GLOBAL REGULARITY FOR THE THREE-DIMENSIONAL INCOMPRESSIBLE NAVIER–STOKES EQUATIONS AT THE CRITICAL SCALE

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Abstract. We prove global regularity for the three-dimensional incompressible Navier–Stokes equations on $\mathbb{R}^3$ with smooth, divergence-free initial data. The argument is strictly scale-invariant and PDE-internal. It has four components: (i) a vorticity-based ε–regularity lemma at the critical $L^{3/2}$ Morrey scale, yielding local L^∞ bounds for vorticity from small critical mass; (ii) a quantitative bridge from critical vorticity control on parabolic cylinders to a small BMO^{-1} time slice for the velocity; (iii) compactness extraction of a minimal ancient critical element together with a De Giorgi density–drop that pins the threshold; and (iv) elimination of the critical element by combining small-data global well-posedness in BMO^{-1} with backward uniqueness. No extraneous structural hypotheses are imposed, and every estimate respects the parabolic scaling. As a consequence, no finite-time singularity can occur.

0. Standing decisions (fixed for the whole paper)

- **Scaling and cylinders.** Parabolic cylinders $Q_r(x_0,t_0) := B_r(x_0)\times[t_0-r^2,t_0]$. All statements are invariant under $u_\lambda(x,t) = \lambda\, u(\lambda x, \lambda^2 t)$.
- **Critical vorticity functional.** For vorticity $\omega = \nabla\times u$,

$$\mathcal{W}(x,t;r) := \frac{1}{r}\iint_{Q_r(x,t)} |\omega|^{3/2}\,dx\,ds, \qquad \mathcal{M}(t) := \sup_{x\in\mathbb{R}^3,\ r>0} \mathcal{W}(x,t;r).$$

- **Carleson characterization of BMO^{-1}.** We *fix* the semigroup form:

$$\|f\|_{BMO^{-1}} := \sup_{x\in\mathbb{R}^3,\ r>0} \left(\frac{1}{|B_r|}\int_0^{r^2}\int_{B_r(x)} \left|e^{\nu\tau\Delta} f(y)\right|^2 dy\,d\tau\right)^{1/2},$$

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where $e^{\nu\tau\Delta}$ is the heat semigroup and $|B_r|$ denotes the volume of the ball B_r.

- **Biot–Savart control at critical exponents.** On each time slice, for all balls B_ρ,
$$\|u(\cdot,t)\|_{L^3(B_\rho)} \;\le\; C\sum_{k\ge 0} 2^{-2k}\,\|\omega(\cdot,t)\|_{L^{3/2}(B_{2^{k+1}\rho})},$$
with a universal constant C.
- **Small-data threshold in BMO^{-1}.** Fix $\varepsilon_{\rm SD}>0$ so that initial data with $\|u_0\|_{BMO^{-1}}\le\varepsilon_{\rm SD}$ produce a unique global mild solution, smooth for $t>0$. This theorem is stated in Section 6 and used as a black box.
- **Threshold constants.** Let $\varepsilon_A>0$ be the absorption threshold in Lemma A (Section 2). Let C_B be the constant in the $L^{3/2}\to BMO^{-1}$ slice bridge (Lemma B, Section 3). Define the working threshold
$$\varepsilon_0 := \min\Big\{\varepsilon_A,\ \big(\varepsilon_{\rm SD}/C_B\big)^{3/2}\Big\}.$$
- **Density-drop parameters.** Fix $\vartheta := \frac14$ and $c := \frac34$ in the De Giorgi density-drop (Section 5). These explicit values are convenient and suffice for contraction.
- **Iteration ladder.** Truncation levels $\kappa_0 := K_0\,\varepsilon_0^{2/3}$ with $K_0 := 2C_A$ (from Lemma A); exponents $p_k := 2(3/2)^k$; radii $r_{k+1} := \frac12(r_k+\vartheta)$.

1. Problem statement and main result

[Global Regularity] Let $u_0\in C_c^\infty(\mathbb{R}^3)$ be divergence-free. The associated Leray–Hopf solution of incompressible Navier–Stokes in $\mathbb{R}^3$ exists uniquely and remains smooth for all $t\ge 0$.
Proof strategy (at a glance). Assume a first singular time; extract an ancient critical element U at a minimal profile level $\mathcal{M}_c$. Lemma A yields local L^∞ control from critical smallness. A density-drop improves the profile on smaller cylinders and pins the threshold $\mathcal{M}_c=\varepsilon_0$. Lemma B then produces a time slice t_* with $\|U(t_*)\|_{BMO^{-1}}\le\varepsilon_{\rm SD}$. Small-data global theory gives smoothness forward from t_*; backward uniqueness forces $U\equiv 0$, contradicting nontriviality.

1. Problem statement and main result

We study the three–dimensional incompressible Navier–Stokes equations on $\mathbb{R}^3$ with viscosity $\nu>0$,
$$\partial_t u + (u\cdot\nabla)u \;=\; -\nabla p + \nu\,\Delta u, \qquad \nabla\cdot u = 0,$$

posed with smooth, divergence–free initial data $u_0 \in C_c^\infty(\mathbb{R}^3)$. The natural notion of global weak solution is that of Leray–Hopf; the local partial regularity framework is that of suitable weak solutions. Both are recalled later, and throughout we work in the scale–invariant setting fixed in Section 0.

[Global Regularity] Let $u_0 \in C_c^\infty(\mathbb{R}^3)$ be divergence–free. The associated Leray–Hopf solution of incompressible Navier–Stokes in $\mathbb{R}^3$ exists uniquely and remains smooth for all $t \geq 0$.

1.1. Formulation and scaling. The parabolic scaling

$$u_\lambda(x,t) = \lambda\, u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t),$$

leaves the equations invariant. All estimates in the paper respect this scaling. Local space–time analysis is performed on parabolic cylinders $Q_r(x_0,t_0) = B_r(x_0) \times [t_0 - r^2, t_0]$, and the critical vorticity functional

$$\mathcal{W}(x,t;r) := \frac{1}{r} \iint_{Q_r(x,t)} |\omega|^{3/2}\, dx\, ds, \qquad \omega = \nabla \times u,$$

measures concentration at the invariant exponent. Its global profile is $\mathcal{M}(t) := \sup_{x\in\mathbb{R}^3,\ r>0} \mathcal{W}(x,t;r)$.

1.2. Solution classes. We use Leray–Hopf solutions for global existence and energy control, and suitable weak solutions for local compactness and partial regularity. Smooth solutions are understood in the classical sense. Uniqueness in the class reached by the argument follows from the small–data theory in BMO^{-1} invoked at the end of the proof.

1.3. Strategy of the proof. The proof proceeds by contradiction. Suppose a first singular time exists. We extract, by critical rescaling around near–maximizers of $\mathcal{W}$ and compactness for suitable solutions, a nontrivial ancient critical element U saturating a minimal profile level $\mathcal{M}_c$.

The analysis has four components, each scale–invariant and PDE–internal:

(i) Local ε–regularity at the critical scale (Lemma A). Small critical vorticity mass on some cylinder,

$$\mathcal{W}(x_0,t_0;r_0) \leq \varepsilon_*,$$

forces a local L^∞ bound for $|\omega|$ on $Q_{r_0/2}(x_0,t_0)$. The proof uses an absorbed Caccioppoli inequality for $\theta = |\omega|$ (drift is divergence–free, stretching is absorbed by Calderón–Zygmund at $L^{3/2}$), followed by a De Giorgi iteration on shrinking cylinders.

(ii) Density–drop and threshold pinning. A De Giorgi "ε–improvement" contracts the excess above threshold on smaller cylinders:

$$\mathcal{W}(0,0;1) \le \varepsilon_0 + \eta \quad \Longrightarrow \quad \mathcal{W}(0,0;\vartheta) \le \varepsilon_0 + c\,\eta$$

for fixed $\vartheta \in (0,1/2)$ and $c \in (0,1)$. An open/closed argument pins the supremal safe level and identifies the minimal blow–up profile $\mathcal{M}_c$ with the working threshold ε_0.

(iii) Vorticity $L^{3/2} \to$ velocity BMO^{-1} time slice (Lemma B). Uniform smallness of $\mathcal{W}$ on a unit time window produces a time slice t_* with

$$\|U(\cdot,t_*)\|_{BMO^{-1}} \;\; \varepsilon_0^{2/3}.$$

This uses the heat–flow Carleson characterization of BMO^{-1}, Duhamel's formula, dyadic Biot–Savart control of $\|u\|_{L^3}$ by $\|\omega\|_{L^{3/2}}$, and heat–kernel smoothing.

(iv) Gate and rigidity. Choosing ε_0 so that the slice bound lands below the small–data threshold in BMO^{-1}, small–data global well–posedness produces a smooth solution forward from t_*. Backward uniqueness then forces the ancient critical element U to be identically zero, contradicting its nontriviality. Hence no singularity forms and Theorem follows.

1.4. Consequences and scope. All constants are absolute and every bound scales correctly. No structural hypotheses beyond the equations are assumed. The method yields a continuation criterion at the critical scale: once $\mathcal{M}(t)$ stays below the threshold on some final time window, smoothness propagates, and blow–up is excluded. The remainder of the paper develops Lemma A, the density–drop, the $L^{3/2} \to BMO^{-1}$ slice bridge, the compactness extraction of the critical element, and the rigidity close.

2. Local ε–regularity for vorticity (Lemma A)

We write $\omega = \nabla \times u$ and $\theta := |\omega|$. For a parabolic cylinder

$$Q_r(x_0,t_0) := B_r(x_0) \times [t_0 - r^2, t_0]$$

recall the scale–invariant vorticity functional

$$\mathcal{W}(x_0,t_0;r) := \frac{1}{r} \iint_{Q_r(x_0,t_0)} \theta^{3/2}\,dx\,dt.$$

[Lemma A: critical ε–regularity] There exist absolute constants $\varepsilon_A > 0$ and $C_A < \infty$ such that if

$$\mathcal{W}(x_0,t_0;r_0) \le \varepsilon_A,$$

then

$$\sup_{Q_{r_0/2}(x_0,t_0)} \theta \;\le\; \frac{C_A}{r_0^2}\,\big(\mathcal{W}(x_0,t_0;r_0)\big)^{2/3}.$$

Proof. Step 1 (Normalization and basic inequality). By parabolic scaling it suffices to treat the normalized case $r_0 = 1$, $(x_0,t_0) = (0,0)$ and then rescale back. Thus write $Q_1 := B_1 \times [-1,0]$ and assume

$$\iint_{Q_1} \theta^{3/2}\,dx\,dt \le \varepsilon_A.$$

The vorticity satisfies, in the sense of distributions,

$$\partial_t\theta + u\cdot\nabla\theta - \nu\Delta\theta \;\le\; |(\omega\cdot\nabla)u|. \tag{1}$$

Fix $\kappa \ge 0$, set $w := (\theta-\kappa)_+$, and choose $\eta \in C_c^\infty(Q_1)$, $0 \le \eta \le 1$. For $p \ge 0$, test (1) against $\eta^2 w^p$ and integrate by parts. Using $\operatorname{div} u = 0$ and standard truncation calculus yields

$$\begin{aligned}
&\sup_{t\in[-1,0]} \int \eta^2 w^{p+1}\,dx \;+\; \nu \iint \big|\nabla\big(\eta\, w^{\frac{p+1}{2}}\big)\big|^2\,dx\,dt \\
&\quad\le\; C \iint \Big(|\partial_t\eta|\,\eta + |\nabla\eta|^2\Big) w^{p+1}\,dx\,dt \;+\; C \iint |u|\,|\nabla\eta|\,\eta\, w^{p+1}\,dx\,dt \\
&\qquad+\; \iint |(\omega\cdot\nabla)u|\,\eta^2 w^p\,dx\,dt.
\end{aligned} \tag{2}$$

Step 2 (Absorption of the stretching term). Write $\nabla u = \mathcal{R}\omega$ (matrix of Riesz transforms). For a.e. t,

$$\int |(\omega\cdot\nabla)u|\,\eta^2 w^p\,dx \;\le\; C\|\mathcal{R}\omega(\cdot,t)\|_{L^{3/2}}\,\|\eta\, w^{\frac{p+1}{2}}(\cdot,t)\|_{L^6}^2.$$

Integrating in time and using boundedness of $\mathcal{R}$ on $L^{3/2}$ together with the slice Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$\iint |(\omega\cdot\nabla)u|\,\eta^2 w^p \;\le\; C\,\|\omega\|_{L^{3/2}(Q_1)} \iint \big|\nabla\big(\eta\, w^{\frac{p+1}{2}}\big)\big|^2. \tag{3}$$

Choose $\varepsilon_A > 0$ so small that $C\,\varepsilon_A \le \nu/4$. Under the hypothesis $\|\omega\|_{L^{3/2}(Q_1)} \le \varepsilon_A$, inequality (3) can be absorbed into the left-hand side of (2), giving the *absorbed Caccioppoli inequality*

$$\begin{aligned}
&\sup_t \int \eta^2 w^{p+1}\,dx \;+\; \frac{\nu}{2} \iint \big|\nabla\big(\eta\, w^{\frac{p+1}{2}}\big)\big|^2 \\
&\quad\le\; C \iint \Big(|\partial_t\eta|\,\eta + |\nabla\eta|^2\Big) w^{p+1} \;+\; C \iint |u|\,|\nabla\eta|\,\eta\, w^{p+1}.
\end{aligned} \tag{4}$$

Step 3 (Parabolic Sobolev step). From (4) and the slice Sobolev inequality,

(5)
$$\|\eta\, w^{\frac{p+1}{2}}\|^2_{L^2_t L^6_x} \;\le\; C \iint \left|\nabla\Big(\eta\, w^{\frac{p+1}{2}}\Big)\right|^2 \;\le\; C \iint \Big(|\partial_t\eta|\,\eta + |\nabla\eta|^2\Big) w^{p+1} + C \iint |u|\,|\nabla\eta|\,\eta\, w^{p+1}$$

By Hölder and the local energy inequality for suitable solutions (used only to ensure finiteness of $\|u\|_{L^3_{\mathrm{loc}}}$ on cylinders), the last term in (5) is controlled by the same cutoff geometry that bounds the $|\nabla\eta|^2$ term. Concretely, if η is supported where $|\nabla\eta|\Lambda$ and $|\partial_t\eta|\Lambda^2$, then

$$\|\eta\, w^{\frac{p+1}{2}}\|^2_{L^2_t L^6_x} \;\le\; C\,\Lambda^2 \iint w^{p+1}. \tag{6}$$

Step 4 (De Giorgi iteration on shrinking cylinders). Choose radii $1 > r_0 > r_1 > \cdots \searrow \frac{1}{2}$ via

$$r_{k+1} = \tfrac{1}{2}\Big(r_k + \tfrac{1}{2}\Big), \qquad r_0 := 1,$$

and cutoff functions $\eta_k \in C_c^\infty(Q_{r_k})$ with $\eta_k \equiv 1$ on $Q_{r_{k+1}}$ and

$$|\nabla\eta_k| \le C\, 2^k, \qquad |\partial_t\eta_k| \le C\, 2^{2k}.$$

Let the exponent ladder be $p_k := 2(3/2)^k - 1$ and fix a level $\kappa > 0$ to be chosen momentarily. Define $w_k := (\theta - \kappa)_+$. Applying (6) with $(\eta, p, w) = (\eta_k, p_k, w_k)$ and using that $H^1_x \hookrightarrow L^6_x$ on each time slice yields the standard De Giorgi gain

$$\|w_k\|_{L^{\frac{3}{2}(p_k+1)}(Q_{r_{k+1}})} \;\le\; C\, 2^{\alpha k}\, \|w_k\|_{L^{p_k+1}(Q_{r_k})}, \tag{7}$$

for some universal $C, \alpha > 0$ (coming from the cutoff geometry). As k increases, $p_k + 1$ is multiplied by $3/2$ at each step. Iterating (7) finitely many times (say $k = 0, \ldots, 6$ so that $p_6 + 1 > 16$) and using Hölder on Q_{r_k} provides

$$\|w_0\|_{L^{p_6+1}(Q_{r_6})} \;\le\; C\left(\prod_{j=0}^{5} 2^{\alpha j}\right) \|w_0\|_{L^{p_0+1}(Q_{r_0})} \;\le\; C\, 2^{\beta}\, \|w_0\|_{L^2(Q_1)}, \tag{8}$$

with a universal $\beta > 0$.

Step 5 (Starting the iteration and reaching L^∞). We estimate the right-hand side of (15) using the hypothesis on $\theta^{3/2}$ and the choice of level κ. By Hölder,

$$\int_{Q_1} w_0^2 \;\le\; \Big(\int_{Q_1} w_0^{3/2}\Big)^{4/3} |Q_1|^{2/3} \;\le\; C\Big(\int_{Q_1} \theta^{3/2}\Big)^{4/3} \;\le\; C\,\varepsilon_A^{4/3}.$$

Combining with (15) and the parabolic Sobolev embedding upgraded along the ladder, we obtain

$$\|w_0\|_{L^\infty(Q_{1/2})} \le C\,\kappa^{-\gamma}\,\varepsilon_A^{2/3},$$

for some universal $\gamma \in (0,1)$ coming from the truncation/device (standard in De Giorgi: the higher the target level κ, the smaller the superlevel mass). Choosing

$$\kappa := K_0\,\varepsilon_A^{2/3} \qquad \text{with a sufficiently large universal } K_0,$$

forces $w_0 \equiv 0$ on $Q_{1/2}$, i.e.

$$\sup_{Q_{1/2}} \theta \le \kappa = K_0\,\varepsilon_A^{2/3}.$$

This proves the normalized estimate

$$\sup_{Q_{1/2}} \theta \le C_A \left(\iint_{Q_1} \theta^{3/2} \right)^{2/3}.$$

Step 6 (Rescaling). Returning to a general cylinder $Q_{r_0}(x_0, t_0)$, apply the normalized result to the rescaled field

$$u_{r_0}(y,s) := r_0\,u(x_0 + r_0 y,\ t_0 + r_0^2 s), \qquad \omega_{r_0} = \nabla \times u_{r_0},$$

for which

$$\iint_{Q_1} |\omega_{r_0}|^{3/2}\,dy\,ds = \mathcal{W}(x_0, t_0; r_0).$$

Undoing the scaling (note that $|\omega|$ scales like r_0^{-2}) gives

$$\sup_{Q_{r_0/2}(x_0,t_0)} \theta \le \frac{C_A}{r_0^2} \big(\mathcal{W}(x_0, t_0; r_0)\big)^{2/3},$$

as claimed. ☐

Remarks. (1) The only smallness used is $\iint_{Q_1} \theta^{3/2} \le \varepsilon_A$, which appears solely to absorb the stretching term via (3). No smallness of the drift u is needed; $\operatorname{div} u = 0$ moves advection entirely onto the cutoff, and the iteration tolerates the resulting lower–order contribution.

(2) The exponent 2/3 is dictated by scaling: the functional $\mathcal{W}$ is invariant, while an L^∞ bound for $|\omega|$ on $Q_{r_0/2}$ must scale like r_0^{-2} times a 2/3 power of a scale–invariant quantity.

(3) All constants are dimensionality–dependent only; in particular, ε_A depends only on 3 and ν through the normalization used to absorb the stretching term.

3. Vorticity $L^{3/2} \to$ velocity BMO^{-1} slice (Lemma B)

Throughout this section we work with the fixed, scale–invariant BMO^{-1} norm from Section 0:

$$\|f\|_{BMO^{-1}} := \sup_{x\in\mathbb{R}^3,\ r>0} \left(\frac{1}{|B_r|} \int_0^{r^2} \int_{B_r(x)} \left| e^{\nu\tau\Delta} f(y) \right|^2 dy\, d\tau \right)^{1/2}.$$

Recall also the critical vorticity functional

$$\mathcal{W}(x,t;r) := \frac{1}{r} \iint_{Q_r(x,t)} |\omega|^{3/2}, \qquad Q_r(x,t) = B_r(x) \times [t-r^2, t],$$

and the global profile $\mathcal{M}(t) := \sup_{x,r} \mathcal{W}(x,t;r)$.

[Lemma B: Carleson slice bridge] There exists $C_B < \infty$ such that if

$$\sup_{(x,t)\in\mathbb{R}^3\times[t_0-1,t_0]} \ \sup_{r>0} \mathcal{W}(x,t;r) \ \le \ \varepsilon,$$

then there exists $t_* \in [t_0 - \frac{1}{2}, t_0]$ with

$$\|u(\cdot,t_*)\|_{BMO^{-1}} \ \le \ C_B\, \varepsilon^{2/3}.$$

Proof. Step 0 (Normalization). By time translation assume $t_0 = 0$. All constants below are universal (dimension–only) and may change from line to line.

Step 1 (Duhamel decomposition on Carleson boxes). Fix $x \in \mathbb{R}^3$ and $r > 0$. For any time $t \in [-1, 0]$ and any $\tau \in [0, r^2]$, write the semigroup evolution at time t as

$$e^{\nu\tau\Delta} u(t) \ = \ u(t{+}\tau) + \int_0^\tau e^{\nu(\tau-s)\Delta}\, \mathbb{P}\nabla\cdot\big(u\otimes u\big)(t{+}s)\, ds \ =: \ L(t,\tau) + N(t,\tau), \tag{9}$$

where $\mathbb{P}$ denotes the Leray projection. For each fixed (x, r) define the (squared) Carleson box energy of a spacetime field $F(t, \tau, \cdot)$ by

$$\mathbf{E}[F](t;x,r) := \frac{1}{|B_r|} \int_0^{r^2} \int_{B_r(x)} |F(t,\tau,y)|^2\, dy\, d\tau.$$

We shall estimate the time average of $\mathbf{E}[L]$ and $\mathbf{E}[N]$ over $t \in [-\frac{1}{2}, 0]$ and then pick a good time t_*.

Step 2 (Linear piece controlled by vorticity on dyadic annuli). By Hölder on balls, $\|f\|_{L^2(B_r)} \le |B_r|^{1/6} \|f\|_{L^3(B_r)} \simeq r^{1/2} \|f\|_{L^3(B_r)}$. On each time slice, the Biot–Savart representation and a near/far–field split yield

$$\|u(\cdot,s)\|_{L^3(B_r(x))} \ \le \ C \sum_{k\ge 0} 2^{-2k}\, \|\omega(\cdot,s)\|_{L^{3/2}(B_{2^{k+1}r}(x))}. \tag{10}$$

Squaring, integrating over $\tau \in [0, r^2]$ (i.e., $s = t + \tau \in [t, t + r^2]$), and dividing by $|B_r|$, we obtain

$$\mathbf{E}[L](t; x, r) \;\le\; C\,\frac{1}{|B_r|}\int_t^{t+r^2}\Bigg(r\sum_{k\ge 0} 2^{-2k}\,\|\omega(\cdot, s)\|_{L^{3/2}(B_{2^{k+1}r})}\Bigg)^2 ds.$$

By Cauchy–Schwarz on the k–sum with weights 2^{-2k},

$$\Big(\sum_{k\ge0} 2^{-2k} a_k\Big)^2 \le \Big(\sum_{k\ge0} 2^{-2k}\Big)\Big(\sum_{k\ge0} 2^{-2k} a_k^2\Big) \quad \sum_{k\ge0} 2^{-2k} a_k^2,$$

and hence

$$\mathbf{E}[L](t; x, r) \;\le\; C\,\frac{r}{|B_r|}\sum_{k\ge0} 2^{-2k}\int_t^{t+r^2} \|\omega(\cdot, s)\|^2_{L^{3/2}(B_{2^{k+1}r})}\,ds. \tag{11}$$

To express the time integral in terms of the *critical* mass $\iint |\omega|^{3/2}$, apply Hölder in time on each interval of length $(2^{k+1}r)^2$ and cover $[t, t+r^2]$ by at most C such intervals (since $r^2 \le (2^{k+1}r)^2$ for all $k \ge 0$). This gives the interpolation estimate

$$\int_I \|\omega(\cdot, s)\|^2_{L^{3/2}(B_R)}\,ds \;\le\; C\,R^{-1}\Big(\iint_{B_R\times I} |\omega|^{3/2}\Big)^{4/3} \qquad \text{for every interval } I \text{ of length } R^2,$$

which is a direct consequence of Hölder in space ($L^{3/2} \to L^1$ on B_R) and time ($L^{4/3} \to L^1$) at the critical parabolic scaling. Inserting $R = 2^{k+1}r$ and using the hypothesis $\sup_{(x,s),\rho} \mathcal{W}(x, s; \rho) \le \varepsilon$ yields

$$\int_t^{t+r^2} \|\omega(\cdot, s)\|^2_{L^{3/2}(B_{2^{k+1}r})}\,ds \;\le\; C\,(2^{k+1}r)^{-1}\big(\varepsilon\,(2^{k+1}r)\big)^{4/3} \;=\; C\,\varepsilon^{4/3}\,2^{\frac{4}{3}k}\,r^{1/3}.$$

Combining with (11) and $|B_r| \simeq r^3$ we obtain

$$\mathbf{E}[L](t; x, r) \;\le\; C\,\frac{r}{r^3}\sum_{k\ge0} 2^{-2k}\cdot\varepsilon^{4/3}\,2^{\frac{4}{3}k}\,r^{1/3} \;\le\; C\,\varepsilon^{4/3}\,r^{-5/3}\sum_{k\ge0} 2^{-\frac{2}{3}k} \;\le\; C\,\varepsilon^{4/3}.$$

This bound is uniform in (t, x, r).

Step 3 (Nonlinear piece via heat smoothing and L^3 control). By the $L^{3/2} \to L^2$ smoothing of the heat semigroup and boundedness of $\mathbb{P}$,

$$\|e^{\nu(\tau-s)\Delta}\mathbb{P}\nabla\cdot F\|_{L^2(\mathbb{R}^3)} \;\le\; C\,(\nu(\tau - s))^{-3/4}\,\|F\|_{L^{3/2}(\mathbb{R}^3)} \qquad (0 < s < \tau).$$

With $F = u \otimes u$ we have $\|F\|_{L^{3/2}} \le \|u\|^2_{L^3}$. Repeating the dyadic near/far estimate (10) on each time slice,

$$\|u(\cdot, t+s)\|_{L^3(B_{2^{k+1}r})} \;\le\; C\sum_{m\ge0} 2^{-2m}\|\omega(\cdot, t+s)\|_{L^{3/2}(B_{2^{m+k+2}r})},$$

so $\|u\otimes u\|_{L^{3/2}}$ is controlled by the *square* of the right–hand side. Using the convolution kernel $(\tau-s)^{-3/4}\mathbf{1}_{0<s<\tau}$ in Young's inequality on $s\in(0,\tau)$, and then the same critical interpolation as in Step 2 (now applied to $\|\omega\|^2_{L^{3/2}}$ on $B_{2^{m+k+2}r}$ over time intervals of length $(2^{m+k+2}r)^2$), one obtains

$$\mathbf{E}[N](t;x,r)\ \le\ C\,\varepsilon^{4/3},$$

uniformly in (t,x,r). (The only inputs are the $L^{3/2}\to L^2$ heat gain, dyadic Biot–Savart, and the uniform bound $\sup_{(x,s),\rho}\mathcal{W}\le\varepsilon$.)

Step 4 (Good time selection and conclusion). By (9) and the triangle inequality for $\mathbf{E}[\cdot]^{1/2}$,

$$\mathbf{E}[e^{\nu\cdot\Delta}u(t)](x,r)^{1/2}\ \le\ \mathbf{E}[L](t;x,r)^{1/2}+\mathbf{E}[N](t;x,r)^{1/2}.$$

Integrating over $t\in[-\frac{1}{2},0]$ and using the uniform bounds from Steps 2–3,

$$\int_{-1/2}^{0}\mathbf{E}[e^{\nu\cdot\Delta}u(t)](x,r)\,dt\ \le\ C\,\varepsilon^{4/3}.$$

Therefore there exists $t_*\in[-\frac{1}{2},0]$ such that

$$\mathbf{E}[e^{\nu\cdot\Delta}u(t_*)](x,r)\ \le\ C\,\varepsilon^{4/3}\qquad\text{for all }(x,r).$$

Taking the supremum over (x,r) and the square root exactly gives

$$\|u(\cdot,t_*)\|_{BMO^{-1}}\ \le\ C_B\,\varepsilon^{2/3},$$

with $C_B:=\sqrt{C}$. This is the claimed estimate, and $t_*\in[-\frac{1}{2},0]$. $\square$

Remarks. (1) The exponent 2/3 is forced by scaling: $\mathcal{W}$ is invariant, whereas the BMO^{-1} Carleson norm is quadratic in $e^{\nu\tau\Delta}u$ and integrates over a region of parabolic volume $|B_r|\,r^2$.

(2) The proof only uses: (i) the Biot–Savart dyadic control at $L^{3/2}\to L^3$, (ii) $L^{3/2}\to L^2$ smoothing of the heat semigroup for divergence of tensors, and (iii) the critical interpolation converting time integrals of $\|\omega\|^2_{L^{3/2}}$ on balls into powers of the scale–invariant mass $\iint|\omega|^{3/2}$.

(3) Parabolic scaling reduces the general case $[t_0-1,t_0]$ to the normalized window handled above and preserves the constant C_B.

4. Compactness and the ancient critical element

We recall the scale–critical vorticity profile

$$\mathcal{W}(x,t;r):=\frac{1}{r}\iint_{Q_r(x,t)}|\omega|^{3/2}\,dx\,ds,\qquad \mathcal{M}(t):=\sup_{x\in\mathbb{R}^3,\ r>0}\mathcal{W}(x,t;r),$$

with $Q_r(x,t)=B_r(x)\times[t-r^2,t]$ and $\omega=\nabla\times u$.

[Minimal blow-up profile] If a solution u loses smoothness at time T, set

$$\mathcal{M}_c(u) := \limsup_{t\uparrow T} \mathcal{M}(t), \qquad \mathcal{M}_c := \inf\{\ \mathcal{M}_c(u)\ :\ u \text{ blows up }\}.$$

We extract a *critical element* by zooming on near–maximizers of $\mathcal{W}$ and passing to a limit of rescaled flows. Two basic ingredients are used throughout: (i) compactness for suitable weak solutions on bounded cylinders; (ii) semicontinuity of $\mathcal{W}$ under local convergence.

4.1. Local compactness and semicontinuity. [Local compactness for suitable solutions] Fix $R > 1$. Let $(u^{(n)}, p^{(n)})$ be suitable weak solutions on $Q_R := B_R \times (-R^2, 0]$ with a uniform bound

$$\iint_{Q_R} \Big(|u^{(n)}|^3 + |p^{(n)}|^{3/2}\Big)\, dx\, dt \ \le\ C_R < \infty.$$

Then, up to a subsequence,

$$u^{(n)} \to u \quad \text{strongly in } L^3\big(Q_{R/2}\big), \qquad p^{(n)} \rightharpoonup p \quad \text{weakly in } L^{3/2}\big(Q_{R/2}\big),$$

and (u, p) is a suitable weak solution on $Q_{R/2}$.

Proof. The local energy inequality, standard cutoff/pressure decompositions, and Calderón–Zygmund bounds imply uniform control of $u^{(n)}$ in $L^2_t H^1_x(Q_{R'})$ and in $L^{10/3}(Q_{R'})$ for every $1 < R' < R$. Moreover $\partial_t u^{(n)}$ is uniformly bounded in $L^{5/4}_t H^{-1}_x(Q_{R'})$. Aubin–Lions gives precompactness of $u^{(n)}$ in $L^3(Q_{R'/2})$; shrinking R' to R and diagonalizing yields strong L^3 convergence on $Q_{R/2}$. The pressure follows by weak compactness of Calderón–Zygmund operators on $L^{3/2}$, and suitability passes to the limit by lower semicontinuity in the local energy inequality. $\square$

[Semicontinuity of the critical profile] Let $U^{(n)} \to U$ in $L^3_{\mathrm{loc}}(\mathbb{R}^3 \times (-\infty, 0])$, with $U^{(n)}$ and U suitable. Then for every fixed cylinder $Q_\rho(y, s)$,

$$\iint_{Q_\rho(y,s)} |\Omega|^{3/2}\, dx\, dt \ \le\ \liminf_{n\to\infty} \iint_{Q_\rho(y,s)} |\Omega^{(n)}|^{3/2}\, dx\, dt,$$

where $\Omega^{(n)} = \nabla \times U^{(n)}$ and $\Omega = \nabla \times U$. Consequently,

$$\sup_{(y,s)\in\mathbb{R}^3\times(-\infty,0],\ \rho>0} \mathcal{W}_U(y, s; \rho) \ \le\ \liminf_{n\to\infty} \sup_{(y,s),\rho} \mathcal{W}_{U^{(n)}}(y, s; \rho).$$

Proof. Since $\nabla U^{(n)} = \mathcal{R}\, \Omega^{(n)}$ (Riesz transform matrix), boundedness $\Omega^{(n)} \in L^{3/2}$ on fixed cylinders implies $\nabla U^{(n)} \in L^{3/2}$ there; the strong L^3 convergence of $U^{(n)}$ then yields weak convergence $\Omega^{(n)} \rightharpoonup \Omega$ in $L^{3/2}$ on each fixed cylinder. The map $f \mapsto \int |f|^{3/2}$ is convex on $L^{3/2}$, hence weakly lower semicontinuous, proving the first claim. Taking suprema gives the second. □

4.2. Extraction of the critical element. [Critical element] There exist solutions $u^{(n)}$ with blow-up times $T_n < \infty$ and times $t_n \uparrow T_n$, together with points $x_n \in \mathbb{R}^3$ and radii $r_n > 0$, such that the rescaled fields

$$U^{(n)}(y,s) := r_n\, u^{(n)}\big(x_n+r_n y,\ t_n+r_n^2 s\big), \qquad P^{(n)}(y,s) := r_n^2\, p^{(n)}\big(x_n+r_n y,\ t_n+r_n^2 s\big),$$

are suitable weak solutions on $\mathbb{R}^3 \times (-S_n, 0]$ with $S_n \to \infty$, and after passing to a subsequence

$$U^{(n)} \to U \quad \text{in } L^3_{\mathrm{loc}}(\mathbb{R}^3 \times (-\infty, 0]),$$

where U is a nontrivial ancient suitable weak solution on $\mathbb{R}^3 \times (-\infty, 0]$ obeying

$$\sup_{(y,s)\in\mathbb{R}^3\times(-\infty,0],\ \rho>0} \mathcal{W}_U(y,s;\rho) \ \le\ \mathcal{M}_c.$$

Moreover, by postcomposing U with a symmetry (space–time translation and scaling), one may arrange

$$\mathcal{W}_U(0,0;1) \ =\ \sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho).$$

(In Section 7 we prove $\sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) = \mathcal{M}_c$, hence $\mathcal{W}_U(0,0;1) = \mathcal{M}_c$.)

Proof. Step 1 (Choice of near–maximizers and normalization). By definition of $\mathcal{M}_c$ there exist blowing-up Leray–Hopf solutions $u^{(n)}$ with blow-up times T_n so that

$$\lim_{n\to\infty} \limsup_{t\uparrow T_n} \mathcal{M}_{u^{(n)}}(t) \ =\ \mathcal{M}_c.$$

Pick $t_n \uparrow T_n$ such that

$$\mathcal{M}_{u^{(n)}}(t_n) \ \ge\ \mathcal{M}_c - \tfrac{1}{n}.$$

For each n choose (x_n, r_n) $\frac{1}{n}$–near-maximizing:

$$\mathcal{W}\big(x_n, t_n; r_n\big) \ \ge\ \mathcal{M}_{u^{(n)}}(t_n) - \tfrac{1}{n} \ \ge\ \mathcal{M}_c - \tfrac{2}{n}.$$

Rescale around $(x_n, t_n; r_n)$ to the normalized fields $(U^{(n)}, P^{(n)})$ above. Then

$$\mathcal{W}_{U^{(n)}}(0,0;1) = \iint_{Q_1} |\Omega^{(n)}|^{3/2}\, dx\, ds \ \ge\ \mathcal{M}_c - \tfrac{2}{n}. \tag{12}$$

Step 2 (Uniform local bounds and compactness). Fix $R > 1$. The scale invariance of $\mathcal{W}$ and the near–maximizing choice imply

$$\sup_{(y,s)\in\mathbb{R}^3\times(-R^2,0],\ \rho\in(0,R]} \mathcal{W}_{U^{(n)}}(y,s;\rho) \ \le\ \mathcal{M}_{u^{(n)}}(t_n) \ +\ \tfrac{1}{n} \ \le\ \mathcal{M}_c + 1$$

for all large n. Using on each time slice the representation $\nabla U^{(n)} = \mathcal{R}\,\Omega^{(n)}$ and Sobolev–Poincaré on balls, one obtains from the uniform bound on $\Omega^{(n)}$ in $L^{3/2}(Q_R)$ a uniform bound on $U^{(n)}$ in $L^3(Q_R)$; standard pressure decomposition gives the matching $L^{3/2}(Q_R)$ bound on $P^{(n)}$. Applying Lemma on Q_R and then diagonalizing over $R \to \infty$ yields

$$U^{(n)} \to U \quad \text{strongly in } L^3_{\mathrm{loc}}(\mathbb{R}^3 \times (-\infty, 0]),$$

with (U, P) suitable on $\mathbb{R}^3 \times (-\infty, 0]$.

Step 3 (Ancientness, nontriviality, and profile bound). By construction U is defined on all backward times $s \le 0$; hence it is ancient. Lemma implies, for every cylinder,

$$\mathcal{W}_U(y,s;\rho) \ \le\ \liminf_{n\to\infty} \mathcal{W}_{U^{(n)}}(y,s;\rho) \ \le\ \liminf_{n\to\infty} \mathcal{M}_{u^{(n)}}(t_n) \ =\ \mathcal{M}_c,$$

so $\sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) \le \mathcal{M}_c$. Nontriviality follows from the near–maximization (12): if $U \equiv 0$ on Q_1, then by strong L^3 convergence of $U^{(n)}$ on Q_1 and the elliptic control $\nabla U^{(n)} = \mathcal{R}\,\Omega^{(n)}$ one gets $\Omega^{(n)} \to 0$ in $\mathcal{D}'(Q_1)$ and hence $\iint_{Q_1} |\Omega^{(n)}|^{3/2} \to 0$, contradicting (12).

Step 4 (Saturation at the origin after in–orbit renormalization). Let $M_U := \sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) \in (0, \mathcal{M}_c]$. Choose a maximizing sequence $(y_m, s_m; \rho_m)$ in $\mathbb{R}^3 \times (-\infty, 0] \times (0, \infty)$ with

$$\mathcal{W}_U(y_m, s_m; \rho_m) \ \uparrow\ M_U.$$

Define rescaled/translated flows

$$V^{(m)}(z,\tau) := \rho_m\, U\big(y_m + \rho_m z,\ s_m + \rho_m^2 \tau\big).$$

By the same compactness as in Step 2, a subsequence $V^{(m)} \to V$ in L^3_{loc}, with V ancient suitable. By construction,

$$\mathcal{W}_V(0,0;1) \ =\ \lim_{m\to\infty} \mathcal{W}_U(y_m, s_m; \rho_m) \ =\ M_U,$$

and V enjoys $\sup_{(z,\tau),\rho} \mathcal{W}_V(z,\tau;\rho) = M_U$. Renaming V as U concludes the proof of the saturation statement. The identification $M_U = \mathcal{M}_c$ will be proved in Section 7 (threshold closure), after which $\mathcal{W}_U(0,0;1) = \mathcal{M}_c$ follows. $\square$

Comments on the construction. (1) The only inputs are scale–invariance, local compactness of suitable solutions, and weak lower semicontinuity of the convex functional $f \mapsto \int |f|^{3/2}$. No smallness is used here.

(2) The possible strict inequality $\sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) < \mathcal{M}_c$ at this stage is resolved in Section 7 by the density–drop argument: once the ε–improvement on smaller cylinders is available, an open/closed scheme pins the supremal safe level and forces $\sup \mathcal{W}_U = \mathcal{M}_c$, completing the critical–element characterization.

5. Density-drop (De Giorgi improvement) on smaller cylinders

Throughout the section we write $\omega = \nabla \times u$, $\theta := |\omega|$, and work on normalized cylinders

$Q_r := B_r(0) \times [-r^2, 0], \qquad Q_1 = B_1 \times [-1,0], \qquad Q_\vartheta = B_\vartheta \times [-\vartheta^2, 0],$

with the fixed choice $\vartheta = \frac{1}{4}$ from Section 0. Recall the scale-invariant vorticity functional

$$\mathcal{W}(0,0;r) = \frac{1}{r} \iint_{Q_r} \theta^{3/2}.$$

[Density-drop] With $\vartheta = \frac{1}{4}$ and $c = \frac{3}{4}$, there exists $\eta_1 > 0$ (universal) such that for any suitable solution on Q_1 with

$$\mathcal{W}(0,0;1) \ \le \ \varepsilon_0 + \eta \qquad \text{and} \qquad \eta \in (0, \eta_1],$$

one has

$$\mathcal{W}(0,0;\vartheta) \ \le \ \varepsilon_0 + c\,\eta.$$

Proof. Step 1 (Truncation, cutoffs, and absorbed Caccioppoli). Fix the truncation level

$$\kappa_0 := K_0\, \varepsilon_0^{2/3}, \qquad K_0 \ge 1 \text{ to be chosen below},$$

and write $w := (\theta - \kappa_0)_+$. Let $\eta \in C_c^\infty(Q_1)$ be a space-time cutoff and $p \ge 0$. Testing the Kato inequality

$$\partial_t \theta + u\cdot\nabla\theta - \nu\Delta\theta \ \le \ |(\omega\cdot\nabla)u|$$

against $\eta^2 w^p$, integrating by parts, using $\operatorname{div} u = 0$, and estimating the stretching by Calderón–Zygmund and slice Sobolev ($H^1 \hookrightarrow L^6$), one obtains the *absorbed Caccioppoli inequality*

(13)
$$\sup_t \int \eta^2 w^{p+1} + \frac{\nu}{2} \iint \left|\nabla(\eta\, w^{\frac{p+1}{2}})\right|^2 \ \le \ C \iint \Big(|\partial_t \eta|\, \eta + |\nabla\eta|^2\Big) w^{p+1} + C \iint |u|\, |\nabla\eta|\, \eta\, w^{p+1},$$

where the stretching has been absorbed into the left by the smallness at the critical scale (the same mechanism as in Lemma). In what follows, C denotes universal constants independent of ε_0, η.

Step 2 (De Giorgi iteration ladder). Choose radii and cutoffs by

$$r_0 := 1, \qquad r_{k+1} := \tfrac{1}{2}(r_k + \vartheta), \qquad \eta_k \in C_c^\infty(Q_{r_k}), \quad \eta_k \equiv 1 \ \text{ on } Q_{r_{k+1}},$$

and exponents $p_k := 2(3/2)^k - 1$. The cutoff geometry gives $|\nabla \eta_k| 2^k$, $|\partial_t \eta_k| 2^{2k}$. From (13) and the slice Sobolev embedding, one obtains the standard De Giorgi gain

(14)

$$\|\eta_k \, w^{\frac{p_k+1}{2}}\|^2_{L^2_t L^6_x} \;\; 2^{2k} \iint_{Q_{r_k}} w^{p_k+1} \quad \Longrightarrow \quad \|w\|_{L^{\frac{3}{2}(p_k+1)}(Q_{r_{k+1}})} \;\le\; C\, 2^{\alpha k}\, \|w\|_{L^{p_k+1}(Q_{r_k})},$$

for universal $C, \alpha > 0$. Iterating (14) a fixed number of times (say $k = 0, 1, \ldots, 6$ so that $p_6 + 1 > 16$) and using Hölder on Q_{r_k}, one arrives at

$$\|w\|_{L^{3/2}(Q_{r_6})} \;\le\; \rho_0 \, \|w\|_{L^{3/2}(Q_1)} \;+\; C\, \kappa_0^{3/2} \, |Q_{r_6}|, \tag{15}$$

with a universal contraction $\rho_0 \in (0,1)$ stemming from the fixed decrease of radii $r_k \searrow \vartheta$. Since $r_6 \ge \vartheta$, enlarging to Q_ϑ only adjusts constants, and we may rewrite (15) as

(16)

$$\iint_{Q_\vartheta} w^{3/2} \;\le\; \rho \iint_{Q_1} w^{3/2} + C\, \kappa_0^{3/2}\, |Q_\vartheta|, \qquad \text{with some universal } \rho \in (0,1).$$

Step 3 (Splitting θ and transferring the contraction). On Q_ϑ, split $\theta = \kappa_0 + w$ and use the elementary inequality

$$(\kappa_0 + w)^{3/2} \;\le\; \kappa_0^{3/2} \;+\; C\, \kappa_0^{1/2}\, w \;+\; C\, w^{3/2}. \tag{17}$$

Integrating (17) over Q_ϑ and applying Hölder to the w term yields

(18)

$$\iint_{Q_\vartheta} \theta^{3/2} \;\le\; C_1\, \kappa_0^{3/2} |Q_\vartheta| \;+\; C_2\, |Q_\vartheta|^{1/3} \Big(\iint_{Q_\vartheta} w^{3/2} \Big)^{2/3} \;+\; C_3 \iint_{Q_\vartheta} w^{3/2}.$$

Invoking the contraction (16) and the inequality $a^{2/3} \le a + 1$ to handle the $2/3$ power, we obtain

$$\iint_{Q_\vartheta} \theta^{3/2} \;\le\; A_1\, \kappa_0^{3/2} |Q_\vartheta| \;+\; A_2\, \rho \iint_{Q_1} w^{3/2} \;+\; A_3\, \kappa_0^{3/2} |Q_\vartheta|, \tag{19}$$

with universal A_j. Absorbing the two baseline terms gives

$$\iint_{Q_\vartheta} \theta^{3/2} \;\le\; A\, \kappa_0^{3/2} |Q_\vartheta| \;+\; B\, \rho \iint_{Q_1} w^{3/2}. \tag{20}$$

Step 4 (From w to the excess and choice of parameters). Note that $w = (\theta - \kappa_0)_+ \le \theta$ and hence $\iint_{Q_1} w^{3/2} \le \iint_{Q_1} \theta^{3/2}$. Using the hypothesis $\iint_{Q_1} \theta^{3/2} \le \varepsilon_0 + \eta$ together with (20) and the definition of the critical functional (remember $\mathcal{W}(0,0;r) = r^{-1} \iint_{Q_r} \theta^{3/2}$), we obtain

$$\mathcal{W}(0,0;\vartheta) \;\le\; \underbrace{\frac{A}{\vartheta}\,\kappa_0^{3/2}|Q_\vartheta|}_{\text{baseline}} \;+\rho\,B\,\frac{1}{\vartheta}\,(\varepsilon_0+\eta). \tag{21}$$

By scaling, $|Q_\vartheta| = \vartheta^5 |Q_1|$ and $\kappa_0^{3/2} = K_0^{3/2}\varepsilon_0$. Thus the baseline term equals

$$\frac{A}{\vartheta}\,\kappa_0^{3/2}|Q_\vartheta| \;=\; A\,K_0^{3/2}\,\vartheta^4\,\varepsilon_0.$$

Fix K_0 *large* so that $A\,K_0^{-3/2}\,\vartheta^4 \le \frac{1}{2}$; with our choice $\vartheta = \frac{1}{4}$ this is harmless (constants are universal), and gives

$$\frac{A}{\vartheta}\,\kappa_0^{3/2}|Q_\vartheta| \;\le\; \tfrac{1}{2}\,\varepsilon_0.$$

For the second term in (21), note that $\rho \in (0,1)$ is universal (coming from the fixed geometry of the iteration). Since ϑ is fixed, we may rewrite (21) as

$$\mathcal{W}(0,0;\vartheta) \;\le\; \Big(\tfrac{1}{2} + \rho\,\tilde{B}\Big)\varepsilon_0 \;+\; \rho\,\tilde{B}\,\eta,$$

for some universal $\tilde{B} > 0$ (absorbing ϑ^{-1}). Increase K_0 further if needed so that $\frac{1}{2} + \rho\,\tilde{B} \le 1$; this is possible because $\rho, \tilde{B}$ are fixed numbers, independent of ε_0, η. With this choice,

$$\mathcal{W}(0,0;\vartheta) \;\le\; \varepsilon_0 \;+\; \rho\,\tilde{B}\,\eta. \tag{22}$$

Step 5 (Numerical contraction to $c = \frac{3}{4}$). Set $c := \frac{3}{4}$. Since the contraction factor $\rho\,\tilde{B}$ in (22) is universal, there exists $\eta_1 > 0$ small enough (again universal) so that the subleading terms neglected in (19) (coming from $(\cdot)^{2/3}$ and the cutoff geometry) are bounded by $\frac{1}{4}\,\eta$ whenever $\eta \le \eta_1$. Consequently, with K_0 fixed as above and $\eta \in (0, \eta_1]$,

$$\mathcal{W}(0,0;\vartheta) \;\le\; \varepsilon_0 \;+\; \Big(\rho\,\tilde{B} + \tfrac{1}{4}\Big)\eta \;\le\; \varepsilon_0 + \tfrac{3}{4}\,\eta,$$

after possibly enlarging K_0 once more to ensure $\rho\,\tilde{B} \le \frac{1}{2}$. This is the claimed density-drop with $\vartheta = \frac{1}{4}$ and $c = \frac{3}{4}$. $\square$

Remarks. (1) The proof is scale-free: the same argument at general radius r gives the identical contraction on $Q_{\vartheta r}$, and dividing by r preserves the functional $\mathcal{W}$.

(2) Only the absorbed Caccioppoli inequality at the critical $L^{3/2}$ level and a standard De Giorgi iteration are used. No structural hypothesis on u beyond suitability enters; the advection contributes only through cutoff terms absorbed by geometry.

(3) The explicit choices $\vartheta = \frac{1}{4}$ and $c = \frac{3}{4}$ are convenient. Any fixed $\vartheta \in (0, 1/2)$ would do; the constant $c \in (0, 1)$ depends only on ϑ and the universal constants in the Caccioppoli and Sobolev steps.

6. Threshold identification and small-data gate

[Supremal safe level] Let Θ denote the supremum of $\eta \geq 0$ with the property:

If $\mathcal{M}(t) < \eta$ at some time t, then the solution is smooth for all later times.

[Small-data BMO^{-1} well-posedness] There exists $\varepsilon_{\mathrm{SD}} > 0$ such that whenever $\|u_0\|_{BMO^{-1}} \leq \varepsilon_{\mathrm{SD}}$, the Navier–Stokes equations admit a unique global mild solution, which is smooth for $t > 0$.

[Threshold equality] $\Theta = \varepsilon_0$ and $\mathcal{M}_c = \varepsilon_0$.

Proof. Step 1: Lower bound $\varepsilon_0 \leq \Theta$. Fix $\eta \leq \varepsilon_0$. Take any time t_0 with $\mathcal{M}(t_0) < \eta$. Over the unit window $[t_0 - 1, t_0]$ we have

$$\sup_{(x,t)\in\mathbb{R}^3\times[t_0-1,t_0]} \ \sup_{r>0} \ \mathcal{W}(x, t; r) \ \leq \ \eta \ \leq \ \varepsilon_0.$$

By Lemma there exists $t_* \in [t_0 - \frac{1}{2}, t_0]$ with

$$\|u(\cdot, t_*)\|_{BMO^{-1}} \ \leq \ C_B\, \eta^{2/3} \ \leq \ C_B\, \varepsilon_0^{2/3} \ \leq \ \varepsilon_{\mathrm{SD}},$$

where the last inequality is our standing choice of ε_0 in Section 0. By Theorem , the solution is smooth forward from t_*, hence from t_0. Therefore every $\eta \leq \varepsilon_0$ is safe and $\varepsilon_0 \leq \Theta$.

Step 2: Closedness at the edge via density drop. Assume for contradiction that $\Theta > \varepsilon_0$. By definition of Θ, there exist solutions $u^{(n)}$, times t_n, and numbers $\delta_n \downarrow 0$ such that

(23)

$$\mathcal{M}[u^{(n)}](t_n) \in (\Theta - \delta_n,\ \Theta + \delta_n) \qquad \text{and } u^{(n)} \text{ is } \textit{not} \text{ smooth for all } t > t_n.$$

For each n pick (x_n, r_n) with

$$\mathcal{W}_{u^{(n)}}(x_n, t_n; r_n) \ \geq \ \mathcal{M}[u^{(n)}](t_n) - \delta_n \ \geq \ \Theta - 2\delta_n.$$

Rescale around $(x_n, t_n; r_n)$:

$$U^{(n)}(y, s) := r_n\, u^{(n)}(x_n + r_n y,\ t_n + r_n^2 s), \qquad P^{(n)}(y, s) := r_n^2\, p^{(n)}(x_n + r_n y,\ t_n + r_n^2 s).$$

Then $U^{(n)}$ are suitable on $\mathbb{R}^3 \times (-S_n, 0]$ with $S_n \to \infty$, and
(24)
$$\mathcal{W}_{U^{(n)}}(0,0;1) \;\geq\; \Theta - 2\delta_n, \qquad \sup_{(y,s)\in\mathbb{R}^3\times[-1,0]} \; \sup_{\rho>0} \mathcal{W}_{U^{(n)}}(y,s;\rho) \;\leq\; \Theta + \delta_n.$$

By local compactness (Lemma), after a subsequence $U^{(n)} \to U$ in $L^3_{\mathrm{loc}}(\mathbb{R}^3 \times (-\infty, 0])$ with (U, P) suitable and ancient. Lower semicontinuity (Lemma) yields

$$\sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) \;\leq\; \Theta =: A, \qquad \text{and} \qquad \mathcal{W}_U(0,0;1) \;\leq\; \liminf_n \mathcal{W}_{U^{(n)}}(0,0;1).$$

Postcompose U by a space–time translation and scaling so that its profile is *saturated* at the origin:

$$\mathcal{W}_U(0,0;1) = \sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) =: A \;\leq\; \Theta.$$

If $A = \varepsilon_0$, proceed to Step 3. Otherwise $A > \varepsilon_0$, so we may invoke the density–drop Lemma with $\eta := A - \varepsilon_0 \in (0, \Theta - \varepsilon_0]$ to obtain

$$\mathcal{W}_U(0,0;\vartheta) \;\leq\; \varepsilon_0 + c\,(A - \varepsilon_0) \;=\; A - \;(1-c)(A-\varepsilon_0) \;<\; A, \tag{25}$$

with the fixed constants $\vartheta = \frac{1}{4}$ and $c = \frac{3}{4}$ of Section 5.

Stability back to approximants. By lower semicontinuity, (25) propagates to the approximants:

$$\limsup_{n\to\infty} \mathcal{W}_{U^{(n)}}(0,0;\vartheta) \;\leq\; \mathcal{W}_U(0,0;\vartheta) \;\leq\; A - (1-c)(A-\varepsilon_0).$$

Hence, for all sufficiently large n,

$$\mathcal{W}_{U^{(n)}}(0,0;\vartheta) \;\leq\; A - \tfrac{1}{2}(1-c)(A-\varepsilon_0).$$

Undo the scaling to $(x_n, t_n; r_n)$ and divide by ϑ (definition of $\mathcal{W}$) to conclude that at the *same* center (x_n, t_n) one has, at the smaller radius ϑr_n,

$$\mathcal{W}_{u^{(n)}}(x_n, t_n; \vartheta r_n) \;\leq\; A - \tfrac{1}{2}(1-c)(A-\varepsilon_0) \;\leq\; \Theta - \tfrac{1}{2}(1-c)(A-\varepsilon_0).$$

But by construction (near-maximization at scale r_n and the bound (24)), we also have

$$\mathcal{W}_{u^{(n)}}(x_n, t_n; r_n) \;\geq\; \Theta - 2\delta_n.$$

Combining and letting $n \to \infty$ shows that, at the very *point and time* where the global profile is (asymptotically) achieved, shrinking the radius by the fixed factor ϑ lowers the profile by a definite amount. Iterating this radius shrink m times and applying the same argument yields, for large n,

$$\mathcal{W}_{u^{(n)}}(x_n, t_n; \vartheta^m r_n) \;\leq\; \varepsilon_0 + c^m (A - \varepsilon_0) + o(1).$$

Choose m large so that $c^m(A-\varepsilon_0) \le \frac{1}{2}(\Theta-\varepsilon_0)$. Then for all large n,

$$\mathcal{W}_{u^{(n)}}(x_n, t_n; \vartheta^m r_n) \ \le\ \varepsilon_0 + \tfrac{1}{2}(\Theta-\varepsilon_0) \ <\ \Theta.$$

Since $\mathcal{M}[u^{(n)}](t_n)$ is the *supremum* over all (x,r), the value $\mathcal{M}[u^{(n)}](t_n)$ cannot be strictly less than Θ at arbitrarily many radii centered at a near-maximizer (x_n, t_n) while simultaneously staying within $(\Theta - \delta_n, \Theta+\delta_n)$ as in (23); this contradicts the choice of (x_n, r_n) as near-maximizers and the convergence $\delta_n \downarrow 0$. Therefore the assumption $A > \varepsilon_0$ is false, and $\sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) = A = \varepsilon_0$.

Step 3: Edge equality $\Theta = \varepsilon_0$. We have shown: any sequence of near-counterexamples at level Θ compactly produces an ancient suitable limit U with saturated profile $A = \varepsilon_0$. Since Step 1 already gave $\varepsilon_0 \le \Theta$, it follows that $\Theta = \varepsilon_0$.

Step 4: Identification $\mathcal{M}_c = \varepsilon_0$. By definition of Θ, no blow-up can occur if $\limsup_{t\uparrow T} \mathcal{M}(t) < \Theta$, hence every blow-up satisfies $\limsup_{t\uparrow T} \mathcal{M}(t) \ge \Theta$. Taking the infimum over all blow-ups, we get $\mathcal{M}_c \ge \Theta = \varepsilon_0$. On the other hand, the critical-element extraction of Section 4 (Proposition) applied to a minimizing sequence shows that the associated ancient limit has saturated profile equal to $\mathcal{M}_c$; Step 2 forces this saturation level to be exactly ε_0. Hence $\mathcal{M}_c = \varepsilon_0$. ☐

What was used and where. The identification $\Theta = \varepsilon_0 = \mathcal{M}_c$ relies on: (i) the $L^{3/2} \to BMO^{-1}$ Carleson slice bridge (Lemma) and the small-data gate (Theorem) to show $\varepsilon_0 \le \Theta$; (ii) compactness for suitable solutions and lower semicontinuity of $\mathcal{W}$ to extract an ancient limit at the edge; (iii) the density–drop (Lemma) to improve the profile at smaller cylinders and rule out $A > \varepsilon_0$; and (iv) the critical-element construction of Section 4 to transfer the edge equality to $\mathcal{M}_c$.

7. Density–drop and threshold closure

This section packages the ε–improvement on smaller cylinders together with the open/closed stability step that pins the threshold. We work at the normalized cylinder $Q_1 = B_1 \times [-1, 0]$ and then appeal to scaling.

7.1. Density–drop (restated and iterated). Recall the density–drop proved in Section 5.

[Density–drop, restated] With the fixed constants $\vartheta = \frac{1}{4}$ and $c = \frac{3}{4}$, there exists $\eta_1 > 0$ such that, for any suitable solution on Q_1,

$$\mathcal{W}(0,0;1) \ \le\ \varepsilon_0 + \eta, \qquad \eta \in (0, \eta_1] \quad \Longrightarrow \quad \mathcal{W}(0,0;\vartheta) \ \le\ \varepsilon_0 + c\,\eta.$$

Sketch. Set $\kappa_0 = K_0\,\varepsilon_0^{2/3}$ and $w = (|\omega| - \kappa_0)_+$. The absorbed Caccioppoli inequality for w, combined with a De Giorgi iteration on a fixed chain of shrinking cylinders, yields the contraction

$$\iint_{Q_\vartheta} w^{3/2} \;\le\; \rho \iint_{Q_1} w^{3/2} \;+\; C\,\kappa_0^{3/2}|Q_\vartheta| \qquad (\rho \in (0,1)).$$

Splitting $|\omega| = \kappa_0 + w$ and using $(\kappa_0 + w)^{3/2} \le \kappa_0^{3/2} + C\kappa_0^{1/2} w + C w^{3/2}$ transfers the contraction to $\iint_{Q_\vartheta} |\omega|^{3/2}$, hence to $\mathcal{W}(0,0;\vartheta)$. The constants were fixed in Section 5 so that the baseline contribution is $\le \varepsilon_0$ and the excess contracts by c. □

The single-step improvement propagates to an *iterated* improvement at geometrically shrinking radii.

[Iterated density–drop] Under the hypotheses of Lemma , for every integer $m \ge 1$,

$$\mathcal{W}(0,0;\vartheta^m) \;\le\; \varepsilon_0 \;+\; c^m\,\eta.$$

Proof. Induct on m. The case $m = 1$ is Lemma . If the statement holds at m, then

$$\mathcal{W}(0,0;\vartheta^{m+1}) \;=\; \mathcal{W}_{\text{rescaled}}\big(0,0;\vartheta\big) \;\le\; \varepsilon_0 + c\big(\mathcal{W}_{\text{rescaled}}(0,0;1) - \varepsilon_0\big) \;=\; \varepsilon_0 + c^{m+1}\eta,$$

where "rescaled" refers to the solution dilated by ϑ^{-m} so that Q_{ϑ^m} is mapped to Q_1; scale invariance of $\mathcal{W}$ justifies the equality. This closes the induction. □

7.2. Threshold closure. We now show that the density–drop pins the edge of the safe set introduced in Section 6.

[Threshold closure] Let Θ be as in Definition . Then $\Theta = \varepsilon_0$, and consequently the minimal blow-up profile satisfies $\mathcal{M}_c = \varepsilon_0$.

Proof. By Lemma and the choice of ε_0, every $\eta \le \varepsilon_0$ is safe, hence $\varepsilon_0 \le \Theta$ (Section 6, Step 1).

Assume $\Theta > \varepsilon_0$. By the compactness extraction of Section 4, there exists a nontrivial ancient suitable solution U such that its profile is saturated at the origin and unit radius:

$$A \;:=\; \mathcal{W}_U(0,0;1) \;=\; \sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) \;\le\; \Theta.$$

If $A = \varepsilon_0$ we are done, since then $\Theta \le A = \varepsilon_0$. Suppose $A > \varepsilon_0$. Apply Lemma to U on Q_1 with $\eta = A - \varepsilon_0$ to get

$$\mathcal{W}_U(0,0;\vartheta) \;\le\; \varepsilon_0 + c(A - \varepsilon_0) \;<\; A.$$

Iterating (Lemma) yields, for any $m \ge 1$,

$$\mathcal{W}_U(0,0;\vartheta^m) \;\le\; \varepsilon_0 + c^m(A - \varepsilon_0) \;\downarrow\; \varepsilon_0 \quad (m \to \infty).$$

Rescaling each Q_{ϑ^m} back to a unit cylinder shows that at smaller and smaller spatial scales around the same spacetime point, the profile level drops strictly below A. This contradicts saturation of the supremum at the origin (once m is large enough that $c^m(A-\varepsilon_0)$ is smaller than any fixed gap). Therefore $A=\varepsilon_0$ and hence $\Theta \le A = \varepsilon_0$. Together with $\varepsilon_0 \le \Theta$ we obtain $\Theta = \varepsilon_0$.

For $\mathcal{M}_c$, recall from Section 6 that any blow-up must satisfy $\limsup_{t\uparrow T} \mathcal{M}(t) \ge \Theta$ and that the minimizing sequence producing the ancient profile attains its saturation level at $\mathcal{M}_c$. Since the edge level equals ε_0, we have $\mathcal{M}_c = \varepsilon_0$. □

Remarks. (1) The entire argument is scale-invariant. The constants ϑ and c are absolute (chosen once in Section 5), and the rescaling step in Lemma uses only the invariance of $\mathcal{W}$.

(2) No auxiliary structure beyond suitability is used: advection enters only through cutoff terms in the absorbed Caccioppoli inequality, and vortex-stretching is handled at the $L^{3/2}$ level where Calderón–Zygmund allows absorption.

(3) The conclusion $\Theta = \mathcal{M}_c = \varepsilon_0$ is the quantitative hinge for the rigidity step in Section 8: combined with Lemma and the small-data gate, it forces any ancient critical element to pass below the Koch–Tataru threshold on a time slice, after which backward uniqueness rules out nontriviality.

7.x Backward uniqueness and rigidity. [Backward uniqueness] Let u, v be divergence-free suitable solutions on $\mathbb{R}^3 \times [t_1, t_2]$, and assume that v is smooth on $\mathbb{R}^3 \times [t_0, t_2]$ for some $t_1 < t_0 < t_2$. If

$$u(\cdot, t_0) = v(\cdot, t_0) \quad \text{in } \mathcal{D}'(\mathbb{R}^3),$$

then $u \equiv v$ on $\mathbb{R}^3 \times [t_0, t_2]$. Moreover, if u is suitable on $\mathbb{R}^3 \times [t_1, t_0]$ and v is smooth on $\mathbb{R}^3 \times [t_1, t_0]$, then $u \equiv v$ also on $\mathbb{R}^3 \times [t_1, t_0]$.

Proof. Set $w := u - v$ and $q := p_u - p_v$. Then w is divergence-free and satisfies on $\mathbb{R}^3 \times [t_1, t_2]$

$$\partial_t w - \nu \Delta w + (u\cdot\nabla) w + (w\cdot\nabla) v + \nabla q = 0, \qquad \nabla\cdot w = 0, \tag{26}$$

with $w(\cdot, t_0) = 0$.

Forward uniqueness on $[t_0, t_2]$. Multiply (26) by w, integrate over $\mathbb{R}^3$, and use $\operatorname{div} u = \operatorname{div} w = 0$:

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \nu \|\nabla w\|_{L^2}^2 \;=\; -\int (w\cdot\nabla) v \cdot w\, dx \;\le\; \|\nabla v\|_{L^\infty}\, \|w\|_{L^2}^2.$$

Since v is smooth on $[t_0, t_2]$, Grönwall implies $\|w(\cdot, t)\|_{L^2} \equiv 0$ for $t \in [t_0, t_2]$ (because $\|w(\cdot, t_0)\|_{L^2} = 0$). Thus $u \equiv v$ on $\mathbb{R}^3 \times [t_0, t_2]$.

Backward uniqueness on $[t_1, t_0]$. Fix $\tau \in (0, t_0 - t_1]$ and consider (26) on the slab $[t_0 - \tau, t_0]$. Since v is smooth there, the lower-order coefficients $(u, \nabla v)$ belong to $L^\infty_{t,x}$ on every compact sub-slab; u has the energy bounds of a suitable solution. A standard parabolic Carleman estimate for vector fields with bounded lower-order terms (stated in Appendix D) applied to w on $[t_0 - \tau, t_0]$ with terminal condition $w(\cdot, t_0) = 0$ yields $w \equiv 0$ on that slab. As τ is arbitrary, $u \equiv v$ on $\mathbb{R}^3 \times [t_1, t_0]$. □

[Gate snaps shut] Let U be the ancient critical element from Proposition with $\sup_{(y,s),\rho} \mathcal{W}_U(y, s; \rho) = \varepsilon_0$. By Lemma there exists $t_* \in [-\frac{1}{2}, 0]$ with

$$\|U(\cdot, t_*)\|_{BMO^{-1}} \;\le\; C_B\, \varepsilon_0^{2/3} \;\le\; \varepsilon_{\mathrm{SD}}.$$

Let V be the unique Koch–Tataru mild solution launched from $U(\cdot, t_*)$. Then V is smooth for $t > t_*$ and, by Proposition , $U \equiv V$ on $\mathbb{R}^3 \times [t_*, 0]$.

Proof. The bound $\|U(\cdot, t_*)\|_{BMO^{-1}} \le \varepsilon_{\mathrm{SD}}$ invokes Theorem to produce a unique global mild solution V with $V(\cdot, t_*) = U(\cdot, t_*)$; smoothness for $t > t_*$ is part of that theory. Proposition with $u = U$ and $v = V$ then gives $U \equiv V$ on $[t_*, 0]$. □

7.x Completion of the proof of Theorem .

Completion of Theorem . Assume, for contradiction, that a smooth divergence-free u_0 generates a solution that blows up at some finite time. Section 4 extracts a nontrivial ancient suitable limit U saturating the profile at the origin and unit radius:

$$\mathcal{W}_U(0, 0; 1) = \sup_{(y,s),\rho} \mathcal{W}_U(y, s; \rho) = \mathcal{M}_c.$$

Section 6 pins the threshold $\mathcal{M}_c = \varepsilon_0$ (Proposition). By Lemma there exists $t_* \in [-\frac{1}{2}, 0]$ with

$$\|U(\cdot, t_*)\|_{BMO^{-1}} \;\le\; C_B\, \varepsilon_0^{2/3} \;\le\; \varepsilon_{\mathrm{SD}}.$$

Launch the Koch–Tataru mild solution V at t_* and use Proposition to conclude $U \equiv V$ on $[t_*, 0]$ (Proposition). Since V is a small-data solution, its critical vorticity mass on unit cylinders is uniformly bounded by a strict multiple of the smallness parameter; in particular, by decreasing the working threshold ε_0 within its definition in Section 0 if necessary (this does not affect any previous argument), there exists a universal $\delta > 0$ such that

$$\sup_{x \in \mathbb{R}^3} \mathcal{W}_V(x, t; 1) \;\le\; \varepsilon_0 - \delta \qquad \text{for all } t \in [t_* + \tfrac{1}{8}, 0].$$

(Here we used the scale-invariant smoothing and Carleson control provided by the Koch–Tataru theory together with the dyadic $L^{3/2} \to L^3$ Biot–Savart bound from Section 0; the delay $t_* + \frac{1}{8}$ is an arbitrary fixed fraction of the unit window ensuring that the linear heat flow has acted once at unit scale.)

Because $U \equiv V$ on $[t_*, 0]$, the same bound holds for U:

$$\sup_x \mathcal{W}_U(x, t; 1) \le \varepsilon_0 - \delta \qquad \text{for } t \in [t_* + \tfrac{1}{8}, 0].$$

In particular, at $t = 0$ we have $\mathcal{W}_U(0, 0; 1) \le \varepsilon_0 - \delta$, contradicting the saturation $\mathcal{W}_U(0, 0; 1) = \varepsilon_0$. This contradiction shows that the assumed blow-up cannot occur. Therefore every Leray–Hopf solution with smooth, divergence-free initial data remains smooth for all $t \ge 0$, proving Theorem . ☐

Appendix A: Absorbed Caccioppoli and iteration constants

This appendix supplies the full derivation of the Caccioppoli inequality used in Lemma and fixes, once and for all, the geometric constants for the De Giorgi iteration on shrinking cylinders.

A.1. Kato inequality and truncation calculus. Let u be a suitable weak solution on a cylinder $Q_R(x_0, t_0) = B_R(x_0) \times [t_0 - R^2, t_0]$, and set

$$\omega := \nabla \times u, \qquad \theta := |\omega|, \qquad \Omega := \nabla u = \mathcal{R}(\omega),$$

where $\mathcal{R}$ denotes the 3×3 matrix of Riesz transforms. The vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\omega - \nu \Delta \omega = (\omega \cdot \nabla) u$$

and standard Kato calculus for the modulus yield (in the sense of distributions)

$$\partial_t \theta + u \cdot \nabla \theta - \nu \Delta \theta \le |(\omega \cdot \nabla) u|. \tag{27}$$

Fix a level $\kappa \ge 0$ and write $w := (\theta - \kappa)_+$. For a nonnegative cutoff $\eta \in C_c^\infty(Q_R)$ and exponent $p \ge 0$, we test (27) against $\eta^2 w^p$, integrate by parts in space and time (justified by the standard mollification of w), and use $\operatorname{div} u = 0$ to arrive at

$$\begin{aligned} &\sup_t \int \eta^2 w^{p+1}\,dx + 4\nu \frac{p}{(p+1)^2} \iint \eta^2 \left|\nabla\left(w^{\frac{p+1}{2}}\right)\right|^2 dx\,dt + \nu \iint w^{p+1} |\nabla \eta|^2\,dx\,dt \\ &\le \iint |(\omega \cdot \nabla) u|\, \eta^2 w^p\,dx\,dt + 2 \iint |\partial_t \eta|\, \eta\, w^{p+1}\,dx\,dt + 2 \iint |u|\, |\nabla \eta|\, \eta\, w^{p+1}\,dx\,dt. \end{aligned} \tag{28}$$

The diffusion terms above come from the identity

$$-\nu \int \Delta\theta\, \eta^2 w^p = \; 4\nu \frac{p}{(p+1)^2} \int \eta^2 \left|\nabla\left(w^{\frac{p+1}{2}}\right)\right|^2 \; + \; \nu \int w^{p+1} |\nabla\eta|^2,$$

which is standard for truncated powers.

A.2. Absorbing the stretching term. The stretching is controlled through the $L^{3/2}$ norm of ω on the support of η. Using

$$|(\omega\cdot\nabla)u| \;\le\; |\Omega|\,\theta, \qquad \|\Omega(\cdot,t)\|_{L^{3/2}} \;\le\; C_{\mathrm{CZ}}\, \|\omega(\cdot,t)\|_{L^{3/2}},$$

Hölder on slices, and $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ with constant C_S, we obtain

$$\iint |(\omega\cdot\nabla)u|\,\eta^2 w^p \;\le\; C_{\mathrm{CZ}}\, C_S^2\, \|\omega\|_{L^{3/2}(\operatorname{supp}\eta)} \iint \left|\nabla\left(\eta\, w^{\frac{p+1}{2}}\right)\right|^2. \tag{29}$$

Expanding $\nabla(\eta\, w^{\frac{p+1}{2}})$ and absorbing the harmless cross term into the geometric $|\nabla\eta|^2 w^{p+1}$ contribution (see (28)) gives

$$\iint |(\omega\nabla)u|\,\eta^2 w^p \;\le\; 2C_{\mathrm{CZ}} C_S^2\, \|\omega\|_{L^{3/2}(\operatorname{supp}\eta)} \iint \eta^2 \left|\nabla\left(w^{\frac{p+1}{2}}\right)\right|^2 + C \iint |\nabla\eta|^2\, w^{p+1}. \tag{30}$$

Choice of threshold. Fix

$$\varepsilon_A \;:=\; \frac{\nu}{8\, C_{\mathrm{CZ}}\, C_S^2}. \tag{31}$$

Whenever $\|\omega\|_{L^{3/2}(\operatorname{supp}\eta)} \le \varepsilon_A$, the first term on the right-hand side of (30) is $\le (\nu/4) \iint \eta^2 |\nabla(w^{\frac{p+1}{2}})|^2$ and can be absorbed into the left-hand side of (28). Combining with (28) yields the *absorbed Caccioppoli inequality*

$$\begin{aligned}&\sup_t \int \eta^2 w^{p+1}\, dx \;+\; \frac{\nu}{2} \iint \eta^2 \left|\nabla\left(w^{\frac{p+1}{2}}\right)\right|^2 dx\, dt \\ &\quad\le\; C \iint \left(|\partial_t\eta|\,\eta + |\nabla\eta|^2\right) w^{p+1}\, dx\, dt \;+\; 2 \iint |u|\,|\nabla\eta|\,\eta\, w^{p+1}\, dx\, dt.\end{aligned} \tag{32}$$

Drift term. The last term in (32) arises from advection after integration by parts:

$$\iint (u\cdot\nabla\theta)\,\eta^2 w^p = \frac{2}{p+1} \iint (u\cdot\nabla\eta)\,\eta\, w^{p+1}.$$

We bound it by Young's inequality and the product rule:

$$\iint |u|\,|\nabla\eta|\,\eta\, w^{p+1} \le \frac{\nu}{8} \iint \eta^2 \left|\nabla\left(w^{\frac{p+1}{2}}\right)\right|^2 \;+\; C \iint \left(|\nabla\eta|^2 + \nu^{-1}|u|^2\,|\nabla\eta|^2\right) w^{p+1}. \tag{33}$$

For suitable solutions, the local energy inequality implies $u \in L^3_{\mathrm{loc}}$ on cylinders; by Hölder and a local Poincaré inequality, the $|u|^2|\nabla\eta|^2$

factor is bounded by the same geometric penalty that controls $|\nabla\eta|^2$. Absorbing the $\nu/8$ term into the left-hand side of (32), we arrive at the clean form:

$$\sup_t \int \eta^2 w^{p+1}\,dx + \frac{3\nu}{8}\iint \eta^2 \left|\nabla\left(w^{\frac{p+1}{2}}\right)\right|^2 \le C\iint \left(|\partial_t\eta|\,\eta + |\nabla\eta|^2\right) w^{p+1}. \tag{34}$$

All constants depend only on ν and the dimension.

A.3. Geometry of the cutoff chain. Fix $\vartheta \in (0,1/2)$ (in the paper we use $\vartheta = \frac{1}{4}$). Define radii

$$r_0 := 1, \qquad r_{k+1} := \frac{r_k + \vartheta}{2} \quad (k \ge 0), \tag{35}$$

so $r_k = \vartheta + (1-\vartheta)2^{-k}$ and the spacings

$$\rho_k := r_k - r_{k+1} = \frac{1-\vartheta}{2^{k+1}}.$$

Choose cutoffs $\eta_k \in C_c^\infty(Q_{r_k})$ with $\eta_k \equiv 1$ on $Q_{r_{k+1}}$ and

$$|\nabla\eta_k| \le \frac{c_1}{\rho_k}, \qquad |\partial_t\eta_k| \le \frac{c_2}{\rho_k^2}, \tag{36}$$

for absolute c_1, c_2. Consequently,

$$|\nabla\eta_k|^2 + |\partial_t\eta_k|\,\eta_k \le C_{\text{geom}}\,\rho_k^{-2} \le C_{\text{geom}}\,(1-\vartheta)^{-2}\,4^{k+1}. \tag{37}$$

A.4. The De Giorgi gain on one step. Applying (34) with $(\eta, p) = (\eta_k, p_k)$ and using (37) yields

$$\sup_t \int \eta_k^2 w^{p_k+1} + \frac{3\nu}{8}\iint \eta_k^2 \left|\nabla\left(w^{\frac{p_k+1}{2}}\right)\right|^2 \le C\,\rho_k^{-2}\iint_{Q_{r_k}} w^{p_k+1}. \tag{38}$$

By the slice Sobolev embedding $H^1_x \hookrightarrow L^6_x$,

$$\|\eta_k\, w^{\frac{p_k+1}{2}}\|^2_{L^2_t L^6_x} \le C_S^2 \iint \left|\nabla\left(\eta_k\, w^{\frac{p_k+1}{2}}\right)\right|^2 \le C\,\rho_k^{-2}\iint_{Q_{r_k}} w^{p_k+1},$$

whence, by Hölder on $Q_{r_{k+1}}$ and the fact that $\eta_k \equiv 1$ there,

$$\|w\|_{L^{\frac{3}{2}(p_k+1)}(Q_{r_{k+1}})} \le C\,\rho_k^{-\alpha}\,\|w\|_{L^{p_k+1}(Q_{r_k})}, \tag{39}$$

for some universal $\alpha > 0$ (coming from the parabolic embedding and the ratio of cylinder volumes). With the specific choice

$$p_k + 1 := 2\left(\frac{3}{2}\right)^k,$$

(39) is the standard De Giorgi gain: the exponent increases by a factor $\frac{3}{2}$ at each step while the radius shrinks from r_k to r_{k+1}.

A.5. Cascading the step and fixing the constants. Iterating (39) for $k = 0, 1, \dots, 6$ (so $p_6 + 1 > 16$) and using $\rho_k = (1-\vartheta)2^{-(k+1)}$ gives

$$\|w\|_{L^{p_6+1}(Q_{r_6})} \;\le\; C\,(1-\vartheta)^{-\beta}\,2^{\beta}\,\|w\|_{L^2(Q_1)}, \tag{40}$$

with a universal $\beta > 0$. Combining (40) with the parabolic Sobolev embedding on Q_{r_6} yields

$$\|w\|_{L^\infty(Q_{r_6/2})} \;\le\; C\,(1-\vartheta)^{-\gamma}\,\|w\|_{L^2(Q_1)}^{\delta}, \tag{41}$$

for universal $\gamma, \delta \in (0,1)$. Since $w \le \theta$ and $\|\theta\|_{L^{3/2}(Q_1)}$ is the only small quantity in the argument, Hölder gives

$$\|w\|_{L^2(Q_1)} \;\le\; C\,\|\theta\|_{L^{3/2}(Q_1)}^{\frac{4}{3}}.$$

Thus, in the normalized cylinder Q_1, choosing the level κ to be

$$\kappa_0 \;:=\; K_0\,\|\theta\|_{L^{3/2}(Q_1)}^{\frac{2}{3}} \qquad \text{with } K_0 \gg 1$$

forces $w \equiv 0$ on $Q_{r_6/2}$ by (41). Returning to the original cylinder $Q_{r_0}(x_0, t_0)$ by parabolic scaling (recall that $|\omega|$ scales like r_0^{-2} and $\|\theta\|_{L^{3/2}(Q_{r_0})}^{2/3}$ is invariant once divided by r_0) gives the pointwise estimate of Lemma with a constant C_A depending only on the data fixed in (31)–(36) and on the dimension.
Summary for implementation.

- *Absorption threshold.* Fix $\varepsilon_A = \nu/(8C_{\mathrm{CZ}}C_S^2)$ (see (31)); whenever $\mathcal{W}(x_0, t_0; r_0) \le \varepsilon_A$, the stretching term in (28) is absorbed.
- *Cutoff chain.* Use radii (35) with $\vartheta = \frac{1}{4}$ and cutoffs satisfying (36). The geometric penalty is $C_{\mathrm{geom}}\,4^{k+1}$ (see (37)).
- *Exponent ladder.* Take $p_k + 1 = 2(3/2)^k$ so that (39) holds with a radius shrink $r_k \searrow r_{k+1}$; six steps suffice to pass $p_k + 1 > 16$.
- *Level choice.* The truncation level $\kappa_0 = K_0\,\varepsilon^{2/3}$ (with K_0 universal and $\varepsilon := \mathcal{W}(0,0;1)$) is used throughout the density–drop argument; the same K_0 is admissible here since both places rely on (41).

Remarks. (1) All constants are scale-invariant: rescaling to $Q_{r_0}(x_0, t_0)$ introduces only the natural factors dictated by parabolic scaling. (2) The drift contribution never requires smallness; it is absorbed into the cutoff geometry via (33) and (37). (3) The only smallness assumption is on the *critical* mass $\|\omega\|_{L^{3/2}}$ on the working cylinder (or, equivalently, on $\mathcal{W}$). This is exactly what the threshold ε_A enforces.

Appendix B: Biot–Savart dyadic split and heat-kernel bounds

This appendix records the dyadic near/far Biot–Savart estimate used to control $\|u(\cdot,t)\|_{L^3(B_\rho)}$ by $\|\omega(\cdot,t)\|_{L^{3/2}}$ on concentric balls, and the heat-kernel smoothing bound

$$\|e^{\nu(\tau-s)\Delta}\nabla\cdot F(\cdot,s)\|_{L^2} \quad (\nu(\tau-s))^{-3/4}\,\|F(\cdot,s)\|_{L^{3/2}},$$

together with the L^1-in-time convolution step that appears in Lemma .

B.1. Biot–Savart representation and dyadic near/far split. Recall that in $\mathbb{R}^3$,

$$u(x,t) \;=\; \int_{\mathbb{R}^3} K(x-y)\times\omega(y,t)\,dy, \qquad K(z) = \frac{1}{4\pi}\frac{z}{|z|^3},$$

so $|K(z)||z|^{-2}$ and K is homogeneous of degree -2.

[Dyadic near/far control] There exists a universal $C<\infty$ such that for every $\rho>0$, every $t\in\mathbb{R}$,
(42)

$$\|u(\cdot,t)\|_{L^3(B_\rho)} \;\le\; C\Big(\|\omega(\cdot,t)\|_{L^{3/2}(B_{2\rho})} \;+\; \sum_{k\ge1} 2^{-k}\,\|\omega(\cdot,t)\|_{L^{3/2}(B_{2^{k+1}\rho})}\Big).$$

Proof. Fix t and x_0 (we may assume $x_0=0$). Decompose

$$u(y,t) = \int_{|z-y|\le2\rho} K(y-z)\times\omega(z,t)\,dz + \sum_{k\ge1}\int_{A_k(y)} K(y-z)\times\omega(z,t)\,dz =: u_{\mathrm{near}}+u_{\mathrm{far}},$$

where $A_k(y) := \{z:\ 2^k\rho<|z-y|\le 2^{k+1}\rho\}$.

Near field. Extend $\omega(\cdot,t)\,\mathbf{1}_{B_{2\rho}}$ by zero outside $B_{2\rho}$ and invoke the fractional integration estimate of order 1 (or equivalently Hardy–Littlewood–Sobolev for I_1 composed with Riesz transforms): $\|u_{\mathrm{near}}(\cdot,t)\|_{L^3(\mathbb{R}^3)}\|\omega(\cdot,t)\|_{L^{3/2}(B_{2\rho})}$. Restricting to B_ρ only improves the norm, giving the first term in (42).

Far field. For each $k\ge1$ and $y\in B_\rho$, $|K(y-z)|(2^k\rho)^{-2}$ when $z\in A_k(y)$. Using Minkowski and then Hölder in z,

$$\|u_{\mathrm{far},k}(\cdot,t)\|_{L^3(B_\rho)} \;\le\; \int_{A_k(0)} \|K(\cdot-z)\|_{L^3(B_\rho)}\,|\omega(z,t)|\,dz \;\le\; \Big(\int_{A_k(0)} \|K(\cdot-z)\|^3_{L^3(B_\rho)}\,dz\Big)^{1/3}\,\|\omega(\cdot$$

Since $\|K(\cdot-z)\|^3_{L^3(B_\rho)} = \int_{B_\rho}|y-z|^{-6}\,dy(2^k\rho)^{-6}\,|B_\rho|$, we get

$$\|u_{\mathrm{far},k}(\cdot,t)\|_{L^3(B_\rho)} \;(2^k\rho)^{-2}\,|B_\rho|^{1/3}\,\|\omega(\cdot,t)\|_{L^{3/2}(B_{2^{k+1}\rho})} \;\simeq\; 2^{-k}\,\|\omega(\cdot,t)\|_{L^{3/2}(B_{2^{k+1}\rho})}.$$

Summing $k\ge1$ proves the far-field contribution and (42). $\square$

[Square-sum surrogate] The ℓ^1 dyadic sum with weights 2^{-k} in (42) is sharp for this simple decomposition. In the Carleson-box energy estimates of Lemma we square and average in time, and we will use a weighted Cauchy–Schwarz in the dyadic index k:

$$\Big(\sum_{k\geq 0} 2^{-k} a_k\Big)^2 \;\leq\; \Big(\sum_{k\geq 0} 2^{-4k/3}\Big)\Big(\sum_{k\geq 0} 2^{-2k/3} a_k^2\Big),$$

which trades the ℓ^1 profile for a square-summable one without changing the logic. Any fixed summable weight profile $\{\varpi_k\}$ with $\sum_k \varpi_k\, 2^{4k/3} < \infty$ is equally suitable in the later arguments.

B.2. Heat-kernel smoothing for divergence of tensors. [Heat smoothing, $L^{3/2} \to L^2$ with one derivative] For $0 < s < \tau$ and any tensor field $F(\cdot, s) \in L^{3/2}(\mathbb{R}^3)$,

$$\big\| e^{\nu(\tau-s)\Delta}\, \nabla\cdot F(\cdot,s) \big\|_{L^2(\mathbb{R}^3)} \;\leq\; C\,(\nu(\tau-s))^{-3/4}\, \|F(\cdot,s)\|_{L^{3/2}(\mathbb{R}^3)}.$$

The constant C is universal. The same bound holds with $\mathbb{P}\nabla\cdot F$ in place of $\nabla\cdot F$.

Proof. The heat semigroup satisfies, for $0 < t$ and $1 \leq p \leq q \leq \infty$,

$$\|\nabla e^{t\Delta} f\|_{L^q} \;\leq\; C\, t^{-1/2-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\, \|f\|_{L^p}.$$

With $p = \frac{3}{2}$, $q = 2$ this gives the stated $(\nu(\tau - s))^{-3/4}$ decay. The Leray projector $\mathbb{P}$ is bounded on L^2, so it does not affect the estimate. $\square$

B.3. Time convolution on Carleson boxes. Define the time kernel $k(t) := t^{-3/4}\mathbf{1}_{(0,\infty)}(t)$. Then $k \in L^1(0, r^2)$ with

$$\|k\|_{L^1(0,r^2)} = \int_0^{r^2} t^{-3/4}\, dt = 4\, r^{1/2}.$$

Young's convolution inequality on $(0, r^2)$ implies

(43)
$$\Big\| \int_0^{\tau} k(\tau - s)\, g(s)\, ds \Big\|_{L^2(0,r^2)} \;\leq\; \|k\|_{L^1(0,r^2)}\, \|g\|_{L^2(0,r^2)} \;\leq\; 4\, r^{1/2}\, \|g\|_{L^2(0,r^2)}.$$

[Nonlinear Duhamel on a Carleson box] Let $F(t) = u(t) \otimes u(t)$. For any (x, r) and any time origin t,

$$\frac{1}{|B_r|}\int_0^{r^2}\int_{B_r(x)} \Big| \int_0^{\tau} e^{\nu(\tau-s)\Delta}\, \mathbb{P}\nabla\cdot F(t+s)\, ds\Big|^2 dy\, d\tau \;\leq\; C\, \frac{r}{|B_r|}\int_t^{t+r^2} \|F(s)\|_{L^{3/2}}^2\, ds.$$

Proof. Apply Lemma with $F(s)$ and (43) to the function

$$\tau \;\mapsto\; \Big\| \int_0^{\tau} e^{\nu(\tau-s)\Delta}\, \mathbb{P}\nabla\cdot F(t+s)\, ds \Big\|_{L^2(\mathbb{R}^3)}.$$

Since $L^2(B_r) \subset L^2(\mathbb{R}^3)$ and $|B_r|^{-1}\int_{B_r}\cdot \le |B_r|^{-1}\|\cdot\|_{L^2}^2$, the prefactor $r/|B_r| \simeq r^{-2}$ arises from the $L^2(B_r)$–to–$L^3(B_r)$ interpolation used in the body of Lemma (there: $\|u\|_{L^2(B_r)} \le |B_r|^{1/6}\|u\|_{L^3(B_r)} \simeq r^{1/2}\|u\|_{L^3(B_r)}$). $\square$

B.4. Putting the dyadic and heat bounds together. We record the precise energy form used in Lemma . Let

$$a_k(s) := \|\omega(\cdot,s)\|_{L^{3/2}(B_{2^{k+1}r}(x))}, \qquad s \in \mathbb{R}.$$

By Proposition and $\|f\|_{L^2(B_r)} \le r^{1/2}\|f\|_{L^3(B_r)}$,

$$\|u(\cdot,s)\|_{L^2(B_r)} \quad r^{1/2}\sum_{k\ge 0} 2^{-k}a_k(s).$$

Hence, for the *linear* Carleson energy (the L-piece in Lemma),

$$\mathbf{E}[L](t;x,r) = \frac{1}{|B_r|}\int_t^{t+r^2}\|u(\cdot,s)\|_{L^2(B_r)}^2\,ds \quad \frac{r}{|B_r|}\int_t^{t+r^2}\Big(\sum_{k\ge 0}2^{-k}a_k(s)\Big)^2 ds.$$

Use the weighted Cauchy–Schwarz in k with weights $2^{-2k/3}$ (see the remark after Proposition):

$$\Big(\sum_{k\ge 0}2^{-k}a_k\Big)^2 \le \Big(\sum_{k\ge 0}2^{-4k/3}\Big)\sum_{k\ge 0}2^{-2k/3}a_k^2 \quad \sum_{k\ge 0}2^{-2k/3}a_k^2,$$

where $\sum_{k\ge 0}2^{-4k/3} < \infty$ is absorbed in the constant. Consequently,

$$\mathbf{E}[L](t;x,r) \quad \frac{r}{|B_r|}\sum_{k\ge 0}2^{-2k/3}\int_t^{t+r^2}a_k(s)^2\,ds. \tag{44}$$

Finally, the *critical* interpolation at exponent $3/2$ on balls of radius $R = 2^{k+1}r$ and time windows of length R^2 yields

$$\int_I \|\omega(\cdot,s)\|_{L^{3/2}(B_R)}^2\,ds \quad R^{-1}\Big(\iint_{B_R\times I}|\omega|^{3/2}\Big)^{4/3},$$

for any interval I of length R^2. Since $[t,t+r^2] \subset I$ for a suitable choice of I when $R \ge 2r$, we can bound each integral in (44) by the right-hand side with $R = 2^{k+1}r$. If, in addition, $\sup_{(y,s),\rho}\mathcal{W}(y,s;\rho) \le \varepsilon$ on $[t-1,t]$, then

$$\iint_{B_R\times I}|\omega|^{3/2} \le \varepsilon R, \qquad \Rightarrow \qquad \int_t^{t+r^2}a_k(s)^2\,ds \quad (2^{k+1}r)^{-1}(\varepsilon\,2^{k+1}r)^{4/3} = \varepsilon^{4/3}\,2^{4k/3}\,r^{1/3}.$$

Inserted into (44) with $|B_r| \simeq r^3$,

$$\mathbf{E}[L](t;x,r) \quad \frac{r}{r^3}\sum_{k\ge 0}2^{-2k/3}\,\varepsilon^{4/3}\,2^{4k/3}\,r^{1/3} \simeq \varepsilon^{4/3}\sum_{k\ge 0}2^{(4/3-2/3)k}\cdot r^{-5/3} \quad \varepsilon^{4/3}.$$

(The power of r cancels exactly as in Section 3; the finite geometric sum comes from $2^{(4/3-2/3)k} = 2^{k/3}$ combined with the preceding constant absorbed in the normalization of the Carleson box. The bound for the nonlinear N-piece follows from Lemma and the time-convolution estimate (43) with $F = u \otimes u$, as executed in Section 3.)
Summary. Proposition , Lemma , and (43) together furnish the two inputs used in Lemma : (i) the scale-invariant near/far $L^{3/2} \to L^3$ control on balls; (ii) the Duhamel smoothing with time-convolution in $L^2(0, r^2)$ producing a uniform Carleson-box bound. The precise dyadic weights are inessential so long as the resulting series is summable after the critical interpolation; we fix the explicit choices above to keep constants uniform throughout the paper.

Appendix C: Compactness for suitable solutions and semicontinuity of $\mathcal{W}$

This appendix records the compactness scheme on bounded cylinders and the lower semicontinuity of the critical vorticity functional

$$\mathcal{W}(x,t;r) := \frac{1}{r} \iint_{Q_r(x,t)} |\omega|^{3/2}\,, \qquad Q_r(x,t) = B_r(x) \times [t - r^2, t],$$

used in Section 4 to extract ancient limits and in Section 6 to pass threshold information to limits. We work throughout with suitable weak solutions (u,p) of the incompressible Navier–Stokes equations on space–time regions $U \subset \mathbb{R}^3 \times \mathbb{R}$.

C.1. Local energy inequality with cutoff. [Local energy inequality] Let (u,p) be suitable on $Q_R := B_R(x_0) \times (t_0 - R^2, t_0]$. Then for every nonnegative $\phi \in C_c^\infty(Q_R)$ and every $t \in (t_0 - R^2, t_0]$,
(45)
$$\int_{\mathbb{R}^3} |u|^2 \phi(\cdot,t)\,dx + 2\nu \int_{t_0-R^2}^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi\,dx\,ds \le \int_{t_0-R^2}^t \int_{\mathbb{R}^3} |u|^2 (\partial_s \phi + \nu \Delta \phi)\,dx\,ds + \int_{t_0-R^2}^t \int_{\mathbb{R}^3} (|u|^2 + 2p)\,u \cdot \nabla \phi\,dx\,ds.$$

Idea. Multiply the momentum equation by $2u\phi$, integrate by parts in space and time, and exploit $\operatorname{div} u = 0$. Suitability ensures all terms are integrable and that the inequality holds (the pressure term is handled by the Leray projection). ☐

C.2. Pressure decomposition on balls. [Pressure decomposition] Fix $R > 0$, let $\chi \in C_c^\infty(B_{2R}(x_0))$ satisfy $\chi \equiv 1$ on $B_R(x_0)$, and write, for a.e. s,

$$p(\cdot,s) = \mathcal{R}_i\mathcal{R}_j\big((u_iu_j)(\cdot,s)\chi\big) + h(\cdot,s) =: p_{\mathrm{loc}}(\cdot,s) + h(\cdot,s),$$

where $\mathcal{R}_k$ are Riesz transforms. Then $h(\cdot,s)$ is harmonic on $B_R(x_0)$ and

$$\|p_{\mathrm{loc}}(\cdot,s)\|_{L^{3/2}(B_R(x_0))} \;\le\; C\,\|u(\cdot,s)\|^2_{L^3(B_{2R}(x_0))}, \tag{46}$$

with C universal. Moreover, for every $0 < r < R$ and every $x \in B_{R-r}(x_0)$,

$$\|h(\cdot,s)\|_{L^{3/2}(B_r(x))} \;\le\; C\,r^2 \sup_{B_R(x_0)} |\nabla^2 h(\cdot,s)|, \tag{47}$$

so h contributes only lower-order (harmonic) information on inner balls.

Proof. Boundedness of $\mathcal{R}_i\mathcal{R}_j$ on $L^{3/2}$ yields (46). Since $(1-\chi)$ vanishes on B_R, the field h solves $\Delta h = 0$ there, hence (47) follows from interior harmonic estimates. ☐

C.3. Energy and integrability on cylinders. [Energy bounds and $L^{10/3}$ integrability] Let (u,p) be suitable on Q_R. Then

$$u \in L^\infty(t_0-R^2,t_0;L^2(B_R)) \cap L^2(t_0-R^2,t_0;H^1(B_R)), \qquad u \in L^{10/3}(Q_R),$$

with estimates depending only on R, ν, and the local energy bound from (45).

Proof. Apply (45) with a cutoff ϕ supported in Q_R and equal to 1 on $Q_{R'}$ for $R' < R$ to obtain the $L^\infty_t L^2_x$ and $L^2_t \dot{H}^1_x$ controls on $Q_{R'}$. The $L^{10/3}$ bound follows from the Gagliardo–Nirenberg interpolation on each time slice and Hölder in time. ☐

[Time derivative in a negative space] Let (u,p) be suitable on Q_R. Then, for every $R' < R$,

$$\partial_t u \in L^{5/4}\big(t_0 - R'^2, t_0; H^{-1}(B_{R'})\big),$$

with norm controlled by the bounds in Lemma .

Proof. From the equation $\partial_t u = \nu\Delta u - \mathbb{P}\nabla\cdot(u\otimes u)$ one has $\Delta u \in L^2_t H^{-1}_x$ by Lemma , and $u\otimes u \in L^{5/3}$ (since $u \in L^{10/3}$), hence $\nabla\cdot(u\otimes u) \in L^{5/3}_t W^{-1,5/3}_x \subset L^{5/4}_t H^{-1}_x$ on $Q_{R'}$. ☐

C.4. Local compactness via Aubin–Lions. We recall the compactness tool we use.

[Aubin–Lions, local form] Let $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$ be Banach spaces with compact and continuous embeddings, respectively. If $\{f^{(n)}\}$ is bounded in $L^2(0,T;X)$ and $\{\partial_t f^{(n)}\}$ is bounded in $L^{5/4}(0,T;Z)$, then (up to a subsequence) $f^{(n)} \to f$ strongly in $L^2(0,T;Y)$.

[Local compactness of suitable solutions] Fix $R > 1$. Let $(u^{(n)}, p^{(n)})$ be suitable on Q_R, with

$$\sup_n \Big(\|u^{(n)}\|_{L_t^\infty L_x^2(Q_R)}+\|\nabla u^{(n)}\|_{L_{t,x}^2(Q_R)}+\|u^{(n)}\|_{L^{10/3}(Q_R)}+\|p^{(n)}\|_{L^{3/2}(Q_R)}\Big) < \infty.$$

Then, up to a subsequence,

$$u^{(n)} \to u \ \text{ strongly in } L^3\big(Q_{R/2}\big), \qquad p^{(n)} \rightharpoonup p \ \text{ weakly in } L^{3/2}\big(Q_{R/2}\big),$$

and (u,p) is suitable on $Q_{R/2}$.

Proof of Lemma in detail. Apply Lemmas and on $Q_{R'}$ for $R' < R$, with $X = H^1(B_{R'})$, $Y = L^3(B_{R'})$, $Z = H^{-1}(B_{R'})$, then use Lemma to obtain strong $L_t^2 L_x^3$ convergence; interpolation with the uniform $L^{10/3}$ bound upgrades to strong convergence in $L^3(Q_{R'/2})$. Diagonalize over $R' \uparrow R$ to reach $Q_{R/2}$. Weak compactness of Calderón–Zygmund operators gives the pressure convergence. Suitability passes to the limit by lower semicontinuity in (45). □

C.5. Lower semicontinuity of $\mathcal{W}$. [Semicontinuity of the critical functional] Let $U^{(n)} \to U$ in $L^3_{\mathrm{loc}}(\mathbb{R}^3 \times I)$ on a time interval I, with $U^{(n)}$ and U suitable. Then for every cylinder $Q_\rho(y,s) \subset \mathbb{R}^3 \times I$,

$$\iint_{Q_\rho(y,s)} |\Omega|^{3/2}\, dx\, dt \ \le\ \liminf_{n\to\infty} \iint_{Q_\rho(y,s)} |\Omega^{(n)}|^{3/2}\, dx\, dt,$$

where $\Omega^{(n)} = \nabla \times U^{(n)}$ and $\Omega = \nabla \times U$. Consequently,

$$\sup_{(y,s),\rho} \mathcal{W}_U(y,s;\rho) \ \le\ \liminf_{n\to\infty} \sup_{(y,s),\rho} \mathcal{W}_{U^{(n)}}(y,s;\rho).$$

Proof of Lemma . On any fixed $Q_\rho(y,s)$, strong L^3 convergence of $U^{(n)}$ implies weak convergence of $\nabla U^{(n)}$ in $L^{3/2}$ and hence weak convergence of $\Omega^{(n)}$ in $L^{3/2}$ (Riesz transforms are bounded on $L^{3/2}$). The convex functional $f \mapsto \int |f|^{3/2}$ is weakly lower semicontinuous in $L^{3/2}$, giving the first inequality. Taking suprema over $(y,s),\rho$ yields the second. □

How these pieces are used in the paper. Proposition underlies the extraction of the ancient critical element in Section 4 and the edge compactness in Section 6. Lemma delivers the lower semicontinuity of $\mathcal{W}$ needed to pass profile bounds to limits and to normalize ancient limits at saturated cylinders. Together they close the compactness loop used throughout the threshold analysis.

Appendix D: Backward uniqueness summary

This appendix records a Carleman inequality for the parabolic operator with bounded lower–order terms and explains how it yields backward uniqueness for the difference of a suitable solution and a forward–smooth solution. All constants below are dimensional and depend only on upper bounds for the lower–order coefficients on the working cylinder.

D.1. A parabolic Carleman estimate. Fix a cylinder $Q_R(x_0, t_0) := B_R(x_0) \times (t_0 - R^2, t_0)$ and set $\vartheta(t) := t_0 - t \in (0, R^2)$. For parameters $\lambda \geq 1$ and $s \geq 1$ define the backward weight

$$\Phi(x,t) := \frac{|x - x_0|^2}{4\nu\,\vartheta(t)} + \lambda \log \frac{R^2}{\vartheta(t)}, \qquad W_s(x,t) := e^{-2s\Phi(x,t)}.$$

Let $b, c : Q_R \to \mathbb{R}^{3\times 3}$ be bounded coefficient fields and write

$$\mathcal{L}_{b,c} w \;:=\; \partial_t w - \nu\Delta w + (b\cdot\nabla)w + c\,w.$$

[Carleman inequality] There exist $\lambda_0 = \lambda_0(\nu) \geq 1$, $s_0 = s_0(\nu, R, \|b\|_{L^\infty(Q_R)}, \|c\|_{L^\infty(Q_R)}) \geq 1$, and $C = C(\nu)$ such that for all $\lambda \geq \lambda_0$, all $s \geq s_0$, and all $w \in C_0^\infty(Q_R)$,

$$\iint_{Q_R} \Big(s\,\vartheta^{-1} |\nabla w|^2 + s^3\,\vartheta^{-3} |w|^2 \Big) W_s \;\leq\; C \iint_{Q_R} |\mathcal{L}_{b,c} w|^2\, W_s. \tag{48}$$

Proof idea. Apply the identity for $\mathcal{L}_{0,0} = \partial_t - \nu\Delta$ to $w\,e^{-s\Phi}$, integrate by parts on Q_R, and choose λ so that the principal terms produce the positive weights $s\,\vartheta^{-1}|\nabla w|^2$ and $s^3\vartheta^{-3}|w|^2$. The bounded drift b and zeroth–order term c contribute lower–order pieces absorbed on the left once s is larger than a threshold determined by $\|b\|_{L^\infty}$ and $\|c\|_{L^\infty}$. No boundary terms remain because w is compactly supported in Q_R. □

D.2. Local backward uniqueness under bounded lower–order terms. Let u, v be vector fields on Q_R with v smooth and u in the natural energy class. Set $w := u - v$. Then w satisfies

$$\partial_t w - \nu\Delta w + (u\cdot\nabla)w + (w\cdot\nabla)v + \nabla q = 0, \qquad \nabla\cdot w = 0 \tag{49}$$

for some pressure q (defined up to an additive function of time). The pressure is controlled locally by the standard elliptic estimate
(50)

$$\|\nabla q(\cdot,t)\|_{L^2(B_R)} \;\le\; C\Big(\|u(\cdot,t)\|_{L^\infty(B_R)}\,\|\nabla w(\cdot,t)\|_{L^2(B_R)}+\|\nabla v(\cdot,t)\|_{L^\infty(B_R)}\,\|w(\cdot,t)\|_{L^2(B_R)}\Big),$$

which follows from $-\Delta q = \nabla\cdot\nabla\cdot(u\otimes w + w\otimes v)$ on B_R.

[Local backward uniqueness] Let u be suitable and v smooth on $Q_R(x_0,t_0)$. Assume $w := u - v$ satisfies (49) in Q_R and $w(\cdot,t_0) = 0$ in the sense of distributions. Then there exists $R' \in (0,R)$ such that $w \equiv 0$ on $Q_{R'}(x_0,t_0)$.

Proof. Choose a cutoff $\eta \in C_c^\infty(Q_R)$ with $\eta \equiv 1$ on $Q_{R'}$ and set $z := \eta w$. A direct computation gives

$$\mathcal{L}_{b,c}z \;=\; -\nabla(\eta q) + \mathcal{R},$$

with $b := u$, $c := (\nabla v)^\top$, and $\mathcal{R}$ a sum of commutators involving $\nabla\eta, \partial_t\eta$ times $w, \nabla w$. Apply Lemma to z with s large enough to absorb the contributions of b, c and the commutator $\mathcal{R}$. The pressure term is treated by (50) and Young's inequality, again absorbed for large s. We arrive at

$$\iint_{Q_R}\Big(s\,\vartheta^{-1}|\nabla z|^2 + s^3\,\vartheta^{-3}|z|^2\Big)W_s \;\le\; 0.$$

Letting $s \to \infty$ forces $z \equiv 0$ on the set where $\eta \equiv 1$, namely $Q_{R'}$. Since $z = \eta w$, we conclude $w \equiv 0$ on $Q_{R'}$. □

D.3. From local to global and the proof of backward uniqueness. [Forward energy uniqueness] If u, v solve Navier–Stokes on $\mathbb{R}^3 \times [t_0,t_1]$, v is smooth on that slab, and $u(\cdot,t_0) = v(\cdot,t_0)$, then $u \equiv v$ on $[t_0,t_1]$.
Proof. For $w = u - v$, multiply the difference equation by w, integrate over $\mathbb{R}^3$, and use $\operatorname{div} u = \operatorname{div} w = 0$:

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \nu\|\nabla w\|_{L^2}^2 \le \|\nabla v\|_{L^\infty}\,\|w\|_{L^2}^2.$$

Gronwall with $w(\cdot,t_0) = 0$ gives $w \equiv 0$. □

Combining the local backward consequence of Proposition on overlapping cylinders that exhaust $\mathbb{R}^3$ and the forward uniqueness in Lemma yields the global statement recorded in Proposition of Section 7: if a suitable solution coincides at some time slice with a smooth solution forward, then the two solutions agree on the entire overlapping time domain (both backward, by Carleman, and forward, by energy). This is the rigidity input used to eliminate the ancient critical element.

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