Evaluation

Online learning

links with optimization and games

Université Paris–Saclay

dilated regularizers on treeplexes

Let $\mathscr{T}$ be a treeplex as defined in the course. Let $(\hat{h}_\theta)_{\theta\in\Theta}$ be a collection of regularizers such that for each $\theta \in \Theta$, $\hat{h}_\theta : \mathbb{R}^{\mathscr{A}_\theta} \to \mathbb{R} \cup \{+\infty\}$ and $\operatorname{dom} \hat{h}_\theta = \Delta(\mathscr{A}_\theta)$. We define $h : \mathbb{R}^\Sigma \to \mathbb{R} \cup \{+\infty\}$ as

$$h(x) = \begin{cases} \displaystyle\sum_{\theta\in\Theta} x_{[p(\theta)]} \cdot \hat{h}_\theta\left(\frac{x_{[\theta]}}{x_{[p(k)]}}\right) & \text{if } x \in \mathscr{T}, \\ +\infty & \text{otherwise,} \end{cases}$$

with convention $0 \times \hat{h}_\theta(0/0) = 0$ for each $\theta \in \Theta$.

1) Prove that h is strictly convex.

2) Prove that h is lower semicontinuous.

3) Let $\theta \in \Theta$ of maximum depth, $a \in \mathscr{A}_\theta$ and define $h_{[\theta a\downarrow]} = 0$. Now assume that the functions $h_{[\theta a\downarrow]}$ have been defined for all $\theta \in \Theta$ of depth $k \geqslant 1$. Let $\theta \in \Theta$ of depth $k-1$ and $a \in \mathscr{A}_\theta$. Define

$$h_{[\theta a\downarrow]}(x) = \begin{cases} \displaystyle\sum_{\theta' \xleftarrow{a} \theta} \left(\hat{h}_{\theta'}(x_{[\theta']}) + \sum_{a' \in \mathscr{A}_{\theta'}} x_{[\theta' a']} \times h_{[\theta' a' \downarrow]}\left(\frac{x_{[\theta' a' \downarrow]}}{x_{[\theta' a']}}\right)\right) & \text{if } x \in \mathscr{T}^{[\theta a\downarrow]}, \\ +\infty & \text{otherwise,} \end{cases}$$

and eventually,

$$h_{[\varnothing\downarrow]}(x) = \begin{cases} \displaystyle\sum_{\theta \leftarrow \varnothing} \left(\hat{h}_\theta(x_{[\theta]}) + \sum_{a \in \mathscr{A}_\theta} x_{[\theta a]} \times h_{[\theta a\downarrow]} \left(\frac{x_{[\theta a\downarrow]}}{x_{[\theta a]}} \right) \right) & \text{if } x \in \mathscr{T}, \\ +\infty & \text{otherwise,} \end{cases}$$

with convention $0 \times \hat{h}_{[\theta a\downarrow]}(0/0) = 0$ for each $(\theta, a) \in \Sigma$, and $0 \times \hat{h}_{[\varnothing\downarrow]}(0/0) = 0$. Prove that for all $x \in \mathscr{T}$ and $\sigma \in \Sigma \cup \{\varnothing\}$,

$$h_{[\sigma\downarrow]}(x) = \sum_{\theta \in \Theta^{[\theta a\downarrow]}} x_{[p(\theta)]} \cdot \hat{h}_\theta \left(\frac{x_{[\theta]}}{x_{[p(\theta)]}} \right).$$

4) Establish an expression for h^* that only involves the functions $\hat{h}_\theta$ for $\theta \in \Theta$ such that $\varnothing \longrightarrow \theta$.

5) Deduce an expression for ∇h^*.

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