Evaluation

Online learning

links with optimization and games

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online newton step

Let $d \geqslant 1$ and $\mathcal{X} \subset \mathbb{R}^d$ a nonempty closed convex set. We consider an online convex optimization problem where the loss functions admit quadratic lower bounds as follows. At step $t \geqslant 0$,

- the Decision Maker chooses $x_t \in \mathcal{X}$
- Nature chooses a loss function ℓ_t such that there exists $g_t \in \partial \ell_t(x_t)$ and M_t a positive semi-definite matrix of size $d \times d$ such that:

$$\forall x \in \mathcal{X}, \quad \ell_t(x) - \ell_t(x_t) \geqslant \langle g_t, x - x_t \rangle + \frac{1}{2} \langle x - x_t, M_t(x - x_t) \rangle .$$

  ℓ_t, g_t and M_t are revealed.

Let $\lambda > 0$. For all $t \geqslant 0$, denote $A_t = \frac{1}{2}\left(\lambda I + \sum_{s=0}^{t-1} M_s\right)$.

1) For each of the three iterations defined below, establish an upper bound on the regret

$$\sum_{t=0}^{T} \left(\ell_t(x_t) - \ell_t(x)\right), \quad x \in \mathcal{X},\ T \geqslant 0.$$

Hint. — Use the lemma from the course involved in the analysis of the Vovk–Azoury–Warmuth algorithm.

(i) Let $x_0 \in \mathscr{X}$ and

$$x_{t+1} = \operatorname*{arg\,min}_{x \in \mathscr{X}} \left\| (x_t - \mathrm{A}_t^{-1} g_t) - x \right\|_{\mathrm{A}_t}, \qquad t \geqslant 0.$$

(ii) Let $x_0 \in \mathscr{X}$ and

$$x_{t+1} = \operatorname*{arg\,min}_{x \in \mathscr{X}} \left\{ \langle -\mathrm{A}_t x_t + g_t, x \rangle - \frac{1}{2} x^\top \mathrm{A}_{t+1} x \right\}, \qquad t \geqslant 0.$$

(iii) Let $y_0 \in \mathbb{R}^d$ and

$$x_t = \operatorname*{arg\,min}_{x \in \mathscr{X}} \left\{ \left\langle -y_0 + \sum_{s=0}^{t-1} g_s, x \right\rangle + \frac{1}{2} \langle x, \mathrm{A}_t x \rangle \right\}, \qquad t \geqslant 0.$$

2) Bonus. — Derive the corresponding regret bounds in the special case of *online portfolio optimization* where $\mathscr{X} = \Delta_d$ and where the loss functions are of the form $\ell_t(x) = -\log \langle r_t, x \rangle$ for some $r_t \in (\mathbb{R}_+^*)^d$.