CS 180: Homework 3

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Discussion 1B

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Problem 1

Given a sequence of requests with start and finish times $(s(i), t(i)) = 1, \ldots, n$, find a set of non-conflicting jobs of maximum possible size. Show that the following algorithm solves the problem correctly:

LATEST START TIME (LST):

(a) Set $R \leftarrow \{1, \ldots, n\}$, and $A \leftarrow \emptyset$.

(b) While $R \neq \emptyset$:

 i. Pick request $i \in R$ with the latest start time.

 ii. Add i to A.

 iii. Remove all requests that conflict with i (including i) from R.

(c) RETURN A.

LEMMA: Our algorithm frees up later than the optimal one.

Suppose that $A = \{i_1, i_2, \ldots, i_k\}$ and $s(i_1) > s(i_2) > \cdots > s(i_k)$.

Suppose an optimal solution is $O = \{j_1, j_2, \ldots, j_m\}$ and $s(j_1) > s(j_2) > \cdots > s(j_m)$.

Jobs in O are ordered in decreasing start time.

LEMMA: $\forall\ l \leq k$, $s(i_l) \geq s(j_l)$ (start time of the l^{th} job under A is later than the start time of the l^{th} job under O).

PROOF: Induction

BASE CASE: $l = 1$. True because $s(i_1)$ was the latest start time.

INDUCTION STEP: Suppose claim is true for l, want to show this is true for $l + 1$.

When we picked i_{l+1}, j_{i+1} also belongs to the set R.

$s(i_l) \geq s(j_l) \geq f(j_{l+1})$

j_{l+1} does not conflict with i_l, so j_{l+1} will never be chosen as the latest start time.

$\therefore$ Lemma is true by induction.

THEOREM: Latest Start Time (LST) finds an optimal set of jobs (maximum possible size)

PROOF: Suppose there exists $O = \{j_1, j_2, \ldots, j_m\}$ such that $m > k$

$A = \{i_1, i_2, \ldots, i_k\}$

$s(i_1) > s(i_2) > \cdots > s(i_k)$

$s(j_1) > s(j_2) > \cdots > s(j_k) > s(j_m)$

By the lemma, $s(i_k) \geq s(j_k)$

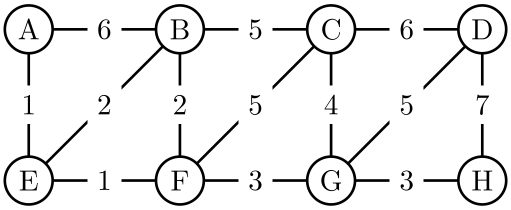
j_{k+1} does not conflict with the jobs in A.

j_{k+1} is not removed

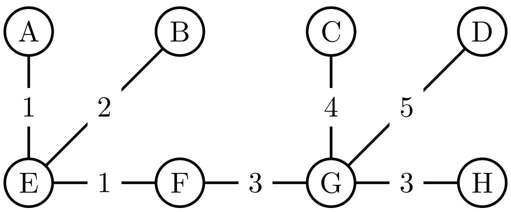
$\therefore$ the set R is not empty

Problem 2

Consider the following graph:



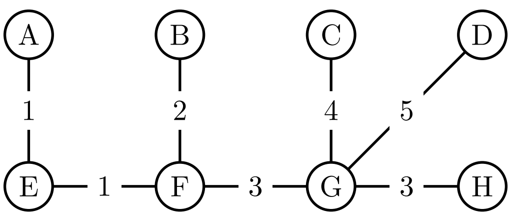
(a) What is the cost of its minimum spanning tree?



Cost = 19

(b) How many minimum spanning trees does it have?

Other MST:



Answer: 2 MSTs

(c) Suppose Kruskal's algorithm is run on this graph. In what order are the edges added to the MST? For each edge in this sequence, give a cut that justifies its addition.

Cut	Edge Added	Cost
{}	A	0
{A}	E	1
{A,E}	F	1
{A,E,F}	B	2
{A,E,F,B}	G	3
{A,E,F,B,G}	H	3
{A,E,F,B,G,H}	C	4
{A,E,F,B,G,H,C}	D	5
{A,E,F,B,G,H,C,D}	∅	19

Problem 3

You are given a connected graph G with n vertices and m edges, and a minimum spanning tree T of the graph. Suppose one of the edge weights $c(e)$ for an edge $e \in T$ is updated. Give an algorithm that runs in time $O(m)$ to test if T still remains the minimum spanning tree of the graph. You may assume that all edge weights are distinct both before and after the update. Explain why your algorithm runs in $O(m)$ time and is correct.

ALGORITHM:

Delete the updated edge $e = \{u, v\}$ from T, this gives us at most two disconnected cuts $S_1 \in T$ and $S_2 \in T$ where $u \in S_1$, $v \in S_2$

For vertex $v \in S_2$:

> Pick an edge f from updated G with least attachment cost to v: $O(m)$
>
> If $f = e$: T still remains the MST.
>
> Else: T is no longer the MST.

This algorithm runs in $O(m)$ time because finding a minimum edge $f \in G$ that connects to v (after deleting e) will take at most m iterations. This is because v can be connected to at most m other vertices in the graph G. All other operations in the algorithm run at $O(1)$ time, making the total runtime $O(m) + O(1) + O(1) = O(m)$.

This algorithm is correct because the deletion of $e = \{u, v\}$ from T will create two cuts $S_1 \in T$ and $S_2 \in T$ in which $u \in S_1$ and $v \in S_2$. All that is necessary to create an MST from these two cuts is to find the minimum edge from S_1 to S_2 in the updated G. We do not need to find the minimum edge from the set of all S_1 vertices to the set of all S_2 vertices because the path from v to the end of S_2 is guaranteed to be in correct MST order due to the **cut property**. For the same reason, all of the edges in S_1 will be in correct MST order. Thus, as long as we choose a minimum edge that connects a vertex in S_1 to $v \in S_2$, we are guaranteed to be left with a MST.

Problem 4

Given an undirected graph $G = (V, E)$, a subset of vertices $I \subset V$ is an independent set in G if no two vertices in I are adjacent to each other. Let $\alpha(G) = \max\{|I| : I \text{ an independent set in } G\}$. Give an efficient algorithm for computing an independent set of maximum size in a tree.

Let $T = (V, E)$ be an acyclic graph on n vertices.

(a) Prove that if u is a leaf in T, then there is a maximum-size independent set in T which contains u. That is, for every leaf in u, there is an independent set I such that $u \in I$ and $|I| = \alpha(T)$.

LEMMA: For a vertex $v \in I$ that is not a leaf but is adjacent to at least one leaf, removing v from I and adding v's leaves to I still gives us a valid independent set.

PROOF: Since T is acyclic, v has at most one parent node and at least one leaf node u.

If we remove v from I and add u to I, we still get a valid independent set because u is adjacent only to v, which no longer appears in the set I.

PROOF BY CONTRADICTION:

Suppose there is a maximum independent set I such that $|I| = \alpha(T)$ and the leaf $u \notin I$.

Since u is a leaf, this implies that there exists a vertex v which is adjacent to u.

If $v \notin I$, then I could not have been the maximum independent set due to the fact that $u \notin I$.

If $v \in I$, then $degree(v) \geq 2$, since it is not a leaf.

Let L be the set containing the leaves of v, where $u \in L$.

$|L| \geq 1$, since $degree(v) \geq 2$.

By the lemma above, we can remove v from I and add the vertices in the set L to the set I.

Since $|L| \geq 1 \rightarrow |I \cup L| \geq |I|$.

$\therefore$ The original set I could not have been the maximum-size independent set in T if $L > 1$.

$\therefore$ There exists an independent set I such that $u \in I$ and $|I| = \alpha(T)$.

(b) Give the graph T as input (in adjacency edge representation), give an algorithm to compute an independent-set of maximum size, $\alpha(T)$, in T. Your algorithm should run in time $O(|V| * |E|)$, prove correctness of your algorithm.

We can use dynamic programming to find the maximum independent set.

Let $OPT(T)$ denote the maximum independent set of T.

Let $OPT(\emptyset) = 0$.

1. Label each vertex in T from $\{1, 2, 3, \ldots, n\}$.

2. Let the vertex labeled s be the last vertex in the T visited using some traversal algorithm (like DFS).

3. Let $A[s]$ denote the list of neighbors of s.

4. We observe that I can either contain the vertex s or not contain s:

    If $s \in I : OPT(T) = (1 + OPT(T - A[s]))$

    Else if $s \notin I : OPT(T) = OPT(T - \{s\})$

5. We can memoize $OPT(T)$ by having an n length array M that stores 1 for every vertex in T that appears in I, and stores 0 otherwise.

FULL ALGORITHM:

Initialize array M of length n where all entries are 0.

Max-Independent-Set-Opt(T):

    If $M[T]$ is empty and $n = 0$ or 1:

        $M[T] = n$

    Else:

        $M[T] =$max$(1 + M[T - A[s]], M[T - \{s\}])$

    Return $M[T]$

PROOF BY INDUCTION:

BASE CASE: $n = 0$, $T = \emptyset$, $I = \emptyset$, $|I| = 0$.

INDUCTION: Suppose the algorithm works for $n = 0$, show that it works for $n + 1$:

    Let $OPT(T)$ be the optimal solution that returns $\alpha(T)$.

Suppose we add 1 vertex v to T. The optimal solution would then become $OPT(T + \{v\})$.

Since $OPT(T)$ was already the optimal solution, adding v will have two possibilities: v will be included in the set I or v will not be included in the set I.

CASE 1: $v \in I$:

If v is added to I, all of v's neighbors must not be in I. The addition of v is represented by $+1$ and the removal of v's neighbors is represented by $T - A[v]$, so the resulting optimal solution is $OPT(T + \{v\}) = (1 + OPT(T - A[v]))$.

CASE 2: $v \notin I$:

If v is not added to I, we only need the optimum solution to the remaining vertices, i.e. T with v removed from it. This is represented by $T - \{v\}$, so the resulting optimal solution is $OPT(T + \{v\}) = OPT(T - \{v\})$.

We take the maximum value of either case to be $\alpha(T)$.

$\therefore$ The algorithm is correct, since the two possible cases are handled correctly.