An Introduction to Reinforcement Learning

From theory to algorithms

Joon Kwon

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Contents

Foreword

As of Fall 2023, this document contains lecture notes from a course given in *Master 2 Mathématiques et intelligence artificielle* in *Université Paris–Saclay*. These are highly incomplete and constantly updated as the lectures are given.

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Introduction

Chapter 1

Markov decision processes

For a finite set I, we denote $\Delta(I)$ the corresponding unit simplex in $\mathbb{R}^I$:

$$\Delta(I) = \left\{ x \in \mathbb{R}^I_+,\ \sum_{i \in I} x_i = 1 \right\}$$

and interpret it as set the probability distributions over I. For $i \in I$, the corresponding Dirac measure is denoted δ_i.

1.1 Definition

Definition 1.1.1. A *finite Markov Decision Process* (MDP) is a 4-tuple $(\mathcal{S}, \mathcal{A}, \mathcal{R}, p)$ where $\mathcal{S}, \mathcal{A}, \mathcal{R}$ are nonmepty finite sets and $p : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \times \mathcal{R} \to [0, 1]$ is such that for all $s, a \in \mathcal{S} \times \mathcal{A}$,

$$\sum_{(s',r) \in \mathcal{S} \times \mathcal{R}} p(s, a, s', r) = 1.$$

The elements of $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{S}$ are respectively called *states*, *actions* and *rewards*.

From now on, we assume that a finite MDP is given. For fixed values $(s, a) \in \mathcal{S} \times \mathcal{A}$, $p(s, a, \,\cdot\,)$ defines a probability distribution on $\mathcal{S} \times \mathcal{R}$, which the following notation emphasizes:

$$p(s', r|s, a) = p(s, a, s', r), \quad (s, a, s', r) \in \mathcal{S} \times \mathcal{A} \times \mathcal{S} \times \mathcal{R}.$$

Definition 1.1.2. Let $t \geqslant 1$. A *history of length* t is a finite sequence of the form

$$(s_0, a_0, r_1, s_1, a_1, r_2, s_2, a_2, \dots, s_{t-1}, a_{t-1}, r_t, s_t) \in (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^t \times \mathcal{S}.$$

By convention, a history of length 0 is an element $s_0 \in \mathcal{S}$. $\mathcal{H}^{(t)}$ denotes the set of histories of length t and $\mathcal{H}^\infty = (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^{\mathbb{N}}$ the set of infinite histories.

1.2 Policies

Definition 1.2.1. A *policy* is a sequence of maps $\pi = (\pi_t)_{t\geqslant 0}$ where $\pi_t : \mathcal{H}^{(t)} \to \Delta(\mathcal{A})$. For each $t \geqslant 0$ and $h^{(t)} \in \mathcal{H}^{(t)}$, denote

$$\pi_t(a|h^{(t)}) := \pi_t(h^{(t)})_a.$$

Π denotes the set of all policies.

Definition 1.2.2. A policy $\pi = (\pi_t)_{t\geqslant 0}$ is

- *deterministic* if for each $t \geqslant 0$ and $h^{(t)} \in \mathcal{H}^{(t)}$, there exists $a \in \mathcal{A}$ such that $\pi_t(h^{(t)})$ is the Dirac distribution in a;
- *Markovian* if for each $t \geqslant 0$, π_t is constant in all its variables but the last: in other words for a fixed value $s_t \in \mathcal{S}$, the map $\pi_t(\,\cdot\,, s_t)$ is constant; π_t can then be represented as $\pi_t : \mathcal{S} \to \Delta(\mathcal{A})$;
- *stationary* if it is Markovian and if $\pi_t = \pi_0$ for all $t \geqslant 0$; π can then be represented as $\pi : \mathcal{S} \to \Delta(\mathcal{A})$ and denoted $\pi(a|s) = \pi(s)_a$ for $(s, a) \in \mathcal{S} \times \mathcal{A}$.

Denote Π_0 (resp. $\Pi_{0,d}$) the set of stationary policies (resp.stationary and deterministic policies). A stationary and deterministic policy can be represented as $\pi : \mathcal{S} \to \mathcal{A}$.

1.3 Induced probability distributions over histories

Proposition 1.3.1. *Let $\mu \in \Delta(\mathcal{S})$ and a policy π. There exists a unique probability measure $\mathbb{P}_{\mu,\pi}$ on $\mathcal{H}^\infty = (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^{\mathbb{N}}$ (equipped with the product sigma-algebra) such that for all $T \geqslant 0$, $a_1, \dots, a_T \in \mathcal{A}$, $s_0, \dots, s_{T+1} \in \mathcal{S}$, and $r_1, \dots, r_{T+1} \in \mathcal{R}$,*

$$\mathbb{P}_{\mu,\pi}\left(\prod_{t=0}^{T} (\{s_t\} \times \{a_t\} \times \{r_{t+1}\}) \times \{s_{T+1}\} \times (\mathcal{S} \times \mathcal{A} \times [0,1])^{\mathbb{N}}\right)$$
$$= \mu(s_0) \prod_{t=0}^{T} \pi_t(a_t|h^{(t)}) p(s_{t+1}, r_{t+1}|s_t, a_t).$$

where for each $1 \leqslant t \leqslant T$, $h^{(t)} = (s_0, a_0, r_1, \dots, s_{t-1}, a_{t-1}, r_t, s_t)$.

Sketch of proof. The above expression defines a value for each set of the form

$$\prod_{t=0}^{T} (\{s_t\} \times \{a_t\} \times \{r_{t+1}\}) \times \{s_{T+1}\} \times (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^{\mathbb{N}}.$$

The map $\mathbb{P}_{\mu,\pi}$ can then be extended to so-called cylinder sets of the form

$$\prod_{t=0}^{T}(\mathcal{S}_t \times \mathcal{A}_t \times \mathcal{R}_{t+1}) \times \mathcal{S}_{T+1} \times (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^{\mathbb{N}},$$

where $\mathcal{S}_0, \dots, \mathcal{S}_{T+1} \subset \mathcal{S}$, $\mathcal{A}_0, \dots, \mathcal{A}_T \subset \mathcal{A}$ and $\mathcal{R}_1, \dots, \mathcal{R}_{T+1} \subset \mathcal{R}$ by summing as follows:

$$\begin{aligned}
&\mathbb{P}_{\mu,\pi}\left(\prod_{t=0}^{T}(\mathcal{S}_t \times \mathcal{A}_t \times \mathcal{R}_{t+1}) \times \mathcal{S}_{T+1} \times (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^{\mathbb{N}}\right) \\
&\quad = \sum_{\substack{s_0 \in \mathcal{S}_0 \\ \vdots \\ s_{T+1} \in \mathcal{S}_{T+1}}} \sum_{\substack{a_0 \in \mathcal{A}_0 \\ \vdots \\ a_T \in \mathcal{A}_T}} \sum_{\substack{r_1 \in \mathcal{R}_1 \\ \vdots \\ r_{T+1} \in \mathcal{R}_{T+1}}} \mu(s_0) \prod_{t=0}^{T} \pi_t(a_t|h^{(t)}) p(s_{t+1}, r_{t+1}|s_t, a_t).
\end{aligned}$$

$\mathbb{P}_{\mu,\pi}$ can then be seen to satisfy the assumptions of Kolmogorov's extension theorem which assures that $\mathbb{P}_{\mu,\pi}$ can be extended to a unique probability measure on $\mathcal{H}^{\infty}$. □

Definition 1.3.2. Let $\mu \in \Delta(\mathcal{S})$ and $\pi \in \Pi$. $\mathbb{P}_{\mu,\pi}$ is called the *probability distribution over histories* induced by initial state distribution μ and policy π.

We introduce some additional notation. Let $\mu \in \Delta(\mathcal{S})$ and $\pi \in \Pi$. We use $\mathbb{E}_{\mu,\pi}[\,\cdot\,]$ as a shorthand for

$$\mathbb{E}_{(S_0, A_0, R_1, \dots) \sim \mathbb{P}_{\mu,\pi}}[\,\cdot\,].$$

If μ is the Dirac in some state $s \in \mathcal{S}$, we write $\mathbb{P}_{s,\pi}$ (resp. $\mathbb{E}_{s,\pi}[\,\cdot\,]$) instead of $\mathbb{P}_{\delta_s,\pi}$ (resp. $\mathbb{E}_{\delta_s,\pi}[\,\cdot\,]$).

Definition 1.3.3. Let $(s,a) \in \mathcal{S} \times \mathcal{A}$, $\pi = (\pi_t)_{t \geqslant 0}$ a policy and $\pi' = (\pi'_t)_{t \geqslant 0}$ defined as

$$\begin{aligned}
\pi'_0(s) &= \delta_a, \\
\pi'_0(s') &= \pi_0(s') \quad \text{for } s' \neq s \\
\pi'_t &= \pi_t \quad \text{for } t \geqslant 1.
\end{aligned}$$

Then, $\mathbb{P}_{s,\pi'}$ is called the probability distribution induced by initial state s, initial action a, and policy π, and is denoted $\mathbb{P}_{s,a,\pi}$.

For $(s,a) \in \mathcal{S} \times \mathcal{A}$ and $\pi \in \Pi$, we also introduce the shorthand

$$\mathbb{E}_{s,a,\pi}[\,\cdot\,] := \mathbb{E}_{(S_0, A_0, R_1, \dots) \sim \mathbb{P}_{s,a,\pi}}[\,\cdot\,].$$

Proposition 1.3.4. *Let $f : \mathcal{S} \times \mathcal{R} \to \mathbb{R}$ and π a stationary policy. Then,* *(i) for all $s \in \mathcal{S}$,*

$$\sum_{(a,s',r)\in\mathcal{A}\times\mathcal{R}\times\mathcal{S}} \pi(a|s)p(s',r|s,a)f(s',r) = \mathbb{E}_{s,\pi}\left[f(S_1,R_1)\right],$$

(ii) for all $(s,a) \in \mathcal{S}\times\mathcal{A}$,

$$\sum_{(s',r)\in\mathcal{S}\times\mathcal{A}} p(s',r|s,a)f(s',r) = \mathbb{E}_{s,a,\pi}\left[f(S_1,R_1)\right].$$

Proof. Using the definition of $\mathbb{P}_{s,\pi}$, and more precisely the expression from Proposition 1.3.1,

$$\begin{aligned}
&\mathbb{E}_{s,\pi}\left[f(S_1,R_1)\right] \\
&\quad= \sum_{(s',r)\in\mathcal{S}\times\mathcal{R}} f(s',r)\times\mathbb{P}_{s,\pi}\left(\mathcal{S}\times\mathcal{A}\times\{r\}\times\left\{s'\right\}\times(\mathcal{A}\times\mathcal{R}\times\mathcal{S})^{\mathbb{N}}\right) \\
&\quad= \sum_{(r,s')\in\mathcal{R}\times\mathcal{S}} f(s',r)\sum_{a\in\mathcal{A}}\pi(a|s)p(r,s'|s,a) \\
&\quad= \sum_{(a,s',r)\in\mathcal{A}\times\mathcal{R}\times\mathcal{S}} \pi(a|s)p(s',r|s,a)f(s',r).
\end{aligned}$$

The other identity is proved similarly. ☐

Proposition 1.3.5. *Let $s \in \mathcal{S}$, π a stationary policy, $f : (\mathcal{R}\times\mathcal{S}\times\mathcal{A})^{\mathbb{N}} \to \mathbb{R}$ a measurable function (with respect to the product sigma-algebra) and random variables $(S'_0, A'_0, R'_1, S'_2, A'_2, R'_2, \dots) \sim \mathbb{P}_{s,\pi}$. Then, almost-surely,*

$$\mathbb{E}_{S'_t,A'_t,\pi}\left[f(R_1,A_2,S_2,\dots)\right] = \mathbb{E}\left[f(R'_{t+1},A'_{t+1},S'_{t+1},\dots)\,\middle|\,S'_t,A'_t\right].$$

Proof. ☐

1.4 Value functions

Definition 1.4.1. (i) A *state-value function* (aka *V-function*) is a function $v : \mathcal{S} \to \mathbb{R}$ or equivalently a vector $v = (v(s))_{s\in\mathcal{S}} \in \mathbb{R}^{\mathcal{S}}$.

(ii) An *action-value function* (aka *Q-function*) is a function $q : \mathcal{S}\times\mathcal{A} \to \mathbb{R}$ or equivalently a vector $q = (q(s,a))_{(s,a)\in\mathcal{S}\times\mathcal{A}} \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$.

Proposition 1.4.2. *Let $(R_t)_{t\geqslant 1}$ be a sequence of random variables with values in $\mathcal{R}$ and $\gamma \in (0,1)$. Then, the series $\sum_{t\geqslant 1}\gamma^{t-1}R_t$ converges almost-surely, and its sum is integrable.*

Proof. $\mathcal{R}$ being a finite subset of $\mathbb{R}$, it holds that $\max_{r\in\mathcal{R}} |r| < +\infty$. Then,

$$\left|\gamma^{t-1} R_t\right| \leqslant \gamma^{t-1} \max_{r\in\mathcal{R}} |r|, \quad \text{a.s.}$$

The result follows the dominated convergence theorem. □

Definition 1.4.3. Let $\pi \in \Pi$ and $\gamma \in (0, 1)$.

(i) The *state-value function of policy* π with discount factor γ is defined as

$$v_\pi^{(\gamma)}(s) = \mathbb{E}_{s,\pi}\left[\sum_{t=1}^{+\infty} \gamma^{t-1} R_t\right], \quad s \in \mathcal{S}.$$

(ii) The *action-value function of policy* π with discount factor γ is defined as

$$q_\pi^{(\gamma)}(s, a) = \mathbb{E}_{s,a,\pi}\left[\sum_{t=1}^{+\infty} \gamma^{t-1} R_t\right], \quad (s, a) \in \mathcal{S} \times \mathcal{A}.$$

We may denote $v_\pi = v_\pi^{(\gamma)}$ and $q_\pi = q_\pi^{(\gamma)}$ when γ is clear from the context.

Chapter 2

Bellman operators & optimality

We assume that $\gamma \in (0, 1)$ in given. The image of an element $x \in X$ by a map $F : X \to Y$ will often be denoted Fx instead of $F(x)$.

2.1 Bellman operators

Definition 2.1.1. Let π be a stationary policy. We define the following operators.

(i) $D^{(\gamma)} : \mathbb{R}^{\mathcal{S}} \to \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ as

$$(D^{(\gamma)}v)(s, a) = \sum_{(s',r)\in\mathcal{S}\times\mathcal{R}} p(s', r|s, a)(r + \gamma v(s')), \quad s \in \mathcal{S},\ a \in \mathcal{A}.$$

(ii) $E_\pi : \mathbb{R}^{\mathcal{S}\times\mathcal{A}} \to \mathbb{R}^{\mathcal{S}}$ as

$$(E_\pi q)(s) = \sum_{a\in\mathcal{A}} \pi(s|a)q(s, a), \quad s \in \mathcal{S}.$$

(iii) $E_* : \mathbb{R}^{\mathcal{S}\times\mathcal{A}} \to \mathbb{R}^{\mathcal{S}}$ as

$$(E_* q)(s) = \max_{a\in\mathcal{A}} q(s, a), \quad s \in \mathcal{S}.$$

(iv) $B_\pi^{(V,\gamma)} = E_\pi \circ D^{(\gamma)}$ (Bellman expectation operator for state-value functions)

(v) $B_*^{(V,\gamma)} = E_* \circ D^{(\gamma)}$ (Bellman optimality operator for state-value functions)

(vi) $B_\pi^{(Q,\gamma)} = D^{(\gamma)} \circ E_\pi$ (Bellman expectation operator for action-value functions)

(vii) $B_*^{(Q,\gamma)} = D^{(\gamma)} \circ E_*$ (Bellman optimality operator for action-value functions)

We will use lighter notation $D, E_\pi, E_*, B_\pi, B_*$ as soon as context prevents confusion. The following expressions follow from the definitions.

Proposition 2.1.2 (Explicit expression of Bellman operators)**.** *Let $v \in \mathbb{R}^{\mathcal{S}}$, $q \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$, and π a stationary policy. Then, the following expressions hold.*

$$(B_\pi v)(s) = \sum_{(a,s',r)\in\mathcal{A}\times\mathcal{S}\times\mathcal{R}} \pi(a|s)p(s',r|s,a)\left(r+\gamma v(s')\right), \quad s\in\mathcal{S},$$

$$(B_* v)(s) = \max_{a\in\mathcal{A}} \sum_{(s',r)\in\mathcal{S}\times\mathcal{R}} p(s',r|s,a)(r+\gamma v(s')), \quad s\in\mathcal{S},$$

$$(B_\pi q)(s,a) = \sum_{(s',r,a')\in\mathcal{S}\times\mathcal{R}\times\mathcal{A}} p(s',r|s,a)\left(r+\gamma\pi(a'|s')q(s',a')\right), \quad (s,a)\in\mathcal{S}\times\mathcal{A},$$

$$(B_* q)(s,a) = \sum_{(s',r)\in\mathcal{S}\times\mathcal{R}} p(s',r|s,a)\left(r+\gamma\max_{a'\in\mathcal{A}} q(s',a')\right), \quad (s,a)\in\mathcal{S}\times\mathcal{A}.$$

Proof. Immediate from the definitions. □

Proposition 2.1.3. *Let $v\in\mathbb{R}^{\mathcal{S}}$, $q\in\mathbb{R}^{\mathcal{S}\times\mathcal{A}}$, $s\in\mathcal{S}$, $a\in\mathcal{A}$ and π a stationary policy. Then,*

$$\begin{aligned}(B_\pi v)(s) &= \mathbb{E}_{s,\pi}\left[R_1+\gamma v(S_1)\right]\\ (B_\pi q)(s,a) &= \mathbb{E}_{s,a,\pi}\left[R_1+\gamma q(S_1,A_1)\right].\end{aligned}$$

Proof. Using the explicit expression from Proposition 2.1.2 and the definition of the probability measure $\mathbb{P}_{s,\pi}$ (see Proposition 1.3.1), we write

$$\begin{aligned}(B_\pi v)(s) &= \sum_{(a,s',r)\in\mathcal{A}\times\mathcal{S}\times\mathcal{R}} \pi(a|s)p(s',r|s,a)(r+\gamma v(s'))\\ &= \sum_{\substack{a\in\mathcal{A}\\ r\in\mathcal{R}}} \mathbb{P}_{s,\pi}\left(\{s\}\times\{a\}\times\{r\}\times\{s'\}\times(\mathcal{S}\times\mathcal{A}\times\mathcal{R})^{\mathbb{N}}\right)\\ &\qquad\times(r+\gamma v(s'))\\ &= \mathbb{E}_{s,\pi}\left[R_1+\gamma v(S_1)\right].\end{aligned}$$

The expression for $B_\pi q$ is proved similary. □

Definition 2.1.4. Let $d, n \geqslant 1$ integers. A map $F:\mathbb{R}^d\to\mathbb{R}^n$ is *monotone* if for all $x, x'\in\mathbb{R}^d$, $x\leqslant x'$ implies $Fx\leqslant Fx'$, where the inequalities are to be understood component-wise.

Proposition 2.1.5. *Operators $D, E_\pi, B_\pi^{(V)}, B_\pi^{(Q)}$ are affine with nonnegative coefficients. E_π is moreover linear. In particular, they are monotone.*

Proof. Immediate from the definitions. □

Proposition 2.1.6. *Let $v \in \mathbb{R}^{\mathcal{S}}$, $q \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$, $s \in \mathcal{S}$ and $a \in \mathcal{A}$. Then,*

(i) $(E_* q)(s,a) = \sup_{\pi\in\Pi_0} (E_\pi q)(s,a) = \sup_{\pi\in\Pi_{0,d}} (E_\pi q)(s,a)$,

(ii) $(B_* v)(s) = \sup_{\pi\in\Pi_0} (B_\pi v)(s) = \sup_{\pi\in\Pi_{0,d}} (B_\pi v)(s)$,

(iii) $(B_* q)(s,a) = \sup_{\pi\in\Pi_0} (B_\pi q)(s,a) = \sup_{\pi\in\Pi_{0,d}} (B_\pi q)(s,a)$.

Proof. (i) is an easy consequence from the definition of E_*. Then (ii) and (iii) follow using the monotonicity from Proposition 2.1.5. □

2.2 Bellman equations

Definition 2.2.1. Let X be a set and $F : X \to X$. An element $x \in X$ is a *fixed point* of F is $Fx = x$.

Theorem 2.2.2 (Banach's fixed point theorem)**.** *Let $0 \leqslant \gamma < 1$, (X, d) a complete metric space, and $F : X \to X$ a γ-Lipschitz map (with respect to distance d). Then, F has a unique fixed point $x_* \in X$ and for all sequence $(x_k)_{k\geqslant 0}$ satisfying $x_{k+1} = Fx_k$ $(k \geqslant 0)$, it holds that*

$$d(x_k, x_*) \leqslant \gamma^k d(x_0, x_*), \quad k \geqslant 0,$$

and thus $x_k \longrightarrow x_*$ as $k \to +\infty$.

Proposition 2.2.3. *Let π be a stationary policy. With respect to the norms $\|\cdot\|_\infty$ in $\mathbb{R}^{\mathcal{S}}$ and $\mathbb{R}^{\mathcal{S}\times\mathcal{A}}$,*

(i) $D^{(\gamma)}$ *is γ-Lipschitz*

(ii) E_π *is 1-Lipschitz*

(iii) E_* *is 1-Lipschitz*

(iv) $B_\pi^{(V,\gamma)}$, $B_*^{(V,\gamma)}$, $B_\pi^{Q,\gamma}$ *and* $B_*^{(Q,\gamma)}$ *are γ-Lipschitz and admit unique fixed points.*

Proof. TODO □

Proposition 2.2.4. *Let π be a stationary policy. Then,*

(i) $v_\pi = E_\pi q_\pi$,

(ii) $q_\pi = Dv_\pi$,

(iii) v_π *is the unique fixed point of* $B_\pi^{(V)}$*, meaning the unique solution to the Bellman expectation equation for state-value functions.*

(iv) q_π *is the unique fixed point of* $B_\pi^{(Q)}$*, meaning the unique solution to the Bellman expectation equation for action-value functions.*

Proof. TODO: consequence of the definitions. □

2.3 Greedy policy

Definition 2.3.1. A stationary and deterministic policy $\pi : \mathcal{S} \to \mathcal{A}$ is

(i) a *greedy policy* with respect to an action-value function $q \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ if for all $s \in \mathcal{S}$,

$$\pi(s) \in \operatorname*{Arg\,max}_{a\in\mathcal{A}} q(s, a),$$

where Arg max denotes the set of maximizers.

(ii) a *greedy policy* with respect to an state-value function $v \in \mathbb{R}^{\mathcal{S}}$ if $\pi \in \Pi_g\,[Dv]$.

$\Pi_g\,[q]$ denotes the set of greedy policies with respect to q and $\Pi_g\,[v]$ is a shorthand for $\Pi_g\,[Dv]$.

Proposition 2.3.2. *For* $v \in \mathbb{R}^{\mathcal{S}}$ *(resp.* $q \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$*),* $\Pi_g\,[v]$ *(resp.* $\Pi_g\,[q]$*) is nonempty.*

Proof. The set of actions $\mathcal{A}$ being finite (and nonempty), $\operatorname{Arg\,max}_{a\in\mathcal{A}} q(s, a)$ is nonempty, and the result follows. □

Notation $\pi_g\,[q]$ (resp. $\pi_g\,[v]$) denotes any element from $\Pi_g\,[q]$ (resp. $\Pi_g\,[v]$).

Proposition 2.3.3. *Let* $v \in \mathbb{R}^{\mathcal{S}}$ *and* $q \in \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$*. Then,*

(i) $E_* q = E_{\pi_g[q]} q$,

(ii) $B_* q = B_{\pi_g[q]} q$.

(iii) $B_* v = B_{\pi_g[v]} v$,

Proof. TODO □

2.4 Optimal value functions & policies

Definition 2.4.1. Let $\gamma \in (0,1)$. The *optimal state-value* and *actions-value functions* with respect to discount factor γ are respectively defined as

$$v_*^{(\gamma)}(s) = \sup_{\pi \in \Pi} v_\pi^{(\gamma)}(s), \quad s \in \mathcal{S},$$
$$q_*^{(\gamma)}(s,a) = \sup_{\pi \in \Pi} q_\pi^{(\gamma)}(s,a), \quad (s,a) \in \mathcal{S} \times \mathcal{A}.$$

As soon as discount factor γ is clear from the context, we may simply use notation v_* and q_*.

Definition 2.4.2. A policy π_* is *optimal* if $v_{\pi_*} = v_*$.

Theorem 2.4.3. *Let v_0 and q_0 the unique fixed points of $B_*^{(V)}$ and $B_*^{(Q)}$ respectively. Then, $\Pi_g[v_0] = \Pi_g[q_0]$ and for π_g in the latter set,*

(i) $v_* = v_0 = v_{\pi_g}$,

(ii) $q_* = q_0 = q_{\pi_g}$,

(iii) $v_* = E_* q_*$,

(iv) $q_* = D v_*$.

Remark 2.4.4. Some important takeaways from the above theorem are the following:

- v_* (resp. q_*) is the unique fixed point of $B_*^{(V)}$ (resp. $B_*^{(Q)}$), meaning the unique solution to the Bellman expectation equation for state-value function (resp. action-value function);
- there exists a stationary and deterministic optimal policy.

Proof. Let us first prove that $q_0 = Dv_0$ and $v_0 = E_* q_0$. Indeed,

$$Dv_0 = DB_* v_0 = DE_* Dv_0 = B_*(Dv_0),$$

therefore, Dv_0 is the unique fixed point of B_*, in other words $q_0 = Dv_0$. Then,

$$E_* q_0 = E_* Dv_0 = B_* v_0 = v_0.$$

Therefore, $\Pi_g[v_0] = \Pi_g[Dv_0] = \Pi_g[q_0]$. We recall that a set of greedy policies is never empty, as stated in Proposition 2.3.2.

Let $\pi_g \in \Pi_g[v_0]$. Then using the property of greedy policies from Proposition 2.3.3, $v_0 = B_* v_0 = B_{\pi_g} v_0$ and $q_0 = B_* q_0 = B_{\pi_g} q_0$. Value functions v_0 and q_0 are therefore the unique fixed points of $B_{\pi_g}^{(V)}$ and $B_{\pi_g}^{(Q)}$, respectively. In other words $v_0 = v_{\pi_g}$ and $q_0 = q_{\pi_g}$, by Proposition 2.2.4.

Therefore, $v_0 = v_{\pi_g} \leqslant \sup_{\pi \in \Pi_{0,d}} v_\pi$ because $\pi_g \in \Pi_{0,d}$ by definition, and similarly $q_0 \leqslant \sup_{\pi \in \Pi_{0,d}} q_\pi$.

Let us now prove that $v_0 \geqslant \sup_{\pi \in \Pi} v_\pi$. Let $\pi = (\pi_t)_{t \geqslant 0}$ be any policy, $s \in \mathcal{S}$, and consider random variables $(S_0, A_0, R_1, S_2, A_2, R_3, \dots) \sim \mathbb{P}_{s,\pi}$. Then for each $t \geqslant 0$,

$$\begin{aligned} v_0(S_t) = (B_* v_0)(S_t) &= \max_{a \in \mathcal{A}} \sum_{(s',r) \in \mathcal{S} \times \mathbb{R}} p(s', r | s, a)(r + \gamma v_0(s')) \\ &\geqslant \sum_{(s',r) \in \mathcal{S} \times \mathbb{R}} p(s', r | S_t, A_t)(r + \gamma v_0(s')) \\ &= \mathbb{E}\left[R_{t+1} + \gamma v_0(S_{t+1}) \,|\, S_t, A_t\right], \end{aligned}$$

where the last equality follows from the definition of $\mathbb{P}_{s,\pi}$. Then using the expression of $(Bv_0)(s)$ from Proposition 2.1.3, and applying the above recursively, we get

$$\begin{aligned} v_0(s) = (Bv_0)(s) &= \mathbb{E}_{s,\pi}\left[R_1 + \gamma v_0(S_1)\right] \\ &\geqslant \mathbb{E}_{s,\pi}\left[R_1 + \gamma \mathbb{E}\left[R_2 + \gamma v_0(S_2) \,|\, S_1, A_1\right]\right] \\ &= \mathbb{E}_{s,\pi}\left[R_1 + \gamma\left(R_2 + \gamma v_0(S_2)\right)\right] \\ &\geqslant \cdots \geqslant \mathbb{E}_{s,\pi}\left[\sum_{t=1}^{+\infty} \gamma^{t-1} R_t\right] \\ &= v_\pi(s). \end{aligned}$$

Therefore, $v_* = \sup_{\pi \in \Pi} v_\pi \leqslant v_0 = v_{\pi_g} \leqslant \sup_{\pi \in \Pi_{0,d}} v_\pi \leqslant \sup_{\pi \in \Pi} v_\pi = v_*$, and the lower and upper bounds being equal, all inequalies are equalities, and the supremums are maximums because they are attained for $\pi_g \in \Pi_{0,d} \subset \Pi$.

Then, we write

$$q_* = \sup_{\pi \in \Pi} q_\pi \geqslant \max_{\pi \in \Pi_{0,d}} q_\pi \geqslant q_{\pi_g} = q_0 = Dv_0 = D\left(\max_{\pi \in \Pi} v_\pi\right) \geqslant \sup_{\pi \in \Pi} Dv_\pi = \sup_{\pi \in \Pi} q_\pi = q_*$$

where the last inequality holds by monotonicity of D from Proposition 2.1.5 (by writing for $\pi \in \Pi$, $D \max_{\pi \in \Pi} v_\pi \geqslant Dv_\pi$ and then taking the supremum over $\pi \in \Pi$) Therefore, all inequalities are equalities are all supremums are maximums. $\square$

Chapter 3

Dynamic programming

3.1 Value iteration for policy evaluation

3.2 Value iteration for control

3.3 Policy iteration for control

3.4 Asynchronous value iteration

Chapter 4

Tabular reinforcement learning

4.1 Asynchronous stochastic approximations

4.2 Stochastic estimators of Bellman equations

4.3 Policy evaluation

4.4 Control

Chapter 5

Value function approximation

Chapter 6

Policy gradient

Chapter 7

Additional methods: actor-critic & model-based