

EXERCICES

ONLINE LINEAR OPTIMIZATION

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Let $d \geq 1$ be an integer and \mathcal{X} a nonempty closed convex subset of \mathbb{R}^d .

EXERCICE 1. — Let $h : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a regularizer, $y, u \in \mathbb{R}^d$ and $x = \nabla h^*(y)$. Prove that

$$\nabla h^*(y + u) = \arg \max_{x' \in \mathbb{R}^d} \{ \langle u, x' \rangle - D_h(x', x; y) \}.$$

EXERCICE 2 (*Dual averaging with time-dependent parameters*). — Prove properties (iii) and (iv) in Proposition 3.2.6 from the lecture notes.

EXERCICE 3 (*Exponential weights algorithm as mirror descent*). — Let $H_{\text{ent}} : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be defined as

$$H_{\text{ent}}(x) = \begin{cases} \sum_{i=1}^d x_i \log x_i & \text{if } x \in (\mathbb{R}_+)^d \\ +\infty & \text{otherwise.} \end{cases}$$

- 1) Prove that H_{ent} is a mirror map compatible with all nonempty closed convex subsets of \mathbb{R}_+^d .
- 2) Let $(u_t)_{t \geq 1}$ be a sequence in \mathbb{R}^d , and for $t \geq 1$,

$$y_t = \sum_{s=1}^{t-1} u_s,$$

$$x_t = \nabla h_{\text{ent}}^*(y_t).$$

Prove that there exists a sequence $(y'_t)_{t \geq 1}$ in \mathbb{R}^d such that $((x_t, y'_t))_{t \geq 1}$ is a sequence of online mirror descent iterates on Δ_d associated with constant mirror map H_{ent} and dual increments $(u_t)_{t \geq 1}$.

EXERCICE 4 (*Iterates based on squared Mahalanobis norms*). — Let $(A_t)_{t \geq 1}$ be a sequence of symmetric positive definite matrices of size $d \times d$, and $(u_t)_{t \geq 1}$ a sequence in \mathbb{R}^d . For each iterates definition below, prove that they are UMD iterates and derive bounds on the regret $\sum_{t=1}^T \langle u_t, x - x_t \rangle$ for $T \geq 1$ and $x \in \mathcal{X}$.

1) Let $x_1 \in \mathcal{X}$ and

$$x_{t+1} = \arg \min_{x \in \mathcal{X}} \|(x_t + A_t^{-1} u_t) - x\|_{A_t}, \quad t \geq 1.$$

2) Let $x_1 \in \mathcal{X}$ and

$$x_{t+1} = \arg \min_{x \in \mathcal{X}} \left\{ -\langle A_t x_t + u_t, x \rangle - \frac{1}{2} x^\top A_{t+1} x \right\}, \quad t \geq 1.$$

3) Let $y_1 \in \mathbb{R}^d$ and

$$x_t = \arg \min_{x \in \mathcal{X}} \left\{ -\left\langle y_1 + \sum_{s=1}^{t-1} u_s, x \right\rangle + \frac{1}{2} \langle x, A_t x \rangle \right\}, \quad t \geq 1.$$

EXERCICE 5 (*Sparse payoff vectors*). — Consider online linear optimization on $\mathcal{X} = \Delta_d$. Let $1 \leq s \leq d$ be an integer, assume that payoff vectors $(u_t)_{t \geq 1}$ are in $[0, 1]^d$ and that for all $t \geq 1$, u_t has at most s nonzero components.

- 1) Using a constant ℓ_p regularizer (or mirror map) for a well-chosen value p , derive the best possible regret bound. Recall that for $1 < p < 2$, $x \mapsto \frac{1}{2} \|x\|_p^2$ is $(p-1)$ -strongly convex with respect to $\|\cdot\|_p$.
- 2) Using time-dependent regularizers (or mirror maps), derive the best possible horizon-free regret bound.

EXERCICE 6 (*A hybrid of mirror descent and dual averaging*). — Let H be a mirror map compatible with \mathcal{X} , $(u_t)_{t \geq 1}$ be a sequence in \mathbb{R}^d , $(\eta_t)_{t \geq 1}$ a positive sequence, and $x_1 \in \mathcal{X} \cap \text{dom } H$. Then define

$$x_{t+1} = \arg \max_{x \in \mathcal{X}} \left\{ \langle \nabla H(x_t) + \eta_t u_t, x \rangle - \frac{\eta_t}{\eta_{t+1}} H(x) \right\}, \quad t \geq 1.$$

One special case of interest is when $(\eta_t)_{t \geq 1}$ is nonincreasing.

- 1) Prove that the above can be seen as UMD iterates.
- 2) Is there cases where the above are MD with step-sizes? MD with parameters?
- 3) Derive regret bounds.

EXERCICE 7 (*Hyperbolic entropy*). — Let $\beta > 0$ and

$$H_\beta(x) = \sum_{i=1}^d \left(x_i \operatorname{arcsinh} \left(\frac{x_i}{\beta} \right) - \sqrt{x_i^2 + \beta^2} \right), \quad x \in \mathbb{R}^d.$$

- 1) Express ∇H_β and ∇H_β^* .
- 2) Prove that H_β is a mirror map compatible with all nonempty closed convex subsets of \mathbb{R}^d .
- 3) Prove that H_β is $(1 + \beta)^{-1}$ -strongly convex with respect to $\|\cdot\|_2$ over B_2 (the closed unit Euclidean ball).
- 4) Prove that H_β is $(1 + \beta d)^{-1}$ -strongly convex with respect to $\|\cdot\|_1$ over B_1 (the closed unit ℓ_1 -ball).
- 5) Prove that $\max_{x \in B_2} D_{H_\beta}(x, 0) \leq 2/\beta$.
- 6) Prove that if $\beta \leq 1$, then $\max_{x \in B_1} D_{H_\beta}(x, 0) \leq \log(3/\beta)$.

EXERCICE 8 (*Exponential weights with step-sizes*). — Let $(u_t)_{t \geq 1}$ be a sequence in \mathbb{R}^d and $(\gamma_t)_{t \geq 1}$ a nonincreasing sequence in \mathbb{R}^d . Consider:

$$x_t = \left(\frac{\exp \left(\left(\sum_{s=1}^{t-1} \gamma_s u_{s,i} \right) \right)}{\sum_{j=1}^d \exp \left(\sum_{s=1}^{t-1} \gamma_s u_{s,j} \right)} \right)_{1 \leq i \leq d}. \quad (1)$$

- 1) Let $T \geq 1$ and $x \in \Delta_d$. Derive a general bound on $\sum_{t=1}^T \langle u_t, x - x_t \rangle$.
- 2) Derive a regret bound in the case where there exists $L > 0$ such that $\|u_t\|_\infty \leq L$ for all $t \geq 1$.
- 3) In the multi-armed bandit problem, consider the variant of EXP3 based on (1), and derive a guarantee with similar assumptions as for EXP3 in the course.

