Sample Questions - 7

Q1. $f_X(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ $\quad f_Y(y) = \dfrac{e^{-\mu}\mu^y}{y!}$

$$Z = X + Y \Rightarrow f_Z(z) = \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!}$$

$$f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} = \frac{f_{X,Y}(x, z-x)}{f_Z(z)} = \frac{f_X(x)f_Y(z-x)}{f_Z(z)}$$

$$= \frac{\dfrac{e^{-\lambda}\lambda^x}{x!}\,\dfrac{e^{-\mu}\mu^{z-x}}{(z-x)!}}{\dfrac{e^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!}} = \frac{z!}{x!(z-x)!}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(\frac{\mu}{\lambda+\mu}\right)^{z-x}$$

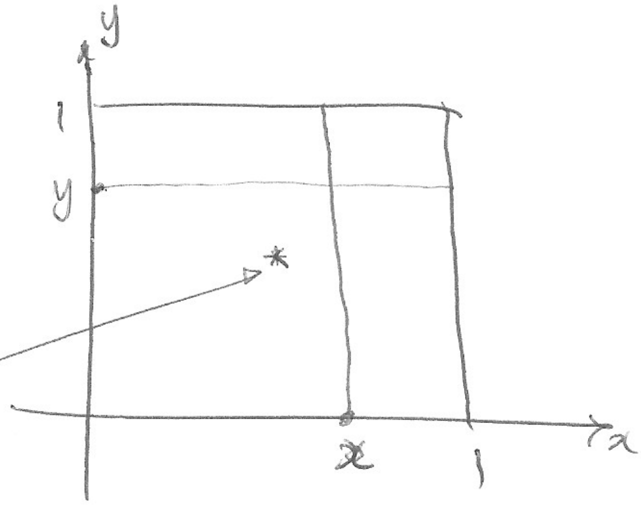
$$= \binom{z}{x} p^x q^{z-x} \qquad p = \frac{\lambda}{\lambda+\mu} \quad q = \frac{\mu}{\lambda+\mu}$$

$$\text{Binomial}\left(z, \frac{\lambda}{\lambda+\mu}\right)$$

Q2.

$$f(x) = 1 \quad f(y) = 1$$

$$f(x,y) = f(x)f(y) = 1$$



Now choose one point in two Dimension.

This has X & Y with joint uniform distribution

$$f(x,y) = \frac{1}{\text{Area}} = \frac{1}{1} = 1$$

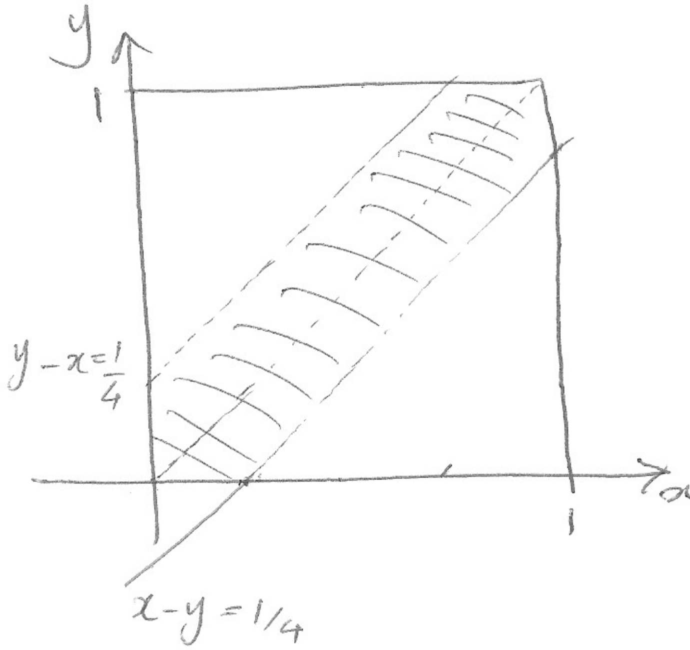
$$f(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (1)\,dy = 1$$

$$f(y) = \int_0^1 f(x,y)\,dx = \int_0^1 (1)\,dx = 1$$

$$f(x,y) = (1)(1) = 1 = f(x,y)$$ X & Y are independent.

So we can use X and Y of one single point in 2D instead of two points X & Y in 1D.

find the dashed area.



$$\left(1 - 2\times\frac{\left(3/4 \times 3/4\right)}{2}\right) \Big/ 1$$

$$= 1 - \frac{9}{16} = \frac{7}{16}$$

Q3. $f_{X,Y}(x,y) = 2e^{-2x-y}$ $x>0$ $y>0$

$$f_X(x) = \int_0^\infty 2e^{-2x-y}\,dy = 2e^{-2x}\int_0^\infty e^{-y}\,dy = 2e^{-2x}$$

$$f_Y(y) = \int_0^\infty 2e^{-2x-y}\,dx = e^{-y}\int_0^\infty 2e^{-2x}\,dx = e^{-y}$$

$$f_X(x)f_Y(y) = 2e^{-2x}\cdot e^{-y} = 2e^{-2x-y} = f_{X,Y}(x,y)$$

$\Rightarrow$ X & Y are independent

$\Rightarrow$ $\text{Cov}(X,Y) = 0$

Q4. $f_X(x) = \lambda e^{-\lambda x}$ $\quad f_Y(y) = \lambda e^{-\lambda y}$

X & Y are independent $\rightarrow f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$

$$\begin{cases} Z = \dfrac{X}{X+Y} = g(X,Y) \\ T = X = h(X,Y) \end{cases} \Rightarrow \begin{cases} Y = \dfrac{T - TZ}{Z} = \dfrac{T}{Z} - T = g^{-1}(Z,T) \\ X = T = h^{-1}(Z,T) \end{cases}$$

$$f_{Z,T}(z,t) = |J| f_{X,Y}\left(g^{-1}(z,t), h^{-1}(z,t)\right) \quad J = \begin{vmatrix} \frac{\partial X}{\partial T} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial Z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{Z}-1 & \frac{-T}{Z^2} \end{vmatrix}$$

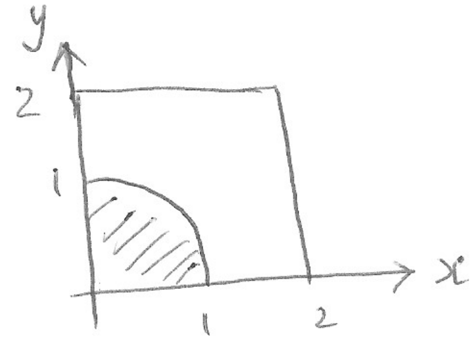
$$= \frac{t}{z^2} \lambda^2 e^{-\lambda \frac{t}{z}} \qquad = \frac{-T}{Z^2}$$

$$f_Z(z) = \int_0^\infty \frac{t}{z^2} \lambda^2 e^{-\frac{\lambda t}{z}} dt = \frac{\lambda^2}{z^2} \int_0^\infty t e^{-\frac{\lambda}{z} t} dt$$

$$= \frac{\lambda^2}{z^2} \frac{1!}{\left(\frac{\lambda}{z}\right)^2} = 1 \qquad 0 < z < 1$$

$\Rightarrow Z \sim U(0,1) \Rightarrow P(Z < a) = a$

Q5. $P(x^2+y^2 \leq 1) = \dfrac{\frac{\pi(1^2)}{4}}{4} = \dfrac{\pi}{16}$



For Uniform distribution in 2D

$$P(A) = \frac{S(A)}{\text{Total } S}$$

Q6. $f(x) = pq^x \quad f(y) = pq^y \qquad x = 0, 1, 2, \ldots \quad y = 0, 1, 2 \ldots$

$$P(X=Y) = \sum_{i=0}^{\infty} P(X=Y=i) = \sum_{i=0}^{\infty} P(X=i) \cdot P(Y=i)$$

$$= \sum_{i=0}^{\infty} pq^i \cdot pq^i = p^2 \sum_{i=1}^{\infty} q^{2i} = \frac{p^2}{1-q^2}$$

$\sum_{i=0}^{\infty} a^i = \dfrac{1}{1-a}$ if $|a|<1$

$$= \frac{p^{\cancel{2}}}{\cancel{(1-q)}(1+q)} = \frac{p}{1+q} = \frac{p}{2-p}$$

Q7. $Y = F(x) = g(X) \Rightarrow g^{-1}(Y) = X = F^{-1}(Y)$

$$f_y(y) = \left|\frac{dg^{-1}(y)}{dy}\right| f_x\left(g^{-1}(y)\right) = \frac{dx}{dy} f_x(x) = \frac{1}{\frac{dy}{dx}} f_x(x)$$

$$= \frac{1}{\frac{dF_X(X)}{dx}} \cdot f_X(x) = \frac{1}{f_X(x)} \cdot f_x(x) = 1$$

As $Y = F(X) \Rightarrow 0 \leqslant y \leq 1$. Then as $f_y(y) = 1$

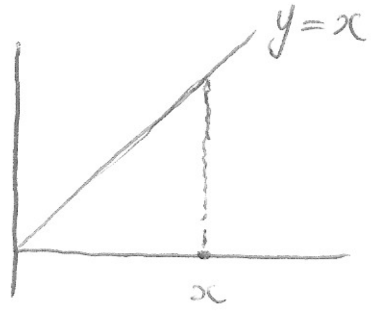
$\Rightarrow Y \sim U(0,1)$

This is a very important result.

Q8. $f(x,y) = ke^{-x} \quad 0<y<x<\infty$

$E[Y|x] = \int y f(y|x)\, dx$ we therefore need $f(y|x)$.

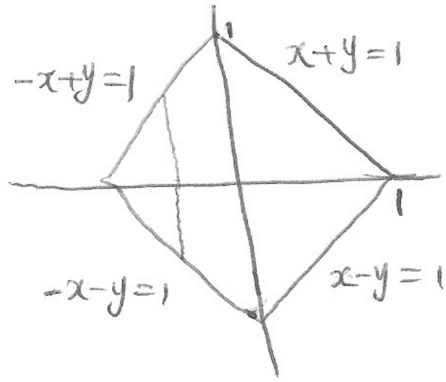
$f(x) = \int_0^x ke^{-x}\, dy = kxe^{-x}$



$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{ke^{-x}}{kxe^{-x}} = \frac{1}{x}$ $\quad 0<y<x$

$\Rightarrow Y|x \sim U(0,x) \quad \Rightarrow E[Y|x] = \frac{x}{2}$

Q9. $f(x,y) = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$



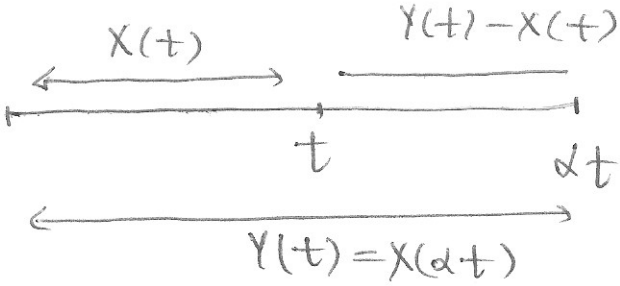
$-1<x<0 \quad f(x) = \int f(x,y)\, dy$

$= \int_{-x-1}^{1+x} \frac{1}{2}\, dy = 1+x$

$0<x<1 \quad f(x) = \int f(x,y)\, dy = \int_{x-1}^{1-x} \frac{1}{2}\, dy = 1-x$

$\Rightarrow f(x) = 1-|x| \quad -1<x<1$

Q10. $X = X(t)$



$$f(x) = \frac{e^{-rt}(rt)^x}{x!}$$

$$f(y) = \frac{e^{-r\alpha t}(r\alpha t)^y}{y!}$$

$X(t)$ & $Y(t) - X(t)$ are obviously independent.

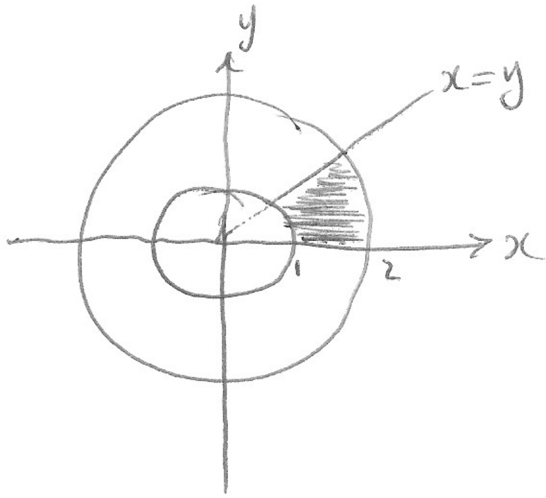
$$\Rightarrow Cov(X(t), Y(t) - X(t)) = 0$$

$$\Rightarrow Cov(X(t), Y(t)) - Cov(X(t), X(t)) = 0$$

$$\Rightarrow Cov(X(t), Y(t)) = cov(X(t), X(t)) = Var(X(t)) = rt$$

$$\rho_{X,Y} = \frac{Cov(X(t), Y(t))}{\sqrt{Var(X(t)) \cdot Var(Y(t))}} = \frac{rt}{\sqrt{rt \cdot r\alpha t}} = \frac{1}{\sqrt{\alpha}}$$

Q11)



$f(x,y) = k(|x| + |y|)$ is symmetric wrt x & y.

The region $1 \leq x^2 + y^2 \leq 2$ is symmetric wrt. the line $x = y$.

⇒ for any point in the dashed area, there are 7 points in other parts of $1 \leq x^2 + y^2 \leq 2$ with same $f(x,y)$.

$\Rightarrow P(0 < Y < X) = \frac{1}{8}$. No calculations Required!

Q12. $f_{X|T}(x|t) = \frac{e^{-nt}(nt)^x}{x!}$ $x=0,1,2,\dots$ $X|T$: # Claims for T years

T: lifetime $T \sim N(\mu, \sigma^2)$

$$E[X] = E[E(X|T)] = E[nT] = nE[T] = n\mu$$

Q13. $$Corr(X_i, \bar{X}) = \frac{Cov(X_i, \bar{X})}{\sqrt{Var(X_i)\,Var(\bar{X})}}$$

$Var(X_i) = \sigma^2$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

$$Cov(X_i, \bar{X}) = Cov\left(X_1, \frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right)$$

$$= \frac{1}{n} Cov(X_1, X_1) = \frac{1}{n} var(X_1) = \frac{\sigma^2}{n}$$

$$\Rightarrow Corr(X_1, X_n) = \frac{\sigma^2/n}{\sqrt{\sigma^2 \cdot \sigma^2/n}} = \frac{1}{\sqrt{n}}$$

Q14)

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$Y = g(X) = X^2$

$X = g^{-1}(Y) = \pm\sqrt{Y}$

As the fuction is not 1-to-1, we'll do $f(y)$ for $+\sqrt{y}$ & $-\sqrt{y}$ separately and then add them.

as the results are the same we double the result for $+\sqrt{y}$.

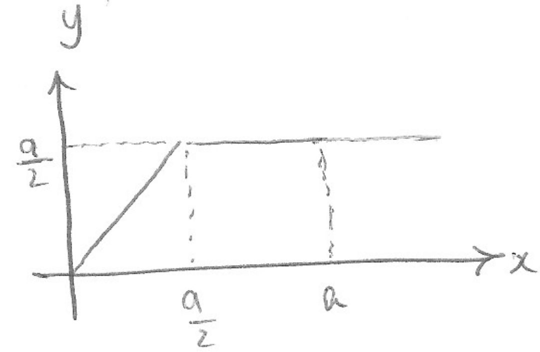
$$f_Y(y) = 2\left|\frac{dg^{-1}(y)}{dy}\right| f_X(g^{-1}(y))$$

$$= 2 \, \frac{1}{2\sqrt{y}} \, \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \qquad y \geq 0$$

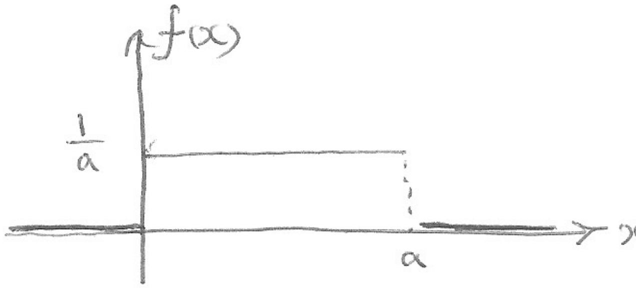
Q15.

$$Y = \begin{cases} X & X < a/2 \\ \frac{a}{2} & X \geq a/2 \end{cases}$$

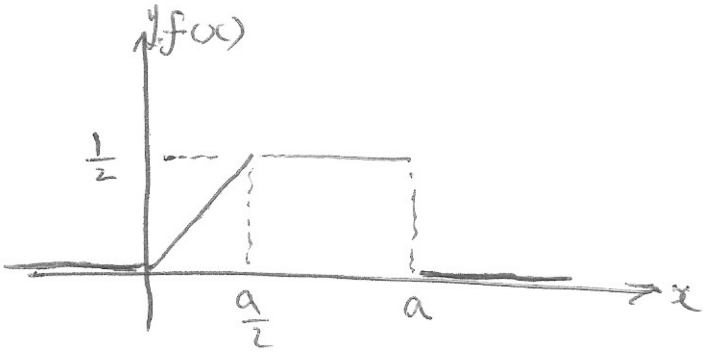


$E[Y] = E[g(X)]$

$$= \int_0^a g(x) f(x) \, dx$$



we therefore need to find the area under the curve in this graph.



$$= \frac{1}{2} \times \frac{a}{2} \times \frac{1}{2} + \frac{1}{2}(a - a/2)$$

$$= \frac{a}{8} + \frac{a}{4} = \frac{3a}{8}.$$

No integral !

Q16.

$$MSE[E[Y|x]] = E[(E[Y|x]-Y)^2|x]$$

$$= E[E^2[Y|x]|x] + E[Y^2|x] - 2E[YE[Y|x]|x]$$

$$= E^2[Y|x] + E[Y^2|x] - 2E[Y|x]E[Y|x]$$

$$= E[Y^2|x] - E^2[Y|x] = Var(Y|x)$$

Q 17.

$$E[Y] = \int_y y f(y)\, dx = \int_y y \int_x f(x,y)\, dx\, dy$$

$$= \int_y y \int_x f(y|x) f(x)\, dx\, dy$$

$$= \int_x f(x) \int_y y f(y|x)\, dy\, dx = \int_x f(x)\, E_y[Y|x]\, dx$$

$$= E_X\left[E_Y[Y|X] \right]$$

1 18.

$$Var(E[Y|X]) = E_X\left[\cancel{E_Y^2[Y|X]}\right] - E_X^2\left[E_Y[Y|X]\right] \quad (1)$$

$$E_X[Var(Y|X)] = E_X\left[E_Y[Y^2|X]\right] - E\left[\cancel{E_Y^2[Y|X]}\right] \quad (2)$$

$\Rightarrow$ Add (1) & (2)

$$Var(E[Y|X]) + E[Var(Y|X)] = E_X\left[E_Y[Y^2|X]\right] - E^2\left[E_Y[Y|X]\right]$$

$$= \int_x f(x)\, E_Y[Y^2|x]\, dx - \left[\int_x f(x)\, E_Y[Y|x]\, dx\right]^2$$

$$= \int_x f(x) \int_y y^2 f(y|x)\, dy\, dx - \left[\int_x f(x) \int_y y f(y|x)\, dy\, dx\right]^2$$

$$= \int_y y^2 \int_x f(x,y)\, dx\, dy - \left[\int_y y \int_x f(y|x) f(x)\, dx\, dy\right]^2$$

$$= \int_y y^2 f(y)\, dy - \left[\int_y y \int_x f(x,y)\, dx\, dy\right]^2$$

$$= \int_y y^2 f(y)\, dy - \left[\int y f(y)\, dy\right]^2 = E[Y^2] - E^2[Y] = Var(Y)$$

① 19. $$y = \sum_{i=1}^{N} x_i$$

$$E[y] = E_N\left[E_y[y \mid N]\right] = E_N\left[N\,E[x]\right] = E[x]E[N]$$

This is very obvious! If we have N (random) customers who each buy €x_i, The mean of Total Sale is mean # of customers by mean Value of Payment/customer

$$Var(y) = E_N\left[Var(y \mid N)\right] + Var_N\left[E[y \mid N]\right]$$

$$= E_N\left[N\,Var(x)\right] + Var_N\left[N\,E[x]\right]$$

$$= Var(x)\,E[N] + E^2[x]\,Var(N)$$

20.

$$P(X=k) = \int_t P(X=k \mid t)\, f(t)\, dt$$

This is Bayes Rule 1 in continuous form for B_i's below

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)$$

$$= \int_t \frac{e^{-\beta t} (\beta t)^k}{k!}\, \alpha e^{-\alpha t}\, dt = \frac{\alpha \beta^k}{k!} \int_0^\infty t^k e^{-(\alpha+\beta)t}\, dt$$

$$= \frac{\alpha \beta^k}{(\alpha+\beta)^{k+1}}$$

Hint: $\displaystyle\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$

Q 21. $E[X] = \underset{T}{E}\left[\underset{X}{E}[X \mid T]\right] = \underset{T}{E}\left[\beta T\right] = \beta \underset{T}{E}[T]$

$= \beta/\alpha$

Hint. $f(x \mid t) = \dfrac{e^{-\beta t} (\beta t)^{x}}{x!}$

$\rightarrow E[x \mid t] = \beta t$, $Var(x \mid t) = \beta t$

Q 22. $Var(X) = E\left[Var(X \mid T)\right] + Var\left[E[X \mid T]\right]$

$= E\left[\beta T\right] + var\left[\beta T\right] = \beta E[T] + \beta^2 Var(T)$

$= \dfrac{\beta}{\alpha} + \dfrac{\beta^2}{\alpha^2}$

Q 23.

$$F_V(v) = P(V \le v) = P(\mathrm{Max}(X_1, \dots X_n) \le v) = \int_{-\infty}^{v} \dots \int_{-\infty}^{v} f(x_1, \dots x_n)\, dx_1 \dots dx_n$$

$$= \left[\int_{-\infty}^{v} f(x_1)\, dx_1 \right]^n = F_X^n(v) \Rightarrow f_V(v) = \frac{dF_V(v)}{dv} = n f_X(v) F_X^{n-1}(v)$$

$$F_U(u) = P(U \le u) = P(\min(X_1, \dots, X_n) \le u) = 1 - P(\min(X_1, \dots X_n) > u)$$

$$= 1 - \int_u^{\infty} \dots \int_u^{\infty} f(x_1, \dots, x_n)\, dx_1\, dx_2 \dots dx_n = 1 - \left[\int_u^{\infty} f(x)\, dx \right]^n$$

$$= 1 - \left[1 - F_X(u) \right]^n \Rightarrow f_U(u) = \frac{dF_U(u)}{du} = n f_X(u) \left[1 - F_X(u) \right]^{n-1}$$

Q 24.

$$f_X(x) = \frac{1}{b-a} \qquad a \le x \le b \qquad F_X(x) = P(X \le x) = \int_a^x \frac{1}{b-a}\, dx$$

$$= \frac{x-a}{b-a} \qquad a \le x \le b$$

$$f_U(u) = n \left(\frac{1}{b-a} \right) \left(1 - \frac{u-a}{b-a} \right)^{n-1} = \frac{n}{(b-a)^n} (b-u)^{n-1} \qquad a \le u \le b$$

$$f_V(v) = n \frac{1}{b-a} \left(\frac{v-a}{b-a} \right)^{n-1} = \frac{n}{(b-a)^n} (v-a)^{n-1} \qquad a \le v \le b$$

Q 25.

$$f_X(x) = \lambda e^{-\lambda x} \qquad F_X(x) = 1 - e^{-\lambda x}$$

$$f_U(u) = n \lambda e^{-\lambda u} \left[1 - (1 - e^{-\lambda u}) \right]^{n-1} = n \lambda e^{-n\lambda u}$$

26. $X = \sigma_x Z_1 + \mu_x \Rightarrow \sigma_x N(0,1) + \mu_x = N(\mu_x, \sigma_x^2)$

$$Y = \sigma_y [\rho Z_1 + \sqrt{1-\rho^2}\, Z_2] + \mu_y$$
$$= \sigma_y [\rho N(0,1) + \sqrt{1-\rho^2}\, N(0,1)] + \mu_y$$
$$= \sigma_y [N(0,\rho^2) + N(0, 1-\rho^2)] + \mu_y$$
$$= N(0, \sigma_y^2 \rho^2) + N(0, \sigma_y^2(1-\rho^2)) + \mu_y$$
$$= N(0, \sigma_y^2) + \mu_y = N(\mu_y, \sigma_y^2)$$

Q 27. $\left\{ \begin{array}{l} \text{Hint: } Cov(aX+b, cY+d) = ac\,Cov(X,Y) \\ Cov(X,Y) = \sigma_X^2 \end{array} \right.$

$$Cov(X,Y) = Cov(\mu_X + \sigma_X Z_1 \, , \, \mu_Y + \sigma_Y [\rho Z_1 + \sqrt{1-\rho^2} Z_2])$$

$$= Cov(\sigma_X Z_1 \, , \, \sigma_Y \rho Z_1 + \sigma_Y \sqrt{1-\rho^2} Z_2)$$

$$= \rho \sigma_{X_1} \sigma_Y \underbrace{\cancel{Cov(Z_1, Z_1)}}_{1} + \sigma_X \sigma_Y \sqrt{1-\rho^2} \underbrace{\cancel{Cov(Z_1, Z_2)}}_{0}$$

$$= \rho \sigma_X \sigma_Y .$$

$$\Rightarrow \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\rho \sigma_X \sigma_Y}{\sigma_X \sigma_Y} = \rho$$

Infact, we could have called $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and ρ other things like a, b, c, d, e, and then using Questions 1, 2, show $\mu_X = a$ $\mu_Y = b$, etc.

But we prefered to use (instead of abc, de) what they exactly are ultimately.

Q 28. Reminder: If $T = g(X,Y)$, $W = h(X,Y)$

$$f_{T,W}(t,w) = |J|\, f_{X,Y}\left(g^{-1}(t,w),\, h^{-1}(t,w)\right)$$

where g^{-1} & h^{-1} are inverse functions of g & h (to express X & Y in terms of T & W) and

$$J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \end{vmatrix}$$

$$X = \mu_X + \sigma_X Z_1 \Rightarrow Z_1 = \frac{X - \mu_X}{\sigma_X}$$

$$Y = \mu_Y + \sigma_Y\left[\rho Z_1 + \sqrt{1-\rho^2}\, Z_2\right] = \mu_Y + \sigma_Y\left[\rho \frac{X-\mu_X}{\sigma_X} + \sqrt{1-\rho^2}\, Z_2\right]$$

$$\Rightarrow Z_2 = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{Y-\mu_Y}{\sigma_Y} - \rho\frac{X-\mu_X}{\sigma_X}\right]$$

Z_1 & Z_2 are x & y in Reminder

X & Y are T & W in Reminder

$$J = \begin{vmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial y} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sigma_X} & 0 \\ \frac{-\rho}{\sigma_X\sqrt{1-\rho^2}} & \frac{1}{\sigma_Y\sqrt{1-\rho^2}} \end{vmatrix} = \frac{1}{\sigma_X \sigma_Y \sqrt{1-\rho^2}}$$

$$f(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{z_1^2 + z_2^2}{2}}$$

$$\Rightarrow f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\, e^{-\left[\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right]}$$

This is bivariate Normal dist Pdf.

Q29. $f(x,y) = |J|\, f(z_1, z_2) = |J| \frac{1}{2\pi} e^{-\frac{z_1^2 + z_2^2}{2}}$

from Q3 $\rightarrow$

$$= \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \frac{1}{1-\rho^2}\left(\frac{y-\mu_y}{\sigma_y} - \rho\frac{x-\mu_x}{\sigma_x}\right)^2\right)\right]$$

from Q 1 $\quad f(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma^2}}$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{1}{\sqrt{2\pi}\,\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_y}{\sigma_y} - \rho\frac{x-\mu_x}{\sigma_x}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_y^2(1-\rho^2)}} \exp\left[-\frac{1}{2(1-\rho^2)\sigma_y^2}\left(y-\mu_y - \frac{\rho\sigma_y}{\sigma_x}(x-\mu_x)\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_y^2(1-\rho^2)}} \exp\left[-\frac{1}{2(1-\rho^2)\sigma_y^2}\left(y-\left(\mu_y + \frac{\rho\sigma_y}{\sigma_x}(x-\mu_x)\right)\right)^2\right]$$

$$\Rightarrow \sim N\left[\mu_{y|x} = \mu_y + \frac{\rho\sigma_y}{\sigma_x}(x-\mu_x),\ \sigma^2_{y|x} = \sigma_y^2(1-\rho^2)\right]$$

$$\Rightarrow E[Y|x] = \mu_y + \frac{\rho\sigma_y}{\sigma_x}(x-\mu_x),\quad Var(Y|x) = \sigma_y^2(1-\rho^2)$$

Q 30. By comparing $f(x,y)$ to the bivariate Normal distribution pdf, it's obvious that $\mu_x = \mu_y = 0$ then we have

$$-\frac{\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}}{2(1-\rho^2)} = -8x^2 - 6xy - 18y^2$$

$$\Rightarrow 2\sigma_x^2(1-\rho^2) = \frac{1}{8} \Rightarrow \sigma_x^2(1-\rho^2) = \frac{1}{4} = Var(X|y) \quad (1)$$

$$2\sigma_y^2(1-\rho^2) = \frac{1}{18} \Rightarrow \sigma_y^2(1-\rho^2) = \frac{1}{9} \quad (2)$$

$$\frac{\rho}{(1-\rho^2)\sigma_x\sigma_y} = -3 \Rightarrow \frac{\rho^2}{(1-\rho^2)^2\sigma_x^2\sigma_y^2} = 9 \quad (3)$$

$$(1)\times(2)\times(3) \Rightarrow \frac{1}{4} = \rho^2 \Rightarrow \rho = -1/2$$

$$\left(\text{not } \frac{1}{2}, \text{because } \frac{\rho}{(1-\rho^2)\sigma_x\sigma_y} < 0\right)$$

$$(1) \Rightarrow \sigma_x^2 = 1/3 \qquad (2) \Rightarrow \sigma_y^2 = \frac{4}{27} \qquad \Rightarrow \sigma_x\sigma_y = \frac{2}{9}$$

I) $$C = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_x\sigma_y} = \frac{1}{2\pi\sqrt{\frac{3}{4}\times\frac{2}{9}}} = \frac{\sqrt{27}}{2\pi}$$

II

$$E(X|Y=4) = \mu_x + \rho\left(\frac{\sigma_x}{\sigma_y}\right)(y-\mu_y) = 0 + \left(-\frac{1}{2}\right)\sqrt{\frac{1/3}{4/27}}(4-0) = -3$$

$$P(X<-2|Y=4) = \Phi\left(\frac{-2-E(X|Y=4)}{\sigma_{X|Y=4}}\right) = \Phi\left(\frac{-2-(-3)}{\frac{1}{2}}\right) = \Phi(2)$$

$$= 0.97725$$