Exercices
Convexity tools
Université Paris–Saclay

Exercice 1 (*Logistic loss*). — Prove that function $x \mapsto \log(1+e^{-x})$ defined on $\mathbb{R}$ is smooth.

Exercice 2 (*Characterization of Lipschitz continuity with subgradients*). — Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, $\|\cdot\|$ a norm on $\mathbb{R}^d$, and $L > 0$. Prove that f is L-Lipschitz on $\operatorname{int} \operatorname{dom} f$ with respect to $\|\cdot\|$ if, and only if:

$$\forall x \in \operatorname{int} \operatorname{dom} f,\ \forall y \in \partial f(x), \quad \|y\|_* \leqslant L.$$

Exercice 3. — Let $\|\cdot\|$ be a norm in $\mathbb{R}^d$, B its closed unit ball, and $\|\cdot\|_*$ its dual norm. Prove that $I_B^* = \|\cdot\|_*$.

Exercice 4. — Let $y \in \mathbb{R}^d$, $a > 0$, $b \in \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. Compute the Legendre–Fenchel transform of the following functions.

a) $x \in \mathbb{R} \mapsto e^x$

b) $x \in \mathbb{R}^d \mapsto \langle y, x \rangle$

c) $x \in \mathbb{R}^d \mapsto \max(1 - \langle y, x \rangle\,,\ 0)$

d) $x \in \mathbb{R}^d \mapsto af(x) + b + \langle y, x \rangle$

EXERCICE 5. — Let $\|\cdot\|$ be a norm in $\mathbb{R}^d$, $\|\cdot\|_*$ its dual norm, $1 < p, q < +\infty$ such that $1/p + 1/q = 1$ and $x, y \in \mathbb{R}^d$. Prove that

$$\langle x, y\rangle \leqslant \frac{1}{p}\|x\|^p + \frac{1}{q}\|y\|_*^q.$$

EXERCICE 6 (*Hyperbolic entropy*). — Let $\beta > 0$ and

$$\mathrm{H}_\beta(x) = \sum_{i=1}^d \left(x_i \operatorname{arcsinh}\left(\frac{x_i}{\beta}\right) - \sqrt{x_i^2 + \beta^2} \right), \qquad x \in \mathbb{R}^d.$$

1) Express $\nabla \mathrm{H}_\beta$ and $\nabla \mathrm{H}_\beta^*$.

2) Prove that H_β is $(1+\beta)^{-1}$-strongly convex with respect to $\|\cdot\|_2$ over B_2 (the closed unit Euclidean ball).

3) Prove that H_β is $(1+\beta d)^{-1}$-strongly convex with respect to $\|\cdot\|_1$ over B_1 (the closed unit ℓ_1-ball).

4) Prove that $\max_{x\in \mathrm{B}_2} \mathrm{D}_{\mathrm{H}_\beta}(x, 0) \leqslant 2/\beta$.

5) Prove that if $\beta \leqslant 1$, then $\max_{x\in \mathrm{B}_1} \mathrm{D}_{\mathrm{H}_\beta}(x, 0) \leqslant \log(3/\beta)$.