MAU22C00: ASSIGNMENT 1
DUE BY FRIDAY, OCTOBER 16 BEFORE MIDNIGHT
UPLOAD SOLUTION ON BLACKBOARD

Please write down clearly both your name and your student ID number on everything you hand in. Please attach a cover sheet with a declaration confirming that you know and understand College rules on plagiarism. Details can be found on `http://tcd-ie.libguides.com/plagiarism/declaration`**.**

1) (10 points) Please carry out the following proof in propositional logic following the proof format in tutorial 1: Hypotheses: $P \to (Q \leftrightarrow \neg R)$, $P \vee \neg S$, $R \to S$, $\neg Q \to \neg R$. Conclusion: $\neg R$.

For each line of the proof, mention which tautology you used giving its number according to the list of tautologies posted in folder Course Documents. Solutions based on truth tables or any other method except for the one specified will be given **NO CREDIT**.

2) (10 points) Prove the following statement: If n is any integer, then $n^2 - 3n$ must be even. (Hint: Cases come in handy here. See tautology #26 for the basis of proofs by cases. This proof follows the format of the one given in lecture that $\sqrt{2}$ is not a rational number.)

3) (10 points) Prove via inclusion in both directions that for any three sets A, B, and C

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

4) (10 points) Let $\mathbb{N} \times \mathbb{N}$ be the Cartesian product of the set of natural numbers with itself consisting of all ordered pairs (x_1, x_2) such that $x_1 \in \mathbb{N}$ and $x_2 \in \mathbb{N}$. We define a relation on its power set $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ as follows: $\forall A, B \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$ $A \sim B$ iff $(A \setminus B) \cup (B \setminus A) = C$ and C is a finite set. Determine whether or not $\sim$ is an equivalence relation and justify your answer by checking each of the three properties in the definition of an equivalence relation. Please note that a set C is finite if it has finitely many elements. In particular, the empty set $\emptyset$ has zero elements and is thus finite.

MAU22C00: ASSIGNMENT 1 SOLUTIONS

1) (10 points) Please carry out the following proof in propositional logic following the proof format in tutorial 1: Hypotheses: $P \to (Q \leftrightarrow \neg R)$, $P \vee \neg S$, $R \to S$, $\neg Q \to \neg R$. Conclusion: $\neg R$.

For each line of the proof, mention which tautology you used giving its number according to the list of tautologies posted in folder Course Documents. Solutions based on truth tables or any other method except for the one specified will be given **NO CREDIT**.

Solution: We wish to prove $\neg R$. We first list the hypotheses:

(1) $P \to (Q \leftrightarrow \neg R)$ hypothesis
(2) $P \vee \neg S$ hypothesis
(3) $R \to S$ hypothesis
(4) $\neg Q \to \neg R$ hypothesis
(5) $\neg S \vee P$ tautology #32 applied to (2)
(6) $S \to P$ tautology #21 applied to (5)
(7) $R \to P$ tautology #14 applied to (3) and (6)
(8) $R \to (Q \leftrightarrow \neg R)$ tautology #14 applied to (1) and (7)
(9) $R \to [(Q \to \neg R) \wedge (\neg R \to Q)]$ tautology #22 applied to (8)
(10) $[R \to (Q \to \neg R)] \wedge [R \to (\neg R \to Q)]$ tautology #25 applied to (9)
(11) $R \to (Q \to \neg R)$ tautology #4 applied to (10)
(12) $R \to (\neg R \to Q)$ tautology #4 applied to (10)
(13) $\neg R \to (Q \to \neg R)$ tautology #8
(14) $R \vee \neg R \to (Q \to \neg R)$ tautology #26 applied to (11) and (13)
(15) $R \vee \neg R$ tautology #1
(16) $Q \to \neg R$ modus ponens (tautology #10) applied to (14) and (15)
(17) $Q \vee \neg Q \to \neg R$ tautology #26 applied to (4) and (16)
(18) $Q \vee \neg Q$ tautology #1
(19) $\neg R$ modus ponens (tautology #10) applied to (17) and (18)

Marking rubric: 10 marks total. 4 marks for getting to a statement involving only Q and R and 6 marks for the rest of the solution.

2) (10 points) Prove the following statement: If n is any integer, then $n^2 - 3n$ must be even. (Hint: Cases come in handy here. See tautology

#26 for the basis of proofs by cases. This proof follows the format of the one given in lecture that $\sqrt{2}$ is not a rational number.)

Solution: We split the argument into cases based on whether n is even or odd:

Case 1: n is even. Therefore, $\exists\, k \in \mathbb{Z}$ such that $n = 2k$. We substitute this expression into $n^2 - 3n$ as follows: $n^2 - 3n = (2k)^2 - 3(2k) = 4k^2 - 6k = 2(2k^2 - 3k)$. Since $k \in \mathbb{Z}$, $2k^2 - 3k$ is likewise an integer since $\mathbb{Z}$ is closed under addition and multiplication. Let $p = 2k^2 - 3k$. $p \in \mathbb{Z}$ and $n^2 - 3n = 2p$. Therefore, $n^2 - 3n$ is even.

Case 2: n is odd. Therefore, $\exists\, k \in \mathbb{Z}$ such that $n = 2k + 1$. We substitute this expression into $n^2 - 3n$ as follows: $n^2 - 3n = (2k+1)^2 - 3(2k + 1) = 4k^2 + 4k + 1 - 6k - 3 = 4k^2 - 2k - 2 = 2(2k^2 - k - 1)$. Once again, since $\mathbb{Z}$ is closed under addition and multiplication, $p = 2k^2 - k - 1 \in \mathbb{Z}$. Therefore, $n^2 - 3n = 2p$ for $p \in \mathbb{Z}$, so $n^2 - 3n$ is even.

We have proven that n is even $\implies$ $n^2 - 3n$ is even and that n is odd $\implies$ $n^2 - 3n$ is even. If n is an integer, n is either even or odd. Therefore, by tautology #26, we conclude that if n is any integer, $n^2 - 3n$ is even. $\square$

Marking rubric: 10 marks total, 5 marks for each case.

3) (10 points) Prove via inclusion in both directions that for any three sets A, B, and C

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C).$$

Venn diagrams, truth tables, or diagrams for simplifying statements in Boolean algebra such as Veitch diagrams are **NOT** acceptable and will not be awarded any credit.

Solution: We first prove "$\subseteq$:" $\forall x \in A \cap (B \setminus C)$, $x \in A$ and $x \in B \setminus C$. Therefore, $x \in A$ and $x \in B$ and $x \notin C$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Since $x \in A$ and $x \notin C$, we conclude $x \notin A \cap C$. Therefore, $x \in A \cap B$ and $x \notin A \cap C$. We conclude that $x \in (A \cap B) \setminus (A \cap C)$.

We now prove "$\supseteq$:" $\forall x \in (A \cap B) \setminus (A \cap C)$, by definition, $x \in A \cap B$ and $x \notin A \cap C$. Since $x \in A \cap B$, $x \in A$ and $x \in B$. Since $x \notin A \cap C$ but $x \in A$, it must be the case that $x \notin C$. Therefore, $x \in B$ and $x \notin C$, so $x \in B \setminus C$, but $x \in A$. Therefore, $x \in A \cap (B \setminus C)$ as needed.

Marking rubric: 10 marks total, 5 marks for each inclusion.

4) (10 points) Let $\mathbb{N} \times \mathbb{N}$ be the Cartesian product of the set of natural numbers with itself consisting of all ordered pairs (x_1, x_2) such that $x_1 \in \mathbb{N}$ and $x_2 \in \mathbb{N}$. We define a relation on its power set $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ as follows: $\forall A, B \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$ $A \sim B$ iff $(A \setminus B) \cup (B \setminus A) = C$ and C

is a finite set. Determine whether or not $\sim$ is an equivalence relation and justify your answer by checking each of the three properties in the definition of an equivalence relation. Please note that a set C is finite if it has finitely many elements. In particular, the empty set $\emptyset$ has zero elements and is thus finite.

Solution: (a) We check whether the relation $\sim$ is reflexive. $(A \setminus A) \cup (A \setminus A) = A \setminus A = \emptyset$. Since $\emptyset$ has zero elements, it is a finite set. We conclude $A \sim A$ is true for all $A \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$, so the relation $\sim$ is reflexive.

(b) We check whether the relation $\sim$ is symmetric. $\forall A, B \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$ if $A \sim B$, then $(A \setminus B) \cup (B \setminus A) = C$ and C is a finite set. Note that $\cup$ is commutative, so $(A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = C$. Since C is finite, $B \sim A$ holds as well, so the relation $\sim$ is symmetric.

(c) We check whether the relation $\sim$ is transitive. $\forall A, B, C \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$ if $A \sim B$ and $B \sim C$, then $(A \setminus B) \cup (B \setminus A) = D$ and D is a finite set and $(B \setminus C) \cup (C \setminus B) = E$ and E is a finite set. We need to check whether $(A \setminus C) \cup (C \setminus A)$ is a finite set. I claim that $(A \setminus C) \cup (C \setminus A) \subseteq D \cup E$. It suffices to prove that $A \setminus C \subseteq D \cup E$ and $C \setminus A \subseteq D \cup E$. To prove $A \setminus C \subseteq D \cup E$, we consider $\forall x \in A \setminus C$. From the definition, $x \in A$ and $x \notin C$. The argument splits into two cases:

Case 1: $x \in B$. Then $x \in B$ and $x \notin C$, so $x \in B \setminus C$, but $(B \setminus C) \cup (C \setminus B) = E$. Therefore, $x \in E$.

Case 2: $x \notin B$. Then $x \in A$ and $x \notin B$, so $x \in A \setminus B$, but $(A \setminus B) \cup (B \setminus A) = D$. Therefore, $x \in D$.

We conclude that if $x \in A \setminus C$, then $x \in D \cup E$. Therefore, $A \setminus C \subseteq D \cup E$. We carry out the same argument with C and A exchanged to conclude that $C \setminus A \subseteq D \cup E$. Putting the two together, we have that $(A \setminus C) \cup (C \setminus A) \subseteq D \cup E$. Now, we know $(A \setminus C) \cup (C \setminus A) = F$ for some $F \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$. So we have shown that $F \subseteq D \cup E$, but we know both D and E are finite sets, which means their union $D \cup E$ must also be a finite set of size at most the sum of the sizes of D and E. Therefore, F is a subset of $D \cup E$, which is a finite set, so F itself must be finite. We conclude that $A \sim C$.

Marking rubric: 10 marks total, 2 marks for proving reflexivity, 2 marks for proving symmetry, and 6 marks for proving transitivity.

MAU22C00: ASSIGNMENT 2
DUE BY THURSDAY, NOVEMBER 12 BEFORE MIDNIGHT
UPLOAD SOLUTION ON BLACKBOARD

Please write down clearly both your name and your student ID number on everything you hand in. Please attach a cover sheet with a declaration confirming that you know and understand College rules on plagiarism. Details can be found on http://tcd-ie.libguides.com/plagiarism/declaration**.**

1) (10 points) Let $A = \mathbb{R}^n = \{(x_1, \dots, x_n) \bigm| x_i \in \mathbb{R},\ 1 \leq i \leq n\}$. For $x, y \in A$, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, xQy if and only if $\forall\, i$, $1 \leq i \leq n$, $x_i = y_i$ or $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ $\forall\, j,\ j < i$. Determine:

(i) Whether or not the relation Q is *reflexive*;
(ii) Whether or not the relation Q is *symmetric*;
(iii) Whether or not the relation Q is *anti-symmetric*;
(iv) Whether or not the relation Q is *transitive*;
(v) Whether or not the relation Q is an *equivalence relation*;
(vi) Whether or not the relation Q is a *partial order*.

Justify your answers.

2) (10 points) Use mathematical induction to prove that for all $n \geq 7$, $n! > 3^n$.

3) (20 points) (a) Let $\{C_n\}_{n=1,2,\dots} = \{C_1, C_2, \dots\}$ be a sequence of sets satisfying that $C_n \subseteq C_{n+1}$ $\forall\, n \geq 1$. Prove by mathematical induction that $C_m \subseteq C_n$ whenever $m < n$.

(b) Recall that the graph of a function $f : A \to B$ is given by

$$\Gamma(f) = \{(x, y) \bigm| x \in A \text{ and } y = f(x)\} \subseteq A \times B.$$

Let $Funct(A, B)$ the set of all functions $f : \tilde{A} \to \tilde{B}$ such that $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$. We define a relation on $Funct(A, B)$ as follows:

$\forall f, g \in Funct(A, B)$ $f \subseteq g$ iff $\Gamma(f) \subseteq \Gamma(g)$.

Prove that this relation is a partial order on $Funct(A, B)$.

(c) Let $\{f_n\}_{n=1,2,\dots} = \{f_1, f_2, \dots\}$ be a sequence of functions in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$. Since functions are in one-to-one correspondence with their graphs, we identify $\bigcup_{n\in\mathbb{N}} f_n$ with $\bigcup_{n\in\mathbb{N}} \Gamma(f_n)$. Using part (a), prove that $\bigcup_{n\in\mathbb{N}} f_n$ is a function and $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A,B)$.

(d) For every $f \in Funct(A,B)$, let $Dom(f)$ be the domain of f, namely if $f : \tilde{A} \to \tilde{B}$ with $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$, $Dom(f) = \tilde{A}$. Prove that $Dom\left(\bigcup_{n\in\mathbb{N}} f_n\right) = \bigcup_{n\in\mathbb{N}} Dom(f_n)$ for every sequence of functions $\{f_n\}_{n=1,2,\dots} = \{f_1, f_2, \dots\}$ in $Funct(A,B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$.

4) (10 points) Let $\mathbb{R}[x]$ be the set of all polynomials in variable x with coefficients in $\mathbb{R}$. In other words,

$$\mathbb{R}[x] = \{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \mid n \in \mathbb{N} \text{ and } a_0, \dots, a_n \in \mathbb{R}\}.$$

(a) Give three examples of elements of $\mathbb{R}[x]$.

(b) Prove that $(\mathbb{R}[x], +)$, $\mathbb{R}[x]$ with addition as the operation, is a semigroup.

(c) Is $(\mathbb{R}[x], +)$ a monoid? Justify your answer.

(d) Does $(\mathbb{R}[x], +)$ have invertible elements? If so, which of its elements are invertible? Justify your answer.

MAU22C00: ASSIGNMENT 2 SOLUTIONS

1) (10 points) Let $A = \mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ 1 \leq i \leq n\}$. For $x, y \in A$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, xQy if and only if $\forall\, i$, $1 \leq i \leq n$, $x_i = y_i$ or $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ $\forall\, j$, $j < i$. Determine:

(i) Whether or not the relation Q is *reflexive*;
(ii) Whether or not the relation Q is *symmetric*;
(iii) Whether or not the relation Q is *anti-symmetric*;
(iv) Whether or not the relation Q is *transitive*;
(v) Whether or not the relation Q is an *equivalence relation*;
(vi) Whether or not the relation Q is a *partial order*.

Justify your answers.

Solution: Note that xQy if and only if one of two mutually exclusive things happens:

(a) $\forall\, i$, $1 \leq i \leq n$, $x_i = y_i$, namely $x = y$

(b) $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ $\forall\, j$, $j < i$.

(i) The relation Q is reflexive because $\forall\, x \in \mathbb{R}^n$ xQx holds (we are in scenario (a) as $x_i = x_i$ $\forall\, i$, $1 \leq i \leq n$).

(ii) To check symmetry, assume xQy for some $x, y \in \mathbb{R}^n$. If we are in scenario (a), namely $x = y$, then clearly yQx will hold as well. If we are in scenario (b), however, it would be impossible for yQx to hold as well since $x_i < y_i$ and $y_i < x_i$ cannot hold simultaneously. To justify properly, it is best to exhibit a counterexample to symmetry by giving an example of $x, y \in \mathbb{R}^n$ such that xQy holds, but yQx is false. Indeed, take $x = (1, 0, \ldots, 0)$ and $y = (2, 0, \ldots, 0)$. Since $x_1 < y_1$ and the condition $x_j = y_j$ $\forall\, j$, $j < 1$ is vacuously satisfied as there are no indices j less than 1, xQy (scenario (b)). Clearly, yQx is false as both scenarios (a) and (b) fail in this case.

(iii) To check anti-symmetry, assume xQy and yQx for some $x, y \in \mathbb{R}^n$. As $x_i < y_i$ and $y_i < x_i$ cannot hold simultaneously, we conclude that xQy and yQx hold because we are in scenario (a). Therefore, $x = y$, so Q is indeed anti-symmetric.

(iv) To check transitivity, assume xQy and yQz for some $x, y, z \in \mathbb{R}^n$. There are two mutually exclusive scenarios for xQy and two mutually exclusive scenarios for yQz. Therefore, we must consider a total of four cases:

Case 1: xQy due to scenario (a) and yQz due to scenario (a). Then $x = y$ and $y = z$, so $x = z$, hence xQz via scenario (a).

Case 2: xQy due to scenario (a) and yQz due to scenario (b). Then $x = y$ and $\exists\, i$ with $1 \leq i \leq n$ such that $y_i < z_i$ and $y_j = z_j\ \forall\, j,\ j < i$. Since $x = y$, the latter implies that $x_i < z_i$ and $x_j = z_j\ \forall\, j,\ j < i$. Therefore, xQz via scenario (b).

Case 3: xQy due to scenario (b) and yQz due to scenario (a). Thus, $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j\ \forall\, j,\ j < i$ and $y = z$. We conclude that $x_i < z_i$ and $x_j = z_j\ \forall\, j,\ j < i$, so xQz via scenario (b).

Case 4: xQy due to scenario (b) and yQz due to scenario (b). xQy due to scenario (b) means $\exists\, i$ with $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j\ \forall\, j,\ j < i$. yQz due to scenario (b) means $\exists\, k$ with $1 \leq k \leq n$ such that $y_k < z_k$ and $y_l = z_l\ \forall\, l,\ l < k$. This case then splits into two subcases based on how i and k relate to each other:

Case 4.1: $i \leq k$. Then $x_i < y_i \leq z_i$, namely $x_i < y_i < z_i$ if $i = k$ and $x_i < y_i = z_i$ if $i < k$. Note that $x_j = y_j\ \forall\, j,\ j < i$ and $y_l = z_l\ \forall\, l,\ l < k$. Since $i \leq k$, $x_j = y_j = z_j\ \forall\, j,\ j < i \leq k$. Therefore, $x_i < z_i$ and $x_j = z_j$ $\forall\, j,\ j < i$, so xQz via scenario (b).

Case 4.2: $k < i$. Then $y_k < z_k$, but $x_k = y_k$ as $k < i$. Therefore, $x_k < z_k$. Moreover, $y_l = z_l\ \forall\, l,\ l < k$ and $x_l = y_l$ also for every l satisfying $l < k$ because $l < k < i$, hence $l < i$. We conclude that $x_l = z_l\ \forall\, l,\ l < k$ and $x_k < z_k$, so xQz via scenario (b).

In all cases and subcases, the conclusion is that xQz, so Q is indeed transitive.

(v) An equivalence relation is a relation that is reflexive, symmetric, and transitive. Q is reflexive and transitive, but it is not symmetric. We thus conclude that Q is not an equivalence relation.

(vi) A partial order is a relation that is reflexive, anti-symmetric, and transitive. Q satisfies all three properties, so it is a partial order.

Marking rubric: 10 marks total. 2 marks for each of the first four parts (1 mark for the answer and 1 mark for the justification); 1 mark for each of the last two parts.

2) (10 points) Use mathematical induction to prove that for all $n \geq 7$, $n! > 3^n$.

Solution: Note that we are asked to prove the statement $n! > 3^n$ for all $n \geq 7$, so this is an induction where the base case takes place at $n = 7$.

Base case: $n = 7$. Then $7! = 7 \cdot 6 \cdot \cdots \cdot 2 \cdot 1 = 5040 > 2187 = 3^7$ as needed.

Inductive case: We assume $n! > 3^n$ and seek to prove $(n+1)! > 3^{n+1}$. Since $(n+1)! = (n+1) \cdot n!$, we multiply $n! > 3^n$ on both sides by $n+1 \geq 7+1 = 8 > 0$ to obtain that $(n+1) \cdot n! > (n+1) \cdot 3^n > 3 \cdot 3^n = 3^{n+1}$ as $n+1 > 3$. We conclude that $(n+1)! > 3^{n+1}$ as needed.

Marking rubric: 10 marks total. 1 mark for figuring out at which n the argument starts, 2 marks for the base case, and 7 marks for the inductive case.

3) (20 points) (a) Let $\{C_n\}_{n=1,2,\ldots} = \{C_1, C_2, \ldots\}$ be a sequence of sets satisfying that $C_n \subseteq C_{n+1}$ $\forall n \geq 1$. Prove by mathematical induction that $C_m \subseteq C_n$ whenever $m < n$.

(b) Recall that the graph of a function $f : A \to B$ is given by

$$\Gamma(f) = \{(x, y) \mid x \in A \text{ and } y = f(x)\} \subseteq A \times B.$$

Let $Funct(A, B)$ the set of all functions $f : \tilde{A} \to \tilde{B}$ such that $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$. We define a relation on $Funct(A, B)$ as follows:
$\forall f, g \in Funct(A, B)$ $f \subseteq g$ iff $\Gamma(f) \subseteq \Gamma(g)$.
Prove that this relation is a partial order on $Funct(A, B)$.

(c) Let $\{f_n\}_{n=1,2,\ldots} = \{f_1, f_2, \ldots\}$ be a sequence of functions in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$. Since functions are in one-to-one correspondence with their graphs, we identify $\bigcup_{n \in \mathbb{N}} f_n$ with $\bigcup_{n \in \mathbb{N}} \Gamma(f_n)$. Using part (a), prove that $\bigcup_{n \in \mathbb{N}} f_n$ is a function and $\bigcup_{n \in \mathbb{N}} f_n \in Funct(A, B)$.

(d) For every $f \in Funct(A, B)$, let $Dom(f)$ be the domain of f, namely if $f : \tilde{A} \to \tilde{B}$ with $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$, $Dom(f) = \tilde{A}$. Prove that $Dom\left(\bigcup_{n \in \mathbb{N}} f_n\right) = \bigcup_{n \in \mathbb{N}} Dom(f_n)$ for every sequence of functions $\{f_n\}_{n=1,2,\ldots} = \{f_1, f_2, \ldots\}$ in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$.

Solution: (a) Our induction starts at m. Therefore:

Base case: $n = m + 1$. By our hypothesis, $C_m \subseteq C_{m+1}$. Therefore, $C_m \subseteq C_n$ as needed.

Inductive case: We assume $C_m \subseteq C_n$ and seek to prove $C_m \subseteq C_{n+1}$. By our hypothesis, $C_n \subseteq C_{n+1}$. We combine this statement with the inductive hypothesis $C_m \subseteq C_n$ to conclude that $C_m \subseteq C_n \subseteq C_{n+1}$. Since inclusion of sets is transitive as we showed in the unit on set theory earlier this term, we conclude that $C_m \subseteq C_{n+1}$ as needed.

(b) To prove that the relation $\subseteq$ on $Funct(A, B)$ is a partial order, we must show it is reflexive, anti-symmetric, and transitive. We do so as follows:

Reflexivity: $\forall f \in Funct(A, B)$ $\Gamma(f) = \Gamma(f)$ from the reflexivity property of the equality of sets. Clearly, $\Gamma(f) = \Gamma(f) \implies \Gamma(f) \subseteq \Gamma(f)$ as $\subseteq$ permits equality of the two sets involved. We conclude that $f \subseteq f$.

Anti-symmetry: For $f, g \in Funct(A, B)$ assume that $f \subseteq g$ and $g \subseteq f$. $f \subseteq g \implies \Gamma(f) \subseteq \Gamma(g)$. $g \subseteq f \implies \Gamma(g) \subseteq \Gamma(f)$. Therefore, both $\Gamma(f) \subseteq \Gamma(g)$ and $\Gamma(g) \subseteq \Gamma(f)$, which is exactly the criterion for equality of sets (inclusion in both directions). We conclude that $\Gamma(f) = \Gamma(g)$. Since there is a one-to-one correspondence between a function in $Funct(A, B)$ and its graph, we conclude that $f = g$.

Transitivity: For $f, g, h \in Funct(A, B)$ assume that $f \subseteq g$ and $g \subseteq h$. $f \subseteq g \implies \Gamma(f) \subseteq \Gamma(g)$. $g \subseteq h \implies \Gamma(g) \subseteq \Gamma(h)$. Therefore, both $\Gamma(f) \subseteq \Gamma(g)$ and $\Gamma(g) \subseteq \Gamma(h)$ hold, and inclusion of sets is transitive as we showed in the unit on set theory earlier this term. We conclude that $\Gamma(f) \subseteq \Gamma(h)$, so $f \subseteq h$ as needed.

(c) We prove that $\displaystyle\bigcup_{n \in \mathbb{N}} f_n$ is a function by contradiction. Assume that $\displaystyle\bigcup_{n \in \mathbb{N}} f_n$ is not a function, hence it fails the definition of a function. Therefore, $\exists\, (x, y_1), (x, y_2) \in \Gamma\left(\displaystyle\bigcup_{n \in \mathbb{N}} f_n\right)$ with $y_1 \neq y_2$. $(x, y_1) \in \Gamma\left(\displaystyle\bigcup_{n \in \mathbb{N}} f_n\right)$ $\implies \exists\, m$ such that $(x, y_1) \in \Gamma(f_m)$. $(x, y_2) \in \Gamma\left(\displaystyle\bigcup_{n \in \mathbb{N}} f_n\right) \implies \exists\, p$ such that $(x, y_2) \in \Gamma(f_p)$. We now distinguish three cases based on how $p, m \in \mathbb{N}$ related to each other:

Case 1: $p = m$. Then $(x, y_1), (x, y_2) \in \Gamma(f_p)$, but $y_1 \neq y_2$. Therefore, f_p is not a function. That is clearly a CONTRADICTION to our hypotheses.

Case 2: $m < p$. By part (a), $\Gamma(f_m) \subseteq \Gamma(f_p)$ as $f_n \subseteq f_{n+1}$ for every $n \geq 1$, hence $\Gamma(f_n) \subseteq \Gamma(f_{n+1})$. Therefore, $(x, y_1), (x, y_2) \in \Gamma(f_p)$, but $y_1 \neq y_2$. Therefore, f_p is not a function. That is clearly a CONTRADICTION to our hypotheses.

Case 3: $p < m$. By part (a), $\Gamma(f_p) \subseteq \Gamma(f_m)$ as $f_n \subseteq f_{n+1}$ for every $n \geq 1$, hence $\Gamma(f_n) \subseteq \Gamma(f_{n+1})$. Therefore, $(x, y_1), (x, y_2) \in \Gamma(f_m)$, but $y_1 \neq y_2$. Therefore, f_m is not a function. Once again, this is a CONTRADICTION to our hypotheses.

In all three cases, we have obtained a contradiction, which completes the proof that $\bigcup_{n\in\mathbb{N}} f_n$ is a function. To prove that $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A, B)$, we must show that the domain of $\bigcup_{n\in\mathbb{N}} f_n$ is a subset of A and that the codomain of $\bigcup_{n\in\mathbb{N}} f_n$ is a subset of B. This assertion is easier to prove than it looks a priori. $\forall (x, y) \in \Gamma\left(\bigcup_{n\in\mathbb{N}} f_n\right)$, $\exists m \in \mathbb{N}^*$ such that $(x, y) \in \Gamma(f_m)$, but $f_m \in Funct(A, B)$, so $x \in A$ and $y \in B$. Clearly, $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A, B)$.

(d) For every sequence of functions $\{f_n\}_{n=1,2,\ldots} = \{f_1, f_2, \ldots\}$ in $Funct(A, B)$ satisfying that $f_n \subseteq f_{n+1}$ for every $n \geq 1$, let the domain of f_n be some A_n such that $A_n \subseteq A$. We will prove that $Dom\left(\bigcup_{n\in\mathbb{N}} f_n\right) = \bigcup_{n\in\mathbb{N}} Dom(f_n)$ via inclusion in both directions.

"$\subseteq$" For all $\forall x \in Dom\left(\bigcup_{n\in\mathbb{N}} f_n\right) \subseteq A$, $\exists! y \in B$ (recall the $\exists!$ the "one and only one quantifier from the beginning of the term) such that $(x, y) \in \Gamma\left(\bigcup_{n\in\mathbb{N}} f_n\right)$ because we have proven in part (c) that $\bigcup_{n\in\mathbb{N}} f_n$ is a function and $\bigcup_{n\in\mathbb{N}} f_n \in Funct(A, B)$. $(x, y) \in \Gamma\left(\bigcup_{n\in\mathbb{N}} f_n\right) \implies \exists m$ such that $(x, y) \in \Gamma(f_m)$, but f_m is a function, so $x \in Dom(f_m) = A_m$. Therefore, $x \in \bigcup_{n\in\mathbb{N}} Dom(f_n)$ as needed.

"$\supseteq$" $\forall x \in \bigcup_{n \in \mathbb{N}} Dom(f_n)$, $\exists m \in \mathbb{N}^*$ and $\exists! y \in B$ such that $(x, y) \in \Gamma(f_m)$, but then $(x, y) \in \Gamma\left(\bigcup_{n \in \mathbb{N}} f_n\right)$. $\bigcup_{n \in \mathbb{N}} f_n$ is a function by part (c). Therefore, $x \in Dom\left(\bigcup_{n \in \mathbb{N}} f_n\right)$ as needed.

Marking rubric: 20 marks total. 4 marks for part (a) (1 mark for the base case and 3 marks for the inductive case). 6 marks for part (b) (2 marks for each of the three properties to be proven). 6 marks for part (c) (3 marks for proving that it is a function and 3 marks for proving that it is in $Funct(A, B)$). 4 marks for part (d) (2 marks for each direction of the inclusion).

4) (10 points) Let $\mathbb{R}[x]$ be the set of all polynomials in variable x with coefficients in $\mathbb{R}$. In other words,

$$\mathbb{R}[x] = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid n \in \mathbb{N} \text{ and } a_0, \ldots, a_n \in \mathbb{R}\}.$$

(a) Give three examples of elements of $\mathbb{R}[x]$.

(b) Prove that $(\mathbb{R}[x], +)$, $\mathbb{R}[x]$ with addition as the operation, is a semi-group.

(c) Is $(\mathbb{R}[x], +)$ a monoid? Justify your answer.

(d) Does $(\mathbb{R}[x], +)$ have invertible elements? If so, which of its elements are invertible? Justify your answer.

Solution: (a) Many examples can be given. Here are three: 0, $\pi x^2 + 1$, and $\sqrt{2}x - 5$.

(b) Adding two polynomials $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ in $\mathbb{R}[x]$ gives us another polynomial in $\mathbb{R}[x]$ because we factor out x^j for every j with $0 \leq j \leq \max\{n, m\}$ and its coefficient will be $a_j + b_j$, which is a real number for every j, since $a_j, b_j \in \mathbb{R}$ and $(\mathbb{R}, +)$ is a semigroup as proven in lecture. Therefore, $+$ on $\mathbb{R}[x]$ is a binary operation. Since $+$ is associative on $\mathbb{R}$ and we gather all coefficients of each x^j in turn, we conclude that $+$ is associative on $\mathbb{R}[x]$. Therefore, $(\mathbb{R}[x], +)$ is a semi-group.

(c) Clearly, $0 \in \mathbb{R}[x]$, and $\forall p \in \mathbb{R}[x]$, $p + 0 = 0 + p = p$. Therefore, the zero polynomial 0 is the identity element of $(\mathbb{R}[x], +)$. We conclude that $(\mathbb{R}[x], +)$ is a monoid.

(d) $\forall\, p \in \mathbb{R}[x]$, $p = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Take $q = -a_n x^n - a_{n-1} x^{n-1} - \cdots - a_1 x - a_0$, where we reverse the signs of all coefficients of p unless they are equal to zero. Clearly, $p + q = 0$, so every element of $\mathbb{R}[x]$ is invertible.

Marking rubric: 10 marks total. 1 mark for part (a) and 3 marks for each of the parts (b)-(d).

MAU22C00: ASSIGNMENT 3
DUE BY FRIDAY, DEC. 18 BEFORE MIDNIGHT
UPLOAD SOLUTION ON BLACKBOARD

Please write down clearly both your name and your student ID number on everything you hand in. Please attach a cover sheet with a declaration confirming that you know and understand College rules on plagiarism. Details can be found on http://tcd-ie.libguides.com/plagiarism/declaration**.**

The assignment may be submitted without penalty until Wednesday, January 6 before midnight.

1) (40 points) Let L be the language over the alphabet $A = \{0, 1\}$ consisting of all words containing an even number of zeroes.

(a) Draw a finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions. Make sure you justify it accepts all strings in the language L and no others.
(b) Devise a regular grammar in normal form that generates the language L. Be sure to specify the start symbol, the non-terminals, and all the production rules.
(c) Write down a regular expression that gives the language L and justify your answer.
(d) Prove from the definition of a regular language that the language L is regular.

2) (20 points) Let M be the language over the alphabet $\{a, r, c\}$ given by $M = \{a^i r^j c^k \mid i, j, k \geq 0 \quad i = 2j - k\}$.

(a) Use the Pumping Lemma to show this language is not regular.
(b) Write down the production rules of a context-free grammar that generates exactly M and justify your answer.

MAU22C00: ASSIGNMENT 3 SOLUTIONS

1) (40 points) Let L be the language over the alphabet $A = \{0, 1\}$ consisting of all words containing an even number of zeroes.

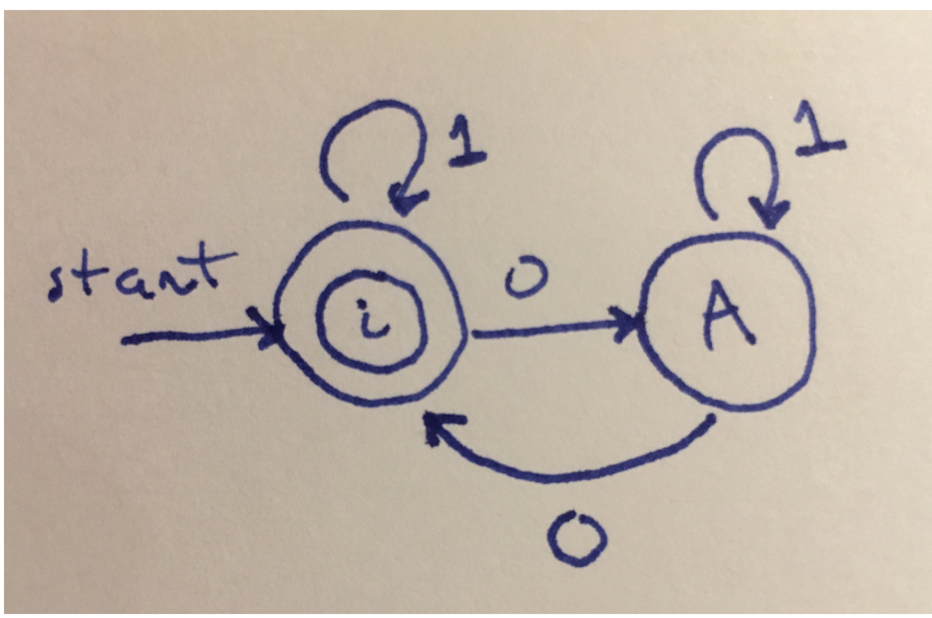
(a) Draw a finite state acceptor that accepts the language L. Carefully label all the states including the starting state and the finishing states as well as all the transitions. Make sure you justify it accepts all strings in the language L and no others.

(b) Devise a regular grammar in normal form that generates the language L. Be sure to specify the start symbol, the non-terminals, and all the production rules.

(c) Write down a regular expression that gives the language L and justify your answer.

(d) Prove from the definition of a regular language that the language L is regular.

Solution: (a) Here is the picture of our finite state acceptor:



Zero is an even number because $0 = 2 \cdot 0$. Therefore, the initial state corresponding to zero zeroes has to be an accepting state. Whichever number of 1's we receive as input does not modify the number of zeroes in the string, so we stay in the initial state. When we receive the input 0, we move to a new state. Let us call it A. This is a non accepting state as it corresponds to 1 zero, which is an odd number of zeroes. Whichever number of 1's we receive as input does not modify the number of zeroes in the string, so we stay in this state A. If we receive the input 0, then we can return to the initial state, which is an accepting state. We will have had 2 zeroes, an even number, and however many 1's, whose number is of no concern. It turns out this is all we need to do. All strings with an even number of zeroes will take us to the initial state, which is an accepting state, and an odd number of zeroes will take us to state A, which is a non-accepting state (convince yourself by trying out strings with 3 zeroes, 4 zeroes, etc.) Please note that other correct solutions with more than two states exist, so your own solution may look different.

Marking rubric: 10 marks total, 6 marks for drawing the finite state acceptor and 4 marks for the justification.

(b) The finite state acceptor has two states $\{i, A\}$, where i is the initial state. Correspondingly, we use two non-terminals in our regular grammar: the start symbol $\langle S\rangle$ corresponding to the initial state i and $\langle A\rangle$ corresponding to state A. We first write the production rules corresponding to the transitions out of the initial state i :

(1) $\langle S\rangle \to 1\langle S\rangle$.
(2) $\langle S\rangle \to 0\langle A\rangle$.

Next, we write the production rules corresponding to the transitions out of state A :

(3) $\langle A\rangle \to 1\langle A\rangle$.
(4) $\langle A\rangle \to 0\langle S\rangle$.
Rules (1)-(4) are of type (i). For each accepting state, we will write down a rule of type (iii). Since there is only one accepting state, i, we have only one such rule:

(7) $\langle S\rangle \to \epsilon$.
Since you are not being asked for a justification, you can simply write down the production rules without any other explanation. Note that if your finite state acceptor in part (a) is more complicated, and you are translating its transitions into production rules of a regular grammar in normal form as we did in lecture, you will end up with more production rules than what is above. As long as your production rules are of a regular grammar in normal form generating the language L, you will receive full marks.

Marking rubric: 10 marks total. 2 marks for each production rule.

(c) $1^* \cup (1^* \circ 0 \circ 1^* \circ 0 \circ 1^*)^*$ is the regular expression that produces L. The 1^* part gives us all strings with no zeroes since as we remarked in part (a), zero is an even number so a string consisting only of 1's has an even number of zeroes. The other part $(1^* \circ 0 \circ 1^* \circ 0 \circ 1^*)^*$ yields all strings with an even number of zeroes. The Kleene star on the outside guarantees that whatever is inside gets repeated m times for $m \in \mathbb{N}$, where m could also be zero. Therefore, what we have inside the parentheses should be the general expression for a string with exactly two zeroes. Indeed, it is. Before the two zeroes we could have a string of 1's hence the first factor of 1^*. We note here that 1^* also yields the empty word, ϵ, corresponding to no 1's before the first zero. In between the two zeroes we could have a string of 1's, hence the second factor of 1^*. After the two zeroes we could have a string of 1's, hence the third factor of 1^*.

Marking rubric: 10 marks total. 1 mark for recognising the need for the 1^* part that gives strings with no zeroes, 5 marks for the block that represents all strings with exactly two zeroes, 1 mark for realising that one needs a Kleene star after that block, and 3 marks for the justification.

(d) Let the alphabet $A = \{0, 1\}$. Recall that the definition of a regular language allows for finite subsets of A^*, the Kleene star, concatenations, and unions. We need to define a finite sequence of languages L_j for $j = 1, \ldots, n$ such that $L_n = L$ and each L_j is a finite subset of A^*, the Kleene star of a previous L_j, a concatenation of two previous L_j's or a union of two previous L_j's. We let $L_1 = \{0\}$ and $L_2 = \{1\}$. Both are finite subsets of A^*. We let $L_3 = L_2^*$. $L_4 = L_3 \circ L_1$. $L_5 = L_4 \circ L_3$. $L_6 = L_5 \circ L_1$. $L_7 = L_6 \circ L_3$. $L_8 = L_7^*$. $L_9 = L_3 \cup L_8 = L$, and we are done. $n = 9$ in this case. Note that this process is constructing step by step the regular expression from part (c). Note also that you may have a different regular expression in part (c) that is equivalent and hence a different number of L_j's and differences in how your L_j's are defined compared to the solution here.

Marking rubric: 10 marks total. 1 mark for demonstrating understanding of the definition of a regular language and 1 mark for each L_j.

2) (20 points) Let M be the language over the alphabet $\{a, r, c\}$ given by $M = \{a^i r^j c^k \mid i, j, k \geq 0 \quad i = 2j - k\}$.

(a) Use the Pumping Lemma to show this language is not regular.
(b) Write down the production rules of a context-free grammar that generates exactly M and justify your answer.

Solution: (a) Note that $i = 2j - k \iff 2j = i + k$. Assume that this language is regular and hence it has a pumping length p. We seek to obtain a contradiction. We can choose any word w we wish in order to obtain the contradiction as long as w has length at least p. We choose $w = a^p r^p c^p$, which is in the language because $2p = p + p$. According to the Pumping Lemma, since $w \in L$ and $|w| \geq p$, there exists a decomposition of w as $w = xuy$ such that $|xu| \leq p$, $|u| > 0$, and $xu^m y \in L$ for every $m \in \mathbb{N}$. Since $w = a^p r^p c^p = xuy$ and $|xu| \leq p$, $x = a^i$ and $u = a^j$ with $j > 0$ and $i + j \leq p$. In other words, both x and u can only be composed of a's, so there is only this one case to consider. We see immediately that $xu^2 y = xuuy = a^{p+j} r^p c^p$, and $2p < p + j + p$ because $j > 0$. We conclude that $xu^2 y \notin L$, which is the contradiction we were seeking. Therefore, L is not a regular language.

Marking rubric: 10 marks total. Depending upon which word w you chose, there might be one case as above or several. No marks will be awarded if you worked with specific numeric values for i, j, and k – a very common mistake.

(b) We note that the case $i = j = k = 0$ corresponding to $a^0 r^0 c^0 = \epsilon \in L$. Proceeding from here, in order for $i = 2j - k \iff 2j = i + k$ to hold, if we add one r, we must add two characters that are either a or c. This gives three possibilities in total:

(i) We add 2 a's and zero c's.

(ii) We add zero a's and 2 c's.

(iii) We add one a and one c.

With these observations in mind, here is a context-free grammar that generates L :

(1) $\langle S\rangle \to \langle A\rangle\langle B\rangle$.
(2) $\langle A\rangle \to \epsilon$.
(3) $\langle B\rangle \to \epsilon$.
(4) $\langle A\rangle \to aa\langle A\rangle r$.
(5) $\langle B\rangle \to r\langle B\rangle cc$.
(6) $\langle A\rangle \to a\langle C\rangle r$.
(7) $\langle B\rangle \to \langle D\rangle c$.
(8) $\langle C\rangle \to \langle A\rangle$.
(9) $\langle D\rangle \to \langle B\rangle$.

Rules (1), (2), and (3) together generate the empty word, which is in L as explained above. (4) takes care of case (i), (5) takes care of case (ii), and rules (6), (7), (8), and (9) together take care of case (iii). Note that without using the two extra non-terminals $\langle C\rangle$ and $\langle D\rangle$, we cannot ensure that rules (6) and (7) are applied as a pair, which is exactly what needs to be done. Rules (8) and (9) return us to the original non-terminals $\langle A \rangle$ and $\langle B \rangle$ since the next addition of one r may need to happen according to case (i) or (ii).

Marking rubric: 10 marks total. 1 mark for the correct rules that yield ϵ, 3 marks for the correct rules that address cases (i) and (ii), 3 marks for the correct rules that address case (iii) (which is harder), and 3 marks for the justification. Please note that your own solution may differ quite a bit from the one above and still be correct.

MAU22C00: ASSIGNMENT 4
DUE FRIDAY, MARCH 12 BEFORE MIDNIGHT
UPLOAD ON BLACKBOARD

Please attach a cover sheet with a declaration confirming that you know and understand College rules on plagiarism. Details can be found on http://tcd-ie.libguides.com/plagiarism/declaration**.**

1) (30 points) Let (V, E) be the graph with vertices a, b, c, d, e, f, g, h, i, j, and k, and edges ab, bc, ac, cd, ch, de, eh, ef, eg, fg, hi, ij, hk, jk, and jh.

(a) Draw this graph.
(b) Write down this graph's incidence table and its incidence matrix.
(c) Write down this graph's adjacency table and its adjacency matrix.
(d) Is this graph complete? Justify your answer.
(e) Is this graph bipartite? Justify your answer.
(f) Is this graph regular? Justify your answer.
(g) Does this graph have any regular subgraph? Justify your answer.
(h) Give an example of an isomorphism φ from the graph (V, E) to itself satisfying that $\varphi(i) = k$.
(i) Is the isomorphism from part (h) unique or can you find another isomorphism ψ that is distinct from φ but also satisfies that $\psi(i) = k$? Justify your answer.
(j) Is this graph connected? Justify your answer.
(k) Does this graph have an Eulerian trail? Justify your answer.
(l) Does this graph have an Eulerian circuit? Justify your answer.
(m) Does this graph have a Hamiltonian circuit? Justify your answer.
(n) Is this graph a tree? Justify your answer.

2) (10 points) Prove that a connected graph (V, E) is a tree if and only if adding an edge between any two vertices in (V, E) creates exactly one circuit.

3) Consider the connected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L and with edges, listed with associated costs, in the following table:

FJ	JK	CH	EI	DJ	AB	EH	FK	BG	GJ	BC	AF
1	1	1	1	2	2	2	2	3	3	3	3
AC	EF	DF	DL	HI	CE	BD	AD	AE	GL	EK	JL
3	4	4	4	5	6	6	6	6	7	7	8

(a) (2 points) Draw the graph and label each edge with its cost.
(b) (9 points) Determine the minimum spanning tree generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.
(c) (9 points) Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex D, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

4) (10 points) Let $(\mathcal{V}, \mathcal{E})$ be the directed graph with vertices A, B, C, D, and E and edges (A, A), (C, A) (A, B), (B, C), (D, C), (C, E), (E, E), and (E, D).
(a) Draw this graph.
(b) Write down this graph's adjacency matrix.
(c) Give an example of an isomorphism φ from the graph $(\mathcal{V}, \mathcal{E})$ to itself such that $\varphi(A) = E$. Note that an isomorphism of directed graphs should also respect the direction of the edges.

5) (10 points) Let R be a relation on a set $V = \{a, b, c, d, e, f\}$ given by

$$R = \{(a, c), (a, d), (c, c), (d, d), (b, b), (e, e), (b, e), (e, f), (f, e), (e, b), (f, f)\}.$$

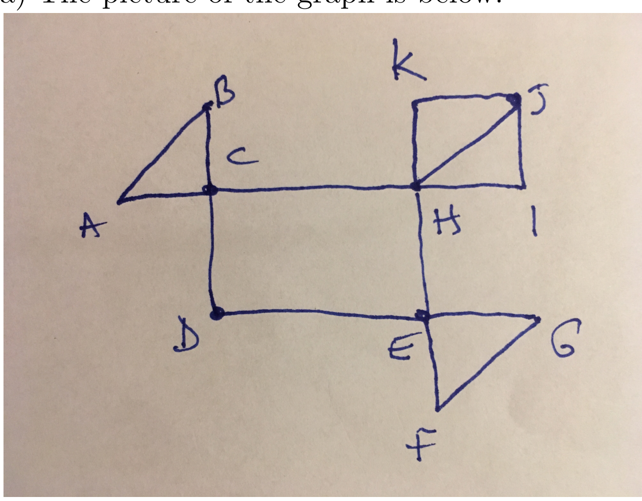
(a) Using the one-to-one correspondence between relations on finite sets and directed graphs, draw the directed graph corresponding to the relation R.
(b) Is R an equivalence relation? Justify your answer.
(c) If R is not an equivalence relation, which ordered pairs would have to be added to R to make it into an equivalence relation?

MAU22C00: ASSIGNMENT 4 SOLUTIONS

1) (30 points) Let (V, E) be the graph with vertices a, b, c, d, e, f, g, h, i, j, and k, and edges ab, bc, ac, cd, ch, de, eh, ef, eg, fg, hi, ij, hk, jk, and jh.

(a) Draw this graph.
(b) Write down this graph's incidence table and its incidence matrix.
(c) Write down this graph's adjacency table and its adjacency matrix.
(d) Is this graph complete? Justify your answer.
(e) Is this graph bipartite? Justify your answer.
(f) Is this graph regular? Justify your answer.
(g) Does this graph have any regular subgraph? Justify your answer.
(h) Give an example of an isomorphism φ from the graph (V, E) to itself satisfying that $\varphi(i) = k$.
(i) Is the isomorphism from part (h) unique or can you find another isomorphism ψ that is distinct from φ but also satisfies that $\psi(i) = k$? Justify your answer.
(j) Is this graph connected? Justify your answer.
(k) Does this graph have an Eulerian trail? Justify your answer.
(l) Does this graph have an Eulerian circuit? Justify your answer.
(m) Does this graph have a Hamiltonian circuit? Justify your answer.
(n) Is this graph a tree? Justify your answer.

1(a) The picture of the graph is below.



(b) (2 points: 1 point each the table and the matrix) The incidence table is

	ab	bc	ac	cd	ch	de	eh	ef	eg	fg	hi	ij	hk	jk	jh
a	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
b	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
c	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
d	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
e	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
f	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0
g	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
h	0	0	0	0	1	0	1	0	0	0	1	0	1	0	1
i	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
j	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
k	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0

and the incidence matrix is

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}.$$

(c) (2 points: 1 point each the table and the matrix) The adjacency table is

	a	b	c	d	e	f	g	h	i	j	k
a	0	1	1	0	0	0	0	0	0	0	0
b	1	0	1	0	0	0	0	0	0	0	0
c	1	1	0	1	0	0	1	0	0	0	0
d	0	0	1	0	1	0	0	1	0	0	0
e	0	0	0	1	0	1	1	1	0	0	0
f	0	0	0	0	1	0	1	0	0	0	0
g	0	0	0	0	1	1	0	0	0	0	0
h	0	0	1	0	1	0	0	0	1	1	1
i	0	0	0	0	0	0	0	1	0	1	0
j	0	0	0	0	0	0	0	1	1	0	1
k	0	0	0	0	0	0	0	1	0	1	0

and the adjacency table is

$$\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}.$$

(d) (2 points: 1 for the answer and 1 for the justification) No, for example edge af is not part of the graph.

(e) (2 points: 1 for the answer and 1 for the justification) No, the graph contains a complete subgraph given by vertices a, b, and c, and edges ab, bc, ac. Given that in this subgraph every vertex is connected to every other vertex, and this complete subgraph has more than two vertices, it cannot be partitioned into two sets so that every edge goes from a vertex in one set to a vertex in the other set.

(f) (2 points: 1 for the answer and 1 for the justification) No, $\deg(a) = 2$, while $\deg(h) = 5$.

(g) (2 points: 1 for the answer and 1 for the justification) Yes, it contains the regular subgraph given by vertices a, b, and c, and edges ab, bc, ac.

(h) (4 points: points decked if not all vertex assignments in the map lead to an isomorphism) $\varphi(a) = b$, $\varphi(b) = a$, $\varphi(c) = c$, $\varphi(d) = d$, $\varphi(e) = e$, $\varphi(f) = g$, $\varphi(g) = f$, $\varphi(h) = h$, $\varphi(i) = k$, $\varphi(k) = i$, and $\varphi(j) = j$.

(i) (2 points: 1 for the answer and 1 for the justification) No, it is not unique. The isomorphism $\varphi(a) = a$, $\varphi(b) = b$, $\varphi(c) = c$, $\varphi(d) = d$, $\varphi(e) = e$, $\varphi(f) = f$, $\varphi(g) = g$, $\varphi(h) = h$, $\varphi(i) = k$, $\varphi(k) = i$, and $\varphi(j) = j$ also exchanges vertices i and k but is distinct from the one given in part (h).

(j) (2 points: 1 for the answer and 1 for the justification) Yes, there is a walk from every vertex to every other vertex.

(k) (2 points: 1 for the answer and 1 for the justification) Yes, as we have exactly two vertices with odd degree ($\deg(h) = 5$ and $\deg(j) = 3$)

while the rest of the vertices have even degrees. Our Eulerian trail would have to start at h and end at j or the other way around.

(l) (2 points: 1 for the answer and 1 for the justification) No, as we would need all vertices to have even degree. Here we have two vertices of odd degree ($\deg(h) = 5$ and $\deg(j) = 3$).

(m) (2 points: 1 for the answer and 1 for the justification) No, since to visit vertices a and b we would have to visit vertex c twice.

(n) (2 points: 1 for the answer and 1 for the justification) No, as there is a circuit $abca$. There are in fact quite a number of circuits.

2) (10 points) Prove that a connected graph (V, E) is a tree if and only if adding an edge between any two vertices in (V, E) creates exactly one circuit.

This statement is an equivalence. To prove it, we will prove each implication separately. Remember that a tree is a connected graph with no cycles (circuits). Since the graph (V, E) is already assumed to be connected, our equivalence amounts to (V, E) contains no circuits if and only if adding an edge between any two vertices in (V, E) creates exactly one circuit.

"$\Leftarrow$" We start with the hypothesis that adding an edge between any two vertices in (V, E) creates exactly one circuit. Remove the added edge to get back to (V, E). Since adding that edge created exactly one circuit, it is clear that (V, E) itself has no circuits, hence (V, E) is a tree.

"$\Rightarrow$" We start with the hypothesis that (V, E) is a tree, hence that it has no circuits. Only an edge that does not already exist in E can be added. Let $u, v \in V$ be distinct vertices $u \neq v$ in V such that there is no edge between them, namely $uv \notin E$. Recall that we proved in lecture that between any two distinct vertices of a tree there exists one and only one path. Let the path in (V, E) between u and v be given by $uw_1w_2 \cdots w_pv$ for some $w_1, w_2, \ldots, w_p$ vertices in V. Clearly, adding the edge uv to E creates one and only one circuit, namely $uw_1w_2 \cdots w_pvu$ because there is no other path between u and v except for $uw_1w_2 \cdots w_pv$. □

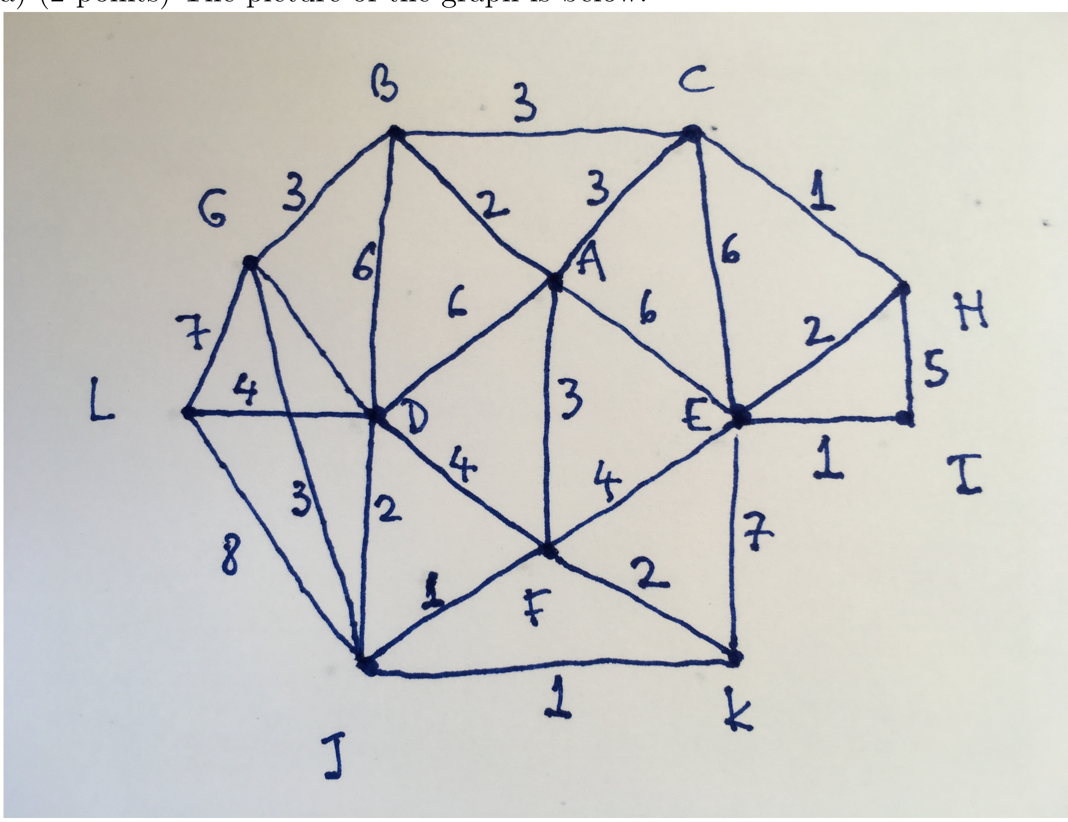
Grading rubric: 10 points, 5 points for each implication in the equivalence.

3) Consider the connected graph with vertices A, B, C, D, E, F, G, H, I, J, K, and L and with edges, listed with associated costs, in the following table:

FJ	JK	CH	EI	DJ	AB	EH	FK	BG	GJ	BC	AF
1	1	1	1	2	2	2	2	3	3	3	3
AC	EF	DF	DL	HI	CE	BD	AD	AE	GL	EK	JL
3	4	4	4	5	6	6	6	6	7	7	8

(a) (2 points) Draw the graph and label each edge with its cost.
(b) (9 points) Determine the minimum spanning tree generated by Kruskal's Algorithm, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.
(c) (9 points) Determine the minimum spanning tree generated by Prim's Algorithm, starting from the vertex D, where that algorithm is applied with the queue specified in the table above. For each step of the algorithm, write down the edge that is added.

3(a) (2 points) The picture of the graph is below:



Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

3(b) The edges are added in the following order: FJ, JK, CH, EI, DJ, AB, EH, BG, GJ, BC, and DL.

Grading rubric: 9 points: roughly 1 point per correct edge in the correct order.

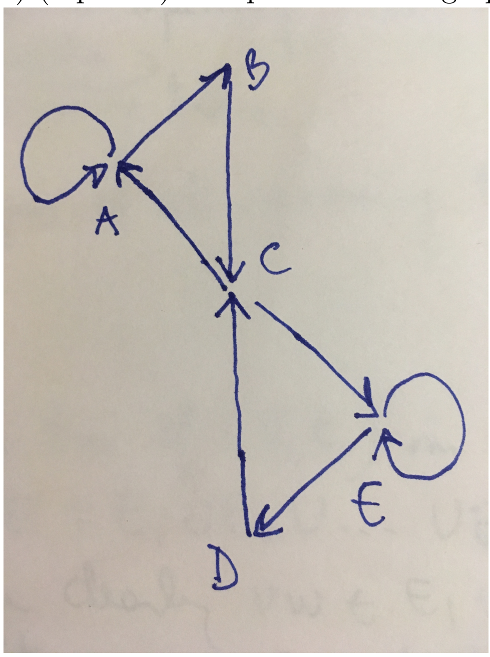
3(c) The edges are added in the following order: DJ, FJ, JK, GJ, BG, AB, BC, CH, EH, EI, and DL.

Grading rubric: 9 points: roughly 1 point per correct edge in the correct order.

4) (10 points) Let $(\mathcal{V}, \mathcal{E})$ be the directed graph with vertices A, B, C, D, and E and edges (A, A), (C, A) (A, B), (B, C), (D, C), (C, E), (E, E), and (E, D).

(a) Draw this graph.
(b) Write down this graph's adjacency matrix.
(c) Give an example of an isomorphism φ from the graph $(\mathcal{V}, \mathcal{E})$ to itself such that $\varphi(A) = E$. Note that an isomorphism of directed graphs should also respect the direction of the edges.

4(a) (2 points) The picture of the graph is below.



Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

4(b) The adjacency matrix is as follows (note that we must stick to the given ordering of vertices; A, B, C, D, E):

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Grading rubric: 2 points: 0 if totally wrong, 1 if mostly fine, 2 if correctly done.

4(c) (6 points) $\varphi(A) = E$, $\varphi(E) = A$, $\varphi(C) = C$, $\varphi(B) = D$, and $\varphi(D) = B$ respects the direction of the edges. Note it takes advantage of the natural symmetry in this directed graph.

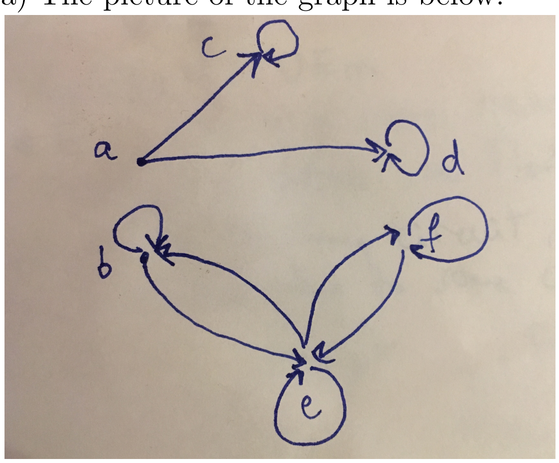
Grading rubric: Roughly 1 point for each correct vertex assignment in the isomorphism map.

5) (10 points) Let R be a relation on a set $V = \{a, b, c, d, e, f\}$ given by

$R = \{(a, c), (a, d), (c, c), (d, d), (b, b), (e, e), (b, e), (e, f), (f, e), (e, b), (f, f)\}$.

(a) Using the one-to-one correspondence between relations on finite sets and directed graphs, draw the directed graph corresponding to the relation R.
(b) Is R an equivalence relation? Justify your answer.
(c) If R is not an equivalence relation, which ordered pairs would have to be added to R to make it into an equivalence relation?

5(a) The picture of the graph is below.



Grading rubric: 2 points: 1 point if the 6 vertices are there, and the extra point for the edges. If only 1, 2 edges are incorrect, full marks given.

5(b) Not an equivalence relation as it is not reflexive (edge (a,a) is missing), not symmetric (edges (c,a) and (d,a) are missing) and transitive (for example, edges (b,e) and (e,f) are there, but edge (b,f) is missing).
Grading rubric: 2 points: 1 point for the answer and 1 points for the justification.

5(c) For reflexivity to be satisfied, add edge (a,a). For symmetry to be satisfied, we add (c,a) and (d,a). To satisfy transitivity in the subgraph with vertices b, e, and f, edges (b,f) and (f,b) need to be added. Given that we have already added (c,a) and (d,a), to achieve transitivity in the subgraph with vertices a, c, and d, we must add (c,d) and (d,c).
Grading rubric: 6 points, roughly a point for each of the seven pairs. No justification is required for this part in fact, so points are awarded strictly for having written down the correct pairs.

MAU22C00: ASSIGNMENT 5
DUE MONDAY, APRIL 12 BEFORE MIDNIGHT
UPLOAD ON BLACKBOARD

Please attach a cover sheet with a declaration confirming that you know and understand College rules on plagiarism. Details can be found on `http://tcd-ie.libguides.com/plagiarism/declaration`**.**

Each of the following problems is worth 5 points:

(1) Let $A = \mathbb{N}\times\mathbb{Z}\times\mathbb{Q}\times\mathbb{C}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

(2) Let A be the the set of points in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy the equation $r^2 = (\sin(\theta)-1)^2$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

(3) Let $A = \{(x, y) \in \mathbb{C}^2 \mid x^6 - 3x^2 + 1 = 0\}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

(4) Let

$$A = \{(x,y) \in \mathbb{R}\times\mathbb{R}^+ \big| 1+xy = 0\} \cap \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-7)^2}{25} + \frac{(y+4)^2}{9} = 1 \right\}.$$

$\mathbb{R}^+$ stands for all positive real numbers. Consider $\mathcal{P}(A)$, the power set of A. Is $\mathcal{P}(A)$ finite, countably infinite or uncountably infinite? Justify your answer.

(5) Let A consist of all 2×2 matrices with entries in the real numbers $\mathbb{R}$ and determinant equal to 1. Is A finite, countably infinite or uncountably infinite? Justify your answer.

(6) Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 2z = 0 \quad \text{and} \quad x + 2y + 3z = 0\}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

(7) Let $A = \{0, 1\}$. Is the language $[(0 \cup \epsilon)^* \circ (1 \cup \epsilon)] \cap (A \circ A)^*$ finite, countably infinite, or uncountably infinite? Justify your answer.

(8) Let A be a countably infinite alphabet. Is A^* finite, countably infinite or uncountably infinite? Justify your answer.

(9) Let $A = \{0, 1, 2, 3, 4, 5\}$. Let the language L consist of all even length strings containing at least three odd letters. Is L finite, countably infinite or uncountably infinite? Justify your answer.

(10) Does there exist a sequence $\{x_1, x_2, x_3, \dots\}$ of languages over a finite alphabet A such that x_i is not a regular language $\forall\, i \geq 1$? Justify your answer.

MAU22C00: ASSIGNMENT 5 SOLUTIONS

Each of the following problems is worth 5 points, 2 points for the answer and 3 points for the justification:

(1) Let $A = \mathbb{N} \times \mathbb{Z} \times \mathbb{Q} \times \mathbb{C}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

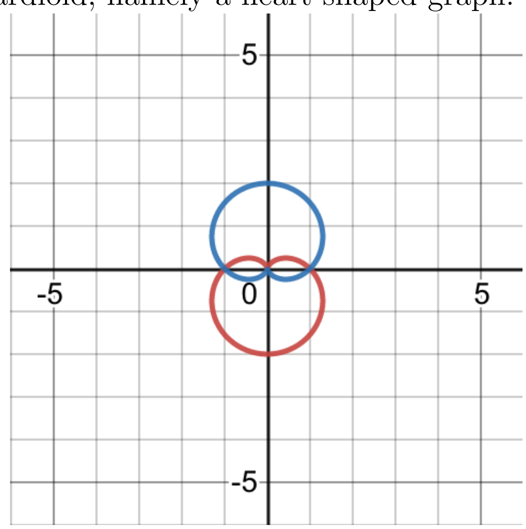
Solution: A is uncountably infinite since it is given by a Cartesian product, which has a factor $\mathbb{C}$ that is uncountably infinite. Here is how we prove that $\mathbb{C}$ is uncountably infinite:

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\} \sim \{(a, b) \mid a, b \in \mathbb{R}\} = \mathbb{R}^2$$

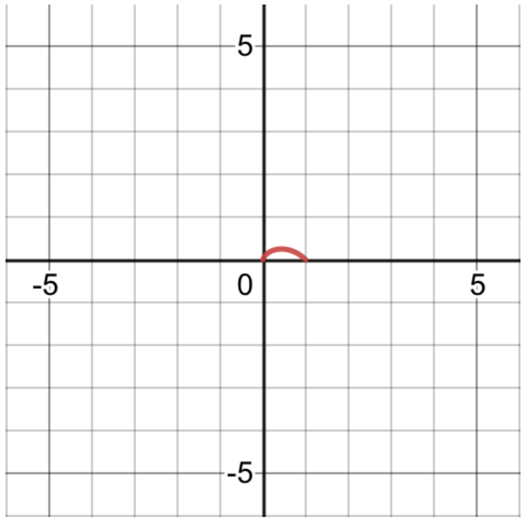
We have already shown that $\mathbb{R}^n$ for $n \geq 1$ is uncountably infinite in tutorial 18. Please refer to the argument given in the solutions to tutorial 18.

(2) Let A be the the set of points in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy the equation $r^2 = (\sin(\theta) - 1)^2$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution: A is uncountably infinite. $r^2 = (\sin(\theta) - 1)^2$ is equivalent to $r = \pm(\sin(\theta) - 1)$. The graph of A is below, where the red graph is $r = 1 - \sin(\theta)$, and the blue graph is $r = \sin(\theta) - 1$. Each of them is a cardioid, namely a heart-shaped graph:



Let us consider only the piece of the red graph $r = 1 - \sin(\theta)$ given by restricting θ to the interval $0 \leq \theta \leq \frac{\pi}{2}$. The graph is below:



Since $\sin(\theta)$ is a bijective function for $\theta \in \left[0, \frac{\pi}{2}\right]$, this red piece as a set in $\mathbb{R}^2$ is in bijective correspondence to $\left[0, \frac{\pi}{2}\right]$, hence with the closed interval $[0, 1]$ (use the bijection $f : \left[0, \frac{\pi}{2}\right] \to [0, 1]$ given by $f(x) = \frac{2}{\pi} x$.) The closed interval $[0, 1]$ contains as a proper set the open interval $(0, 1)$. We proved in lecture that $(0, 1)$ is uncountably infinite. Therefore, $[0, 1]$ must be uncountably infinite as it has an uncountably infinite subset. Thus, $\left[0, \frac{\pi}{2}\right]$ is uncountably infinite, hence that red piece of the cardioid is uncountably infinite. That red piece is a subset of A, however, which means that A has an uncountably infinite subset. We conclude that A must be uncountably infinite.

(3) Let $A = \{(x, y) \in \mathbb{C}^2 \mid x^6 - 3x^2 + 1 = 0\}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

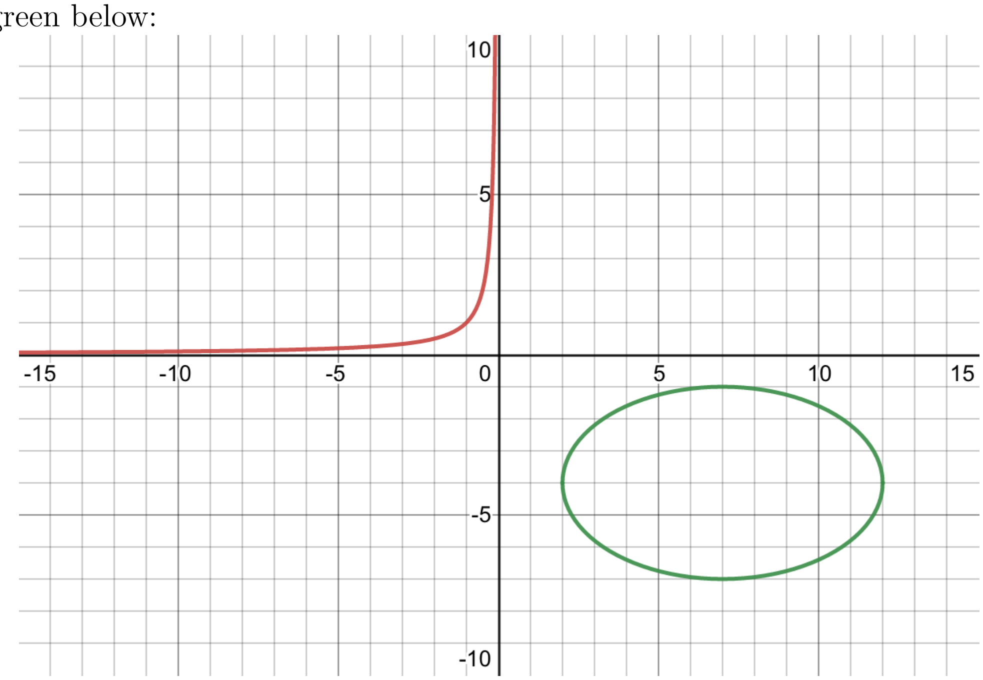
Solution: A is uncountably infinite. By the Fundamental Theorem of Algebra, there are at most six $x \in \mathbb{C}$ that satisfy the equation $x^6 - 3x^2 + 1 = 0$. Recall that the Fundamental Theorem of Algebra tells us that the number of roots of a polynomial equation counted with multiplicity equals the degree of the polynomial, which is 6 here. There is no condition on y, however, which means A is the Cartesian product of a finite set with $\mathbb{C}$. As explained in the solution of problem (1), $\mathbb{C}$ is uncountably infinite, and since A is the Cartesian product containing one uncountably infinite factor, it is itself uncountably infinite.

(4) Let

$$A = \{(x,y) \in \mathbb{R}\times\mathbb{R}^+ \big| 1+xy=0\} \cap \left\{(x,y) \in \mathbb{R}^2 \;\Big|\; \frac{(x-7)^2}{25} + \frac{(y+4)^2}{9} = 1\right\}.$$

$\mathbb{R}^+$ stands for all positive real numbers. Consider $\mathcal{P}(A)$, the power set of A. Is $\mathcal{P}(A)$ finite, countably infinite or uncountably infinite? Justify your answer.

Solution: $\{(x,y) \in \mathbb{R}\times\mathbb{R}^+ \mid 1+xy=0\}$ is the part of the hyperbola $1+xy=0$ satisfying that $y \geq 0$, namely the branch of the hyperbola in the second quadrant. This graph is red below. $\frac{(x-7)^2}{25} + \frac{(y+4)^2}{9} = 1$ is an ellipse with center at the point with coordinates $(7,-4)$, horizontal radius 5 (the square root of 25), and vertical radius 3 (the square root of 9). This ellipse is entirely inside the fourth quadrant. Its graph is green below:



As you can see, the intersection of the red graph with the green one is empty, so $A = \emptyset$. Therefore, the power set of it $\mathcal{P}(A)$ must contain exactly one element, namely $\{\emptyset\}$, the set containing the empty set as we saw in the unit on set theory. Thus, $\mathcal{P}(A)$ is finite.

(5) Let A consist of all 2×2 matrices with entries in the real numbers $\mathbb{R}$ and determinant equal to 1. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution: A consists of elements that are matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying that $ad - bc = 1$ (this is the condition that the determinant is 1 as you hopefully remember from linear algebra, which you studied with Meriel last year). In particular, every element of the form

$$\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

for $a \in \mathbb{R} \setminus \{0\}$ is in A. Let the set of all such elements be called B. Note that B is in one-to-one correspondence with $\mathbb{R} \setminus \{0\}$, but the open interval $(0, 1)$, which we proved to be uncountably infinite in lecture is a subset of $\mathbb{R} \setminus \{0\}$. Therefore, $\mathbb{R} \setminus \{0\}$ is uncountably infinite, and likewise B is uncountably infinite. Therefore, since $B \subset A$, A contains an uncountably infinite subset and hence must be uncountably infinite itself.

(6)] Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 2z = 0 \quad \text{and} \quad x + 2y + 3z = 0\}$. Is A finite, countably infinite or uncountably infinite? Justify your answer.

Solution: As you hopefully remember from Meriel's linear algebra, $3x-y+2z = 0$ determines a plane perpendicular to the vector $(3, -1, 2)$, and $x + 2y + 3z = 0$ determines a plane perpendicular to the vector $(1, 2, 3)$. As these two vectors are linearly independent, the two planes are not parallel but rather must intersect in a line. Therefore, A is a line in $\mathbb{R}^3$, which is in bijective correspondence to any other line in $\mathbb{R}^3$. Consider the x-axis in $\mathbb{R}^3$ (which is a line), and remember that it contains the interval $(0, 1)$, which we proved to be uncountably infinite, as a subset. Therefore, the x-axis is uncountably infinite, and hence the line A is likewise uncountably infinite.

(7) Let $A = \{0, 1\}$. Is the language $[(0 \cup \epsilon)^* \circ (1 \cup \epsilon)] \cap (A \circ A)^*$ finite, countably infinite, or uncountably infinite? Justify your answer.

Solution: As you showed in a tutorial, any regular expression gives a regular language that is either finite or countably infinite, and since our language is given by a regular expression, it can only be finite or countably infinite. Note that

$$(0^* \circ \epsilon) \cap (A \circ A)^* = 0^* \cap (A \circ A)^* = \{0^{2m} \mid m \in \mathbb{N}\}$$

is a sublanguage of our language, which is countably infinite because it can be put into one-to-one correspondence with $\mathbb{N}$, which is countably infinite, via the bijection $f(m) = 0^{2m}$. We conclude that the language given by $[(0 \cup \epsilon)^* \circ (1 \cup \epsilon)] \cap (A \circ A)^*$, which is at most countably infinite, contains a countably infinite sublanguage, so it must be countably infinite.

(8) Let A be a countably infinite alphabet. Is A^* finite, countably infinite or uncountably infinite? Justify your answer.

Solution: $A^* = \bigcup_{j=0}^{\infty} A^j$, where A^j is the set of words of length j over the alphabet A. Note that A^j is in one-to-one correspondence with the Cartesian product of A with itself j times. When $j = 0$, $A^0 = \{\epsilon\}$, the set containing just the empty word, which is finite, but if $j \geq 1$, A^j is the Cartesian product of the countably infinite set A with itself j times, which is countably infinite by a theorem proven in lecture. Therefore, $A^* = \{\epsilon\} \cup \bigcup_{j=1}^{\infty} A^j$ is the union of one element with a countably infinite union of countably infinite sets, which is countably infinite by the theorem we proved in lecture. We conclude that A^* is countably infinite.

(9) Let $A = \{0, 1, 2, 3, 4, 5\}$. Let the language L consist of all even length strings containing at least three odd letters. Is L finite, countably infinite or uncountably infinite? Justify your answer.

Solution: Since A is finite, A^* is countably infinite as proven in lecture. $L \subset A^*$, so L is either finite or countably infinite. We will prove that L is countably infinite by showing it contains a countably infinite subset. This is once again a sandwich argument. Consider $L' = \{1^{2m}3^{2m}5^{2m} \mid m \in \mathbb{N}^*\}$. Clearly, $L' \subset L$, and L' is in one-to-one correspondence with the countably infinite set $\mathbb{N}^*$ via the bijection $f : \mathbb{N}^* \to L'$ given by $f(m) = 1^{2m}3^{2m}5^{2m}$. Therefore, L contains a countably infinite subset and hence must be countably infinite.

(10) Does there exist a sequence $\{x_1, x_2, x_3, \ldots\}$ of languages over a finite alphabet A such that x_i is not a regular language $\forall\, i \geq 1$? Justify your answer.

Solution: Yes, such a sequence exists, and in fact, uncountably many such sequences exist. Some of you constructed the sequence explicitly. I will give here a formal proof for the existence of such a sequence. Let B be the set of all languages over A. B is uncountably infinite as proven in lecture. Let C be the set of all regular languages over A. C is countably infinite as proven in lecture. As we showed in lecture in the proof that $\mathbb{R}$ is uncountably infinite, the set difference between an uncountably infinite set and a countably infinite one must be uncountably infinite. Therefore, the set $B \setminus C$ of non-regular languages over the alphabet A must be uncountably infinite. Since $B \setminus C$ is uncountably infinite, it is clearly non-empty. Therefore, $\exists\, x_1 \in B \setminus C$. Now, consider $B \setminus (C \cup \{x_1\})$. We have taken an element out of an uncountably infinite set. Therefore, $B \setminus (C \cup \{x_1\})$ is uncountably infinite, hence non-empty. Therefore, $\exists\, x_2 \in B \setminus (C \cup \{x_1\})$ and so on. Inductively, $B \setminus (C \cup \{x_1, x_2, \ldots, x_{n-1}\})$ is uncountably infinite, hence non-empty. Therefore, $\exists\, x_n \in B \setminus (C \cup \{x_1, x_2, \ldots, x_{n-1}\})$. By construction, each x_i is not a regular language. Therefore, $\{x_1, x_2, x_3, \ldots\}$ is a sequence of the kind we were trying to construct, so such a sequence exists.

MAU22C00: ASSIGNMENT 6
DUE FRIDAY, APRIL 23 BEFORE MIDNIGHT
UPLOAD ON BLACKBOARD

Please write down clearly both your name and your student ID number on everything you hand in. Please attach a cover sheet with a declaration confirming that you know and understand College rules on plagiarism. Details can be found on http://tcd-ie.libguides.com/plagiarism/declaration.

This assignment may be uploaded onto Blackboard up to Friday, April 30 before midnight without any lateness penalty.

1) (20 points)

(a) Let L be the language over the alphabet $A = \{a, l, p\}$ consisting of all words containing both a and l. Write down the algorithm of a Turing machine that decides L. Process the following strings according to your algorithm: p, al, pap, pla, and $aappla$.

(b) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

2) (10 points) Let the alphabet $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Write down the algorithm of an enumerator that prints out EXACTLY ONCE every string in the language

$$L = \{3m + 1 \mid m \in \mathbb{N}\}$$

that is EVEN.

3) (20 points)

(a) The emptiness testing problem for phrase structure grammars (PSG's) is given by the language

$$E_{PSG} = \{\langle G \rangle \mid G \text{ is a phrase structure grammar and } L(G) = \emptyset\}.$$

Is it possible to modify the marking argument used in lecture to prove that the emptiness testing problem for context-free grammars E_{CFG} is Turing-decidable in order to prove that E_{PSG} is also Turing-decidable? Justify your answer.

(b) Is E_{PSG} a finite set, a countably infinite set, or an uncountably infinite set? Justify your answer. You may assume without proof the fact that if G is a phrase structure grammar, then $L(G)$ is a Turing-recognisable language.

MAU22C00: ASSIGNMENT 6 SOLUTIONS

1) (20 points)

(a) Let L be the language over the alphabet $A = \{a, l, p\}$ consisting of all words containing both a and l. Write down the algorithm of a Turing machine that decides L. Process the following strings according to your algorithm: p, al, pap, pla, and $aappppla$.

(b) Write down the transition diagram of the Turing machine from part (a) carefully labelling the initial state, the accept state, the reject state, and all the transitions specified in your algorithm.

Solution: 1)

(a) Here is the algorithm for deciding L, which is actually a regular language:

(1) If there is a blank in the first cell, then REJECT. If p is in the first cell, then move right. If a is in the first cell, then move right and go to step 3. If l is in the first cell, then move right and go to step 4.

(2) If a is in the current cell, then move right and go to step 3. If l is in the current cell, then move right and go to step 4. If there is a blank in the current cell, then REJECT.

(3) If a or p is in the current cell, then move right and repeat this step. If l is in the current cell, then ACCEPT. If there is a blank in the current cell, then REJECT.

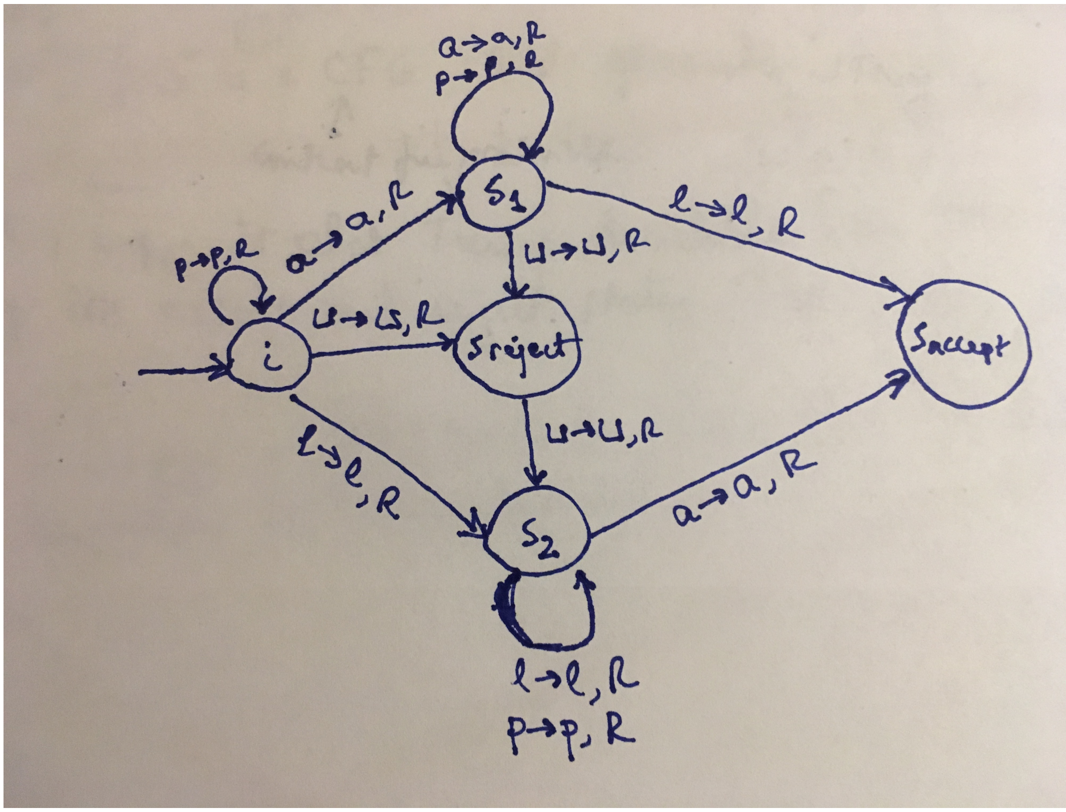
(4) If l or p is in the current cell, then move right and repeat this step. If a is in the current cell, then ACCEPT. If there is a blank in the current cell, then REJECT.

We will use a dot over a character to denote where the tape head is located. Here is how the following strings are treated:

- $\dot{p}\textvisiblespace \to p\dot{\textvisiblespace} \to$ REJECT at step 2.
- $\dot{a}l\textvisiblespace \to a\dot{l}\textvisiblespace \to$ ACCEPT at step 3.
- $\dot{p}ap\textvisiblespace \to p\dot{a}p\textvisiblespace \to pa\dot{p}\textvisiblespace \to pap\dot{\textvisiblespace} \to$ REJECT at step 3.
- $\dot{p}la\textvisiblespace \to p\dot{l}a\textvisiblespace \to pl\dot{a}\textvisiblespace \to$ ACCEPT at step 4.
- $\dot{a}appppla\textvisiblespace \to a\dot{a}ppppla\textvisiblespace \to aa\dot{p}pppla\textvisiblespace \to aap\dot{p}ppla\textvisiblespace \to aapp\dot{p}pla\textvisiblespace \to$ $aapppp\dot{l}a\textvisiblespace \to$ ACCEPT at step 3.

(b) Here is the transition diagram for

$T = (\{i, s_1, s_2, s_{\text{accept}}, s_{\text{reject}}\}, \{a, l, p\}, \{a, l, p, \sqcup\}, t, i, s_{\text{accept}}, s_{\text{reject}}) :$



Grading Rubric & Remarks: 6 marks for the algorithm, 4 marks for processing the strings and 10 marks for (b). If your algorithm from (a) is incorrect, but your transition diagram in (b) is faithful to what you wrote, no additional marks will be taken off.

For (b), large, clear, well labelled diagrams are a necessity. If it cannot be read, it cannot be awarded marks! Correct, consistent notation is also required.

2) (10 points) Let the alphabet $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Write down the algorithm of an enumerator that prints out EXACTLY ONCE every string in the language

$$L = \{3m + 1 \mid m \in \mathbb{N}\}$$

that is EVEN.

Solution: Let us order the input tape as $\{w_1, w_2, w_3, \dots\} = \{0, 1, 2, 3, 4, \dots\}$. Note that for $3m+1$ to be even, m has to be odd so that $3m$ is odd and

$3m + 1$ is the sum of two odd numbers, hence even. Since m is odd, we can represent m as $2n + 1$ for $n \in \mathbb{N}$. We plug $2n + 1$ into $3m + 1$ to obtain $3(2n + 1) + 1 = 6n + 3 + 1 = 6n + 4$.

$E =$ Given the input tape $\{w_1, w_2, w_3, \dots\}$

(1) If $i = 6n + 5$ for $n \in \mathbb{N}$, then print w_i.

We have $i = 6n + 5$ instead of $i = 6n + 4$ because $i = w_i + 1$, so $w_i = i - 1$.

Grading Rubric: 10 points; other correct solutions exist besides this one.

3) (20 points)

(a) The emptiness testing problem for phrase structure grammars (PSG's) is given by the language

$$E_{PSG} = \{\langle G\rangle \;\big|\; G \text{ is a phrase structure grammar and } L(G) = \emptyset\}.$$

Is it possible to modify the marking argument used in lecture to prove that the emptiness testing problem for context-free grammars E_{CFG} is Turing-decidable in order to prove that E_{PSG} is also Turing-decidable? Justify your answer.

(b) Is E_{PSG} a finite set, a countably infinite set, or an uncountably infinite set? Justify your answer. You may assume without proof the fact that if G is a phrase structure grammar, then $L(G)$ is a Turing-recognisable language.

Solution: 3 (a) In lecture, we proved that the emptiness testing problem for context-free grammars (CFG's) given by

$$E_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$$

is Turing-decidable by defining the following Turing machine:

$M =$ on input $\langle G\rangle$, where G is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variable get marked:
3. Mark any non-terminal $\langle T\rangle$ if G contains a production rule $\langle T\rangle \to u_1 \cdots u_k$, and each symbol (terminal or non-terminal) $u_1, \dots, u_k$ has already been marked.
4. If start symbol $\langle S\rangle$ is not marked, accept; otherwise, reject.

As we can see from step 4, if $\langle S\rangle$ is marked, then the context free grammar will end up generating at least one string as all terminals have already been marked in step 1. Therefore, $L(G) \neq \emptyset$, and we reject G. For a phrase structure grammar, each production rule has the form $v_1 \cdots v_s \to u_1 \cdots u_k$, where $v_1, \ldots, v_s$ consists of terminals and non-terminals and is such that there is at least one non-terminal among them. If we could modify the above argument, then step 3 would read:

3. Mark any terminals and non-terminals $v_1, \ldots, v_s$ if the phrase structure grammar G contains a production rule $v_1 \cdots v_s \to u_1 \cdots u_k$, and each symbol (terminal or non-terminal) $u_1, \ldots, u_k$ has already been marked.

The issue is that we could have the following valid phrase structure grammar:

(1) $\langle S\rangle 0\langle S\rangle \to 10100$.
(2) $\langle S\rangle \to 0\langle S\rangle$.

Rule (2) produces $0^p\langle S\rangle$ for $p \geq 1$, and rule (1) never gets to be applied as we never obtain any string with a substring given by the left-hand side of rule (1), namely $\langle S\rangle 0\langle S\rangle$. As a result, this phrase structure grammar produces the empty language, hence it should be accepted by our Turing machine with that modified step 3. Obviously, the right-hand side of rule (1) only consists of terminals, so by the modified step 3, we would mark all terminals and non-terminals on the left-hand side of rule (1) including the start symbol $\langle S\rangle$. This phrase structure grammar would thus be incorrectly rejected at step 4. Therefore, the marking argument used for CFG's cannot work for PSG's. In fact, E_{PSG} is not Turing-decidable, but it is via far more advanced methods than presented in this module that this assertion is proven.

(b) Given that if G is a phrase structure grammar $L(G)$ is a Turing-recognisable language, there are at most as many languages generated by phrase structure grammars as there are Turing-recognisable languages. We proved in lecture that the set of Turing-recognisable languages is countably infinite. Therefore, E_{PSG} is a subset of a countably infinite set, hence either finite or countably infinite. To rule out that it could be finite, we notice that for any $n \in \mathbb{N}^*$, we can write down a phrase structure grammar G_n with n production rules such that $L(G_n) = \emptyset$. Since $\mathbb{N}^*$ is countably infinite and $\{G_1, G_2, \ldots\} \subset E_{PSG}$, we conclude that E_{PSG} must be countably infinite.

Grading Rubric: 10 points per part. Everyone got full marks for part (a) due to the confusion created by the original statement. For part (b), proving E_{PSG} is the subset of a countably infinite set is worth 5 points. The rest of the argument is worth the remaining 5 points.