Chapter 2

Basic Building Blocks.

2.1 The term structure of interest rates

Definition2.1

$B(t,T)$: Zero coupon bond price(value at time t of a security paying 1 monetary at time T) or Discount factor.
T : Maturity.
$T-t$: time to maturity.

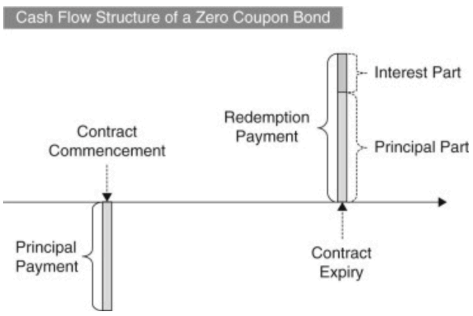

FIGURE 2.1

Definition2.2

$V(t,\mathbb{T},k)$: A coupon bond Pays k at each coupon date $T_i \in \mathbb{T}$ for $1 \le i < n$ and pays 1+k at the bond maturity T_n

$\mathbb{T}=\{T_0,T_1,\ldots,T_n\}$: Tenor structure of the coupon bond.

Index of the next date in the tenor structure:

$$\eta(t)=\min\left\{0<i\le n \mid t<T_i\right\}$$

The value of the coupon bond:

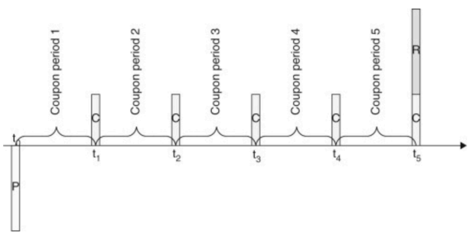

FIGURE2.3

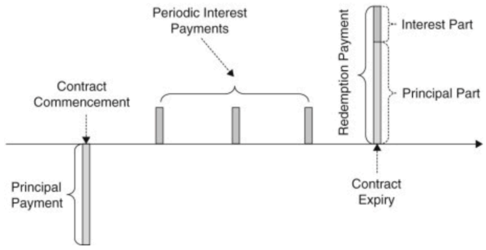

FIGURE2.2

$$V(t,\mathbb{T},k)=B\left(t,T_n\right)+k\sum_{i=\eta(t)}^{n}B\left(t,T_i\right)$$

Definition2.3

$y(t,T)$: The continuously compounded yield is given by $B(t,T)=\exp\{-(T-t)y(t,T)\}$

Definition2.4

$r(t)$ **:** Continuously compounded short rate

$$r(t)=\lim_{T\searrow t}y(t,T)=-\left.\frac{\partial\ln B(t,T)}{\partial T}\right|_{T\searrow t}$$

Definition2.5

The value of saving account $\beta(t,T)$ is given by

$$\beta(t,T)=\exp\left\{\int_t^T r(s)ds\right\}$$

Definition2.6

A forward contract is the right and the obligation to buy (respectively to sell) an asset $S(T)$ at time T for the forward price $F(S,t,T)$

	Forward	Futrues
Where	Free	Exchange
Time	Free	Not Free
How	1:1	Competitive
Rule	×	o

FIGURE2.4

Proposition 2.7

$F(S,t,T) = \dfrac{S(t)}{B(t,T)}$.: The forward price.

Proof

If we do the forward contract then at time T our cash flow will be $S(T) - F(s,t,T)$. However if we borrow money at time t, the cash flow at time T is $S(T) - S(t)B(t,T)$. By the law of one price, the forward price can be calculated like above.

Definition 2.8

The continuously compounded forward yield $f(t,T_1,T_2)$ is given by

$$B\left(t,T_2\right)/B\left(t,T_1\right) = \exp\left\{-\left(T_2-T_1\right)f\left(t,T_1,T_2\right)\right\}$$

Definition 2.9

The instantaneous forward rate $f(t,T)$ is given by

$$f(t,T) = -\left.\frac{\partial \ln B(t,u)}{\partial u}\right|_{u=T}$$

$$\text{or } B(t,T) = \exp\left\{-\int_t^T f(t,u)du\right\}$$

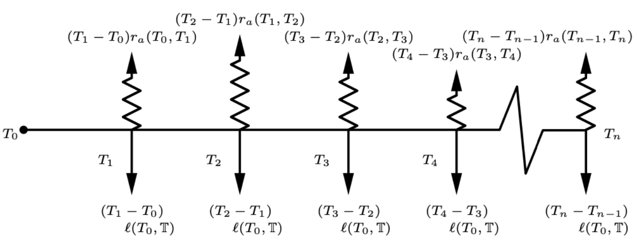

FIGURE 2.5

Proposition 2.10

$l(T_0,\mathbb{T})$: swap yield of contracted in T_0

The swap yield $l(T_0,\mathbb{T})$ of a swap contracted in T_0 must equal the coupon on a coupon bond quoted at par.

Remark 2.11

The swap rate can be represented in terms of zero coupon bond.(Linear combination).

$$0 = \left(1 - B\left(T_0,T_n\right)\right) - \ell\left(T_0,\mathbb{T}\right)\sum_{i=1}^{n}\left(T_i-T_{i-1}\right)B\left(T_0,T_i\right)$$

$$\Leftrightarrow \quad \ell\left(T_0,\mathbb{T}\right) = \frac{1-B\left(T_0,T_n\right)}{\sum_{i=1}^{N}\left(T_i-T_{i-1}\right)B\left(T_0,T_i\right)}$$

The sum

$$\sum_{i=1}^{N}\left(T_i-T_{i-1}\right)B\left(T_0,T_i\right)$$

Is often canoed the present value of a basis point(PVBP).

Note) BP is price change of bond per 0.01 change in yield.

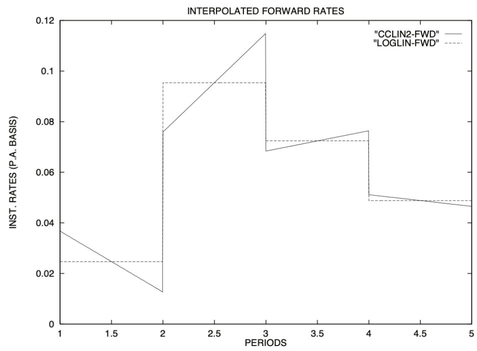

Figure 2.3 *Interpolated instantaneous forward rates*

Theorem 2.12 (Vasicek)

The term structure $f(0,t)$ of instantaneous forward rates That satisfies the maximum smoothness criterion

$$\min \int_0^T \left(f''(0,s)\right)^2 ds$$

$f(0,t) = c_i t^4 + b_i t + a_i$

for $t_{i-1} < t \le t_i$, with $i = 1,2,\cdots,m+1$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$

The coefficients $a_i, b_i, c_i, i = 1,2,\cdots,m+1$ satisfy the equations,

$$c_i t_i^4 + b_i t_i + a_i = c_{i+1}t_i^4 + b_{i+1}t_i + a_{i+1}, \quad i = 1,2,\cdots,m$$

$$4c_i t_i^3 + b_i = \quad 4c_{i+1}t_i^3 + b_{i+1}, \quad i = 1,2,\cdots,m$$

$$\frac{1}{5}c_i\left(t_i^5 - t_{i-1}^5\right) + \frac{1}{2}b_i\left(t_i^2 - t_{i-1}^2\right) + a_i\left(t_i - t_{i-1}\right) = -\log\left(\frac{B\left(0,t_i\right)}{B\left(0,t_{i-1}\right)}\right)$$

and $c_{m+1} = 0$

$$f'(0,T) = 0 \quad \Leftrightarrow \quad b_{m+1} = 0 \text{ and } a_1 = r(0)$$

To apply the interpolation method given swap yields $l(0,t_i)$,

$$1 = \exp\left\{-\int_0^{t_i} f(0,s)ds\right\} + \sum_{j=1}^{i}\ell\left(0,t_j\right)\cdot\exp\left\{-\int_0^{t_j} f(0,s)ds\right\}$$