2025-07-20

Rules:
$\alpha_{k,i}$ represents the coefficients associated with the kth derivative and ith term in the polynomial.
We can represent $\alpha_{k,i}$ as follows:

$\alpha_{k,i} \in \mathbb{R}^{k \times i}$

$$\alpha_{k,i} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \cdots & \alpha_{0,i} \\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,i} \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{k,0} & \alpha_{k,1} & \alpha_{k,2} & \cdots & \alpha_{k,i} \end{pmatrix}$$

Now we can generate a random assortment of matrices that represent ODEs for which we wish to find recurrence relations. Let us use this matrix to solve the following Legendre ODE: $(1-x^2)y'' - 2xy' + l(l+1)y = 0$
Example

$$\alpha = \begin{pmatrix} l(l+1) & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + l(l+1)a_n = 0$$

Now, these recurrence relations can be computed with a dot product. For the above example, you do a dot product.

$$\begin{pmatrix} a_n & a_{n-1} & a_{n-2} \\ (n+1)a_{n+1} & na_n & (n-1)a_{n-1} \\ (n+2)(n+1)a_{n+2} & (n+1)na_{n+1} & n(n-1)a_n \end{pmatrix} \cdot \begin{pmatrix} l(l+1) & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} = (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + l(l+1)a_n$$

Here is the general matrix that you need to consider for the julia script we are making.

$$\begin{pmatrix} a_n & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots \\ (n+1)a_{n+1} & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & (n-3)a_{n-3} & \cdots \\ (n+2)(n+1)a_{n+2} & (n+1)na_{n+1} & n(n-1)a_n & (n-1)(n-2)a_{n-1} & (n-2)(n-3)a_{n-2} & \cdots \\ (n+3)(n+2)(n+1)a_{n+3} & (n+2)(n+1)na_{n+2} & (n+1)n(n-1)a_{n+1} & n(n-1)(n-2)a_n & (n-1)(n-2)(n-3)a_{n-1} & \cdots \\ (n+4)(n+3)(n+2)(n+1)a_{n+4} & (n+3)(n+2)(n+1)na_{n+3} & (n+2)(n+1)n(n-1)a_{n+2} & (n+1)n(n-1)(n-2)a_{n+1} & n(n-1)(n-2)(n-3)a_n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This matrix follows the following pattern:

$$\sum_{i=0}^{\infty} \alpha_{k,i} \prod_{j=1}^{k} (n+j-i)a_{n+k-i}$$