CS 112: Homework 5

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Discussion 1A

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Continuous Random Variables

Problem 1. There are N machines where N is a large number. Each machine has one of three possible states and changes states (independently) according to a Markov Chain with transition probabilities

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of machines are in each state?

$$\pi = \pi \mathbf{P}$$

$$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 = 0.35$$

$$\pi_1 = 0.41$$

$$\pi_2 = 0.24$$

$$\boxed{\pi = \begin{bmatrix} 0.35 & 0.41 & 0.24 \end{bmatrix}}$$

Problem 2. Suppose that a node in the computer network is either a host (H), a router (R) or an access point (AP). If the current hop is a host, the next hop will be a host, router, or AP with probabilities 0.5, 0.4, and 0.1, respectively. If the current hop is a router, the next hop will be a host, router or AP with probabilities 0.3, 0.4, and 0.3. If the current hop is an AP, the next hop will be a host, router, or AP with probabilities 0.2, 0.3, 0.5.

a) Find the transition probability matrix $\mathbf{P}$.

$$\boxed{\mathbf{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}}$$

b) Generally, what are the proportions of these three types of nodes?

$$\pi = \pi \mathbf{P}$$

$$\pi_0 = 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2$$

$$\pi_1 = 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 = \frac{21}{62}$$

$$\pi_1 = \frac{23}{62}$$

$$\pi_2 = \frac{18}{62}$$

$$\boxed{\pi = \begin{bmatrix} \frac{21}{62} & \frac{23}{62} & \frac{18}{62} \end{bmatrix}}$$

Problem 3. A system is performing a sequence of tasks. Whether or not the current task will successfully be performed depends on the results of the previous two tasks. Suppose that the previous two takes are both successfully performed, then the current task will be successful with probability 0.7; if the last task was successfull, but the one before last one is a failure, then the current one will be successful with probability 0.5; if the last one was a failure but the one before last one was successful, then the current one will be successful with probability 0.4. If both the last one and the one before the last one failed, then the current one will be successful with probability 0.2.

a) Define appropriate states in order to make the above model a Markov Chain.

Let S = success, and F = failure.

- State 0 (SS): both success
- State 1 (FS): success in last, fail one before last
- State 2 (SF): fail in last, success one before last
- State 3 (FF): both fail

b) Find the transition probabilty matrix $\mathbf{P}$ for the states defined in part (a).

$$\boxed{\mathbf{P} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}}$$

Problem 4. Suppose that there are two computer vision algorithms for detecting objects. According to the statistics, algorithm A has a correct detection rate of 0.7, and algorithm B has a correct detection rate of 0.6. If the algorithm used this time yields correct detection result, then we will use algorithm A next time; if the algorithm used this time yields incorrect detection result, then we will select algorithm B next time. If the algorithm selected for the first time is equally likely to be algorithm A or B, then what is the probability that the algorithm used for the fourth time is algorithm A?

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

$$\mathbf{P}^2 = \begin{bmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{bmatrix}$$

$$\mathbf{P}^3 = \begin{bmatrix} 0.667 & 0.333 \\ 0.666 & 0.334 \end{bmatrix}$$

$$\boxed{\frac{1}{2}\left[P_{11}^3 + P_{21}^3\right] = 0.6665}$$

Problem 5. In a mobile ad hoc network (a network formed by the hosts themselves without the help of infrastructures), three out of every four laptops are connected to a cellphone, while one of every file cellphones is connected to a laptop. What fraction of devices are laptops?

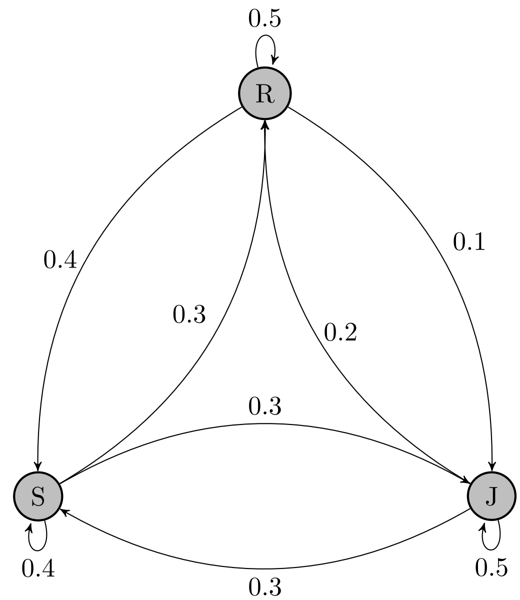
$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.75 & 0.25 \end{bmatrix}$$

$$\pi = \begin{bmatrix} \frac{15}{19} & \frac{4}{19} \end{bmatrix}$$

$$\boxed{\frac{4}{19} \text{ are laptops}}$$

Problem 6. An instruction for a RISC cpu is either R-type (R), Store/Load (S), or Jump-type (J). If the current instruction is R-type, then the next instruction will be R, S, or J with probabilities 0.5, 0.4, and 0.1, respectively. If the current instruction is Store/Load, then the probabilities for the next instruction to be R, S, or J are 0.3, 0.4, and 0.3, respectively. If the current instruction is Jump-type, then the next instruction will be R, S, or J with probabilities 0.2, 0.3, 0.5, respectively.

a) Draw the state transition diagram.



b) Find the transition probability matrix $\mathbf{P}$.

$$\boxed{\mathbf{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}}$$

c) In the long run, what proportion of instructions are of each type?

$$\pi = \pi\mathbf{P}$$

$$\pi_0 = 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2$$

$$\pi_1 = 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 = \frac{21}{62}$$

$$\pi_1 = \frac{23}{62}$$

$$\pi_2 = \frac{18}{62}$$

$$\boxed{\pi = \begin{bmatrix} \frac{21}{62} & \frac{23}{62} & \frac{18}{62} \end{bmatrix}}$$

Problem 7. A workstation tries to transmit frames through Ethernet. Suppose that whether or not collision occurs in the current transmission depends on the result of the last two transmissions the workstation had. That is, suppose that if collisions have occured in both of the past two transmissions, then with probability 0.7 a collision will occur in the current transmission; if a collision occurs in last transmission but not the transmission before the last one, then a collision will occur in the current transmission with probability 0.5; if a collision occured in the transmission before the last one but not the last one, then one will occur in the current transmission with probability 0.4; if there have been no collisions in the past two transmissions, then a collision will occur in the current transmission with probability 0.2.

a) Denote appropriate states in order to make the above model a Markov Chain.

   Let C = collision, and N = no collision.

   - State 0 (CC): both collide
   - State 1 (NC): collide in last, no collide one before last
   - State 2 (CN): no collide in last, collide one before last
   - State 3 (NN): both don't collide

b) Find the transition probability matrix **P** for the states defined in part (a).

$$\boxed{\mathbf{P} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}}$$

c) What fraction of frames cause a collision?

$$\pi = \pi \mathbf{P}$$

$$\pi_0 = 0.7\pi_0 + 0.5\pi_1$$

$$\pi_1 = 0.4\pi_2 + 0.2\pi_3$$

$$\pi_2 = 0.3\pi_0 + 0.5\pi_1$$

$$\pi_3 = 0.6\pi_2 + 0.8\pi_3$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_0 = \frac{5}{20}$$

$$\pi_1 = \frac{3}{20}$$

$$\pi_2 = \frac{3}{20}$$

$$\pi_3 = \frac{9}{20}$$

$$\boxed{\pi = \begin{bmatrix} \frac{5}{20} & \frac{3}{20} & \frac{3}{20} & \frac{9}{20} \end{bmatrix}}$$