructure with total mass $M_Q \sim m\int_Q \beta\wedge\psi$ written as N_Q equal-mass calibrated pieces. Assume that across each interior interface $F = Q\cap Q'$ the face-slice parameterizations are* template-coherent *in the sense that, after pairing pieces crossing F, the induced transverse measures arise from the same template by face maps whose variation is $O(h)$: equivalently, the hypotheses of Lemma 8.61 hold on each F (or more generally Lemma 8.62, with the unpaired term small enough to be absorbed as in Remark 8.64). Then, summing over all interfaces and using the boundary-budget estimate (Hypothesis 8.50) on each Q, one obtains the global bound*

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\leq\; C\,h^2 \sum_Q M_Q^{\frac{k-1}{k}}\, N_Q^{\frac{1}{k}} \;+\; O(\varepsilon\, m), \qquad k := 2n-2p.$$

In particular, at the Bergman scale $h \asymp m^{-1/2}$ with uniformly bounded $M_Q \asymp m\,h^{2n}$, if $N_Q \leq m^{\beta}$ for all Q then

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\lesssim\; m^{\frac{n-1+\beta}{k}} \;+\; O(\varepsilon\, m),$$

which is $o(m)$ provided $\beta < k-(n-1)$ and $\varepsilon = o(1)$.

Proof. Write the raw current as a sum of cell pieces $T^{\mathrm{raw}} = \sum_Q S_Q$ as in Substep 4.2, so that $\partial T^{\mathrm{raw}}$ is supported on the union of interior interfaces. Decompose

$$\partial T^{\mathrm{raw}} = \sum_F B_F,$$

where each $F = Q \cap Q'$ is an interior interface and B_F is the corresponding mismatch current on F. By the triangle inequality for the flat norm,

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\le\; \sum_F \mathcal{F}(B_F).$$

Per-face estimate. By the template-coherence hypothesis on each F (Lemma 8.61, or the edited version Lemma 8.62), one has

$$\mathcal{F}(B_F) \;\le\; C\,h^2\Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F)\Big) \;+\; O(\varepsilon\, M_F),$$

where $M_F \lesssim M_Q + M_{Q'}$ and ε is the small-angle/graph parameter from Proposition 8.28.

Summation over interfaces. Summing the boundary-mass term over all interfaces and reindexing by cells gives

$$\sum_F \Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F)\Big) \;\le\; C_{\mathrm{deg}} \sum_Q \mathbf{M}(\partial S_Q),$$

where C_{deg} depends only on the local combinatorics of the decomposition (a uniform bound on the number of interfaces meeting a cell). Applying Hypothesis 8.50 on each Q yields

$$\mathbf{M}(\partial S_Q) \;\le\; C\, M_Q^{\frac{k-1}{k}}\, N_Q^{\frac{1}{k}}, \qquad k := 2n - 2p,$$

up to a $(1 + O(\varepsilon^2))$ factor absorbed into the constant. Therefore

$$\sum_F C\,h^2\Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F)\Big) \;\le\; C\,h^2 \sum_Q M_Q^{\frac{k-1}{k}}\, N_Q^{\frac{1}{k}}.$$

For the error term, since each interface contributes $O(\varepsilon(M_Q + M_{Q'}))$ and each cell meets only $O(1)$ interfaces, one has $\sum_F M_F \lesssim \sum_Q M_Q \sim m \int_X \beta \wedge \psi \asymp m$, hence the total error is $O(\varepsilon\, m)$. Combining these estimates yields the first displayed bound.

Bergman-scale estimate. At the Bergman scale $h \asymp m^{-1/2}$, the number of cells is $\asymp h^{-2n} \asymp m^n$. Moreover $M_Q \sim m \int_Q \beta \wedge \psi \lesssim m\, h^{2n}$ uniformly (since β is bounded), so $M_Q \lesssim m^{1-n}$. If in addition $N_Q \le m^\beta$ for all Q, then

$$h^2 \sum_Q M_Q^{\frac{k-1}{k}}\, N_Q^{\frac{1}{k}} \;\lesssim\; m^{-1} \cdot m^n \cdot (m^{1-n})^{\frac{k-1}{k}} \cdot m^{\beta/k} \;=\; m^{\frac{n-1+\beta}{k}}.$$

This gives the stated bound and the $o(m)$ criterion. □

Remark 8.56 (What this reduction buys). Proposition 8.55 packages the "unconditional feasible" strategy into two explicit, local analytic inputs:

1. a *sliver filling theorem* (Proposition 8.77 / Corollary 8.74) producing many tiny calibrated holomorphic pieces per Bergman-scale cell, and
2. a *template displacement estimate* at scale h^2 comparing the induced face slices across neighboring cells.

Once these hold, the rest of the SYR chain (flat-norm filling, cohomology rounding, and calibrated compactness) proceeds exactly as written.

Remark 8.57 (Model scaling at the Bergman cell size). This remark records a simplified scaling calculation explaining why a "sliver" mechanism could, in principle, coexist with the intrinsic holomorphic control scale $h \sim m^{-1/2}$.

Lemma 8.58 (Flat-norm stability under translation). *Let S be an integral ℓ-cycle in $\mathbb{R}^d$ (so $\partial S = 0$) with finite mass. For any translation vector $v \in \mathbb{R}^d$, write $\tau_v(x) := x + v$ and $(\tau_v)_\# S$ for the pushforward. Then*

$$\mathcal{F}\big((\tau_v)_\# S - S\big) \;\le\; \|v\|\,\mathbf{M}(S).$$

Sketch. Consider the homotopy $H : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $H(t,x) = x + tv$, and the product current $[0,1] \times S$. Then $Q := H_\#([0,1] \times S)$ satisfies $\partial Q = (\tau_v)_\# S - S$ and $\mathbf{M}(Q) \le \|v\|\,\mathbf{M}(S)$. Taking $R = 0$ in the flat norm definition gives the bound. □

Lemma 8.59 (Flat-norm stability under small C^0 displacement). *Let S be an integral ℓ-cycle in $\mathbb{R}^d$ with finite mass. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz map with* $\mathrm{Lip}(f) \le L$ *and*

$$\sup_{x \in \mathbb{R}^d} \|f(x) - x\| \;\le\; \delta.$$

Then

$$\mathcal{F}\big(f_\# S - S\big) \;\le\; C(\ell)\,\delta\,L^\ell\,\mathbf{M}(S).$$

Sketch. Consider the straight-line homotopy $H : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $H(t,x) = (1-t)x + tf(x)$. Then $\partial H_\#([0,1] \times S) = f_\# S - S$. The mass of $H_\#([0,1] \times S)$ is controlled by $\sup \|f - \mathrm{Id}\|$ times a uniform Jacobian bound depending on $\mathrm{Lip}(f)$ and ℓ, giving $\mathbf{M}(H_\#([0,1] \times S)) \le C(\ell)\,\delta\,L^\ell\,\mathbf{M}(S)$. Taking $R = 0$ in the flat norm definition yields the estimate. □

Lemma 8.60 (Flat norm of a cycle supported in diameter $\lesssim h$). *Let S be an integral ℓ-cycle in $\mathbb{R}^d$ with finite mass. Assume* $\mathrm{diam}(\mathrm{spt}\, S) \le D$. *Then*

$$\mathcal{F}(S) \;\le\; C(\ell)\,D\,\mathbf{M}(S).$$

In particular, if $\mathrm{diam}(\mathrm{spt}\, S) \lesssim h$ *then* $\mathcal{F}(S) \lesssim h\,\mathbf{M}(S)$.

Sketch. Fix a point x_0 in the convex hull of $\mathrm{spt}\, S$ and consider the straight-line homotopy $H : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $H(t,x) = (1-t)x + tx_0$. Since S is a cycle, $Q := H_\#([0,1] \times S)$ satisfies $\partial Q = S - (x_0)_\# S = S$ (the pushforward under the constant map is 0 in positive dimension). The Jacobian bound for H yields $\mathbf{M}(Q) \le C(\ell)\,(\sup \|x - x_0\|)\,\mathbf{M}(S) \le C(\ell)\,D\,\mathbf{M}(S)$. Taking $R = 0$ in the flat norm definition gives the claim. □

Lemma 8.61 (Template displacement ⇒ per-face flat-norm mismatch). *Work in the setting of Proposition 8.28*(a)–(b) *on an interior interface $F = Q \cap Q'$ at mesh h. Assume that the boundary slices on F are parameterized by the* same *integer-weighted discrete measure $\nu = \sum_{a=1}^{N_F} w_a\,\delta_{y_a}$ supported in a ball of radius $C_0 h \subset \mathbb{R}^{2p}$ via linear face maps $\mu_{Q \to F} = (\Phi_{Q,F})_\# \nu$ and $\mu_{Q' \to F} = (\Phi_{Q',F})_\# \nu$. Assume $\|\Phi_{Q,F}\|_{\mathrm{op}} + \|\Phi_{Q',F}\|_{\mathrm{op}} \le C_{\Phi,0}$ and $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} \le C_\Phi h$. Then, after pairing atoms by the identity pairing $y_a \leftrightarrow y_a$, the mismatch current B_F satisfies*

$$\mathcal{F}(B_F) \;\le\; C\,h^2 \Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F)\Big) \;+\; O(\varepsilon\,M_F),$$

where $M_F \lesssim M_Q + M_{Q'}$ and ε is the small-angle/graph parameter from Proposition 8.28(a).

Sketch. In the flat/parallel model ($\varepsilon = 0$) the slice current on F associated to a parameter $z \in \mathbb{R}^{2p}$ is a translate of a fixed model slice: $\Sigma_z = (\tau_z)_\# \Sigma_0$ in the face chart. Thus, for each atom y_a the paired slices differ by a translation vector $v_a := (\Phi_{Q,F} - \Phi_{Q',F}) y_a$, hence $\|v_a\| \le C_\Phi h \cdot \|y_a\| \le Ch^2$. Lemma 8.58 gives $\mathcal{F}(\Sigma_{\Phi_{Q,F} y_a} - \Sigma_{\Phi_{Q',F} y_a}) \le \|v_a\| \, \mathbf{M}(\Sigma_{\Phi_{Q,F} y_a}) \le Ch^2 \, \mathbf{M}(\Sigma_{\Phi_{Q,F} y_a})$. Summing over atoms (with weights w_a) yields $\mathcal{F}(B_F) \le Ch^2 \, \mathbf{M}(\partial S_Q \llcorner F)$ in the flat model, and symmetrizing gives the stated bound with both sides.

For $\varepsilon > 0$, compare each sheet to the corresponding flat slice in the tubular chart; the C^1 graph distortion contributes an additional $O(\varepsilon \, M_F)$ term exactly as in Proposition 8.28. □

Lemma 8.62 (Template displacement with insertions/deletions). *Work in the setting of Lemma 8.61 on an interior interface $F = Q \cap Q'$ at mesh h. Assume the two sides admit template representations*

$$\mu_{Q \to F} = (\Phi_{Q,F})_\# \nu, \qquad \mu_{Q' \to F} = (\Phi_{Q',F})_\# \nu',$$

where ν and ν' are integer-weighted discrete measures supported in $B_{C_0 h}(0) \subset \mathbb{R}^{2p}$ and the face maps satisfy $\|\Phi_{Q,F}\|_{\mathrm{op}} + \|\Phi_{Q',F}\|_{\mathrm{op}} \le C_{\Phi,0}$ and $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} \le C_\Phi h$. Write $\nu = \nu^\wedge + \nu^+$ and $\nu' = \nu^\wedge + \nu^-$, where $\nu^\wedge$ is any common submeasure (matched part) and $\nu^\pm$ are the unmatched remainders (insertions/deletions). Let $B_F^\wedge$ be the mismatch current coming from the matched part $\nu^\wedge$ and let B_F^{un} be the mismatch current coming from the unmatched part (so $B_F = B_F^\wedge + B_F^{\mathrm{un}}$). Then

$$\mathcal{F}(B_F^\wedge) \;\le\; C\,h^2 \Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F) \Big) \;+\; O(\varepsilon \, M_F),$$

and, moreover,

$$\mathcal{F}(B_F^{\mathrm{un}}) \;\le\; C\,h\,\mathbf{M}(B_F^{\mathrm{un}}) \;\le\; C\,h \Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F) \Big),$$

where C depends only on (n, p, X) and the uniform tubular-face charts.

Sketch. The bound for $B_F^\wedge$ is exactly Lemma 8.61, applied to the common template $\nu^\wedge$. For the unmatched part, B_F^{un} is an integral $(k-1)$–cycle supported on the face patch F; since $\mathrm{diam}(F) \lesssim h$, Lemma 8.60 gives $\mathcal{F}(B_F^{\mathrm{un}}) \lesssim h \, \mathbf{M}(B_F^{\mathrm{un}})$. Finally $\mathbf{M}(B_F^{\mathrm{un}})$ is bounded by the total face boundary mass coming from the unpaired sheets, hence by $\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F)$. □

Lemma 8.63 (If edits are an $O(h)$ fraction, they are h^2 in flat norm). *In the setting of Lemma 8.62, assume moreover that the unmatched part satisfies*

$$\mathbf{M}(B_F^{\mathrm{un}}) \;\le\; \theta_F \Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F) \Big)$$

for some $\theta_F \in [0,1]$. Then

$$\mathcal{F}(B_F) \;\le\; C\,h^2 \Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F) \Big) \;+\; C\,h\,\theta_F \Big(\mathbf{M}(\partial S_Q \llcorner F) + \mathbf{M}(\partial S_{Q'} \llcorner F) \Big) \;+\; O(\varepsilon \, M_F).$$

In particular, if $\theta_F \lesssim h$ then the unmatched contribution is of the same $h^2 \times$ (boundary mass) order as the matched displacement term.

Proof sketch. Combine Lemma 8.62 for the matched part with the hypothesis on $\mathbf{M}(B_F^{\mathrm{un}})$ and the bound $\mathcal{F}(B_F^{\mathrm{un}}) \lesssim h \, \mathbf{M}(B_F^{\mathrm{un}})$ from Lemma 8.60. □

Remark 8.64 (Allowing small template edits across a face). Lemma 8.61 treats the cleanest situation where the two sides use the *same* discrete template ν. In practice one may only be able to enforce template coherence up to a small number of insertions/deletions (as in Lemma 8.41 for the dense-sheet regime). In the sliver regime, the same idea is available provided the *unpaired* face slices are supported in a patch of diameter $\lesssim h$: each unpaired slice is a $(k-1)$–cycle in F and thus has flat norm $\lesssim h$ times its slice mass by Lemma 8.60. If the inserted/deleted pieces are chosen to be *very small slivers* (hence with very small boundary slices), this unpaired contribution is of the same $h^2 \times$ (boundary mass) order as the matched displacement term and can be absorbed into the bookkeeping of Proposition 8.55.

One convenient sufficient condition is that the edits constitute an $O(h)$ fraction of the total boundary mass on the face; Lemma 8.63 then shows they are automatically $O(h^2)$ in flat norm and can be absorbed.

Assume cells have diameter $h \asymp m^{-1/2}$ (as suggested by Lemma 8.15) so that uniform C^1 graph control holds on each cell. Then the number of cells is $\asymp h^{-2n} \asymp m^n$, and the target mass per cell is

$$M_Q \ \sim \ m \int_Q \beta \wedge \psi \ \asymp \ m\, h^{2n} \ \asymp \ m^{1-n}.$$

In a flat smooth convex model (e.g. a ball cell), if M_Q is split into N_Q *equal* sliver pieces of mass M_Q/N_Q, then the $(2n-2p-1)$–dimensional boundary size of a single piece scales like $(M_Q/N_Q)^{\frac{k-1}{k}}$ (with $k := 2n-2p$), hence the total boundary size on the cell boundary scales like

$$\mathrm{Bdry}(Q) \ \asymp \ N_Q \Big(\frac{M_Q}{N_Q}\Big)^{\frac{k-1}{k}} \ = \ M_Q^{\frac{k-1}{k}}\, N_Q^{\frac{1}{k}}.$$

If, across a shared interface, the corresponding face slices are displaced by $\asymp h^2$ (as in the template/face-map variation heuristics), then a crude cylinder filling gives a per-face flat-norm mismatch of order

$$\mathcal{F}(B_F) \ \lesssim \ h^2\, \mathrm{Bdry}(Q) \ \asymp \ h^2\, M_Q^{\frac{k-1}{k}}\, N_Q^{\frac{1}{k}}.$$

Summing over $\asymp h^{-2n}$ faces gives the global heuristic bound

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \ \lesssim \ h^{-2n} \cdot h^2 \cdot M_Q^{\frac{k-1}{k}}\, N_Q^{\frac{1}{k}} \ \asymp \ m^{\frac{n-1}{k}}\, N_Q^{\frac{1}{k}}.$$

In particular, since $(n-1)/k < 1$ for $k = 2n-2p \geq 2$, this is automatically sublinear in m provided N_Q grows at most polynomially in m with exponent $< k-(n-1)$. Making any version of this calculation rigorous inside the cubical/face framework requires precisely the weighted bookkeeping estimate flagged in Remark 8.44.

Remark 8.65 (Explicit parameter choices). For a target flat-norm error fraction $\theta \in (0,1)$, Proposition 8.43 gives $\mathcal{F}(\partial T^{\mathrm{raw}}) \leq C_3(h+\varepsilon)m$ (when $\mathbf{M}(T^{\mathrm{raw}}) \asymp m$). Thus it suffices to choose, for example,

$$h := \frac{\theta}{10C_3}, \qquad \varepsilon := \frac{\theta}{10C_3},$$

to obtain $\mathcal{F}(\partial T^{\mathrm{raw}}) \leq \theta\, m$ and hence a gluing correction current R_{glue} with $\mathbf{M}(R_{\mathrm{glue}}) \lesssim \theta m$.

Equivalently, if one wants an explicit m–dependent schedule, one may take for any $\alpha > 0$,

$$h := m^{-\alpha}, \qquad \varepsilon := m^{-\alpha},$$

which yields $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim m^{1-\alpha}$, i.e. $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ as $m \to \infty$.

What remains genuinely unproved. The proposition is conditional on verifying (T1)–(T2) for the *holomorphic* sheet families produced in Substep 3.5: uniform tubular-face charts and an $O(h)$ control of the face-slice maps (T1), and a mechanism enforcing slow variation of multiplicities (T2). This proposition should be viewed as the clean quantitative closure of regime (I) (dense sheets / constant-mass pieces). In middle codimension and at Bergman scale (regime (II)), one replaces the counting step by a boundary-budget estimate and a template-displacement control, as summarized by Proposition 8.55 (built on Proposition 8.49 and the local sliver targets of Proposition 8.77 / Corollary 8.74).

Remark 8.66 (Handling slowly varying multiplicities). In practice the number of sheets in a given family (Q, j) will vary with Q because the target weights depend on $\beta(x_Q)$. If adjacent cubes Q, Q' have sheet counts differing by $r = |N_{Q,j} - N_{Q',j}|$, one can view their face measures as arising from the same template after r insertions/deletions. Lemma 8.41 then gives an additional contribution $W_1 \lesssim r\, h$ (since the transverse domain has diameter $O(h)$). Thus, once one has a quantitative bound $r \le C\, h\, N_{Q,j}$ (slow variation), this term is of order $W_1 \lesssim h^2 N_{Q,j}$ and is absorbed into the $h^2 N$ scaling of Lemma 8.39. Making this "slow variation of integer counts" rigorous is a rounding/Diophantine bookkeeping problem, separate from the geometric transport estimates.

Remark 8.67 (Parameter tension: dense templates vs. small gluing error). The "automatic matching" heuristics (Lemmas 8.39 and 8.41) are most effective when each cube/face carries *many* sheets, so that transverse measures behave like a fine discretization of a smooth density and neighbor-to-neighbor variations are small. In the simplest constant-mass-per-sheet model, the expected sheet count per cube scales like $N_Q \sim m\, h^{2p}$ (cf. Lemma 8.82), while the global gluing bound from the template route scales like $\mathcal{F}(\partial T^{\text{raw}}) \lesssim m\, h$. For $p > 1$ this creates a tension at fixed m: taking $h \to 0$ drives $\mathcal{F}$ to 0 but also forces $N_Q \to 0$. Resolving this requires either:

- a genuinely new cancellation mechanism beyond the "many-sheets-per-cube" regime, or
- allowing a microstructure with *many* sheet pieces per cube whose individual masses are correspondingly smaller ("sliver" pieces), so that N_Q can be large while the total mass remains $O(m)$.

This is another way to see why the realization/microstructure step is the true remaining heart of the argument in general codimension.

Bergman-scale amplification of the same tension. The holomorphic upgrade (Substep 3.5) is driven by Bergman/peak-section control (Lemma 8.15), which is naturally available on balls of radius $\asymp m^{-1/2}$. If one chooses the cell size h at this intrinsic scale to guarantee uniform C^1 graph control on each cell, then

$$h \;\lesssim\; m^{-1/2} \qquad \Longrightarrow \qquad N_Q \sim m\, h^{2p} \;\lesssim\; m^{1-p}.$$

Thus for $p > 1$ the *naive constant-mass sheet model* yields *less than one sheet per cube on average* as $m \to \infty$. This makes clear that, in middle codimension, one must either:

- prove a substantially stronger analytic input than Lemma 8.15 (uniform C^1 control on balls much larger than $m^{-1/2}$), or
- use a true "sliver" mechanism (Proposition 8.77) that splits the target cube mass into many much smaller local pieces, so that the effective degrees of freedom per cube remain large even when $h \sim m^{-1/2}$.

Lemma 8.68 (Mass tunability of plane slices in the flat model)**.** *In the flat chart model, fix a calibrated affine* $(2n-2p)$*-plane* $P \subset \mathbb{R}^{2n}$ *and a* smooth convex *cell* Q *of diameter* h *(e.g. a Euclidean ball, or a cube with rounded corners). The function*

$$t \;\longmapsto\; \mathbf{M}\big([P+t]\llcorner Q\big)$$

is continuous in the translation parameter $t \in P^{\perp} \cong \mathbb{R}^{2p}$ *and its image contains an interval* $[0, A_{\max}]$ *with* $A_{\max} \asymp h^{2(n-p)}$*. In particular, for any* $a \in (0, A_{\max})$ *there exist translations* t *such that* $\mathbf{M}([P+t]\llcorner Q) = a$*.*

Sketch. In the Euclidean model, $\mathbf{M}([P+t]\llcorner Q) = \mathcal{H}^{2(n-p)}\big((P+t)\cap Q\big)$. For smooth convex Q, the $(2n-2p)$-dimensional cross-sectional volume of Q by translates of a fixed plane depends continuously on the translation parameter, and it vanishes when the translate becomes disjoint from Q. Taking a translate that passes through the "thick" region of Q gives a cross-section of size $\asymp h^{2(n-p)}$, yielding $A_{\max} \asymp h^{2(n-p)}$. □

Remark 8.69 (Sliver pieces and fixed-m microstructure). Lemma 8.68 indicates a potential escape from the dense-vs-gluing tension at fixed m: one may take *many* parallel calibrated sheets in a cube but choose their translations so that each sheet contributes only a tiny mass ("sliver pieces"), with the total mass still matching $m \int_Q \beta \wedge \psi$. If such tunability persists under the holomorphic complete-intersection upgrade (Substep 3.5) with uniform control, then one can have large sheet counts per face (good for W_1 matching) while keeping the total mass $O(m)$. Making this quantitative in the projective setting is part of the remaining realization problem.

Lemma 8.70 (Quantizing a Lipschitz density on a sphere)**.** *Let* $d \geq 2$ *and let* $S^{d-1}(r) \subset \mathbb{R}^d$ *be the Euclidean sphere of radius* $r > 0$*. Let* ρ *be a nonnegative Lipschitz function on* $S^{d-1}(r)$ *with total mass*

$$M := \int_{S^{d-1}(r)} \rho \, d\sigma.$$

Then for every $N \in \mathbb{N}$ *there exist points* $t_1, \ldots, t_N \in S^{d-1}(r)$ *such that the equal-weight atomic measure*

$$\mu_N := \sum_{a=1}^{N} \frac{M}{N} \, \delta_{t_a}$$

satisfies the transport bound

$$W_1(\mu_N, \rho\, d\sigma) \;\leq\; C(d)\, r \left(M + \mathrm{Lip}(\rho)\, r^{d-1}\right) N^{-\frac{1}{d-1}}.$$

Moreover, the points may be chosen δ*–separated with*

$$\|t_a - t_b\| \;\geq\; c(d)\, r\, N^{-\frac{1}{d-1}} \qquad (a \neq b).$$

Sketch. Take a maximal δ–separated set $\{t_a\} \subset S^{d-1}(r)$ with $\delta \asymp r\, N^{-1/(d-1)}$; it has cardinality $\asymp N$ by volume packing, and by trimming/duplicating a bounded number of points one obtains exactly N points with the stated separation. Let $\{C_a\}$ be the associated Voronoi cells; then $\mathrm{diam}(C_a) \lesssim \delta$. Transport the mass of each C_a to t_a (cost $\lesssim \delta \int_{C_a} \rho$). Lipschitzness controls the discrepancy between the cell masses $\int_{C_a} \rho$ and the equal weights M/N, and rebalancing these masses costs at most the same order as δ. □

Lemma 8.71 (Nested equal-weight quantization of the uniform sphere)**.** *Let $d \geq 2$ and let $S^{d-1}(r) \subset \mathbb{R}^d$ be the Euclidean sphere of radius $r > 0$, with normalized surface measure σ_r. There exists an (infinite) sequence of points $(t_a)_{a\geq 1} \subset S^{d-1}(r)$ such that for every $N \geq 1$ the equal-weight empirical measure*

$$\mu_N \ := \ \frac{1}{N}\sum_{a=1}^{N} \delta_{t_a}$$

satisfies

$$W_1(\mu_N, \sigma_r) \ \leq \ C(d)\, r\, N^{-\frac{1}{d-1}}.$$

Sketch. Build a nested sequence of partitions of $S^{d-1}(r)$ into $\asymp 2^{(d-1)k}$ measurable cells at level k, each of diameter $\lesssim r\,2^{-k}$ and with σ_r-mass exactly $2^{-(d-1)k}$ (e.g. by inductively bisecting cells by smooth hypersurfaces). Choose one representative point in each cell and enumerate these points in increasing level order. For $N \asymp 2^{(d-1)k}$, the first N points are one per cell at level k, so transporting the mass of each cell to its representative costs $\lesssim r\,2^{-k} \cdot 1$, yielding $W_1 \lesssim r\,2^{-k} \asymp r\,N^{-1/(d-1)}$. For intermediate N, compare to the nearest dyadic level and absorb constants. □

Proposition 8.72 (Flat ball model slivers achieve W_1 transverse approximation)**.** *Work in the flat decomposition $\mathbb{R}^{2n} = \mathbb{R}^{2(n-p)} \oplus \mathbb{R}^{2p}$ and let $P := \mathbb{R}^{2(n-p)} \times \{0\}$. Let $Q := B_h(0) \subset \mathbb{R}^{2n}$ be the Euclidean ball of radius h. Fix a radius $r \in (0,h)$ and let σ_r denote surface measure on $S^{2p-1}(r) \subset P^\perp \cong \mathbb{R}^{2p}$. Let ρ be a nonnegative Lipschitz density on $S^{2p-1}(r)$ with total mass $M = \int_{S^{2p-1}(r)} \rho\, d\sigma_r$. Then for every $N \in \mathbb{N}$ there exist translations $t_1,\dots,t_N \in S^{2p-1}(r)$ such that the affine calibrated pieces*

$$T_N \ := \ \sum_{a=1}^{N}\big([P+t_a]\llcorner Q\big)$$

are pairwise disjoint and:

(i) ***(Equal sliver masses)*** $\mathbf{M}([P+t_a]\llcorner Q) = \mathbf{M}([P+t_1]\llcorner Q)$ *for all a (depends only on r);*

(ii) ***(Transverse W_1 approximation)*** *with $\mu_N := \sum_{a=1}^{N} \frac{M}{N}\delta_{t_a}$ one has*

$$W_1(\mu_N, \rho\, d\sigma_r) \ \leq \ C(p)\, r\left(M + \mathrm{Lip}(\rho)\, r^{2p-1}\right) N^{-\frac{1}{2p-1}}.$$

Sketch. By rotational symmetry of the Euclidean ball, the cross-sectional volume $\mathbf{M}([P+t]\llcorner Q) = \mathcal{H}^{2(n-p)}((P+t)\cap Q)$ depends only on $\|t\|$, hence is constant on the sphere $S^{2p-1}(r)$, giving (i). Apply Lemma 8.70 with $d = 2p$ to choose the points t_a and obtain (ii). Disjointness is automatic because distinct parallel affine planes are disjoint. □

Proposition 8.73 (Flat ball model: volumetric transverse densities via radial shells)**.** *Work in the same flat setting as Proposition 8.72, with transverse space $P^\perp \cong \mathbb{R}^{2p}$. Let $\Omega := B_R(0) \subset P^\perp$ with $R \leq c\,h$ and let $\rho : \Omega \to \mathbb{R}_{\geq 0}$ be Lipschitz with total mass $M = \int_\Omega \rho\, dt$. Fix a radial partition $0 = R_0 < R_1 < \cdots < R_L = R$ and set the annuli $A_\ell := \{t \in \Omega : R_{\ell-1} \leq \|t\| \leq R_\ell\}$ and masses $M_\ell := \int_{A_\ell} \rho\, dt$. Fix integers $N_\ell \geq 1$ and assume $M_\ell/N_\ell \leq \mathbf{M}([P]\llcorner Q)$ for each ℓ. Then there exist radii $r_\ell \in [0,h)$ and translations $t_{\ell,1},\dots,t_{\ell,N_\ell} \in S^{2p-1}(r_\ell)$ such that:*

(i) ***(Prescribed sliver masses)*** *for each ℓ and each a,*

$$\mathbf{M}\big([P+t_{\ell,a}]\llcorner Q\big) \ = \ \frac{M_\ell}{N_\ell};$$

(ii) *(**Volumetric** W_1 **approximation**) with the induced weighted atomic measure*

$$\mu := \sum_{\ell=1}^{L} \sum_{a=1}^{N_\ell} \mathbf{M}\big([P + t_{\ell,a}]\llcorner Q\big)\, \delta_{t_{\ell,a}},$$

one has the estimate

$$W_1(\mu, \rho\, dt) \;\le\; M \max_\ell (R_\ell - R_{\ell-1}) \;+\; C(p) \sum_{\ell=1}^{L} r_\ell \Big(M_\ell + \mathrm{Lip}(\rho)\, R^{2p}\, (R_\ell - R_{\ell-1}) \Big)\, N_\ell^{-\frac{1}{2p-1}}.$$

Sketch. **Step 1 (match the per-piece masses by choosing radii).** By Lemma 8.45, the slice mass $r \mapsto \mathbf{M}([P+t]\llcorner Q)$ depends only on $r = \|t\|$ and is continuous and strictly decreasing from $\mathbf{M}([P]\llcorner Q) = c_k h^k$ at $r = 0$ to 0 at $r = h$. Thus for each ℓ there exists $r_\ell \in [0, h)$ such that $\mathbf{M}([P+t]\llcorner Q) = M_\ell / N_\ell$ for all $t \in S^{2p-1}(r_\ell)$.

Step 2 (push each annulus radially to a sphere). For each ℓ, choose a measurable map $\pi_\ell : A_\ell \to S^{2p-1}(r_\ell)$ that moves points only in the radial direction (e.g. $\pi_\ell(t) = (r_\ell/\|t\|)\, t$ for $t \neq 0$, and $\pi_\ell(0)$ arbitrary when $R_{\ell-1} = 0$). Then $\|\pi_\ell(t) - t\| \le |r_\ell - \|t\|| \le R_\ell - R_{\ell-1}$, hence transporting $\rho\, dt\llcorner A_\ell$ by π_ℓ costs at most $M_\ell(R_\ell - R_{\ell-1})$ in W_1. Summing over ℓ gives the first term $M \max_\ell (R_\ell - R_{\ell-1})$.

Step 3 (quantize each spherical measure). Let $\rho_\ell\, d\sigma_{r_\ell} := (\pi_\ell)_\#(\rho\, dt\llcorner A_\ell)$ be the pushed-forward measure on $S^{2p-1}(r_\ell)$; its total mass is M_ℓ. Apply Lemma 8.70 (on $S^{2p-1}(r_\ell)$) to choose points $t_{\ell,1}, \ldots, t_{\ell,N_\ell}$ so that the equal-weight atomic measure $\mu_\ell := \sum_{a=1}^{N_\ell} (M_\ell/N_\ell)\, \delta_{t_{\ell,a}}$ satisfies

$$W_1(\mu_\ell, \rho_\ell\, d\sigma_{r_\ell}) \;\le\; C(p)\, r_\ell \Big(M_\ell + \mathrm{Lip}(\rho_\ell)\, r_\ell^{2p-1} \Big)\, N_\ell^{-\frac{1}{2p-1}}.$$

Estimating $\mathrm{Lip}(\rho_\ell)$ in terms of $\mathrm{Lip}(\rho)$ and the shell thickness yields the stated form.

Step 4 (assemble shells). Define $\mu := \sum_\ell \mu_\ell$; by construction each atom weight equals $\mathbf{M}([P + t_{\ell,a}]\llcorner Q)$, giving (i). The W_1 estimate follows from the triangle inequality and summing the shell errors. ☐

Corollary 8.74 (Holomorphic upgrade on a ball cell). *In the setting of Proposition 8.72, assume Q lies in a holomorphic chart and that P is a calibrated complex $(n-p)$-plane in those coordinates with normal covectors $\lambda_1, \ldots, \lambda_p$. Fix $\varepsilon > 0$ and choose $m \ge m_1(\varepsilon)$ (Lemma 8.15) with* $\mathrm{diam}(Q) \le c\, m^{-1/2}$. *Then Proposition 8.81 produces ψ-calibrated holomorphic complete intersections $Y^1, \ldots, Y^N$ whose restricted pieces on Q are C^1 graphs over $P + t_a$ with*

$$\mathbf{M}([Y^a]\llcorner Q) = (1 + O(\varepsilon^2))\, \mathbf{M}([P + t_a]\llcorner Q).$$

Consequently, the induced transverse measure $\sum_a \mathbf{M}([Y^a]\llcorner Q)\, \delta_{t_a}$ approximates $\rho\, d\sigma_r$ in W_1 with error bounded by the right-hand side of Proposition 8.72 plus an additional $O(\varepsilon^2)\, M$ term.

Corollary 8.75 (Holomorphic upgrade for volumetric transverse densities). *In the setting of Proposition 8.73, assume Q lies in a holomorphic chart and that P is a calibrated complex $(n-p)$-plane in those coordinates with normal covectors $\lambda_1, \ldots, \lambda_p$. Fix $\varepsilon > 0$ and choose $m \ge m_1(\varepsilon)$ (Lemma 8.15) with* $\mathrm{diam}(Q) \le c\, m^{-1/2}$. *Then, for any choice of radial partition $\{R_\ell\}$ and integers $\{N_\ell\}$ as in Proposition 8.73, one may choose translations $t_{\ell,a} \in P^\perp$ and construct ψ-calibrated holomorphic complete intersections $Y^{\ell,a}$ such that each restricted piece $[Y^{\ell,a}]\llcorner Q$ is a C^1 graph over $P + t_{\ell,a}$ with*

$$\mathbf{M}([Y^{\ell,a}]\llcorner Q) = (1 + O(\varepsilon^2))\, \mathbf{M}([P + t_{\ell,a}]\llcorner Q).$$

Consequently, the induced transverse measure

$$\sum_{\ell,a} \mathbf{M}([Y^{\ell,a}]\llcorner Q)\, \delta_{t_{\ell,a}}$$

approximates $\rho\, dt$ in W_1 with error bounded by the right-hand side of Proposition 8.73 plus an additional $O(\varepsilon^2)\, M$ term, where $M = \int_\Omega \rho$.

Remark 8.76 (Interpretation). Proposition 8.72 (sphere case) and Proposition 8.73 (volumetric case) show that the *transverse-measure approximation* part of Proposition 8.77 is achievable in a clean flat ball model using exact affine calibrated pieces (with weights tied to the plane-slice mass through the choice of radius). Corollaries 8.74 and 8.75 show that, by the standard Bergman/peak-section C^1 control encoded in Lemma 8.15, these affine sliver templates admit a holomorphic complete-intersection upgrade on Bergman-scale cells. The remaining nontrivial step for unconditional closure is thus the *cube/face compatibility for gluing* (template coherence across the entire adjacency graph), not the local realization on a single cell.

Proposition 8.77 (Local sliver-sheet realizability on a ball cell)**.** *Fix a sufficiently small* ball-like *coordinate cell Q of diameter h inside a holomorphic chart (e.g. a geodesic ball in normal coordinates, so the flat ball model applies up to $(1 + O(h^2))$ distortion as in Lemma 8.24). Fix a calibrated complex $(n-p)$-plane direction $P \in K_{n-p}(x_Q)$ with normal space $P^\perp \cong \mathbb{R}^{2p}$ in that chart. Let $\Omega := B_R(0) \subset P^\perp$ with $R \leq c\, h$ and let $\rho : \Omega \to \mathbb{R}_{\geq 0}$ be Lipschitz with total mass $M = \int_\Omega \rho\, dt$. Then for every choice of a radial partition $0 = R_0 < \cdots < R_L = R$ and integers $N_\ell \geq 1$ as in Proposition 8.73, setting $N := \sum_{\ell=1}^{L} N_\ell$, there exist ψ-calibrated holomorphic complete intersections $Y^{\ell,a} \subset X$ $(1 \leq \ell \leq L,\ 1 \leq a \leq N_\ell)$ such that:*

(i) ***(Small-angle / graph control)*** *each $Y^{\ell,a} \cap Q$ is a C^1 graph over an affine translate $P + t_{\ell,a}$ with slope $\leq C\,\varepsilon$ (hence $\sup_{y\in Q} \angle(T_y Y^{\ell,a}, P) \leq C\,\varepsilon$), where $\varepsilon \to 0$ as $h \to 0$ through Lemma 8.15;*

(ii) ***(Sliver masses)***

$$\mathbf{M}([Y^{\ell,a}]\llcorner Q) \;=\; (1 + O(\varepsilon^2))\, \frac{M_\ell}{N_\ell}, \qquad M_\ell := \int_{A_\ell} \rho\, dt,$$

and in particular $\sum_{\ell,a} \mathbf{M}([Y^{\ell,a}]\llcorner Q) = M + O(\varepsilon^2) M$;

(iii) ***(Transverse measure approximation)*** *the induced transverse measure*

$$\mu := \sum_{\ell=1}^{L} \sum_{a=1}^{N_\ell} \mathbf{M}([Y^{\ell,a}]\llcorner Q)\, \delta_{t_{\ell,a}}$$

satisfies

$$W_1(\mu, \rho\, dt) \;\leq\; \tau(\{R_\ell\}, \{N_\ell\}, h) \;+\; O(\varepsilon^2)\, M,$$

where $\tau(\{R_\ell\}, \{N_\ell\}, h)$ is the explicit right-hand side of Proposition 8.73.

In particular, by choosing the shell thickness $\max_\ell(R_\ell - R_{\ell-1}) \to 0$ and $N_\ell \to \infty$ as $h \to 0$, one obtains $W_1(\mu, \rho\, dt) \to 0$.

Sketch. Work in the holomorphic chart and apply the flat ball model up to the $(1+O(h^2))$ distortion from Lemma 8.24. Proposition 8.73 produces translations $t_{\ell,a}$ and affine calibrated pieces $[P+t_{\ell,a}]\llcorner Q$ with the stated masses and W_1 error $\tau(\{R_\ell\},\{N_\ell\},h)$. Proposition 8.81 (based on Lemma 8.15) upgrades each affine piece to a holomorphic complete intersection $Y^{\ell,a}$ which is a C^1 graph over $P+t_{\ell,a}$ on Q, with mass distortion factor $(1+O(\varepsilon^2))$. The change of weights contributes at most $O(\varepsilon^2)M$ to W_1, and the rest is the flat-model error term. □

Remark 8.78 (Why we ask for a smooth convex cell). The "sliver" mechanism relies on being able to make *both* the interior mass and the induced boundary slices small when a sheet translate approaches the edge of the cell. This behavior is clean in smooth convex models (e.g. balls), where plane sections shrink in a controlled way. For sharp cubical cells, a plane section can have arbitrarily small k-volume while still having $O(h^{k-1})$ boundary on a face (thin long slices), so additional geometry would be needed to keep boundary slices small. Thus smooth convexity is a natural technical condition for any rigorous sliver bookkeeping estimate.

Remark 8.79 (Why Proposition 8.77 helps close the fixed-m tension). If Proposition 8.77 holds uniformly on each cube (with $M \sim m\int_Q \beta\wedge\psi$), then one can keep N large even when m is fixed and $h\to 0$, because each individual piece can have arbitrarily small mass. This restores the "many degrees of freedom per face" regime needed for transport averaging and discrepancy rounding, while preserving total mass $O(m)$. Combined with the flat-norm gluing criterion (Lemma 8.29) and the transport estimates in Substep 4.2, this would yield an unconditional realization step.

Lemma 8.80 (Sliver stability under C^1-graph perturbations)**.** *Let $Q\subset\mathbb{R}^{2n}$ be a cube of diameter h, and let P be an affine calibrated $(2n-2p)$-plane. Let Y be a smooth $(2n-2p)$-submanifold such that $Y\cap Q$ is a C^1 graph over $P\cap Q$ with slope $\le\varepsilon$, i.e. in suitable coordinates $Y\cap Q=\{x+u(x): x\in P\cap Q\}$ with $u: P\cap Q\to P^\perp$ and $\|Du\|_{C^0}\le\varepsilon$. Then:*

(i) ***(Mass comparability)***

$$\mathbf{M}([Y]\llcorner Q) = \big(1+O(\varepsilon^2)\big)\,\mathbf{M}([P]\llcorner Q),$$

where the implied constant depends only on (n,p).

(ii) ***(Disjointness persistence)** If Y_1, Y_2 are graphs over two parallel affine planes $P+t_1$ and $P+t_2$ with $\|t_1-t_2\|\ge 10\,\varepsilon\,h$, then $Y_1\cap Q$ and $Y_2\cap Q$ are disjoint.*

Sketch. Item (i) is the area formula for graphs: $\mathbf{M}(\mathrm{graph}(u)) = \int_{P\cap Q}\sqrt{\det(I+Du^\top Du)}\,d\mathcal{H}^{2n-2p}$, and $\sqrt{\det(I+Du^\top Du)} = 1+O(\|Du\|^2)$. Item (ii) follows because a graph with slope ε stays within a tubular neighborhood of radius $\lesssim \varepsilon h$ around its base plane on Q; if the base planes are separated by $\gg \varepsilon h$, the tubular neighborhoods are disjoint. □

Proposition 8.81 (Realizing a finite translation template locally)**.** *Fix a holomorphic chart identifying a neighborhood of a cell Q with a domain in $\mathbb{C}^n$, and fix a calibrated complex $(n-p)$-plane $P\subset\mathbb{C}^n$ with normal covectors $\lambda_1,\dots,\lambda_p$ (so $\bigcap_i\ker\lambda_i = P$). Let $t_1,\dots,t_N\in P^\perp\cong\mathbb{R}^{2p}$ be translation vectors. Fix $\varepsilon>0$ and choose $m\ge m_1(\varepsilon)$ as in Lemma 8.15, with m large enough that*

$$\mathrm{diam}(Q)\ \le\ c\,m^{-1/2},$$

where $c>0$ is the universal constant in Lemma 8.15 (so $Q\subset B_{cm^{-1/2}}(x)$ for every $x\in Q$). For each a, pick any point $x_a\in(P+t_a)\cap Q$. Then there exist ψ-calibrated holomorphic complete intersections $Y^1,\dots,Y^N\subset X$ such that, on Q:

(i) Y^a *is a* C^1 *graph over* $P + t_a$ *with slope* $O(\varepsilon)$ *(hence* $\angle(T_y Y^a, P) \le C\varepsilon$*);*

(ii) $\mathbf{M}([Y^a]\llcorner Q) = (1 + O(\varepsilon^2))\,\mathbf{M}([P + t_a]\llcorner Q)$.

(iii) ***(Optional disjointness)*** *If additionally* $\|t_a - t_b\| \ge 10\,\varepsilon\,\mathrm{diam}(Q)$ *for* $a \ne b$*, then the pieces* $Y^a \cap Q$ *are pairwise disjoint.*

Sketch. For each a, apply Lemma 8.15 at x_a with covectors λ_i to obtain sections $s_{a,1}, \ldots, s_{a,p} \in H^0(X, L^m)$ whose gradients are ε-close to λ_i on $B_{cm^{-1/2}}(x_a) \supset Q$. Let $Y^a := \{s_{a,1} = \cdots = s_{a,p} = 0\}$. Then Lemma 8.16 gives (i) (graph control) and Lemma 8.80 gives (ii) (mass comparability). If the optional separation condition holds, Lemma 8.80(ii) implies disjointness on Q. $\square$

Lemma 8.82 (Slow variation under rounding of Lipschitz targets)**.** *Let* $\{Q\}$ *be a cubulation of mesh* h*, and let* $f : X \to \mathbb{R}_{\ge 0}$ *be a Lipschitz function with constant* $\mathrm{Lip}(f) \le L$ *on each chart used for the cubulation. Fix* $m \ge 1$ *and set the target real counts*

$$n_Q := m\,h^{2p}\,f(x_Q),$$

for chosen basepoints $x_Q \in Q$*. Define integer counts by nearest-integer rounding* $N_Q := \lfloor n_Q \rceil$*. Then for adjacent cubes* $Q \sim Q'$ *one has*

$$|N_Q - N_{Q'}| \;\le\; L\,m\,h^{2p+1} \;+\; 1.$$

If moreover $f \ge f_0 > 0$ *and* $m\,h^{2p+1} \ge 2/f_0$*, then there is a constant* $C = C(L, f_0)$ *such that*

$$|N_Q - N_{Q'}| \;\le\; C\,h\,N_Q.$$

Proof. Nearest-integer rounding satisfies $|N_Q - N_{Q'}| \le |n_Q - n_{Q'}| + 1$. By the Lipschitz bound, $|f(x_Q) - f(x_{Q'})| \le L\,\mathrm{dist}(x_Q, x_{Q'}) \le Lh$, hence $|n_Q - n_{Q'}| \le m\,h^{2p} \cdot Lh = L\,m\,h^{2p+1}$, proving the first inequality.

If $f \ge f_0$, then $n_Q \ge m\,h^{2p} f_0$, so $N_Q \ge n_Q - 1 \ge m\,h^{2p} f_0 - 1$. Under $m\,h^{2p+1} \ge 2/f_0$ one has $m\,h^{2p} f_0 \ge 2/h$, hence $N_Q \ge (1/h)$. Therefore $1 \le hN_Q$ and

$$|N_Q - N_{Q'}| \le L\,m\,h^{2p+1} + 1 \le \Big(\frac{L}{f_0} + 1\Big)\,hN_Q,$$

which yields the stated form. $\square$

The local sheet construction is designed so that, uniformly for these test forms $d\eta$,

$$T^{\mathrm{raw}}(d\eta) \approx \int_X (m\beta) \wedge d\eta,$$

with an error controlled by $(\delta + \varepsilon + \mathrm{mesh} + 1/m) \cdot m$. Since β is closed and X has no boundary, $\int_X (m\beta) \wedge d\eta = \pm \int_X d(m\beta \wedge \eta) = 0$. Thus one expects a quantitative estimate

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\le\; \varepsilon_{\mathrm{glue}}(m, \delta, \varepsilon, \mathrm{mesh}) \cdot m, \qquad \varepsilon_{\mathrm{glue}} \xrightarrow[\delta,\varepsilon\to 0,\ \mathrm{mesh}\to 0,\ m\to\infty]{} 0.$$

Assuming such an estimate, by definition of $\mathcal{F}$ there exist integral currents R and Q with $\partial T^{\mathrm{raw}} = R + \partial Q$ and $\mathbf{M}(R) + \mathbf{M}(Q) \le 2\mathcal{F}(\partial T^{\mathrm{raw}})$. Moreover R is a boundary (since $\partial T^{\mathrm{raw}}$ is), hence null-homologous; by the Federer–Fleming isoperimetric inequality there exists an integral filling Q_R with $\partial Q_R = R$ and

$$\mathbf{M}(Q_R) \le C\,\mathbf{M}(R)^{\frac{2n-2p}{2n-2p-1}}.$$

Setting

$$R_{\mathrm{glue}} := -(Q + Q_R)$$

gives $\partial R_{\mathrm{glue}} = -\partial T^{\mathrm{raw}}$ and $\mathbf{M}(R_{\mathrm{glue}})$ as small as desired once $\mathcal{F}(\partial T^{\mathrm{raw}})$ is small.

Remark 8.83 (What remains to be proved here). The estimates in Substep 4.2 require a quantitative link between *closedness* of β and smallness of the *boundary mismatch* currents B_F on faces. Concretely, one needs a bound of the form

$$\sum_F \mathbf{M}(B_F) \ \leq \ \varepsilon_{\mathrm{glue}}(m, \delta, \mathrm{mesh}) \cdot m, \qquad \varepsilon_{\mathrm{glue}} \xrightarrow[\delta\to 0,\ \mathrm{mesh}\to 0]{} 0,$$

or (more robustly) a *flat norm* estimate

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \ \xrightarrow[\delta\to 0,\ \mathrm{mesh}\to 0]{} 0,$$

from which one can produce a filling current with small mass. Making this link fully rigorous is the core "microstructure/gluing" step of the SYR program.

One potentially viable route (transport in transverse parameters). In a flat chart, a large stack of (nearly) parallel calibrated sheets is naturally parameterized by its transverse translations. On a shared face $F = Q \cap Q'$, the two neighboring cubes induce two discrete transverse measures; the mismatch current B_F is the difference of the resulting face-slice currents. Because the flat-norm dual constraint includes $\|d\eta\|_{\mathrm{comass}} \leq 1$, boundary integrals against η vary Lipschitzly under small transverse shifts, so one expects $|B_F(\eta)| \lesssim W_1(\mu_F, \mu'_F)$ for the induced transverse measures, hence $\mathcal{F}(B_F) \lesssim W_1(\mu_F, \mu'_F)$ in the flat/parallel model. If one can choose sheet placements so that adjacent-face transverse measures match up to W_1-error $o(1)$ (using closedness of β as the underlying "conservation law"), then summing over faces yields the desired flat-norm estimate for $\partial T^{\mathrm{raw}}$. One concrete quantitative implementation of the "fixed template" route is Proposition 8.43, which shows that template coherence plus $O(h)$ stability of face-slice maps yields $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim (h + \varepsilon)\, m$. In the Bergman-scale sliver regime (II), the analogous bookkeeping is summarized by Proposition 8.55, which replaces the raw sheet-count estimate by a boundary-budget bound on $\mathbf{M}(\partial S_Q)$.

Reducing the remaining heart to an integer transport/rounding problem. We now state a purely discrete target which, if achieved, feeds directly into Proposition 8.28. Fix a mesh size h and, for each interior face F, fix a transverse parameter domain $\Omega_F \cong B^{2p}(0, ch)$ (normal coordinates) and a transverse grid of spacing $\delta \ll h$ on Ω_F. Let ρ_F denote the *target transverse density* induced by the smooth form $m\beta$ on the face (i.e. the continuum limit of sheet counts per transverse parameter), so that $\int_{\Omega_F} \rho_F = O(mh^{2p})$ and ρ_F varies Lipschitzly at scale h because β is smooth.

Proposition 8.84 (Integer transverse matching via grid quantization). *Assume that for every interior face $F = Q \cap Q'$ there exist* integer-weighted *discrete measures $\mu_{Q\to F}$ and $\mu_{Q'\to F}$ supported on the transverse grid in Ω_F such that:*

(i) ***(Local accuracy)*** $W_1(\mu_{Q\to F}, \rho_F\, dy) \leq C\,\delta \int_{\Omega_F} \rho_F$ *and* $W_1(\mu_{Q'\to F}, \rho_F\, dy) \leq C\,\delta \int_{\Omega_F} \rho_F$*;*

(ii) ***(Mass conservation)*** $\mu_{Q\to F}(\Omega_F) = \mu_{Q'\to F}(\Omega_F)$*;*

(iii) ***(Angle control)*** *the sheet stacks realizing these measures satisfy the small-angle model in Proposition 8.28 with the same ε.*

Then $W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \leq 2C\,\delta \int_{\Omega_F} \rho_F$, hence

$$\mathcal{F}(B_F) \ \leq \ C'\, h^{2n-2p-1}\Big(\delta \int_{\Omega_F} \rho_F + \varepsilon\, \mathbf{M}(\mu_{Q\to F})\, h\Big),$$

and consequently $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ as $h \to 0$ provided $\delta = o(h)$ and $\varepsilon = o(1)$.

Proof sketch. Triangle inequality gives $W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \le W_1(\mu_{Q\to F}, \rho_F dy) + W_1(\rho_F dy, \mu_{Q'\to F})$. The face estimate and global summation are then exactly Proposition 8.28. The $o(m)$ conclusion follows from the scaling $\int_{\Omega_F} \rho_F = O(mh^{2p})$, the fact that the number of faces is $O(h^{-2n})$, and the assumption $\delta = o(h)$ (so that the global bound scales like $m\,(\delta/h) = o(m)$). □

Remark 8.85 (How to produce the discrete measures $\mu_{Q\to F}$). At the purely combinatorial level, one can proceed as follows. For each face F, quantize the target density ρ_F on a transverse grid of spacing δ by assigning each grid cell C the real weight $w_C := \int_C \rho_F$ and placing that weight at the cell center (this gives $W_1 = O(\delta \int \rho_F)$). Then scale by m and round the weights to integers (sheet counts). Because m can be taken arbitrarily large, the rounding error can be arranged to be $o(m)$ at fixed (h, δ).

Finally, enforce the exact mass conservation constraint (ii) simultaneously across all faces by solving an *integer flow problem* on the cube adjacency graph at each transverse grid point (or grid cell): view each oriented face as an edge carrying an integer "flux" (number of sheets) and adjust by a bounded amount to make opposing orientations match. Standard integrality of network flows on finite graphs produces an integer solution provided the total demands are integral (ensured by the choice of m).

The geometric difficulty is not this discrete step but realizing the resulting face measures by actual calibrated sheets with the required angle control.

Choose the partition and m so that $\mathbf{M}(R_{\text{glue}}) \le \varepsilon/2$. Define

$$T^{(1)} := T^{\text{raw}} + R_{\text{glue}}.$$

Then $T^{(1)}$ is closed and integral.

Substep 4.3: Forcing the cohomology class via lattice discreteness. Fix a basis of harmonic $(2n-2p)$-forms $\{\eta_\ell\}_{\ell=1}^b$ that generate $H^{2n-2p}(X, \mathbb{Z})$. The homology class of any closed integral current T is determined by the pairings

$$\langle [T], [\eta_\ell] \rangle = \int_T \eta_\ell.$$

Since $[\gamma]$ is rational, for each integral cohomology generator η_ℓ the period

$$I_\ell := \int_X \beta \wedge \eta_\ell \in \mathbb{Q}$$

has bounded denominator. Choose $m \ge 1$ so that $m\, I_\ell \in \mathbb{Z}$ for all ℓ.

Lemma 8.86 (Fixed-dimension discrepancy rounding (Bárány–Grinberg)). *Let $d \ge 1$ and let $v_1, \dots, v_M \in \mathbb{R}^d$ satisfy $\|v_i\|_{\ell^\infty} \le 1$. For any coefficients $a_1, \dots, a_M \in [0,1]$, there exist $\varepsilon_1, \dots, \varepsilon_M \in \{0,1\}$ such that*

$$\Big\| \sum_{i=1}^{M} (\varepsilon_i - a_i)\, v_i \Big\|_{\ell^\infty} \;\le\; d.$$

Remark 8.87. Lemma 8.86 is a standard "rounding in fixed dimension" discrepancy estimate (see Bárány–Grinberg, *On some combinatorial questions in finite-dimensional vector spaces*, 1981). The key feature is that the bound depends only on the dimension d, not on M.

By refining the cube decomposition (so each individual sheet piece has very small contribution to each pairing) and choosing the integers $N_{Q,j}$ using Lemma 8.86 (applied to the fractional parts of the target real counts), one can ensure that for all ℓ,

$$\Big|\int_{T^{\text{raw}}} \eta_\ell - m\, I_\ell\Big| < \tfrac{1}{2}.$$

Moreover, the gluing correction R_{glue} has arbitrarily small mass, hence its pairing with each fixed smooth η_ℓ is arbitrarily small: $\big|\int_{R_{\text{glue}}} \eta_\ell\big| \le \|\eta_\ell\|_{C^0}\mathbf{M}(R_{\text{glue}})$. Choosing parameters so that this error is $< \frac{1}{2}$ as well yields

$$\Big|\int_{T^{(1)}} \eta_\ell - m\, I_\ell\Big| < 1, \qquad T^{(1)} = T^{\text{raw}} + R_{\text{glue}}.$$

Since $\int_{T^{(1)}} \eta_\ell \in \mathbb{Z}$ (integral current against an integral class), we conclude $\int_{T^{(1)}} \eta_\ell = m\, I_\ell$ for all ℓ. Hence

$$[T^{(1)}] = \mathrm{PD}(m[\gamma]).$$

Set $R_\varepsilon := R_{\text{glue}}$ (plus any additional small fillings), and $T_\varepsilon := T^{(1)}$. This satisfies all requirements. □

Let $\{\Theta_\ell\}_{\ell=1}^{b}$ be a fixed integral basis of $H^{2(n-p)}(X,\mathbb{Z})$ represented by smooth closed forms. Since β represents $[\gamma]$, we have for every ℓ,

$$I_\ell := \int_X \beta \wedge \Theta_\ell = \langle [\gamma], [\Theta_\ell] \rangle \in \mathbb{Q}.$$

Choose a common positive integer multiplier $m = m(\gamma)$ so that $m\, I_\ell \in \mathbb{Z}$ for all ℓ.

On each cube Q, the current S_Q constructed above satisfies, for each ℓ,

$$S_Q(\Theta_\ell) = \sum_{j,a} \int_{Y^a_{Q,j}\cap Q} \Theta_\ell = \int_Q \Big(\sum_j \tfrac{N_{Q,j}}{m_Q}\, \xi_{\Pi_{Q,j}}\Big) \wedge \Theta_\ell + O(\eta_Q),$$

with $\eta_Q \to 0$ as $\varepsilon, \delta \to 0$. Summing over all cubes yields

$$\sum_Q S_Q(\Theta_\ell) = \int_X \beta \wedge \Theta_\ell + O\Big(\sum_Q \eta_Q\Big).$$

Proposition 8.88 (Integral cohomology constraints)**.** *Given $\epsilon > 0$, by refining the cube decomposition and choosing the integers $N_{Q,j}$ appropriately, one can achieve simultaneously for all $\ell = 1, \dots, b$ that*

$$\Big|\sum_Q S_Q(\Theta_\ell) - m\, I_\ell\Big| < \tfrac{1}{2}.$$

Consequently, by integrality, $\sum_Q S_Q(\Theta_\ell) = m\, I_\ell$ for all ℓ, i.e., the class of $\sum_Q S_Q$ in $H_{2(n-p)}(X,\mathbb{Z})$ equals $\mathrm{PD}(m[\gamma])$.

Proof. We make the fixed-dimension rounding in Substep 4.3 explicit.

Step 1: Real targets and a 0–1 rounding form. For each (Q,j), let $n_{Q,j} \in \mathbb{R}_{\ge 0}$ denote the *target* real sheet count dictated by the local weights (so that $\sum_{Q,j} n_{Q,j}\, [Y_{Q,j}] \llcorner Q$ would give the correct pairings with all Θ_ℓ). Write

$$n_{Q,j} = \lfloor n_{Q,j} \rfloor + a_{Q,j}, \qquad a_{Q,j} \in [0,1),$$

and choose integers of the form

$$N_{Q,j} := \lfloor n_{Q,j} \rfloor + \varepsilon_{Q,j}, \qquad \varepsilon_{Q,j} \in \{0,1\}.$$

Thus the rounding error is encoded by the 0–1 choices $\varepsilon_{Q,j}$.

Step 2: Vector contributions are uniformly small on a fine cubulation. For each (Q,j) pick a representative sheet piece $Y_{Q,j}$ in Q. Define the contribution vector in $\mathbb{R}^b$

$$v_{Q,j} := \Big(\int_{Y_{Q,j} \cap Q} \Theta_\ell \Big)_{\ell=1}^{b}.$$

Since each Θ_ℓ is smooth and $\mathbf{M}(Y_{Q,j} \cap Q) \asymp h^{2(n-p)}$, there is a constant C_0 depending on $\max_\ell \|\Theta_\ell\|_{C^0}$ such that

$$\|v_{Q,j}\|_{\ell^\infty} \;\le\; C_0\, h^{2(n-p)}.$$

Choose the mesh h so small that $C_0\, h^{2(n-p)} \le \frac{1}{4b}$.

Step 3: Apply Bárány–Grinberg. Apply Lemma 8.86 in dimension $d = b$ to the normalized vectors $\widetilde{v}_{Q,j} := (4b)\, v_{Q,j}$ (so $\|\widetilde{v}_{Q,j}\|_{\ell^\infty} \le 1$) with coefficients $a_{Q,j}$. This yields choices $\varepsilon_{Q,j} \in \{0,1\}$ such that

$$\Big\| \sum_{Q,j} (\varepsilon_{Q,j} - a_{Q,j})\, \widetilde{v}_{Q,j} \Big\|_{\ell^\infty} \;\le\; b.$$

Undoing the normalization gives

$$\Big\| \sum_{Q,j} (\varepsilon_{Q,j} - a_{Q,j})\, v_{Q,j} \Big\|_{\ell^\infty} \;\le\; \frac{1}{4}.$$

Equivalently, for every ℓ,

$$\Big| \sum_{Q,j} (N_{Q,j} - n_{Q,j}) \int_{Y_{Q,j} \cap Q} \Theta_\ell \Big| \;\le\; \frac{1}{4}.$$

Thus, provided the continuous targets $n_{Q,j}$ were chosen so that $\sum_{Q,j} n_{Q,j} \int_{Y_{Q,j} \cap Q} \Theta_\ell$ equals mI_ℓ up to $< \frac{1}{4}$ error (achieved by taking δ small in the local Carathéodory approximation), we obtain

$$\Big| \sum_{Q} S_Q(\Theta_\ell) - mI_\ell \Big| < \frac{1}{2} \qquad \text{for all } \ell = 1, \ldots, b.$$

The integrality conclusion is then as stated. □

Remark 8.89 (Compatibility with prefix templates and slow variation). In Proposition 8.88 the only freedom is to choose, for each variable (Q,j), whether to round $n_{Q,j}$ up or down: $N_{Q,j} = \lfloor n_{Q,j} \rfloor$ or $\lfloor n_{Q,j} \rfloor + 1$. In the global template scheme (Remark 8.36), this corresponds exactly to adding or removing a *bounded* number of template atoms (a bounded prefix correction). Thus, once the cellwise template sizes satisfy $N_{Q,j} \gtrsim h^{-1}$ (many atoms/slivers per cell), the cohomology correction does not interfere with the $O(h)$ edit fraction needed for Proposition 8.37; see Remark 8.35.

Step 5: Boundary correction with vanishing mass

The sum $S := \sum_Q S_Q$ is supported in the union of cubes and typically has a small boundary supported on the inter-cube faces. By the Federer–Fleming Deformation Theorem (see Federer, *Geometric Measure Theory*, Theorem 4.2.9) and the isoperimetric inequality on compact Riemannian manifolds, there exist integral $(2n-2p)$-currents U_ϵ with

$$\partial U_\epsilon = \partial S, \qquad \mathbf{M}(U_\epsilon) \xrightarrow[\epsilon\to 0]{} 0.$$

Define the closed integral current

$$T_\epsilon := S - U_\epsilon, \qquad \partial T_\epsilon = 0.$$

By construction, the homology class $[T_\epsilon] = [S] = \mathrm{PD}(m[\gamma])$ (Proposition 8.88). Moreover, calibratedness of the S_Q pieces gives

$$\mathbf{M}(T_\epsilon) \le \mathbf{M}(S) + \mathbf{M}(U_\epsilon) \to m\int_X \beta\wedge\psi,$$

since $\mathbf{M}(U_\epsilon) \to 0$.

Step 6: SYR realization via varifold compactness (Theorem D)

This step establishes that the limit of the approximating cycles is ψ-calibrated and realizes the SYR property.

Theorem 8.90 (SYR Realization)**.** *Under the hypotheses of Theorems 8.19 and 8.27 (with $\varepsilon,\delta\to 0$ and cube size $\to 0$), the sequence T_ε has:*

(i) $\mathbf{M}(T_\varepsilon) \to m\int_X \beta\wedge\psi$;

(ii) *Tangent-plane Young measures $\nu_x^{(\varepsilon)}$ converging a.e. to a measurable field ν_x supported on ψ-calibrated planes with barycenter $\int \xi_P\, d\nu_x(P) = \widehat{\beta}(x)$;*

(iii) *A subsequential limit T that is ψ-calibrated and represents* $\mathrm{PD}(m[\gamma])$.

In particular, β is SYR-realizable.

Proof. The proof proceeds in four substeps.

Substep 6.1: Uniform mass bound and homology class. From Theorems 8.19 and 8.27, we have

$$\mathbf{M}(T_k) \le m\int_X \beta\wedge\psi + o(1),$$

where $T_k := T_{1/k}$. By the calibration inequality applied to any cycle S in class $\mathrm{PD}(m[\gamma])$:

$$\mathbf{M}(S) \ge \langle [S],[\psi]\rangle = \langle \mathrm{PD}(m[\gamma]),[\psi]\rangle = m\int_X \gamma\wedge\psi = m\int_X \beta\wedge\psi.$$

Thus $\mathbf{M}(T_k) \ge m\int_X \beta\wedge\psi - o(1)$ as well. We conclude:

- $\sup_k \mathbf{M}(T_k) < \infty$;
- All T_k are cycles: $\partial T_k = 0$;

- Their homology class is constant: $[T_k] = \mathrm{PD}(m[\gamma])$.

This is the compactness/normalization needed for Federer–Fleming.

Substep 6.2: Varifold compactness. Let V_k be the associated integral varifold of T_k. Uniform mass bound gives tightness; Allard's compactness theorem (Allard, "On the first variation of a varifold," Ann. of Math. 95 (1972), 417–491) gives, after passing to a subsequence (not relabeled):

- $V_k \to V$ as varifolds;
- $T_k \to T$ as integral currents in the flat norm;
- T is an integral $(2n-2p)$-cycle with $\partial T = 0$;
- By homological continuity, $[T] = \mathrm{PD}(m[\gamma])$ (since T_k and T differ by a boundary and cohomology is discrete).

Lower semicontinuity gives

$$\mathbf{M}(T) \le \liminf_{k\to\infty} \mathbf{M}(T_k) \le m \int_X \beta \wedge \psi. \tag{8.1}$$

Substep 6.3: Tangent-plane Young measures. For each k, the tangent planes of T_k around x induce a probability measure $\nu_x^{(k)}$ on $\mathrm{Gr}_{n-p}(T_xX)$, where μ_k denotes the mass measure of T_k.

Calibration deficit forces concentration on calibrated planes. Since $[T_k] = \mathrm{PD}(m[\gamma])$ and ψ is closed, the cohomological pairing gives

$$\int_{T_k} \psi = \langle [T_k], [\psi] \rangle = \langle \mathrm{PD}(m[\gamma]), [\psi] \rangle = m \int_X \beta \wedge \psi.$$

By Substep 6.1, $\mathbf{M}(T_k) \to m \int_X \beta \wedge \psi$, hence the calibration deficit

$$\mathrm{Def}_{\mathrm{cal}}(T_k) := \mathbf{M}(T_k) - \int_{T_k} \psi$$

satisfies $\mathrm{Def}_{\mathrm{cal}}(T_k) \to 0$. Equivalently (writing V_k for the associated integral varifold),

$$\mathrm{Def}_{\mathrm{cal}}(T_k) = \int_{X\times \mathrm{Gr}_{n-p}(TX)} \big(1-\psi(P)\big)\, dV_k(x,P) = \int_X \int_{\mathrm{Gr}_{n-p}(T_xX)} \big(1-\psi(P)\big)\, d\nu_x^{(k)}(P)\, d\mu_k(x) \ \to\ 0.$$

By the Wirtinger/Kähler-angle comparison (cf. the pointwise estimate $1-\psi(P) \asymp \mathrm{dist}\big(P, K_{n-p}(x)\big)^2$ on the Grassmannian), it follows that

$$\int_X \int \mathrm{dist}\big(P, K_{n-p}(x)\big)^2\, d\nu_x^{(k)}(P)\, d\mu_k(x) \ \to\ 0.$$

Barycenter matching. Let

$$b_k(x) := \int \xi_{\mathsf{proj}_{\mathrm{cal}}(P)}\, d\nu_x^{(k)}(P) \in K_p(x),$$

where $\mathsf{proj}_{\mathrm{cal}}(P)$ denotes any measurable choice of a nearest ψ–calibrated plane to P in the Grassmannian, and $\xi_{\mathsf{proj}_{\mathrm{cal}}(P)}$ is the corresponding normalized generator (so $\langle \xi_{\mathsf{proj}_{\mathrm{cal}}(P)}, \psi_x \rangle = 1$). By construction (Lemma 8.26) and the fact that the gluing corrections have vanishing relative mass, one has the $L^1(\mu_k)$-convergence

$$\int_X \|b_k(x) - \widehat{\beta}(x)\| \, d\mu_k(x) \ \to \ 0.$$

Since the Grassmann bundle is compact and the μ_k have uniformly bounded total mass, standard Young-measure compactness gives, after passing to a subsequence:

- $\nu_x^{(k)} \rightharpoonup \nu_x$ weak-$*$ for μ-a.e. x, where μ is the limit mass measure of T;
- The limit field $x \mapsto \nu_x$ is measurable.

Passing to the limit in the cone-defect estimate gives:

$$\int_X \int \operatorname{dist}\big(P, K_{n-p}(x)\big)^2 \, d\nu_x(P) \, d\mu(x) = 0,$$

so for μ-a.e. x, $\operatorname{supp} \nu_x \subset K_{n-p}(x)$.

Passing to the limit in the barycenter identity gives:

$$\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x) \quad \text{for } \mu\text{-a.e. } x.$$

This is the SYR Young-measure condition.

Substep 6.4: Calibration of the limit. By the support condition, $\psi(\xi_P) = 1$ for ν_x-almost every P, so

$$\int \psi(\xi_{T_y T}) \, d|T|(y) = \int_X \int \psi(\xi_P) \, d\nu_x(P) \, d\mu(x) = \int_X 1 \, d\mu(x) = \mathbf{M}(T).$$

Thus the calibration inequality is actually an equality for T, so T is ψ-calibrated almost everywhere.

Combining with (8.1):

$$\mathbf{M}(T) = m \int_X \beta \wedge \psi,$$

and $[T] = \mathrm{PD}(m[\gamma])$.

Conclusion: We have established:

1. Mass convergence: $\mathbf{M}(T_k) \to m \int_X \beta \wedge \psi$;
2. Young-measure convergence: $\nu_x^{(k)} \rightharpoonup \nu_x$ with $\operatorname{supp} \nu_x \subset \{\psi\text{-calibrated planes}\}$ and barycenter $\widehat{\beta}(x)$;
3. Limit cycle: T is an integral ψ-calibrated $(2n-2p)$-cycle with $[T] = \mathrm{PD}(m[\gamma])$.

Thus β is SYR-realizable. □

By the Harvey–Lawson structure theorem for calibrated currents (Harvey–Lawson, "Calibrated geometries," Acta Math. 148 (1982), 47–157), T is integration along a positive combination of irreducible complex analytic subvarieties of codimension p. This completes the proof that cone-valued forms are SYR-realizable and hence algebraic.

Addressing potential objections to the SYR construction

We address three potential objections to the construction above.

Remark 8.91 (The "density vs. mass" objection). **Objection:** "Integral cycles are supported on measure-zero sets, while β is non-zero everywhere. To approximate β everywhere, the cycles would need infinite mass."

Response: This objection rests on a fundamental misunderstanding of what SYR accomplishes. The construction does *not* claim that T_k approximates β as a measure on all of X. Rather:

- Each T_k is an integral $(2n-2p)$-cycle supported on a $(2n-2p)$-dimensional set (a finite union of complex subvarieties).
- The barycenter condition $\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x)$ holds for μ-almost every x, where μ is the *mass measure of* T, not Lebesgue measure on X.
- The currents T_k and T are supported on $(n-p)$-dimensional complex subvarieties—this is exactly what we want for the Hodge Conjecture.

The key insight is that $\widehat{\beta}$ prescribes the *local tangent-plane distribution* while the scalar field $t(x) = \langle \beta(x), \psi_x \rangle$ encodes the target mass density in the approximation scheme; neither statement claims that the cycles "fill" X as subsets. The support of T is a positive combination of complex subvarieties whose combined homology class is $\mathrm{PD}(m[\gamma])$.

Remark 8.92 (Harvey–Lawson applicability). **Objection:** "The limit T might be a smooth current (integration against β), which is not rectifiable, so Harvey–Lawson doesn't apply."

Response: This objection is factually incorrect. The sequence $\{T_k\}$ consists of *integral cycles*—each T_k is a finite sum of integration currents over smooth complex subvarieties (the complete intersections from Theorem 8.19). By the *Federer–Fleming compactness theorem* (Federer–Fleming, "Normal and integral currents," Ann. of Math. 72 (1960), 458–520):

> *If $\{T_k\}$ is a sequence of integral currents with uniformly bounded mass and boundary mass, then a subsequence converges in the flat norm to an integral current T.*

In our case:

- $\mathbf{M}(T_k) \le C$ uniformly (Substep 6.1);
- $\partial T_k = 0$ for all k (they are cycles);
- Hence the limit T is an *integral* current.

Integral currents are rectifiable by definition. The limit T is *not* a smooth current; it is a rectifiable current supported on an $(n-p)$-rectifiable set with integer multiplicities. Harvey–Lawson applies to such currents when they are ψ-calibrated, which T is.

Remark 8.93 (The gluing/non-integrability objection). **Objection:** "The plane field $x \mapsto \beta(x)$ is generically non-integrable. Local sheets cannot be glued without accumulating mass."

Response: This objection conflates two different things:

(a) *Integrating a plane field* into a single foliation (which requires the Frobenius condition);

(b) *Building many separate calibrated sheets* whose tangent planes locally approximate a given decomposition.

The construction does (b), not (a). We are *not* trying to find a submanifold whose tangent planes equal $\beta(x)$ everywhere—that would indeed require integrability. Instead:

- On each cube Q, we decompose $\beta(x_Q)$ as a convex combination of calibrated planes via Carathéodory.
- We build finitely many *separate, disjoint* calibrated complete intersections through Q, each with a *constant* tangent plane (up to ε-error on the small cube).
- The complete intersections are algebraic subvarieties—they exist by Bertini's theorem, regardless of whether β is integrable.

The non-integrability of β as a plane field is irrelevant because we never integrate it. The "gluing" step (Theorem 8.27, Substep 4.2) uses Federer–Fleming to fill boundary mismatches. The key estimate is formulated in *flat norm*:

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\leq\; \varepsilon_{\mathrm{glue}}(m, \delta, \varepsilon, \mathrm{mesh}) \cdot m,$$

This is the robust target because the individual face mismatches can have large mass even when there is strong cancellation. Concretely, by the dual characterization of $\mathcal{F}$ and Stokes, for every smooth $(2n-2p-1)$-form η with $\|\eta\|_{\mathrm{comass}} \leq 1$ and $\|d\eta\|_{\mathrm{comass}} \leq 1$ one has

$$\partial T^{\mathrm{raw}}(\eta) = T^{\mathrm{raw}}(d\eta) \approx \int_X (m\beta) \wedge d\eta.$$

Since β is closed and X has no boundary, $\int_X (m\beta) \wedge d\eta = \pm \int_X d(m\beta \wedge \eta) = 0$. Thus the remaining task is to make the approximation error quantitative in terms of $(\delta, \varepsilon, \mathrm{mesh}, m)$; see Remark 8.83. Once $\mathcal{F}(\partial T^{\mathrm{raw}})$ is small, the correction current R_{glue} is produced by the flat-norm decomposition and the Federer–Fleming isoperimetric inequality as in Substep 4.2. The smoothness of β is essential here—it ensures the local decompositions are compatible across cube boundaries.

Remark 8.94 (Why the construction succeeds). The SYR construction succeeds because it exploits three key facts:

1. **Algebraic density:** By Bergman kernel asymptotics, any calibrated plane at any point can be approximated by the tangent plane of an algebraic complete intersection (Proposition 8.17).
2. **Carathéodory decomposition:** Any cone-valued form $\beta(x)$ is a finite convex combination of calibrated planes, with uniformly bounded number of terms (Lemma 8.11).
3. **Federer–Fleming compactness:** Integral cycles with bounded mass converge to integral cycles, preserving rectifiability.

The construction builds integral cycles T_k that are finite unions of algebraic subvarieties. The limit T is again an integral current (by Federer–Fleming), and it is ψ-calibrated (by the mass equality argument in Substep 6.4). Harvey–Lawson then identifies T as a positive sum of complex subvarieties.

Critically, the form β is *never* the limit current. The limit T is an algebraic cycle whose *existence* is guaranteed by compactness, whose *homology class* is $\mathrm{PD}(m[\gamma])$ by construction, and whose *calibrated structure* follows from the mass equality.

Automatic SYR: summary theorem

Theorem 8.95 (Automatic SYR for cone-valued forms (conditional on microstructure))**.** *Let (X, ω) be a smooth projective Kähler manifold of complex dimension n, and let $1 \leq p \leq n$. Every smooth closed cone-valued (p,p)-form β representing a rational Hodge class $[\gamma]$ satisfies the Stationary Young-measure Realizability property: there exist ψ-calibrated integral $(2n-2p)$-cycles T_k with $\partial T_k = 0$ and*

(i) $\mathbf{M}(T_k) \to m \int_X \beta \wedge \psi$,

(ii) *the tangent-plane Young measures of T_k converge a.e. to a measurable field ν_x supported on complex $(n-p)$-planes with barycenter $\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x)$,*

(iii) $[T_k] = \mathrm{PD}(m[\gamma])$ *for some fixed $m \in \mathbb{N}$ independent of k.*

Consequently, there exists a ψ-calibrated integral current T representing $\mathrm{PD}(m[\gamma])$. By Harvey–Lawson, T is integration along a positive sum of complex analytic subvarieties; hence $[\gamma]$ is algebraic.

Proof. Assume the Microstructure/flat-norm gluing theorem (Theorem 1.5). Apply it to the effective class $[\gamma]$ and its cone-valued representative β. This produces a sequence of closed integral ψ-calibrated currents in the desired homology class with the stated mass and Young-measure properties. Harvey–Lawson and Chow then yield algebraicity. □

Signed decomposition: the unconditional step

The preceding machinery applies to *effective* classes—those admitting cone-valued representatives. The following lemma shows that *every* rational Hodge class reduces to this case.

Definition 8.96 (Effective class)**.** A cohomology class $\gamma \in H^{2p}(X, \mathbb{R}) \cap H^{p,p}(X)$ is called *effective* if there exists a smooth closed (p,p)–form β representing γ such that $\beta(x) \in K_p(x)$ for all $x \in X$.

Lemma 8.97 (Positivity of the Kähler power)**.** *The (p,p)–form ω^p is strictly positive: for all $x \in X$, $\omega^p(x) \in \operatorname{int} K_p(x)$. In the Hermitian model, $\omega^p(x)$ corresponds to a positive definite matrix $W(x)$ with $\lambda_{\min}(W(x)) \geq c_0 > 0$ for some constant $c_0 > 0$ depending only on (X, ω).*

Proof. At each point x, choose unitary coordinates so that $\omega(x) = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$. Then $\omega^p(x)$ is a positive linear combination of simple (p,p)–forms, each corresponding to a rank-one PSD matrix in the Hermitian model. The sum is strictly positive definite. By compactness of X and smoothness of ω, the minimum eigenvalue is uniformly bounded below. □

Lemma 8.98 (Signed Decomposition)**.** *Let $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be any rational Hodge class. Then there exist effective classes γ^+ and γ^- such that*

$$\gamma \;=\; \gamma^+ - \gamma^-.$$

Moreover, both γ^+ and γ^- are rational Hodge classes, and γ^- can be taken to be a positive rational multiple of $[\omega^p]$.

Proof. Let α be any smooth closed (p,p)–form representing γ. In the Hermitian model at each $x \in X$, $\alpha(x)$ corresponds to a Hermitian matrix $A(x)$. Define

$$M \;:=\; \sup_{x \in X} \bigl|\lambda_{\min}(A(x))\bigr| \;<\; \infty,$$

which is finite by compactness of X and smoothness of α.

By Lemma 8.97, $\omega^p(x)$ corresponds to $W(x)$ with $\lambda_{\min}(W(x)) \geq c_0 > 0$. Choose $N > M/c_0$. Then for all $x \in X$:

$$\lambda_{\min}\big(A(x) + N \cdot W(x)\big) \;\geq\; \lambda_{\min}(A(x)) + N \cdot \lambda_{\min}(W(x)) \;\geq\; -M + Nc_0 \;>\; 0.$$

Thus $A(x) + N \cdot W(x)$ is positive definite, hence $\alpha(x) + N \cdot \omega^p(x) \in K_p(x)$ for all $x \in X$.

Define $\gamma^+ := \gamma + N \cdot [\omega^p]$ and $\gamma^- := N \cdot [\omega^p]$. Then $\gamma = \gamma^+ - \gamma^-$ by construction, γ^+ is effective (represented by the cone-valued form $\alpha + N \cdot \omega^p$), γ^- is effective (represented by $N \cdot \omega^p$), and both are rational Hodge classes since $[\omega^p] = c_1(L)^p$ is rational for the ample bundle L. □

Lemma 8.99 (γ^- is algebraic)**.** *On a smooth projective variety $X \subset \mathbb{P}^M$ with hyperplane class $H = c_1(\mathcal{O}(1)|_X)$, the class $[\omega^p] = H^p$ is algebraic, represented by a complete intersection of p generic hyperplane sections.*

Proof. By Bertini's theorem, for generic hyperplanes $H_1, \dots, H_p$ in $\mathbb{P}^M$, the intersection $Z := X \cap H_1 \cap \cdots \cap H_p$ is a smooth subvariety of codimension p in X. Its fundamental class $[Z] \in H_{2n-2p}(X, \mathbb{Z})$ satisfies $\mathrm{PD}([Z]) = H^p = [\omega^p]$. Thus $[\omega^p]$ is algebraic, and $\gamma^- = N \cdot [\omega^p]$ is algebraic for any rational $N > 0$. □

Theorem 8.100 (Effective classes are algebraic (conditional on microstructure))**.** *Let $\gamma^+ \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be an effective rational Hodge class on a smooth projective Kähler manifold. Then γ^+ is algebraic.*

Proof. Since γ^+ is effective, it admits a cone-valued representative β with $\beta(x) \in K_p(x)$ for all x. By Theorem 8.95, β is SYR–realizable. Thus there exists a sequence of integral cycles T_k with $[T_k] = \mathrm{PD}(m[\gamma^+])$ and $\mathbf{M}(T_k) \to c_0$. By Theorem 8.1, a subsequence converges to a ψ–calibrated integral current T, which by Harvey–Lawson is a positive sum of complex analytic subvarieties, hence algebraic by Chow's theorem. □

Main theorem: Hodge conjecture (conditional on microstructure)

Theorem 8.101 (Hodge Conjecture for rational (p,p) classes (conditional))**.** *Let X be a smooth projective Kähler manifold. Every rational Hodge class $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ is algebraic.*

Proof. By Lemma 8.98, write $\gamma = \gamma^+ - \gamma^-$ where γ^+ and $\gamma^- = N[\omega^p]$ are both effective rational Hodge classes.

By Lemma 8.99, γ^- is algebraic: it is represented by a complete intersection Z^-.

By Theorem 8.100, γ^+ is algebraic: it is represented by an algebraic cycle Z^+ obtained from the calibration–coercivity/SYR construction.

Therefore:

$$\gamma \;=\; \gamma^+ - \gamma^- \;=\; [Z^+] - [Z^-],$$

where $Z^+ - Z^-$ denotes the formal difference in the group of algebraic cycles tensored with $\mathbb{Q}$. Hence γ is algebraic. □

Corollary 8.102 (Full Hodge conjecture (conditional))**.** *Every rational (p,p) class on a smooth projective Kähler manifold is represented by an algebraic cycle.*

Proof. This is exactly Theorem 8.101. □

Remark 8.103 (Why signed decomposition is the key)*.* The signed decomposition sidesteps the fundamental obstruction that the harmonic representative γ_{harm} of a general Hodge class need not be cone-valued. For classes like $[\pi_1^*\omega_1]-[\pi_2^*\omega_2]$ on a product surface, the harmonic form has indefinite signature everywhere. We do *not* claim that every Hodge class has a cone-valued representative; we only use that every Hodge class is a *difference* of two that do. This is trivially achieved by adding a large multiple of $[\omega^p]$, which is strictly positive.