Exercices
Convexity tools
Université Paris–Saclay

Exercice 1 (*Logistic loss*). — Prove that function $x \mapsto \log(1 + e^{-x})$ defined on $\mathbb{R}$ is smooth.

Exercice 2 (*Characterization of Lipschitz continuity with subgradients*). — Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, $\|\cdot\|$ a norm on $\mathbb{R}^d$, and $L > 0$. Prove that f is L-Lipschitz on $\operatorname{int} \operatorname{dom} f$ with respect to $\|\cdot\|$ if, and only if:

$$\forall x \in \operatorname{int} \operatorname{dom} f, \ \forall y \in \partial f(x), \qquad \|y\|_* \leqslant L.$$

Exercice 3. — Let $\|\cdot\|$ be a norm in $\mathbb{R}^d$, B its closed unit ball, and $\|\cdot\|_*$ its dual norm. Prove that $I_B^* = \|\cdot\|_*$.

Exercice 4. — Let $y \in \mathbb{R}^d$, $a > 0$, $b \in \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. Compute the Legendre–Fenchel transform of the following functions:

a) $x \in \mathbb{R} \mapsto e^x$

b) $x \in \mathbb{R}^d \mapsto \langle y, x \rangle$

c) $x \in \mathbb{R}^d \mapsto \max(1 - \langle y, x \rangle \,,\ 0)$

d) $x \in \mathbb{R}^d \mapsto a f(x) + b + \langle y, x \rangle$

Exercice 5. — Let $\|\cdot\|$ be a norm in $\mathbb{R}^d$, $\|\cdot\|_*$ its dual norm, $1 < p, q < +\infty$ such that $1/p + 1/q = 1$ and $x, y \in \mathbb{R}^d$. Prove that

$$\langle x, y \rangle \leqslant \frac{1}{p} \|x\|^p + \frac{1}{q} \|y\|_*^q.$$

Exercice 6. — Let $1 < p \leqslant 2$. Prove that function $x \mapsto \frac{1}{2}\|x\|_p^2$ defined on $\mathbb{R}^d$ is $(p-1)$-strongly convex with respect to $\|\cdot\|_p$.

Exercice 7 (*Entropic regularizer*). — Let $h_{\text{ent}} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$h_{\text{ent}}(x) = \begin{cases} \sum_{i=1}^d x_i \log x_i & \text{if } x \in \Delta_d \\ +\infty & \text{otherwise,} \end{cases}$$

with convention $0 \log 0 = 0$.

1) Prove that h_{ent} is lower semicontinuous and convex.

2) Express the Bregman divergence associated with h_{ent}.

3) Prove that h_{ent} is 1-strongly convex with respect to $\|\cdot\|_1$.