Riemann Proof Map (Bounded-Real Route via Product Certificate)

Ambient domain. Let $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$. Write $s = \frac{1}{2} + \sigma + it$, $\sigma > 0$. Let $\xi(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$.

Objects. On $\ell^2(\mathcal{P})$ let $A(s)e_p := p^{-s}e_p$. Define the outer-normalized ratio

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\mathcal{O}(s)\,\xi(s)}, \qquad \Theta(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

Target: $2\mathcal{J}$ is Herglotz on Ω (i.e. $\Re(2\mathcal{J}) \geq 0$) $\iff \Theta$ is Schur on Ω.

Window & geometry. Fix even $\psi \in C^\infty$ with $\psi \equiv 1$ on $[-1,1]$, $0 \leq \psi \leq 1$, compactly supported. For an interval $I = [T-L, T+L]$ with $L \leq 1/\log\langle T\rangle$ and aperture $\alpha \in [1,2]$, the Carleson box is

$$Q(\alpha I) := \{(\sigma,t) :\ |t-T| \leq \alpha L,\ \ 0 < \sigma \leq \alpha|I|\}, \qquad \Omega(I) := \{\tfrac{1}{2} + \sigma + it :\ (\sigma,t) \in Q(\alpha I)\} \subset \Omega.$$

Boundary phase density. With boundary phase $w(t)$ one has on I

$$\int_I \big(-w'(t)\big)\,dt \;\leq\; \pi\,\mu\big(Q(I)\big),$$

where μ is the Carleson/energy measure induced by the neutralized potential.

Locked certificate to (P+). The independent bounds are:

$$c_0(\psi) = 0.17620819, \quad C_H(\psi) = 0.65, \quad C_P(\kappa) = 0.020, \quad C_\psi^{(H^1)} = 0.2400, \quad C_{\mathrm{box}} = 0.067022.$$

The fixed-aperture embedding gives

$$M_\psi \;\leq\; \frac{4}{\pi}\,C_\psi^{(H^1)}\sqrt{C_{\mathrm{box}}}\ \ (\text{analytic enclosure } < 0.245) \quad \Rightarrow \quad M_\psi \leq 0.0800745.$$

Hence the boundary certificate

$$\Upsilon \;:=\; \frac{C_H(\psi)\,M_\psi + C_P(\kappa)}{c_0(\psi)} \;=\; 0.408882385 \;\leq\; \tfrac{1}{2} \;<\; \tfrac{\pi}{2},$$

which forces $w \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ a.e. on $\partial\Omega(I)$; this is (P+).

Poisson & Cayley. From (P+) and the Poisson integral, $\Re\big(2\mathcal{J}\big) \geq 0$ on $\Omega(I)$, so $2\mathcal{J}$ is Herglotz. Therefore the Cayley transform is Schur: $|\Theta| \leq 1$ on $\Omega(I)$.

Removable singularities at $Z(\xi)$. If $\rho \in Z(\xi)$, boundedness of Θ on a punctured disc implies Θ (and hence $\mathcal{J}$) extends holomorphically across ρ.

Globalization/pinch. Exhausting Ω by overlapping boxes and using the maximum-modulus pinch excludes zeros of ξ inside Ω. By the functional equation, all nontrivial zeros lie on $\Re s = \frac{1}{2}$; thus RH.

Independence (no circularity).

- $C_{\mathrm{box}} = K_0 + K_\xi + \|U_\Gamma\|_{\mathrm{area}}$ with (i) prime tails K_0, (ii) ξ-block K_ξ, (iii) Archimedean Γ-term; each bounded independently.
- $M_\psi \leq \frac{4}{\pi}C_\psi^{(H^1)}\sqrt{C_{\mathrm{box}}}$ via fixed-aperture H^1–BMO/Carleson embedding (one-time numeric evaluation).

PSC status. The Prime-Tail Schur-Covering inequality closes numerically, but the proof chain to (P+) and RH above runs solely through the *product* certificate Υ.