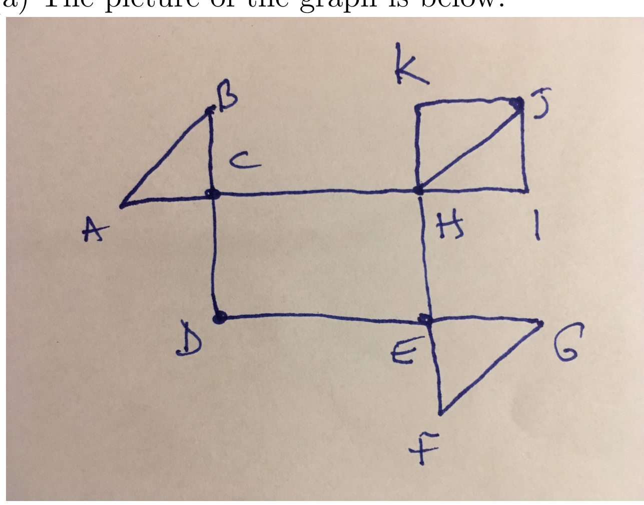
MAU22C00: ASSIGNMENT 4 SOLUTIONS

1) (30 points) Let (V, E) be the graph with vertices a, b, c, d, e, f, g, h, i, j, and k, and edges ab, bc, ac, cd, ch, de, eh, ef, eg, fg, hi, ij, hk, jk, and jh.

(a) Draw this graph.
(b) Write down this graph's incidence table and its incidence matrix.
(c) Write down this graph's adjacency table and its adjacency matrix.
(d) Is this graph complete? Justify your answer.
(e) Is this graph bipartite? Justify your answer.
(f) Is this graph regular? Justify your answer.
(g) Does this graph have any regular subgraph? Justify your answer.
(h) Give an example of an isomorphism φ from the graph (V, E) to itself satisfying that $\varphi(i) = k$.
(i) Is the isomorphism from part (h) unique or can you find another isomorphism ψ that is distinct from φ but also satisfies that $\psi(i) = k$? Justify your answer.
(j) Is this graph connected? Justify your answer.
(k) Does this graph have an Eulerian trail? Justify your answer.
(l) Does this graph have an Eulerian circuit? Justify your answer.
(m) Does this graph have a Hamiltonian circuit? Justify your answer.
(n) Is this graph a tree? Justify your answer.

1(a) The picture of the graph is below.



(b) (2 points: 1 point each the table and the matrix) The incidence table is

	ab	bc	ac	cd	ch	de	eh	ef	eg	fg	hi	ij	hk	jk	jh
a	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
b	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
c	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
d	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
e	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
f	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0
g	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
h	0	0	0	0	1	0	1	0	0	0	1	0	1	0	1
i	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
j	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
k	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0

and the incidence matrix is

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}.$$

(c) (2 points: 1 point each the table and the matrix) The adjacency table is

	a	b	c	d	e	f	g	h	i	j	k
a	0	1	1	0	0	0	0	0	0	0	0
b	1	0	1	0	0	0	0	0	0	0	0
c	1	1	0	1	0	0	1	0	0	0	0
d	0	0	1	0	1	0	0	1	0	0	0
e	0	0	0	1	0	1	1	1	0	0	0
f	0	0	0	0	1	0	1	0	0	0	0
g	0	0	0	0	1	1	0	0	0	0	0
h	0	0	1	0	1	0	0	0	1	1	1
i	0	0	0	0	0	0	0	1	0	1	0
j	0	0	0	0	0	0	0	1	1	0	1
k	0	0	0	0	0	0	0	1	0	1	0

and the adjacency table is

$$\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}.$$

(d) (2 points: 1 for the answer and 1 for the justification) No, for example edge af is not part of the graph.

(e) (2 points: 1 for the answer and 1 for the justification) No, the graph contains a complete subgraph given by vertices a, b, and c, and edges ab, bc, ac. Given that in this subgraph every vertex is connected to every other vertex, and this complete subgraph has more than two vertices, it cannot be partitioned into two sets so that every edge goes from a vertex in one set to a vertex in the other set.

(f) (2 points: 1 for the answer and 1 for the justification) No, $\deg(a) = 2$, while $\deg(h) = 5$.

(g) (2 points: 1 for the answer and 1 for the justification) Yes, it contains the regular subgraph given by vertices a, b, and c, and edges ab, bc, ac.

(h) (4 points: points decked if not all vertex assignments in the map lead to an isomorphism) $\varphi(a) = b$, $\varphi(b) = a$, $\varphi(c) = c$, $\varphi(d) = d$, $\varphi(e) = e$, $\varphi(f) = g$, $\varphi(g) = f$, $\varphi(h) = h$, $\varphi(i) = k$, $\varphi(k) = i$, and $\varphi(j) = j$.

(i) (2 points: 1 for the answer and 1 for the justification) No, it is not unique. The isomorphism $\varphi(a) = a$, $\varphi(b) = b$, $\varphi(c) = c$, $\varphi(d) = d$, $\varphi(e) = e$, $\varphi(f) = f$, $\varphi(g) = g$, $\varphi(h) = h$, $\varphi(i) = k$, $\varphi(k) = i$, and $\varphi(j) = j$ also exchanges vertices i and k but is distinct from the one given in part (h).

(j) (2 points: 1 for the answer and 1 for the justification) Yes, there is a walk from every vertex to every other vertex.

(k) (2 points: 1 for the answer and 1 for the justification) Yes, as we have exactly two vertices with odd degree ($\deg(h) = 5$ and $\deg(j) = 3$)

while the rest of the vertices have even degrees. Our Eulerian trail would have to start at h and end at j or the other way around.

(l) (2 points: 1 for the answer and 1 for the justification) No, as we would need all vertices to have even degree. Here we have two vertices of odd degree ($\deg(h) = 5$ and $\deg(j) = 3$).

(m) (2 points: 1 for the answer and 1 for the justification) No, since to visit vertices a and b we would have to visit vertex c twice.

(n) (2 points: 1 for the answer and 1 for the justification) No, as there is a circuit $abca$. There are in fact quite a number of circuits.

2) (10 points) Prove that a connected graph (V, E) is a tree if and only if adding an edge between any two vertices in (V, E) creates exactly one circuit.

This statement is an equivalence. To prove it, we will prove each implication separately. Remember that a tree is a connected graph with no cycles (circuits). Since the graph (V, E) is already assumed to be connected, our equivalence amounts to (V, E) contains no circuits if and only if adding an edge between any two vertices in (V, E) creates exactly one circuit.

"$\Leftarrow$" We start with the hypothesis that adding an edge between any two vertices in (V, E) creates exactly one circuit. Remove the added edge to get back to (V, E). Since adding that edge created exactly one circuit, it is clear that (V, E) itself has no circuits, hence (V, E) is a tree.

"$\Rightarrow$" We start with the hypothesis that (V, E) is a tree, hence that it has no circuits. Only an edge that does not already exist in E can be added. Let $u, v \in V$ be distinct vertices $u \neq v$ in V such that there is no edge between them, namely $uv \notin E$. Recall that we proved in lecture that between any two distinct vertices of a tree there exists one and only one path. Let the path in (V, E) between u and v be given by $uw_1w_2 \cdots w_pv$ for some $w_1, w_2, \ldots, w_p$ vertices in V. Clearly, adding the edge uv to E creates one and only one circuit, namely $uw_1w_2 \cdots w_pvu$ because there is no other path between u and v except for $uw_1w_2 \cdots w_pv$. $\square$

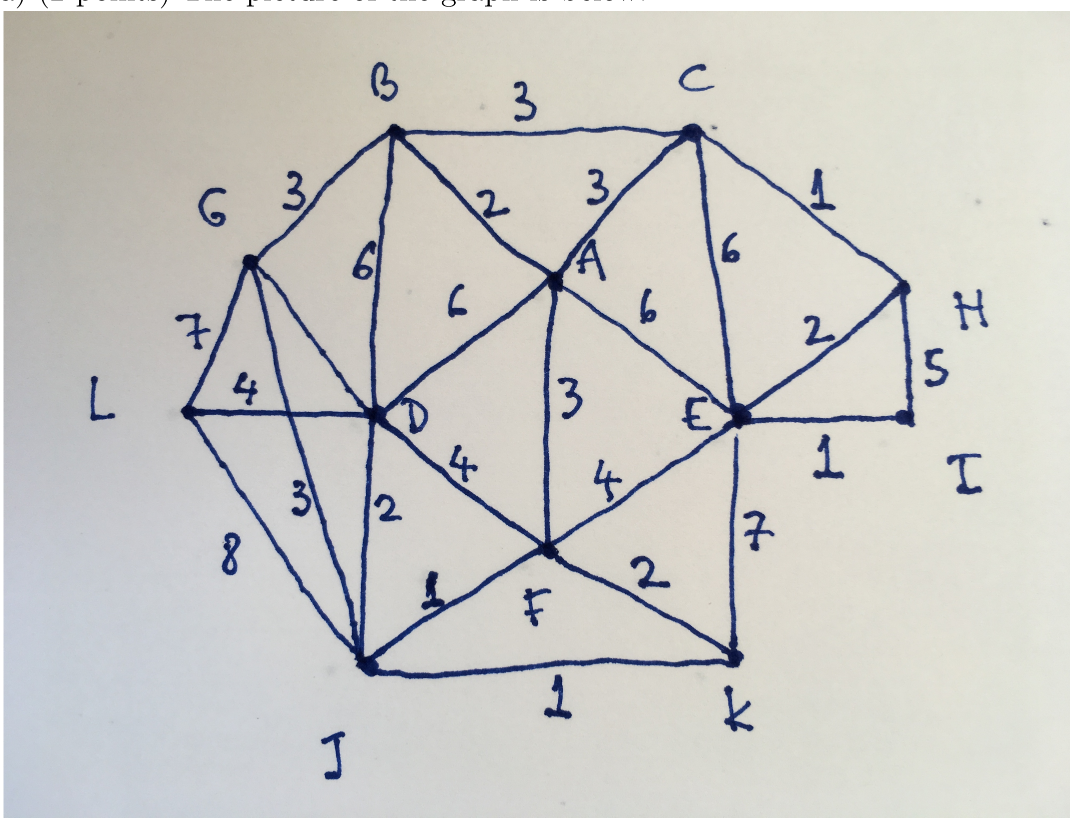
Grading rubric: 10 points, 5 points for each implication in the equivalence.

3) Consider the connected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L and with edges, listed with associated costs, in the following table:

FJ	JK	CH	EI	DJ	AB	EH	FK	BG	GJ	BC	AF
1	1	1	1	2	2	2	2	3	3	3	3
AC	EF	DF	DL	HI	CE	BD	AD	AE	GL	EK	JL
3	4	4	4	5	6	6	6	6	7	7	8

(a) (2 points) Draw the graph and label each edge with its cost.
(b) (9 points) Determine the minimum spanning tree generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.
(c) (9 points) Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex D, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

3(a) (2 points) The picture of the graph is below:



Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

3(b) The edges are added in the following order: FJ, JK, CH, EI, DJ, AB, EH, BG, GJ, BC, and DL.

Grading rubric: 9 points: roughly 1 point per correct edge in the correct order.

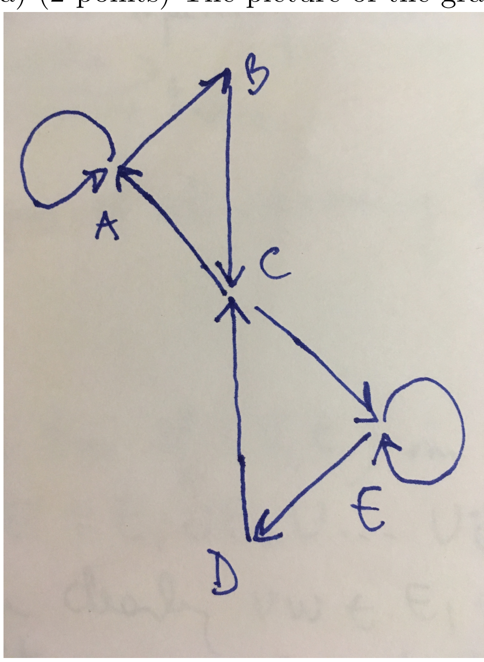
3(c) The edges are added in the following order: DJ, FJ, JK, GJ, BG, AB, BC, CH, EH, EI, and DL.

Grading rubric: 9 points: roughly 1 point per correct edge in the correct order.

4) (10 points) Let $(\mathcal{V}, \mathcal{E})$ be the directed graph with vertices A, B, C, D, and E and edges (A, A), (C, A) (A, B), (B, C), (D, C), (C, E), (E, E), and (E, D).

(a) Draw this graph.
(b) Write down this graph's adjacency matrix.
(c) Give an example of an isomorphism φ from the graph $(\mathcal{V}, \mathcal{E})$ to itself such that $\varphi(A) = E$. Note that an isomorphism of directed graphs should also respect the direction of the edges.

4(a) (2 points) The picture of the graph is below.



Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

4(b) The adjacency matrix is as follows (note that we must stick to the given ordering of vertices; A, B, C, D, E):

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

4(c) (6 points) $\varphi(A) = E$, $\varphi(E) = A$, $\varphi(C) = C$, $\varphi(B) = D$, and $\varphi(D) = B$ respects the direction of the edges. Note it takes advantage of the natural symmetry in this directed graph.

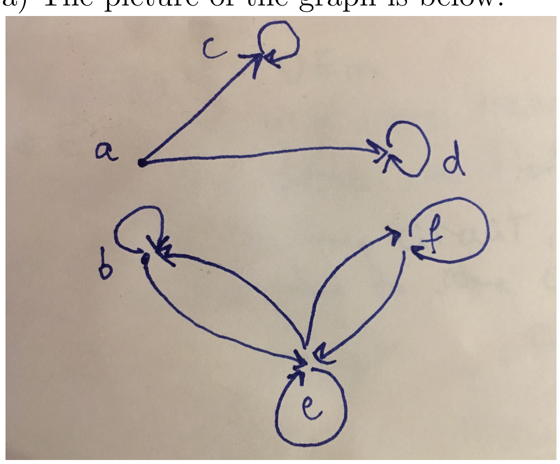
Grading rubric: Roughly 1 point for each correct vertex assignment in the isomorphism map.

5) (10 points) Let R be a relation on a set $V = \{a, b, c, d, e, f\}$ given by

$R = \{(a, c), (a, d), (c, c), (d, d), (b, b), (e, e), (b, e), (e, f), (f, e), (e, b), (f, f)\}$.

(a) Using the one-to-one correspondence between relations on finite sets and directed graphs, draw the directed graph corresponding to the relation R.
(b) Is R an equivalence relation? Justify your answer.
(c) If R is not an equivalence relation, which ordered pairs would have to be added to R to make it into an equivalence relation?

5(a) The picture of the graph is below.



Grading rubric: 2 points: 1 point if the 6 vertices are there, and the extra point for the edges. If only 1, 2 edges are incorrect, full marks given.

5(b) Not an equivalence relation as it is not reflexive (edge (a,a) is missing), not symmetric (edges (c,a) and (d,a) are missing) and transitive (for example, edges (b,e) and (e,f) are there, but edge (b,f) is missing).
Grading rubric: 2 points: 1 point for the answer and 1 points for the justification.

5(c) For reflexivity to be satisfied, add edge (a,a). For symmetry to be satisfied, we add (c,a) and (d,a). To satisfy transitivity in the subgraph with vertices b, e, and f, edges (b,f) and (f,b) need to be added. Given that we have already added (c,a) and (d,a), to achieve transitivity in the subgraph with vertices a, c, and d, we must add (c,d) and (d,c).
Grading rubric: 6 points, roughly a point for each of the seven pairs. No justification is required for this part in fact, so points are awarded strictly for having written down the correct pairs.