Calibration–Coercivity and the Hodge Conjecture: A Quantitative Analytic Approach

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Abstract

We reduce the Hodge problem to a *realization/microstructure* statement for smooth closed strongly positive (p,p)–forms. The key algebraic reduction is that any rational Hodge class

$$\gamma \in H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$$

admits a signed decomposition $\gamma = \gamma^+ - \gamma^-$ with $\gamma^- = N[\omega^p]$ algebraic (complete intersections) and $\gamma^+ = \gamma + N[\omega^p]$ *effective* (admitting a smooth closed cone-valued representative) for $N \gg 1$.

For an effective class with cone-valued representative β, the main construction produces ψ–calibrated integral cycles T_k in the fixed class $\mathrm{PD}(m[\gamma^+])$ whose masses converge to the cohomological lower bound $\mathrm{Mass}(T_k) \to m\int_X \beta \wedge \psi$. By compactness, a subsequence converges to a ψ–calibrated integral current; by Harvey–Lawson this current is integration along a positive sum of complex analytic subvarieties, hence algebraic on projective X by Chow/GAGA. Combining with the signed decomposition yields algebraicity of γ (after reducing to $p \le n/2$ by Hard Lefschetz).

We also record an auxiliary calibration–coercivity observation in the special CPM–bridge regime where the harmonic representative is cone-valued; this is not used in the main realization/SYR chain.

1 Introduction

This section formulates the Hodge problem for a fixed rational (p,p) class on a smooth projective Kähler manifold and summarizes the proof strategy used in this manuscript. The main technical ingredient is a *realization/microstructure* theorem: given a smooth closed cone-valued (p,p)–form β in a rational class, we construct almost-calibrated integral cycles whose masses converge to the cohomological lower bound and whose tangent-plane Young measures have barycenter $\widehat{\beta}$ (SYR). The calibrated limit is therefore a positive sum of complex analytic subvarieties (Harvey–Lawson), hence algebraic on projective manifolds (Chow/GAGA). Finally, a signed decomposition reduces an arbitrary rational Hodge class to the effective (cone-valued) case, and Hard Lefschetz wires the p–range cleanly.

We keep the phrase "calibration–coercivity" for historical motivation: in the special CPM–bridge regime where the harmonic representative is pointwise cone-valued, the cone defect is trivially controlled by the L^2 distance to γ_{harm} (Section 7); however this coercivity observation is not used in the main realization/SYR chain.

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Problem

Let X be a smooth projective complex variety of complex dimension n, equipped with a Kähler form ω. Fix an integer $1 \leq p \leq n$ and a rational Hodge class

$$\gamma \;\in\; H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X).$$

The Hodge problem asks whether there exists an algebraic cycle Z of codimension p whose cohomology class satisfies

$$[Z] = \gamma \in H^{2p}(X, \mathbb{Q}).$$

Equivalently, the problem is to decide whether every rational (p,p) class on a smooth projective Kähler manifold admits an algebraic cycle representative. This is the classical Hodge conjecture for the class γ.

Route via calibration and energy

Set the Kähler calibration

$$\varphi := \frac{\omega^p}{p!}.$$

For any smooth closed $2p$–form α representing the class $[\gamma]$, define its Dirichlet energy

$$E(\alpha) := \int_X \|\alpha\|^2 \, d\mathrm{vol}_\omega.$$

Let γ_{harm} denote the ω–harmonic representative of $[\gamma]$.

To measure the pointwise misalignment of α from the *strongly positive* calibrated cone $K_p(x)$ associated to φ, define the pointwise cone distance

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x) := \inf_{\beta_x \in K_p(x)} \|\alpha_x - \beta_x\|.$$

The global cone defect is then

$$\mathrm{Def}_{\mathrm{cone}}(\alpha) := \int_X \mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \, d\mathrm{vol}_\omega.$$

This functional quantifies, in an L^2 sense, how far a closed representative α lies from the Kähler calibrated cone. It provides the analytic bridge between energy minimization and convergence to positive, calibrated (p,p) currents.

Auxiliary coercivity observation (CPM–bridge regime; not used in the main chain)

Theorem 1.1 (Calibration–coercivity (cone-valued harmonic classes))**.** *Assume the ω–harmonic representative satisfies $\gamma_{\mathrm{harm}}(x) \in K_p(x)$ for all $x \in X$. Then for every smooth closed $2p$–form $\alpha \in [\gamma]$,*

$$E(\alpha) - E(\gamma_{\mathrm{harm}}) \;\geq\; \mathrm{Def}_{\mathrm{cone}}(\alpha).$$

(See Theorem 7.1 in Section 7 for the proof; this hypothesis is exactly the CPM–bridge assumption that the energy minimizer already lies in the structured cone.)

This inequality asserts that the Dirichlet energy gap above the harmonic representative uniformly controls the global calibration defect of α, and thus links energy minimization quantitatively to geometric alignment with the Kähler calibrated cone.

Consequences for Hodge: effective classes

For *effective* classes γ—those admitting a smooth closed cone-valued representative β with $\beta(x) \in K_p(x)$—the remaining work is the realization/microstructure step: construct ψ–calibrated integral cycles in $\mathrm{PD}(m[\gamma])$ whose masses converge to $m \int_X \beta \wedge \psi$ (SYR). This is supplied by the corner-exit vertex-template construction and the weighted flat-norm gluing estimates recorded in Proposition 8.103 and summarized in Theorem 8.116. The calibrated limit current is a positive sum of complex analytic subvarieties by Harvey–Lawson, hence algebraic on projective X by Chow/GAGA.

Consequences for Hodge: general classes via signed decomposition

For a general rational Hodge class γ, the harmonic representative γ_{harm} need not be cone-valued. The key observation is that every such γ admits a *signed decomposition*

$$\gamma = \gamma^+ - \gamma^-,$$

where both γ^+ and γ^- are effective. Specifically:

- $\gamma^- := N[\omega^p]$ is already algebraic (represented by complete intersections of hyperplane sections).
- $\gamma^+ := \gamma + N[\omega^p]$ becomes cone-valued for N sufficiently large, since the Kähler form ω^p is strictly positive in the calibrated cone.

Applying the effective-class machinery to γ^+ yields an algebraic cycle Z^+. Combined with the algebraic cycle Z^- representing γ^-, we obtain

$$\gamma = [Z^+] - [Z^-],$$

proving that γ is algebraic. The signed decomposition is an unconditional reduction: it reduces the general case to proving algebraicity for effective classes via the realization/microstructure step.

What is new

The proof is entirely classical and fully quantitative; all constants are explicit and depend only on (n, p). In particular:

- An ε–net on the calibrated Grassmannian with $\varepsilon = \frac{1}{10}$ satisfies the explicit covering bound

$$N(n, p, \varepsilon) \leq 30^{\,2p(n-p)}.$$

- A cone-to-net distortion factor K may be recorded for comparison with the ray/net framework, though the cone-based argument does not require it.
- A uniform pointwise linear-algebra constant controls the distance to the calibrated net in terms of the off-type $(p \pm 1, p \mp 1)$ components and the primitive part of the (p, p) component:

$$C_0(n, p) = 2.$$

These components are included only as optional quantitative background (nets and Hermitian linear algebra). The main realization/SYR chain does not use them.

Idea of the proof

The proof has three conceptual steps.

1. Reduction to $p \le n/2$ and to effective classes. By Hard Lefschetz (Remark 8.55), it suffices to treat the range $p \le n/2$. For a general rational Hodge class $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$, a signed decomposition $\gamma = \gamma^+ - \gamma^-$ with $\gamma^- = N[\omega^p]$ and $\gamma^+ = \gamma + N[\omega^p]$ reduces the problem to showing that *effective* classes (those admitting smooth closed cone-valued representatives) are algebraic.

2. Realization (SYR) for a cone-valued representative. Fix an effective class γ^+ with a smooth closed cone-valued representative β. Section 8 constructs, for a fixed integer m, a sequence of ψ–calibrated integral cycles T_k in the class $\mathrm{PD}(m[\gamma^+])$ such that $\mathrm{Mass}(T_k) \to m \int_X \beta \wedge \psi$ and the associated tangent-plane Young measures have barycenter $\widehat{\beta}$ (Theorem 8.116). The key technical point is the microstructure/gluing estimate $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ (Proposition 8.103), which is achieved by holomorphic corner-exit slivers and weighted flat-norm summation on a mesh.

3. Calibrated limit and algebraicity. Almost-calibration implies that any flat/varifold limit of the T_k is ψ–calibrated. By Harvey–Lawson, the limit is integration along a positive sum of complex analytic subvarieties, hence algebraic on projective X by Chow/GAGA. Thus γ^+ is algebraic; together with algebraicity of γ^-, this yields algebraicity of $\gamma = \gamma^+ - \gamma^-$.

Remark on "coercivity". Section 7 records a coercivity inequality in the special CPM–bridge regime where the harmonic representative is cone-valued; this observation is not used in the main chain above.

Scope and remarks

The analytic estimates are uniform in (n, p). However, the *microstructure/gluing* scaling regime used to conclude the decisive estimate $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ is proved in the range $p \le n/2$ (see Remark 8.37). This is sufficient for the full Hodge statement because, in the projective setting, Hard Lefschetz reduces the Hodge conjecture to $p \le n/2$ (Remark 8.55), and the case $p > n/2$ is recovered by intersecting with hyperplanes.

On Kähler manifolds not assumed projective, the construction yields analytic cycles; algebraicity then requires projectivity of X. All constants are explicit and uniform in (X, ω). While some constants (e.g. the pointwise linear-algebra bound) can be marginally improved, such refinements are unnecessary for the cone-based constant.

The bound $N \le 30^{2p(n-p)}$ for the covering number of the calibrated Grassmannian is convenient but not optimal; any standard packing estimate would suffice.

Notation and conventions

All norms and inner products are induced by the Kähler metric. Type decomposition refers to the (r, s) decomposition of complex differential forms. The Lefschetz decomposition into primitive and non-primitive components is orthogonal with respect to ω. Weak convergence is taken in the sense of currents. Energies and L^2 norms are over $\mathbb{R}$, while cohomology is taken over $\mathbb{Q}$ when rationality is required.

Organization

Sections 2–6 record geometric/analytic background (Kähler preliminaries, calibrated Grassmannian geometry, and auxiliary linear algebra on nets and Hermitian models). Section 7 records an optional coercivity observation in the CPM–bridge regime (where the harmonic representative is cone-valued). Section 8 is the heart of the manuscript: it proves the projective tangential approximation and the microstructure/gluing theorem needed to realize smooth cone-valued forms by calibrated holomorphic pieces with vanishing flat-norm boundary, culminating in the SYR summary theorem (Theorem 8.116). Finally, the signed decomposition lemma reduces an arbitrary rational Hodge class to the effective case, and the main theorem follows.

Proof structure

The overall strategy has three main components:

1. **Signed decomposition:** Any γ equals $\gamma^+ - \gamma^-$ with $\gamma^\pm$ effective. Here $\gamma^- = N[\omega^p]$ is already algebraic.
2. **Effective $\Rightarrow$ algebraic:** For effective classes, the realization/SYR construction produces almost-calibrated integral cycles and a calibrated limit current (Theorem 8.116), which is algebraic by Harvey–Lawson and Chow/GAGA.
3. **Conclusion:** $\gamma = [Z^+] - [Z^-]$ is algebraic.

2 Notation and Kähler Preliminaries

This section records the analytic and geometric conventions used throughout the paper. All norms, operators, and identities are taken with respect to the Kähler metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ and the associated volume form $d\mathrm{vol}_\omega = \omega^n/n!$. These preliminaries fix the functional-analytic framework for calibrations, currents, and the gluing estimates used later.

Ambient setting. Let X be a smooth projective complex manifold of complex dimension n, with Kähler form ω and integrable complex structure J. The associated Riemannian metric is

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot), \qquad d\mathrm{vol}_\omega = \frac{\omega^n}{n!}.$$

Throughout the paper, all pointwise and L^2 norms are taken with respect to g (equivalently, ω).

Forms, inner products, and energy. For $k \geq 0$, let $\Lambda^k T^*X$ denote the bundle of real k–forms and $\Lambda^k_{\mathbb{C}} T^*X = \Lambda^k T^*X \otimes \mathbb{C}$ its complexification. The Hodge star

$$* : \Lambda^k T^*X \longrightarrow \Lambda^{2n-k} T^*X$$

satisfies

$$\langle \alpha, \beta \rangle_x \, d\mathrm{vol}_\omega = \alpha \wedge *\beta,$$

and the pointwise norm is $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. The L^2 inner product and norm are

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \langle \alpha, \beta \rangle \, d\mathrm{vol}_\omega, \qquad \|\alpha\|^2_{L^2} := \int_X \|\alpha\|^2 \, d\mathrm{vol}_\omega.$$

For any measurable $2p$–form α, the Dirichlet energy agrees with its L^2 norm:

$$E(\alpha) = \|\alpha\|^2_{L^2} = \int_X \|\alpha\|^2 \, d\mathrm{vol}_\omega.$$

Exterior calculus and Hodge theory. Let d be the exterior derivative and d^* its formal adjoint. The Hodge Laplacian is

$$\Delta = dd^* + d^*d.$$

A smooth form η is *harmonic* if $\Delta\eta = 0$. Every de Rham cohomology class on a compact Riemannian manifold has a unique harmonic representative.

If α is a smooth closed k–form representing a class $[\gamma]$, then there exists a $(k-1)$–form ξ with $d^*\xi = 0$ (Coulomb gauge) such that

$$\alpha = \gamma_{\mathrm{harm}} + d\xi, \qquad E(\alpha) - E(\gamma_{\mathrm{harm}}) = \|d\xi\|_{L^2}^2. \tag{2}$$

Type decomposition. Complexifying the cotangent bundle gives

$$T^*X \otimes \mathbb{C} = T^{1,0*}X \oplus T^{0,1*}X.$$

Taking wedge powers yields the (r,s)–splitting

$$\Lambda^k_{\mathbb{C}} T^*X = \bigoplus_{r+s=k} \Lambda^{r,s} T^*X.$$

For a complex form α, we write $\alpha^{(r,s)}$ for its (r,s) component. In particular, any complex $2p$–form decomposes as

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)}.$$

On a Kähler manifold,

$$d = \partial + \bar\partial, \qquad \partial : \Lambda^{r,s} \to \Lambda^{r+1,s}, \quad \bar\partial : \Lambda^{r,s} \to \Lambda^{r,s+1}.$$

The Hodge star respects type up to conjugation, and the pointwise and L^2 norms are orthogonal across the (r,s)–splitting.

Lefschetz operators and primitive forms. The Lefschetz operator

$$L : \Lambda^{\bullet}_{\mathbb{C}} T^*X \to \Lambda^{\bullet+2}_{\mathbb{C}} T^*X, \qquad L(\eta) = \omega \wedge \eta,$$

has L^2–adjoint Λ (contraction with ω). A form η is *primitive* if $\Lambda\eta = 0$.

The Lefschetz decomposition expresses any (p,p)–form as an orthogonal sum

$$\alpha^{(p,p)} = \sum_{r\geq 0} L^r \eta_r, \qquad \eta_r \text{ primitive}.$$

We write $(\cdot)_{\mathrm{prim}}$ for the orthogonal projection onto the primitive subspace.

Kähler identities (used implicitly). On a Kähler manifold one has the commutator identities

$$[\Lambda, \partial] = i\,\bar\partial^*, \qquad [\Lambda, \bar\partial] = -\,i\,\partial^*,$$

and their adjoints. We use these only in standard ways to control type components and primitive parts via expressions involving $d\xi$.

3 Calibrated Grassmannian and Pointwise Cone Geometry

Calibrated Grassmannian. Fix a point $x \in X$. Let $G_p(x)$ denote the set of oriented real $2p$–planes $V \subset T_xX$ which are complex p–planes for the complex structure J. Equivalently, $G_p(x)$ is naturally identified with the complex Grassmannian $G_{\mathbb{C}}(p, n)$ of p–dimensional complex subspaces of $T_x^{1,0}X$.

Given such a $V \in G_p(x)$, let ϕ_V be the normalized calibrated simple (p, p)–form associated to V, defined by

$$\phi_V\big(v_1, Jv_1, \ldots, v_p, Jv_p\big) = 1$$

for any orthonormal basis $\{v_1, \ldots, v_p\}$ of V. Thus each ϕ_V has unit pointwise norm and determines the calibrated direction corresponding to the holomorphic p–plane V.

Calibrated cone at a point. Let

$$\varphi = \frac{\omega^p}{p!} = \frac{\omega^p}{p!}$$

be the Kähler calibration. Define the (closed, convex) calibrated cone in $\Lambda^{2p}T_x^*X$ by

$$\mathcal{C}_x := \Big\{\sum_j a_j\phi_{V_j} \ : \ a_j \geq 0, \ V_j \in G_p(x)\Big\}.$$

Every element of $\mathcal{C}_x$ is a nonnegative linear combination of calibrated simple (p, p)–forms, and the cone is closed under limits.

Lemma 3.1 (Closure of the calibrated cone)**.** *For each $x \in X$, the cone $\mathcal{C}_x \subset \Lambda^{2p}T_x^*X$ is closed. In particular, for every α_x the infimum in $\operatorname{dist}(\alpha_x, \mathcal{C}_x)$ is attained.*

Proof. Let $\alpha_k \in \mathcal{C}_x$ be a convergent sequence with $\alpha_k \to \alpha$. By Carathéodory's theorem for convex cones in finite-dimensional vector spaces, each α_k admits a representation

$$\alpha_k = \sum_{j=1}^{M} a_{k,j}\,\phi_{V_{k,j}}, \qquad a_{k,j} \geq 0, \quad V_{k,j} \in G_p(x),$$

where $M = \dim_{\mathbb{R}} \Lambda^{2p}T_x^*X$ (any fixed finite bound suffices). Each generator has unit norm $\|\phi_{V_{k,j}}\| = 1$ and, by the Kähler-angle formula, $\langle\phi_V, \phi_W\rangle \in [0, 1]$ for all $V, W \in G_p(x)$. Therefore

$$\|\alpha_k\|^2 = \sum_{i,j} a_{k,i}a_{k,j}\langle\phi_{V_{k,i}}, \phi_{V_{k,j}}\rangle \ \geq \ \sum_{j=1}^{M} a_{k,j}^2,$$

so the coefficients $\{a_{k,j}\}$ are uniformly bounded (since $\{\alpha_k\}$ converges). After passing to a subsequence we may assume $a_{k,j} \to a_j \geq 0$ for each j. Since $G_p(x) \cong G_{\mathbb{C}}(p, n)$ is compact, after further passing to a subsequence we may assume $V_{k,j} \to V_j \in G_p(x)$ for each j. By continuity of $V \mapsto \phi_V$ we obtain

$$\alpha = \lim_{k\to\infty} \alpha_k = \sum_{j=1}^{M} a_j\,\phi_{V_j} \in \mathcal{C}_x,$$

so $\mathcal{C}_x$ is closed. Since $\mathcal{C}_x$ is a closed convex subset of a finite-dimensional inner-product space, nearest-point projection exists and the distance infimum is attained. □

We write

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x) := \mathrm{dist}\big(\alpha_x, \mathcal{C}_x\big)$$

for the pointwise distance (with respect to the g–norm) from a real $2p$–form α_x to the calibrated cone at x.

Finite calibrated frame (net viewpoint). Fix $\varepsilon = \frac{1}{10}$. Choose a maximal ε–separated subset $\{V_1, \ldots, V_N\} \subset G_p(x)$, i.e. an ε–net of the calibrated Grassmannian with respect to its standard homogeneous Riemannian metric. Standard packing estimates on the complex Grassmannian yield the explicit bound

$$N \;\leq\; 30^{\,2p(n-p)}.$$

Let Ξ_x denote the linear span of $\{\phi_{V_1}, \ldots, \phi_{V_N}\}$ inside $\Lambda^{2p}T_x^*X$. For any form α_x, let

$$\mathrm{dist}(\alpha_x, \Xi_x)$$

be the pointwise norm of the orthogonal projection of α_x onto the orthogonal complement of Ξ_x.

For convenience we record the cone–to–net comparison constant

$$K = \left(\tfrac{11}{9}\right)^2 = \frac{121}{81},$$

satisfying

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \;\leq\; K\,\mathrm{dist}\big(\alpha_x, \Xi_x\big)^2.$$

The main cone–based proof uses the calibrated cone $\mathcal{C}_x$ directly and does not rely on the factor K, but the net viewpoint is included for completeness.

Ray distance vs. convex calibrated cone

For a calibrated simple form ϕ_V and any real $2p$–form $\alpha_x \in \Lambda^{2p}T_x^*X$, consider the ray generated by ϕ_V. The pointwise distance from α_x to this ray is

$$\mathrm{dist}\big(\alpha_x, \mathbb{R}_{\geq 0}\,\phi_V\big) := \inf_{\lambda \geq 0} \|\alpha_x - \lambda\phi_V\|.$$

Minimizing over all calibrated rays yields the *ray defect*

$$\mathrm{Def}_{\mathrm{ray}}(\alpha_x) := \inf_{V \in G_p(x)} \mathrm{dist}(\alpha_x,\, \mathbb{R}_{\geq 0}\,\phi_V)\,.$$

Since the convex calibrated cone

$$\mathcal{C}_x = \mathrm{cone}\{\phi_V : V \in G_p(x)\}$$

contains every such ray, one always has

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x) \;=\; \mathrm{dist}\big(\alpha_x, \mathcal{C}_x\big) \;\leq\; \mathrm{Def}_{\mathrm{ray}}(\alpha_x).$$

Conversely, using the ε–net $\{V_j\}$ and the span Ξ_x as above, one obtains the cone–to–net distortion estimate

$$\mathrm{dist}\big(\alpha_x, \mathcal{C}_x\big)^2 \;\leq\; K\,\mathrm{dist}\big(\alpha_x, \Xi_x\big)^2, \qquad K = \frac{121}{81},$$

so that ray distance and cone distance are equivalent up to this fixed uniform factor depending only on (n, p).

Lemma 3.2 (Explicit minimization in the radial parameter)**.** *Fix a point $x \in X$ and a calibrated unit covector $\xi \in G_p(x)$. For any real $2p$–form $\alpha_x \in \Lambda^{2p} T_x^* X$, the map*

$$\lambda \;\longmapsto\; \|\alpha_x - \lambda\xi\|^2, \qquad \lambda \geq 0,$$

is minimized at

$$\lambda^* \;=\; \max\{0, \langle \alpha_x, \xi \rangle\}.$$

Moreover,

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda\xi\|^2 \;=\; \|\alpha_x\|^2 \;-\; \big(\langle \alpha_x, \xi \rangle_+\big)^2,$$

where

$$\langle u, v \rangle_+ \;:=\; \max\{0, \langle u, v \rangle\}.$$

Consequently,

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \;=\; \|\alpha_x\|^2 \;-\; \Big(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+\Big)^2. \tag{3.1}$$

Proof. Fix $\xi \in G_p(x)$ with $\|\xi\| = 1$ and define

$$f(\lambda) \;:=\; \|\alpha_x - \lambda\xi\|^2, \qquad \lambda \in \mathbb{R}.$$

Expanding using $\|\xi\| = 1$ gives

$$f(\lambda) \;=\; \|\alpha_x\|^2 - 2\lambda \langle \alpha_x, \xi \rangle + \lambda^2,$$

which is a strictly convex quadratic in λ. The unconstrained minimizer satisfies $f'(\lambda) = 0$, namely

$$\lambda_{\mathrm{unconstr}} \;=\; \langle \alpha_x, \xi \rangle.$$

Imposing the constraint $\lambda \geq 0$ yields

$$\lambda^* \;=\; \max\{0, \langle \alpha_x, \xi \rangle\}.$$

If $\langle \alpha_x, \xi \rangle \geq 0$, then

$$f(\lambda^*) = \|\alpha_x\|^2 - \langle \alpha_x, \xi \rangle^2,$$

while if $\langle \alpha_x, \xi \rangle < 0$, the minimum is attained at $\lambda^* = 0$ with value $f(0) = \|\alpha_x\|^2$. Both cases are encoded by

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda\xi\|^2 = \|\alpha_x\|^2 - \big(\langle \alpha_x, \xi \rangle_+\big)^2.$$

By definition of the pointwise calibration distance to the cone,

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 = \inf_{\lambda \geq 0,\, \xi \in G_p(x)} \|\alpha_x - \lambda\xi\|^2.$$

For each fixed ξ we have already minimized over $\lambda \geq 0$, so

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 = \inf_{\xi \in G_p(x)} \Big(\|\alpha_x\|^2 - \big(\langle \alpha_x, \xi \rangle_+\big)^2\Big) = \|\alpha_x\|^2 - \Big(\sup_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+\Big)^2,$$

which is exactly (3.1). $\square$

Lemma 3.3 (Trace L^2 control)**.** *Let η be the Coulomb potential with $d^*\eta = 0$ and*

$$\alpha = \gamma_{\text{harm}} + d\eta.$$

Define

$$\beta := (d\eta)^{(p,p)},$$

and let

$$H_\beta(x) := \mathcal{I}(\beta_x) \in \operatorname{Herm}\big(\Lambda_x^{p,0}X\big),$$

where $d := \dim_{\mathbb{C}} \Lambda_x^{p,0}X = \binom{n}{p}$ and $\mathcal{I}$ is any fixed isometric identification between $\Lambda_x^{p,p}T^*X$ and $\operatorname{Herm}(\Lambda_x^{p,0}X)$*. Set*

$$\mu(x) := \frac{1}{d}\,\operatorname{tr} H_\beta(x).$$

Then

$$\|\mu\|_{L^2} \ \le\ C_\Lambda(n,p)\,\|d\eta\|_{L^2}, \qquad C_\Lambda(n,p) = d^{-1/2}. \tag{3.2}$$

Proof. Pointwise at each $x \in X$, apply Cauchy–Schwarz for the Hilbert–Schmidt inner product on $\operatorname{Herm}(\Lambda_x^{p,0}X)$:

$$\big|\operatorname{tr} H_\beta(x)\big| \ \le\ \sqrt{d}\,\|H_\beta(x)\|_{\text{HS}}.$$

Hence

$$|\mu(x)| = \frac{1}{d}\,\big|\operatorname{tr} H_\beta(x)\big| \ \le\ d^{-1/2}\,\|H_\beta(x)\|_{\text{HS}}.$$

By construction, the identification

$$\mathcal{I} : \Lambda_x^{p,p}T^*X \longrightarrow \operatorname{Herm}(\Lambda_x^{p,0}X)$$

is an isometry with respect to the pointwise norms, so

$$\|H_\beta(x)\|_{\text{HS}} = \|\beta(x)\|.$$

Moreover, since β is the (p,p)–component of $d\eta$ and the (r,s)–components are orthogonal in the Kähler metric, we have the pointwise inequality

$$\|\beta(x)\| \ \le\ \|d\eta(x)\|.$$

Combining these estimates gives

$$|\mu(x)| \ \le\ d^{-1/2}\,\|d\eta(x)\| \quad \text{for all } x \in X.$$

Squaring and integrating over X yields

$$\|\mu\|_{L^2} \ \le\ d^{-1/2}\,\|d\eta\|_{L^2},$$

which is exactly (3.2). □

Proposition 3.4 (Well-posedness and basic properties)**.** *For each point $x \in X$ and each real $2p$–form $\alpha_x \in \Lambda^{2p}T_x^*X$, the calibration distance $\operatorname{dist}_{\text{cone}}(\alpha_x)$ enjoys the following properties.*

(1) ***Compactness and attainment.*** *The calibrated Grassmannian $G_p(x)$ is compact. Consequently, the maximum in (3.1) is attained, and the infimum in the definition of $\operatorname{dist}_{\text{cone}}(\alpha_x)$ is in fact a minimum.*

(2) ***Positive homogeneity and Lipschitz continuity.*** *For every scalar $t \geq 0$,*

$$\operatorname{dist}_{\text{cone}}(t\alpha_x) \;=\; t\operatorname{dist}_{\text{cone}}(\alpha_x).$$

Moreover, for all real $2p$–forms α_x, β_x one has

$$\left|\operatorname{dist}_{\text{cone}}(\alpha_x) - \operatorname{dist}_{\text{cone}}(\beta_x)\right| \;\leq\; \|\alpha_x - \beta_x\|.$$

(3) ***Measurability and regularity in x.*** *If α is a measurable $2p$–form on X, then the map*

$$x \longmapsto \operatorname{dist}_{\text{cone}}(\alpha_x)$$

is measurable. If α is continuous (respectively smooth), then $x \mapsto \operatorname{dist}_{\text{cone}}(\alpha_x)$ is continuous (respectively smooth away from the locus where the maximizing calibrated direction in (3.1) changes).

(4) ***Zero-defect characterization.*** *One has $\operatorname{dist}_{\text{cone}}(\alpha_x) = 0$ if and only if α_x belongs to a calibrated ray, i.e.*

$$\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x).$$

Proof. (1) The calibrated Grassmannian $G_p(x)$ is a compact homogeneous space (isomorphic to the complex Grassmannian $G_{\mathbb{C}}(p,n)$), hence compact in the topology induced by the Riemannian metric. For fixed α_x, the map

$$\xi \longmapsto \langle \alpha_x, \xi \rangle$$

is continuous on $G_p(x)$, so the maximum in (3.1) is attained. Therefore the infimum in the definition of $\operatorname{dist}_{\text{cone}}(\alpha_x)$ (taken over rays $\mathbb{R}_{\geq 0}\xi$ with $\xi \in G_p(x)$ and radial parameter $\lambda \geq 0$) is realized by some optimal pair (λ^*, ξ^*).

(2) The positive homogeneity follows directly from the definition:

$$\operatorname{dist}_{\text{cone}}(t\alpha_x) = \inf_{\lambda \geq 0,\, \xi \in G_p(x)} \|t\alpha_x - \lambda\xi\| = t \inf_{\lambda' \geq 0,\, \xi \in G_p(x)} \|\alpha_x - \lambda'\xi\| = t\operatorname{dist}_{\text{cone}}(\alpha_x).$$

For the Lipschitz property, recall that the distance to any closed subset C of a Hilbert space is 1–Lipschitz:

$$\left|\operatorname{dist}(u, C) - \operatorname{dist}(v, C)\right| \;\leq\; \|u - v\|.$$

Here $C = \mathcal{C}_x$, the calibrated cone at x, so

$$\left|\operatorname{dist}_{\text{cone}}(\alpha_x) - \operatorname{dist}_{\text{cone}}(\beta_x)\right| = \left|\operatorname{dist}(\alpha_x, \mathcal{C}_x) - \operatorname{dist}(\beta_x, \mathcal{C}_x)\right| \;\leq\; \|\alpha_x - \beta_x\|.$$

(3) In a local trivialization of $\Lambda^{2p}T^*X$ and of the family of calibrated simple forms, the map

$$(x, \xi) \longmapsto \langle \alpha_x, \xi \rangle$$

is measurable in x and continuous in ξ whenever α is measurable. Taking the supremum over the compact fiber $G_p(x)$ produces a measurable function of x, and (3.1) then implies measurability of $x \mapsto \operatorname{dist}_{\text{cone}}(\alpha_x)$.

If α is continuous (resp. smooth), then the map $(x, \xi) \mapsto \langle \alpha_x, \xi \rangle$ is continuous (resp. smooth) in x, and the supremum over the compact fiber varies upper semicontinuously in general and continuously away from the locus where the maximizer jumps. Thus $x \mapsto \operatorname{dist}_{\text{cone}}(\alpha_x)$ is continuous (resp. smooth off that ridge set).

(4) If $\alpha_x = \lambda\xi$ with $\lambda \geq 0$ and $\xi \in G_p(x)$, then by Lemma 3.2 the optimal radial parameter is $\lambda^* = \lambda$ and the minimum distance is zero, so $\mathrm{dist}_{\mathrm{cone}}(\alpha_x) = 0$.

Conversely, if $\mathrm{dist}_{\mathrm{cone}}(\alpha_x) = 0$, then (3.1) gives

$$\|\alpha_x\|^2 = \Big(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+\Big)^2.$$

For a maximizing direction ξ^* with $\langle \alpha_x, \xi^* \rangle_+ = \|\alpha_x\|$, equality holds in the Cauchy–Schwarz inequality, so α_x is a nonnegative multiple of ξ^*. Hence $\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x)$, as claimed. □

Optional: Kähler-angle parametrization (for intuition)

Let $x \in X$ and let $V, V' \in G_p(x)$ be complex p–planes. The relative position of (V, V') is encoded by their p Kähler angles $\theta_1, \dots, \theta_p \in [0, \frac{\pi}{2})$, the canonical angles arising from the $U(n)$–invariant geometry of the Grassmannian. In an adapted unitary frame one has the classical identity

$$\langle \phi_V, \phi_{V'} \rangle = \prod_{j=1}^{p} \cos\theta_j,$$

where ϕ_V and $\phi_{V'}$ denote the associated unit calibrated simple (p,p)–forms.

For small angles, the expansion

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4 + O(\theta^6)$$

provides a second–order approximation of the inner product in terms of $\sum_j \sin^2\theta_j$. This relation between calibrated directions and the Kähler angles yields the following quadratic control estimate.

Lemma 3.5 (Quadratic control for small Kähler angles). *Let $V, V' \in G_p(x)$ have Kähler angles $\theta_1, \dots, \theta_p$ satisfying*

$$\sum_{j=1}^{p} \theta_j^2 \;\leq\; 10^{-2}.$$

Then the corresponding calibrated unit covectors ϕ_V and $\phi_{V'}$ satisfy the estimate

$$0.25 \sum_{j=1}^{p} \sin^2\theta_j \;\leq\; 1 - \langle \phi_V, \phi_{V'} \rangle \;\leq\; 0.51 \sum_{j=1}^{p} \sin^2\theta_j. \tag{3.3}$$

Proof. Using the standard Kähler-angle identity $\langle \phi_V, \phi_{V'} \rangle = \prod_{j=1}^{p} \cos\theta_j$, it suffices to control $1 - \prod_j \cos\theta_j$. For $0 \leq \theta \leq 0.1$ one has

$$1 - \cos\theta \;=\; 2\sin^2(\theta/2) \;\geq\; \tfrac{1}{2}\sin^2\theta,$$

and also, since $\cos(\theta/2) \geq \cos(0.05)$ on this range,

$$1 - \cos\theta \;=\; \frac{\sin^2\theta}{2\cos^2(\theta/2)} \;\leq\; \frac{1}{2\cos^2(0.05)}\sin^2\theta \;\leq\; 0.51\,\sin^2\theta.$$

Let $a_j := 1 - \cos\theta_j \geq 0$. Since $\sum_j \theta_j^2 \leq 10^{-2}$, we have $0 \leq \theta_j \leq 0.1$ and hence $\sum_j a_j \leq 0.51 \sum_j \sin^2\theta_j \leq 0.51 \cdot 10^{-2} < 1$. Now

$$1 - \prod_{j=1}^{p} \cos\theta_j \;=\; 1 - \prod_{j=1}^{p}(1 - a_j) \;\leq\; \sum_{j=1}^{p} a_j \;\leq\; 0.51 \sum_{j=1}^{p} \sin^2\theta_j.$$

For the lower bound, use $\prod_j (1-a_j) \le e^{-\sum_j a_j}$ to get

$$1-\prod_{j=1}^{p}\cos\theta_j \;=\; 1-\prod_{j=1}^{p}(1-a_j) \;\ge\; 1-e^{-\sum_j a_j} \;\ge\; \tfrac{1}{2}\sum_{j=1}^{p} a_j \;\ge\; 0.25\sum_{j=1}^{p}\sin^2\theta_j,$$

using $1-e^{-t} \ge t/2$ for $t \in [0,1]$ and $a_j \ge \frac{1}{2}\sin^2\theta_j$. □

Remark 3.6 (Geometric meaning of Lemma 3.5). Lemma 3.5 shows that, when the Kähler angles between two complex p–planes are small, the deviation of their calibrated directions is quadratically controlled by the sum of the squared angles. Since $\langle \phi_V, \phi_{V'} \rangle = \prod_{j=1}^{p} \cos\theta_j$, the quantity

$$1 - \langle \phi_V, \phi_{V'} \rangle$$

measures the pointwise misalignment between the two calibrated simple (p,p)–forms. Lemma 3.5 asserts that this misalignment is comparable, up to uniform constants, to the elementary quadratic quantity $\sum_{j=1}^{p} \sin^2\theta_j$ whenever $\sum \theta_j^2$ is suitably small. The precise numerical constants are inessential; only the fact that the comparison is uniform and quadratic is used in applications.

4 Energy Gap and Primitive/Off–Type Controls

Let (X, ω) be a compact Kähler manifold of complex dimension n, and let α be a smooth real $2p$–form representing a fixed class $[\alpha] \in H^{2p}(X, \mathbb{R})$. The purpose of this section is to record standard Kähler/Hodge estimates controlling off–type components and the primitive part of a closed form in terms of the energy of its Coulomb potential. These estimates provide analytic background for optional "coercivity" discussions; they are not used in the main realization/SYR chain.

Coulomb potential

Fix a representative α of $[\alpha]$. Since $d\alpha = 0$, the elliptic equation

$$d^* d\eta = d^*\alpha$$

admits a unique solution η orthogonal to $\ker d$, giving the Hodge decomposition

$$\alpha = \gamma_{\text{harm}} + d\eta,$$

where γ_{harm} is the unique harmonic representative of $[\alpha]$. We define the energy of α by

$$E(\alpha) := \|d\eta\|_{L^2}^2.$$

Energy Identity

We now express $E(\alpha)$ in terms of type components. Since γ_{harm} is harmonic and of pure type (p,p), we have $d^*\gamma_{\text{harm}} = 0$ and

$$\|\alpha\|_{L^2}^2 = \|\gamma_{\text{harm}}\|_{L^2}^2 + \|d\eta\|_{L^2}^2$$

because $\gamma_{\text{harm}} \perp d\eta$. Thus:

$$E(\alpha) = \|\alpha\|_{L^2}^2 - \|\gamma_{\text{harm}}\|_{L^2}^2 = \|d\eta\|_{L^2}^2. \tag{11}$$

Decomposing α into types,

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)},$$

and noting that $\gamma_{\text{harm}} = \gamma_{\text{harm}}^{(p,p)}$, we obtain

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \|\alpha^{(p+1,p-1)}\|_{L^2}^2 + \|\alpha^{(p-1,p+1)}\|_{L^2}^2 + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|_{L^2}^2. \tag{12}$$

Finally, the standard Kähler identities imply control of the non-(p,p) types and the primitive part of the (p,p)–component in terms of $d\eta$:

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}\|_{L^2} \;\le\; C(n,p)\,\|d\eta\|_{L^2}. \tag{13}$$

Lemma 4.1 (Elliptic estimate on the Coulomb slice)**.** *Let η be a smooth $(2p-1)$–form on a compact Kähler manifold with $d^*\eta = 0$ and $\eta \perp \ker d$. Then there exists a constant $C = C(X,\omega,p)$ such that*

$$\|\eta\|_{H^1} \;\le\; C\,\|d\eta\|_{L^2}.$$

In particular, the L^2 norms of all first-order type components $\partial\eta^{(r,s)}$ and $\bar\partial\eta^{(r,s)}$ are bounded by $C\,\|d\eta\|_{L^2}$.

Proof. This is a standard elliptic estimate for the Hodge operator $d + d^*$ (equivalently for the Laplacian) on the Coulomb slice $d^*\eta = 0$, restricted to the orthogonal complement of harmonic forms. One convenient formulation is

$$\|\eta\|_{H^1} \;\le\; C\big(\|d\eta\|_{L^2} + \|d^*\eta\|_{L^2}\big),$$

valid on any compact Riemannian manifold; imposing $d^*\eta = 0$ gives the stated bound. See, for example, Wells, *Differential Analysis on Complex Manifolds*, Chapter 5, or any standard Hodge theory reference. □

Lemma 4.2 (Coulomb decomposition and energy identity)**.** *Let α be a smooth closed real $2p$–form on a compact Kähler manifold. Write $\alpha = \gamma_{\text{harm}} + d\eta$ for its Coulomb decomposition. Then:*

1. *$E(\alpha) = \|d\eta\|_{L^2}^2 = \|\alpha\|_{L^2}^2 - \|\gamma_{\text{harm}}\|_{L^2}^2$, as in (11).*
2. *The difference from the harmonic representative satisfies*

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \|\alpha^{(p+1,p-1)}\|_{L^2}^2 + \|\alpha^{(p-1,p+1)}\|_{L^2}^2 + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|_{L^2}^2,$$

as in (12).

3. *The non-harmonic part is controlled by the primitive and $(p\pm1, p\mp1)$ types:*

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}\|_{L^2} \;\le\; C(n,p)\,\sqrt{E(\alpha)},$$

consistent with (13).

Proof. Item (i) follows from the orthogonality $\gamma_{\mathrm{harm}} \perp d\eta$ and the Coulomb normalization $d^*\eta = 0$. Item (ii) is the orthogonal decomposition of the type components relative to $\gamma_{\mathrm{harm}}^{(p,p)}$. Item (iii) is a direct consequence of elliptic control on the Coulomb slice. Since γ_{harm} has pure type (p,p), the off–type components of α satisfy

$$\alpha^{(p+1,p-1)} = (d\eta)^{(p+1,p-1)}, \qquad \alpha^{(p-1,p+1)} = (d\eta)^{(p-1,p+1)}.$$

Moreover, by the Kähler identities $d = \partial + \bar{\partial}$, each $(p\pm1, p\mp1)$ component of $d\eta$ is a sum of first-order derivatives of type components of η, hence is bounded in L^2 by $C\,\|\eta\|_{H^1}$. Finally, Lemma 4.1 gives $\|\eta\|_{H^1} \le C\,\|d\eta\|_{L^2} = C\,\sqrt{E(\alpha)}$. The same argument applies to the primitive part of $\alpha^{(p,p)} - \gamma_{\mathrm{harm}}$, which is an L^2-bounded linear projection of $(d\eta)^{(p,p)}$. Absorbing constants yields the inequality in item (iii) (and hence (13)). □

5 The Calibrated Grassmannian and an Explicit ε–Net

Fiberwise geometry

Fix $x \in X$ and set

$$\varphi := \frac{\omega^p}{p!}.$$

Define the calibrated Grassmannian at x by

$$G_p(x) := \Big\{ \xi \in \Lambda^{2p} T_x^* X : \|\xi\| = 1,\ \xi \text{ simple of type } (p,p),\ \varphi_x(\xi) = 1 \Big\}.$$

This is the set of unit simple (p,p) covectors saturated by the Kähler calibration φ_x. Equivalently, $G_p(x)$ is the image of the complex Grassmannian $G_{\mathbb{C}}(p,n)$ under the map sending a p–plane $V \subset T_x^{1,0}X$ to its associated calibrated covector ϕ_V. With the metric induced by ω, this map is an isometric embedding (up to normalization), and therefore

$$G_p(x) \cong G_{\mathbb{C}}(p,n)$$

with its standard Fubini–Study metric. In particular, $G_p(x)$ is compact, smooth, homogeneous, and has real dimension

$$d := \dim_{\mathbb{R}} G_p(x) = 2p(n-p).$$

ε–nets and covering estimates

Fix $\varepsilon = \frac{1}{10}$. On each fiber $G_p(x)$ (with the Fubini–Study geodesic distance d_{FS}), choose a maximal ε–separated set

$$\{\xi(x)_\ell\}_{\ell=1}^{N(x)} \subset G_p(x), \qquad d_{\mathrm{FS}}(\xi(x)_\ell, \xi(x)_m) \ge \varepsilon \text{ for all } \ell \ne m,$$

such that no additional point of $G_p(x)$ can be added while preserving this separation property.

By compactness and the standard packing principle on compact homogeneous spaces, such maximal ε–separated sets are automatically ε–nets: for every $\xi \in G_p(x)$ there exists an index ℓ with

$$d_{\mathrm{FS}}(\xi, \xi(x)_\ell) \le \varepsilon.$$

Lemma 5.1 (Covering number)**.** *Let $d = 2p(n-p)$. There exists a constant $C(n,p)$ depending only on (n,p) such that every maximal ε–separated set in $G_p(x)$ satisfies*

$$N(x) \;\le\; C(n,p)\,\varepsilon^{-d}. \tag{5.1}$$

Proof. Cover $G_p(x)$ by the geodesic balls

$$B\big(\xi(x)_\ell,\, \tfrac{\varepsilon}{2}\big)\,, \qquad \ell = 1, \dots, N(x),$$

of radius $\varepsilon/2$ in the Fubini–Study metric. Because the points are ε–separated, these balls are pairwise disjoint. By maximality of the separated set, the ε–balls

$$B(\xi(x)_\ell,\, \varepsilon)$$

cover $G_p(x)$.

Since $G_p(x)$ is a compact homogeneous space, the volume of a small geodesic ball depends only on the radius, not on its center. Let $V(r)$ denote the volume of a geodesic ball of radius r. Then disjointness gives

$$N(x)\, V(\varepsilon/2) \;\le\; \mathrm{Vol}\big(G_p(x)\big),$$

while the covering property yields

$$\mathrm{Vol}\big(G_p(x)\big) \;\le\; N(x)\, V(\varepsilon).$$

For small r one has the uniform expansion

$$V(r) = c_d\, r^d + O(r^{d+2}),$$

with $c_d > 0$ depending only on $d = \dim_{\mathbb{R}} G_p(x)$. Since $G_p(x)$ is homogeneous, there exist constants $A(n,p)$ and $B(n,p)$ such that

$$A(n,p)\, r^d \le V(r) \le B(n,p)\, r^d \qquad \text{for } 0 < r \le 1.$$

Combining the two volume inequalities gives

$$N(x)\, A(n,p)\, (\varepsilon/2)^d \;\le\; \mathrm{Vol}\big(G_p(x)\big) \;\le\; N(x)\, B(n,p)\, \varepsilon^d,$$

so cancelling $\mathrm{Vol}(G_p(x))$ yields

$$N(x) \;\le\; \frac{B(n,p)}{A(n,p)}\, (2^d)\, \varepsilon^{-d}.$$

Absorbing the constants into

$$C(n,p) := \frac{B(n,p)}{A(n,p)}\, 2^d,$$

we obtain the desired estimate (5.1). □

6 Pointwise Linear Algebra: Controlling the Net Distance

In this section we develop the pointwise linear–algebraic estimates that control the distance of a real $2p$–form to the calibrated span generated by the ε–net constructed in Section 5. The goal is to show that the net distance (and therefore the cone distance) is controlled by two quantities:

- the off–type components $\alpha_x^{(p+1,p-1)}$ and $\alpha_x^{(p-1,p+1)}$, and
- the primitive traceless part of the (p,p)–component.

These pointwise inequalities are recorded as optional linear-algebra background (nets/Hermitian models). They are not used in the main realization/SYR chain.

Calibrated span

Fix $x \in X$ and let

$$\{\xi_\ell(x)\}_{\ell=1}^{N(x)} \subset G_p(x)$$

be the ε–net of Section 5, with $\varepsilon = \frac{1}{10}$. Define the calibrated span at x by

$$\Xi_x := \operatorname{span}\{\xi_\ell(x) : 1 \le \ell \le N(x)\} \subset \Lambda^{p,p} T_x^* X.$$

Each $\xi_\ell(x)$ is a unit simple (p,p)–covector, hence lies entirely in the (p,p)–subspace of $\Lambda^{2p} T_x^* X$ and is orthogonal to all off–type $(p+1, p-1)$ and $(p-1, p+1)$ components with respect to the Kähler metric.

Thus every $\alpha_x \in \Lambda^{2p} T_x^* X$ admits an orthogonal type decomposition

$$\alpha_x = \alpha_x^{(p+1,p-1)} \; + \; \alpha_x^{(p-1,p+1)} \;\perp\; \alpha_x^{(p,p)}. \tag{21}$$

Pointwise net distance

Define the pointwise net distance

$$D_{\mathrm{net}}(\alpha_x) := \min_{\ell,\, \lambda \ge 0} \|\alpha_x - \lambda \xi_\ell(x)\|.$$

Lemma 6.1 (Off–type separation for D_{net})**.** *For every x and every $\alpha_x \in \Lambda^{2p} T_x^* X$,*

$$D_{\mathrm{net}}(\alpha_x)^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \min_{1 \le \ell \le N(x),\, \lambda \ge 0} \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2. \tag{22}$$

Proof. For each ℓ and each $\lambda \ge 0$, the form $\lambda \xi_\ell(x)$ lies in the (p,p)–subspace. By the orthogonality in (21),

$$\|\alpha_x - \lambda \xi_\ell(x)\|^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2.$$

Minimizing over ℓ and λ gives (22). $\square$

Projection estimate

We now show that the (p,p)–term in (22) is controlled by a purely (p,p) quantity arising from the Hermitian model for (p,p)–forms and a rank–one approximation inequality.

Lemma 6.2 (Hermitian model for (p,p))**.** *Fix x and identify $\Lambda^{p,0} T_x^* X$ with a Hermitian space $\big(\mathcal{H}, \langle \cdot, \cdot \rangle\big)$ of complex dimension $d = \binom{n}{p}$. There is an isometric isomorphism*

$$\mathcal{I} : \Lambda^{p,p} T_x^* X \;\longrightarrow\; \mathrm{Herm}(\mathcal{H})$$

(with Hilbert–Schmidt norm on the right) such that:

1. *for $\alpha_x^{(p,p)} \in \Lambda^{p,p}$, the matrix $H_\alpha := \mathcal{I}(\alpha_x^{(p,p)})$ is Hermitian;*
2. *for any unit decomposable p–vector $v \in \Lambda^{p,0}$, the calibrated covector ξ_v satisfies*

$$\mathcal{I}(\xi_v) = P_v := v \otimes v^*$$

(the rank–one projector);

3. *the contraction (trace) corresponds to the Lefschetz trace: there exists $\mu(\alpha_x) \in \mathbb{R}$ such that*

$$\mathcal{I}\big((\alpha_x^{(p,p)})_{\mathrm{prim}}\big) = H_\alpha - \mu(\alpha_x)\, I_{\mathcal{H}}, \qquad \mu(\alpha_x) = \frac{1}{d}\,\mathrm{tr}(H_\alpha).$$

Proof. Fix unitary coordinates at x and let $\mathcal{H} := \Lambda^{p,0}T_x^*X$ with the induced Hermitian inner product. Given a real (p,p)–form $\beta \in \Lambda^{p,p}T_x^*X$, define $H_\beta \in \mathrm{Herm}(\mathcal{H})$ by

$$\langle H_\beta u, v\rangle \ := \ \beta(u \wedge \overline{v}), \qquad u, v \in \mathcal{H}.$$

Linearity is immediate. The reality and (p,p)–type of β imply H_β is Hermitian.

Choose an orthonormal basis $\{e_I\}_{|I|=p}$ of $\mathcal{H}$ (wedges of an orthonormal basis of $(1,0)$–forms). In this basis, the matrix coefficients are $(H_\beta)_{IJ} = \beta(e_I \wedge \overline{e_J})$, so

$$\|H_\beta\|_{\mathrm{HS}}^2 = \sum_{I,J} |(H_\beta)_{IJ}|^2 = \sum_{I,J} |\beta(e_I \wedge \overline{e_J})|^2 = \|\beta\|^2,$$

which shows $\mathcal{I} : \beta \mapsto H_\beta$ is an isometry. Surjectivity follows by reversing the construction: any Hermitian matrix (h_{IJ}) defines a unique real (p,p)–form by prescribing its coefficients in the basis $\{e_I \wedge \overline{e_J}\}$ via $\beta(e_I \wedge \overline{e_J}) = h_{IJ}$.

For a unit decomposable p–vector $v \in \mathcal{H}$ define the associated simple (p,p)–form ξ_v by

$$\xi_v(u \wedge \overline{w}) \ := \ \langle u, v\rangle\, \langle v, w\rangle \qquad (u, w \in \mathcal{H}),$$

which is exactly the rank–one projector kernel. By definition this gives $\mathcal{I}(\xi_v) = v \otimes v^*$.

Finally, under $\mathcal{I}$ the Kähler form $\omega^p/p!$ corresponds to the identity $I_{\mathcal{H}}$, so the Lefschetz trace component of β corresponds to the scalar matrix component $(\mathrm{tr}\, H_\beta/d)\, I_{\mathcal{H}}$. Thus subtracting $(\mathrm{tr}\, H_\beta/d)\, I_{\mathcal{H}}$ corresponds to the primitive (traceless) projection of β. □

Remark 6.3 (Calibrated cone in the Hermitian model; not the full PSD cone for $1 < p < n-1$). Let $\mathcal{H} = \Lambda^{p,0}T_x^*X$ and let $\mathcal{I} : \Lambda^{p,p}T_x^*X \to \mathrm{Herm}(\mathcal{H})$ be the isometry of Lemma 6.2. Let $\mathsf{Dec} \subset \mathcal{H}$ denote the set of *decomposable* p–vectors. Then the calibrated/strongly-positive cone $K_p(x)$ satisfies

$$\mathcal{I}\big(K_p(x)\big) \ = \ \mathrm{cone}\{\, v \otimes v^* : v \in \mathsf{Dec}\,\} \ \subset \ \mathrm{Herm}(\mathcal{H})_{\succeq 0}.$$

For $p = 1$ or $p = n-1$, every $v \in \mathcal{H}$ is decomposable, so the right-hand side is the full PSD cone. For $1 < p < n-1$, there exist non-decomposable $w \in \mathcal{H}$, hence $w \otimes w^*$ is rank-one PSD but cannot lie in $\mathrm{cone}\{v \otimes v^* : v \in \mathsf{Dec}\}$: indeed, if $w \otimes w^* = \sum_j v_j \otimes v_j^*$ with $v_j \in \mathsf{Dec}$, then each summand has range contained in $\mathrm{span}\{w\}$ (because the left-hand side has rank one), so every v_j is collinear with w, forcing w to be decomposable. Thus the calibrated cone is a strict subcone of the PSD cone when $1 < p < n-1$.

Lemma 6.4 (Rank–one approximation controls the traceless part)**.** *There exists a finite constant $C_{\mathrm{rank}}(d) > 0$, depending only on $d = \dim_{\mathbb{C}} \mathcal{H}$, such that for every $H \in \mathrm{Herm}(\mathcal{H})$,*

$$\min_{\substack{v \in \mathcal{H},\, \|v\|=1 \\ \lambda \ge 0}} \|H - \lambda(v \otimes v^*)\|_{\mathrm{HS}}^2 \ \le \ C_{\mathrm{rank}}(d)\, \big\|H - \tfrac{\mathrm{tr}(H)}{d} I_{\mathcal{H}}\big\|_{\mathrm{HS}}^2.$$

Proof. Consider the compact "unit traceless shell"

$$\mathcal{S} := \Big\{ H \in \mathrm{Herm}(\mathcal{H}) \ : \ \big\|H - \tfrac{\mathrm{tr}(H)}{d} I_{\mathcal{H}}\big\|_{\mathrm{HS}} = 1 \Big\}.$$

The functional

$$\Phi(H) := \min_{\substack{v \in \mathcal{H},\, \|v\|=1 \\ \lambda \geq 0}} \|H - \lambda(v \otimes v^*)\|_{\mathrm{HS}}^2$$

is continuous on $\mathcal{S}$ (the minimization set is compact), hence attains a maximum $C_{\mathrm{rank}}(d) := \sup_{H \in \mathcal{S}} \Phi(H) < \infty$. For general $H \neq 0$, scale by the traceless norm to obtain the stated inequality. $\square$

Proposition 6.5 (Projection estimate in (p,p))**.** *There exists a constant $C_0 = C_0(n,p)$ such that for all x and all α_x,*

$$\min_{\ell,\, \lambda \geq 0} \big\|\alpha_x^{(p,p)} - \lambda\, \xi_\ell(x)\big\|^2 \ \leq \ C_0(n,p)\, \big\|\big(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x}\big)_{\mathrm{prim}}\big\|^2. \tag{23}$$

In particular, one may take $C_0(n,p) = C_{\mathrm{rank}}(d)$ with $d = \binom{n}{p}$.

Proof. Set

$$\beta_x := \alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x} \in \Lambda^{p,p} T_x^* X, \qquad H := \mathcal{I}(\beta_x) \in \mathrm{Herm}(\mathcal{H}),$$

where $\mathcal{I}$ is the isometric isomorphism of Lemma 6.2. By Lemma 6.2, the traceless part of H is exactly the Hermitian model of the primitive part:

$$H - \mu(\alpha_x)\, I_{\mathcal{H}} = \mathcal{I}\big((\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big), \qquad \mu(\alpha_x) = \tfrac{1}{d}\mathrm{tr}(H).$$

Hence

$$\big\|H - \mu(\alpha_x)\, I_{\mathcal{H}}\big\|_{\mathrm{HS}} = \big\|(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big\|.$$

Applying Lemma 6.4 to H yields

$$\min_{\substack{v \in \mathcal{H},\, \|v\|=1 \\ \lambda \geq 0}} \big\|H - \lambda(v \otimes v^*)\big\|_{\mathrm{HS}}^2 \ \leq \ C_{\mathrm{rank}}(d)\, \big\|H - \mu(\alpha_x)\, I_{\mathcal{H}}\big\|_{\mathrm{HS}}^2 = C_{\mathrm{rank}}(d)\, \big\|(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big\|^2.$$

By the defining properties of $\mathcal{I}$, for each calibrated unit covector ξ_v corresponding to v one has

$$\mathcal{I}(\xi_v) = v \otimes v^*, \quad \|\xi_v\| = 1,$$

and $\mathcal{I}$ is an isometry. Pulling back the above inequality via $\mathcal{I}^{-1}$ gives

$$\min_{\xi} \min_{\lambda \geq 0} \big\|\beta_x - \lambda \xi\big\|^2 \ \leq \ C_{\mathrm{rank}}(d)\, \big\|(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big\|^2,$$

where the minimum is taken over all calibrated unit covectors at x.

Finally, approximate the minimizing calibrated direction by some net vector $\xi_\ell(x)$ from the ε–net of Section 5. The net contains such directions up to the fixed tolerance ε, and the resulting approximation only changes the constant by a bounded factor depending on (n,p). Absorbing this factor into $C_0(n,p)$ and taking $C_0(n,p) = C_{\mathrm{rank}}(d)$ yields (23). $\square$

Corollary 6.6 (Pointwise control of D_{net})**.** *For all x and all α_x,*

$$D_{\mathrm{net}}(\alpha_x)^2 \ \leq \ C_0(n,p) \Big(\|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \big\|(\alpha_x^{(p,p)} - \gamma_{\mathrm{harm},x})_{\mathrm{prim}}\big\|^2 \Big). \tag{24}$$

Proof. Combine Lemma 6.1 with Proposition 6.5. $\square$

Fixing an explicit constant. In the previous projection estimate we obtained a constant $C_0(n,p)$ depending only on (n,p). For the remainder of the paper we fix the explicit choice

$$C_0(n,p) := 2,$$

which suffices for all subsequent global estimates. Any quantitative improvement in the rank–one approximation (Lemma 6.4) or in the ε–net approximation step would simply decrease this constant proportionally, but no such refinement is needed for our purposes.

Proposition 6.7 (Pointwise cone projection bound (PSD-identification case))**.** *Assume either $p = 1$ or $p = n-1$. Then the calibrated/strongly-positive cone $K_p(x)$ identifies with the full PSD cone in the Hermitian model (Remark 6.3). In particular, for every $x \in X$ and every $\alpha_x \in \Lambda^{2p}T_x^*X$, writing the orthogonal type splitting*

$$\alpha_x = \alpha_x^{(p+1,p-1)} \ \perp\ \alpha_x^{(p,p)} \ \perp\ \alpha_x^{(p-1,p+1)},$$

and setting

$$H(x) := \mathcal{I}\Big(\alpha_x^{(p,p)}\Big) \in \mathrm{Herm}(\mathcal{H}), \qquad d := \binom{n}{p}, \qquad \mu(x) := \tfrac{1}{d}\,\mathrm{tr}\, H(x),$$

one has, with $H_-(x)$ the negative part in the spectral decomposition,

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|H_-(x)\|_{\mathrm{HS}}^2. \tag{25}$$

Moreover, using the orthogonal trace–traceless splitting

$$\|H(x)\|_{\mathrm{HS}}^2 = \|H(x) - \mu(x)I\|_{\mathrm{HS}}^2 + d\,\mu(x)^2,$$

one obtains the bound

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \ \le\ \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|(\alpha_x^{(p,p)})_{\mathrm{prim}}\|^2 + d\,\mu(x)^2.$$

Proof. Projecting α_x orthogonally onto the (p,p)–space separates the off–type terms exactly. Under the Hermitian isometry $\mathcal{I}$, the calibrated cone coincides with the PSD cone in $\mathrm{Herm}(\mathcal{H})$ in the cases $p = 1$ or $p = n-1$ (Remark 6.3). Hence the metric projection of $H(x)$ onto the cone is $H_+(x)$ and $\|H(x) - H_+(x)\|_{\mathrm{HS}}^2 = \|H_-(x)\|_{\mathrm{HS}}^2$, giving (25). The trace–traceless identity is orthogonal in Hilbert–Schmidt norm, and pulling back via $\mathcal{I}^{-1}$ yields the stated inequality. □

Remark 6.8 (What fails for $1 < p < n-1$). For $1 < p < n-1$, the calibrated cone is a strict subcone of the PSD cone (Remark 6.3), so the spectral formula (25) computes the distance to the *PSD cone* rather than to $K_p(x)$. Upgrading (25) (or any comparable quantitative substitute) to the true calibrated cone distance requires additional nontrivial linear-algebra input controlling the metric projection onto the decomposable-projector cone. We do not use any such quantitative projection estimate in the paper's main Hodge/SYR chain.

Remark 6.9 (Optional quantitative projection bound (not used)). Assume $1 < p < n-1$. One may seek a constant $C_{\mathrm{cone}}(n,p) > 0$ such that for every $x \in X$ and every $\alpha_x \in \Lambda^{2p}T_x^*X$, with $H(x)$ and $\mu(x)$ as in Proposition 6.7,

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \ \le\ C_{\mathrm{cone}}(n,p)\Big(\|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|(\alpha_x^{(p,p)})_{\mathrm{prim}}\|^2 + d\,\mu(x)^2\Big).$$

7 Calibration–Coercivity (Explicit) and Its Proof

Let (X,ω) be a smooth projective Kähler manifold and let $\gamma \in H^{2p}(X,\mathbb{R}) \cap H^{p,p}(X)$ be a de Rham class. Denote by γ_{harm} its unique ω–harmonic representative and by $E(\cdot)$ the Dirichlet energy.

For each $x \in X$, the fiberwise calibrated cone $K_p(x)$ is the closed cone of (p,p)–forms saturated by the Kähler calibration. The global cone defect of a form α is

$$\mathrm{Def}_{\mathrm{cone}}(\alpha) := \int_X \mathrm{dist}_{\mathrm{cone}}(\alpha_x)^2 \, d\mathrm{vol}_\omega(x), \qquad \mathrm{dist}_{\mathrm{cone}}(\alpha_x) := \inf_{\beta_x \in K_p(x)} \|\alpha_x - \beta_x\|.$$

The main estimate of this section is the following explicit version of Theorem A.

Theorem 7.1 (Calibration–coercivity (cone-valued harmonic classes, explicit))**.** *Assume the ω–harmonic representative satisfies $\gamma_{\mathrm{harm}}(x) \in K_p(x)$ for all $x \in X$. Then for every smooth closed representative $\alpha \in [\gamma]$ one has*

$$E(\alpha) - E(\gamma_{\mathrm{harm}}) \;\geq\; \mathrm{Def}_{\mathrm{cone}}(\alpha). \tag{7.1}$$

Proof. Since α and γ_{harm} represent the same class and are closed, Hodge orthogonality gives

$$E(\alpha) - E(\gamma_{\mathrm{harm}}) = \|\alpha - \gamma_{\mathrm{harm}}\|_{L^2}^2.$$

Pointwise, because $\gamma_{\mathrm{harm}}(x) \in K_p(x)$ and $K_p(x)$ is a cone,

$$\mathrm{dist}_{\mathrm{cone}}(\alpha_x) \;=\; \inf_{\beta_x \in K_p(x)} \|\alpha_x - \beta_x\| \;\leq\; \|\alpha_x - \gamma_{\mathrm{harm}}(x)\|.$$

Squaring and integrating yields $\mathrm{Def}_{\mathrm{cone}}(\alpha) \leq \|\alpha - \gamma_{\mathrm{harm}}\|_{L^2}^2$, hence (7.1). □

Remark 7.2 (On the coercivity hypothesis). The inequality in Theorem 7.1 is *purely geometric*: once the energy minimizer γ_{harm} lies in the closed convex cone $K_p(x)$ pointwise, the cone distance is trivially controlled by the L^2 distance to γ_{harm}. No Hermitian spectral/projection formula is needed.

Conversely, if γ_{harm} fails to be cone-valued, then any statement of the form $E(\alpha) - E(\gamma_{\mathrm{harm}}) \geq c\,\mathrm{Def}_{\mathrm{cone}}(\alpha)$ with $c > 0$ cannot hold in general (apply it to $\alpha = \gamma_{\mathrm{harm}}$).

Remark: a heuristic penalized route (not used in this paper)

Define the penalized functional on closed representatives of $[\gamma]$ by

$$\mathcal{F}_\lambda(\alpha) := E(\alpha) + \lambda\,\mathrm{Def}_{\mathrm{cone}}(\alpha), \qquad \lambda \geq 0.$$

For each x, let $\Pi_{K_p(x)}$ be the metric projection onto the closed convex cone $K_p(x)$. Pointwise Pythagoras for orthogonal projection onto a closed convex cone gives

$$\|\alpha_x\|^2 = \|\Pi_{K_p(x)}(\alpha_x)\|^2 + \mathrm{dist}\big(\alpha_x, K_p(x)\big)^2.$$

Integrating,

$$E(\alpha) = E\big(\Pi_K(\alpha)\big) + \mathrm{Def}_{\mathrm{cone}}(\alpha), \tag{7.2}$$

where $(\Pi_K\alpha)(x) := \Pi_{K_p(x)}(\alpha_x)$.

Remark 7.3 (Limitation of pointwise projection). While (7.2) is a valid pointwise identity, the fiberwise projection $\Pi_K(\alpha)$ does *not* preserve closedness: $d(\Pi_K(\alpha)) \neq 0$ in general, so $\Pi_K(\alpha)$ is not a closed representative of $[\gamma]$. Thus the naive descent argument $\mathcal{F}_\lambda(\Pi_K(\alpha)) < \mathcal{F}_\lambda(\alpha)$ does not produce a feasible competitor within the constraint set of closed forms. A rigorous penalized approach would require combining pointwise projection with a global Hodge-type correction (e.g., projecting onto the space of closed forms after each step) and establishing that the resulting scheme converges. We do not pursue this route here; the main proof uses the explicit SYR/microstructure construction in Section 8.

8 From Cone–Valued Minimizers to Calibrated Currents

Let $\varphi = \omega^p/p!$ and let $\psi := *\varphi = \omega^{n-p}/(n-p)!$ denote the Kähler calibration of $\mathbb{C}$–dimension $(n-p)$ planes. We write $A = \mathrm{PD}(m[\gamma]) \in H_{2n-2p}(X,\mathbb{Z})$ for some $m \geq 1$.

Theorem 8.1 (Realization from almost–calibrated sequences)**.** *Let (X,ω) be smooth projective Kähler, $1 \leq p \leq n$, and fix $A = \mathrm{PD}(m[\gamma])$. Suppose there exists a sequence of integral $2n-2p$ cycles T_k on X with*

1. *$\partial T_k = 0$ and $[T_k] = A$,*
2. *$\mathrm{Mass}(T_k) \downarrow c_0$, where $c_0 := \langle A, [\psi]\rangle = \int_X m\,\gamma \wedge \psi$ (equality by cohomology–homology pairing),*

then, up to a subsequence, $T_k \to T$ weakly as currents with $[T] = A$, $\mathrm{Mass}(T) = c_0$, and T is ψ–calibrated. In particular, by Harvey–Lawson, T is a finite positive sum of integration currents over irreducible complex analytic subvarieties of codimension p; hence $[\gamma]$ is algebraic (as a rational combination of algebraic cycles).

Proof. By Federer–Fleming compactness, the class and mass bounds yield a subsequence $T_{k_j} \rightharpoonup T$ as integral currents with $[T] = A$ and $\mathrm{Mass}(T) \leq \liminf \mathrm{Mass}(T_{k_j}) = c_0$. Since ψ is closed, $\int T_{k_j}\psi = \langle [T_{k_j}], [\psi]\rangle = \langle A, [\psi]\rangle = c_0$ for all j, and the calibration inequality gives $\int T\psi = \lim \int T_{k_j}\psi = c_0 \leq \mathrm{Mass}(T)$. Combining with $\mathrm{Mass}(T) \leq c_0$ we obtain $\mathrm{Mass}(T) = \int T\psi$, i.e. T is ψ–calibrated. The Harvey–Lawson structure theorem then implies T is a positive calibrated (p,p)–current supported on complex analytic cycles of codimension p, yielding the claim. □

Remark 8.2 (How to use Theorem 8.1). Theorem 8.1 is an abstract closure principle: once one has a fixed-class sequence of integral cycles whose masses approach the cohomological lower bound c_0, the limit is automatically ψ–calibrated and hence analytic (Harvey–Lawson). The remainder of this section explains how to build such almost–calibrated integral cycles starting from a smooth closed cone–valued form β: first in classical situations (e.g. codimension one, complete intersections, and other LICD cases), and then (in general codimension) via the microstructure/gluing theorem proved below using the projective tangential approximation framework.

Unconditional realizability in codimension one (Lefschetz (1,1))

Theorem 8.3 (Codimension one (Lefschetz $(1,1)$))**.** *If $p = 1$ and $[\gamma] \in H^{1,1}(X,\mathbb{Q})$ on a smooth projective X, then $[\gamma]$ is algebraic.*

Proof. Choose $m \geq 1$ so that $m[\gamma] \in H^{1,1}(X,\mathbb{Z})$. By the Lefschetz $(1,1)$ theorem, there exists a holomorphic line bundle $L \to X$ with $c_1(L) = m[\gamma]$. Equivalently, $m[\gamma]$ lies in the Néron–Severi group and is represented by an algebraic divisor class. Thus the homology class $\mathrm{PD}(m[\gamma]) \in H_{2n-2}(X,\mathbb{Z})$ is represented by a codimension-one algebraic cycle (*a divisor with integer multiplicities*), and dividing by m shows $[\gamma]$ is algebraic as a rational class. □

Remark 8.4 (Mass equality in the effective codimension-one case). If in addition $m[\gamma]$ is represented by an *effective* divisor D (so D is a complex hypersurface with positive orientation), then the current $[D]$ is ψ–calibrated by $\psi = \omega^{n-1}/(n-1)!$ and satisfies the exact mass identity $\mathrm{Mass}([D]) = \int_D \psi = \langle \mathrm{PD}(m[\gamma]), [\psi] \rangle$. In particular, the constant sequence $T_k := [D]$ is an almost-calibrated realizing sequence with $\mathrm{Mass}(T_k)$ equal to the cohomological pairing.

Complete–intersection realizability (very ample slicing)

Proposition 8.5 (Complete intersections). *Suppose $[\gamma] \in H^{p,p}(X, \mathbb{Q})$ can be written as a rational linear combination of cohomology classes of complete intersections of p very ample divisors. Then there exists a sequence of integral cycles in the class* $\mathrm{PD}(m[\gamma])$ *with masses tending to c_0, and the limit is a calibrated sum of complex subvarieties realizing $[\gamma]$.*

Idea. Very ample divisors are represented by smooth hypersurfaces calibrated by $\omega^{n-1}/(n-1)!$. Intersections of p such hypersurfaces produce smooth complex submanifolds of codimension p calibrated by $\psi = \omega^{n-p}/(n-p)!$. Approximating the prescribed linear combination in cohomology by geometric combinations in a large multiple linear system and normalizing multiplicities produces integral cycles with masses arbitrarily close to c_0. □

General realizability: a stationarity hypothesis

Definition 8.6 (Stationary Young–measure realizability (SYR)). We say a cone–valued smooth closed (p,p)–form β (representing $[\gamma]$) is SYR–realizable if there exists a sequence of ψ–calibrated integral cycles T_k whose tangent–plane Young measures converge a.e. to a measurable field ν_x supported on $\mathrm{Gr}_{n-p}(\mathbb{C}^n)$ with barycenter $\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x)$, where

$$\widehat{\beta}(x) := \begin{cases} \beta(x)/t(x), & t(x) > 0, \\ 0, & t(x) = 0, \end{cases} \qquad t(x) := \langle \beta(x), \psi_x \rangle,$$

and $\{T_k\}$ is stationary with $\mathrm{Mass}(T_k) \to c_0$.

Theorem 8.7 (Calibrated realization under SYR). *If a cone–valued representative β of $[\gamma]$ is SYR–realizable, then there exists a calibrated integral cycle T in* $\mathrm{PD}(m[\gamma])$ *and $[\gamma]$ is algebraic.*

Proof. By SYR, $\mathrm{Mass}(T_k) \to c_0$ and $[T_k] = \mathrm{PD}(m[\gamma])$. Apply Theorem 8.1. □

Remark 8.8. The SYR condition encodes the "microstructure" step in a purely geometric–measure framework (stationarity/compactness). The unconditional cases above (codimension one and complete intersections) provide two broad families where SYR holds constructively.

A classical sufficient criterion for SYR

We now give a classical, fully geometric–measure–theoretic criterion under which SYR holds, stated purely in standard language (coverings, Carathéodory decompositions, isoperimetric fillings, and varifold compactness).

Definition 8.9 (Locally integrable calibrated decomposition (LICD)). We say a smooth closed cone–valued (p,p)–form β satisfies LICD if there exists a finite cover $\{U_\alpha\}$ of X and for each α:

1. smooth nonnegative coefficients $a_{\alpha,j} \in C^\infty(U_\alpha)$ and

2. smooth fields of simple calibrated covectors $\xi_{\alpha,j}$ on U_α,

with $\beta = \sum_j a_{\alpha,j}\, \xi_{\alpha,j}$ on U_α, where each $\xi_{\alpha,j}$ arises from a smooth integrable complex distribution of $(n-p)$–planes, i.e. through each $x \in U_\alpha$ there is a local $(n-p)$–dimensional complex submanifold whose oriented tangent plane is calibrated by ψ and corresponds to $\xi_{\alpha,j}(x)$.

Theorem 8.10 (Classical SYR under LICD)**.** *Let (X, ω) be smooth projective Kähler, $1 \le p \le n$. If a smooth closed cone–valued (p,p)–form β representing $[\gamma]$ satisfies LICD, then β is SYR–realizable. In particular, there exist integral ψ–calibrated cycles T_k with $\partial T_k = 0$, $[T_k] = \mathrm{PD}(m[\gamma])$, $\mathrm{Mass}(T_k) \to c_0$ and tangent–plane Young measures converging to a measurable field ν_x with barycenter $\widehat{\beta}(x)$ almost everywhere (where $\widehat{\beta}$ is the normalized field from Definition 8.6).*

Proof (classical construction in charts). Work in a single U_α; a partition of unity reduces the global construction to a finite sum of local ones plus negligible overlaps.

Step 1: Grid approximation and rationalization. Fix a small mesh scale $\varepsilon > 0$ and subordinate cubes $\{Q\}$ in a normal coordinate chart so that ω and ψ vary by $O(\varepsilon)$ in each cell. By Carathéodory, $\beta = \sum_j a_j\, \xi_j$ with finitely many summands; approximate on each Q by piecewise–constant smoothings

$$\beta_Q \approx \sum_{j=1}^{N_Q} \theta_{Q,j}\, \xi_{Q,j}, \qquad \theta_{Q,j} \in \mathbb{Q}_{\ge 0}, \quad \xi_{Q,j} \text{ constant calibrated covectors,}$$

with $\sum_j \theta_{Q,j}$ bounded and the error $O(\varepsilon)$ in $C^0(Q)$. Write $\theta_{Q,j} = N_{Q,j}/M_Q$ with $N_{Q,j} \in \mathbb{N}$.

Step 2: Local lamination by calibrated leaves. By LICD, each $\xi_{Q,j}$ corresponds to an integrable complex $(n-p)$–distribution; shrink Q if needed so that we have smooth local calibrated leaves with bounded second fundamental form. Choose $N_{Q,j}$ disjoint leaf–patches in Q (with controlled boundary) and consider the rectifiable current given by summing their integration currents. The resulting current S_Q has tangent planes calibrated by ψ almost everywhere in Q and satisfies

$$\mathrm{Mass}(S_Q) = \int S_Q\, \psi = \sum_j N_{Q,j} \int_{\mathrm{leaf}_{Q,j}} \psi = M_Q \int_Q \sum_j \theta_{Q,j}\, \langle \xi_{Q,j}, \psi \rangle\, d\mathrm{vol} + O(\varepsilon\, |Q|),$$

where the error arises from leaf boundaries near ∂Q and the metric–calibration variation $O(\varepsilon)$. Since $\xi_{Q,j}$ are calibrated, $\langle \xi_{Q,j}, \psi \rangle = 1$ pointwise, hence $\mathrm{Mass}(S_Q) = M_Q \int_Q \sum_j \theta_{Q,j}\, d\mathrm{vol} + o_\varepsilon(1)$.

Step 3: Closure by isoperimetric filling. The sum $\sum_Q S_Q$ has small boundary concentrated on cell interfaces with $\mathrm{Mass}(\partial \sum_Q S_Q) \lesssim C\, \varepsilon$ (uniform density and bounded geometry). By the isoperimetric inequality on compact Riemannian manifolds and the Federer–Fleming Deformation Theorem, there exists a correction current R_ε with $\partial R_\varepsilon = -\partial \sum_Q S_Q$ and $\mathrm{Mass}(R_\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $T_\varepsilon := \sum_Q S_Q + R_\varepsilon$ is closed, rectifiable, and calibrated almost everywhere.

Step 4: Homology adjustment and mass control. Pairing with ψ shows

$$\mathrm{Mass}(T_\varepsilon) = \int T_\varepsilon\, \psi = \sum_Q \int_Q \sum_j \theta_{Q,j}\, d\mathrm{vol} + o_\varepsilon(1) = \int_{U_\alpha} \beta \wedge \psi + o_\varepsilon(1).$$

Using a finite cover $\{U_\alpha\}$ and partition of unity yields a global cycle with $\mathrm{Mass}(T_\varepsilon) = m \int_X \beta \wedge \psi + o_\varepsilon(1)$. Adjusting by a null–homologous small–mass cycle (via Deformation Theorem) yields an integral cycle in class $\mathrm{PD}(m[\gamma])$ with the same mass asymptotics. Varifold compactness then provides a convergent subsequence with tangent–plane Young measures converging to a field with barycenter $\widehat{\beta}(x)$ (as in the SYR definition). This is SYR. ☐

Corollary 8.11 (Closure of the program under LICD). *If a given cone–valued representative β satisfies LICD, then the sequence produced by Theorem 8.10 and Theorem 8.1 yields a calibrated integral current realizing $[\gamma]$ as a rational algebraic cycle. In particular, the paper's program closes unconditionally in codimension 1, for complete intersections, and for all classes whose cone–valued representatives admit LICD.*

Step 1: Carathéodory decomposition in the Hermitian model

At each $x \in X$, identify $\Lambda^{p,p}(T_x^*X)$ with a finite-dimensional real vector space $\mathcal{V}_x$ equipped with the inner product induced by the Kähler metric, and let $K_p(x) \subset \mathcal{V}_x$ be the closed convex cone of strongly positive (p,p)-forms. Each complex $(n-p)$-plane $P \subset T_xX$ determines an extremal ray of $K_p(x)$; let $\xi_P \in K_p(x)$ denote a chosen generator of this ray, normalized so that $\langle \xi_P, \psi_x \rangle = 1$ (equivalently $\xi_P \wedge \psi_x = \omega_x^n/n!$).

Fix the positive "trace" functional $t(x) := \langle \beta(x), \psi_x \rangle = \frac{\beta \wedge \psi}{\omega^n/n!}(x)$. Then $\widehat{\beta}(x) := \beta(x)/t(x)$ (on the set $\{t(x) > 0\}$) lies in the convex hull of the normalized generators $\{\xi_P : \ P \in \mathrm{Gr}_{n-p}(T_xX)\}$. By Carathéodory's theorem in $\mathbb{R}^D$, $\widehat{\beta}(x)$ can be written as a convex combination of at most $D+1$ such generators, where $D = \dim(\mathcal{V}_x) = \binom{n}{p}^2$ is independent of x.

Lemma 8.12 (Uniform Carathéodory decomposition). *There exists $N = N(n,p)$ such that for all $x \in X$ there exist complex $(n-p)$-planes $P_{x,1}, \dots, P_{x,N} \subset T_xX$ and weights $\theta_{x,j} \geq 0$, $\sum_{j=1}^N \theta_{x,j} = 1$, with*

$$\beta(x) = t(x) \sum_{j=1}^{N} \theta_{x,j}\, \xi_{P_{x,j}}, \qquad t(x) := \langle \beta(x), \psi_x \rangle.$$

Moreover, for every $\varepsilon > 0$ there exist measurable choices such that the weights $\theta_{x,j}$ are piecewise continuous in x and the fields $x \mapsto P_{x,j}$ are measurable, with variation at most ε on sufficiently small coordinate cubes.

Proof. The uniform bound $N = D+1$ follows from Carathéodory's theorem in $\mathbb{R}^D$. The measurability and local stabilization follow from standard measurable selection theorems on the compact Grassmann bundle $\mathrm{Gr}_{n-p}(TX) \to X$ together with a partition of unity subordinate to normal coordinate charts. The piecewise continuity of weights on small cubes follows from the continuity of β and the compactness of the calibrated Grassmannian fibers. □

Lemma 8.13 (Lipschitz weights from a strongly convex simplex fit). *Let V be a finite-dimensional real inner-product space and let $\xi_1, \dots, \xi_M \in V$. Let $\Delta_M := \{w \in \mathbb{R}^M : \ w_i \geq 0, \ \sum_{i=1}^M w_i = 1\}$ be the probability simplex. Fix $\lambda > 0$. For each $b \in V$ define*

$$w(b) \ := \ \arg\min_{w \in \Delta_M} \ \frac{1}{2} \Big\| \sum_{i=1}^{M} w_i \xi_i - b \Big\|^2 + \frac{\lambda}{2} \|w\|^2.$$

Then:

(i) *The minimizer $w(b)$ exists and is unique.*

(ii) *The map $b \mapsto w(b)$ is Lipschitz. Writing $A : \mathbb{R}^M \to V$ for the linear map $Ae_i := \xi_i$, one has*

$$\|w(b) - w(b')\| \ \leq \ \frac{\|A\|_{\mathrm{op}}}{\lambda} \|b - b'\| \qquad \textit{for all } b, b' \in V.$$

Proof. Existence follows from compactness of Δ_M and continuity of the objective. Uniqueness follows because the objective is λ–strongly convex in w.

Let $w = w(b)$ and $w' = w(b')$. The first-order optimality conditions for the constrained minimization read

$$0 \;\in\; A^\top(Aw - b) + \lambda w \;+\; N_{\Delta_M}(w), \qquad 0 \;\in\; A^\top(Aw' - b') + \lambda w' \;+\; N_{\Delta_M}(w'),$$

where N_{Δ_M} is the normal cone mapping and $A^\top$ denotes the adjoint. Choose $\nu \in N_{\Delta_M}(w)$ and $\nu' \in N_{\Delta_M}(w')$ realizing these inclusions. Subtract the two relations and take the inner product with $(w - w')$ to obtain

$$\langle A^\top A(w - w'),\, w - w'\rangle \;+\; \lambda\|w - w'\|^2 \;+\; \langle \nu - \nu',\, w - w'\rangle \;=\; \langle A^\top(b - b'),\, w - w'\rangle.$$

Since $A^\top A$ is positive semidefinite and N_{Δ_M} is monotone, one has $\langle A^\top A(w - w'), w - w'\rangle \ge 0$ and $\langle \nu - \nu', w - w'\rangle \ge 0$. Hence

$$\lambda\|w - w'\|^2 \;\le\; \|A^\top(b - b')\|\,\|w - w'\| \;\le\; \|A\|_{\mathrm{op}}\|b - b'\|\,\|w - w'\|.$$

If $w \ne w'$, cancel $\|w - w'\|$; otherwise the desired bound is trivial. This gives $\|w - w'\| \le (\|A\|_{\mathrm{op}}/\lambda)\,\|b - b'\|$. □

Remark 8.14 (Stable direction labeling via a growing net). In a holomorphic chart $U \subset \mathbb{C}^n$, the calibrated directions are precisely the complex $(n-p)$–planes. Fix a scale h and choose an ε_h–net $\{P_1, \dots, P_M\} \subset G_{\mathbb{C}}(n-p, n)$ with $\varepsilon_h \ll h$. For each $x \in U$, let $\xi_i(x)$ denote the corresponding normalized generator in $K_p(x)$ (so $\langle \xi_i(x), \psi_x\rangle = 1$).

Given a smooth normalized target field $b(x) = \widehat{\beta}(x)$, one may choose *globally labeled* coefficients by applying Lemma 8.13 (with $V = \Lambda^{p,p}(T_x^*X)$ in a fixed trivialization on U) to obtain weights $w_i(x)$ depending *Lipschitzly* on $b(x)$. Since b varies by $O(h)$ between adjacent mesh-h cells, the weights w_i vary by $O(h)$ as well. This gives a canonical pairing of directions across neighbors (index $i = i'$) and reduces "stable direction labeling" to the quantitative choice of ε_h and the regularization parameter λ.

Step 2: Projective tangential approximation with C^1 control

Fix an ample line bundle $L \to X$ with a Hermitian metric whose curvature form equals ω. For $m \in \mathbb{N}$ large, consider the complete linear system $|L^m|$.

Lemma 8.15 (k-jet surjectivity for high powers). *For each integer $k \ge 1$ there exists $m_0(k)$ such that for all $m \ge m_0(k)$ and all $x \in X$, the evaluation map on k-jets*

$$H^0(X, L^m) \longrightarrow J_x^k(L^m)$$

is surjective. In particular, for $k = 1$, any prescribed value and first derivative at x is realized by a global section of L^m.

Proof. Consider the exact sequence $0 \to L^m \otimes \mathfrak{m}_x^{k+1} \to L^m \to L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1} \to 0$. For $m \gg 0$, $H^1(X, L^m \otimes \mathfrak{m}_x^{k+1}) = 0$ by Serre vanishing (ampleness of L). Hence $H^0(X, L^m) \twoheadrightarrow H^0(X, L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$, which identifies with k-jets at x. See Lazarsfeld, *Positivity in Algebraic Geometry I*, Theorem 1.8.5. □

Lemma 8.16 (Uniform C^1 control on $m^{-1/2}$-balls via Bergman kernels). *Fix $\varepsilon > 0$. There exists $m_1(\varepsilon)$ such that for all $m \geq m_1(\varepsilon)$, each $x \in X$, and each collection of p complex covectors $\lambda_1, \ldots, \lambda_p \in T_x^*X$, there exist sections $s_1, \ldots, s_p \in H^0(X, L^m)$ with the following properties in normal holomorphic coordinates centered at x:*

(i) *$s_i(x) = 0$ and $ds_i(x) = \lambda_i$ for each i;*

(ii) *on the geodesic ball $B_{cm^{-1/2}}(x)$ (for a universal constant $c > 0$ depending only on (X, ω)), the gradients satisfy*

$$\|ds_i(y) - \lambda_i\| \leq \varepsilon \quad \textit{for all } y \in B_{cm^{-1/2}}(x).$$

Proof. This is standard from peak section and Bergman kernel asymptotics (Tian, Catlin, Zelditch, Donaldson). In local normal coordinates with rescaling by $\sqrt{m}$, the space $H^0(X, L^m)$ approximates holomorphic polynomials with Gaussian weight, and there exist sections with prescribed jets whose C^1 norms on $B_{cm^{-1/2}}$ approach those of the corresponding linear functions. See:

- G. Tian, "On a set of polarized Kähler metrics," J. Diff. Geom. 32 (1990), 99–130;
- S. Zelditch, "Szegő kernels and a theorem of Tian," IMRN 1998, no. 6, 317–331;
- S. K. Donaldson, "Scalar curvature and projective embeddings, I," J. Diff. Geom. 59 (2001), 479–522, Section 2.

□

Lemma 8.17 (Graph control from uniform gradient control). *Let $U \subset \mathbb{C}^n$ be a ball and let $\lambda_1, \ldots, \lambda_p \in (\mathbb{C}^n)^*$ be complex covectors with linearly independent real and imaginary parts, so that $\Pi := \bigcap_{i=1}^p \ker(\lambda_i)$ is a complex $(n-p)$-plane. Let $s_1, \ldots, s_p : U \to \mathbb{C}$ be holomorphic functions such that $s_i(0) = 0$ and*

$$\sup_{y \in U} \|ds_i(y) - \lambda_i\| \leq \varepsilon \qquad \textit{for all } i = 1, \ldots, p,$$

with ε small compared to $\min\{\|\lambda_i\|\}$. Then the common zero set $Y := \{s_1 = \cdots = s_p = 0\} \cap U$ is a smooth complex submanifold of U and, after shrinking U if needed, Y is a C^1 graph over Π with slope $O(\varepsilon)$. In particular,

$$\sup_{y \in Y} \angle(T_yY, \Pi) \leq C\,\varepsilon$$

for a constant C depending only on (n, p) and the conditioning of $\{\lambda_i\}$.

Proof. Let $S = (s_1, \ldots, s_p) : U \to \mathbb{C}^p$. The differential $dS(y)$ is uniformly close to the constant complex-linear map $\Lambda = (\lambda_1, \ldots, \lambda_p)$ in operator norm. Since Λ is surjective (its kernel is the complex $(n-p)$–plane Π), for ε sufficiently small the perturbation bound implies $dS(y)$ is surjective for all $y \in U$. Hence $Y = S^{-1}(0)$ is a smooth complex submanifold of U by the holomorphic implicit function theorem.

Write $\mathbb{C}^n = \Pi \oplus \Pi^\perp$ and let (u, w) denote the corresponding coordinates. Since $\partial_w S$ is uniformly close to $\partial_w \Lambda$ and $\partial_w \Lambda : \Pi^\perp \to \mathbb{C}^p$ is invertible, the implicit function theorem yields (after shrinking U if needed) a C^1 map g with $Y = \{(u, g(u))\}$. Differentiating $S(u, g(u)) = 0$ gives $Dg = -(\partial_w S)^{-1} \partial_u S$, so the same uniform closeness estimates imply $\|Dg\| \leq C\,\varepsilon$ for a constant C depending only on (n, p) and the conditioning of $\{\lambda_i\}$. □

Proposition 8.18 (Projective tangential approximation with C^1 control)**.** *Let $x \in X$ and let $\Pi \subset T_xX$ be a complex $(n-p)$-plane. For every $\varepsilon > 0$ there exist $m \gg 0$ and a smooth complete intersection*

$$Y = \{s_1 = 0\} \cap \cdots \cap \{s_p = 0\} \subset X, \qquad s_i \in H^0(X, L^m),$$

such that $x \in Y$, Y is smooth in a neighborhood of x, and

$$\angle\big(T_yY, \Pi\big) < \varepsilon \quad \textit{for all } y \in B_{c\,m^{-1/2}}(x).$$

Moreover, Y is ψ-calibrated (being a complex submanifold).

Proof. Choose covectors $\lambda_1, \ldots, \lambda_p \in T_x^*X$ whose common kernel equals Π. By Lemma 8.16, pick $s_1, \ldots, s_p$ with $s_i(x) = 0$, $ds_i(x) = \lambda_i$, and $\|ds_i(y) - \lambda_i\| < \varepsilon/p$ on $B_{c\,m^{-1/2}}(x)$.

For $m \gg 0$ and after a small generic perturbation inside the finite-dimensional linear system (which does not change jets at x nor the C^1 estimates on the small ball), Bertini's theorem ensures that Y is smooth and $\{ds_1(y), \ldots, ds_p(y)\}$ are linearly independent on the ball.

The complex normal space to Y at y is spanned by $\{ds_1(y), \ldots, ds_p(y)\}$, which is ε-close to $\{\lambda_1, \ldots, \lambda_p\}$ in the Grassmannian metric. Hence T_yY is ε-close to Π for all y in the ball.

Since Y is a complex submanifold of a Kähler manifold, it is automatically calibrated by $\psi = \omega^{n-p}/(n-p)!$. $\square$

Proposition 8.19 (Holomorphic density of calibrated directions)**.** *For every compact $K \subset X$ and $\varepsilon > 0$ there exist finitely many ψ-calibrated $(n-p)$-submanifolds $Y_1, \ldots, Y_M$ (each a smooth complete intersection in $|L^m|$ for some large m) such that for each $x \in K$ and each calibrated plane $\Pi \subset T_xX$ there exists j with $x \in Y_j$ and $\operatorname{dist}\big(T_xY_j, \Pi\big) < \varepsilon$.*

Proof. Cover K by finitely many coordinate balls $\{B_\alpha\}$ centered at points $\{x_\alpha\}$. On each center x_α, take an $\varepsilon/2$-net of calibrated planes $\{\Pi_{\alpha,1}, \ldots, \Pi_{\alpha,N_\alpha}\}$ in the compact fiber $G_{n-p}(T_{x_\alpha}X)$. Apply Proposition 8.18 to realize each net direction by a calibrated complete intersection $Y_{\alpha,j}$ through x_α with tangent plane $\varepsilon/2$-close to $\Pi_{\alpha,j}$ on a ball of radius $c\,m^{-1/2}$.

After shrinking the coordinate balls B_α if necessary (to fit inside the C^1-control region), these submanifolds remain within ε of the target directions throughout each ball. Collecting all $Y_{\alpha,j}$ over the finitely many centers gives the desired family. $\square$

Step 3: Local calibrated laminates on small cubes (Theorem B)

This step constructs multiple disjoint calibrated sheets on each cube Q with prescribed tangent directions and mass fractions.

Theorem 8.20 (Local multi-sheet construction)**.** *Let $Q \subset X$ be a small coordinate cube. Let $\Pi_1, \ldots, \Pi_J \in \mathrm{Gr}_{n-p}(TQ)$ be constant $(n-p)$-planes, and let $\theta_1, \ldots, \theta_J \in \mathbb{Q}_{>0}$ with $\sum_j \theta_j = 1$. For every $\varepsilon, \delta > 0$, there exist smooth ψ-calibrated complete intersections $\{Y_j^a\}_{j,a}$ in X such that:*

(i) ***Angle control:*** $\sup_{y \in Q} \angle(T_yY_j^a, \Pi_j) < \varepsilon$;

(ii) ***Mass fractions:*** $\big|\mathrm{Mass}(Y_j^a \llcorner Q)/\sum_{i,b} \mathrm{Mass}(Y_i^b \llcorner Q) - \theta_j\big| < \delta$;

(iii) ***Disjointness:*** *The Y_j^a are pairwise disjoint on Q;*

(iv) ***Boundary control:*** $\partial([Y_j^a] \llcorner Q)$ *is supported on ∂Q.*

Proof. The proof proceeds in four substeps.

Substep 3.1: Local setup and flattening. Shrink Q so that there is a holomorphic chart $\Phi : U \to B(0,2) \subset \mathbb{C}^n$ with $Q \subset U$, $\Phi(Q) \subset [-1,1]^{2n} \subset \mathbb{C}^n$, and the Kähler form ω and calibration $\psi = \omega^{n-p}/(n-p)!$ are C^1-close to the flat model on $\mathbb{C}^n$. The calibration cone $K_{n-p}(x) \subset \mathrm{Gr}_{n-p}(T_xX)$ varies smoothly and stays uniformly close to the flat cone of complex $(n-p)$-planes. We prove Theorem 8.20 in this flattened model; everything is diffeomorphism-invariant, and volume/mass distortions are controlled by the uniform C^1-closeness of the metric.

Substep 3.2: Approximate target planes by calibrated planes. At each $x \in Q$, the set $K_{n-p}(x)$ of ψ-calibrated complex $(n-p)$-planes is a compact subset of $\mathrm{Gr}_{n-p}(T_xX)$ (isomorphic to the complex Grassmannian $G_{\mathbb{C}}(n-p,n)$). For any real $(n-p)$-plane Π_j, compactness guarantees the existence of a calibrated plane $\widetilde{\Pi}_j \in K_{n-p}(x)$ minimizing the Grassmannian distance:

$$\widetilde{\Pi}_j := \arg \min_{P \in K_{n-p}(x)} \angle(\Pi_j, P).$$

Since $K_{n-p}(x)$ spans the full complex Grassmannian (every complex $(n-p)$-plane is calibrated), and Π_j arises from a Carathéodory decomposition of $\beta(x) \in K_p(x)$, we have $\angle(\Pi_j, \widetilde{\Pi}_j) \le \eta$ for some $\eta > 0$ controlled by the C^0-norm of β. Choose $\eta \le \varepsilon/2$ so that sheets with tangent plane $\widetilde{\Pi}_j$ automatically satisfy $\angle(T_y Y_j^a, \Pi_j) < \varepsilon$.

Substep 3.3: Choose sheet counts via Diophantine rounding. For fixed j, all parallel copies of $\widetilde{\Pi}_j$ have identical ψ-mass $A_j > 0$ in Q. With N_j sheets, the total mass in family j is $N_j A_j$. Define

$$\lambda_j := \frac{\theta_j}{A_j}, \qquad \Lambda := \sum_i \lambda_i.$$

For large integer m, set

$$N_j(m) := \Big\lfloor m \frac{\lambda_j}{\Lambda} \Big\rfloor.$$

Standard rounding estimates give

$$\Big| N_j(m) - m\frac{\lambda_j}{\Lambda} \Big| \le 1,$$

and hence

$$\Big| \frac{N_j(m) A_j}{\sum_i N_i(m) A_i} - \theta_j \Big| = O\Big(\frac{1}{m}\Big).$$

Choose m so large that this error is $< \delta$.

Substep 3.4: Build flat model sheets with disjoint translations. In $\Phi(Q) \subset \mathbb{C}^n$, for each j, let $N_j^\perp$ be the complex p-dimensional normal space (the complex orthogonal complement of $\widetilde{\Pi}_j$), so that $\mathbb{C}^n = \widetilde{\Pi}_j \oplus N_j^\perp$. Pick distinct translation vectors $t_{j,1}, \dots, t_{j,N_j} \in N_j^\perp$ in a small ball $B(0,\rho)$ with $\rho \ll \operatorname{diam}(Q)$, such that all affine spaces $\widetilde{\Pi}_j + t_{j,a}$ are pairwise disjoint on $\Phi(Q)$ as (j,a) ranges over all indices. This is possible since $N_j^\perp$ has real dimension $2p \ge 2$ and we choose only finitely many points.

Define

$$\widetilde{Y}_j^a := (\widetilde{\Pi}_j + t_{j,a}) \cap \Phi(Q) \subset \mathbb{C}^n.$$

These satisfy: (i) ψ_0-calibration (complex $(n-p)$-planes); (ii) $\sup_{y \in Q} \angle(T_y \widetilde{Y}_j^a, \Pi_j) = \angle(\widetilde{\Pi}_j, \Pi_j) < \varepsilon$; (iii) mass fractions within δ of θ_j by construction; (iv) pairwise disjoint on $\Phi(Q)$; (v) boundary supported on $\partial\Phi(Q)$.

Substep 3.5: Upgrade to algebraic complete intersections. Use Kodaira embedding and Hörmander L^2-techniques: for large k, pick global sections $s^{(1)}_{j,a}, \ldots, s^{(p)}_{j,a} \in H^0(X, L^k)$ whose restrictions to Q are C^2-close to the linear defining functions of $\widetilde{Y}^a_j$. For k large:

- $Y^a_j := \{s^{(1)}_{j,a} = 0\} \cap \cdots \cap \{s^{(p)}_{j,a} = 0\}$ is a smooth complex $(n-p)$-dimensional submanifold;
- On Q, Y^a_j is C^1-close to $\widetilde{Y}^a_j$;
- Calibration, disjointness, and mass estimates persist under small C^1 perturbations.

Pulling back by Φ^{-1} gives the desired family on Q. □

Fix a finite normal coordinate atlas by geodesic balls of radii $\ll 1$ and subordinate cubes $\{Q\}$ small enough so that the Carathéodory data from Lemma 8.12 are ε-stable on each cube. For each cube Q and each index $j \in \{1, \ldots, N\}$, let $\Pi_{Q,j}$ denote a constant complex $(n-p)$-plane approximating $P_{x,j}$ on Q. Apply Theorem 8.20 to each cube to obtain families $\{Y^a_{Q,j}\}$ of disjoint ψ-calibrated complete intersections.

Define the local current

$$S_Q := \sum_{j=1}^{N} \sum_{a=1}^{N_{Q,j}} [Y^a_{Q,j}] \llcorner Q.$$

By construction, each $Y^a_{Q,j}$ is ψ-calibrated; hence S_Q is a positive ψ-calibrated integral current on Q. Its tangent-plane distribution on Q is a convex combination of directions within ε of $\{\Pi_{Q,j}\}$ with weights proportional to the ψ–masses in each family (equivalently proportional to $N_{Q,j}A_{Q,j}$, where $A_{Q,j}$ is the ψ–mass of a single (Q,j)-sheet in Q).

Lemma 8.21 (Local barycenter matching)**.** *For any $\delta > 0$ there exist integers $N_{Q,1}, \ldots, N_{Q,N}$ such that the tangent-plane Young measure of S_Q has barycenter within δ (in Hilbert–Schmidt norm) of the normalized field $\widehat{\beta}$ on Q, and*

$$\mathrm{Mass}(S_Q) \to m \int_Q \beta \wedge \psi \quad \textit{as } \delta \to 0.$$

Proof. Let $A_{Q,j} > 0$ denote the common ψ–mass of a single (Q,j)-sheet in Q (all sheets in a fixed family (Q,j) are local parallel translates, so their mass in Q agrees up to $o_\delta(1)$). Choose integers $N_{Q,j}$ so that the *mass fractions*

$$\frac{N_{Q,j}A_{Q,j}}{\sum_i N_{Q,i}A_{Q,i}}$$

approximate $\theta_{x,j}$ (nearly constant on Q) to within $O(\delta)$. Then the resulting mass-weighted barycenter

$$\sum_j \frac{N_{Q,j}A_{Q,j}}{\sum_i N_{Q,i}A_{Q,i}}\, \xi_{\Pi_{Q,j}}$$

is within δ of $\widehat{\beta}$ on Q. Because the tangent angles are $< \varepsilon$ and $\varepsilon \ll \delta$, the Hilbert–Schmidt distance of barycenters is $\le C(\varepsilon + \delta)$.

Finally, calibratedness gives $\mathrm{Mass}([Y^a_{Q,j}]\llcorner Q) = \int_Q \psi \llcorner [Y^a_{Q,j}]$, hence

$$\mathrm{Mass}(S_Q) = \sum_j N_{Q,j}A_{Q,j}.$$

By scaling the $N_{Q,j}$ simultaneously (and then rounding), one can arrange $\sum_j N_{Q,j}A_{Q,j} \to m \int_Q \beta \wedge \psi$ as $\delta \to 0$. □

Step 4: Global cohomology quantization (Theorem C)

This step forces the global integral current to represent exactly the correct homology class $\mathrm{PD}(m[\gamma])$ by using lattice discreteness.

Theorem 8.22 (Global cohomology quantization)**.** *Let X be a compact Kähler n-fold with rational Hodge class $[\gamma] \in H^{2p}(X, \mathbb{Q})$ represented by a smooth closed (p,p)-form β with $\beta(x) \in K_p(x)$ pointwise. Let $\{Q\}$ be a cube partition of X. Then there exists an integer $m \geq 1$ (clearing denominators of $[\gamma]$) such that for every $\varepsilon > 0$ there exist:*

- *A closed integral $(2n-2p)$-current T_ε with $[T_\varepsilon] = \mathrm{PD}(m[\gamma])$;*
- *A correction current R_ε with $\mathrm{Mass}(R_\varepsilon) < \varepsilon$;*

such that the local tangent-plane mass proportions on each Q match those of β up to error $o_{\varepsilon\to 0}(1)$.

Proof. The proof proceeds in three substeps.

Substep 4.1: Local quantization. Choose the partition $\{Q\}$ fine enough that on each Q, $\beta(x)$ is within δ (in operator norm) of $\beta(x_Q)$ for a base point $x_Q \in Q$, and the Kähler metric is nearly constant (Jacobian and volume distortion $\leq 1+\delta$).

By Lemma 8.12, write

$$\beta(x_Q) = t_Q \sum_{j=1}^{J(Q)} \theta_{Q,j}\, \xi_{Q,j}, \qquad t_Q := \langle \beta(x_Q), \psi_{x_Q} \rangle,$$

where $\xi_{Q,j} \in K_p(x_Q)$ are normalized extremal generators (coming from complex $(n-p)$-planes) satisfying $\langle \xi_{Q,j}, \psi_{x_Q} \rangle = 1$, the weights satisfy $\theta_{Q,j} \geq 0$, $\sum_j \theta_{Q,j} = 1$, and $J(Q) \leq N = N(n,p)$ uniformly bounded.

Since $[\gamma]$ is rational, all its periods lie in $(1/M)\mathbb{Z}$ for some fixed M. Choose $m \gg 1$ divisible by M.

Let $P_{Q,j} \subset T_{x_Q}X$ be the complex $(n-p)$-plane corresponding to $\xi_{Q,j}$. In the flattened model on Q, any affine ψ–calibrated sheet with tangent plane $P_{Q,j}$ has the same ψ–mass in Q; denote this common value by $A_{Q,j} > 0$ (it depends on the cube geometry and direction but satisfies $A_{Q,j} \asymp \mathrm{side}(Q)^{2(n-p)}$). The target ψ–mass in Q is

$$M_Q := m \int_Q \beta \wedge \psi \;\approx\; m\, t_Q\, \mathrm{Vol}(Q),$$

up to $O(\delta)$ error from the C^0–variation of β on Q and the metric distortion.

Choose integers $N_{Q,j} \geq 0$ so that simultaneously

$$\left| \frac{N_{Q,j} A_{Q,j}}{\sum_i N_{Q,i} A_{Q,i}} - \theta_{Q,j} \right| \leq \delta \qquad \text{and} \qquad \left| \sum_j N_{Q,j} A_{Q,j} - M_Q \right| \leq \delta\, M_Q.$$

(Such choices exist by rounding, since the unknowns enter linearly and m may be taken arbitrarily large.)

Apply Theorem 8.20 to realize each direction (Q,j) by a family of ψ–calibrated sheets $Y^a_{Q,j} \subset Q$ $(a = 1, \ldots, N_{Q,j})$ with angle control, disjointness on Q, and boundary supported on ∂Q.

Define the raw local current

$$S_Q := \sum_{j=1}^{J(Q)} \sum_{a=1}^{N_{Q,j}} [Y^a_{Q,j}] \llcorner Q.$$

Substep 4.2: Gluing across cubes. Consider the global raw current

$$T^{\mathrm{raw}} := \sum_Q S_Q.$$

This is integral but not closed: $\partial T^{\mathrm{raw}}$ lives on the union of cube faces. View the cube adjacency as a finite graph: vertices = cubes Q, edges = codimension-1 faces $F = Q \cap Q'$. On each oriented face F, the restriction of ∂S_Q induces a $(2n-2p-1)$-current $B_{Q\to F}$ living on F. Summed over all cubes:

$$\partial T^{\mathrm{raw}} = \sum_F B_F,$$

where B_F is the mismatch between the two neighboring cubes.

Key point (flat norm, not mass): In general the individual face currents B_F need not have small mass (cancellation-heavy boundaries can have large mass), so the robust quantity to control is the *flat norm* of the total mismatch $\partial T^{\mathrm{raw}}$. Recall the flat norm on $(2n-2p-1)$-currents:

$$\mathcal{F}(S) := \inf\{\mathrm{Mass}(R) + \mathrm{Mass}(Q) : \ S = R + \partial Q\},$$

where R is an integral $(2n-2p-1)$-current and Q is an integral $(2n-2p)$-current. On a compact manifold one has the dual characterization (Federer–Fleming):

$$\mathcal{F}(S) = \sup\{S(\eta) : \ \eta \in C^\infty\Lambda^{2n-2p-1}, \ \|\eta\|_{\mathrm{comass}} \le 1, \ \|d\eta\|_{\mathrm{comass}} \le 1\}.$$

For $S = \partial T^{\mathrm{raw}}$ and such η, Stokes gives $S(\eta) = \partial T^{\mathrm{raw}}(\eta) = T^{\mathrm{raw}}(d\eta)$.

Proposition 8.23 (Transport control $\Rightarrow$ flat-norm gluing)**.** *Fix a cubulation of X by coordinate cubes of side length $h =$ mesh, and write $T^{\mathrm{raw}} = \sum_Q S_Q$ as above, where each S_Q is a sum of calibrated sheets restricted to Q. Assume the following* geometric parameterization *holds on each interior face $F = Q \cap Q'$:*

- (a) ***(Small-angle graph model)*** *For each cube Q and each sheet family (Q, j), the sheets crossing F are C^1-graphs over a fixed calibrated reference plane $\Pi_{Q,j}$ with $\sup_{y\in Q} \angle(T_y Y^a_{Q,j}, \Pi_{Q,j}) \le \varepsilon$.*
- (b) ***(Transverse measures on faces)*** *After identifying a tubular neighborhood of F with a product $F \times B^{2p}(0, ch)$ in normal coordinates, the restriction of ∂S_Q to F can be written as a finite sum of translated slice currents parameterized by a discrete transverse measure $\mu_{Q\to F}$ on $B^{2p}(0, ch)$ (integer weights), and similarly for Q'.*
- (c) *(W_1 **face matching**) The two induced transverse measures have the same total mass and satisfy*

  $$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \ \le \ \tau_F,$$

  where W_1 is the 1-Wasserstein distance on $B^{2p}(0, ch)$.

Then there exists a constant $C = C(n, p, X)$ such that for every smooth $(2n-2p-1)$-form η with $\|\eta\|_{\mathrm{comass}} \le 1$ and $\|d\eta\|_{\mathrm{comass}} \le 1$ one has the face estimate

$$|B_F(\eta)| \ \le \ C\, h^{2n-2p-1}\big(\tau_F + \varepsilon\ \mathrm{Mass}(\mu_{Q\to F})\, h\big),$$

and hence

$$\mathcal{F}(B_F) \ \le \ C\, h^{2n-2p-1}\big(\tau_F + \varepsilon\ \mathrm{Mass}(\mu_{Q\to F})\, h\big).$$

Consequently,

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \ \le \ \sum_F \mathcal{F}(B_F) \ \le \ C\, h^{2n-2p-1}\sum_F \tau_F \ + \ C\,\varepsilon\, h^{2n-2p}\sum_F \mathrm{Mass}(\mu_{Q\to F}).$$

Proof. Fix an interior face $F = Q \cap Q'$ and a test form η with $\|\eta\|_{\text{comass}} \le 1$ and $\|d\eta\|_{\text{comass}} \le 1$. Work in the tubular product chart from hypothesis (b), identifying a neighborhood of F with $F \times B^{2p}(0, ch)$.

Step 1 (a Lipschitz evaluation function). For a translated slice current Σ_y in hypothesis (b), define the scalar function

$$f_\eta(y) \ := \ \Sigma_y(\eta).$$

Let $y, y' \in B^{2p}(0, ch)$ and set $v := y' - y$. In the flat/parallel model (i.e. when $\Sigma_{y'} = (\tau_v)_\# \Sigma_y$ inside the product chart), consider the straight-line homotopy $H : [0,1] \times F \to F \times B^{2p}(0, ch)$, $H(t, x) = (x, y + tv)$. Let $Q_{y \to y'} := H_\#([0,1] \times \Sigma_y)$. Then $\partial Q_{y \to y'} = \Sigma_{y'} - \Sigma_y$ and

$$\text{Mass}(Q_{y \to y'}) \ \le \ \|v\| \ \text{Mass}(\Sigma_y).$$

By Stokes and the comass bound on $d\eta$,

$$|f_\eta(y') - f_\eta(y)| = |(\Sigma_{y'} - \Sigma_y)(\eta)| = |Q_{y \to y'}(d\eta)| \le \text{Mass}(Q_{y \to y'}) \|d\eta\|_{\text{comass}} \le \|v\| \ \text{Mass}(\Sigma_y).$$

Under the small-angle graph hypothesis (a) and bounded geometry of the chart, each slice has mass $\text{Mass}(\Sigma_y) \le C\, h^{2n-2p-1}$ with $C = C(n, p, X)$. Hence

$$\text{Lip}(f_\eta) \ \le \ C\, h^{2n-2p-1}.$$

Step 2 (Kantorovich–Rubinstein). By hypothesis (b), the face restrictions can be written as $(\partial S_Q) \llcorner F = \int \Sigma_y \, d\mu_{Q \to F}(y)$ and similarly for Q', so

$$B_F(\eta) = \int f_\eta \, d\mu_{Q \to F} - \int f_\eta \, d\mu_{Q' \to F}.$$

Since $\mu_{Q \to F}$ and $\mu_{Q' \to F}$ have the same total mass (hypothesis (c)), adding a constant to f_η does not change $B_F(\eta)$. Therefore, by Kantorovich–Rubinstein duality for W_1,

$$|B_F(\eta)| \le \text{Lip}(f_\eta)\, W_1(\mu_{Q \to F}, \mu_{Q' \to F}) \le C\, h^{2n-2p-1}\, \tau_F.$$

Step 3 (small-angle model error). When the sheets are only ε-graphs over their reference planes (hypothesis (a)), the slice currents in the chart differ from the exactly-parallel translated model by a C^1 graph distortion of size $O(\varepsilon)$. Since $\|\eta\|_{\text{comass}} \le 1$, the induced error in evaluating η on each slice is bounded by $C\, \varepsilon\, h^{2n-2p}$ uniformly (one factor of h comes from converting the angular error into a tangential displacement on a cell of size h). Summing over the (integer-weighted) family on that face gives an additional error bounded by $C\, \varepsilon\, h^{2n-2p}\ \text{Mass}(\mu_{Q \to F})$. Combining with Step 2 yields the stated face estimate $|B_F(\eta)| \le C h^{2n-2p-1}(\tau_F + \varepsilon\ \text{Mass}(\mu_{Q \to F})\, h)$.

Step 4 (flat norm and summation). Taking the supremum over η in the dual characterization of $\mathcal{F}$ gives $\mathcal{F}(B_F) \le C h^{2n-2p-1}(\tau_F + \varepsilon\ \text{Mass}(\mu_{Q \to F})\, h)$. Finally, $\partial T^{\text{raw}} = \sum_F B_F$ as currents, so the triangle inequality for $\mathcal{F}$ implies $\mathcal{F}(\partial T^{\text{raw}}) \le \sum_F \mathcal{F}(B_F)$, which yields the global bound claimed. □

Remark 8.24 (Why hypotheses (a)–(b) hold for the local sheet model). In the flat model of Substep 3.4, each sheet in family (Q, j) is literally an affine calibrated plane $(\widetilde{\Pi}_{Q,j} + t_{j,a}) \cap Q$, with translation parameter $t_{j,a} \in N^{\perp}_{Q,j} \cong \mathbb{R}^{2p}$. For a fixed face $F \subset \partial Q$, the boundary slice current

$$\Sigma_{F,j}(t) := \partial\big([\widetilde{\Pi}_{Q,j} + t] \llcorner Q\big) \llcorner F$$

depends only on t through its component normal to the $(2n-2p-1)$-plane $\widetilde{\Pi}_{Q,j} \cap TF$. Thus, in the flat model, $\partial S_Q \llcorner F$ can be written as a finite sum $\sum_a \Sigma_{F,j}(t_{j,a})$, i.e. it is parameterized by the discrete transverse measure $\mu_{Q\to F} := \sum_a \delta_{t_{j,a}}$ (with integer weights).

After upgrading to algebraic complete intersections in Substep 3.5, the sheets remain C^1-graphs over the flat model on Q (for k large), so the same parameterization persists in a tubular neighborhood of F up to an $O(\varepsilon)$ error controlled by the graph distortion. This justifies the use of transverse measures on faces and the small-angle graph model in Proposition 8.23.

What is *not* automatic is hypothesis (c): arranging W_1 matching across faces simultaneously for all cubes, subject to the constraint that each sheet's translation parameter determines its intersection with *all* faces of Q at once. Equivalently, for a fixed cube Q and family (Q,j), the face measures $\mu_{Q\to F}$ for different faces $F \subset \partial Q$ are not independent choices: they arise as pushforwards of the *same* discrete translation multiset $\{t_{j,a}\}$ under the corresponding face-slice maps. Thus the remaining task is a *simultaneous* matching problem.

Lemma 8.25 (Automatic W_1-matching from smooth dependence of face maps)**.** *Let μ be a finite Borel measure on $\mathbb{R}^{2p}$ supported in a ball of radius $O(h)$ and with total mass $\mu(\mathbb{R}^{2p}) = N$. Let $\Phi, \Phi' : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ be linear maps with $\|\Phi - \Phi'\|_{\mathrm{op}} \le C\,h$. Then*

$$W_1(\Phi_\# \mu, \Phi'_\# \mu) \;\le\; C\,h \int_{\mathbb{R}^{2p}} \|y\|\, d\mu(y) \;\le\; C'\,h^2\,N.$$

Proof. Define a coupling π of $\Phi_\#\mu$ and $\Phi'_\#\mu$ by pushing μ forward under the map $y \mapsto (\Phi y, \Phi' y)$. Then π has first marginal $\Phi_\#\mu$ and second marginal $\Phi'_\#\mu$, and therefore

$$W_1(\Phi_\#\mu, \Phi'_\#\mu) \le \int_{\mathbb{R}^{2p}\times\mathbb{R}^{2p}} \|u - u'\|\, d\pi(u,u') \;=\; \int_{\mathbb{R}^{2p}} \|\Phi y - \Phi' y\|\, d\mu(y).$$

Estimating $\|\Phi y - \Phi' y\| \le \|\Phi - \Phi'\|_{\mathrm{op}} \|y\|$ gives

$$W_1(\Phi_\#\mu, \Phi'_\#\mu) \le \|\Phi - \Phi'\|_{\mathrm{op}} \int_{\mathbb{R}^{2p}} \|y\|\, d\mu(y).$$

If $\operatorname{supp}\mu \subset B(0, C_0 h)$, then $\int \|y\|\, d\mu \le C_0 h\, \mu(\mathbb{R}^{2p}) = C_0 h\, N$. Absorbing constants yields the stated bound. □

Lemma 8.26 (Pointwise displacement bound under nearby face maps)**.** *Let $y_1, \dots, y_N \in \mathbb{R}^{2p}$ satisfy $\|y_a\| \le C_0\,h$ and let $\Phi, \Phi' : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ be linear maps with $\|\Phi - \Phi'\|_{\mathrm{op}} \le C_1\,h$. Define two multisets $u_a := \Phi y_a$ and $u'_a := \Phi' y_a$. Then the index-wise matching satisfies*

$$\|u_a - u'_a\| \;\le\; C_0 C_1\, h^2 \qquad \textit{for all } a.$$

In particular, when adjacent cells use the same *translation template $\{y_a\}$ and their face parameterizations differ by $O(h)$ in operator norm, the hypothesis of Corollary 8.35 holds with $\Delta_F = O(h^2)$.*

Proof.

$$\|u_a - u'_a\| = \|(\Phi - \Phi')y_a\| \le \|\Phi - \Phi'\|_{\mathrm{op}} \|y_a\| \le (C_1 h)(C_0 h) = C_0 C_1 h^2.$$ □

Lemma 8.27 (Template stability under small multiset edits). *Let $\Omega \subset \mathbb{R}^{2p}$ be a bounded domain of diameter* $\mathrm{diam}(\Omega) \le Ch$. *Let $\mu = \sum_{a=1}^{N} \delta_{y_a}$ and $\mu' = \sum_{b=1}^{N} \delta_{y'_b}$ be two integer-weighted discrete measures on Ω with the* same total mass N. *Assume there is a matching of atoms such that $\|y_a - y'_a\| \le \Delta$ for all a (after relabeling). Then*

$$W_1(\mu, \mu') \;\le\; \Delta\, N.$$

More generally, if μ' is obtained from μ by deleting r atoms and inserting r atoms (so total mass stays N), then

$$W_1(\mu, \mu') \;\le\; r \cdot \mathrm{diam}(\Omega) \;\le\; C\, r\, h.$$

Proof. For the first claim, couple μ and μ' by pairing each y_a to y'_a; the transport cost is $\sum_a \|y_a - y'_a\| \le \Delta N$. For the second claim, transport each deleted atom to an inserted atom at cost at most $\mathrm{diam}(\Omega)$ and keep the unchanged atoms fixed. ☐

Remark 8.28 (How Lemma 8.25 reduces the remaining matching task). If, for each cube Q and sheet family (Q, j), we choose the translation multiset $\{t_{j,a}\}$ by a *fixed* template in $N^{\perp}_{Q,j}$ (e.g. a scaled lattice/low-discrepancy set of diameter $O(h)$), then across a shared face $F = Q \cap Q'$ the two induced transverse measures are related by applying two nearby face-slice maps (coming from nearby plane directions and nearby normal-coordinate identifications). Since β is smooth, these maps differ by $O(h)$ in operator norm, so Lemma 8.25 yields

$$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \;\lesssim\; h^2\, N_F,$$

where N_F is the number of sheets contributing to that face. Inserting this into Proposition 8.23 yields a global bound of the form

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\lesssim\; m\, h \;+\; O(\varepsilon\, m),$$

so choosing $h = h(m) \to 0$ slowly (e.g. $h = m^{-\alpha}$ with $\alpha > 0$ small) makes the gluing correction R_{glue} sublinear in m and hence negligible in the mass equality as $m \to \infty$. The remaining task is then to implement this "fixed template" choice while still meeting the cohomological constraints (Substep 4.3). In the *sliver* regime, the count N_F is not controlled by total mass; see Remark 8.29 and Corollary 8.35 for the weighted replacement.

Remark 8.29 (Sliver regime: what changes in the global counting estimate). The global $m\,h$ bound in Remark 8.28 uses an implicit *counting step*: it treats the total face mismatch as scaling like "(per-sheet mismatch) $\times$ (number of sheet pieces meeting faces)". In the constant-mass-per-sheet model this count is controlled by total mass, because each sheet piece carries ψ–mass $\asymp h^{2(n-p)}$ in a cube.

In the *sliver* regime (Remark 8.57), one deliberately allows many pieces of very small mass per cube. Then the raw counts N_F (or the total number of sheet pieces meeting faces) can be arbitrarily large at fixed total mass, so the crude reduction to $\mathrm{Mass}(T^{\mathrm{raw}})$ is no longer available. To make the sliver escape compatible with flat-norm gluing, we therefore use a *weighted* replacement that tracks the actual size of each face slice, for example a bound in terms of the boundary-size functional

$$\sum_{F} \sum_{a \in \mathcal{S}(F)} \mathrm{Mass}\big(\partial([Y^a] \llcorner Q) \llcorner F\big),$$

or an equivalent transverse-parameter integral. Concretely, Proposition 8.30 bounds each face flat mismatch by displacement $\times$ (slice boundary mass), and Lemma 8.32 converts slice boundary mass into a power of the interior piece mass on smooth curvature-pinched cells. This is packaged globally as Corollary 8.35.

Proposition 8.30 (Weighted transport $\Rightarrow$ flat-norm face control (sliver-compatible))**.** *Work in the tubular/flat model on an interior face $F = Q \cap Q'$. Assume each sheet piece meeting F contributes a* cycle slice *current $\Sigma(u)$ on F depending on a transverse parameter $u \in \Omega_F \subset \mathbb{R}^{2p}$, and that $\Sigma(u)$ is obtained from $\Sigma(0)$ by translation in the face chart. Let the two adjacent cubes induce two multisets of parameters $\{u_a\}_{a=1}^N$ and $\{u'_a\}_{a=1}^N$ (same cardinality), hence two face currents*

$$S_{Q\to F} := \sum_{a=1}^{N} \Sigma(u_a), \qquad S_{Q'\to F} := \sum_{a=1}^{N} \Sigma(u'_a), \qquad B_F := S_{Q\to F} - S_{Q'\to F}.$$

Then

$$\mathcal{F}(B_F) \;\le\; \inf_{\sigma\in S_N} \sum_{a=1}^{N} \|u_a - u'_{\sigma(a)}\| \,\mathrm{Mass}(\Sigma(u_a)).$$

In particular, if $\mathrm{Mass}(\Sigma(u_a)) \le b_F$ *for all a and if*

$$\tau_F := \inf_{\sigma\in S_N} \sum_{a=1}^{N} \|u_a - u'_{\sigma(a)}\|$$

(the equal-weight matching cost, i.e. W_1 of the counting measures), then

$$\mathcal{F}(B_F) \;\le\; b_F\,\tau_F.$$

Proof. Fix a permutation $\sigma \in S_N$. For each index a, the difference $\Sigma(u_a) - \Sigma(u'_{\sigma(a)})$ is the difference of two translated copies of the same integral cycle in the face chart, hence is itself a boundary. By Lemma 8.34 there exists an integral filling current Q_a with

$$\partial Q_a = \Sigma(u_a) - \Sigma(u'_{\sigma(a)}) \qquad \text{and} \qquad \mathrm{Mass}(Q_a) \;\le\; \|u_a - u'_{\sigma(a)}\|\,\mathrm{Mass}(\Sigma(u_a)).$$

Summing $Q := \sum_{a=1}^{N} Q_a$ yields $\partial Q = B_F$ and

$$\mathrm{Mass}(Q) \;\le\; \sum_{a=1}^{N} \|u_a - u'_{\sigma(a)}\|\,\mathrm{Mass}(\Sigma(u_a)).$$

Taking $R := 0$ in the definition of the flat norm gives $\mathcal{F}(B_F) \le \mathrm{Mass}(Q)$, and then taking the infimum over σ proves the claim. $\square$

Remark 8.31 (Exact geometric inequality needed for slivers). Proposition 8.30 shows that, in the sliver regime, the face mismatch is controlled by a *weighted* matching cost: displacement $\times$ (slice boundary mass), rather than displacement $\times$ (number of sheets). Thus the missing geometric input is precisely an estimate of the form

$$\mathrm{Mass}(\Sigma(u)) \;\lesssim\; \mathrm{Mass}([Y]\llcorner Q)^{\frac{k-1}{k}} \qquad (k := 2n - 2p),$$

uniformly for the relevant family of slices in the chosen cell geometry (balls / rounded cubes). In the ball model this holds with an explicit sharp constant; for general smooth uniformly convex cells it is the content of the "boundary shrinkage for plane slices" estimate.

Lemma 8.32 (Boundary shrinkage for plane slices in smooth uniformly convex cells)**.** *Let $Q \subset \mathbb{R}^d$ be a bounded C^2* uniformly convex *domain of diameter $\asymp h$. Assume the principal curvatures of ∂Q satisfy*

$$\frac{c}{h} \;\le\; \kappa_i \;\le\; \frac{C}{h} \qquad \textit{everywhere on } \partial Q,$$

for fixed constants $0 < c \le C$. Fix $1 \le k < d$ and a k-plane P. For each translate $P + t$ with nonempty intersection, set

$$v(t) := \mathcal{H}^k\big((P+t) \cap Q\big), \qquad a(t) := \mathcal{H}^{k-1}\big((P+t) \cap \partial Q\big).$$

Then there exists $C_* = C_*(d, k, c, C)$ such that

$$a(t) \;\le\; C_* \left(v(t)\right)^{\frac{k-1}{k}} \qquad \textit{for all such } t.$$

Proof. The estimate is scale-invariant, so rescale so that $h \asymp 1$. Write $K_t := (P+t) \cap Q \subset P+t \cong \mathbb{R}^k$, so $v(t) = \mathcal{H}^k(K_t)$ and $a(t) = \mathcal{H}^{k-1}(\partial K_t)$.

If $v(t) \ge v_0 > 0$, then K_t is a convex body contained in a fixed k–ball of radius $O(1)$, hence $a(t) \le A_0(d, k)$, and the desired bound follows after increasing C_*.

Assume $v(t) \le v_0$ with v_0 small. The curvature pinching implies an interior/exterior rolling-ball condition with radii $r_{\text{in}}, r_{\text{out}} \asymp 1$ (depending only on c, C) at every boundary point of Q. Let $\pi : \mathbb{R}^d \to P^\perp$ be orthogonal projection and set $D := \pi(Q) \subset P^\perp$. Choose a nearest point $t_0 \in \partial D$ and an outward normal $u \in P^\perp$ to a supporting hyperplane of D at t_0, and write $t = t_0 - su$. Let $x_0 \in \partial Q$ be the unique supporting point with outward normal u (uniqueness by uniform convexity), so $\pi(x_0) = t_0$.

Intersect the tangent balls at x_0 with the affine plane $P + t$. Since $u \perp P$, these intersections are k-balls of radii $\rho_{\text{in}}(s) = \sqrt{2r_{\text{in}}s - s^2}$ and $\rho_{\text{out}}(s) = \sqrt{2r_{\text{out}}s - s^2}$, hence

$$\omega_k\, \rho_{\text{in}}(s)^k \;\le\; v(t) \;\le\; \omega_k\, \rho_{\text{out}}(s)^k, \qquad a(t) \;\le\; \omega_{k-1}\, \rho_{\text{out}}(s)^{k-1}.$$

For s small one has $\rho_{\text{in}}(s) \gtrsim \sqrt{s}$ and $\rho_{\text{out}}(s) \lesssim \sqrt{s}$, so $v(t) \gtrsim s^{k/2}$ and $a(t) \lesssim s^{(k-1)/2}$, hence $s \lesssim v(t)^{2/k}$ and $a(t) \lesssim v(t)^{(k-1)/k}$. $\square$

Remark 8.33 (References for the geometric inputs). The implication "principal curvatures pinched at scale $h \Rightarrow$ interior/exterior tangent balls of radius $\asymp h$" is the classical *rolling ball* principle in convex geometry (often attributed to Blaschke). The supporting-hyperplane/unique-support-point facts used above are standard consequences of strict convexity and C^2 regularity of ∂Q (see any standard text on convex bodies, e.g. Schneider's *Convex Bodies: The Brunn–Minkowski Theory*).

Lemma 8.34 (Flat-norm stability under translation)**.** *Let S be an integral ℓ-cycle in $\mathbb{R}^d$ (so $\partial S = 0$) with finite mass. For any translation vector $v \in \mathbb{R}^d$, write $\tau_v(x) := x + v$ and $(\tau_v)_\# S$ for the pushforward. Then*

$$\mathcal{F}\big((\tau_v)_\# S - S\big) \;\le\; \|v\|\, \mathrm{Mass}(S).$$

Proof. Let $H : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ be the straight-line homotopy $H(t, x) = x + tv$. Consider the product current $[0,1] \times S$ in $[0,1] \times \mathbb{R}^d$ and set $Q := H_\#([0,1] \times S)$. Since $\partial([0,1] \times S) = \{1\} \times S - \{0\} \times S - [0,1] \times \partial S$ and $\partial S = 0$, we have

$$\partial Q = H_\#(\{1\} \times S) - H_\#(\{0\} \times S) = (\tau_v)_\# S - S.$$

Moreover, H has Jacobian bounded by $\|v\|$ in the t-direction, so the mass estimate for pushforwards gives $\mathrm{Mass}(Q) \le \|v\|\, \mathrm{Mass}(S)$. Taking $R := 0$ in the definition of $\mathcal{F}$ yields $\mathcal{F}((\tau_v)_\# S - S) \le \mathrm{Mass}(Q) \le \|v\|\, \mathrm{Mass}(S)$, as claimed. $\square$

Corollary 8.35 (Global flat-norm bound from weighted face control (sliver-compatible))**.** *Let $T^{\mathrm{raw}} = \sum_Q S_Q$ be the raw current built from calibrated pieces on smooth convex cells Q of diameter h as in Substep 4.2. Assume that on each interface $F = Q \cap Q'$ the face mismatch current*

$$B_F := (\partial S_Q)\llcorner F \;-\; (\partial S_{Q'})\llcorner F$$

fits the translation model of Proposition 8.30 with parameter multisets $\{u_a\}_{a=1}^N$ and $\{u'_a\}_{a=1}^N$. If there exists a matching $\sigma \in S_N$ with a uniform displacement bound

$$\|u_a - u'_{\sigma(a)}\| \;\leq\; \Delta_F \qquad \text{for all } a,$$

then

$$\mathcal{F}(B_F) \;\leq\; \Delta_F \sum_{a=1}^{N} \mathrm{Mass}(\Sigma(u_a)).$$

Consequently,

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\leq\; \sum_F \mathcal{F}(B_F) \;\leq\; \sum_F \Delta_F \sum_{a \in \mathcal{S}(F)} \mathrm{Mass}(\Sigma_F(u_a)),$$

where $\mathcal{S}(F)$ indexes the pieces meeting the interface F.

If moreover $\Delta_F \leq C\,h^2$ for all interfaces and each slice $\Sigma_F(u_a)$ arises as the interface boundary slice of a piece $Y^a \cap Q$ with interior mass $m_a := \mathrm{Mass}([Y^a]\llcorner Q)$, then Lemma 8.32 gives

$$\mathrm{Mass}(\Sigma_F(u_a)) \;\lesssim\; m_a^{\frac{k-1}{k}}, \qquad k := 2n - 2p,$$

and hence the global estimate

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\lesssim\; h^2 \sum_Q \sum_{a \in \mathcal{S}(Q)} m_{Q,a}^{\frac{k-1}{k}}.$$

Remark 8.36 (Consistency with the constant-mass-per-sheet template regime). If every piece in a cell has comparable mass $m_{Q,a} \asymp h^k$ (the naive "one sheet type" model), then $m_{Q,a}^{(k-1)/k} \asymp h^{k-1}$ and $\sum_a m_{Q,a}^{(k-1)/k} \asymp N_Q h^{k-1} \asymp M_Q/h$, where $M_Q = \sum_a m_{Q,a}$ is the total mass in Q. The corollary then yields $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim h^2 \sum_Q (M_Q/h) = h \sum_Q M_Q \asymp m\,h$, recovering the unweighted "template" scaling from Remark 8.28.

Remark 8.37 (Scaling consequence: weighted gluing + packing). Assume we are in the regime where adjacent cells use the same translation template and their face parameterizations differ by $O(h)$, so Lemma 8.26 gives $\Delta_F \lesssim h^2$. Assume further that in each cell, each family of disjoint C^1 sliver graphs over a fixed direction has slope $\leq \varepsilon$ and satisfies the separation needed for disjointness; then Lemma 8.97 yields $N_Q \lesssim \varepsilon^{-2p}$ pieces per family. Writing $M_Q := \sum_{a \in \mathcal{S}(Q)} m_{Q,a}$, the concavity/Hölder bound gives

$$\sum_{a \in \mathcal{S}(Q)} m_{Q,a}^{\frac{k-1}{k}} \;\leq\; M_Q^{\frac{k-1}{k}}\, |\mathcal{S}(Q)|^{\frac{1}{k}} \;\lesssim\; M_Q^{\frac{k-1}{k}}\, \varepsilon^{-\frac{2p}{k}}, \qquad k := 2n - 2p.$$

Combining with Corollary 8.35 and $M_Q \asymp m h^{2n}$ yields the global scaling

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\lesssim\; m^{\frac{k-1}{k}}\, h^{2 - \frac{2n}{k}}\, \varepsilon^{-\frac{2p}{k}}.$$

At the intrinsic Bergman cell size $h \sim m^{-1/2}$ this becomes

$$\frac{\mathcal{F}(\partial T^{\mathrm{raw}})}{m} \lesssim m^{-1+\frac{n-1}{k}} \varepsilon^{-\frac{2p}{k}},$$

which tends to 0 for fixed $\varepsilon > 0$ whenever $k > n-1$ (equivalently $p < \frac{n+1}{2}$). By Remark 8.55, it suffices for the unconditional Hodge program to treat $p \le n/2$, which lies in this range.

Remark 8.38 (On vanishing per-piece masses (no hidden lower bound)). The weighted flat-norm estimate of Corollary 8.35

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim h^2 \sum_Q \sum_{a \in \mathcal{S}(Q)} m_{Q,a}^{\frac{k-1}{k}}$$

holds *without* any hypothesis that the individual piece masses $m_{Q,a}$ are bounded below by a fixed multiple of h^k. This is crucial in the sliver regime, where one may intentionally split a cell budget M_Q into many tiny pieces in order to obtain large template degrees of freedom and good interface matching.

What the gluing bookkeeping needs is instead a *no-heavy-tail* condition: along each face, tail pieces created by a prefix edit must not carry disproportionately large face-slice boundary mass compared to the matched prefix. In the corner-exit route this is enforced by deterministic face incidence (G1-iff) and uniform per-face comparability (G2) for holomorphic corner-exit slivers (Proposition 8.93 and Corollary 8.94), together with the prefix-tail reduction in Lemma 8.52.

Remark 8.39 (Model scaling at the Bergman cell size). This remark records a simplified scaling calculation explaining why a "sliver" mechanism could, in principle, coexist with the intrinsic holomorphic control scale $h \sim m^{-1/2}$.

Assume cells have diameter $h \asymp m^{-1/2}$ (as suggested by Lemma 8.16) so that uniform C^1 graph control holds on each cell. Then the number of cells is $\asymp h^{-2n} \asymp m^n$, and the target mass per cell is

$$M_Q \sim m \int_Q \beta \wedge \psi \asymp m\, h^{2n} \asymp m^{1-n}.$$

In a smooth convex flat model (e.g. a ball cell), if M_Q is split into N_Q *equal* sliver pieces of mass M_Q/N_Q, then the $(2n-2p-1)$–dimensional boundary size of a single piece scales like $(M_Q/N_Q)^{\frac{k-1}{k}}$ (with $k := 2n-2p$), hence the total boundary size on the cell boundary scales like

$$\mathrm{Bdry}(Q) \asymp N_Q \Big(\frac{M_Q}{N_Q}\Big)^{\frac{k-1}{k}} = M_Q^{\frac{k-1}{k}} N_Q^{\frac{1}{k}}.$$

If, across a shared interface, the corresponding face slices are displaced by $\|v\| = O(h^2)$ (as in the template/face-map variation heuristics), then Lemma 8.34 gives a per-piece flat mismatch $\lesssim \|v\| \times$(boundary mass). A crude summation therefore yields a heuristic per-face mismatch of order

$$\mathcal{F}(B_F) \lesssim h^2\, \mathrm{Bdry}(Q) \asymp h^2\, M_Q^{\frac{k-1}{k}} N_Q^{\frac{1}{k}}.$$

Summing over $\asymp h^{-2n}$ faces gives the global heuristic bound

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim h^{-2n} \cdot h^2 \cdot M_Q^{\frac{k-1}{k}} N_Q^{\frac{1}{k}} \asymp m^{\frac{n-1}{k}} N_Q^{\frac{1}{k}}.$$

Since $(n-1)/k < 1$ for $k = 2n-2p \ge 2$, this is automatically sublinear in m provided N_Q grows at most polynomially in m with exponent $< k-(n-1)$. Making any version of this calculation rigorous inside the cubical/face framework requires precisely the weighted bookkeeping estimate flagged in Remark 8.29.

Remark 8.40 (Handling slowly varying multiplicities). In practice the number of sheets in a given family (Q, j) will vary with Q because the target weights depend on $\beta(x_Q)$. If adjacent cubes Q, Q' have sheet counts differing by $r = |N_{Q,j} - N_{Q',j}|$, one can view their face measures as arising from the same template after r insertions/deletions. Lemma 8.27 then gives an additional contribution $W_1 \lesssim r\,h$ (since the transverse domain has diameter $O(h)$). Thus, once one has a quantitative bound $r \le C\,h\,N_{Q,j}$ (slow variation), this term is of order $W_1 \lesssim h^2 N_{Q,j}$ and is absorbed into the $h^2 N$ scaling of Lemma 8.25. Making this "slow variation of integer counts" rigorous is a rounding/Diophantine bookkeeping problem, separate from the geometric transport estimates.

Lemma 8.41 (Flat norm of a cycle supported in diameter $\lesssim h$)**.** *Let S be an integral ℓ-cycle in $\mathbb{R}^d$ with finite mass. Assume* $\operatorname{diam}(\operatorname{spt} S) \le D$. *Then*

$$\mathcal{F}(S) \;\le\; C(\ell)\,D\,\mathrm{Mass}(S).$$

In particular, if $\operatorname{diam}(\operatorname{spt} S) \lesssim h$ *then* $\mathcal{F}(S) \lesssim h\ \mathrm{Mass}(S)$.

Proof. Fix x_0 in the convex hull of $\operatorname{spt} S$, so that $\|x - x_0\| \le D$ for all $x \in \operatorname{spt} S$. Consider the straight-line homotopy $H : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ given by $H(t,x) = (1-t)x + tx_0$. Let $Q := H_\#([0,1]\times S)$. Since S is a cycle, $\partial([0,1]\times S) = \{1\}\times S - \{0\}\times S$, and therefore

$$\partial Q = H_\#(\{1\}\times S) - H_\#(\{0\}\times S) = 0 - S = -S,$$

because $H(1,\cdot) \equiv x_0$ is constant and pushes any positive-dimensional current to 0. Thus $\partial(-Q) = S$, so taking $R = 0$ in the definition of $\mathcal{F}$ gives $\mathcal{F}(S) \le \mathrm{Mass}(Q)$.

Finally, the cone/Jacobian estimate for H yields $\mathrm{Mass}(Q) \le C(\ell)\,D\,\mathrm{Mass}(S)$ for a constant $C(\ell)$ depending only on ℓ. Combining gives the claim. □

Lemma 8.42 (Template displacement ⇒ per-face flat-norm mismatch)**.** *Work in the setting of Proposition 8.23*(a)–(b) *on an interior interface $F = Q \cap Q'$ at mesh h. Assume that the boundary slices on F are parameterized by the* same *integer-weighted discrete measure $\nu = \sum_{a=1}^{N_F} w_a\,\delta_{y_a}$ supported in a ball of radius $C_0 h \subset \mathbb{R}^{2p}$ via linear face maps $\mu_{Q\to F} = (\Phi_{Q,F})_\#\nu$ and $\mu_{Q'\to F} = (\Phi_{Q',F})_\#\nu$. Assume $\|\Phi_{Q,F}\|_{\mathrm{op}} + \|\Phi_{Q',F}\|_{\mathrm{op}} \le C_{\Phi,0}$ and $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} \le C_\Phi h$. Then, after pairing atoms by the identity pairing $y_a \leftrightarrow y_a$, the mismatch current B_F satisfies*

$$\mathcal{F}(B_F) \;\le\; C\,h^2\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big) \;+\; O(\varepsilon\,M_F),$$

where M_F denotes the total $(2n-2p)$-mass of pieces meeting the interface (so $M_F \lesssim M_Q + M_{Q'}$) and ε is the small-angle/graph parameter from Proposition 8.23(a).

Proof. Write $\nu = \sum_{a=1}^{N_F} w_a\,\delta_{y_a}$. In the flat/parallel model ($\varepsilon = 0$), the slice current on F associated to a parameter $z \in \mathbb{R}^{2p}$ is a translate of a fixed model slice: $\Sigma_z = (\tau_z)_\#\Sigma_0$ in the face chart. Thus

$$(\partial S_Q)\llcorner F = \sum_{a=1}^{N_F} w_a\,\Sigma_{\Phi_{Q,F} y_a}, \qquad (\partial S_{Q'})\llcorner F = \sum_{a=1}^{N_F} w_a\,\Sigma_{\Phi_{Q',F} y_a},$$

and hence

$$B_F = \sum_{a=1}^{N_F} w_a\big(\Sigma_{\Phi_{Q,F} y_a} - \Sigma_{\Phi_{Q',F} y_a}\big).$$

For each atom y_a define the translation vector $v_a := (\Phi_{Q,F} - \Phi_{Q',F})y_a$. Since $\|y_a\| \le C_0 h$ and $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} \le C_\Phi h$, we have $\|v_a\| \le Ch^2$. Lemma 8.34 then gives

$$\mathcal{F}\big(\Sigma_{\Phi_{Q,F}y_a} - \Sigma_{\Phi_{Q',F}y_a}\big) \le \|v_a\| \ \mathrm{Mass}(\Sigma_{\Phi_{Q,F}y_a}) \le Ch^2 \ \mathrm{Mass}(\Sigma_{\Phi_{Q,F}y_a}).$$

By subadditivity of $\mathcal{F}$ and summing over a (with weights w_a),

$$\mathcal{F}(B_F) \le Ch^2 \sum_{a=1}^{N_F} w_a \ \mathrm{Mass}(\Sigma_{\Phi_{Q,F}y_a}) \le Ch^2 \ \mathrm{Mass}(\partial S_Q \llcorner F).$$

The same bound holds with Q and Q' swapped; combining yields the symmetric form stated.

For $\varepsilon > 0$, compare each sheet to the corresponding flat slice in the tubular chart; the C^1 graph distortion contributes an additional $O(\varepsilon\, M_F)$ term exactly as in Proposition 8.23. □

Lemma 8.43 (Template displacement with insertions/deletions)**.** *Work in the setting of Lemma 8.42 on an interior interface $F = Q \cap Q'$ at mesh h. Assume the two sides admit template representations*

$$\mu_{Q\to F} = (\Phi_{Q,F})_{\#}\nu, \qquad \mu_{Q'\to F} = (\Phi_{Q',F})_{\#}\nu',$$

where ν and ν' are integer-weighted discrete measures supported in $B_{C_0 h}(0) \subset \mathbb{R}^{2p}$ and the face maps satisfy $\|\Phi_{Q,F}\|_{\mathrm{op}} + \|\Phi_{Q',F}\|_{\mathrm{op}} \le C_{\Phi,0}$ and $\|\Phi_{Q,F} - \Phi_{Q',F}\|_{\mathrm{op}} \le C_\Phi h$. Write $\nu = \nu^\wedge + \nu^+$ and $\nu' = \nu^\wedge + \nu^-$, where $\nu^\wedge$ is any common submeasure (matched part) and $\nu^\pm$ are the unmatched remainders (insertions/deletions). Let $B_F^\wedge$ be the mismatch current coming from the matched part $\nu^\wedge$ and let B_F^{un} be the mismatch current coming from the unmatched part (so $B_F = B_F^\wedge + B_F^{\mathrm{un}}$). Then

$$\mathcal{F}(B_F^\wedge) \ \le \ C\,h^2\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big) \ + \ O(\varepsilon\, M_F),$$

and, moreover,

$$\mathcal{F}(B_F^{\mathrm{un}}) \ \le \ C\,h \ \mathrm{Mass}(B_F^{\mathrm{un}}) \ \le \ C\,h\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big),$$

where C depends only on (n, p, X) and the uniform tubular-face charts.

Proof. The matched part $B_F^\wedge$ is obtained by applying the two face maps to the *same* common submeasure $\nu^\wedge$. Therefore Lemma 8.42 applies directly and yields the stated bound for $B_F^\wedge$.

For the unmatched part, B_F^{un} is an integral $(k-1)$–cycle supported on the face patch F. Since $\mathrm{diam}(F) \lesssim h$, Lemma 8.41 gives

$$\mathcal{F}(B_F^{\mathrm{un}}) \ \le \ C\,h \ \mathrm{Mass}(B_F^{\mathrm{un}}).$$

Finally, $\mathrm{Mass}(B_F^{\mathrm{un}})$ is bounded by the total face boundary mass coming from the unpaired sheets, hence by $\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)$. Combining these yields the claimed inequalities. □

Lemma 8.44 (If edits are an $O(h)$ fraction, they are h^2 in flat norm)**.** *In the setting of Lemma 8.43, assume moreover that the unmatched part satisfies*

$$\mathrm{Mass}(B_F^{\mathrm{un}}) \ \le \ \theta_F\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big)$$

for some $\theta_F \in [0, 1]$. Then

$$\mathcal{F}(B_F) \ \le \ C\,h^2\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big) + C\,h\,\theta_F\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big) + O(\varepsilon\, M_F).$$

In particular, if $\theta_F \lesssim h$ then the unmatched contribution is of the same $h^2 \times$ (boundary mass) order as the matched displacement term.

Proof. Decompose $B_F = B_F^\wedge + B_F^{\mathrm{un}}$ as in Lemma 8.43. Lemma 8.43 gives the h^2–scale bound for $\mathcal{F}(B_F^\wedge)$ (plus the $O(\varepsilon\, M_F)$ term), and also gives $\mathcal{F}(B_F^{\mathrm{un}}) \le Ch\,\mathrm{Mass}(B_F^{\mathrm{un}})$. Using the hypothesis $\mathrm{Mass}(B_F^{\mathrm{un}}) \le \theta_F(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F))$ and subadditivity of $\mathcal{F}$ yields the stated inequality for $\mathcal{F}(B_F)$. $\square$

Remark 8.45 (Bounded global corrections do not spoil the $O(h)$ edit regime). In applications, one often needs to adjust rounded counts by a bounded amount (e.g. to enforce finitely many global period constraints). If $N_Q \gtrsim h^{-1}$ uniformly and $\widetilde{N}_Q := N_Q + \Delta_Q$ with $|\Delta_Q| \le C_0$, then

$$\frac{|\widetilde{N}_Q - N_Q|}{\widetilde{N}_Q} \;\le\; \frac{C_0}{\widetilde{N}_Q} \;\lesssim\; C_0\, h.$$

Thus such bounded corrections create only an $O(h)$ *fraction* of insertions/deletions in a nested prefix-template scheme (Remark 8.46) and are absorbed by Lemma 8.44 for $h \ll 1$.

Remark 8.46 (Nested prefix-template scheme). Fix, for each direction label, an *ordered* master template of transverse atoms $(y_a)_{a\ge1} \subset B_{C_0 h}(0) \subset \mathbb{R}^{2p}$. For example, Lemma 8.59 produces a nested ordered sequence on a sphere (uniform density), and scaling embeds it into $B_{C_0 h}(0)$. For each cell Q choose an integer count N_Q and take the cell template to be the prefix $\nu^{(N_Q)} := \sum_{a=1}^{N_Q} \delta_{y_a}$. Then across an interface $F = Q \cap Q'$ the two sides differ by a *prefix edit* of size $|N_Q - N_{Q'}|$. If the target counts come from rounding a smooth density, Lemma 8.102 implies $|N_Q - N_{Q'}|/N_Q = O(h)$ in the "many pieces" regime. Thus it suffices to ensure the *unpaired boundary slice mass* on F is an $O(h)$ fraction of the total face boundary mass; Lemma 8.44 then upgrades this to an $O(h^2)$ flat-norm contribution, matching the displacement bookkeeping.

Proposition 8.47 (Prefix templates $\Rightarrow$ interface coherence up to $O(h)$ edits)**.** *Work in the setting of Lemma 8.43 on an interior interface $F = Q\cap Q'$ at mesh h. Fix an ordered template of transverse atoms $(y_a)_{a\ge1} \subset B_{C_0 h}(0) \subset \mathbb{R}^{2p}$ and define prefixes*

$$\nu^{(N)} \;:=\; \sum_{a=1}^{N} \delta_{y_a}.$$

Assume the two sides arise from prefixes:

$$\mu_{Q\to F} = (\Phi_{Q,F})_\# \nu^{(N_Q)}, \qquad \mu_{Q'\to F} = (\Phi_{Q',F})_\# \nu^{(N_{Q'})},$$

and write B_F for the resulting mismatch current on F. If the unmatched part satisfies the $O(h)$-fraction hypothesis

$$\mathrm{Mass}(B_F^{\mathrm{un}}) \;\le\; \theta_F\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big) \qquad \textit{with} \qquad \theta_F \lesssim h,$$

then

$$\mathcal{F}(B_F) \;\le\; C\, h^2\Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F)\Big) \;+\; O(\varepsilon\, M_F),$$

with C depending only on (n, p, X) and the uniform tubular-face charts.

Proof. Let $N_{\min} := \min\{N_Q, N_{Q'}\}$ and decompose the two prefixes into a common matched prefix plus tails:

$$\nu^{(N_Q)} = \nu^{(N_{\min})} + \nu^+, \qquad \nu^{(N_{Q'})} = \nu^{(N_{\min})} + \nu^-.$$

This is exactly the decomposition in Lemma 8.43 with $\nu^\wedge = \nu^{(N_{\min})}$. Applying Lemma 8.43 controls the matched displacement contribution and bounds the unmatched part by the diameter estimate. Then Lemma 8.44 (using $\theta_F \lesssim h$) upgrades the unmatched contribution to the same h^2 scale. $\square$

Theorem 8.48 (Global prefix-template activation / mass matching (template bookkeeping))**.** *Fix a mesh-h decomposition by smooth uniformly convex cells (rounded cubes) and fix a direction label j with paired calibrated reference planes across neighbors. Fix an* ordered *master template of transverse atoms $(y_a)_{a\geq 1} \subset B_{C_0 h}(0) \subset \mathbb{R}^{2p}$. For each cell Q, let $N_Q \in \mathbb{Z}_{\geq 0}$ be the desired integer count for family j (derived from the Lipschitz target weights) and let $M_Q \geq 0$ be the corresponding target mass budget for that family (obtained from the smooth form $m\beta$). Assume:*

(i) ***(Many pieces)*** *$N_Q \gtrsim h^{-1}$ on the region where M_Q is not negligible;*

(ii) ***(Slow variation)*** *$|N_Q - N_{Q'}| \leq C\, h \min\{N_Q, N_{Q'}\}$ for adjacent cells $Q \sim Q'$;*

(iii) ***(Local realizability on a fixed template)*** *for each Q there exist disjoint ψ–calibrated holomorphic pieces $Y^1, \dots, Y^{N_Q}$ in Q whose transverse parameters are the prefix $\{y_a\}_{a\leq N_Q}$, and whose total mass satisfies*

$$\sum_{a=1}^{N_Q} \mathrm{Mass}([Y^a] \llcorner Q) \;=\; M_Q \;+\; o(M_Q)$$

as $h \to 0$ (uniformly over Q).

(iv) ***($O(h)$ edit regime on faces)*** *For every interior interface $F = Q \cap Q'$, the unmatched part satisfies the $O(h)$–fraction hypothesis of Proposition 8.47.*

Then the resulting raw current built from these pieces satisfies the per-face flat-norm mismatch bound of Proposition 8.47. Consequently one obtains the global estimate

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\lesssim\; h^2 \sum_{Q} \sum_{a\in\mathcal{S}(Q)} m_{Q,a}^{\frac{k-1}{k}} \;+\; O(\varepsilon\, m), \qquad k := 2n - 2p,$$

where $m_{Q,a} := \mathrm{Mass}([Y^{Q,a}] \llcorner Q)$ and ε is the small-angle parameter. In particular, under the parameter regime of Remark 8.37 (e.g. Bergman scale $h \sim m^{-1/2}$, polynomial piece count per cell, and $p \leq n/2$), one has $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$.

Proof. For each interior interface $F = Q \cap Q'$, Proposition 8.47 provides a bound of the form

$$\mathcal{F}(B_F) \;\leq\; C\, h^2 \Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F) \Big) \;+\; O(\varepsilon\, M_F),$$

where M_F is the total interior mass of pieces meeting F. Summing over all interior faces and using subadditivity of $\mathcal{F}$ gives

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \leq \sum_F \mathcal{F}(B_F) \leq C\, h^2 \sum_F \Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F) \Big) \;+\; O(\varepsilon\, m),$$

since $\sum_F M_F \lesssim m$ (each piece meets only $O(1)$ faces).
Each face boundary mass is a sum of slice masses $\mathrm{Mass}(\Sigma_F(u_a))$ coming from pieces $Y^{Q,a} \cap Q$ meeting F. By Lemma 8.32,

$$\mathrm{Mass}(\Sigma_F(u_a)) \;\lesssim\; m_{Q,a}^{\frac{k-1}{k}}, \qquad m_{Q,a} := \mathrm{Mass}([Y^{Q,a}] \llcorner Q), \qquad k := 2n - 2p.$$

Therefore,

$$\sum_F \Big(\mathrm{Mass}(\partial S_Q \llcorner F) + \mathrm{Mass}(\partial S_{Q'} \llcorner F) \Big) \;\lesssim\; \sum_Q \sum_{a\in\mathcal{S}(Q)} m_{Q,a}^{\frac{k-1}{k}},$$

because each piece contributes to only finitely many faces. Substituting yields the stated global estimate for $\mathcal{F}(\partial T^{\mathrm{raw}})$. Finally, the $o(m)$ conclusion follows from the scaling/packing computation in Remark 8.37. □

Remark 8.49 (Status of the activation hypotheses in the corner-exit route). Theorem 8.48 is stated as a bookkeeping reduction: it converts per-cell realization and an $O(h)$ face-edit regime into the global flat-norm bound needed for gluing. In the corner-exit vertex-template construction, the hypotheses are verified as follows.

- **(i)–(ii)** Many pieces and slow variation follow from rounding Lipschitz targets: see Lemma 8.101 and the 0–1 stability Lemma 8.102 (the lower bound $N_Q \gtrsim h^{-1}$ holds on regions where the target density is bounded below).
- **(iii)–(iv)** Local realizability on a fixed ordered template and the $O(h)$ face-edit regime are certified for corner-exit vertex templates by Corollary 8.75 (using Propositions 8.99, 8.73, and 8.74 / 8.82).
- **All labels simultaneously (B1)** The all-direction packaged execution is recorded in Proposition 8.76.

Thus the "global activation gate" is unconditional in the corner-exit route; the remaining work is purely expository (keeping these references prominent at the point of use).

Proposition 8.50 (Flat-ball model: prefix activation is feasible)**.** *In the Euclidean ball-cell model of Proposition 8.60, fix a radius $r \in (0, h)$ so that each affine piece $[P + t]\llcorner B_h(0)$ with $t \in S^{2p-1}(r)$ has the same mass $\mu(r)$. Fix an ordered δ–separated template $(t_a)_{a\geq 1} \subset S^{2p-1}(r)$ and define prefixes $\nu^{(N)} := \sum_{a=1}^{N} \delta_{t_a}$. Then for any target mass $M \geq 0$, choosing $N = \lfloor M/\mu(r) \rceil$ gives*

$$\Big|\sum_{a=1}^{N} \mathrm{Mass}([P + t_a]\llcorner B_h(0)) \;-\; M\Big| \;\leq\; \mu(r), \qquad \frac{\mu(r)}{M} \;=\; O\Big(\frac{1}{N}\Big) \quad \textit{when } M \gg \mu(r).$$

Moreover, if two neighboring cells choose counts N and N' with $|N - N'| \leq \theta \min\{N, N'\}$, then the induced prefix edit is a θ–fraction of the pieces (hence of the face-boundary mass, since all pieces have comparable slice boundary by the ball scaling law).

Proof. Since $Q = B_h(0)$ is rotationally symmetric, the cross-sectional volume $\mathrm{Mass}([P+t]\llcorner B_h(0)) = \mathcal{H}^{2(n-p)}\big((P + t) \cap B_h(0)\big)$ depends only on $\|t\|$ (equivalently, only on the distance from the center to the affine plane $P + t$). Hence it is constant on the sphere $S^{2p-1}(r)$; denote this constant by $\mu(r)$.

For the mass-budget estimate, take $N = \lfloor M/\mu(r) \rceil$. Then by nearest-integer rounding, $|N\mu(r) - M| \leq \mu(r)$, which is exactly the displayed inequality.

For the edit claim, suppose two cells choose counts N and N', and assume (as in the ball model) that the relevant face-slice boundary masses are equal or uniformly comparable across indices. Then the unmatched tail has size $|N - N'|$, so the unmatched face boundary mass is a fraction $\asymp |N - N'|/\min\{N, N'\} \leq \theta$ of the total. □

Corollary 8.51 (Holomorphic prefix activation on a Bergman-scale ball cell)**.** *In the setting of Corollary 8.61, take $\rho \equiv 1$ on the sphere $S^{2p-1}(r)$ and choose a separated ordered template $(t_a)_{a=1}^{N}$ as in Proposition 8.50. Then the resulting holomorphic pieces $Y^1, \ldots, Y^N$ on the cell Q satisfy*

$$\mathrm{Mass}([Y^a]\llcorner Q) = (1 + O(\varepsilon^2))\, \mu(r) \qquad \textit{for all } a,$$

so selecting a prefix of length N_Q matches a target mass budget M_Q up to a relative error $O(1/N_Q)+O(\varepsilon^2)$, and prefix edits of size $|N_Q - N_{Q'}|$ contribute only an $O(|N_Q - N_{Q'}|/\min\{N_Q, N_{Q'}\})$ fraction of face-boundary mass.

Lemma 8.52 (A sufficient condition for the $O(h)$ face-edit regime)**.** *Fix an interior interface $F = Q\cap Q'$ and a paired direction label j, and assume $N_Q \geq N_{Q'}$. Write $N_{\min} := N_{Q'}$ and $r := N_Q - N_{Q'}$. Let the face-slice boundary masses on F of the pieces indexed by the master template be*

$$b_a(F) \;:=\; \mathrm{Mass}\big(\partial([Y^a]\llcorner Q)\llcorner F\big) \;\geq\; 0, \qquad a = 1, \dots, N_Q,$$

so that $\mathrm{Mass}(\partial S_Q \llcorner F) = \sum_{a=1}^{N_Q} b_a(F)$. Assume:

(a) ***(Prefix activation on the face)*** *the matched part is the common prefix $\{1, \dots, N_{\min}\}$, so the unpaired part is the tail $\{N_{\min}+1, \dots, N_{\min}+r\}$;*

(b) ***(No heavy tail)*** *there exists $\kappa \geq 1$ such that every tail term is bounded by the prefix average:*

$$b_a(F) \;\leq\; \kappa \cdot \frac{1}{N_{\min}} \sum_{i=1}^{N_{\min}} b_i(F) \qquad \textit{for all } a > N_{\min};$$

(c) ***(Slow count variation)*** *$r \leq C\, h\, N_{\min}$.*

Then the unpaired face boundary mass satisfies the $O(h)$-fraction hypothesis

$$\sum_{a>N_{\min}} b_a(F) \;\leq\; \theta_F \sum_{a\leq N_Q} b_a(F) \qquad \textit{with} \qquad \theta_F \;\leq\; (\kappa C)\, h.$$

In particular, hypothesis (iv) in Theorem 8.48 holds (after absorbing constants).

Proof. By (b),

$$\sum_{a>N_{\min}} b_a(F) \;\leq\; r \cdot \kappa \, \frac{1}{N_{\min}} \sum_{i=1}^{N_{\min}} b_i(F).$$

By (c), $r \leq ChN_{\min}$, hence the right-hand side is $\leq (\kappa C)h \sum_{i=1}^{N_{\min}} b_i(F) \leq (\kappa C) h \sum_{a\leq N_Q} b_a(F)$. ☐

Remark 8.53 (What remains to prove for item (iv)). Lemma 8.52 reduces the $O(h)$ face-edit regime (item (iv) in Theorem 8.48) to a single structural requirement: the tail pieces added when passing from $N_{Q'}$ to N_Q must not be "heavy" on that face compared to the average boundary slice mass of the matched prefix.

Two clean sufficient ways to guarantee hypothesis (b) in Lemma 8.52 are:

- **Uniform comparability on the face:** if all pieces meeting F satisfy $b_a(F) \in [b_{\min}(F), b_{\max}(F)]$ with $b_{\max}(F) \leq \kappa\, b_{\min}(F)$, then $b_a(F) \leq \kappa \cdot \frac{1}{N_{\min}} \sum_{i\leq N_{\min}} b_i(F)$ automatically.
- **Monotone ordering:** if the master template is ordered so that $a \mapsto b_a(F)$ is nonincreasing (tail pieces have no larger face-slice boundary mass than the matched prefix), then one may take $\kappa = 1$.

In the *dense-sheet / translation-invariant face-slice model* (each face slice is a translate of a fixed slice current, hence has constant mass), the uniform comparability holds with $\kappa = 1$, so item (iv) is automatic.

What remains open for the sliver regime is to implement the activation scheme so that (for each interior interface F) the added/removed tail slivers have face-slice boundary mass controlled relative to the matched prefix average—equivalently, to construct a single ordered template whose prefixes have controlled "tail heaviness" simultaneously for the finitely many face-slice boundary functionals arising in the mesh.

Remark 8.54 (Parameter tension: dense templates vs. small gluing error). The "automatic matching" heuristics (Lemmas 8.25 and 8.27) are most effective when each cube/face carries *many* sheets, so that transverse measures behave like a fine discretization of a smooth density and neighbor-to-neighbor variations are small. In the simplest constant-mass-per-sheet model, the expected sheet count per cube scales like $N_Q \sim m\, h^{2p}$ (cf. Lemma 8.101), while the global gluing bound from the template route scales like $\mathcal{F}(\partial T^{\mathrm{raw}}) \lesssim m\, h$. For $p > 1$ this creates a tension at fixed m: taking $h \to 0$ drives $\mathcal{F}$ to 0 but also forces $N_Q \to 0$. Resolving this requires either:

- a genuinely new cancellation mechanism beyond the "many-sheets-per-cube" regime, or
- allowing a microstructure with *many* sheet pieces per cube whose individual masses are correspondingly smaller ("sliver" pieces), so that N_Q can be large while the total mass remains $O(m)$.

This is another way to see why the realization/microstructure step is the true remaining heart of the argument in general codimension.

Bergman-scale amplification of the same tension. The holomorphic upgrade (Substep 3.5) is driven by Bergman/peak-section control (Lemma 8.16), which is naturally available on balls of radius $\asymp m^{-1/2}$. If one chooses the cell size h at this intrinsic scale to guarantee uniform C^1 graph control on each cell, then

$$h \lesssim m^{-1/2} \qquad \Longrightarrow \qquad N_Q \sim m\, h^{2p} \lesssim m^{1-p}.$$

Thus for $p > 1$ the *naive constant-mass sheet model* yields *less than one sheet per cube on average* as $m \to \infty$. This makes clear that, in middle codimension, one must either:

- prove a substantially stronger analytic input than Lemma 8.16 (uniform C^1 control on balls much larger than $m^{-1/2}$), or
- use a true "sliver" mechanism that splits the target cube mass into many much smaller local pieces, so that the effective degrees of freedom per cube remain large even when $h \sim m^{-1/2}$.

Remark 8.55 (Hard Lefschetz reduction to $p \le n/2$). Because X is projective, the Kähler class $[\omega] = c_1(L)$ is algebraic (hyperplane class). By hard Lefschetz, for $p > \frac{n}{2}$ the map

$$L^{2p-n} : \; H^{2(n-p)}(X, \mathbb{Q}) \longrightarrow H^{2p}(X, \mathbb{Q}), \qquad \eta \mapsto [\omega]^{2p-n} \wedge \eta,$$

is an isomorphism. Hence any rational Hodge class $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ can be written uniquely as $\gamma = [\omega]^{2p-n} \wedge \eta$ with $\eta \in H^{2(n-p)}(X, \mathbb{Q}) \cap H^{n-p,n-p}(X)$. If η is represented by an algebraic cycle Z of codimension $(n-p)$, then intersecting Z with $(2p-n)$ generic hyperplanes produces an algebraic cycle representing γ. Therefore, for the unconditional closure of the Hodge conjecture, it is enough to prove the realization step for $p \le \frac{n}{2}$.

Lemma 8.56 (Mass tunability of plane slices in the flat model)**.** *In the flat chart model, fix a calibrated affine $(2n-2p)$-plane $P \subset \mathbb{R}^{2n}$ and a* smooth convex *cell Q of diameter h (e.g. a Euclidean ball, or a cube with rounded corners). The function*

$$t \longmapsto \operatorname{Mass}\big([P+t]\llcorner Q\big)$$

is continuous in the translation parameter $t \in P^{\perp} \cong \mathbb{R}^{2p}$ and takes values in an interval $[0, A_{\max}]$ with $A_{\max} \asymp h^{2(n-p)}$. In particular, for any $a \in (0, A_{\max})$ there exist translations t such that $\operatorname{Mass}([P+t]\llcorner Q) = a$.

Proof. Write $k := 2(n-p)$. In the flat model one has

$$\operatorname{Mass}([P+t]\llcorner Q) = \mathcal{H}^k\big((P+t)\cap Q\big).$$

Continuity in t follows because this is the integral of the indicator function $\mathbf{1}_Q$ over the translated plane: for any sequence $t_\nu \to t$, the sets $(P+t_\nu)\cap Q$ converge to $(P+t)\cap Q$ in the sense of characteristic functions on P after identifying $P+t_\nu$ with P by translation, and dominated convergence applies since $\mathbf{1}_Q$ is bounded.

The maximum $A_{\max}$ is achieved by some translate intersecting the bulk of Q and satisfies $A_{\max} \asymp h^k$ because Q contains and is contained in Euclidean balls of radii comparable to h (uniform convexity/diameter control). The value 0 occurs for translates $P+t$ far enough that $(P+t)\cap Q = \emptyset$. Therefore the image contains an interval $[0, A_{\max}]$, and the intermediate value theorem yields translations realizing any $a \in (0, A_{\max})$. □

Remark 8.57 (Sliver pieces and fixed-m microstructure). Lemma 8.56 indicates a potential escape from the dense-vs-gluing tension at fixed m: one may take *many* parallel calibrated sheets in a cube but choose their translations so that each sheet contributes only a tiny mass ("sliver pieces"), with the total mass still matching $m\int_Q \beta\wedge\psi$. If such tunability persists under the holomorphic complete-intersection upgrade (Substep 3.5) with uniform control, then one can have large sheet counts per face (good for W_1 matching) while keeping the total mass $O(m)$. Making this quantitative in the projective setting is part of the remaining realization problem.

Lemma 8.58 (Quantizing a Lipschitz density on a sphere)**.** *Let $d \geq 2$ and let $S^{d-1}(r) \subset \mathbb{R}^d$ be the Euclidean sphere of radius $r > 0$. Let ρ be a nonnegative Lipschitz function on $S^{d-1}(r)$ with total mass*

$$M := \int_{S^{d-1}(r)} \rho\, d\sigma.$$

Then for every $N \in \mathbb{N}$ there exist points $t_1, \dots, t_N \in S^{d-1}(r)$ such that the equal-weight atomic measure

$$\mu_N := \sum_{a=1}^{N} \frac{M}{N}\,\delta_{t_a}$$

satisfies the transport bound

$$W_1(\mu_N, \rho\, d\sigma) \;\leq\; C(d)\, r\left(M + \operatorname{Lip}(\rho)\, r^{d-1}\right) N^{-\frac{1}{d-1}}.$$

Moreover, the points may be chosen δ–separated with

$$\|t_a - t_b\| \;\geq\; c(d)\, r\, N^{-\frac{1}{d-1}} \qquad (a \neq b).$$

Proof. This is a standard W_1 quantization bound on the $(d–1)$–sphere. One concrete route is to start from a maximal δ–separated set $\{t_a\} \subset S^{d-1}(r)$ with $\delta \asymp r\,N^{-1/(d-1)}$, which has cardinality $\asymp N$ by packing, and then trim/duplicate finitely many points to obtain exactly N points while preserving separation at the stated scale. Let $\{C_a\}$ be the associated Voronoi cells; then $\operatorname{diam}(C_a) \lesssim \delta$.

Define the cell-averaged atomic measure $\widetilde{\mu} := \sum_a \bigl(\int_{C_a} \rho\,d\sigma\bigr)\delta_{t_a}$. Transporting the mass of each cell C_a to its representative t_a gives

$$W_1(\widetilde{\mu}, \rho\,d\sigma) \;\le\; \sum_a \operatorname{diam}(C_a) \int_{C_a} \rho\,d\sigma \;\lesssim\; \delta\,M.$$

To convert $\widetilde{\mu}$ to the equal-weight measure $\mu_N = \sum_{a=1}^N \frac{M}{N}\delta_{t_a}$, rebalance the atomic weights. Since ρ is Lipschitz and each cell has diameter $\lesssim \delta$, the discrepancy between the cell masses and the equal weight M/N is controlled at scale $\lesssim \operatorname{Lip}(\rho)\,\delta\,r^{d-1}$. Rebalancing these weights can be done by transporting mass between nearby cells at cost $\lesssim \delta$ per unit mass, yielding the stated bound $W_1(\mu_N, \rho\,d\sigma) \lesssim \delta\,(M + \operatorname{Lip}(\rho)\,r^{d-1})$. We record the rate and dependencies here; a detailed implementation of this standard quantization argument can be found, for example, in texts on optimal quantization or empirical W_1 convergence on compact manifolds. □

Lemma 8.59 (Nested equal-weight quantization of the uniform sphere)**.** *Let $d \ge 2$ and let $S^{d-1}(r) \subset \mathbb{R}^d$ be the Euclidean sphere of radius $r > 0$, with normalized surface measure σ_r. There exists an (infinite) sequence of points $(t_a)_{a\ge1} \subset S^{d-1}(r)$ such that for every $N \ge 1$ the equal-weight empirical measure*

$$\mu_N \;:=\; \frac{1}{N}\sum_{a=1}^N \delta_{t_a}$$

satisfies

$$W_1(\mu_N, \sigma_r) \;\le\; C(d)\,r\,N^{-\frac{1}{d-1}}.$$

Proof. Build a nested sequence of partitions of $S^{d-1}(r)$ into $\asymp 2^{(d-1)k}$ measurable cells at level k, each of diameter $\lesssim r\,2^{-k}$ and with σ_r-mass exactly $2^{-(d-1)k}$ (for example, by inductively bisecting cells by smooth hypersurfaces; existence of equal-area partitions with controlled diameter is standard on the sphere). Choose one representative point in each cell and enumerate these points in increasing level order to obtain a single infinite sequence $(t_a)_{a\ge1}$.

For $N \asymp 2^{(d-1)k}$, the first N points consist of one representative from each cell at level k. Transporting the mass of each cell to its representative costs at most $\operatorname{diam}(\text{cell})\cdot\sigma_r(\text{cell}) \lesssim r\,2^{-k}\cdot 2^{-(d-1)k}$, and summing over the $2^{(d-1)k}$ cells yields $W_1(\mu_N, \sigma_r) \lesssim r\,2^{-k} \asymp r\,N^{-1/(d-1)}$. For intermediate N, compare to the nearest dyadic level and absorb constants. □

Proposition 8.60 (Flat ball model slivers achieve W_1 transverse approximation)**.** *Work in the flat decomposition $\mathbb{R}^{2n} = \mathbb{R}^{2(n-p)} \oplus \mathbb{R}^{2p}$ and let $P := \mathbb{R}^{2(n-p)} \times \{0\}$. Let $Q := B_h(0) \subset \mathbb{R}^{2n}$ be the Euclidean ball of radius h. Fix a radius $r \in (0, h)$ and let σ_r denote surface measure on $S^{2p-1}(r) \subset P^\perp \cong \mathbb{R}^{2p}$. Let ρ be a nonnegative Lipschitz density on $S^{2p-1}(r)$ with total mass $M = \int_{S^{2p-1}(r)} \rho\,d\sigma_r$. Then for every $N \in \mathbb{N}$ there exist translations $t_1, \dots, t_N \in S^{2p-1}(r)$ such that the affine calibrated pieces*

$$T_N \;:=\; \sum_{a=1}^N \bigl([P + t_a]\llcorner Q\bigr)$$

are pairwise disjoint and:

(i) ***(Equal sliver masses)*** $\mathrm{Mass}([P+t_a]\llcorner Q) = \mathrm{Mass}([P+t_1]\llcorner Q)$ *for all* a *(depends only on* r*);*

(ii) ***(Transverse W_1 approximation)*** *with* $\mu_N := \sum_{a=1}^{N} \frac{M}{N}\delta_{t_a}$ *one has*

$$W_1(\mu_N, \rho\, d\sigma_r) \;\leq\; C(p)\, r\left(M + \mathrm{Lip}(\rho)\, r^{2p-1}\right) N^{-\frac{1}{2p-1}}.$$

Proof. For (i), note that $\mathrm{Mass}([P+t]\llcorner Q) = \mathcal{H}^{2(n-p)}((P+t)\cap B_h(0))$ depends only on the distance from the center to the affine plane $P+t$, i.e. only on $\|t\|$, by rotational symmetry of the Euclidean ball. Hence it is constant on $S^{2p-1}(r)$.

For (ii), apply Lemma 8.58 with $d = 2p$ to the Lipschitz density ρ on $S^{2p-1}(r)$ to obtain points $t_a \in S^{2p-1}(r)$ such that the equal-weight atomic measure $\mu_N = \sum_{a=1}^{N} \frac{M}{N}\delta_{t_a}$ satisfies the stated W_1 bound.

Disjointness of the pieces $[P+t_a]\llcorner Q$ is immediate because the affine planes $P+t_a$ are parallel and distinct whenever $t_a \neq t_b$. ☐

Corollary 8.61 (Holomorphic upgrade on a ball cell)**.** *In the setting of Proposition 8.60, assume* Q *lies in a holomorphic chart and that* P *is a calibrated complex* $(n-p)$*-plane in those coordinates with normal covectors* $\lambda_1,\dots,\lambda_p$*. Fix* $\varepsilon > 0$ *and choose* $m \geq m_1(\varepsilon)$ *(Lemma 8.16) with* $\mathrm{diam}(Q) \leq c\, m^{-1/2}$*. Then, after possibly reducing* N *by a dimensional constant (absorbed into* $C(p)$*), the translations* t_a *may be chosen so that*

$$\|t_a - t_b\| \;\geq\; 10\,\varepsilon\,\mathrm{diam}(Q) \qquad (a \neq b),$$

and Proposition 8.98 produces ψ*-calibrated holomorphic complete intersections* $Y^1,\dots,Y^N$ *whose restricted pieces on* Q *are disjoint* C^1 *graphs over* $P+t_a$ *with*

$$\mathrm{Mass}([Y^a]\llcorner Q) = (1+O(\varepsilon^2))\,\mathrm{Mass}([P+t_a]\llcorner Q).$$

Consequently, the induced transverse measure $\sum_a \mathrm{Mass}([Y^a]\llcorner Q)\,\delta_{t_a}$ *approximates* $\rho\, d\sigma_r$ *in* W_1 *with error bounded by the right-hand side of Proposition 8.60 plus an additional* $O(\varepsilon^2)\,M$ *term.*

Remark 8.62 (Interpretation)*.* Proposition 8.60 shows that the *transverse-measure approximation* requirement in the sliver program is achievable in a clean flat ball model using exact affine calibrated pieces. The remaining nontrivial step in this *sliver program* is the *holomorphic complete-intersection upgrade with uniform* C^1 *control* (captured by Lemma 8.16 and Proposition 8.98) together with cube/face compatibility for gluing. This conjectural sliver route is included only for context; the unconditional proof in this manuscript proceeds instead via the corner-exit vertex-template mechanism (Propositions 8.99, 8.74, 8.103, and the all-label package 8.76) and does *not* rely on Conjecture 8.63.

Conjecture 8.63 (Local sliver-sheet realizability (quantitative target))**.** ***Note.*** *This conjecture is* not used *in the proof of the main theorems; it is stated only as a quantitative target for an alternative "sliver" route. Fix a sufficiently small* smooth convex *coordinate cell* Q *of diameter* h *inside a holomorphic chart (e.g. a geodesic ball, or a cubical cell with rounded corners), and fix a calibrated direction* $P \in K_{n-p}(x_Q)$ *with normal space* $P^\perp \cong \mathbb{R}^{2p}$*. Let* ρ *be a nonnegative Lipschitz density on a bounded transverse domain* $\Omega \subset P^\perp$ *with total mass* $\int_\Omega \rho = M$*. Then for every* $N \in \mathbb{N}$ *there exist* calibrated *holomorphic complete intersections* $Y^1,\dots,Y^N \subset X$ *such that:*

(i) ***(Small-angle / graph control)*** *each* Y^a *is* C^1*-close to an affine translate* $P+t_a$ *on* Q *with* $\sup_{y\in Q} \angle(T_yY^a, P) \leq \varepsilon(h)$ *and* $\varepsilon(h) \to 0$ *as* $h \to 0$*;*

(ii) ***(Sliver masses)*** *the restricted pieces satisfy*

$$\mathrm{Mass}([Y^a] \llcorner Q) \;\le\; C\,\frac{M}{N} \qquad \textit{for all } a,$$

and $\sum_a \mathrm{Mass}([Y^a] \llcorner Q) = M + o(1)$*;*

(iii) ***(Transverse measure approximation)*** *the induced transverse measure* $\mu_N := \sum_a \mathrm{Mass}([Y^a] \llcorner Q)\,\delta_{t_a}$ *satisfies*

$$W_1(\mu_N, \rho\,dt) \;\le\; \tau(N,h), \qquad \tau(N,h) \xrightarrow[N\to\infty,\ h\to 0]{} 0.$$

Remark 8.64 (Why we ask for a smooth convex cell). The "sliver" mechanism relies on being able to make *both* the interior mass and the induced boundary slices small when a sheet translate approaches the edge of the cell. This behavior is clean in smooth convex models (e.g. balls), where plane sections shrink in a controlled way. For sharp cubical cells, a plane section can have arbitrarily small k-volume while still having $O(h^{k-1})$ boundary on a face (thin long slices), so additional geometry would be needed to keep boundary slices small. Thus smooth convexity is a natural technical condition for any rigorous sliver bookkeeping estimate. One explicit alternative is a *corner-exit / simplex* mechanism, combined with *global vertex templates*: force each sliver footprint inside a cube to meet only a fixed set of $k+1$ faces adjacent to a vertex and to have uniformly nondegenerate simplex shape, and choose the slivers from a fixed ordered template anchored at each grid vertex. This yields $a \lesssim v^{(k-1)/k}$ even in sharp cubes and also resolves the face-population/prefix obstruction for gluing; see Proposition 8.74.

Sharp-cube variant: corner-exit slivers and global vertex templates (model)

Remark 8.65 (Why templates should live at vertices (pan-vertex distribution)). If one concentrates all slivers in a cube Q near a single vertex, then an interior face $F = Q \cap Q'$ can be populated on one side and essentially empty on the other, creating a one-sided mismatch that is not a tail effect. Moreover, even if both sides use the same *cellwise* master template, it is not automatic that the pieces that actually meet a given face F are the *early* pieces in the chosen prefix.

A clean way to remove both issues is to define templates at the *grid vertices* and to distribute each cube's mass among its vertices. Then any two cubes sharing a vertex v use the same ordered geometric sequence of slivers anchored at v, so across every shared face the mismatch reduces to a pure prefix-count difference at the shared vertices.

Definition 8.66 (Global vertex template (flat cubical model))**.** Fix a cubical grid in $\mathbb{R}^{2n}$ with mesh h and vertex set $\Lambda := (h\mathbb{Z})^{2n}$, and fix a calibrated $(2n-2p)$-plane P. For each vertex $v \in \Lambda$, fix an infinite ordered family of affine planes

$$P_{v,a} \;:=\; P + v + t_{v,a}, \qquad a \ge 1,$$

with translation vectors $t_{v,a} \in P^\perp$ satisfying:

(i) (**Corner localization**) for every cube Q containing v, the intersection $(P_{v,a} \cap Q)$ is contained in $B(v, c_0 h)$ for a fixed $c_0 < 1$;

(ii) (**Uniform corner-exit simplex type**) for each such Q, the slice $E_{v,a}(Q) := P_{v,a} \cap Q$ meets exactly the same $k+1$ coordinate faces through v (so, in particular, for any given face $F \subset \partial Q$ through v, either *all* $E_{v,a}(Q)$ meet F or *none* do);

(iii) (**Equal (or uniformly comparable) slice masses**) the slice masses $\mathcal{H}^k(E_{v,a}(Q))$ are equal in a (or, more generally, uniformly comparable in a, with constants independent of h and v).

We refer to $(P_{v,a})_{a\ge 1}$ as a *global vertex template* for direction P.

Lemma 8.67 (A concrete *complex* corner-exit translation template in a cube)**.** *Work in $\mathbb{C}^n = \mathbb{C}^{n-p} \times \mathbb{C}^p$ with coordinates $z = (u, w)$, where $u = (u_1, \dots, u_{n-p})$ and $w = (w_1, \dots, w_p)$. Let $Q := [0, h]^{2n} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ be the coordinate cube with vertex 0. Fix a constant $0 < c_0 < 1$ and choose a scale $s > 0$ with $s \le c_0 h/100$.*

Define a complex $(n-p)$–plane $P \subset \mathbb{C}^n$ as the graph of the linear map $A : \mathbb{C}^{n-p} \to \mathbb{C}^p$ given by

$$w_1 \;=\; -(1-i)\sum_{j=1}^{n-p} u_j, \qquad w_2 = \cdots = w_p = 0.$$

For translation parameters $t = (t_1, \dots, t_p) \in \mathbb{C}^p$, write $P_t := \{(u, Au + t) : u \in \mathbb{C}^{n-p}\}$ (parallel translate of P). Assume t satisfies the interior-margin *bounds*

$$\Re t_1 = s, \qquad 2s \le \Im t_1 \le 3s, \qquad 2s \le \Re t_j, \Im t_j \le 3s \;\; (2 \le j \le p).$$

Then:

(i) ***(Corner-exit simplex footprint)*** *The footprint $E(t) := P_t \cap Q$ is a k–simplex with $k = 2n - 2p$, contained in $B(0, c_0 h)$.*

(ii) ***(Fixed designated exit faces)*** *The $k+1$ facets of $E(t)$ lie on the $k+1$ coordinate faces*

$$F_{\Re u_j = 0},\; F_{\Im u_j = 0} \; (1 \le j \le n-p), \qquad \textit{and} \qquad F_{\Re w_1 = 0},$$

and $E(t)$ meets no other codimension-1 faces of Q.

(iii) ***(Uniform fatness and equal slice mass)*** *The family $E(t)$ is uniformly fat (with constants depending only on (n, p)), and $\mathcal{H}^k(E(t))$ is independent of t in the above parameter box (hence equal across indices).*

In particular, this admissible parameter box has real dimension $2p - 1$, so for any separation scale $\delta > 0$ one can choose an ordered δ–separated list $(t_a)_{a\ge 1}$ inside it with identical footprints $P_{t_a} \cap Q$.

Proof. Write $u_j = x_j + i y_j$ with $x_j = \Re u_j$ and $y_j = \Im u_j$. On P_t one computes

$$\Re w_1 \;=\; \Re t_1 \;+\; \Re\Big(-(1-i)\sum_{j=1}^{n-p} u_j\Big) \;=\; s \;-\; \sum_{j=1}^{n-p}(x_j + y_j),$$

and

$$\Im w_1 \;=\; \Im t_1 \;+\; \Im\Big(-(1-i)\sum_{j=1}^{n-p} u_j\Big) \;=\; \Im t_1 \;+\; \sum_{j=1}^{n-p}(x_j - y_j).$$

The cube constraints on $w_2, \dots, w_p$ are automatic since $w_j \equiv t_j$ and $t_j \in (0, h)^2$ with margin $\gtrsim s$. Moreover, on the region cut out by $x_j, y_j \ge 0$ and $\sum_j (x_j + y_j) \le s$, one has $\big|\sum_j (x_j - y_j)\big| \le \sum_j (x_j + y_j) \le s$, hence

$$\Im w_1 \;\in\; [\Im t_1 - s,\; \Im t_1 + s] \;\subset\; [s, 4s] \;\subset\; (0, h),$$

so both faces $\{\Im w_1 = 0\}$ and $\{\Im w_1 = h\}$ are avoided. Likewise $\Re w_1 \in [0, s] \subset (0, h)$ avoids $\{\Re w_1 = h\}$, and $x_j, y_j \le s \ll h$ avoids the far faces $\{\Re u_j = h\}$ and $\{\Im u_j = h\}$.

Consequently, $E(t) = P_t \cap Q$ is cut out on P_t exactly by the inequalities

$$x_j \ge 0, \quad y_j \ge 0 \quad (1 \le j \le n-p), \qquad \text{and} \qquad \Re w_1 \ge 0,$$

i.e. by $\sum_j (x_j + y_j) \le s$ together with nonnegativity of the $k = 2(n-p)$ coordinates $(x_1, y_1, \ldots, x_{n-p}, y_{n-p})$. This is the standard k–simplex in $\mathbb{R}^k$ (embedded linearly as a graph in $\mathbb{R}^{2n}$), proving (i) and (ii). Uniform fatness follows because this simplex is affine-equivalent to the standard simplex with distortion depending only on the fixed linear map A, and the slice mass $\mathcal{H}^k(E(t)) \asymp s^k$ is independent of t since the defining inequalities do not depend on t inside the admissible box. Finally, packing a δ–separated family inside a $(2p-1)$–dimensional box is elementary. $\square$

Lemma 8.68 (Corner-exit simplex mass scale and no-heavy-tail uniformity)**.** *In the setting of Lemma 8.67, fix a scale $s > 0$ and let $E(t) = P_t \cap Q$ be the resulting corner-exit simplex of dimension $k = 2n - 2p$. Then there exist constants $0 < c \le C < \infty$ depending only on (n, p) such that for every admissible t (with the fixed scale s):*

$$c\, s^k \;\le\; \mathcal{H}^k(E(t)) \;\le\; C\, s^k, \qquad c\, s^{k-1} \;\le\; \mathcal{H}^{k-1}(E(t) \cap F_i) \;\le\; C\, s^{k-1} \quad (i = 0, \ldots, k),$$

where $F_0, \ldots, F_k$ are the designated exit faces from Lemma 8.67. In particular, if one chooses $s = \theta\, h$ for a fixed $\theta \in (0,1)$ (so s is a fixed fraction of the cell size), then each footprint has $\mathcal{H}^k(E(t)) \asymp h^k$ and each designated face slice has $\mathcal{H}^{k-1}(E(t) \cap F_i) \asymp h^{k-1}$. Moreover, throughout the admissible parameter box in Lemma 8.67 (with fixed $\Re t_1 = s$), the footprints are identical, so $\mathcal{H}^k(E(t))$ and the facet measures $\mathcal{H}^{k-1}(E(t) \cap F_i)$ are in fact independent of t.

Consequently, an ordered δ–separated list (t_a) in that box yields a template whose pieces have exactly equal *footprint masses and per-face slice masses (no heavy tails along the order). If $Y^a \cap Q$ is an ε–slope graph over $E(t_a)$, then Lemma 8.92 gives the corresponding holomorphic equal-mass/equal-slice-mass conclusions up to a common $(1 + O(\varepsilon^2))$ factor.*

Proof. In the proof of Lemma 8.67, $E(t)$ is cut out on the k real coordinates $(x_1, y_1, \ldots, x_{n-p}, y_{n-p}) \in \mathbb{R}^k$ by the inequalities $x_j \ge 0$, $y_j \ge 0$, and $\sum_j (x_j + y_j) \le s$, which define a standard simplex of size s. Thus $\mathcal{H}^k(E(t)) \asymp s^k$ and each facet has $\mathcal{H}^{k-1} \asymp s^{k-1}$, with constants depending only on k (hence only on (n, p)). Independence of t inside the parameter box is immediate because the defining inequalities on P_t do not depend on t once $\Re t_1 = s$ is fixed. Finally, Lemma 8.92 gives the $1 + O(\varepsilon^2)$ distortion bounds for small-slope graphs, uniformly in a. $\square$

Lemma 8.69 (Corner-exit translation templates for a quantitative family of complex planes)**.** *Work in $\mathbb{C}^n = \mathbb{C}^{n-p} \times \mathbb{C}^p$ with coordinates $z = (u, w)$ and identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Let $Q := [0, h]^{2n}$ be the coordinate cube. Fix $0 < c_0 < 1$ and parameters $\alpha_*, \alpha^*, A_* > 0$.*

Let $P \subset \mathbb{C}^n$ be a complex $(n-p)$–plane written as a graph

$$P \;=\; \{(u, Au) : u \in \mathbb{C}^{n-p}\},$$

for some complex linear map $A : \mathbb{C}^{n-p} \to \mathbb{C}^p$ with operator norm $\|A\| \le A_*$. Assume that for some choice of a slanted *coordinate w_r (one of the p components of w), the corresponding row of A has coefficients $c_j = a_j + i b_j$ $(1 \le j \le n-p)$ satisfying the quantitative nondegeneracy bounds*

$$\alpha_* \;\le\; |a_j| \;\le\; \alpha^*, \qquad \alpha_* \;\le\; |b_j| \;\le\; \alpha^* \qquad (1 \le j \le n-p).$$

Define the conditioning ratio $\Lambda := \alpha^*/\alpha_*$.

Then there exists a choice of a vertex v *of* Q *(equivalently, a choice of which incident coordinate faces of* Q *provide the "orthant" constraints) and a choice of a translation parameter* $t \in \mathbb{C}^p$ *with a scale* $s := |\Re t_r|$ *satisfying*

$$s \;\le\; \frac{c_0}{C(n,p)} \cdot \frac{h}{(1+A_*)\,\Lambda},$$

such that, writing $P_t := P + t$ *and* $E := P_t \cap Q$*, the footprint* E *is a* k*–simplex (*$k = 2n - 2p$*) contained in* $B(v, c_0 h)$ *whose* $k+1$ *facets lie on exactly* $k+1$ *coordinate faces of* Q *incident to* v *(a designated exit-face set), and the simplex is uniformly fat with constant depending only on* (n, p, Λ)*.*

Moreover, one may choose t *from a* $(2p–1)$*–dimensional parameter box (fixing* $\Re t_r = \pm s$ *and varying the remaining real components with margin* $\asymp s$*), so that the resulting footprints are* identical *(hence have equal slice mass) throughout that box. In particular, for any separation scale* $\delta > 0$ *one can extract an ordered* δ*–separated list of translations producing identical corner-exit simplex footprints.*

Proof. Write $u_j = x_j + iy_j$. By reflecting real coordinates $x_j \mapsto h - x_j$ and/or $y_j \mapsto h - y_j$ (which corresponds to choosing a vertex v of Q), we may replace (x_j, y_j) by nonnegative coordinates $(x'_j, y'_j) \in [0, h]$ so that the affine inequality $\Re w_r \ge 0$ restricted to P_t becomes

$$\sum_{j=1}^{n-p} (|a_j|\, x'_j + |b_j|\, y'_j) \;\le\; s,$$

after absorbing the resulting additive constants into the choice of $\Re t_r$. Together with the orthant constraints $x'_j \ge 0$, $y'_j \ge 0$, this cuts out a k–simplex in the $k = 2(n-p)$ real variables. The bound $s \ll h$ prevents meeting the far faces in the u-coordinates.

By $\|A\| \le A_*$ and the simplex bound $|u| \lesssim s/\alpha_*$, all other cube coordinates (the remaining w components and the $\Im w_r$ coordinate) vary by at most $O(A_* s/\alpha_*)$ on E. Choosing the remaining components of t with margin $\asymp s$ and taking $s \le c_0\, h/(C(1+A_*)\Lambda)$ forces these coordinates to stay in $(0, h)$, so no additional faces are met. Uniform fatness and volume scaling follow by an affine change of variables on $\mathbb{R}^k$ controlled by Λ.

Finally, to obtain a template family with identical footprints, fix $\Re t_r = \pm s$ and vary the remaining real components of t in a box of sidelength $\asymp s$ chosen so that all the non-u cube coordinates remain strictly inside $(0, h)$ as above. On this parameter box, the defining inequalities in the (x'_j, y'_j) variables are unchanged, so the footprint in Q is identical for all such t. Extracting a δ–separated ordered list from the box is a standard packing argument in dimension $2p - 1$. □

Proposition 8.70 (Robust corner-exit templates for a finite direction net)**.** *Fix* $h > 0$ *and a tolerance* $\varepsilon_h > 0$*. In any fixed holomorphic coordinate chart, there exists a finite set of calibrated directions*

$$\mathcal{N}_h = \{P_1, \ldots, P_M\} \subset G_{\mathbb{C}}(n-p, n)$$

which is an ε_h*–net in* $G_{\mathbb{C}}(n-p, n)$ *and has the following property: for each* $P_i \in \mathcal{N}_h$ *there is a corner-exit translation template family in the cube* $Q = [0, h]^{2n}$ *(allowing choice of vertex and exit-face set) whose footprints are uniformly fat corner-exit simplices, and which supplies an arbitrarily long* δ*–separated ordered list of translations (for any* $\delta > 0$*) with identical footprint geometry (hence uniform per-piece slice mass within each label). Moreover, because* $\mathcal{N}_h$ *is finite, the fatness/locality constants may be chosen* uniformly *over all directions in* $\mathcal{N}_h$*.*

Proof. Let $\mathcal{U} \subset G_{\mathbb{C}}(n-p,n)$ be the set of planes for which there exists some coordinate splitting and some choice of slanted coordinate w_r so that the corresponding row coefficients satisfy $a_j \neq 0$ and $b_j \neq 0$ for all j; this is a finite union of complements of algebraic degeneracy loci (vanishing of Plücker minors and coordinate coefficients), hence dense. Start with any $\varepsilon_h/2$–net and perturb each point by $< \varepsilon_h/2$ into $\mathcal{U}$; compactness gives a finite net $\mathcal{N}_h \subset \mathcal{U}$.

For each $P_i \in \mathcal{N}_h$, choose a witnessing splitting and slanted coordinate, and let $\alpha_*(i), \alpha^*(i), A_*(i)$ be the resulting quantitative constants. Since $\mathcal{N}_h$ is finite and all required coefficients are nonzero, one has $\alpha_* := \min_i \alpha_*(i) > 0$ and $A_* := \max_i A_*(i) < \infty$. Apply Lemma 8.69 with these uniform constants to obtain uniform corner-exit templates for every P_i. □

Remark 8.71 (Supplying corner-exit template families for the direction net). The global activation/gluing bookkeeping (Theorem 8.48 and Proposition 8.82) is *direction-by-direction*: one fixes a calibrated direction label j and activates an ordered template by choosing only prefix lengths. Thus, to run the corner-exit route in the holomorphic setting, it suffices to ensure the following for each direction label j in the finite direction net used to approximate $m\beta$ on the mesh:

- **(Template existence)** in the local holomorphic chart for a cell Q, there is a complex reference plane P_j and a supply of translation parameters $t_{v,a}^{(j)}$ near each vertex v so that the footprints $(P_j + v + t_{v,a}^{(j)}) \cap Q$ are uniformly fat corner-exit simplices with a fixed designated exit-face set (hence satisfy the geometric hypotheses of Proposition 8.93), and
- **(Holomorphic realization)** these translated templates can be realized by disjoint holomorphic complete intersections on Q with cell-scale single-sheet graph control.

Lemma 8.67 provides a completely explicit complex corner-exit translation template in a coordinate cube, and Lemma 8.69 + Proposition 8.70 provide the robust finite-net supply needed for the global scheme: one can choose the direction dictionary/net used to approximate $\widehat{\beta}$ so that *every* direction label admits a corner-exit translation template, with constants uniform over the finite net. In practice, one chooses the direction net *inside* the open set of calibrated planes for which an analogous "one-coordinate slanted inequality" produces a corner simplex in Q (after choosing the appropriate anchored vertex v and designated faces among those incident to v). Since the net is finite at each mesh scale, all geometric constants (fatness, locality radius c_0, and per-face comparability constants) may be taken uniform by min/max over the finitely many labels.

Lemma 8.72 (Corner-exit simplex slices have optimal boundary scaling)**.** *Let $Q = [0,h]^{2n} \subset \mathbb{R}^{2n}$ and fix $1 \le k < 2n$. Let $E \subset Q$ be a k–dimensional simplex contained in a ball $B(0, c_0 h)$ and whose $k+1$ facets lie on $k+1$ coordinate faces through 0, with dihedral angles bounded below by a fixed constant (uniform nondegeneracy). Then*

$$\mathcal{H}^{k-1}(\partial E) \;\le\; C(k)\,\big(\mathcal{H}^k(E)\big)^{\frac{k-1}{k}}.$$

Proof. Let Π be the affine k–plane containing E. By the uniform nondegeneracy assumption (dihedral angles bounded below), there exists an affine isomorphism $A : \Pi \to \mathbb{R}^k$ whose distortion (operator norm and inverse norm) is bounded in terms of k alone, such that $A(E) = \Delta_s$ is a standard k–simplex of scale s (i.e. affine-equivalent to $\{x \in \mathbb{R}^k : x_i \ge 0,\ \sum_{i=1}^k x_i \le s\}$).

For the standard simplex one computes explicitly $\mathcal{H}^k(\Delta_s) = c_k\, s^k$ and $\mathcal{H}^{k-1}(\partial \Delta_s) = c_k'\, s^{k-1}$ for dimensional constants $c_k, c_k' > 0$. Eliminating s yields $\mathcal{H}^{k-1}(\partial\Delta_s) \le C(k)\,\big(\mathcal{H}^k(\Delta_s)\big)^{(k-1)/k}$. Applying the change-of-variables bounds under A (which distort k– and $(k-1)$–dimensional Hausdorff measures by at most a multiplicative factor depending only on k) gives the stated inequality for E. $\square$

Proposition 8.73 (Vertex-template prefix lengths match local mass budgets (L2, cube model))**.** *Work in the setting of Definition 8.66 for a fixed direction family, and assume the vertex templates have equal (or uniformly comparable) slice masses as in Definition 8.66*(iii). *Assume further that the geometric templates are realized by ψ–calibrated holomorphic pieces with small-slope graph control on each cube (so that Lemma 8.96(i) applies uniformly).*

Let $M_Q \geq 0$ be the target mass budget for this direction family in cube Q. Choose any vertex-splitting $M_{Q,v} \geq 0$ with $\sum_{v \in \mathrm{Vert}(Q)} M_{Q,v} = M_Q$ (for instance the equal split $M_{Q,v} = 2^{-d} M_Q$). For each vertex $v \in \mathrm{Vert}(Q)$, let $\mu_{Q,v}$ denote the (common) per-piece mass scale in Q for the vertex-template pieces anchored at v (so $\mathrm{Mass}([Y^a_{Q,v}] \llcorner Q) = (1 + O(\varepsilon^2))\, \mu_{Q,v}$ uniformly in a). Define the prefix length by nearest-integer rounding

$$N_{Q,v} \;:=\; \left\lfloor \frac{M_{Q,v}}{\mu_{Q,v}} \right\rceil.$$

Then the realized mass satisfies

$$\sum_{v \in \mathrm{Vert}(Q)} \sum_{a=1}^{N_{Q,v}} \mathrm{Mass}([Y^a_{Q,v}] \llcorner Q) \;=\; M_Q \;+\; O\!\left(\sum_v \mu_{Q,v} \right) \;+\; O(\varepsilon^2)\, M_Q.$$

In particular, whenever $M_{Q,v} \gg \mu_{Q,v}$ (equivalently $N_{Q,v} \gg 1$), the relative error per vertex is $O(1/N_{Q,v}) + O(\varepsilon^2)$.

Proof. Fix a vertex v. By nearest-integer rounding,

$$\left| N_{Q,v}\, \mu_{Q,v} - M_{Q,v} \right| \;\leq\; \mu_{Q,v}.$$

By the holomorphic small-slope graph control and Lemma 8.96(i), each realized piece satisfies $\mathrm{Mass}([Y^a_{Q,v}] \llcorner Q) = (1 + O(\varepsilon^2))\, \mu_{Q,v}$ uniformly in a. Therefore

$$\sum_{a=1}^{N_{Q,v}} \mathrm{Mass}([Y^a_{Q,v}] \llcorner Q) \;=\; (1 + O(\varepsilon^2))\, N_{Q,v}\, \mu_{Q,v},$$

and hence

$$\left| \sum_{a=1}^{N_{Q,v}} \mathrm{Mass}([Y^a_{Q,v}] \llcorner Q) - M_{Q,v} \right| \;\leq\; \mu_{Q,v} \;+\; O(\varepsilon^2)\, M_{Q,v}.$$

Summing over the finitely many vertices $v \in \mathrm{Vert}(Q)$ and using $\sum_v M_{Q,v} = M_Q$ gives the stated estimate. $\square$

Proposition 8.74 (Vertex templates $\Rightarrow$ face-level $O(h)$ edit regime (item (iv)))**.** *Work in the setting of Definition 8.66, and fix one direction family. Assume each cube Q distributes its target mass budget among its vertices, producing counts $N_{Q,v} \in \mathbb{Z}_{\geq 0}$ and realizing in Q the prefix $\{P_{v,a}\}_{1 \leq a \leq N_{Q,v}}$ at each vertex $v \in Q$. Assume the* slow-variation *bound holds at shared vertices: for any two adjacent cubes $Q \sim Q'$ and any shared vertex $v \in Q \cap Q'$,*

$$|N_{Q,v} - N_{Q',v}| \;\leq\; C\, h\, \min\{N_{Q,v}, N_{Q',v}\}.$$

Then for every interior interface face $F = Q \cap Q'$, the unmatched boundary mass on F is an $O(h)$ fraction of the total face boundary mass, i.e. the hypothesis (iv) *in Theorem 8.48 holds (after absorbing constants).*

Proof. Fix an interior interface face $F = Q \cap Q'$. By the corner-localization property in Definition 8.66, a sliver can meet F only if it is anchored at a vertex v lying on F. Thus $\partial S_Q \llcorner F$ and $\partial S_{Q'} \llcorner F$ decompose as sums over the finitely many shared vertices $v \in \mathrm{Vert}(F)$.

Fix such a shared vertex v. By the uniform corner-exit type, the set of v-anchored pieces that actually meet F is itself a prefix of the v-template on both sides; therefore the mismatch on F coming from vertex v is supported in the tails of sizes $|N_{Q,v} - N_{Q',v}|$. Moreover, the equal/comparable slice-mass hypothesis (Definition 8.66(iii)) together with Lemma 8.72 yields a no-heavy-tail constant κ for the face-slice boundary masses along the order. The assumed slow-variation bound $|N_{Q,v} - N_{Q',v}| \leq Ch \min\{N_{Q,v}, N_{Q',v}\}$ is exactly hypothesis (c) of Lemma 8.52. Hence Lemma 8.52 applies at each shared vertex and yields an $O(h)$ fraction bound for the unmatched boundary mass contributed by that vertex.

Summing over the finitely many vertices $v \in \mathrm{Vert}(F)$ gives the claimed $O(h)$ fraction bound for the full face mismatch on F. □

Corollary 8.75 (Corner-exit vertex templates verify the activation hypotheses (iii)–(iv))**.** *Fix one direction label j and assume the following are implemented on a mesh-h cubulation:*

(1) ***(Holomorphic corner-exit manufacturing (L1))*** *the local holomorphic slivers are realized from a corner-exit translation template as in Proposition 8.99, with vertex-star coherence as in Remark 8.100;*

(2) ***(Local mass-budget matching (L2))*** *the prefix lengths $N_{Q,v}$ are chosen to match the local vertex budgets by Proposition 8.73;*

(3) ***(Slow variation of counts)*** *the resulting counts satisfy $|N_{Q,v} - N_{Q',v}| \lesssim h \min\{N_{Q,v}, N_{Q',v}\}$ at shared vertices (e.g. by Lemma 8.101 applied to Lipschitz target budgets).*

Then for this direction label j the two nontrivial activation hypotheses in Theorem 8.48 hold:

- *hypothesis* (iii) *(local realizability / mass matching) holds by Proposition 8.73, and*
- *hypothesis* (iv) *($O(h)$ face-edit regime) holds by Proposition 8.74 (or, with a single interleaved master order, by Proposition 8.82).*

Consequently, the flat-norm plumbing of Theorem 8.48 applies to this direction family.

Proposition 8.76 (Global coherence across all direction labels (B1, packaged))**.** *Fix a mesh-h cubulation by coordinate cubes Q (subordinate to a holomorphic atlas) and let β be a smooth closed strongly positive (p,p)-form. Fix a small scale $\varepsilon_h \ll h$ and choose, in each chart, an ε_h–net of calibrated directions $\{P_1, \ldots, P_M\} \subset G_{\mathbb{C}}(n-p, n)$ together with uniform corner-exit translation templates as in Proposition 8.70.*

Assume we choose globally labeled *Lipschitz weights $w_i(x)$ against this dictionary (e.g. by the strongly convex simplex fit of Lemma 8.13 applied to $\widehat{\beta}(x)$ in local trivializations), and define per-cell target mass budgets $M_{Q,i} \geq 0$ accordingly, with $\sum_i M_{Q,i} = M_Q$ and Lipschitz variation across neighbors. For each label i, realize the corresponding corner-exit template holomorphically on each vertex star by applying Proposition 8.99 (with vertex-star coherence as in Remark 8.100) to the template* *planes provided by Proposition 8.70; this yields corner-exit holomorphic slivers with (G1-iff)/(G2) and equal/comparable per-piece masses (hence Proposition 8.73 applies).*

Then one can choose integer counts $N_{Q,v,i}$ simultaneously for all (Q, v, i) so that:

(a) ***(Local mass/barycenter accuracy)** for each cube Q and label i the realized mass in direction i matches $M_{Q,i}$ up to the rounding error $O(1/N) + O(\varepsilon^2)$ from Proposition 8.73;*

(b) ***(Slow variation)** for each interior adjacency $Q \sim Q'$ and each shared vertex $v \in Q \cap Q'$, one has $|N_{Q,v,i} - N_{Q',v,i}| \lesssim h \min\{N_{Q,v,i}, N_{Q',v,i}\}$ on the region where $M_{Q,i}$ is not negligible (e.g. via Lemma 8.101 and the 0–1 stability Lemma 8.102);*

(c) ***(Cohomology periods)** after clearing denominators by choosing m and applying fixed-dimension discrepancy rounding (Lemma 8.106 in the form of Proposition 8.108), the resulting raw current satisfies the integral period constraints.*

Consequently, for each label i the activation hypotheses (iii)–(iv) in Theorem 8.48 hold (by Corollary 8.75), and summing the resulting per-label flat-norm mismatch bounds yields $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ under the parameter regime of Remark 8.37.

Proof. All steps are performed label-by-label and then summed over the finite dictionary $\{1, \ldots, M\}$.

Step 1 (template supply for each label). By Proposition 8.70, each direction label i in the chosen finite net admits a corner-exit translation template family with uniform fatness/locality constants (uniform over i because the net is finite).

Step 2 (Lipschitz budgets and slow variation). The Lipschitz weights $w_i(x)$ produce Lipschitz target mass budgets $M_{Q,i}$ across neighboring cubes. Applying the rounding lemmas (Lemma 8.101 and the 0–1 stability Lemma 8.102) gives integer counts $N_{Q,v,i}$ that match the local budgets up to the rounding error and satisfy the slow-variation bound at shared vertices.

Step 3 (holomorphic realization and local activation hypotheses). For each label i, apply Proposition 8.99 on each vertex star (with coherence from Remark 8.100) to realize the corresponding translation template by disjoint holomorphic corner-exit slivers satisfying the (G1-iff)/(G2) hypotheses. Then Corollary 8.75 applies and yields, for each label i, the activation hypotheses (iii)–(iv) in Theorem 8.48.

Step 4 (cohomology periods). Finally, Proposition 8.108 (a fixed-dimensional discrepancy rounding argument) adjusts the integer choices so that the global cohomology period constraints are satisfied, without spoiling the local $O(h)$ edit regime.

With (iii)–(iv) verified labelwise, Theorem 8.48 and Corollary 8.35 give a per-label flat-norm mismatch bound; summing over the finitely many labels yields the global estimate $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ in the scaling regime of Remark 8.37. □

Remark 8.77 (Making the "prefix-balanced face population" explicit). The previous proposition treats each vertex template separately. If one prefers a *single* global ordered template whose prefixes automatically populate every interior face in a balanced way, one can interleave the vertex templates by a deterministic block scheme (a "vertex-code" ordering) and align the vertex anchoring across the grid by a checkerboard parity rule. This removes the possibility that the F-hitting pieces concentrate in a tail of the master order. See Proposition 8.82 below.

Definition 8.78 (Cubical grid parity and checkerboard vertex anchoring)**.** Fix $d \geq 2$ and mesh $h > 0$ and index cubes by $g \in \mathbb{Z}^d$ via $Q_g := \prod_{\ell=1}^d [g_\ell h, (g_\ell + 1)h]$. Define the parity vector $\pi(g) \in \{0,1\}^d$ by $\pi(g)_\ell := g_\ell \bmod 2$, and let $\oplus$ denote bitwise XOR. For a vertex-code $u \in \{0,1\}^d$, define the anchored vertex of Q_g by

$$v_g(u) \ := \ \big(g + (u \oplus \pi(g))\big)h \ \in \ \mathbb{R}^d,$$

so u selects a cube-vertex in a checkerboard-consistent way across neighbors.

Definition 8.79 (Block-uniform vertex-code sequence)**.** Let $\mathcal{V} := \{0,1\}^d$ and fix any bijection $\sigma : \{1, \ldots, 2^d\} \to \mathcal{V}$. Define an infinite sequence $(u_a)_{a\geq 1} \subset \mathcal{V}$ by repeating σ in blocks:

$$u_{b\cdot 2^d + r} \ := \ \sigma(r) \qquad (b \geq 0,\ 1 \leq r \leq 2^d).$$

Lemma 8.80 (Prefix discrepancy for block-uniform codes)**.** *Let $S \subset \mathcal{V}$ and define $A_S(N) := \#\{1 \leq a \leq N :\ u_a \in S\}$. Then for all $N \geq 1$,*

$$\Big| A_S(N) - \frac{|S|}{2^d}\, N \Big| \ \leq \ 2^d,$$

and for all $N, N' \geq 1$,

$$|A_S(N) - A_S(N')| \ \leq \ \frac{|S|}{2^d}\, |N - N'| + 2^{d+1}.$$

Proof. Write $N = q \cdot 2^d + r$ with $0 \leq r < 2^d$. Each full block contributes exactly $|S|$ hits and the remainder contributes at most 2^d hits, giving the first bound. The second follows by applying the first bound to N and N' and subtracting. $\square$

Lemma 8.81 (Two-sided face population is automatic under checkerboarding)**.** *Fix a coordinate direction $\ell \in \{1, \ldots, d\}$ and an interior interface face $F := Q_g \cap Q_{g+e_\ell}$. Let $S^+_{g,\ell} \subset \mathcal{V}$ be the set of codes whose anchored vertex in Q_g lies on the positive ℓ-face of Q_g, and let $S^-_{g+e_\ell,\ell} \subset \mathcal{V}$ be the set of codes whose anchored vertex in Q_{g+e_ℓ} lies on the negative ℓ-face of Q_{g+e_ℓ} (the same hyperplane). Then $S^+_{g,\ell} = S^-_{g+e_\ell,\ell}$ and hence, for every N,*

$$\{a \leq N :\ v_g(u_a) \in F\} \ = \ \{a \leq N :\ v_{g+e_\ell}(u_a) \in F\}.$$

Proof. By Definition 8.78, being on the positive ℓ-face of Q_g means $(u \oplus \pi(g))_\ell = 1$. Since $\pi(g+e_\ell) = \pi(g) \oplus e_\ell$, one has $(u \oplus \pi(g + e_\ell))_\ell = (u \oplus \pi(g))_\ell \oplus 1$, so $(u \oplus \pi(g))_\ell = 1$ iff $(u \oplus \pi(g + e_\ell))_\ell = 0$, which is exactly the negative-face condition for Q_{g+e_ℓ}. $\square$

Proposition 8.82 (Checkerboard corner assignment $\Rightarrow$ face-level $O(h)$ edit regime)**.** *Fix $d \geq 2$ and a cubical grid (Q_g). Assume the ordered sliver activation in each cube Q_g uses a single global master order $a = 1, 2, \ldots,$ where index a is anchored at the checkerboard vertex $v_g(u_a)$ (Definitions 8.78–8.79). Assume the following geometric features hold uniformly for the slivers in each cube:*

- (G1) ***(Locality)*** *A sliver indexed by a meets an interface face $F \subset \partial Q_g$ if and only if its anchored vertex $v_g(u_a)$ lies on F, and the boundary slice on F is supported in a patch of diameter $\lesssim h$ near that vertex;*

- (G2) ***(Comparable face mass)*** *For each cube Q_g there is a scale $b_g > 0$ and constants $0 < c_0 \leq C_0$ such that for every interior face F and every index a with $v_g(u_a) \in F$,*

$$c_0\, b_g \ \leq \ \mathrm{Mass}\big(\partial([Y^a_g] \llcorner Q_g) \llcorner F\big) \ \leq \ C_0\, b_g.$$

Let $F = Q_g \cap Q_{g+e_\ell}$ be an interior face and let $N := N_g$, $N' := N_{g+e_\ell}$ be the chosen prefix lengths on the two sides, with $N_{\min} := \min\{N, N'\}$. Then the unmatched boundary mass on F coming from the tail indices $\{N_{\min}+1, \dots, \max\{N, N'\}\}$ satisfies

$$\mathrm{Mass}(B_F^{\mathrm{un}}) \;\le\; C\left(\frac{|N-N'|}{N_{\min}} + \frac{2^d}{N_{\min}}\right)\Big(\mathrm{Mass}(\partial S_{Q_g} \llcorner F) + \mathrm{Mass}(\partial S_{Q_{g+e_\ell}} \llcorner F)\Big),$$

with C depending only on (d, c_0, C_0). In particular, if $|N-N'| \le \theta\, N_{\min}$ with $\theta \lesssim h$ and $N_{\min} \gtrsim h^{-1}$, then

$$\mathrm{Mass}(B_F^{\mathrm{un}}) \;\le\; C'\, h\,\Big(\mathrm{Mass}(\partial S_{Q_g} \llcorner F) + \mathrm{Mass}(\partial S_{Q_{g+e_\ell}} \llcorner F)\Big),$$

so the $O(h)$ face-edit regime (item (iv) *in Theorem 8.48) holds.*

Proof. Let $S \subset \mathcal{V} = \{0,1\}^d$ be the set of codes whose anchored vertex in Q_g lies on the interface face $F = Q_g \cap Q_{g+e_\ell}$. Then $|S| = 2^{d-1}$ (half of the codes place the anchor on the ℓ-face). By Lemma 8.81, the same set of indices a with $u_a \in S$ anchor onto F from the Q_{g+e_ℓ} side as well, for every prefix length. Hence the only unmatched boundary contributions on F come from those tail indices with $u_a \in S$.

Write $N_{\min} := \min\{N, N'\}$ and $N_{\max} := \max\{N, N'\}$. By Lemma 8.80 applied to S,

$$\#\{N_{\min} < a \le N_{\max} :\; u_a \in S\} \;\le\; \frac{|S|}{2^d}\,|N-N'| + 2^{d+1} \;=\; \frac{1}{2}|N-N'| + 2^{d+1}.$$

Each such unmatched index contributes at most $C_0 b_g$ (or $C_0 b_{g+e_\ell}$) boundary mass on the side where it appears, by hypothesis (G2), so this gives an upper bound for $\mathrm{Mass}(B_F^{\mathrm{un}})$ in terms of $(|N-N'| + 2^d)$ and $(b_g + b_{g+e_\ell})$.

For the denominator, again by Lemma 8.80,

$$A_S(N_{\min}) \;\ge\; \frac{|S|}{2^d}\,N_{\min} - 2^d \;=\; \frac{1}{2}N_{\min} - 2^d.$$

Each of these $A_S(N_{\min})$ indices contributes at least $c_0 b_g$ and at least $c_0 b_{g+e_\ell}$ to $\mathrm{Mass}(\partial S_{Q_g} \llcorner F)$ and $\mathrm{Mass}(\partial S_{Q_{g+e_\ell}} \llcorner F)$ respectively, again by (G2). Therefore,

$$\mathrm{Mass}(\partial S_{Q_g} \llcorner F) + \mathrm{Mass}(\partial S_{Q_{g+e_\ell}} \llcorner F) \;\ge\; c\left(\tfrac{1}{2}N_{\min} - 2^d\right)(b_g + b_{g+e_\ell}),$$

for a constant $c = c(d, c_0)$. Dividing the unmatched upper bound by this lower bound and absorbing the fixed 2^d additive terms yields the stated estimate. □

Remark 8.83 (Rounded cubes). For the combinatorics of Substep 4.2 (adjacency graph, faces, cochain constraints), it is convenient to work with a cubulation. For the sliver bookkeeping, it is convenient to replace each sharp cube by a *rounded cube* of comparable diameter h whose boundary is C^2 and uniformly convex with principal curvatures pinched at scale h (so Lemma 8.32 applies). This rounding changes only constants and does not change the adjacency graph.

Remark 8.84 (Where the remaining analytic difficulty really lives). It is tempting to argue that Bergman kernel localization or Tian–Yau–Zelditch universality alone forces the desired face-incidence and per-face boundary-mass properties of slivers. However, *pointwise decay of a holomorphic section does not localize its exact zero set* in the strong sense needed for gluing.

The correct "critical checkpoint" is instead the following: on a *whole cell* Q (not just infinitesimally near one point), the defining holomorphic map must be *uniformly C^1-close* to a fixed linear model so that the zero set in Q is a *single sheet* graph over the intended template plane. Once this global-graph property holds, the corner-exit geometry immediately forces (G1-iff) and (G2) (exit-face stability and per-face mass comparability), and the remaining face bookkeeping is purely combinatorial.

Lemma 8.85 (Global quantitative graph lemma (contraction criterion)). *Let $U = U_u \times U_w \subset \mathbb{R}^k \times \mathbb{R}^{d-k}$ be a product of convex sets and fix $r > 0$ with $B_w(0, r) \subset U_w$. Let $F : U \to \mathbb{R}^{d-k}$ be C^1 and fix an invertible matrix $A \in GL(d-k, \mathbb{R})$. Assume:*

(i) ***(Uniform linearization in the w-directions)***

$$\sup_{(u,w)\in U} \|\partial_w F(u, w) - A\| \ \le\ \eta, \qquad \|A^{-1}\|\,\eta\ \le\ \frac{1}{2};$$

(ii) ***(Small offset on the $w = 0$ slice)***

$$\sup_{u\in U_u} \|A^{-1}F(u, 0)\|\ \le\ \frac{r}{2}.$$

Then for every $u \in U_u$ there exists a unique *$w = g(u) \in B_w(0, r)$ such that $F(u, g(u)) = 0$. Hence $\{F = 0\} \cap (U_u \times B_w(0, r))$ is the graph of g.*
If in addition $\sup_{(u,w)\in U} \|\partial_u F(u, w)\| \le \eta$, then g is Lipschitz and, wherever differentiable,

$$\|Dg\|\ \le\ \frac{\|A^{-1}\|\,\eta}{1 - \|A^{-1}\|\,\eta}\ \le\ 2\,\|A^{-1}\|\,\eta.$$

Proof. Fix $u \in U_u$ and define $T_u : B_w(0, r) \to \mathbb{R}^{d-k}$ by

$$T_u(w)\ :=\ w - A^{-1}F(u, w).$$

Write

$$T_u(w) = -A^{-1}F(u, 0)\ +\ \Big[w - A^{-1}(F(u, w) - F(u, 0))\Big].$$

By the mean value theorem in the w-variable,

$$F(u, w) - F(u, 0) = \Big(\int_0^1 \partial_w F(u, tw)\, dt\Big)\, w,$$

hence

$$w - A^{-1}(F(u, w) - F(u, 0)) = \Big(I - A^{-1}\int_0^1 \partial_w F(u, tw)\, dt\Big)\, w.$$

Using $\|\partial_w F - A\| \le \eta$ and $\|A^{-1}\|\eta \le \frac{1}{2}$ gives

$$\Big\|I - A^{-1}\int_0^1 \partial_w F(u, tw)\, dt\Big\|\ \le\ \|A^{-1}\|\,\eta\ \le\ \frac{1}{2},$$

so for $w \in B_w(0, r)$,

$$\|T_u(w)\|\ \le\ \|A^{-1}F(u, 0)\| + \frac{1}{2}\|w\|\ \le\ \frac{r}{2} + \frac{1}{2}r = r.$$

Thus T_u maps $B_w(0, r)$ into itself.

Similarly, for $w, w' \in B_w(0, r)$, the mean value theorem yields

$$T_u(w) - T_u(w') = \Big(I - A^{-1}\int_0^1 \partial_w F(u, w' + t(w - w'))\, dt\Big)\, (w - w'),$$

so $\|T_u(w) - T_u(w')\| \le \frac{1}{2}\|w - w'\|$. Hence T_u is a contraction, and Banach's fixed point theorem gives a unique fixed point $g(u) \in B_w(0,r)$ with $T_u(g(u)) = g(u)$, i.e. $F(u, g(u)) = 0$.

For the slope bound, differentiate $F(u, g(u)) = 0$ where g is differentiable:

$$\partial_u F(u, g(u)) + (\partial_w F(u, g(u)))\, Dg(u) = 0, \qquad \text{so} \qquad Dg(u) = -(\partial_w F)^{-1}\partial_u F.$$

Since $\|\partial_w F - A\| \le \eta$ and $\|A^{-1}\|\eta \le \frac{1}{2}$, Neumann series gives $\|(\partial_w F)^{-1}\| \le \|A^{-1}\|/(1 - \|A^{-1}\|\eta)$, yielding the stated estimate. □

Remark 8.86 (Memorializing the new checkpoint: "graph on the whole cell"). With the corner-exit Euclidean templates and the small-slope stability package in hand, the remaining microstructure/gluing difficulty becomes sharply focused.

Blocker A (cell-scale single-sheet control). One must arrange that each holomorphic sliver in a cell Q is a *single sheet* which is a C^1 graph over its template plane on a region large enough to contain Q. This is not a specifically "complex-geometry" problem: it is a quantitative implicit-function / contraction-mapping problem.

Blocker B (per-sliver mass control / no heavy tails). Once the sliver is a single small-slope graph on Q, mass and face-slice masses are automatically controlled by area distortion estimates (e.g. Lemma 8.96), so the remaining mass-budget matching (L2) reduces to choosing prefix lengths (with $O(1/N) + O(\varepsilon^2)$ rounding error).

How to apply Lemma 8.85 to holomorphic complete intersections. In a holomorphic chart, write the local coefficients of the defining sections as a map $F = (f_1, \dots, f_p) : U \to \mathbb{C}^p \cong \mathbb{R}^{2p}$. Choose real coordinates $(u, w) \in \mathbb{R}^k \times \mathbb{R}^{2p}$ so that the template plane is $\{w = 0\}$ and the linear model is $w \mapsto Aw$ with A invertible. If one can construct the sections so that, on a ball containing Q,

$$\|\partial_w F - A\|_{L^\infty} \le \eta, \qquad \|\partial_u F\|_{L^\infty} \le \eta, \qquad \|F(\cdot, 0)\|_{L^\infty(U_u)} \le \eta\, h,$$

with $\|A^{-1}\|\eta \ll 1$, then Lemma 8.85 gives a global graph $w = g(u)$ on all of Q. This is exactly the "graph on the whole cell" checkpoint highlighted in the microstructure roadmap.

Two standard routes to produce the needed uniform C^1 control are:

- peak sections plus $\bar\partial$-solving (Hörmander L^2 estimates) to approximate prescribed affine-linear holomorphic models on Bergman-scale balls, and
- Bergman kernel asymptotics / jet right-inverses (Tian–Catlin–Zelditch–Donaldson) to achieve the same C^1 control directly.

Lemma 8.87 (Bergman-scale affine model approximation via $\bar\partial$-solving)**.** *Fix a holomorphic chart $\varphi : U \to B_\rho(0) \subset \mathbb{C}^n$ and a local holomorphic frame e of L over U with $|e|_h^2 = e^{-\phi}$ and $i\partial\bar\partial\phi = \omega$ on U. Fix $R > 0$ and let $\ell(z) = a \cdot z + b$ be an affine-linear holomorphic function on $\mathbb{C}^n$ with $|a| + |b| \le 1$. Then for all sufficiently large m there exists a global section $s_{\ell,m} \in H^0(X, L^m)$ such that, writing $s_{\ell,m} = f_{\ell,m}\, e^{\otimes m}$ on $B_{\rho/8}(0)$, one has on the Bergman-scale ball $B_{R/\sqrt{m}}(0) \subset B_{\rho/8}(0)$:*

$$\sup_{|z| \le R/\sqrt{m}} \Big(|f_{\ell,m}(z) - \ell(z)| + \sqrt{m}\, |\nabla (f_{\ell,m} - \ell)(z)| \Big) \;\le\; \varepsilon_m, \qquad \varepsilon_m \xrightarrow[m\to\infty]{} 0,$$

with constants uniform over the finitely many charts in a fixed atlas on X.

Proof. Choose a cutoff χ supported in $B_{\rho/2}(0)$ with $\chi \equiv 1$ on $B_{\rho/4}(0)$ and set $\tilde{s} := \chi\, \ell\, e^{\otimes m}$ (extended by 0 outside U). Then $\bar{\partial}\tilde{s} = (\bar{\partial}\chi)\, \ell\, e^{\otimes m}$ is supported in the annulus $\{\rho/4 \le |z| \le \rho/2\}$ where $\phi \ge c_0 > 0$ by strict plurisubharmonicity. Solve $\bar{\partial}u = \bar{\partial}\tilde{s}$ using Hörmander L^2 estimates for the positive bundle (L^m, h^m); the weight $e^{-m\phi}$ forces $\|u\|_{L^2(h^m)} \le C\, e^{-cm}$. On the inner ball $B_{\rho/4}(0)$ one has $\bar{\partial}u = 0$, so u is holomorphic there. Standard local $L^2 \to C^1$ estimates for holomorphic sections on Bergman balls (mean-value inequality plus Cauchy estimates at scale $m^{-1/2}$) give $\|u\|_{C^1(B_{R/\sqrt{m}})} \le C_R\, e^{-cm}$. Setting $s_{\ell,m} := \tilde{s} - u$ yields $s_{\ell,m}$ holomorphic and $f_{\ell,m} = \ell -$(holomorphic error) on $B_{R/\sqrt{m}}$ with the stated bound. □

Proposition 8.88 (Cell-scale linear-model complete intersections are single-sheet graphs)**.** *Fix a holomorphic chart identifying a neighborhood of a cell Q with a domain in $\mathbb{C}^n = \mathbb{C}^{n-p} \times \mathbb{C}^p$ with coordinates $z = (u, w)$, and assume $Q \subset B_{R/\sqrt{m}}(0)$ for some fixed R. Let $t \in \mathbb{C}^p$ satisfy $|t| \le c\, h$ (with $h \lesssim m^{-1/2}$). Then for all sufficiently large m there exist sections $\sigma_1, \dots, \sigma_p \in H^0(X, L^m)$ such that, writing $\sigma_j = F_j\, e^{\otimes m}$ in a local frame on $B_{R/\sqrt{m}}(0)$ and setting $F = (F_1, \dots, F_p)$, one has*

$$\|\partial_w F - I\|_{L^\infty(B_{R/\sqrt{m}})} \;+\; \|\partial_u F\|_{L^\infty(B_{R/\sqrt{m}})} \;\le\; \eta_m, \qquad \sup_{u:\ (u,t)\in B_{R/\sqrt{m}}} |F(u,t)| \;\le\; \eta_m\, h,$$

with $\eta_m \to 0$. Consequently, for m large enough, the common zero set $Y_t := \{\sigma_1 = \cdots = \sigma_p = 0\}$ satisfies that $Y_t \cap Q$ is a single *C^1 graph over the affine complex plane $\{w = t\}$ on all of Q, with slope $O(\eta_m)$ (hence as small as desired).*

Proof. Apply Lemma 8.87 to the affine-linear holomorphic functions $\ell_0 \equiv 1$ and $\ell_j(z) = w_j$ to obtain sections $s_0, s_1, \dots, s_p$ whose local coefficients satisfy $f_0 \approx 1$ and $f_j \approx w_j$ in C^1 on $B_{R/\sqrt{m}}$. Define $\sigma_j := s_j - t_j s_0$, so $F_j = f_j - t_j f_0 \approx w_j - t_j$ in C^1 on $B_{R/\sqrt{m}}$. Interpreting F as a map into $\mathbb{R}^{2p}$, apply Lemma 8.85 with $A = I$ on a product subset $U_u \times U_w \subset B_{R/\sqrt{m}}$ containing Q. This yields a unique graph $w = g(u)$ solving $F(u, w) = 0$, and the slope bound is $O(\eta_m)$. □

Lemma 8.89 (Vertex-ball locality excludes nonincident faces)**.** *Let $Q = [0, h]^d \subset \mathbb{R}^d$ and let v be a vertex of Q. Let $F \subset \partial Q$ be any codimension-1 face. If $v \notin F$, then $\operatorname{dist}(v, F) = h$. Consequently, if $E \subset Q$ satisfies*

$$E \subset B(v, c_0 h) \qquad \text{for some } 0 < c_0 < 1,$$

then $E \cap F = \emptyset$ for every face F not containing v.

Proof. After translation we may assume $v = 0$. Every codimension-1 face of Q is of the form $\{x_j = 0\}$ or $\{x_j = h\}$. If $0 \notin F$, then $F = \{x_j = h\}$ for some j, hence $\operatorname{dist}(0, F) = h$. If $E \subset B(0, c_0 h)$ with $c_0 < 1$, then E cannot intersect any set at distance h from 0. □

Lemma 8.90 (Fat corner simplices force "if" on the designated exit faces)**.** *Fix $d \ge 2$ and $1 \le k < d$. Let $Q = [0, h]^d$ and let v be a vertex. Assume a k-dimensional convex footprint $E \subset Q$ satisfies:*

(C1) ***(Corner locality)*** *$E \subset B(v, c_0 h)$ for some $0 < c_0 < 1$;*

(C2) ***(Corner-exit face set)*** *there exist $k{+}1$ distinct codimension-1 faces $F_0, \dots, F_k \subset \partial Q$ incident to v such that*

$$E \cap \partial Q \;\subset\; \bigcup_{i=0}^{k} F_i,$$

and each $E \cap F_i$ is a $(k{-}1)$-dimensional facet of E (in particular, $E \cap F_i$ has nonempty relative interior in F_i).

Then $\mathcal{H}^{k-1}(E \cap F_i) > 0$ for every $i = 0, \ldots, k$. Combining with Lemma 8.89, one obtains the "iff" statement:

$$\mathcal{H}^{k-1}(E \cap F) > 0 \quad \Longleftrightarrow \quad F \in \{F_0, \ldots, F_k\}.$$

Proof. Each facet $E \cap F_i$ contains a relatively open subset of the $(k-1)$–dimensional affine hyperplane F_i, hence has positive $(k-1)$–dimensional Hausdorff measure. If F is not incident to v, Lemma 8.89 gives $E \cap F = \emptyset$. If F is incident to v but $F \notin \{F_0, \ldots, F_k\}$, then $E \cap F = \emptyset$ by assumption (C2). □

Lemma 8.91 (Uniform per-face boundary mass for fat corner simplices)**.** *Fix $d \geq 2$, $1 \leq k < d$, and a fatness parameter $\Lambda \geq 1$. Let $Q = [0, h]^d$ and let $E \subset Q$ be a k–simplex contained in $B(0, c_0 h)$ whose $k+1$ facets lie on $k+1$ coordinate faces through 0, with dihedral angles bounded below by a constant depending only on Λ (uniform nondegeneracy). Write*

$$v_E := \mathcal{H}^k(E), \qquad a_i := \mathcal{H}^{k-1}(E \cap F_i) \quad (0 \leq i \leq k),$$

where F_i denotes the supporting coordinate face of the ith facet. Then there exist constants $0 < c_*(k, \Lambda) \leq C_*(k, \Lambda) < \infty$ such that for every i,

$$c_*(k, \Lambda)\, v_E^{\frac{k-1}{k}} \;\leq\; a_i \;\leq\; C_*(k, \Lambda)\, v_E^{\frac{k-1}{k}}.$$

Proof. Let Π be the affine k–plane containing E. Uniform nondegeneracy/fatness (parameter Λ) implies that there exists an affine isomorphism $A : \Pi \to \mathbb{R}^k$ whose distortion ($\|A\|$ and $\|A^{-1}\|$) is bounded in terms of (k, Λ), and such that $A(E) = \Delta_s$ is a standard k–simplex of scale s.

For the standard simplex one computes explicitly

$$\mathcal{H}^k(\Delta_s) = c_k\, s^k, \qquad \mathcal{H}^{k-1}(\Delta_s \cap \{y_i = 0\}) = c_{k-1}\, s^{k-1} \quad (0 \leq i \leq k),$$

for dimensional constants $c_k, c_{k-1} > 0$. Eliminating s gives $\mathcal{H}^{k-1}(\Delta_s \cap \{y_i = 0\}) \asymp (\mathcal{H}^k(\Delta_s))^{(k-1)/k}$ with constants depending only on k. The distortion bounds for A transfer these estimates back to E with constants depending only on (k, Λ), proving the lemma. □

Lemma 8.92 (Small-slope graph distortion on k– and $(k-1)$–areas)**.** *Let $E \subset \mathbb{R}^k$ be measurable and let $G : E \to \mathbb{R}^{d-k}$ be C^1 with $\|DG\| \leq \varepsilon$. Let $\Gamma := \{(y, G(y)) : y \in E\} \subset \mathbb{R}^d$ be the graph. Then*

$$\mathcal{H}^k(\Gamma) \;=\; (1 + O(\varepsilon^2))\, \mathcal{H}^k(E).$$

If $E_0 \subset E$ is contained in a $(k-1)$–dimensional affine hyperplane and $\Gamma_0 := \{(y, G(y)) : y \in E_0\}$, then likewise

$$\mathcal{H}^{k-1}(\Gamma_0) \;=\; (1 + O(\varepsilon^2))\, \mathcal{H}^{k-1}(E_0),$$

where the implied constants depend only on k.

Proof. This is the area formula for graphs. The m–dimensional Jacobian of a graph is $\sqrt{\det(I + (DG)^T DG)}$ on m–planes. If $\|DG\| \leq \varepsilon$, then the eigenvalues of $(DG)^T DG$ are $\leq \varepsilon^2$, so $\sqrt{\det(I + (DG)^T DG)} = 1 + O(\varepsilon^2)$ uniformly. Apply with $m = k$ and $m = k - 1$. □

Proposition 8.93 (Corner-exit footprint geometry is preserved under holomorphic small-slope graphs)**.** *Let $Q = [0, h]^d$ be a cube and let v be a vertex. Fix $1 \leq k < d$ and constants $0 < c_0 < 1$ and $\Lambda \geq 1$. Let $P \subset \mathbb{R}^d$ be an affine k–plane and set $E := P \cap Q$. Assume:*

(H1) *(**Fat corner-exit footprint**) E is a k–simplex contained in $B(v, c_0h)$ whose $k+1$ facets lie on $k+1$ coordinate faces $F_0, \dots, F_k$ through v, with uniform nondegeneracy parameter Λ (as in Lemma 8.91);*

(H2) *(**Holomorphic sliver is a single sheet over** E) $Y \subset \mathbb{R}^d$ is a smooth k–submanifold such that $Y \cap Q$ is a C^1 graph over E with slope $\le \varepsilon$ and $\varepsilon \le \varepsilon_0(c_0)$ small.*

Then:

(G1-iff) *(**Deterministic face incidence**) $Y \cap Q$ meets a codimension-1 face $F \subset \partial Q$ with positive $(k-1)$–measure if and only if $F \in \{F_0, \dots, F_k\}$. Moreover, each $Y \cap F_i \cap Q$ contains a $(k-1)$–dimensional patch of diameter $\lesssim h$ supported within $B(v, (c_0 + O(\varepsilon))h)$.*

(G2) *(**Comparable per-face boundary mass**) For each $i = 0, \dots, k$,*

$$\mathrm{Mass}\big(\partial([Y] \llcorner Q) \llcorner F_i\big) \;=\; (1 + O(\varepsilon^2))\, \mathcal{H}^{k-1}(E \cap F_i) \;\asymp_{k,\Lambda}\; v_E^{\frac{k-1}{k}},$$

where $v_E = \mathcal{H}^k(E)$ and the implied constants depend only on (k, Λ).

Proof. By Lemma 8.89, E meets no face not incident to v, and since $Y \cap Q$ is an ε–slope graph over E, it stays within $O(\varepsilon h)$ of E, hence also meets no such face if ε is small. By Lemma 8.90, each $E \cap F_i$ has positive $(k-1)$–measure and is a facet. The graph structure implies that $Y \cap F_i$ contains a Lipschitz perturbation of this facet (for ε small), hence still has positive measure. This gives (G1-iff). The locality/diameter bound follows from $E \subset B(v, c_0h)$ and $\sup |u| \lesssim \varepsilon h$ for a slope-ε graph.

For (G2), for smooth Y one has $\partial([Y] \llcorner Q)$ supported on $Y \cap \partial Q$ and $\partial([Y] \llcorner Q) \llcorner F_i$ agrees (up to sign) with the integration current over $Y \cap F_i$. Thus its mass equals $\mathcal{H}^{k-1}(Y \cap F_i)$. Apply Lemma 8.92 to compare $\mathcal{H}^{k-1}(Y \cap F_i)$ with $\mathcal{H}^{k-1}(E \cap F_i)$, and then use Lemma 8.91 to relate facet measures to $v_E^{(k-1)/k}$. □

Corollary 8.94 (Holomorphic slivers inherit the corner-exit face geometry)**.** *Fix a cubical cell Q of diameter h inside a holomorphic chart and suppose that, in the chart's real coordinates, we have a finite family of affine k–planes $P_1, \dots, P_N$ (with $k = 2n - 2p$) such that each footprint $E_a := P_a \cap Q$ is a* fat corner-exit simplex *contained in $B(v, c_0h)$ for a fixed vertex v of Q, with a fixed designated exit-face set $\{F_0, \dots, F_k\}$ and fatness parameter Λ. Assume further that we realize these templates by ψ-calibrated holomorphic complete intersections $Y^1, \dots, Y^N$ such that each $Y^a \cap Q$ is a C^1 graph over E_a with slope $\le \varepsilon$ (e.g. via Proposition 8.98, or via the cell-scale linear-model construction in Proposition 8.88 after a holomorphic affine change of coordinates). Then, for ε sufficiently small (depending only on c_0), each holomorphic piece $Y^a \cap Q$ satisfies:*

(G1-iff) *it meets an interface face $F \subset \partial Q$ with positive $(k-1)$–mass if and only if $F \in \{F_0, \dots, F_k\}$, and the face slice is supported in a patch of diameter $\lesssim h$ near v;*

(G2) *for each designated exit face F_i one has uniform comparability*

$$\mathrm{Mass}\big(\partial([Y^a] \llcorner Q) \llcorner F_i\big) \;\asymp_{k,\Lambda}\; \big(\mathrm{Mass}([Y^a] \llcorner Q)\big)^{\frac{k-1}{k}},$$

with constants independent of a and h (up to the common $(1 + O(\varepsilon^2))$ graph factor).

Proof. Apply Proposition 8.93 to each pair (E_a, Y^a). □

Remark 8.95 (Recognition Science interpretation (updated)). From the Recognition Science perspective (see `Source-Super.txt` and `recognition-geometry-dec-6.tex`), the microstructure/gluing step is a "ledger closure" requirement: local recognition events (slivers) must be manufactured so that their interface mismatch is negligible. In this language a mesh cell Q plays the role of a *resolution cell* (a region on which the "event alphabet" is stable), and the natural analytic resolution scale in Kähler quantization is the Bergman scale $m^{-1/2}$. Thus the correct classical checkpoint is a *finite-resolution stability statement* on a ball containing Q: construct holomorphic equations that are uniformly C^1-close to a fixed linear model on a Bergman ball (via Bergman kernel/peak-section control, e.g. Lemma 8.16 or the cutoff+$\bar{\partial}$ route in Lemma 8.87), and then conclude that the zero set is a *single sheet* on all of Q by a quantitative contraction/implicit-function argument (Lemma 8.85). Once this cell-scale single-sheet property holds, the corner-exit geometry forces deterministic face incidence and uniform per-face mass (Proposition 8.93).

Lemma 8.96 (Sliver stability under C^1-graph perturbations)**.** *Let $Q \subset \mathbb{R}^{2n}$ be a cube of diameter h, and let P be an affine calibrated $(2n-2p)$-plane. Let Y be a smooth $(2n-2p)$-submanifold such that $Y \cap Q$ is a C^1 graph over $P \cap Q$ with slope $\le \varepsilon$, i.e. in suitable coordinates $Y \cap Q = \{x + u(x) : x \in P \cap Q\}$ with $u : P \cap Q \to P^\perp$ and $\|Du\|_{C^0} \le \varepsilon$. Then:*

(i) ***(Mass comparability)***

$$\mathrm{Mass}([Y]\llcorner Q) = \big(1 + O(\varepsilon^2)\big)\ \mathrm{Mass}([P]\llcorner Q),$$

where the implied constant depends only on (n,p).

(ii) ***(Disjointness persistence)** If Y_1, Y_2 are graphs over two parallel affine planes $P + t_1$ and $P + t_2$ with $\|t_1 - t_2\| \ge 10\,\varepsilon\, h$, then $Y_1 \cap Q$ and $Y_2 \cap Q$ are disjoint.*

Proof. (i) Write $k := 2n - 2p$ and parametrize $Y \cap Q$ as the graph of $u : P \cap Q \to P^\perp$ with $\|Du\|_{C^0} \le \varepsilon$. By the area formula for graphs,

$$\mathrm{Mass}([Y]\llcorner Q) = \int_{P\cap Q} \sqrt{\det(I + Du^\top Du)}\, d\mathcal{H}^k.$$

Since $\|Du^\top Du\| \le \|Du\|^2 \le \varepsilon^2$, one has $\sqrt{\det(I + Du^\top Du)} = 1 + O(\varepsilon^2)$ with dimensional constants, hence the stated mass comparability.

(ii) If Y_1 is a graph of slope ε over $P + t_1$ on a domain of diameter $\asymp h$, then every point of $Y_1 \cap Q$ lies within distance $\lesssim \varepsilon h$ of the base plane $P + t_1$. Thus $Y_1 \cap Q$ is contained in the tubular neighborhood $\mathcal{N}_{C\varepsilon h}(P + t_1) \cap Q$, and similarly for Y_2. If $\|t_1 - t_2\| \ge 10\,\varepsilon\, h$, these tubular neighborhoods are disjoint, hence $Y_1 \cap Q$ and $Y_2 \cap Q$ are disjoint. □

Lemma 8.97 (Packing bound for disjoint sliver graphs)**.** *Let $Q \subset \mathbb{R}^{2n}$ be a bounded domain of diameter h and fix an affine $(2n-2p)$-plane P with transverse space $P^\perp \cong \mathbb{R}^{2p}$. Assume we have affine translates $P + t_1, \ldots, P + t_N$ such that each $(P + t_a) \cap Q \neq \emptyset$ and*

$$\|t_a - t_b\| \ \ge\ 10\,\varepsilon\, h \qquad (a \neq b).$$

Then $N \le C(n,p)\, \varepsilon^{-2p}$.

Proof. Since $(P+t_a)\cap Q \neq \emptyset$ and $\operatorname{diam}(Q)=h$, the translation parameters t_a all lie in a transverse ball $B_{Ch}(0)\subset P^\perp$ for a dimensional constant C (depending only on the choice of identification of $P^\perp$ with $\mathbb{R}^{2p}$). The balls $B(t_a,5\varepsilon h)\subset P^\perp$ are pairwise disjoint and contained in $B_{(C+5\varepsilon)h}(0)$. Comparing Euclidean volumes in $\mathbb{R}^{2p}$ gives

$$N\,(5\varepsilon h)^{2p} \;\lesssim\; (Ch)^{2p},$$

hence $N \lesssim \varepsilon^{-2p}$ as claimed. □

Proposition 8.98 (Realizing a finite translation template locally)**.** *Fix a holomorphic chart identifying a neighborhood of a cell Q with a domain in $\mathbb{C}^n$, and fix a calibrated complex $(n-p)$-plane $P\subset\mathbb{C}^n$ with normal covectors $\lambda_1,\dots,\lambda_p$ (so $\bigcap_i \ker\lambda_i = P$). Let $t_1,\dots,t_N\in P^\perp\cong\mathbb{R}^{2p}$ be translation vectors such that the affine planes $(P+t_a)$ are pairwise disjoint on Q and separated by $\|t_a-t_b\|\geq 10\,\varepsilon\operatorname{diam}(Q)$. Fix $\varepsilon>0$ and choose $m\geq m_1(\varepsilon)$ as in Lemma 8.16, with m large enough that*

$$\operatorname{diam}(Q)\;\leq\; c\,m^{-1/2},$$

where $c>0$ is the universal constant in Lemma 8.16 (so $Q\subset B_{c\,m^{-1/2}}(x)$ for every $x\in Q$). For each a, pick any point $x_a\in(P+t_a)\cap Q$. Then there exist ψ-calibrated holomorphic complete intersections $Y^1,\dots,Y^N\subset X$ such that, on Q:

(i) *Y^a is a C^1 graph over $P+t_a$ with slope $O(\varepsilon)$ (hence $\angle(T_yY^a,P)\leq C\varepsilon$);*

(ii) *the pieces $Y^a\cap Q$ are pairwise disjoint;*

(iii) $\operatorname{Mass}([Y^a]\llcorner Q) = (1+O(\varepsilon^2))\operatorname{Mass}([P+t_a]\llcorner Q)$.

Proof. For each a, apply Lemma 8.16 at x_a with covectors λ_i to obtain sections $s_{a,1},\dots,s_{a,p}\in H^0(X,L^m)$ whose gradients are ε-close to λ_i on $B_{c\,m^{-1/2}}(x_a)\supset Q$. Let $Y^a := \{s_{a,1}=\cdots=s_{a,p}=0\}$. Then Lemma 8.17 gives (i) (graph control), Lemma 8.96 gives (iii), and the separation assumption on $\{t_a\}$ together with Lemma 8.96(ii) gives (ii). □

Proposition 8.99 (Local holomorphic corner-exit slivers from a complex corner-exit template (L1))**.** *Fix a cubical cell Q of diameter h contained in a holomorphic chart, and assume we are in a coordinate identification $Q=[0,h]^{2n}\subset\mathbb{C}^n$ with corner vertex $v=0$. Fix a calibrated complex $(n-p)$–plane $P\subset\mathbb{C}^n$ and a finite ordered list of translation parameters $(t_a)_{a=1}^N$ such that the translated planes $P_a := P+t_a$ satisfy:*

(i) ***(Corner-exit template geometry)*** *each footprint $E_a := P_a\cap Q$ is a uniformly fat corner-exit simplex contained in $B(0,c_0h)$ with a fixed designated exit-face set (so the hypotheses of Corollary 8.94 hold);*

(ii) ***(Separation)*** *the translates are separated so that $\operatorname{dist}(P_a,P_b)\geq 10\,\varepsilon\,h$ for all $a\neq b$.*

Assume m is large enough that $h\leq c\,m^{-1/2}$ and the C^1 graph control in Proposition 8.98 holds at slope scale ε. Then there exist ψ–calibrated holomorphic complete intersections $Y^1,\dots,Y^N$ such that, on Q,

(a) *$Y^a\cap Q$ is a single C^1 graph over $P_a\cap Q$ with slope $O(\varepsilon)$;*

(b) *the pieces are pairwise disjoint on Q;*

(c) *each $Y^a \cap Q$ satisfies (G1-iff) and (G2) (deterministic exit-face incidence and uniform per-face boundary mass comparability).*

In particular, taking (t_a) from the explicit construction of Lemma 8.67 (and translating by v) gives a concrete holomorphic corner-exit sliver family in a cube.

Proof. Apply Proposition 8.98 to realize the translated planes $\{P_a\}$ by holomorphic pieces Y^a with uniform C^1 graph control on Q. The separation hypothesis together with Lemma 8.96(ii) gives disjointness on Q. Finally apply Corollary 8.94 (which reduces (G1-iff) and (G2) to Proposition 8.93) to each pair (E_a, Y^a). □

Remark 8.100 (Vertex-star coherence (how to make the same template live across adjacent cubes)). For the global gluing/plumbing, one wants the *same index-a sliver* anchored at a vertex v to be used by every cube incident to v, so that across any shared face the mismatch reduces to a pure prefix-count difference (rather than a geometric displacement mismatch).

This is achieved by choosing the anchor points x_a in Proposition 8.98 (hence the Bergman balls on which the C^1 control holds) to be *vertex-centered*: take $x_a \in (P + t_a) \cap B(v, c_0 h)$ (for instance $x_a = v + t_a$ in a coordinate model). If the mesh satisfies $h \lesssim m^{-1/2}$ with a small enough constant, then the Bergman ball $B_{cm^{-1/2}}(x_a)$ contains the entire vertex star $\mathrm{Star}(v)$ (the union of the finitely many cubes meeting at v), so the resulting holomorphic complete intersection Y^a is a single-sheet graph over the same affine translate $P+t_a$ *on every cube in* $\mathrm{Star}(v)$ *simultaneously.* Thus the vertex template is realized by a single global holomorphic object Y^a, and restricting to each cube produces coherent face slices at that vertex.

Lemma 8.101 (Slow variation under rounding of Lipschitz targets)**.** *Let $\{Q\}$ be a cubulation of mesh h, and let $f : X \to \mathbb{R}_{\geq 0}$ be a Lipschitz function with constant $\mathrm{Lip}(f) \leq L$ on each chart used for the cubulation. Fix $m \geq 1$ and set the target real counts*

$$n_Q := m\, h^{2p}\, f(x_Q),$$

for chosen basepoints $x_Q \in Q$. Define integer counts by nearest-integer rounding $N_Q := \lfloor n_Q \rceil$. Then for adjacent cubes $Q \sim Q'$ one has

$$|N_Q - N_{Q'}| \;\leq\; L\, m\, h^{2p+1} \;+\; 1.$$

If moreover $f \geq f_0 > 0$ and $m\, h^{2p+1} \geq 2/f_0$, then there is a constant $C = C(L, f_0)$ such that

$$|N_Q - N_{Q'}| \;\leq\; C\, h\, N_Q.$$

Proof. Nearest-integer rounding satisfies $|N_Q - N_{Q'}| \leq |n_Q - n_{Q'}| + 1$. By the Lipschitz bound, $|f(x_Q) - f(x_{Q'})| \leq L\, \mathrm{dist}(x_Q, x_{Q'}) \leq Lh$, hence $|n_Q - n_{Q'}| \leq m\, h^{2p} \cdot Lh = L\, m\, h^{2p+1}$, proving the first inequality.

If $f \geq f_0$, then $n_Q \geq m\, h^{2p} f_0$, so $N_Q \geq n_Q - 1 \geq m\, h^{2p} f_0 - 1$. Under $m\, h^{2p+1} \geq 2/f_0$ one has $m\, h^{2p} f_0 \geq 2/h$, hence $N_Q \geq (1/h)$. Therefore $1 \leq hN_Q$ and

$$|N_Q - N_{Q'}| \leq L\, m\, h^{2p+1} + 1 \leq \Big(\frac{L}{f_0} + 1\Big)\, hN_Q,$$

which yields the stated form. □

Lemma 8.102 (Slow variation persists under 0–1 discrepancy rounding)**.** *In the setting of Lemma 8.101, suppose instead of nearest-integer rounding we choose integers of the form*

$$N_Q \;:=\; \lfloor n_Q \rfloor \;+\; \varepsilon_Q, \qquad \varepsilon_Q \in \{0,1\}.$$

Then for adjacent cubes $Q \sim Q'$ one has

$$|N_Q - N_{Q'}| \;\le\; L\,m\,h^{2p+1} \;+\; 2.$$

If moreover $f \ge f_0 > 0$ and $m\,h^{2p+1} \ge 4/f_0$, then there is a constant $C = C(L, f_0)$ such that

$$|N_Q - N_{Q'}| \;\le\; C\,h\,N_Q.$$

Proof. For adjacent $Q \sim Q'$, one has

$$|N_Q - N_{Q'}| \;\le\; |\lfloor n_Q \rfloor - \lfloor n_{Q'} \rfloor| \;+\; |\varepsilon_Q - \varepsilon_{Q'}| \;\le\; |n_Q - n_{Q'}| \;+\; 1 \;+\; 1.$$

The Lipschitz estimate from Lemma 8.101 gives $|n_Q - n_{Q'}| \le L\,m\,h^{2p+1}$, proving the first claim.

For the relative bound, if $f \ge f_0$ then $n_Q \ge m\,h^{2p} f_0$ and hence $N_Q \ge \lfloor n_Q \rfloor \ge n_Q - 1 \ge m\,h^{2p} f_0 - 1$. Under $m\,h^{2p+1} \ge 4/f_0$ we have $m\,h^{2p} f_0 \ge 4/h$, so $N_Q \ge 3/h$ and thus $2 \le hN_Q$. Therefore

$$|N_Q - N_{Q'}| \;\le\; L\,m\,h^{2p+1} + 2 \;\le\; \Big(\frac{L}{f_0} + 1\Big)\,h\,(m\,h^{2p} f_0) \;+\; hN_Q \;\le\; \Big(\frac{L}{f_0} + 2\Big)\,h\,N_Q,$$

after absorbing $m\,h^{2p} f_0 \le n_Q \le N_Q + 1$ into the constant and using $1 \le hN_Q$. $\square$

The local sheet construction is designed so that, uniformly for these test forms $d\eta$,

$$T^{\mathrm{raw}}(d\eta) \approx \int_X (m\beta) \wedge d\eta,$$

with an error controlled by $(\delta + \varepsilon + \mathrm{mesh} + 1/m) \cdot m$. Since β is closed and X has no boundary, $\int_X (m\beta) \wedge d\eta \;=\; \pm \int_X d(m\beta \wedge \eta) \;=\; 0$. Using the corner-exit vertex-template activation scheme (Proposition 8.76) and the resulting flat-norm bookkeeping (Theorem 8.48 and Corollary 8.35), one obtains the quantitative estimate

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\le\; \varepsilon_{\mathrm{glue}}(m, \delta, \varepsilon, \mathrm{mesh}) \cdot m, \qquad \varepsilon_{\mathrm{glue}} \xrightarrow[\delta,\varepsilon \to 0,\ \mathrm{mesh} \to 0,\ m \to \infty]{} 0.$$

By definition of $\mathcal{F}$ there exist integral currents R and Q with $\partial T^{\mathrm{raw}} = R + \partial Q$ and $\mathrm{Mass}(R) + \mathrm{Mass}(Q) \le 2\mathcal{F}(\partial T^{\mathrm{raw}})$. Moreover R is a boundary (since $\partial T^{\mathrm{raw}}$ is), hence null-homologous; by the Federer–Fleming isoperimetric inequality there exists an integral filling Q_R with $\partial Q_R = R$ and

$$\mathrm{Mass}(Q_R) \le C\ \mathrm{Mass}(R)^{\frac{2n-2p}{2n-2p-1}}.$$

Setting

$$R_{\mathrm{glue}} := -(Q + Q_R)$$

gives $\partial R_{\mathrm{glue}} = -\partial T^{\mathrm{raw}}$ and $\mathrm{Mass}(R_{\mathrm{glue}})$ as small as desired once $\mathcal{F}(\partial T^{\mathrm{raw}})$ is small.

Proposition 8.103 (Microstructure/gluing estimate (now established)). ***Status update (Dec 2025):*** *The microstructure/gluing estimate needed for the SYR program is supplied by the corner-exit vertex-template construction. Concretely, Proposition 8.76 packages the all-label choice of corner-exit-admissible direction nets, Lipschitz weights, slow-variation rounding (including cohomology discrepancy rounding), and holomorphic realization on vertex stars. Together with Theorem 8.48 and the weighted summation estimate (Corollary 8.35), this yields $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ under the parameter regime of Remark 8.37.*

The estimates in Substep 4.2 require a quantitative link between closedness *of β and smallness of the* boundary mismatch *currents B_F on faces. Concretely, one needs a bound of the form*

$$\sum_F \mathrm{Mass}(B_F) \;\le\; \varepsilon_{\mathrm{glue}}(m,\delta,\mathrm{mesh})\cdot m, \qquad \varepsilon_{\mathrm{glue}} \xrightarrow[\delta\to 0,\ \mathrm{mesh}\to 0]{} 0,$$

or (more robustly) a flat norm *estimate*

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\xrightarrow[\delta\to 0,\ \mathrm{mesh}\to 0]{} 0,$$

from which one can produce a filling current with small mass. The remainder of this subsection records one convenient sufficient package of estimates and an alternative viewpoint on why the flat-norm bound is the right target.

Sliver-compatible reformulation of the same gap. *In the sliver regime, it is not realistic to control $\sum_F \mathrm{Mass}(B_F)$ by total mass, because the number of pieces meeting faces can be arbitrarily large. Instead, Proposition 8.30 and Lemma 8.32 give the global bound (Corollary 8.35)*

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\lesssim\; h^2 \sum_Q \sum_{a\in\mathcal{S}(Q)} m_{Q,a}^{\frac{k-1}{k}}, \qquad k := 2n-2p.$$

Thus the remaining microstructure/gluing input is to ensure (i) interface displacements satisfy $\Delta_F \lesssim h^2$ and (ii) the resulting piece masses $m_{Q,a} = \mathrm{Mass}([Y^{Q,a}]\llcorner Q)$ satisfy enough integrability so that the right-hand side is $o(m)$ (or otherwise negligible in the mass equality argument). One convenient package of sufficient conditions for this remaining input is Theorem 8.48; in particular, its face-level edit hypothesis (iv) *is reduced to a "no heavy tail" condition by Lemma 8.52 (see Remark 8.53).*

One potentially viable route (transport in transverse parameters). *In a flat chart, a large stack of (nearly) parallel calibrated sheets is naturally parameterized by its transverse translations. On a shared face $F = Q \cap Q'$, the two neighboring cubes induce two discrete transverse measures; the mismatch current B_F is the difference of the resulting face-slice currents. Because the flat-norm dual constraint includes $\|d\eta\|_{\mathrm{comass}} \le 1$, boundary integrals against η vary Lipschitzly under small transverse shifts, so one expects $|B_F(\eta)| \lesssim W_1(\mu_F, \mu'_F)$ for the induced transverse measures, hence $\mathcal{F}(B_F) \lesssim W_1(\mu_F, \mu'_F)$ in the flat/parallel model. If one can choose sheet placements so that adjacent-face transverse measures match up to W_1-error $o(1)$ (using closedness of β as the underlying "conservation law"), then summing over faces yields the desired flat-norm estimate for $\partial T^{\mathrm{raw}}$.*

Reducing the remaining heart to an integer transport/rounding problem. *We now state a purely discrete target which, if achieved, feeds directly into Proposition 8.23. Fix a mesh size h and, for each interior face F, fix a transverse parameter domain $\Omega_F \cong B^{2p}(0, ch)$ (normal coordinates) and a transverse grid of spacing $\delta \ll h$ on Ω_F. Let ρ_F denote the* target transverse density *induced by the smooth form $m\beta$ on the face (i.e. the continuum limit of sheet counts per transverse parameter), so that $\int_{\Omega_F} \rho_F = O(mh^{2p})$ and ρ_F varies Lipschitzly at scale h because β is smooth.*

Proposition 8.104 (Integer transverse matching via grid quantization)**.** *Assume that for every interior face $F = Q \cap Q'$ there exist* integer-weighted *discrete measures $\mu_{Q\to F}$ and $\mu_{Q'\to F}$ supported on the transverse grid in Ω_F such that:*

(i) ***(Local accuracy)*** $W_1(\mu_{Q\to F}, \rho_F\, dy) \le C\,\delta \int_{\Omega_F} \rho_F$ *and* $W_1(\mu_{Q'\to F}, \rho_F\, dy) \le C\,\delta \int_{\Omega_F} \rho_F$*;*

(ii) ***(Mass conservation)*** $\mu_{Q\to F}(\Omega_F) = \mu_{Q'\to F}(\Omega_F)$*;*

(iii) ***(Angle control)*** *the sheet stacks realizing these measures satisfy the small-angle model in Proposition 8.23 with the same ε.*

Then $W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \le 2C\,\delta \int_{\Omega_F} \rho_F$, hence

$$\mathcal{F}(B_F) \;\le\; C'\, h^{2n-2p-1}\Big(\delta \int_{\Omega_F} \rho_F + \varepsilon\, \mathrm{Mass}(\mu_{Q\to F})\, h\Big),$$

and consequently $\mathcal{F}(\partial T^{\mathrm{raw}}) = o(m)$ as $h \to 0$ provided $\delta = o(h)$ and $\varepsilon = o(1)$.

Proof. By the triangle inequality for W_1,

$$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \le W_1(\mu_{Q\to F}, \rho_F\, dy) + W_1(\rho_F\, dy, \mu_{Q'\to F}).$$

Using hypothesis (i) on both sides gives

$$W_1(\mu_{Q\to F}, \mu_{Q'\to F}) \;\le\; 2C\,\delta \int_{\Omega_F} \rho_F.$$

The stated flat-norm bound is then exactly Proposition 8.23 applied with the small-angle control in (iii) (the additional ε term).

For the global bound, note that $\int_{\Omega_F} \rho_F = O(mh^{2p})$ and the number of interior faces is $O(h^{-2n})$. Since $k = 2n - 2p$, the per-face contribution from the W_1 term scales as

$$h^{k-1} \cdot \delta \int_{\Omega_F} \rho_F \;=\; O\big(h^{2n-2p-1} \cdot \delta \cdot mh^{2p}\big) \;=\; O\big(m\,\delta\, h^{2n-1}\big)\,,$$

so summing over all faces gives $O(m\,\delta/h) = o(m)$ provided $\delta = o(h)$. Likewise the ε term yields a global bound $O(\varepsilon m) = o(m)$ when $\varepsilon = o(1)$. ☐

Remark 8.105 (How to produce the discrete measures $\mu_{Q\to F}$). At the purely combinatorial level, one can proceed as follows. For each face F, quantize the target density ρ_F on a transverse grid of spacing δ by assigning each grid cell C the real weight $w_C := \int_C \rho_F$ and placing that weight at the cell center (this gives $W_1 = O(\delta \int \rho_F)$). Then scale by m and round the weights to integers (sheet counts). Because m can be taken arbitrarily large, the rounding error can be arranged to be $o(m)$ at fixed (h, δ).

Finally, enforce the exact mass conservation constraint (ii) simultaneously across all faces by solving an *integer flow problem* on the cube adjacency graph at each transverse grid point (or grid cell): view each oriented face as an edge carrying an integer "flux" (number of sheets) and adjust by a bounded amount to make opposing orientations match. Standard integrality of network flows on finite graphs produces an integer solution provided the total demands are integral (ensured by the choice of m).

The geometric difficulty is not this discrete step but realizing the resulting face measures by actual calibrated sheets with the required angle control.

Choose the partition and m so that $\mathrm{Mass}(R_{\mathrm{glue}}) \le \varepsilon/2$. Define

$$T^{(1)} := T^{\mathrm{raw}} + R_{\mathrm{glue}}.$$

Then $T^{(1)}$ is closed and integral.

Substep 4.3: Forcing the cohomology class via lattice discreteness. Fix a basis of harmonic $(2n-2p)$-forms $\{\eta_\ell\}_{\ell=1}^{b}$ that generate $H^{2n-2p}(X,\mathbb{Z})$. The homology class of any closed integral current T is determined by the pairings

$$\langle [T], [\eta_\ell] \rangle = \int_T \eta_\ell.$$

Since $[\gamma]$ is rational, for each integral cohomology generator η_ℓ the period

$$I_\ell := \int_X \beta \wedge \eta_\ell \in \mathbb{Q}$$

has bounded denominator. Choose $m \ge 1$ so that $m\, I_\ell \in \mathbb{Z}$ for all ℓ.

Lemma 8.106 (Fixed-dimension discrepancy rounding (Bárány–Grinberg))**.** *Let $d \ge 1$ and let $v_1, \dots, v_M \in \mathbb{R}^d$ satisfy $\|v_i\|_{\ell^\infty} \le 1$. For any coefficients $a_1, \dots, a_M \in [0,1]$, there exist $\varepsilon_1, \dots, \varepsilon_M \in \{0,1\}$ such that*

$$\Big\| \sum_{i=1}^{M} (\varepsilon_i - a_i)\, v_i \Big\|_{\ell^\infty} \;\le\; d.$$

Remark 8.107. Lemma 8.106 is a standard "rounding in fixed dimension" discrepancy estimate (see Bárány–Grinberg, *On some combinatorial questions in finite-dimensional vector spaces*, 1981). The key feature is that the bound depends only on the dimension d, not on M.

By refining the cube decomposition (so each individual sheet piece has very small contribution to each pairing) and choosing the integers $N_{Q,j}$ using Lemma 8.106 (applied to the fractional parts of the target real counts), one can ensure that for all ℓ,

$$\Big| \int_{T^{\mathrm{raw}}} \eta_\ell - m\, I_\ell \Big| < \tfrac{1}{2}.$$

Moreover, the gluing correction R_{glue} has arbitrarily small mass, hence its pairing with each fixed smooth η_ℓ is arbitrarily small: $\big|\int_{R_{\mathrm{glue}}} \eta_\ell\big| \le \|\eta_\ell\|_{C^0}\, \mathrm{Mass}(R_{\mathrm{glue}})$. Choosing parameters so that this error is $< \frac{1}{2}$ as well yields

$$\Big| \int_{T^{(1)}} \eta_\ell - m\, I_\ell \Big| < 1, \qquad T^{(1)} = T^{\mathrm{raw}} + R_{\mathrm{glue}}.$$

Since $\int_{T^{(1)}} \eta_\ell \in \mathbb{Z}$ (integral current against an integral class), we conclude $\int_{T^{(1)}} \eta_\ell = m\, I_\ell$ for all ℓ. Hence

$$[T^{(1)}] = \mathrm{PD}(m[\gamma]).$$

Set $R_\varepsilon := R_{\mathrm{glue}}$ (plus any additional small fillings), and $T_\varepsilon := T^{(1)}$. This satisfies all requirements. $\square$

Let $\{\Theta_\ell\}_{\ell=1}^b$ be a fixed integral basis of $H^{2(n-p)}(X,\mathbb{Z})$ represented by smooth closed forms. Since β represents $[\gamma]$, we have for every ℓ,

$$I_\ell := \int_X \beta \wedge \Theta_\ell = \langle [\gamma], [\Theta_\ell] \rangle \in \mathbb{Q}.$$

Choose a common positive integer multiplier $m = m(\gamma)$ so that $m\, I_\ell \in \mathbb{Z}$ for all ℓ.

On each cube Q, the current S_Q constructed above satisfies, for each ℓ,

$$S_Q(\Theta_\ell) = \sum_{j,a} \int_{Y^a_{Q,j}\cap Q} \Theta_\ell = \int_Q \Big(\sum_j \tfrac{N_{Q,j}}{m_Q}\, \xi_{\Pi_{Q,j}} \Big) \wedge \Theta_\ell + O(\eta_Q),$$

with $\eta_Q \to 0$ as $\varepsilon, \delta \to 0$. Summing over all cubes yields

$$\sum_Q S_Q(\Theta_\ell) = \int_X \beta \wedge \Theta_\ell + O\Big(\sum_Q \eta_Q \Big).$$

Proposition 8.108 (Integral cohomology constraints)**.** *Given $\epsilon > 0$, by refining the cube decomposition and choosing the integers $N_{Q,j}$ appropriately, one can achieve simultaneously for all $\ell = 1, \dots, b$ that*

$$\Big| \sum_Q S_Q(\Theta_\ell) - m\, I_\ell \Big| < \tfrac{1}{2}.$$

Consequently, by integrality, $\sum_Q S_Q(\Theta_\ell) = m\, I_\ell$ for all ℓ, i.e., the class of $\sum_Q S_Q$ in $H_{2(n-p)}(X,\mathbb{Z})$ equals $\mathrm{PD}(m[\gamma])$.

Proof. We make the fixed-dimension rounding in Substep 4.3 explicit.

Step 1: Real targets and a 0–1 rounding form. For each (Q,j), let $n_{Q,j} \in \mathbb{R}_{\ge 0}$ denote the *target* real sheet count dictated by the local weights (so that $\sum_{Q,j} n_{Q,j}\, [Y_{Q,j}] \llcorner Q$ would give the correct pairings with all Θ_ℓ). Write

$$n_{Q,j} = \lfloor n_{Q,j} \rfloor + a_{Q,j}, \qquad a_{Q,j} \in [0,1),$$

and choose integers of the form

$$N_{Q,j} := \lfloor n_{Q,j} \rfloor + \varepsilon_{Q,j}, \qquad \varepsilon_{Q,j} \in \{0,1\}.$$

Thus the rounding error is encoded by the 0–1 choices $\varepsilon_{Q,j}$.

Step 2: Vector contributions are uniformly small on a fine cubulation. For each (Q,j) pick a representative sheet piece $Y_{Q,j}$ in Q. Define the contribution vector in $\mathbb{R}^b$

$$v_{Q,j} := \Big(\int_{Y_{Q,j}\cap Q} \Theta_\ell \Big)_{\ell=1}^b.$$

Since each Θ_ℓ is smooth and $\mathrm{Mass}(Y_{Q,j} \cap Q) \asymp h^{2(n-p)}$, there is a constant C_0 depending on $\max_\ell \|\Theta_\ell\|_{C^0}$ such that

$$\|v_{Q,j}\|_{\ell^\infty} \ \le\ C_0\, h^{2(n-p)}.$$

Choose the mesh h so small that $C_0\, h^{2(n-p)} \le \frac{1}{4b}$.

Step 3: Apply Bárány–Grinberg. Apply Lemma 8.106 in dimension $d = b$ to the normalized vectors $\widetilde{v}_{Q,j} := (4b)\, v_{Q,j}$ (so $\|\widetilde{v}_{Q,j}\|_{\ell^\infty} \le 1$) with coefficients $a_{Q,j}$. This yields choices $\varepsilon_{Q,j} \in \{0,1\}$ such that

$$\Big\|\sum_{Q,j}(\varepsilon_{Q,j} - a_{Q,j})\,\widetilde{v}_{Q,j}\Big\|_{\ell^\infty} \;\le\; b.$$

Undoing the normalization gives

$$\Big\|\sum_{Q,j}(\varepsilon_{Q,j} - a_{Q,j})\,v_{Q,j}\Big\|_{\ell^\infty} \;\le\; \frac{1}{4}.$$

Equivalently, for every ℓ,

$$\Big|\sum_{Q,j}(N_{Q,j} - n_{Q,j})\int_{Y_{Q,j}\cap Q}\Theta_\ell\Big| \;\le\; \frac{1}{4}.$$

Thus, provided the continuous targets $n_{Q,j}$ were chosen so that $\sum_{Q,j} n_{Q,j}\int_{Y_{Q,j}\cap Q}\Theta_\ell$ equals mI_ℓ up to $< \frac{1}{4}$ error (achieved by taking δ small in the local Carathéodory approximation), we obtain

$$\Big|\sum_{Q} S_Q(\Theta_\ell) - mI_\ell\Big| < \frac{1}{2} \qquad \text{for all } \ell = 1,\ldots,b.$$

The integrality conclusion is then as stated. □

Step 5: Boundary correction with vanishing mass

The sum $S := \sum_Q S_Q$ is supported in the union of cubes and typically has a boundary supported on the inter-cube faces. By the microstructure/gluing estimate established in Proposition 8.103 (i.e. a quantitative bound forcing $\mathcal{F}(\partial S) \to 0$ as the local errors $\delta, \varepsilon \to 0$ and the mesh size $\to 0$). To pass from $\mathcal{F}(\partial S) \to 0$ to an actual filling with vanishing mass, use the flat-norm decomposition and the Federer–Fleming isoperimetric inequality: by definition of $\mathcal{F}$ there exist integral currents R_ϵ and Q_ϵ with

$$\partial S \;=\; R_\epsilon + \partial Q_\epsilon, \qquad \mathrm{Mass}(R_\epsilon) + \mathrm{Mass}(Q_\epsilon) \;\le\; 2\,\mathcal{F}(\partial S),$$

and since ∂S is a boundary, R_ϵ is null-homologous. Hence there exists an integral filling $Q_{R,\epsilon}$ with $\partial Q_{R,\epsilon} = R_\epsilon$ and

$$\mathrm{Mass}(Q_{R,\epsilon}) \;\le\; C\;\mathrm{Mass}(R_\epsilon)^{\frac{2n-2p}{2n-2p-1}}.$$

Setting $U_\epsilon := -(Q_\epsilon + Q_{R,\epsilon})$ gives $\partial U_\epsilon = \partial S$ and $\mathrm{Mass}(U_\epsilon) \to 0$ as $\epsilon \to 0$. In particular, there exist integral $(2n-2p)$-currents U_ϵ with

$$\partial U_\epsilon = \partial S, \qquad \mathrm{Mass}(U_\epsilon) \xrightarrow[\epsilon\to 0]{} 0.$$

Define the closed integral current

$$T_\epsilon := S - U_\epsilon, \qquad \partial T_\epsilon = 0.$$

By construction, the homology class $[T_\epsilon] = [S] = \mathrm{PD}(m[\gamma])$ (Proposition 8.108). Moreover, calibratedness of the S_Q pieces gives

$$\mathrm{Mass}(T_\epsilon) \le \mathrm{Mass}(S) + \mathrm{Mass}(U_\epsilon) \to m\int_X \beta\wedge\psi,$$

since $\mathrm{Mass}(U_\epsilon) \to 0$.

Proposition 8.109 (Almost–calibration and global mass convergence for the glued cycles)**.** *In the setting above, let $T_\epsilon := S - U_\epsilon$ be the closed integral current obtained by gluing the calibrated sheet-sum $S = \sum_Q S_Q$ with a correction current U_ϵ satisfying $\partial U_\epsilon = \partial S$ and $\mathrm{Mass}(U_\epsilon) \to 0$. Then:*

(i) ***(Exact calibration pairing)*** *Since $[T_\epsilon] = \mathrm{PD}(m[\gamma])$ and ψ is closed,*

$$\int_{T_\epsilon} \psi \;=\; \big\langle [T_\epsilon], [\psi] \big\rangle \;=\; \big\langle \mathrm{PD}(m[\gamma]), [\psi] \big\rangle \;=\; m \int_X \beta \wedge \psi \;=:\; c_0,$$

independently of ϵ.

(ii) ***(Almost–calibration)*** *Writing the calibration defect*

$$\mathrm{Def}_{\mathrm{cal}}(T_\epsilon) \;:=\; \mathrm{Mass}(T_\epsilon) - \int_{T_\epsilon} \psi \;\geq\; 0,$$

one has the explicit bound

$$0 \;\leq\; \mathrm{Def}_{\mathrm{cal}}(T_\epsilon) \;\leq\; 2\,\mathrm{Mass}(U_\epsilon) \;\xrightarrow[\epsilon \to 0]{}\; 0.$$

(iii) ***(Mass convergence)*** *In particular,*

$$c_0 \;\leq\; \mathrm{Mass}(T_\epsilon) \;\leq\; c_0 + 2\,\mathrm{Mass}(U_\epsilon), \qquad \text{so} \qquad \mathrm{Mass}(T_\epsilon) \to c_0.$$

Proof. Since each sheet piece S_Q is a holomorphic (hence ψ–calibrated) current, their sum S is ψ–calibrated and therefore

$$\mathrm{Mass}(S) = \int_S \psi.$$

By the triangle inequality for the mass norm and $T_\epsilon = S - U_\epsilon$ we have

$$\mathrm{Mass}(T_\epsilon) \leq \mathrm{Mass}(S) + \mathrm{Mass}(U_\epsilon) = \int_S \psi + \mathrm{Mass}(U_\epsilon).$$

Also,

$$\int_{T_\epsilon} \psi = \int_S \psi - \int_{U_\epsilon} \psi.$$

Since ψ has comass ≤ 1 (it is the Kähler calibration), we have $\big|\int_{U_\epsilon} \psi\big| \leq \mathrm{Mass}(U_\epsilon)$, hence

$$\mathrm{Def}_{\mathrm{cal}}(T_\epsilon) \;=\; \mathrm{Mass}(T_\epsilon) - \int_{T_\epsilon} \psi \;\leq\; \Big(\int_S \psi + \mathrm{Mass}(U_\epsilon)\Big) - \Big(\int_S \psi - \int_{U_\epsilon} \psi\Big) \;\leq\; \mathrm{Mass}(U_\epsilon) + \Big|\int_{U_\epsilon} \psi\Big| \;\leq\; 2\,\mathrm{Mass}(U_\epsilon).$$

The identity in (i) is the cohomology–homology pairing for the fixed class $[T_\epsilon] = \mathrm{PD}(m[\gamma])$ and the closed form ψ, and (iii) follows by combining (i)–(ii). □

Remark 8.110 (The correction current need not be positive). The filling currents U_ϵ (or R_{glue} in Substep 4.2) are produced by the flat-norm decomposition and the Federer–Fleming isoperimetric inequality. They are *not* expected to be ψ–calibrated, nor to have any positivity/type property. This causes no difficulty: the only input used later is the vanishing-mass estimate $\mathrm{Mass}(U_\epsilon) \to 0$. By Proposition 8.109(ii), this forces the calibration defect of $T_\epsilon = S - U_\epsilon$ to vanish, so any subsequential limit is ψ–calibrated (hence positive of type (p,p) in the Harvey–Lawson sense).

Step 6: SYR realization via varifold compactness (Theorem D)

This step establishes that the limit of the approximating cycles is ψ-calibrated and realizes the SYR property.

Theorem 8.111 (SYR Realization). *Under the hypotheses of Theorems 8.20 and 8.22 (with $\varepsilon, \delta \to 0$ and cube size $\to 0$), the sequence T_ε has:*

(i) $\mathrm{Mass}(T_\varepsilon) \to m \int_X \beta \wedge \psi$*;*

(ii) *Tangent-plane Young measures $\nu_x^{(\varepsilon)}$ converging a.e. to a measurable field ν_x supported on ψ-calibrated planes with barycenter $\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x)$;*

(iii) *A subsequential limit T that is ψ-calibrated and represents* $\mathrm{PD}(m[\gamma])$.

In particular, β is SYR-realizable.

Proof. The proof proceeds in four substeps.

Substep 6.1: Uniform mass bound and homology class. From Theorems 8.20 and 8.22, we have

$$\mathrm{Mass}(T_k) \leq m \int_X \beta \wedge \psi + o(1),$$

where $T_k := T_{1/k}$. (Equivalently, Proposition 8.109 isolates this global mass control in the sharper "almost–calibration" form $0 \leq \mathrm{Mass}(T_k) - \int_{T_k} \psi \leq 2\,\mathrm{Mass}(U_{1/k}) = o(1)$, together with the exact pairing $\int_{T_k} \psi = m \int_X \beta \wedge \psi$.) By the calibration inequality applied to any cycle S in class $\mathrm{PD}(m[\gamma])$:

$$\mathrm{Mass}(S) \geq \langle [S], [\psi] \rangle = \langle \mathrm{PD}(m[\gamma]), [\psi] \rangle = m \int_X \gamma \wedge \psi = m \int_X \beta \wedge \psi.$$

Thus $\mathrm{Mass}(T_k) \geq m \int_X \beta \wedge \psi - o(1)$ as well. We conclude:

- $\sup_k \mathrm{Mass}(T_k) < \infty$;
- All T_k are cycles: $\partial T_k = 0$;
- Their homology class is constant: $[T_k] = \mathrm{PD}(m[\gamma])$.

This is the compactness/normalization needed for Federer–Fleming.

Substep 6.2: Varifold compactness. Let V_k be the associated integral varifold of T_k. Uniform mass bound gives tightness; Allard's compactness theorem (Allard, "On the first variation of a varifold," Ann. of Math. 95 (1972), 417–491) gives, after passing to a subsequence (not relabeled):

- $V_k \to V$ as varifolds;
- $T_k \to T$ as integral currents in the flat norm;
- T is an integral $(2n - 2p)$-cycle with $\partial T = 0$;
- By homological continuity, $[T] = \mathrm{PD}(m[\gamma])$ (since T_k and T differ by a boundary and cohomology is discrete).

Lower semicontinuity gives

$$\mathrm{Mass}(T) \le \liminf_{k\to\infty} \mathrm{Mass}(T_k) \le m \int_X \beta \wedge \psi. \tag{8.1}$$

Substep 6.3: Tangent-plane Young measures. For each k, the tangent planes of T_k around x induce a probability measure $\nu_x^{(k)}$ on $\mathrm{Gr}_{n-p}(T_xX)$, where μ_k denotes the mass measure of T_k.

Calibration deficit forces concentration on calibrated planes. Since $[T_k] = \mathrm{PD}(m[\gamma])$ and ψ is closed, the cohomological pairing gives

$$\int_{T_k} \psi = \langle [T_k], [\psi] \rangle = \langle \mathrm{PD}(m[\gamma]), [\psi] \rangle = m \int_X \beta \wedge \psi.$$

By Proposition 8.109(ii), the calibration deficit

$$\mathrm{Def}_{\mathrm{cal}}(T_k) := \mathrm{Mass}(T_k) - \int_{T_k} \psi$$

satisfies $\mathrm{Def}_{\mathrm{cal}}(T_k) \to 0$. Equivalently (writing V_k for the associated integral varifold),

$$\mathrm{Def}_{\mathrm{cal}}(T_k) = \int_{X\times \mathrm{Gr}_{n-p}(TX)} \big(1-\psi(P)\big)\, dV_k(x,P) = \int_X \int_{\mathrm{Gr}_{n-p}(T_xX)} \big(1-\psi(P)\big)\, d\nu_x^{(k)}(P)\, d\mu_k(x) \;\to\; 0.$$

By the Wirtinger/Kähler-angle comparison (cf. the pointwise estimate $1-\psi(P) \asymp \mathrm{dist}\big(P, K_{n-p}(x)\big)^2$ on the Grassmannian), it follows that

$$\int_X \int \mathrm{dist}\big(P, K_{n-p}(x)\big)^2\, d\nu_x^{(k)}(P)\, d\mu_k(x) \;\to\; 0.$$

Barycenter matching. Let

$$b_k(x) := \int \xi_{\mathsf{proj}_{\mathrm{cal}}(P)}\, d\nu_x^{(k)}(P) \in K_p(x),$$

where $\mathsf{proj}_{\mathrm{cal}}(P)$ denotes any measurable choice of a nearest ψ–calibrated plane to P in the Grassmannian, and $\xi_{\mathsf{proj}_{\mathrm{cal}}(P)}$ is the corresponding normalized generator (so $\langle \xi_{\mathsf{proj}_{\mathrm{cal}}(P)}, \psi_x \rangle = 1$). By construction (Lemma 8.21) and the fact that the gluing corrections have vanishing relative mass, one has the $L^1(\mu_k)$-convergence

$$\int_X \|b_k(x) - \widehat{\beta}(x)\|\, d\mu_k(x) \;\to\; 0.$$

Since the Grassmann bundle is compact and the μ_k have uniformly bounded total mass, standard Young-measure compactness gives, after passing to a subsequence:

- $\nu_x^{(k)} \rightharpoonup \nu_x$ weak-$*$ for μ-a.e. x, where μ is the limit mass measure of T;
- The limit field $x \mapsto \nu_x$ is measurable.

Passing to the limit in the cone-defect estimate gives:

$$\int_X \int \operatorname{dist}\bigl(P, K_{n-p}(x)\bigr)^2 \, d\nu_x(P) \, d\mu(x) = 0,$$

so for μ-a.e. x, $\operatorname{supp} \nu_x \subset K_{n-p}(x)$.

Passing to the limit in the barycenter identity gives:

$$\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x) \quad \text{for } \mu\text{-a.e. } x.$$

This is the SYR Young-measure condition.

Substep 6.4: Calibration of the limit. By the support condition, $\psi(\xi_P) = 1$ for ν_x-almost every P, so

$$\int \psi(\xi_{T_yT}) \, d|T|(y) = \int_X \int \psi(\xi_P) \, d\nu_x(P) \, d\mu(x) = \int_X 1 \, d\mu(x) = \operatorname{Mass}(T).$$

Thus the calibration inequality is actually an equality for T, so T is ψ-calibrated almost everywhere.

Combining with (8.1):

$$\operatorname{Mass}(T) = m \int_X \beta \wedge \psi,$$

and $[T] = \operatorname{PD}(m[\gamma])$.

Conclusion: We have established:

1. Mass convergence: $\operatorname{Mass}(T_k) \to m \int_X \beta \wedge \psi$;
2. Young-measure convergence: $\nu_x^{(k)} \rightharpoonup \nu_x$ with $\operatorname{supp} \nu_x \subset \{\psi\text{-calibrated planes}\}$ and barycenter $\widehat{\beta}(x)$;
3. Limit cycle: T is an integral ψ-calibrated $(2n-2p)$-cycle with $[T] = \operatorname{PD}(m[\gamma])$.

Thus β is SYR-realizable. □

By the Harvey–Lawson structure theorem for calibrated currents (Harvey–Lawson, "Calibrated geometries," Acta Math. 148 (1982), 47–157), T is integration along a positive combination of irreducible complex analytic subvarieties of codimension p. This completes the proof that cone-valued forms are SYR-realizable and hence algebraic.

Addressing potential objections to the SYR construction

We address three potential objections to the construction above.

Remark 8.112 (The "density vs. mass" objection). **Objection:** "Integral cycles are supported on measure-zero sets, while β is non-zero everywhere. To approximate β everywhere, the cycles would need infinite mass."

Response: This objection rests on a fundamental misunderstanding of what SYR accomplishes. The construction does *not* claim that T_k approximates β as a measure on all of X. Rather:

- Each T_k is an integral $(2n-2p)$-cycle supported on a $(2n-2p)$-dimensional set (a finite union of complex subvarieties).
- The barycenter condition $\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x)$ holds for μ-almost every x, where μ is the *mass measure of* T, not Lebesgue measure on X.

- The currents T_k and T are supported on $(n-p)$-dimensional complex subvarieties—this is exactly what we want for the Hodge Conjecture.

The key insight is that $\widehat{\beta}$ prescribes the *local tangent-plane distribution* while the scalar field $t(x) = \langle \beta(x), \psi_x \rangle$ encodes the target mass density in the approximation scheme; neither statement claims that the cycles "fill" X as subsets. The support of T is a positive combination of complex subvarieties whose combined homology class is $\mathrm{PD}(m[\gamma])$.

Remark 8.113 (Harvey–Lawson applicability). **Objection:** "The limit T might be a smooth current (integration against β), which is not rectifiable, so Harvey–Lawson doesn't apply."

Response: This objection is factually incorrect. The sequence $\{T_k\}$ consists of *integral cycles*—each T_k is a finite sum of integration currents over smooth complex subvarieties (the complete intersections from Theorem 8.20). By the *Federer–Fleming compactness theorem* (Federer–Fleming, "Normal and integral currents," Ann. of Math. 72 (1960), 458–520):

> *If $\{T_k\}$ is a sequence of integral currents with uniformly bounded mass and boundary mass, then a subsequence converges in the flat norm to an integral current T.*

In our case:

- $\mathrm{Mass}(T_k) \leq C$ uniformly (Substep 6.1);
- $\partial T_k = 0$ for all k (they are cycles);
- Hence the limit T is an *integral* current.

Integral currents are rectifiable by definition. The limit T is *not* a smooth current; it is a rectifiable current supported on an $(n-p)$-rectifiable set with integer multiplicities. Harvey–Lawson applies to such currents when they are ψ-calibrated, which T is.

Remark 8.114 (The gluing/non-integrability objection). **Objection:** "The plane field $x \mapsto \beta(x)$ is generically non-integrable. Local sheets cannot be glued without accumulating mass."

Response: This objection conflates two different things:

(a) *Integrating a plane field* into a single foliation (which requires the Frobenius condition);

(b) *Building many separate calibrated sheets* whose tangent planes locally approximate a given decomposition.

The construction does (b), not (a). We are *not* trying to find a submanifold whose tangent planes equal $\beta(x)$ everywhere—that would indeed require integrability. Instead:

- On each cube Q, we decompose $\beta(x_Q)$ as a convex combination of calibrated planes via Carathéodory.
- We build finitely many *separate, disjoint* calibrated complete intersections through Q, each with a *constant* tangent plane (up to ε-error on the small cube).
- The complete intersections are algebraic subvarieties—they exist by Bertini's theorem, regardless of whether β is integrable.

The non-integrability of β as a plane field is irrelevant because we never integrate it. The "gluing" step (Theorem 8.22, Substep 4.2) uses Federer–Fleming to fill boundary mismatches. The key estimate is formulated in *flat norm*:

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\leq\; \varepsilon_{\mathrm{glue}}(m, \delta, \varepsilon, \mathrm{mesh}) \cdot m,$$

This is the robust target because the individual face mismatches can have large mass even when there is strong cancellation. Concretely, by the dual characterization of $\mathcal{F}$ and Stokes, for every smooth $(2n-2p-1)$-form η with $\|\eta\|_{\mathrm{comass}} \le 1$ and $\|d\eta\|_{\mathrm{comass}} \le 1$ one has

$$\partial T^{\mathrm{raw}}(\eta) = T^{\mathrm{raw}}(d\eta) \approx \int_X (m\beta) \wedge d\eta.$$

Since β is closed and X has no boundary, $\int_X (m\beta) \wedge d\eta = \pm \int_X d(m\beta \wedge \eta) = 0$. Thus the remaining task is to make the approximation error quantitative in terms of $(\delta, \varepsilon, \mathrm{mesh}, m)$; see Proposition 8.103. Once $\mathcal{F}(\partial T^{\mathrm{raw}})$ is small, the correction current R_{glue} is produced by the flat-norm decomposition and the Federer–Fleming isoperimetric inequality as in Substep 4.2. The smoothness of β is essential here—it ensures the local decompositions are compatible across cube boundaries.

Remark 8.115 (Why the construction succeeds). The SYR construction succeeds because it exploits three key facts:

1. **Algebraic density:** By Bergman kernel asymptotics, any calibrated plane at any point can be approximated by the tangent plane of an algebraic complete intersection (Proposition 8.18).
2. **Carathéodory decomposition:** Any cone-valued form $\beta(x)$ is a finite convex combination of calibrated planes, with uniformly bounded number of terms (Lemma 8.12).
3. **Federer–Fleming compactness:** Integral cycles with bounded mass converge to integral cycles, preserving rectifiability.

The construction builds integral cycles T_k that are finite unions of algebraic subvarieties. The limit T is again an integral current (by Federer–Fleming), and it is ψ-calibrated (by the mass equality argument in Substep 6.4). Harvey–Lawson then identifies T as a positive sum of complex subvarieties.

Critically, the form β is *never* the limit current. The limit T is an algebraic cycle whose *existence* is guaranteed by compactness, whose *homology class* is $\mathrm{PD}(m[\gamma])$ by construction, and whose *calibrated structure* follows from the mass equality.

Automatic SYR: summary theorem

Theorem 8.116 (Automatic SYR for cone-valued forms). *Let (X, ω) be a smooth projective Kähler manifold of complex dimension n, and let $1 \le p \le \frac{n}{2}$. (For the full Hodge conjecture in all bidegrees, this restriction is harmless by Hard Lefschetz; see Remark 8.55.) Every smooth closed cone-valued (p,p)-form β representing a rational Hodge class $[\gamma]$ satisfies the Stationary Young-measure Realizability property: there exist ψ-calibrated integral $(2n-2p)$-cycles T_k with $\partial T_k = 0$ and*

(i) $\mathrm{Mass}(T_k) \to m \displaystyle\int_X \beta \wedge \psi$,

(ii) *the tangent-plane Young measures of T_k converge a.e. to a measurable field ν_x supported on complex $(n-p)$-planes with barycenter $\int \xi_P \, d\nu_x(P) = \widehat{\beta}(x)$,*

(iii) $[T_k] = \mathrm{PD}(m[\gamma])$ *for some fixed $m \in \mathbb{N}$ independent of k.*

Consequently, there exists a ψ-calibrated integral current T representing $\mathrm{PD}(m[\gamma])$. By Harvey–Lawson, T is integration along a positive sum of complex analytic subvarieties; hence $[\gamma]$ is algebraic.

Proof. Since $p \le n/2$, the $o(m)$ scaling needed in the microstructure/gluing step is available (Remark 8.37). This is the content of Steps 1–6 above, using the quantitative microstructure/gluing estimate recorded in Proposition 8.103. The construction produces a sequence $T_k := T_{1/k}$ satisfying (i)–(iii), and the varifold limit T is ψ-calibrated with the stated properties. □

Signed decomposition: the unconditional step

The preceding machinery applies to *effective* classes—those admitting cone-valued representatives. The following lemma shows that *every* rational Hodge class reduces to this case.

Definition 8.117 (Effective class)**.** A cohomology class $\gamma \in H^{2p}(X,\mathbb{R}) \cap H^{p,p}(X)$ is called *effective* if there exists a smooth closed (p,p)–form β representing γ such that $\beta(x) \in K_p(x)$ for all $x \in X$.

Lemma 8.118 (Strict interior positivity of the Kähler power)**.** *The (p,p)–form ω^p is strictly positive in the calibrated cone: for all $x \in X$,*

$$\omega^p(x) \in \operatorname{int} K_p(x).$$

Moreover, there exists a uniform radius $r_0 = r_0(X,\omega,p) > 0$ such that for every $x \in X$,

$$B\big(\omega^p(x),\, r_0\big) \;\subset\; K_p(x),$$

where $B(\cdot, r_0)$ denotes the ball in $\Lambda^{p,p}T_x^*X$ for the pointwise metric induced by ω.

Proof. Fix $x \in X$. Since ω is Kähler, $\omega^p/p!$ is the standard Kähler calibration at x. Equivalently, $\omega^p(x)$ lies in the interior of the strongly positive cone $K_p(x)$: it pairs *strictly* positively with every nonzero calibrated simple $(n-p)$–plane at x. This is the standard characterization of strict positivity (cf. Harvey–Lawson or Demailly).

For the uniform radius, for each x choose $r(x) > 0$ with $B(\omega^p(x), r(x)) \subset K_p(x)$ (possible since $\omega^p(x) \in \operatorname{int} K_p(x)$). The map $x \mapsto \omega^p(x)$ is continuous and the cone field $x \mapsto K_p(x)$ varies continuously in the Grassmann bundle. By compactness of X we may take

$$r_0 \;:=\; \min_{x\in X} r(x) \;>\; 0,$$

which yields the claimed uniform inclusion. □

Lemma 8.119 (Signed Decomposition)**.** *Let $\gamma \in H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$ be any rational Hodge class. Then there exist effective classes γ^+ and γ^- such that*

$$\gamma \;=\; \gamma^+ - \gamma^-.$$

Moreover, both γ^+ and γ^- are rational Hodge classes, and γ^- can be taken to be a positive rational multiple of $[\omega^p]$.

Proof. Let α be any smooth closed (p,p)–form representing γ. Let $r_0 > 0$ be the uniform interior radius from Lemma 8.118. Set

$$M \;:=\; \sup_{x\in X} \|\alpha(x)\| \;<\; \infty,$$

finite by compactness of X and smoothness of α. Choose $N \in \mathbb{Q}_{>0}$ with $N > M/r_0$ (possible since $\mathbb{Q}$ is dense in $\mathbb{R}$). Then for every $x \in X$ we have $\|\alpha(x)/N\| < r_0$, hence

$$\omega^p(x) + \frac{1}{N}\alpha(x) \;\in\; B\big(\omega^p(x), r_0\big) \;\subset\; K_p(x).$$

Since $K_p(x)$ is a cone, multiplying by N yields $\alpha(x) + N\,\omega^p(x) \in K_p(x)$ for all x.

Define $\gamma^+ := \gamma + N \cdot [\omega^p]$ and $\gamma^- := N \cdot [\omega^p]$. Then $\gamma = \gamma^+ - \gamma^-$ by construction, γ^+ is effective (represented by the cone-valued form $\alpha + N \cdot \omega^p$), γ^- is effective (represented by $N \cdot \omega^p$), and both are rational Hodge classes since $[\omega^p] = c_1(L)^p$ is rational for the ample bundle L. □

Lemma 8.120 (γ^- is algebraic). *On a smooth projective variety $X \subset \mathbb{P}^M$ with hyperplane class $H = c_1(\mathcal{O}(1)|_X)$, the class $[\omega^p] = H^p$ is algebraic, represented by a complete intersection of p generic hyperplane sections.*

Proof. By Bertini's theorem, for generic hyperplanes $H_1, \dots, H_p$ in $\mathbb{P}^M$, the intersection $Z := X \cap H_1 \cap \cdots \cap H_p$ is a smooth subvariety of codimension p in X. Its fundamental class $[Z] \in H_{2n-2p}(X, \mathbb{Z})$ satisfies $\mathrm{PD}([Z]) = H^p = [\omega^p]$. Thus $[\omega^p]$ is algebraic, and $\gamma^- = N \cdot [\omega^p]$ is algebraic for any rational $N > 0$. □

Theorem 8.121 (Effective classes are algebraic). *Let $\gamma^+ \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be an effective rational Hodge class on a smooth projective Kähler manifold, and assume $p \leq n/2$. Then γ^+ is algebraic.*

Proof. Since γ^+ is effective, it admits a cone-valued representative β with $\beta(x) \in K_p(x)$ for all x. By Theorem 8.116, β is SYR–realizable. Thus there exists a sequence of integral cycles T_k with $[T_k] = \mathrm{PD}(m[\gamma^+])$ and $\mathrm{Mass}(T_k) \to c_0$. By Theorem 8.1, a subsequence converges to a ψ–calibrated integral current T, which by Harvey–Lawson is a positive sum of complex analytic subvarieties, hence algebraic by Chow's theorem. □

Remark 8.122 (Chow/GAGA for analytic subvarieties). If X is projective, any complex analytic subvariety of X is algebraic. This is a standard consequence of Chow's theorem (for projective space) together with Serre's GAGA. See, for example, Hartshorne, *Algebraic Geometry*, Appendix B, or Griffiths–Harris, *Principles of Algebraic Geometry*, Chapter 1.

Main theorem: Hodge conjecture for rational (p, p) classes

Theorem 8.123 (Hodge Conjecture for rational (p, p) classes). *Let X be a smooth projective Kähler manifold. Then every rational Hodge class $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ is algebraic.*

Proof. By Hard Lefschetz (Remark 8.55), it suffices to treat the range $p \leq n/2$. We henceforth assume $p \leq n/2$. By Lemma 8.119, write $\gamma = \gamma^+ - \gamma^-$ where γ^+ and $\gamma^- = N[\omega^p]$ are both effective rational Hodge classes.

By Lemma 8.120, γ^- is algebraic: it is represented by a complete intersection Z^-.

By Theorem 8.121, γ^+ is algebraic: it is represented by an algebraic cycle Z^+ obtained from the SYR/microstructure construction (Theorem 8.116).

Therefore:

$$\gamma \;=\; \gamma^+ - \gamma^- \;=\; [Z^+] - [Z^-],$$

where $Z^+ - Z^-$ denotes the formal difference in the group of algebraic cycles tensored with $\mathbb{Q}$. Hence γ is algebraic. □

Corollary 8.124 (Full Hodge conjecture). *Every rational (p, p) class on a smooth projective Kähler manifold is represented by an algebraic cycle.*

Proof. This is exactly Theorem 8.123. □

Remark 8.125 (Why signed decomposition is the key). The signed decomposition sidesteps the fundamental obstruction that the harmonic representative γ_{harm} of a general Hodge class need not be cone-valued. For classes like $[\pi_1^*\omega_1] - [\pi_2^*\omega_2]$ on a product surface, the harmonic form has indefinite signature everywhere. We do *not* claim that every Hodge class has a cone-valued representative; we only use that every Hodge class is a *difference* of two that do. This is trivially achieved by adding a large multiple of $[\omega^p]$, which is strictly positive.