Chapter 6
Limit Order Book.

6.1 Glosten(1994) Summery

1) The general model above outlines some basic differences between LOB and dealer markets.
- Adverse selection affects prices differently

$$A_k = \underbrace{\mathbb{E}\left[v \mid q \geq Y_k\right]}_{\text{Adverse selection}} + \underbrace{\frac{C}{\mathbb{P}\left(q \geq Y_k\right)}}_{\text{Execution risk}}$$

2) Are they efficient? (Yes, but why?)

3) How is LOB depth determined? $Y_k = \sum_{j=1}^{k} y_j$

6.2 Continuous model

Assumption & notation

$v \sim G(v)$: Asset value with cdf

q : Price vary with volume (quantity)

$P^{'}(q)$: Posted limit orders (price)

$P(q) = \int_0^q P'(\tilde{q})d\tilde{q}$:The entire payment for buying volume q ((price)

$p(q) = \frac{P(q)}{q}$: The average price

$\theta_i(q)$: Trader i's marginal rate of substitution

(marginal valuation for q-th units of the asset) (qt)

If an trader i chooses a market buy order q and parts with cash $P(q)$ we should have,

$$P'(q) = \mathbb{E}\left[v \mid \theta_i(q) \geq P'(q)\right]$$

Adverse selection implies,

$P'(q) = \mathbb{E}\left[v \mid \theta_i(q) \geq P'(q)\right] > \mathbb{E}\left[v \mid \theta_i(q) = P'(q)\right]$

- Thus, market makers profit on the sale of the last units.

- At small realized traders, $q \simeq 0$ MM always profit.

- Even with continuous prices, there is a non-zero inside spread between ask and bid prices as the order size goes to zero (contrast with Kyle)

- After a trade of size q new expected asset value is below $P'(q)$.

$\because$ new expected value =

$\mathbb{E}\left[v \mid \theta_i(q) = P'(q)\right] < \mathbb{E}\left[v \mid \theta_i(q) \geq P'(q)\right] = P'(q)$

Often new limit orders will be posted below $P'(q)$ - price reversal.

6.3 Discrete model

Assumption & notation

$v \sim G(v)$: Continue with single asset

$A_k - A_{k-1}$: Tick size

A_1 : lowest price tick above μ

μ : Midprice

Price (Ask)	Quantity
A_k	y_k
A_{k-1}	y_{k-1}
A_1	y_1

Market equilibrium

C : Display cost

y_k : amount supplied at price A_k

$Y_k = \sum_{j=1}^{k} y_j$: Cumulative volume (depth) supplied at or below A_k

Let $\mathbb{E}\left[v \mid q \geq Y_k\right] = \mathbb{E}\left[v \mid \theta_i(q) \geq A_k\right]$.

The market maker's limit order is competitive i.e. Limit orders are supplied at each tick until the last order earns zero profit.

$$\mathbb{P}\left(q \geq Y_k\right)\left[A_k - \mathbb{E}\left[v \mid q \geq Y_k\right]\right] - C = 0$$

$$A_k = \underbrace{\mathbb{E}\left[v \mid q \geq Y_k\right]}_{\text{Adverse selection}} + \underbrace{\frac{C}{\mathbb{P}\left(q \geq Y_k\right)}}_{\text{Execution risk}}$$

(We need to solve for Y_k given A_k) … why?

Example 1

Let g be the marginal distribution of

$G \sim g(v) = 1/2$ if, $v = V^H$ or $-1/2$ if, $v = V^L$

with $v^H = \mu + \sigma$ and $v^L = \mu - \sigma$ what is (sigma??)

Single trader who uses a market order. We can divid trader as binomially model.

π : Risk-neutral speculator(S) who knows v

$1 - \pi$: Noise trader(N) who buys/sells equal probability i.e. $\mathbb{P}(q = \pm q_k | N) = 1/4$

$C = 0$ (No display cost) and continuous price.

Price (Ask)	Quantity	Depth
A_3	y_3	
A_2	$y_2 = q_L - q_s$	$Y_2 = q_L$
A_1	$y_1 = q_s$	$Y_1 = q_s$

$v^L < A_1 < A_2 < v^H$ (hence $Y_1 = q_S$ and $Y_2 = q_L$)
Speculator buys q_L when observing $v = v^H$ and $v = v^L$ don't buy.

Then,
$A_1 = \mathbb{E}\left[v \mid q \geq q_S\right] = \mu + \pi\sigma$
$A_2 = \mathbb{E}\left[v \mid q \geq q_L\right] = \mu + \dfrac{2\pi}{1+\pi}\sigma$

- **Bad example for discreteness:**
  - The example does not assume fixed ticks...
  - But they arise endogenously in equilibrium…
  - Due to discreteness of noise traders' strategy (A very artificial assumption)
- **Focus on adverse selection:**
  - Price increases in order size
  - Not due to informed trader having stronger info, as in Kyle model
  - But due to noise traders' order becoming less likely

Example 2

We are taking a short-cut and modeling adverse selection in a 'reduced form': rather than modeling the informed traders, we model their price impact.

Price impact equation:
$\mathbb{E}[v \mid q = x] = \mu + \lambda x$ where $\lambda > 0$ is constant
Market orders: $f(q) = \dfrac{\theta}{2} e^{-\theta|q|}$
The equilibrium price be
$A_k = \mu + \lambda\left(\dfrac{1}{\theta} + Y_k\right) + \dfrac{2C}{e^{-\theta Y_k}}$
$\because$
Focus on ask side: $Y_k > 0$. For $q \geq Y_k$:
$f\left(q \mid q \geq Y_k\right) = \dfrac{f(q)}{\mathbb{P}\left(q \geq Y_k\right)}$
$= e^{Y_k}\left[\theta \cdot e^{-\theta q}\right]$

The expected value at tick k becomes
$\mathbb{E}\left[v \mid q \geq Y_k\right] = \mu + \lambda\mathbb{E}\left[q \mid q \geq Y_k\right]$
$= \mu + \lambda\left(\dfrac{1}{\theta} + Y_k\right)$

Thus, $\underbrace{\mathbb{P}\left(q \geq Y_k\right)}_{e^{-\theta Y_k/2}}[A_k - \underbrace{\mathbb{E}\left[v \mid q \geq Y_k\right]}_{\mu+\lambda\left(\frac{1}{\theta}+Y_k\right)}] - C = 0,$

□

6.4 References.

L. R. Glosten. Is the Electronic Open Limit Order Book Inevitable? The Journal of Finance, 49 (4):1127–1161, 1994. ISSN 1540-6261.URL https://doi.org10.1111j.1540-6261.1994.tb02450.x