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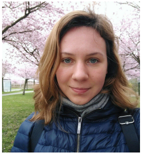
An Improved Algorithm for Adversarial Linear Contextual Bandits via Reduction

Tim van Erven

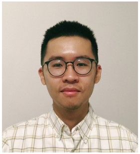
University of Amsterdam

Joint work with:

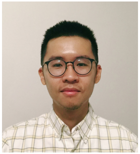
Jack Mayo



Julia Olkhovskaya



Chen-Yu Wei



Chen-Yu Wei

Outline

Setting

1. Adversarial Bandits
2. Adversarial Linear Contextual Bandits

Main Results

3. General Setting
4. First-order Bounds, Initial Setting

Approach: Reduction to Non-contextual Linear Bandits

5. Reduction: The Basic Idea
6. Approximating Ω
7. Side-Result: Efficient Robust Linear Bandit Algorithm
8. Controlling the Difference between Ψ and $\hat{\Psi}$
9. Restricting π_t to be a Linear Policy

Adversarial Bandits

For $t = 1, \ldots, T$:

1. Learner choses (randomized) arm $a_t \in \{1, \ldots, K\}$
2. Loss value $\ell_t(a_t)$ is revealed

Regret w.r.t. arm *a*: $$R_T(a) = \mathbb{E}\left[\sum_{t=1}^{T} \ell_t(a_t)\right] - \sum_{t=1}^{T} \ell_t(a)$$

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- Oblivious adversary: losses $\ell_t(a)$ fixed a priori for all t, a
- Expectation w.r.t. learner's randomness

Adversarial Contextual Bandits

For $t = 1, \ldots, T$:

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Regret w.r.t. policy π:
$$R_T(\pi) = \mathbb{E}\Big[\sum_{t=1}^{T} \ell_t(a_t) - \sum_{t=1}^{T} \ell_t(\pi)\Big]$$
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Adversarial losses, stochastic contexts [Neu and Olkhovskaya, 2020]:

- **Linear losses:** $\ell_t(a) = \langle X_t, \theta_{t,a} \rangle \in [-1, +1]$, where $\theta_{t,a}$ fixed a priori
- **I.i.d. contexts**: $X_t \sim \mathcal{D}$

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- **I.i.d. contexts**: $X_t \sim \mathcal{D}$
- (Opposite setting with fixed loss function and adversarial contexts also considered in literature.)

Adversarial Contextual Bandits More Abstractly I

Incorporate contexts $X_t \in \mathbb{R}^p$ into actions a such that
$\ell_t(a) = \langle X_t, \theta_{t,a} \rangle = \langle a, \theta_t \rangle$:

$$\theta_t = \begin{pmatrix} \theta_{t,1} \\ \theta_{t,2} \\ \vdots \\ \theta_{t,K} \end{pmatrix} \in \mathbb{R}^{p \times K} \qquad a = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{p \times K}$$

Then every round we receive a **random action set** (with K actions):

$$\mathcal{A}_t = \left\{ \begin{pmatrix} X_t \\ \\ \\ \end{pmatrix}, \begin{pmatrix} \\ X_t \\ \\ \end{pmatrix}, \ldots, \begin{pmatrix} \\ \\ \\ X_t \end{pmatrix} \right\} \subset \mathbb{R}^{p \times K}$$

Adversarial Contextual Bandits More Abstractly II

For $t = 1, \dots, T$:

1. Draw **action set** $\mathcal{A}_t \subset \mathbb{R}^d$ i.i.d. from $\mathcal{D}$
2. Learner choses (randomized) action $a_t \in \mathcal{A}_t$
3. Loss value $\ell_t(a_t) = \langle a_t, \theta_t \rangle \in [-1, +1]$ is revealed

Regret w.r.t. policy π: $$R_T(\pi) = \mathbb{E}\left[\sum_{t=1}^{T} \ell_t(a_t) - \sum_{t=1}^{T} \ell_t(\pi)\right]$$

$$\ell_t(\pi) = \langle \pi(\mathcal{A}_t), \theta_t \rangle \quad \text{for } \pi(\mathcal{A}_t) \in \mathrm{conv}(\mathcal{A}_t)$$

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Optimal policy is linear:

$$\min_\pi \mathbb{E}\left[\sum_{t=1}^T \langle \pi(\mathcal{A}_t), \theta_t \rangle\right] = \min_\pi \mathbb{E}\left[\langle \pi(\mathcal{A}), \sum_{t=1}^T \theta_t \rangle\right]$$

$$\pi^*(\mathcal{A}) = \pi_\phi(\mathcal{A}) := \arg\min_{a \in \mathcal{A}} \langle a, \phi \rangle \quad \text{for } \phi = \sum_{t=1}^T \theta_t$$

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Main Results I: General Setting

- d: dimension of the actions, $\mathcal{A}_t \subset \mathbb{R}^d$
- K: maximum number of actions, $|\mathcal{A}_t| \leq K$
- C: maximum number of linear constraints that describe $\mathrm{conv}(\mathcal{A}_t)$
- Simulator: free access to independent samples $\mathcal{A} \sim \mathcal{D}$

Algorithm	**Regret**[1]	**Runtime**	**Simulator**
Dai et al. [2023]	$\min\{d\sqrt{T}, \sqrt{dT \log K}\}$	$\mathrm{poly}(d, K, T)$	yes
Liu et al. [2023]	$d\sqrt{T}$	$K \cdot T^{\Omega(d)}$	no
Liu et al. [2023]	$d^2\sqrt{T}$	$\mathrm{poly}(d, K, T)$	no
Ours	$d^{1.5}\sqrt{T \log K}$	$\mathrm{poly}(d, C, T)$	no
Ours	$d\sqrt{T}$	$\mathrm{poly}(d, C, T)$	yes

[1] Up to poly-logarithmic factors in d and T

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- Always $C \leq K + 1$, but in many combinatorial problems $C = \text{poly}(d)$ and $K = 2^{\Omega(d)}$
  - Example: in shortest path with d edges, set of all paths can be described by a linear program with $O(d)$ constraints, but number of paths can be of order $2^{\Omega(d)}$.

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Ours	$d^{1.5}\sqrt{T\log K}$	$\text{poly}(d, C, T)$	no
Ours	$d\sqrt{L^*}$	$\text{poly}(d, C, T)$	yes

$$L^* = \min_{\pi} \mathbb{E}\Big[\sum_{t=1}^{T} \langle \pi(\mathcal{A}_t), \theta_t \rangle\Big] \leq T$$

Assuming $\ell_t(a) \in [0, 1]$

Main Results II: First-order Bounds, Initial Setting

- p: dimension of contexts, $X_t \in \mathbb{R}^p$
- K: number of actions, $|\mathcal{A}_t| = K$
- Simulator: free access to independent context samples $X \sim \mathcal{D}$

$$L^* = \min_{\pi} \mathbb{E}\Big[\sum_{t=1}^{T} \langle \pi(\mathcal{A}_t), \theta_t \rangle\Big]$$

Algorithm	Regret[1]	Runtime	Simulator	Note
Neu and Olkhovskaya [2020]	$\sqrt{KpT}$	poly(p, K, T)	yes	
Olkhovskaya et al. [2023]	$K\sqrt{pL^*}$	$\Theta\Big(T\big(\frac{T}{K^2p}\big)^{Kp}\Big)$	no	
Olkhovskaya et al. [2023]	$K\sqrt{pL^*}$	poly(p, K, T)	yes	$\star$
Ours	$Kp\sqrt{L^*}$	poly(p, K, T)	yes	

Strong assumption $\star$: contexts X_t have log-concave distribution

[1] Up to poly-logarithmic factors

Outline

Reduction: The Basic Idea

Expected loss for policy π in round t is:

$$\mathop{\mathbb{E}}_{\mathcal{A}_t}\left[\langle \pi(\mathcal{A}_t), \theta_t \rangle\right] = \left\langle \mathop{\mathbb{E}}_{\mathcal{A}_t}[\pi(\mathcal{A}_t)], \theta_t \right\rangle = \left\langle \Psi(\pi), \theta_t \right\rangle,$$

where **$\Psi(\pi)$ is the mean action** for π:

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Possibilities for expected loss:

$$\langle y, \theta_t \rangle \qquad \text{for } y \in \Omega = \{\Psi(\pi) \mid \pi \in \Pi\}$$

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Reduction:

1. Let linear bandit algorithm choose $y_t \in \Omega$ in round t
2. Play π_t such that $\Psi(\pi_t) = y_t$
3. Provide unbiased loss estimate $\langle \pi_t(\mathcal{A}_t), \theta_t \rangle$ as feedback to linear bandit algorithm

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NB Hanna et al. [2023] introduced this reduction for different setting of **stochastic** linear contextual bandits, but their techniques do not carry over.

Approximating Ω

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$$\Omega = \{\Psi(\pi) \mid \pi \in \Pi\}, \qquad \Psi(\pi) = \mathop{\mathbb{E}}_{\mathcal{A}}[\pi(\mathcal{A})]$$

Issue: Ψ and Ω depend on **unknown distribution $\mathcal{D}$ of $\mathcal{A}$**

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Issue: Ψ and Ω depend on **unknown distribution $\mathcal{D}$ of $\mathcal{A}$**

Using simulator: Given separate sample $\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_N$ from $\mathcal{D}$:

$$\hat{\Omega} = \{\hat{\Psi}(\pi) \mid \pi \in \Pi\}, \qquad \hat{\Psi}(\pi) = \frac{1}{N} \sum_{i=1}^{N} \pi(\tilde{\mathcal{A}}_i)$$

Approximating Ω

Reduction:

1. Let linear bandit algorithm choose $y_t \in \hat{\Omega}$ in round t
2. Play π_t such that $\hat{\Psi}(\pi_t) = y_t$
3. Provide biased loss estimate $\langle \pi_t(\mathcal{A}_t), \theta_t \rangle$ as feedback to linear bandit algorithm

$$\Omega = \{\Psi(\pi) \mid \pi \in \Pi\}, \qquad \Psi(\pi) = \mathop{\mathbb{E}}_{\mathcal{A}}[\pi(\mathcal{A})]$$

Using simulator: Given separate sample $\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_N$ from $\mathcal{D}$:

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- Need computationally efficient adversarial linear bandit algorithm that is robust to biased stochastic feedback

Side-Result: Efficient Robust Linear Bandit Algorithm

Definition (α-misspecification-robust linear bandit algorithm)

Given random feedback $f_t(y_t) \in [-1, +1]$ with bias at most some **known** $\epsilon \geq 0$:

$$\left|\mathbb{E}_t[f_t(y_t)] - \langle y_t, \theta_t \rangle\right| \leq \epsilon,$$

the algorithm achieves regret at most

$$\mathbb{E}\left[\sum_{t=1}^{T} \langle y_t, \theta_t \rangle\right] \leq \min_{y \in \hat{\Omega}} \sum_{t=1}^{T} \langle y, \theta_t \rangle + \tilde{O}\left(d\sqrt{T} + \alpha\sqrt{d}\epsilon T\right).$$

- [Liu et al., 2024a]: optimal $\alpha = 1$, but runtime scales with number of actions K
- New alg: $\alpha = \sqrt{d}$, and poly(d, C, T) runtime
  - Version of continuous exponential weights similar to Ito et al. [2020] with bonuses like [Lee et al., 2020, Zimmert and Lattimore, 2022, Liu et al., 2024b]
  - Also achieves first-order bound

Controlling the Difference between Ψ and $\hat{\Psi}$

Are we there yet?

Controlling the Difference between Ψ and $\hat{\Psi}$

Suppose, with high probability,

$$|\langle \Psi(\pi), \theta_t \rangle - \langle \hat{\Psi}(\pi), \theta_t \rangle| \leq \epsilon \quad \text{for } \pi \in \{\pi^*, \pi_t\},\ t = 1, \ldots, T$$

Controlling the Difference between Ψ and $\hat{\Psi}$

Suppose, with high probability,

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This would resolve the following **remaining issues:**

1. Controlling bias:

$$|\mathop{\mathbb{E}}_{\mathcal{A}_t}[\langle \pi_t(\mathcal{A}_t), \theta_t \rangle] - \langle y_t, \theta_t \rangle| = |\langle \Psi(\pi_t), \theta_t \rangle - \langle \hat{\Psi}(\pi_t), \theta_t \rangle| \leq \epsilon$$

Controlling the Difference between Ψ and $\hat{\Psi}$

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2. Linear bandit gives regret bound w.r.t. $y^* \in \hat{\Omega}$ instead of Ω:

$$\begin{aligned}
\mathbb{E}\Big[\sum_{t=1}^{T} \big(\langle y_t, \theta_t \rangle - \langle y^*, \theta_t \rangle\big)\Big] &\geq \mathbb{E}\Big[\sum_{t=1}^{T} \big(\langle \hat{\Psi}(\pi_t), \theta_t \rangle - \langle \hat{\Psi}(\pi^*), \theta_t \rangle\big)\Big] \\
&\geq \mathbb{E}\Big[\sum_{t=1}^{T} \big(\langle \Psi(\pi_t), \theta_t \rangle - \langle \Psi(\pi^*), \theta_t \rangle\big)\Big] - 2T\epsilon
\end{aligned}$$

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$$\Psi(\pi) = \mathop{\mathbb{E}}_{\mathcal{A}}[\pi(\mathcal{A})] \qquad\qquad \hat{\Psi}(\pi) = \frac{1}{N} \sum_{i=1}^{N} \pi(\tilde{\mathcal{A}}_i)$$

Lemma (Uniform Convergence over Linear Policies)

Let $\pi_\phi(\mathcal{A}) := \arg\min_{a \in \mathcal{A}} \langle a, \phi \rangle$ *be a linear policy. Then, w.p.* $\geq 1 - \delta$,

$$\sup_{\phi} \left| \langle \Psi(\pi_\phi), \theta_t \rangle - \langle \hat{\Psi}(\pi_\phi), \theta_t \rangle \right| \leq 2\sqrt{\frac{2d \ln(NK^2)}{N}} + \sqrt{\frac{2 \ln(4/\delta)}{N}}.$$

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- Union bound over $t = 1, \ldots, T$ is cheap: δ / T instead of δ

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- We know π^* is always a linear policy, but algorithm's choices π_t may not be! **Problem!**

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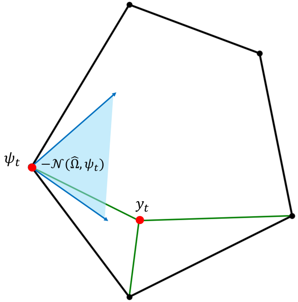
- Union bound over $t = 1, \ldots, T$ is cheap: δ/T instead of δ
- We know π^* is always a linear policy, but algorithm's choices π_t may not be! **Problem!**
- Can we solve this by extending to uniform convergence over all policies π? No, does not hold! So need to ensure π_t is linear policy.

Restricting π_t to be a Linear Policy

- $\hat{\Omega} \subset \mathbb{R}^d$ is a polytope
- Lemma: for every vertex v of $\hat{\Omega}$, there exists a linear policy π_ϕ that maps to it: $\hat{\Psi}(\pi_\phi) = v$.
  - In fact, this holds for any interior point ϕ of the negative normal cone $-\mathcal{N}(\hat{\Omega}, v)$ at v.



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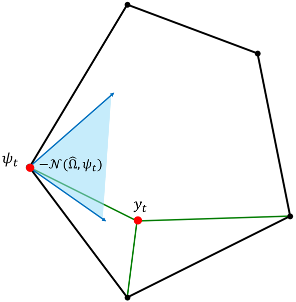

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- By Carathéodory's theorem, y_t is a convex combination of $m \leq d + 1$ vertices $v_1, \ldots, v_m$:

$$y_t = \sum_{j=1}^{m} \lambda_j v_j \qquad \text{for } \lambda = (\lambda_1, \ldots, \lambda_m) \in \Delta_m \tag{1}$$

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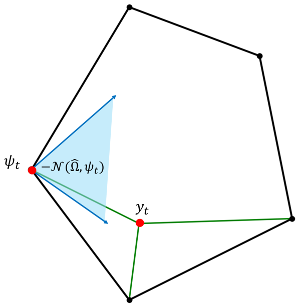

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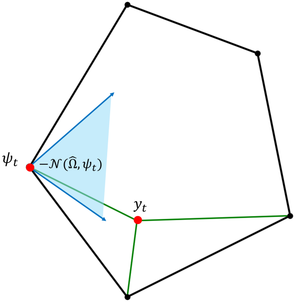

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- Computation: We can both find the decomposition (1) and the interior point ϕ of the normal cone in poly(d, N, C) time, because we can construct an efficient separation oracle for $\hat{\Omega}$.

Putting It All Together

Theorem

Given access to an α-misspecification robust linear bandit algorithm, we obtain

$$R_T(\pi) = \tilde{O}\left(d\sqrt{T} + \alpha T d\sqrt{\frac{\log(NKT)}{N}}\right).$$

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Without simulator access:

- Run in epochs of lengths 2^i for $i = 0, 1, 2, 3, \ldots$
- In epoch i use $N = \Theta(2^i)$ samples from all previous epochs to construct $\hat{\Omega}$

$$R_T(\pi) = \tilde{O}\big(d\sqrt{T} + \alpha d \sqrt{T \log(KT)}\big)$$

Conclusion

Highlights:

- First algorithm for this setting that handles combinatorial action sets efficiently
- Efficient reduction from contextual to (misspecified) non-contextual linear bandits
- Handle resulting misspecification in linear bandit algorithm

Open Questions:

- Improve computation to match Neu and Valko [2014]? For semi-bandit feedback, they only require a linear optimization oracle for each action set instead of a polynomial number of constraints.
- Improve regret to $\tilde{O}(d\sqrt{T})$ with polynomial-time algorithm without a simulator?
- Improve first-order bound to $\tilde{O}(\sqrt{pKL^*})$ in initial setting?

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