

# An Introduction to Reinforcement Learning

*From theory to algorithms*

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# Foreword

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# Introduction

Reinforcement learning deals with problems where an agent sequentially interacts with a dynamic environment, which yields a sequence of rewards. We aim at finding the decision rule for the agent which yields the highest cumulative reward. We first study the case where characteristics of the environments are known, and then turn to techniques for dealing with unknown environments, which must then be progressively learnt through repeated interaction.

Reinforcement learning achieves great success in various applications: super-human algorithm for Go, robotics, finance, protein structure prediction, to name a few. Because it is so successful in practice, many resources are practice-oriented.

In these lectures, we first aim at a very rigorous presentation of the basic notions and tools. These building blocks will then be used to define algorithms, and establish theoretical guarantees for some of them.

# Chapter 1

## Markov decision processes

The framework for reinforcement learning is the Markov decision process, which is a repeated interaction between an agent and a dynamic environment, which can be informally described as follows.

We are given three finite nonempty sets  $\mathcal{S}$ ,  $\mathcal{A}$  and  $\mathcal{R}$ , the latter being a subset of  $\mathbb{R}$ . The environment chooses an initial *state*  $S_0 \in \mathcal{S}$  and reveals it to the agent. The agent then chooses an *action*  $A_0 \in \mathcal{A}$ , possibly at random. The environment then draws  $(R_1, S_1) \in \mathcal{R} \times \mathcal{S}$  according to a probability distribution that depends on  $S_0$  and  $A_0$ . The *reward*  $R_1$  and the new state  $S_1$  are revealed to the agent. The agent then chooses action  $A_2 \in \mathcal{A}$ , possibly at random. The environment then draws  $(R_2, S_2) \in \mathcal{R} \times \mathcal{S}$  according to a probability distribution which depends on  $S_0$  and  $A_0$ , and so on.

The total reward of the agent is defined as  $\sum_{t=1}^{+\infty} \gamma^{t-1} R_t$ , where  $0 < \gamma < 1$  is a given *discount factor*. The goal is to find the decision rule for the agent that yields the highest expected total reward.

Note that at stage  $t \geq 1$ , the choice of actions  $A_t$  by the agent may depend on all previously observed information, meaning  $(S_0, A_0, R_1, \dots, R_t, S_t)$ .

Depending on the problem, the dynamics of the environment (which maps a state-action pair to a probability distribution over reward-state pairs) may be known or not.

This chapter presents basic notions regarding MDPs, in a formal fashion.

For a finite set  $I$ , we denote  $\Delta(I)$  the corresponding unit simplex in  $\mathbb{R}^I$ :

$$\Delta(I) = \left\{ x \in \mathbb{R}_+^I, \sum_{i \in I} x_i = 1 \right\}$$

and interpret it as set the probability distributions over  $I$ . For  $i \in I$ , the corresponding Dirac measure is denoted  $\delta_i$ .

## 1.1 Formal definition

**Definition 1.1.1.** A *finite Markov Decision Process* (MDP) is a 4-tuple  $(\mathcal{S}, \mathcal{A}, \mathcal{R}, p)$  where  $\mathcal{S}, \mathcal{A}, \mathcal{R}$  are nonempty finite sets and  $\mathcal{R} \subset \mathbb{R}$ , and  $p : \mathcal{S} \times \mathcal{A} \times \mathcal{R} \times \mathcal{S} \rightarrow [0, 1]$  is such that for all  $s, a \in \mathcal{S} \times \mathcal{A}$ ,

$$\sum_{(r, s') \in \mathcal{R} \times \mathcal{S}} p(s, a, r, s') = 1.$$

The elements of  $\mathcal{S}$ ,  $\mathcal{A}$  and  $\mathcal{R}$  are respectively called *states*, *actions* and *rewards*. The following notation will be used:

$$p(r, s' | s, a) = p(s, a, r, s'), \quad (s, a, r, s') \in \mathcal{S} \times \mathcal{A} \times \mathcal{R} \times \mathcal{S}.$$

The knowledge of  $\mathcal{S}$  and  $\mathcal{A}$  is always assumed, but  $\mathcal{R}$  and  $p$  may not be known, depending on the context.

From now on, we assume that a finite MDP is given.

*Remark 1.1.2.* For fixed values  $(s, a) \in \mathcal{S} \times \mathcal{A}$ ,  $p(s, a, \cdot)$  defines a probability distribution on  $\mathcal{R} \times \mathcal{S}$ , which justifies notation  $p(\cdot | s, a)$ .

**Definition 1.1.3.** Let  $t \geq 1$ . A *history of length  $t$*  is a finite sequence of the form

$$(s_0, a_0, r_1, s_1, a_1, r_2, s_2, a_2, \dots, s_{t-1}, a_{t-1}, r_t, s_t) \in (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^t \times \mathcal{S}.$$

By convention, a history of length 0 is an element  $s_0 \in \mathcal{S}$ .  $\mathcal{H}^{(t)}$  denotes the set of histories of length  $t$  and  $\mathcal{H}^\infty = (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^\mathbb{N}$  the set of infinite histories.

*Remark 1.1.4.* Histories of length  $t$  correspond to the information observed by the agent at step  $t$  before choosing its action.

## 1.2 Policies

We now define policies, which are the formalization of decision rules for the agent. We first consider general policies, which allow for random decisions, as well as decision rules that depend on all available information (from the beginning of the interaction to the present state).

**Definition 1.2.1.** A *policy* is a sequence of maps  $\pi = (\pi^{(t)})_{t \geq 0}$  where  $\pi^{(t)} : \mathcal{H}^{(t)} \rightarrow \Delta(\mathcal{A})$ . For each  $t \geq 0$  and  $h^{(t)} \in \mathcal{H}^{(t)}$ , denote

$$\pi^{(t)}(a | h^{(t)}) := \pi^{(t)}(h^{(t)})_a.$$

$\Pi$  denotes the set of all policies.

*Remark 1.2.2.* When using policy  $\pi$ ,  $\pi^{(t)}(a|h^{(t)})$  is interpreted as the probability of the agent choosing action  $a$  at time  $t$  after having observed history  $h^{(t)}$ .

**Definition 1.2.3.** A policy  $\pi = (\pi^{(t)})_{t \geq 0}$  is

- *deterministic* if for each  $t \geq 0$  and  $h^{(t)} \in \mathcal{H}^{(t)}$ , there exists  $a \in \mathcal{A}$  such that  $\pi^{(t)}(h^{(t)})$  is the Dirac distribution in  $a$ ;
- *Markovian* if for each  $t \geq 0$ ,  $\pi^{(t)}$  is constant in all its variables but the last: in other words for a fixed value  $s_t \in \mathcal{S}$ , the map  $\pi^{(t)}(\cdot, s_t)$  is constant;  $\pi^{(t)}$  can then be represented as  $\pi^{(t)} : \mathcal{S} \rightarrow \Delta(\mathcal{A})$ ;
- *stationary* if it is Markovian and if for all  $t \geq 0$ ,  $\pi^{(t)} = \pi^{(0)}$ ;  $\pi$  can then be represented as  $\pi : \mathcal{S} \rightarrow \Delta(\mathcal{A})$  and denoted  $\pi(a|s) = \pi(s)_a$  for  $(s, a) \in \mathcal{S} \times \mathcal{A}$ .

Denote  $\Pi_0$  (resp.  $\Pi_{0,d}$ ) the set of stationary policies (resp. stationary and deterministic policies). A stationary and deterministic policy can be represented as  $\pi : \mathcal{S} \rightarrow \mathcal{A}$ .

In the next chapter, we will establish that there exists a stationary and deterministic optimal policy, and focus on stationary policies. We will however continue working with non-deterministic strategies, as they will later prove handy for *exploring* an unknown environment.

### 1.3 Induced probability distributions over histories

As soon as an MDP, a policy  $\pi$ , and an initial state distribution  $\mu$  are given, the interaction produces random variables  $S_0, A_0, R_1, S_1, A_1, R_2, \dots$ . This is formalized by the proposition below.

We first introduce the following notation. For  $T \geq 0$  and  $h^{(T)} = (s_0, a_0, r_1, \dots, r_T, s_T)$ , we consider the following associated subset of  $\mathcal{H}^\infty$ :

$$\text{Cyl } h^{(T)} = \{s_0\} \times \{a_0\} \times \{r_1\} \times \dots \times \{r_T\} \times \{s_T\} \times (\mathcal{A} \times \mathcal{R} \times \mathcal{S})^\mathbb{N}.$$

**Proposition 1.3.1.** *Let  $\mu \in \Delta(\mathcal{S})$  and a policy  $\pi$ . There exists a unique probability measure  $\mathbb{P}_{\mu, \pi}$  on  $\mathcal{H}^\infty = (\mathcal{S} \times \mathcal{A} \times \mathcal{R})^\mathbb{N}$  (equipped with the product  $\sigma$ -algebra) such that for all  $T \geq 0$ , and all  $h^{(T)} = (s_0, a_0, r_1, \dots, r_T, s_T) \in \mathcal{H}^{(T)}$ ,*

$$\mathbb{P}_{\mu, \pi}(\text{Cyl } h^{(T)}) = \mu(s_0) \prod_{t=0}^{T-1} \pi^{(t)}(a_t|h^{(t)}) p(r_{t+1}, s_{t+1}|s_t, a_t).$$

where for each  $0 \leq t \leq T$ ,  $h^{(t)} = (s_0, a_0, r_1, \dots, s_{t-1}, a_{t-1}, r_t, s_t)$ .

*Sketch of proof.* The above expression defines associates a value for each set of the form  $\text{Cyl } h^{(T)}$  for  $T \geq 0$  and  $h^{(T)} \in \mathcal{H}^{(T)}$ . The map  $\mathbb{P}_{\mu,\pi}$  can then be extended to so-called cylinder sets of the form

$$\prod_{t=0}^T (\mathcal{S}_t \times \mathcal{A}_t \times \mathcal{R}_{t+1}) \times \mathcal{S}_{T+1} \times (\mathcal{A} \times \mathcal{R} \times \mathcal{S})^{\mathbb{N}},$$

where  $\mathcal{S}_0, \dots, \mathcal{S}_{T+1} \subset \mathcal{S}$ ,  $\mathcal{A}_0, \dots, \mathcal{A}_T \subset \mathcal{A}$  and  $\mathcal{R}_1, \dots, \mathcal{R}_{T+1} \subset \mathcal{R}$  by summing as follows:

$$\begin{aligned} \mathbb{P}_{\mu,\pi} & \left( \prod_{t=0}^T (\mathcal{S}_t \times \mathcal{A}_t \times \mathcal{R}_{t+1}) \times \mathcal{S}_{T+1} \times (\mathcal{A} \times \mathcal{R} \times \mathcal{S})^{\mathbb{N}} \right) \\ &= \sum_{\substack{s_0 \in \mathcal{S}_0 \\ \vdots \\ s_{T+1} \in \mathcal{S}_{T+1}}} \sum_{\substack{a_0 \in \mathcal{A}_0 \\ \vdots \\ a_T \in \mathcal{A}_T}} \sum_{\substack{r_1 \in \mathcal{R}_1 \\ \vdots \\ r_{T+1} \in \mathcal{R}_{T+1}}} \mu(s_0) \prod_{t=0}^T \pi^{(t)}(a_t | h^{(t)}) p(s_{t+1}, r_{t+1} | s_t, a_t). \end{aligned}$$

$\mathbb{P}_{\mu,\pi}$  can then be seen to satisfy the assumptions of Kolmogorov's extension theorem which assures that  $\mathbb{P}_{\mu,\pi}$  can be extended to a unique probability measure on  $\mathcal{H}^\infty$ .  $\square$

*Remark 1.3.2.* In particular, Proposition 1.3.1 implies that a measure on  $\mathcal{H}^\infty$  coincide with  $\mathbb{P}_{\mu,\pi}$  as soon as they coincide on sets of the form  $\text{Cyl } h^{(T)}$ . This will be used in the proofs of Propositions 1.3.4 and 1.3.5 below.

**Definition 1.3.3.** Let  $\mu \in \Delta(\mathcal{S})$ ,  $\pi \in \Pi$ ,  $s \in \mathcal{S}$  and  $a \in \mathcal{A}$ .

- (i)  $\mathbb{P}_{\mu,\pi}$  from Proposition 1.3.1 is called the *probability distribution over histories* induced by initial state distribution  $\mu$  and policy  $\pi$ .
- (ii) We write  $\mathbb{P}_{s,\pi}$  instead of  $\mathbb{P}_{\delta_s,\pi}$ , which is called the probability distribution over histories induced by initial state  $s$  and policy  $\pi$ .
- (iii) Let  $\tilde{\pi} = (\tilde{\pi}^{(t)})_{t \geq 0}$  be defined as

$$\begin{aligned} \tilde{\pi}^{(0)}(s) &= \delta_a, \\ \tilde{\pi}^{(0)}(s') &= \pi^{(0)}(s') \quad \text{for } s' \neq s \\ \tilde{\pi}^{(t)} &= \pi^{(t)} \quad \text{for } t \geq 1. \end{aligned}$$

$\mathbb{P}_{s,\tilde{\pi}}$  is then called the probability distribution induced by initial state  $s$ , initial action  $a$ , and policy  $\pi$ , and is denoted  $\mathbb{P}_{s,a,\pi}$ .

The following shorthands will be used:

$$\begin{aligned} \mathbb{E}_{\mu,\pi} [\cdot] &= \mathbb{E}_{(S_0, A_0, R_1, \dots) \sim \mathbb{P}_{\mu,\pi}} [\cdot] \\ \mathbb{E}_{s,\pi} [\cdot] &= \mathbb{E}_{(S_0, A_0, R_1, \dots) \sim \mathbb{P}_{s,\pi}} [\cdot] \\ \mathbb{E}_{s,a,\pi} [\cdot] &= \mathbb{E}_{(S_0, A_0, R_1, \dots) \sim \mathbb{P}_{s,a,\pi}} [\cdot]. \end{aligned}$$



$\mathbb{P}_{s,a,\pi}$  corresponds to the interaction where the initial state is  $s$ , initial action is  $a$  (deterministically), and decision rule is given by  $\pi$  only for  $t \geq 1$ . In general, it cannot be defined as  $\mathbb{P}_{s,a}$  conditioned on the event  $\{A_0 = a\}$  because the probability  $\pi(a|s)$  of this event may be zero.

**Proposition 1.3.4.** *Let  $\pi = (\pi^{(t)})_{t \geq 0}$  be a policy and  $s \in \mathcal{S}$ . Then,*

$$\mathbb{P}_{s,\pi} = \sum_{a \in \mathcal{A}} \pi^{(0)}(a|s) \cdot \mathbb{P}_{s,a,\pi}.$$

*Proof.* It is sufficient to prove the identity between those two measures on the sets  $\text{Cyl } h^{(T)}$  that appear in the statement of Proposition 1.3.1, because they would then uniquely extend to all measurable subsets of  $\mathcal{H}^\infty$ .

Let  $T \geq 0$  and  $h^{(T)} = (s_0, a_0, r_1, \dots, r_T, s_T) \in \mathcal{H}^{(T)}$ , and denote  $h^{(t)} := (s_0, a_0, r_1, \dots, r_t, s_t)$  for  $0 \leq t \leq T$ . If  $s_0 \neq s$ , then the measures of the identity are zero when evaluated at  $\text{Cyl } h^{(T)}$ . We now assume  $s_0 = s$ .

Fix  $a \in \mathcal{A}$  and consider  $\tilde{\pi}$  defined as in Definition 1.3.3. Then,

$$\begin{aligned} \pi^{(0)}(a|s) \cdot \mathbb{P}_{s,a,\pi} \left( \text{Cyl } h^{(T)} \right) &= \pi^{(0)}(a|s) \prod_{t=0}^{T-1} \tilde{\pi}^{(t)}(a_t|h^{(t)}) p(r_{t+1}, s_{t+1}|s_t, a_t) \\ &= \mathbb{1}_{\{s_0 = s\}} \prod_{t=0}^{T-1} \pi^{(t)}(a_t|h^{(t)}) p(r_{t+1}, s_{t+1}|s_t, a_t) \\ &= \mathbb{1}_{\{a_0 = a\}} \cdot \mathbb{P}_{s,\pi} \left( \text{Cyl } h^{(T)} \right). \end{aligned}$$

Summing over  $a \in \mathcal{A}$  then gives

$$\sum_{a \in \mathcal{A}} \pi^{(0)}(a|s) \cdot \mathbb{P}_{s,a,\pi} \left( \text{Cyl } h^{(T)} \right) = \mathbb{P}_{s,\pi} \left( \text{Cyl } h^{(T)} \right).$$

□

The following proposition demonstrates that a given stationary policy induces a distribution over histories that has a Markov property in the following sense: for all  $t \geq 0$ , the distribution of  $(S_t, A_t, R_{t+1}, \dots)$  conditionally on  $\{S_t = s\}$  has the same as the distribution of  $(S_0, A_0, R_1, \dots)$  when the latter has initial state  $s$ .

**Proposition 1.3.5** (Markov property). *Let  $s, s' \in \mathcal{S}$ ,  $a, a' \in \mathcal{A}$ ,  $\pi$  a stationary policy,  $f : \mathcal{H}^\infty \rightarrow \mathbb{R}$  a bounded measurable function (with respect to the product  $\sigma$ -algebra), random variables  $(S'_0, A'_0, R'_1, S'_2, A'_2, R'_2, \dots)$  with distribution  $\mathbb{P}_{s,\pi}$  or  $\mathbb{P}_{s',\pi}$ , and  $t \geq 0$ .*

- (i) *If  $\mathbb{P}[S'_t = s'] > 0$ , the distribution of  $(S'_t, A'_t, R'_{t+1}, S'_{t+1}, \dots)$  conditionally on  $\{S'_t = s'\}$  is  $\mathbb{P}_{s',\pi}$ .*

(ii) *Almost-surely,*

$$\mathbb{E}_{S'_t, \pi} [f(S_0, A_0, R_1, \dots)] = \mathbb{E} [f(S'_t, A'_t, R'_{t+1}, \dots) \mid S'_t].$$

(iii) *If  $\mathbb{P}[S'_t = s', A'_t = a'] > 0$ , the distribution of  $(S'_t, A'_t, R'_{t+1}, S'_{t+1}, \dots)$  conditionnaly on  $\{S'_t = s', A'_t = a'\}$  is  $\mathbb{P}_{s', a', \pi}$ .*

(iv) *Almost-surely,*

$$\mathbb{E}_{S'_t, A'_t, \pi} [f(S_0, A_0, R_1, \dots)] = \mathbb{E} [f(S'_t, A'_t, R'_{t+1}, \dots) \mid S'_t, A'_t].$$

*Proof.* Let us assume  $\mathbb{P}[S'_t = s'] > 0$ . To prove (i), thanks to Proposition 1.3.1, it is enough to prove that  $\mathbb{P}[\cdot \mid S'_t = s']$  and  $\mathbb{P}_{s', \pi}$  coincide on sets on the form  $\text{Cyl } h^{(T)}$ . Let  $T \geq t$  and  $(s_t, a_t, r_{t+1}, \dots, r_T, s_T) \in \mathcal{H}^{(T-t)}$ . Using the expression from the statement of Proposition 1.3.1,

$$\begin{aligned} & \mathbb{P}[S'_t = s_t, A'_t = a_t, R'_{t+1} = r_{t+1}, \dots, R_T = r_T, S_T = s_T \mid S'_t = s'] \\ &= \frac{\mathbb{P}[S'_t = s', S'_t = s_t, A'_t = a_t, R'_{t+1} = r_{t+1}, \dots, R_T = r_T, S_T = s_T]}{\mathbb{P}[S'_t = s']} \\ &= \frac{\mathbb{1}_{\{s_0 = s'\}} \times \sum_{(s_0, \dots, s_{t-1}) \in \mathcal{S}^t} \delta_s(s_0) \prod_{\tau=0}^{T-1} \pi(a_\tau) p(r_{\tau+1}, s_{\tau+1} \mid s_\tau, a_\tau)}{\sum_{(s_0, \dots, s_{t-1}) \in \mathcal{S}^t} \delta_s(s_0) \prod_{t=0}^{T-1} \pi(a_\tau) p(r_{\tau+1}, s_{\tau+1} \mid s_\tau, a_\tau)} \\ &= \mathbb{1}_{\{s_0 = s'\}} \times \prod_{\tau=t}^{T-1} \pi(a_\tau) p(r_{\tau+1}, s_{\tau+1} \mid s_\tau, a_\tau) \\ &= \mathbb{P}_{s', \pi}[S_0 = s_t, A_0 = a_t, R_1 = r_{t+1}, \dots, R_{T-t} = r_T, S_{T-t} = s_T], \end{aligned}$$

and (i) follows.

We now turn to (ii). By definition of the conditionnal expectation,  $\mathbb{E}[f(S'_t, A'_t, R'_{t+1}, \dots) \mid S'_t]$  designates any random variable, measurable with respect to  $S'_t$  and with expectation equal to  $\mathbb{E}[f(S'_t, A'_t, R'_{t+1}, \dots)]$ . Let us prove that  $\mathbb{E}_{S'_t, \pi}[f(S_0, A_0, R_1, \dots)]$  indeed satisfy those properties. It is obviously measurable with respect to  $S'_t$ , as a deterministic function of the value of  $S'_t$ . Regarding the expectation, we can write

$$\begin{aligned} \mathbb{E}[\mathbb{E}_{S'_t, \pi}[f(S_0, A_0, R_1, \dots)]] &= \sum_{s' \in \mathcal{S}} \mathbb{P}[S'_t = s'] \times \mathbb{E}_{s', \pi}[f(S_0, A_0, R_1, \dots)] \\ &= \sum_{\substack{s' \in \mathcal{S} \\ \mathbb{P}[S'_t = s'] > 0}} \mathbb{P}[S'_t = s'] \times \mathbb{E}[f(S'_t, A'_t, R'_{t+1}, \dots) \mid S'_t = s'] \\ &= \mathbb{E}[f(S'_t, A'_t, R'_{t+1}, \dots)]. \end{aligned}$$

(iii) and (iv) are proved similarly.  $\square$

## 1.4 Value functions

We now introduce value functions which are fundamental tools for solving MDPs. The *optimal* value function, defined in the next chapter, associates to each state the best possible average reward than can be obtained starting from that state. Almost all algorithms aim at getting close to the optimal value function through iterative updates.

**Definition 1.4.1.** (i) A *state-value function* (aka *V-function*) is a function  $v : \mathcal{S} \rightarrow \mathbb{R}$  or equivalently a vector  $v = (v(s))_{s \in \mathcal{S}} \in \mathbb{R}^{\mathcal{S}}$ .  
(ii) An *action-value function* (aka *Q-function*) is a function  $q : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$  or equivalently a vector  $q = (q(s, a))_{(s, a) \in \mathcal{S} \times \mathcal{A}} \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ .

We equip both spaces with the  $\ell^\infty$  norm:

$$\|v\|_\infty = \max_{s \in \mathcal{S}} |v(s)|, \quad \|q\|_\infty = \max_{(s, a) \in \mathcal{S} \times \mathcal{A}} |q(s, a)|,$$

and with component-wise inequalities:

$$\begin{aligned} v \leq v' &\iff \forall s \in \mathcal{S}, v(s) \leq v'(s), \\ q \leq q' &\iff \forall (s, a) \in \mathcal{S} \times \mathcal{A}, q(s, a) \leq q'(s, a). \end{aligned}$$

**Lemma 1.4.2.** Let  $(R_t)_{t \geq 1}$  be a sequence of random variables with values in  $\mathcal{R}$  and  $\gamma \in (0, 1)$ . Then, the series  $\sum_{t \geq 1} \gamma^{t-1} R_t$  converges almost-surely, and its sum is integrable.

*Proof.*  $\mathcal{R}$  being a finite subset of  $\mathbb{R}$ , it holds that  $\max_{r \in \mathcal{R}} |r| < +\infty$ . Then,

$$|\gamma^{t-1} R_t| \leq \gamma^{t-1} \max_{r \in \mathcal{R}} |r|, \quad \text{a.s.}$$

The result follows from the dominated convergence theorem.  $\square$

**Definition 1.4.3.** Let  $\pi \in \Pi$  and  $\gamma \in (0, 1)$ .

(i) The *state-value function of policy  $\pi$*  with discount factor  $\gamma$  is defined as

$$v_\pi^{(\gamma)}(s) = \mathbb{E}_{s, \pi} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R_t \right], \quad s \in \mathcal{S}.$$

(ii) The *action-value function of policy  $\pi$*  with discount factor  $\gamma$  is defined as

$$q_\pi^{(\gamma)}(s, a) = \mathbb{E}_{s, a, \pi} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R_t \right], \quad (s, a) \in \mathcal{S} \times \mathcal{A}.$$

We may denote  $v_\pi = v_\pi^{(\gamma)}$  and  $q_\pi = q_\pi^{(\gamma)}$  when  $\gamma$  is clear from the context.

*Remark 1.4.4.*  $v_\pi(s)$  corresponds to the expected total reward starting from state  $s$  and following policy  $\pi$ .

## Chapter 2

# Bellman operators & optimality

This chapter introduces Bellman operators, which are the fundamental tools for solving MDPs. We then define optimal value functions and policies, and characterize them with the help of the Bellman operators.

We assume that  $\gamma \in (0, 1)$  is given. The image of an element  $x \in X$  by a map  $F : X \rightarrow Y$  will often be denoted  $Fx$  instead of  $F(x)$ .

### 2.1 Bellman operators

**Definition 2.1.1.** Let  $\pi$  be a stationary policy. We define the following operators.

(i)  $D^{(\gamma)} : \mathbb{R}^{\mathcal{S}} \rightarrow \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$  as

$$(D^{(\gamma)}v)(s, a) = \sum_{(r, s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s, a)(r + \gamma v(s')), \quad s \in \mathcal{S}, a \in \mathcal{A}.$$

(ii)  $E_{\pi} : \mathbb{R}^{\mathcal{S} \times \mathcal{A}} \rightarrow \mathbb{R}^{\mathcal{S}}$  as

$$(E_{\pi}q)(s) = \sum_{a \in \mathcal{A}} \pi(s|a)q(s, a), \quad s \in \mathcal{S}.$$

(iii)  $E_{*} : \mathbb{R}^{\mathcal{S} \times \mathcal{A}} \rightarrow \mathbb{R}^{\mathcal{S}}$  as

$$(E_{*}q)(s) = \max_{a \in \mathcal{A}} q(s, a), \quad s \in \mathcal{S}.$$

(iv)  $B_{\pi}^{(V, \gamma)} = E_{\pi} \circ D^{(\gamma)}$  (Bellman expectation operator for state-value functions)

- (v)  $B_*^{(V,\gamma)} = E_* \circ D^{(\gamma)}$  (Bellman optimality operator for state-value functions)
- (vi)  $B_\pi^{(Q,\gamma)} = D^{(\gamma)} \circ E_\pi$  (Bellman expectation operator for action-value functions)
- (vii)  $B_*^{(Q,\gamma)} = D^{(\gamma)} \circ E_*$  (Bellman optimality operator for action-value functions)

We will use lighter notation  $D, E_\pi, E_*, B_\pi, B_*$  as soon as context prevents confusion. The following expressions follow from the definitions.

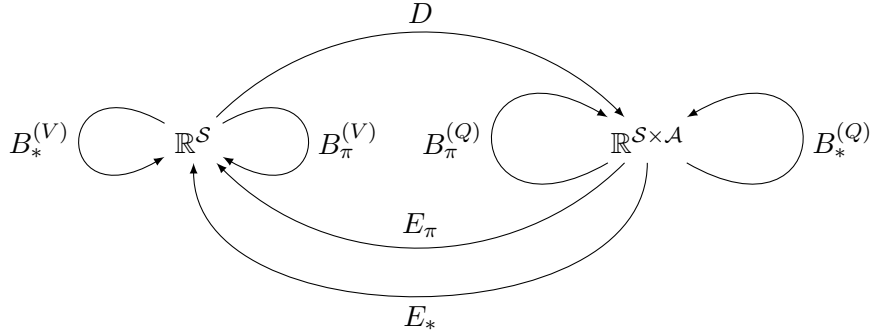


Figure 2.1: Operators  $D, E_\pi, E_*, B_\pi^{(V)}, B_*^{(V)}, B_\pi^{(Q)}$  and  $B_*^{(Q)}$ .

**Proposition 2.1.2** (Explicit expression of Bellman operators). *Let  $v \in \mathbb{R}^{\mathcal{S}}$ ,  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ , and  $\pi$  a stationary policy. Then, the following expressions hold.*

$$\begin{aligned}
 (B_\pi v)(s) &= \sum_{(a,r,s') \in \mathcal{A} \times \mathcal{R} \times \mathcal{S}} \pi(a|s) p(r, s'|s, a) (r + \gamma v(s')), \quad s \in \mathcal{S}, \\
 (B_* v)(s) &= \max_{a \in \mathcal{A}} \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} p(r, s'|s, a) (r + \gamma v(s')), \quad s \in \mathcal{S}, \\
 (B_\pi q)(s, a) &= \sum_{(r,s',a') \in \mathcal{R} \times \mathcal{S} \times \mathcal{A}} p(r, s'|s, a) (r + \gamma \pi(a'|s') q(s', a')), \quad (s, a) \in \mathcal{S} \times \mathcal{A}, \\
 (B_* q)(s, a) &= \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} p(r, s'|s, a) \left( r + \gamma \max_{a' \in \mathcal{A}} q(s', a') \right), \quad (s, a) \in \mathcal{S} \times \mathcal{A}.
 \end{aligned}$$

*Proof.* Immediate from the definitions.  $\square$

**Proposition 2.1.3** (Bellman operators as expectations). *Let  $v \in \mathbb{R}^{\mathcal{S}}$ ,  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ ,  $\pi$  a stationary policy,  $s \in \mathcal{S}$ ,  $a \in \mathcal{A}$ . Then,*

$$(i) \quad (Dv)(s, a) = \mathbb{E}_{s,a,\pi} [R_1 + \gamma v(S_1)],$$

$$\begin{aligned}
(ii) \quad (E_\pi q)(s) &= \mathbb{E}_{s,\pi} [q(s, A_0)], \\
(iii) \quad (B_\pi v)(s) &= \mathbb{E}_{s,\pi} [R_1 + \gamma v(S_1)], \\
(iv) \quad (B_\pi q)(s, a) &= \mathbb{E}_{s,a,\pi} [R_1 + \gamma q(S_1, A_1)], \\
(v) \quad (B_* v)(s) &= \max_{a \in \mathcal{A}} \mathbb{E}_{s,a,\pi} [R_1 + \gamma v(S_1)], \\
(vi) \quad (B_* q)(s, a) &= \mathbb{E}_{s,a,\pi} \left[ R_1 + \gamma \max_{a' \in \mathcal{A}} q(S_1, a') \right].
\end{aligned}$$

*Proof.* Let us prove (i). Let  $\tilde{\pi}$  the policy associated with  $(s, a)$  used in Definition 1.3.3 to define  $\mathbb{P}_{s,a,\pi}$ . Using the definition of the probability measure  $\mathbb{P}_{s,\pi}$  (see Proposition 1.3.1),

$$\begin{aligned}
\mathbb{E}_{s,a,\pi} [R_1 + \gamma v(S_1)] &= \mathbb{E}_{s,\tilde{\pi}} [R_1 + \gamma v(S_1)] \\
&= \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} (r + \gamma v(s')) \\
&\quad \times \mathbb{P}_{s,\pi'} \left( \mathcal{S} \times \mathcal{A} \times \{r\} \times \{s'\} \times (\mathcal{R} \times \mathcal{S} \times \mathcal{A})^{\mathbb{N}} \right) \\
&= \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s, a) (r + \gamma v(s')) \\
&= (Dv)(s, a)
\end{aligned}$$

We now turn to (ii).

$$\begin{aligned}
\mathbb{E}_{s,\pi} [q(s, A_0)] &= \sum_{a \in \mathcal{A}} q(s, a) \times \mathbb{P}_{s,a} \left( \mathcal{S} \times \{a\} \times (\mathcal{R} \times \mathcal{S} \times \mathcal{A})^{\mathbb{N}} \right) \\
&= \sum_{a \in \mathcal{A}} q(s, a) \pi(a | s) = (E_\pi q)(s).
\end{aligned}$$

We now deduce (iii) using Proposition 1.3.4:

$$\begin{aligned}
(B_\pi v)(s) &= (E_\pi(Dv))(s) = \sum_{a \in \mathcal{A}} \pi(a | s) \mathbb{E}_{s,a,\pi} [R_1 + \gamma v(S_1)] \\
&= \mathbb{E}_{s,\pi} [R_1 + \gamma v(S_1)].
\end{aligned}$$

For (iv), we combine (i) and (ii) with the help of the Markov property from Proposition 1.3.5; let  $(S'_0, A'_0, R'_1, \dots) \sim \mathbb{P}_{s,a,\pi}$ , then

$$\begin{aligned}
(B_\pi q)(s, a) &= (D(E_\pi q))(s, a) = \mathbb{E} [R'_1 + \gamma (E_\pi q)(S'_1)] \\
&= \mathbb{E} [R'_1 + \gamma \cdot \mathbb{E}_{S'_1,\pi} [q(S'_0, A_0)]] \\
&= \mathbb{E} [R'_1 + \gamma \cdot \mathbb{E} [q(S'_1, A'_1) | S'_1]] \\
&= \mathbb{E} [R'_1 + \gamma \cdot q(S'_1, A'_1)].
\end{aligned}$$

Finally, (v) and (vi) follow by composition.  $\square$

*Remark 2.1.4.* If for each  $s \in \mathcal{S}$ ,  $v(s)$  is interpreted as an estimate of the total reward obtained starting from state  $s$  and using policy  $\pi$ ,  $(B_\pi v)(s)$  is then an alternative estimate, as it is the expectation, when starting from state  $s$  of the actual first reward  $R_1$ , plus  $\lambda v(S_1)$  which is an estimate of remaining discounted rewards, as estimated by  $v$ . A similar interpretation holds for  $B_\pi q$ . We will see that the latter estimate is in some sense better: the Bellman operators will thus be used to iteratively *update* the estimates.

**Definition 2.1.5.** Let  $d, n \geq 1$  integers. A map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^n$  is *monotone* if for all  $x, x' \in \mathbb{R}^d$ ,  $x \leq x'$  implies  $Fx \leq Fx'$ , where the inequalities are to be understood component-wise.

**Proposition 2.1.6.** Let  $\pi$  be a stationary policy. Then, operators  $D$ ,  $E_\pi$ ,  $B_\pi^{(V)}$  and  $B_\pi^{(Q)}$  are affine with nonnegative coefficients.  $E_\pi$  is moreover linear. In particular, they are monotone.

*Proof.* Immediate from the definitions.  $\square$

**Proposition 2.1.7.** Let  $v \in \mathbb{R}^{\mathcal{S}}$ ,  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ ,  $s \in \mathcal{S}$  and  $a \in \mathcal{A}$ . Then,

- (i)  $(E_* q)(s) = \max_{\pi \in \Pi_0} (E_\pi q)(s) = \max_{\pi \in \Pi_{0,d}} (E_\pi q)(s),$
- (ii)  $(B_* v)(s) = \max_{\pi \in \Pi_0} (B_\pi v)(s) = \max_{\pi \in \Pi_{0,d}} (B_\pi v)(s),$
- (iii)  $(B_* q)(s, a) = \max_{\pi \in \Pi_0} (B_\pi q)(s, a) = \max_{\pi \in \Pi_{0,d}} (B_\pi q)(s, a).$

*Proof.* (i) Let  $s \in \mathcal{S}$  and  $\pi \in \Pi_0$ .

$$\begin{aligned} (E_\pi q)(s) &= \sum_{a \in \mathcal{A}} \pi(a|s) q(s, a) \leq \sum_{a \in \mathcal{A}} \pi(a|s) \max_{a' \in \mathcal{A}} q(s, a') \\ &= (E_* q)(s) \sum_{a \in \mathcal{A}} \pi(a|s) = (E_* q)(s). \end{aligned}$$

Taking the supremum over  $\pi \in \Pi_0$  yields

$$\sup_{\pi \in \Pi_0} (E_\pi q)(s) \leq (E_* q)(s).$$

Besides, for each  $s \in \mathcal{S}$ , there exists a maximizer of  $q(s, \cdot)$  (because the number of values is finite). Let  $\pi_{0,d}(\cdot|s)$  be a Dirac at one of the maximizers. This defines a stationary and deterministic policy  $\pi_{0,d}$ , which satisfies  $(E_{\pi_{0,d}} q)(s) = \max_{a \in \mathcal{A}} q(s, a)$  for all  $s \in \mathcal{S}$ . We then write for  $s \in \mathcal{S}$ ,

$$\begin{aligned} \sup_{\pi \in \Pi_0} (E_\pi q)(s) &\leq (E_* q)(s) = \max_{a \in \mathcal{A}} q(s, a) = (E_{\pi_{0,d}} q)(s) \\ &\leq \sup_{\pi \in \Pi_{0,d}} (E_\pi q)(s) \leq \sup_{\pi \in \Pi_0} (E_\pi q)(s). \end{aligned}$$

The above lowest and highest quantities are the same. Therefore, all inequalities are equalities, and the supremums are maximums because they are attained by  $\pi_{0,d}$ .

Then (ii) and (iii) follow from the monotonicity from Proposition 2.1.6.  $\square$

## 2.2 Bellman equations

**Definition 2.2.1.** Let  $X$  be a set and  $F : X \rightarrow X$ . An element  $x \in X$  is a *fixed point* of  $F$  is  $Fx = x$ .

The fixed points of Bellman operators will be of particular interest. They are often written in the form of the so-called Bellman equations: for a given stationary policy  $\pi$ , a state-value function  $v \in \mathbb{R}^{\mathcal{S}}$  is a fixed point of  $B_{\pi}^{(V)}$  if, and only if:

$$v(s) = \sum_{(a,r,s') \in \mathcal{A} \times \mathcal{R} \times \mathcal{S}} \pi(a|s) p(r, s'|s, a) (r + \gamma v(s')), \quad s \in \mathcal{S}.$$

The above is called the *Bellman expectation equation* for state-value functions. Similarly,  $v$  is the fixed point of  $B_{*}^{(V)}$  if, and only if:

$$v(s) = \max_{a \in \mathcal{A}} \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} p(r, s'|s, a) (r + \gamma v(s')), \quad s \in \mathcal{S},$$

which is called the Bellman *optimality equation*. The corresponding equations for action-value functions are similarly defined. We establish below that these equations have unique solutions and that they correspond respectively to  $v_{\pi}$  and  $v_{*}$ , where  $v_{*}$  is the value function associated with an optimal policy.

**Definition 2.2.2.** Let  $\eta > 0$ ,  $(X, d)$  and  $(Y, d')$  be metric spaces. A map  $F : X \rightarrow Y$  is a  $\eta$ -contraction if  $\gamma \in [0, 1)$  and  $F$  is  $\eta$ -Lipschitz continuous.

**Theorem 2.2.3** (Banach's fixed point theorem). *Let  $0 \leq \eta < 1$ ,  $(X, d)$  a complete metric space, and  $F : X \rightarrow X$  a  $\eta$ -contraction. Then,  $F$  has a unique fixed point  $x_{*} \in X$  and for all sequence  $(x_k)_{k \geq 0}$  satisfying  $x_{k+1} = Fx_k$  ( $k \geq 0$ ), it holds that*

$$d(x_k, x_{*}) \leq \eta^k d(x_0, x_{*}), \quad k \geq 0,$$

and thus  $x_k \rightarrow x_{*}$  as  $k \rightarrow +\infty$ .

*Proof.* For all  $k \geq 1$ , using the Lipschitz continuity of  $F$ ,

$$d(x_{k+1}, x_k) = d(Fx_k, Fx_{k-1}) \leq \eta d(x_k, x_{k-1}),$$



which by a simple induction implies

$$d(x_{k+1}, x_k) \leq \eta^k d(x_1, x_0),$$

from which we deduce that  $(x_k)_{k \geq 0}$  is a Cauchy sequence and thus admits a limit  $x_* \in X$ . Map  $F$  is continuous because of its Lipschitz property, and taking the limit in the identity  $x_{k+1} = Fx_k$  yields  $x_* = Fx_*$ , in other words,  $x_*$  is indeed of fixed point of  $F$ . If  $x_{**} \in X$  is also a fixed point, it holds that

$$d(x_*, x_{**}) = d(Fx_*, Fx_{**}) \leq \eta d(x_*, x_{**}),$$

which, because  $0 \leq \eta < 1$ , is only possible when  $x_* = x_{**}$ . The fixed point is therefore unique.

Besides, for all  $k \geq 0$ ,

$$d(x_{k+1}, x_*) = d(Fx_k, Fx_*) \leq \eta d(x_k, x_*),$$

which by a simple induction yields

$$d(x_k, x_*) \leq \eta^k d(x_0, x_*).$$

□

*Remark 2.2.4.* The above convergence is guaranteed *regardless* of the initial point  $x_0$ .

**Proposition 2.2.5.** *Let  $\pi$  be a stationary policy. With respect to the norms  $\|\cdot\|_\infty$  in  $\mathbb{R}^{\mathcal{S}}$  and  $\mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ ,*

- (i)  $D^{(\gamma)}$  is a  $\gamma$ -contraction,
- (ii)  $E_\pi$  is 1-Lipschitz continuous,
- (iii)  $E_*$  is 1-Lipschitz continuous,
- (iv)  $B_\pi^{(V, \gamma)}$ ,  $B_*^{(V, \gamma)}$ ,  $B_\pi^{(Q, \gamma)}$  and  $B_*^{(Q, \gamma)}$  are  $\gamma$ -contractions and admit unique fixed points.

*Proof.* Let  $v, v' \in \mathbb{R}^{\mathcal{S}}$ .

$$\begin{aligned} \|Dv' - Dv\|_\infty &= \max_{(s, a) \in \mathcal{S} \times \mathcal{A}} |Dv'(s, a) - Dv(s, a)| \\ &= \max_{(s, a) \in \mathcal{S} \times \mathcal{A}} \left| \sum_{(r, s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s, a) \gamma (v'(s') - v(s)) \right| \\ &\leq \max_{(s, a) \in \mathcal{S} \times \mathcal{A}} \gamma \|v' - v\|_\infty \sum_{(r, s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s, a) \\ &= \gamma \|v' - v\|_\infty, \end{aligned}$$

where the last inequality follows from  $p(\cdot | s, a)$  being a probability distribution over  $\mathcal{R} \times \mathcal{S}$ , which proves (i).

Let  $q, q' \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$  and  $\pi$  a stationary policy.

$$\begin{aligned} \|E_\pi q' - E_\pi q\|_\infty &= \max_{s \in \mathcal{A}} \left| \sum_{a \in \mathcal{A}} \pi(a|s) |q'(s, a) - q(s, a)| \right| \\ &\leq \max_{s \in \mathcal{A}} \sum_{a \in \mathcal{A}} \pi(a|s) \|q' - q\|_\infty \\ &= \|q' - q\|_\infty, \end{aligned}$$

where the last inequality follows from  $\pi(\cdot | s)$  being a probability distribution over  $\mathcal{A}$ .

Let  $s \in \mathcal{S}$ . If  $(E_* q')(s) \geq (E_* q)(s)$ , then

$$\begin{aligned} |(E_* q')(s) - (E_* q)(s)| &= (E_* q')(s) - (E_* q)(s) \\ &= \max_{a' \in \mathcal{A}} q'(s, a') - \max_{a \in \mathcal{A}} q(s, a) \\ &\leq \max_{a' \in \mathcal{A}} \{q'(s, a') - q(s, a')\} \\ &\leq \max_{a' \in \mathcal{A}} |q'(s, a') - q(s, a')| \\ &\leq \|q' - q\|_\infty. \end{aligned}$$

Similarly, if  $(E_* q')(s) \leq (E_* q)(s)$ , then

$$|E_* q'(s) - E_* q(s)| \leq \|q' - q\|_\infty.$$

Taking the maximum over  $s \in \mathcal{S}$  yields (iii):

$$\|E_* q' - E_* q\|_\infty \leq \|q' - q\|_\infty.$$

The Lipschitz property (iv) of Bellman operators then follow by composition.  $\square$

**Proposition 2.2.6.** *Let  $\pi$  be a stationary policy. Then,*

$$(i) \quad v_\pi = E_\pi q_\pi,$$

$$(ii) \quad q_\pi = Dv_\pi,$$

$$(iii) \quad v_\pi \text{ is the unique fixed point of } B_\pi^{(V)},$$

$$(iv) \quad q_\pi \text{ is the unique fixed point of } B_\pi^{(Q)}.$$

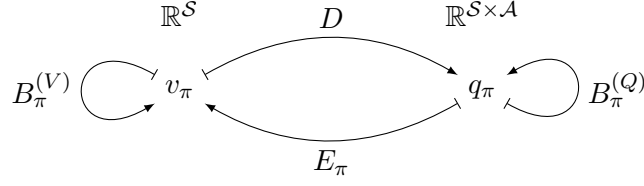


Figure 2.2: Relations between  $v_\pi$ ,  $q_\pi$ ,  $D$ ,  $E_\pi$ ,  $B_\pi^{(V)}$  and  $B_\pi^{(Q)}$ .

*Proof.* Let  $s \in \mathcal{S}$ . We prove (i) using Proposition 1.3.4:

$$\begin{aligned} (E_\pi q_\pi)(s) &= \sum_{a \in \mathcal{A}} \pi(a|s) q(s, a) = \sum_{a \in \mathcal{A}} \pi(a|s) \cdot \mathbb{E}_{s,a,\pi} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R_t \right] \\ &= \mathbb{E}_{s,\pi} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R_t \right] = v_\pi. \end{aligned}$$

We now turn to (ii). Let  $a \in \mathcal{A}$ . Let  $(S'_0, A'_0, R'_1, \dots) \sim \mathbb{P}_{s,a,\pi}$ . Then, using the expression of the Bellman operator as an expectation (from Proposition 2.1.3), we write

$$\begin{aligned} (Dv_\pi)(s, a) &= \mathbb{E}_{s,a,\pi} [R_1 + \gamma v_\pi(S_1)] \\ &= \mathbb{E} [R'_1 + \gamma v_\pi(S'_1)] \\ &= \mathbb{E} \left[ R'_1 + \gamma \cdot \mathbb{E}_{S'_1,\pi} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R_t \right] \right] \\ &= \mathbb{E} \left[ R'_1 + \gamma \cdot \mathbb{E} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R'_{t+1} \mid S'_1 \right] \right] \\ &= \mathbb{E} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R_t \right] = v_\pi, \end{aligned}$$

where for the fourth equality we used the Markov property for  $\mathbb{P}_{s,a,\pi}$  from Proposition 1.3.5.

Combining (i) and (ii) together with Banach's fixed point theorem from Theorem (2.2.3) yields (iii) and (iv).  $\square$

*Remark 2.2.7.* In other words,  $v_\pi$  (resp.  $q_\pi$ ) is the unique solution of the Bellman expectation equation for state-value function (resp. action-value functions).

## 2.3 Greedy policies

**Definition 2.3.1.** A stationary and deterministic policy  $\pi : \mathcal{S} \rightarrow \mathcal{A}$  is

- (i) a *greedy policy* with respect to an action-value function  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$  if for all  $s \in \mathcal{S}$ ,

$$\pi(s) \in \operatorname{Arg max}_{a \in \mathcal{A}} q(s, a),$$

where  $\operatorname{Arg max}$  denotes the set of maximizers;

- (ii) a *greedy policy* with respect to a state-value function  $v \in \mathbb{R}^{\mathcal{S}}$  if it is greedy with respect to  $Dv$ .

$\Pi_g[q]$  denotes the set of greedy policies with respect to  $q$  and  $\Pi_g[v]$  is a shorthand for  $\Pi_g[Dv]$ . Notation  $\pi_g[q]$  (resp.  $\pi_g[v]$ ) denotes any element from  $\Pi_g[q]$  (resp.  $\Pi_g[v]$ ).

*Remark 2.3.2.*  $\pi_g[q]$  corresponds to a policy which selects actions by simply comparing values of the action-value function  $q$ . In the case of  $\pi_g[v]$ , the action selection is based on a *one-step look-ahead*, as it rewrites as follows using Proposition 2.1.3:

$$\pi_g(s) \in \operatorname{Arg max}_{a \in \mathcal{A}} \mathbb{E}_{s,a,\pi} [R_1 + \gamma v(S_1)].$$

**Proposition 2.3.3.** For  $v \in \mathbb{R}^{\mathcal{S}}$  (resp.  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ ),  $\Pi_g[v]$  (resp.  $\Pi_g[q]$ ) is nonempty.

*Proof.* The set of actions  $\mathcal{A}$  being finite (and nonempty),  $\operatorname{Arg max}_{a \in \mathcal{A}} q(s, a)$  is nonempty, and the result follows.  $\square$

**Proposition 2.3.4.** Let  $v \in \mathbb{R}^{\mathcal{S}}$  and  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ . Then,

$$(i) \quad E_* q = E_{\pi_g[q]} q,$$

$$(ii) \quad B_* q = B_{\pi_g[q]} q.$$

$$(iii) \quad B_* v = B_{\pi_g[v]} v,$$

*Proof.* Let  $s \in \mathcal{S}$  and  $\pi \in \Pi_g[q]$ . By definition of a greedy policy,

$$(E_* q)(s) = \max_{a \in \mathcal{A}} q(s, a) = q(s, \pi(s)) = \sum_{a \in \mathcal{A}} \pi(s|a) q(s, a) = (E_\pi q)(s).$$

Then,  $B_*^{(Q)} = D \circ E_* = D \circ E_\pi = B_\pi$  and  $B_*^{(V)} = E_* \circ D = E_\pi \circ D = B_\pi$ .  $\square$

## 2.4 Optimal value functions and policies

**Definition 2.4.1.** Let  $\gamma \in (0, 1)$ . The *optimal state-value* and *actions-value functions* for discount factor  $\gamma$  are respectively defined as

$$\begin{aligned} v_*^{(\gamma)}(s) &= \sup_{\pi \in \Pi} v_\pi^{(\gamma)}(s), \quad s \in \mathcal{S}, \\ q_*^{(\gamma)}(s, a) &= \sup_{\pi \in \Pi} q_\pi^{(\gamma)}(s, a), \quad (s, a) \in \mathcal{S} \times \mathcal{A}. \end{aligned}$$

As soon as discount factor  $\gamma$  is clear from the context, we may simply use notation  $v_*$  and  $q_*$ .

*Remark 2.4.2.*  $v_*$  and  $q_*$  are well-defined because  $v_\pi$  and  $q_\pi$  can be easily seen to be bounded by  $(1 - \gamma)^{-1} \max_{r \in \mathbb{R}} |r|$ .

**Definition 2.4.3.** A policy  $\pi_*$  is *optimal* if  $v_{\pi_*} = v_*$ .

**Theorem 2.4.4.** Let  $v_0$  and  $q_0$  the unique fixed points of  $B_*^{(V)}$  and  $B_*^{(Q)}$  respectively. Then,  $\Pi_g[v_0] = \Pi_g[q_0]$  and for  $\pi_g$  in the latter set,

- (i)  $v_* = v_0 = v_{\pi_g}$ ,
- (ii)  $q_* = q_0 = q_{\pi_g}$ ,
- (iii)  $v_* = E_* q_*$ ,
- (iv)  $q_* = D v_*$ .

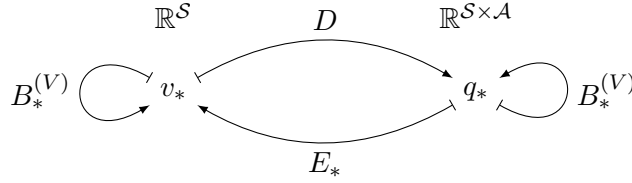


Figure 2.3: Relations between  $v_*$ ,  $q_*$ ,  $D$ ,  $E_*$ ,  $B_*^{(V)}$  and  $B_*^{(Q)}$ .

*Remark 2.4.5.* Some important takeaways from the above theorem are the following:

- $v_*$  (resp.  $q_*$ ) is the unique fixed point of  $B_*^{(V)}$  (resp.  $B_*^{(Q)}$ ), meaning the unique solution to the Bellman optimality equation for state-value function (resp. action-value function);
- there exists a stationary and deterministic optimal policy.

*Proof.* Let us first prove that  $q_0 = D v_0$  and  $v_0 = E_* q_0$ . Indeed,

$$D v_0 = D B_* v_0 = D E_* D v_0 = B_*(D v_0),$$

therefore,  $D v_0$  is the unique fixed point of  $B_*$ , in other words  $q_0 = D v_0$ . Then,

$$E_* q_0 = E_* D v_0 = B_* v_0 = v_0.$$

Therefore,  $\Pi_g[v_0] = \Pi_g[D v_0] = \Pi_g[q_0]$ . We recall that a set of greedy policies is never empty, as stated in Proposition 2.3.3.

Let  $\pi_g \in \Pi_g[v_0]$ . Then using the property of greedy policies from Proposition 2.3.4,  $v_0 = B_* v_0 = B_{\pi_g} v_0$  and  $q_0 = B_* q_0 = B_{\pi_g} q_0$ . Value functions  $v_0$

and  $q_0$  are therefore the unique fixed points of  $B_{\pi_g}^{(V)}$  and  $B_{\pi_g}^{(Q)}$ , respectively. In other words  $v_0 = v_{\pi_g}$  and  $q_0 = q_{\pi_g}$ , by Proposition 2.2.6.

Therefore,  $v_0 = v_{\pi_g} \leq \sup_{\pi \in \Pi_{0,d}} v_\pi$  because  $\pi_g \in \Pi_{0,d}$  by definition, and similarly  $q_0 \leq \sup_{\pi \in \Pi_{0,d}} q_\pi$ .

Let us now prove that  $v_0 \geq \sup_{\pi \in \Pi} v_\pi$ . Let  $\pi = (\pi^{(t)})_{t \geq 0}$  be any policy,  $s \in \mathcal{S}$ , and consider random variables  $(S_0, A_0, R_1, S_1, A_1, R_2, \dots) \sim \mathbb{P}_{s,\pi}$ . Let  $t \geq 0$ ,

$$v_0(S_t) = (B_* v_0)(S_t) = \max_{a \in \mathcal{A}} (Dv_0)(S_t, a) \geq (Dv_0)(S_t, A_t).$$

Let us rewrite this last quantity. Let  $(s_0, a_0) \in \mathcal{S}$  such that  $\mathbb{P}[S_t = s_0, A_t = a_0] > 0$ . Then, using the definition of  $\mathbb{P}_{s,\pi}$  (meaning the expression from Proposition 1.3.1),

$$\begin{aligned} (Dv_0)(s_0, a_0) &= \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s_0, a_0) (r + \gamma v_0(s')) \\ &= \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} \frac{\mathbb{P}[R_{t+1} = r, S_{t+1} = s', S_t = s_0, A_t = a_0]}{\mathbb{P}[S_t = s_0, A_t = a_0]} (r + \gamma v_0(s')) \\ &= \mathbb{E}[R_{t+1} + \gamma v_0(S_{t+1}) | S_t = s_0, A_t = a_0]. \end{aligned}$$

We deduce

$$(Dv_0)(S_t, A_t) = \mathbb{E}[R_{t+1} + \gamma v_0(S_{t+1}) | S_t, A_t],$$

and

$$v_0(S_t) \geq \mathbb{E}[R_{t+1} + \gamma v_0(S_{t+1}) | S_t, A_t].$$

Then using the expresion of  $B_*$  from Proposition 2.1.7 and the expression of  $B_\pi$  from Proposition 2.1.3, applying the above recursively, we get

$$\begin{aligned} v_0(s) &= (B_* v_0)(s) = \sup_{\pi' \in \Pi} (B_{\pi'} v_0)(s) \\ &\geq (B_\pi v_0)(s) = \mathbb{E}[R_1 + \gamma v_0(S_1)] \\ &\geq \mathbb{E}[R_1 + \gamma \mathbb{E}[R_2 + \gamma v_0(S_2) | S_1, A_1]] \\ &= \mathbb{E}[R_1 + \gamma R_2 + \gamma^2 v_0(S_2)] \\ &\geq \dots \geq \mathbb{E}_{s,\pi} \left[ \sum_{t=1}^T \gamma^{t-1} R_t + \gamma^T v_0(S_T) \right]. \end{aligned}$$

Using the dominated convergence theorem, we can take the limit in the above last expectation, as for all  $T \geq 1$ , almost-surely,

$$\begin{aligned} \left| \sum_{t=1}^T \gamma^{t-1} R_t + \gamma^T v_0(S_T) \right| &\leq \sum_{t=1}^T \gamma^{t-1} |\mathcal{R}| + \gamma^T \|v_0\|_\infty \\ &\leq |\mathcal{R}| \sum_{t=1}^{+\infty} \gamma^{t-1} + \|v_0\|_\infty, \end{aligned}$$

which provides an integrable upper bound, where  $|\mathcal{R}| := \max_{r \in \mathcal{R}} |r|$ . Hence,

$$v_0(s) \geq \mathbb{E} \left[ \sum_{t=1}^{+\infty} \gamma^{t-1} R_t \right] = v_\pi(s).$$

Therefore,

$$v_* = \sup_{\pi \in \Pi} v_\pi \leq v_0 = v_{\pi_g} \leq \sup_{\pi \in \Pi_{0,d}} v_\pi \leq \sup_{\pi \in \Pi} v_\pi = v_*,$$

and the lower and upper bounds being equal, all inequalities are equalities, and the supremums are maximums because they are attained for  $\pi_g \in \Pi_{0,d} \subset \Pi$ .

Then, we write

$$\begin{aligned} q_* &= \sup_{\pi \in \Pi} q_\pi \geq \max_{\pi \in \Pi_{0,d}} q_\pi \geq q_{\pi_g} = q_0 = Dv_0 \\ &= D \left( \max_{\pi \in \Pi} v_\pi \right) \geq \sup_{\pi \in \Pi} Dv_\pi = \sup_{\pi \in \Pi} q_\pi = q_*, \end{aligned}$$

where the last inequality holds by monotonicity of  $D$  from Proposition 2.1.6 (by writing for  $\pi \in \Pi$ ,  $D(\max_{\pi \in \Pi} v_\pi) \geq Dv_\pi$  and then taking the supremum over  $\pi \in \Pi$ ). Therefore, all inequalities are equalities and all supremums are maximums.  $\square$

## Chapter 3

# Dynamic programming

The properties of the Bellman operators established in the previous chapter allow the construction and analysis of dynamic programming algorithms (DP), meaning algorithms that solve MDPs with known dynamics. Starting from Chapter 4, we will study reinforcement learning, which is solving MDPs with either unknown dynamics, and/or by approximating the problem in some way. Most reinforcement learning methods (RL) are sample<sup>1</sup> variants of dynamic programming algorithms.

### 3.1 Value iteration

*Policy evaluation* is the computation of the value function  $v_\pi$  or  $q_\pi$  of a policy  $\pi$ . Many dynamic programming and reinforcement learning algorithms use policy evaluation as an intermediate step in finding the optimal policy. The (synchronous) value iteration for policy evaluation computes  $v_\pi$  (or  $q_\pi$ ), in the case of a stationary policy, by iterating the Bellman expectation operator  $B_\pi^{(V)}$  (resp.  $B_\pi^{(Q)}$ ). *Synchronous* means that all values (for each state, or each state-action pair) are updated simultaneously using the values from the current iterate.

In the context of MDPs, *control* is the computation of an optimal optimal policy. The (synchronous) value iteration for control approximately computes  $v_*$  (resp.  $q_*$ ) by iterating the Bellman expectation operator  $B_*^{(V)}$  (resp.  $B_*^{(Q)}$ ) and then considers a greedy policy.

**Definition 3.1.1** (Synchronous value iteration). Let  $\pi$  be a stationary policy,  $(v_k)_{k \geq 0}$  and  $(q_k)_{k \geq 0}$  two sequences in  $\mathbb{R}^{\mathcal{S}}$  and  $\mathbb{R}^{\mathcal{S} \times \mathcal{A}}$  respectively.

- (i)  $(v_k)_{k \geq 0}$  (resp.  $(q_k)_{k \geq 0}$ ) is a *synchronous state-value* (resp. *action-value*) iteration for the evaluation of  $\pi$  if for all  $k \geq 0$ ,

$$v_{k+1} = B_\pi v_k, \quad (\text{resp. } q_{k+1} = B_\pi q_k)$$

---

<sup>1</sup>as in *sampling*



- (ii)  $(v_k)_{k \geq 0}$  (resp.  $(q_k)_{k \geq 0}$ ) is a *synchronous state-value* (resp. *action-value*) iteration for control if for all  $k \geq 0$ ,

$$v_{k+1} = B_* v_k, \quad (\text{resp. } q_{k+1} = B_* q_k).$$

**Algorithm 1:** Synchronous value iteration for state-value functions for the evaluation of  $\pi$ .

**Input:** Initial value function  $v \in \mathbb{R}^{\mathcal{S}}$ , number of iterations  $n \geq 1$ .  
**for**  $k = 1, \dots, n$  **do**  
     $v' \leftarrow v$   
    **for**  $s \in \mathcal{S}$  **do**  
         $v(s) \leftarrow \sum_{(a,r,s') \in \mathcal{A} \times \mathcal{S} \times \mathcal{R}} \pi(a|s) p(r, s'|s, a) (r + \gamma v'(s'))$   
**return**  $v, \pi_g[v]$

**Algorithm 2:** Synchronous value iteration for state-value functions for control.

**Input:** Initial value function  $v \in \mathbb{R}^{\mathcal{S}}$ , number of iterations  $n \geq 1$ .  
**for**  $k = 1, \dots, n$  **do**  
     $v' \leftarrow v$   
    **for**  $s \in \mathcal{S}$  **do**  
         $v(s) \leftarrow \max_{a \in \mathcal{A}} \sum_{(r,s') \in \mathcal{S} \times \mathcal{R}} p(r, s'|s, a) (r + \gamma v'(s'))$   
**return**  $v, \pi_g[v]$

**Proposition 3.1.2** (Equivalence between synchronous state-value and action-value iterations). *Let  $\pi$  be a stationary policy,  $(v_k)_{k \geq 0}$  and  $(q_k)_{k \geq 0}$  two sequences in  $\mathbb{R}^{\mathcal{S}}$  and  $\mathbb{R}^{\mathcal{S} \times \mathcal{A}}$  respectively. Consider the following assertions.*

- |  |  |
|--|--|
| (a) $\forall k \geq 0, \quad v_{k+1} = B_\pi v_k;$ | (e) $\forall k \geq 0, \quad q_k = D v_k;$     |
| (b) $\forall k \geq 0, \quad q_{k+1} = B_\pi q_k;$ | (f) $\forall k \geq 0, \quad v_k = E_\pi q_k;$ |
| (c) $\forall k \geq 0, \quad v_{k+1} = B_* v_k;$   | (g) $\forall k \geq 0, \quad v_k = E_* q_k.$   |
| (d) $\forall k \geq 0, \quad q_{k+1} = B_* q_k;$   |  |

Then,

**Algorithm 3:** Synchronous value iteration for action-value functions for the evaluation of  $\pi$ .

**Input:** Initial value function  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ , number of iterations  $n \geq 1$ .  
**for**  $k = 1, \dots, n$  **do**  
     $q' \leftarrow q$   
    **for**  $s \in \mathcal{S}$  **do**  
        **for**  $a \in \mathcal{A}$  **do**  
             $q(s, a) \leftarrow \sum_{(r, s', a') \in \mathcal{R} \times \mathcal{S} \times \mathcal{A}} p(r, s' | s, a) (r + \gamma \pi(a' | s') q'(s', a'))$   
**return**  $q$

**Algorithm 4:** Synchronous value iteration for action-value functions for control.

**Input:** Initial value function  $q \in \mathbb{R}^{\mathcal{S} \times \mathcal{A}}$ , number of iterations  $n \geq 1$ .  
**for**  $k = 1, \dots, n$  **do**  
     $q' \leftarrow q$   
    **for**  $s \in \mathcal{S}$  **do**  
        **for**  $a \in \mathcal{A}$  **do**  
             $q(s, a) \leftarrow \sum_{(r, s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s, a) \left( r + \gamma \max_{a' \in \mathcal{A}} q'(s', a') \right)$   
**return**  $q$

- (i) (a) and (e) imply (b),                      (iii) (c) and (e) imply (d),  
(ii) (b) and (f) imply (a),                      (iv) (d) and (g) imply (c).

*Proof.* Assume (a) and (e). Then for all  $k \geq 0$ ,

$$B_\pi q_k = DE_\pi Dv_k = DB_\pi v_k = Dv_{k+1} = q_{k+1},$$

and (b) holds. The other implications are proved similarly.  $\square$

**Proposition 3.1.3** (Linear convergence of synchronous value iteration).  
*Let  $\pi$  be a stationary policy.*

- If  $(v_k)_{k \geq 0}$  and  $(q_k)_{k \geq 0}$  are synchronous state-value (resp. action-value) iterations for the evaluation of policy  $\pi$ , then for all  $k \geq 0$ ,

$$\begin{aligned} \|v_k - v_\pi\|_\infty &\leq \gamma^k \|v_0 - v_\pi\|_\infty, \\ \|q_k - q_\pi\|_\infty &\leq \gamma^k \|q_0 - q_\pi\|_\infty. \end{aligned}$$

- If  $(v_k)_{k \geq 0}$  and  $(q_k)_{k \geq 0}$  are synchronous state-value (resp. action-value) iterations for control, then for all  $k \geq 0$ ,

$$\begin{aligned} \|v_k - v_*\|_\infty &\leq \gamma^k \|v_0 - v_*\|_\infty, \\ \|q_k - q_*\|_\infty &\leq \gamma^k \|q_0 - q_*\|_\infty. \end{aligned}$$

*Proof.* We know from Proposition 2.2.6 and Theorem 2.4.4 that  $v_\pi$  (resp.  $q_\pi, v_*, q_*$ ) is the unique fixed point of Bellman operator  $B_\pi^{(V)}$  (resp.  $B_\pi^{(Q)}, B_*^{(V)}, B_*^{(Q)}$ ). The latter is a  $\gamma$ -contraction with respect to  $\|\cdot\|_\infty$  according to Proposition 2.2.5. The Banach's fixed point theorem (Theorem 2.2.3) then applies and gives the result.  $\square$

*Remark 3.1.4* (Computational complexity and memory requirements). The above results demonstrate that both algorithms for policy evaluation (resp. control) are equivalent in terms of output solutions. However, memory requirements of the state-value counterpart are lower by a factor  $|\mathcal{A}|$ . There is therefore no reason to choose action-value iteration in the context of dynamic programming. In reinforcement learning however, the additional stored values of the latter will be of great help.

## 3.2 Policy iteration

**Proposition 3.2.1** (Greedy policy improvement). *Let  $\pi$  be a stationary policy and  $\pi_g \in \Pi_g[v_\pi]$ . Then,*

- (i)  $v_{\pi_g} \geq v_\pi$ , (iii)  $v_{\pi_g} = v_\pi$  implies  $v_\pi = v_*$ ,  
(ii)  $q_{\pi_g} \geq q_\pi$ , (iv)  $q_{\pi_g} = q_\pi$  implies  $q_\pi = q_*$ .

*Proof.* Using the fact that  $v_\pi$  is a fixed point of  $B_\pi$  (Proposition 2.2.6), the property  $B_* = \sup_{\pi_0 \in \Pi_0} B_{\pi_0}$  from Proposition 2.1.7 and the property of greedy policies from Proposition 2.3.4,

$$v_\pi = B_\pi v_\pi \leq B_* v_\pi = B_{\pi_g} v_\pi.$$

Then, applying on both sides operator  $B_{\pi_g}$ , which is monotone thanks to Proposition 2.1.6, we get  $B_{\pi_g} v_\pi \leq B_{\pi_g}^2 v_\pi$ . Therefore,  $v_\pi \leq B_{\pi_g}^k v_\pi$  for all  $k \geq 1$ , and by Proposition 3.1.3, taking the limit as  $k \rightarrow +\infty$  gives (i). Besides, using the monotonicity of  $D$  from Proposition 2.1.6, together with Proposition 2.2.6 gives (ii):

$$q_\pi = Dv_\pi \leq Dv_{\pi_g} = q_{\pi_g}.$$

We now turn to (iii) and assume  $v_{\pi_g} = v_\pi$ . Using Propositions 2.2.6 and 2.3.4, we write  $v_\pi = v_{\pi_g} = B_{\pi_g} v_{\pi_g} = B_{\pi_g} v_\pi = B_* v_\pi$ . Thus,  $v_\pi$  is a fixed point of  $B_*$ , and  $v_\pi = v_*$  by Theorem 2.4.4, which proves (iii). (iv) is proved similarly.  $\square$

**Definition 3.2.2** (Policy iteration). A sequence  $(\pi_k)_{k \geq 0}$  of stationary policies is a *policy iteration* if  $\pi_{k+1} \in \Pi_g[v_{\pi_k}]$  for all  $k \geq 0$ .

*Remark 3.2.3* (Policy iteration is an idealized algorithm). Except in situations where  $v_{\pi_k}$  can be computed exactly, policy iteration is only an idealized algorithm because each step would involve the computation of  $v_{\pi_k}$  by iterating  $B_{\pi_k}$  infinitely. A practical variant, where  $B_{\pi_k}$  is only iterated a finite number of times is presented in Algorithm 5 and discussed in Remark 3.2.6 below.

*Remark 3.2.4* (Equivalent definition from action-value functions). Policy iteration can be written with action-value functions:

$$\pi_{k+1} \in \Pi_g[q_{\pi_k}],$$

which is equivalent to the above, because by definition of greedy policies for state-value functions gives:

$$\Pi_g[v_{\pi_k}] = \Pi_g[Dv_{\pi_k}] = \Pi_g[q_{\pi_k}],$$

where we used Proposition 2.2.6 for the last equality. Unlike value iterations, the corresponding algorithm is exactly the same even regarding the computational and memory requirements, because determining a greedy policy in  $\Pi_g[v_{\pi_k}]$  requires by definition the computation of  $Dv_{\pi_k} = q_{\pi_k}$ .

**Algorithm 5:** An approximate policy iteration

**Input:** Initial stationary and deterministic policy  $\pi$ , initial value function  $v \in \mathbb{R}^{\mathcal{S}}$ , number of inner iterations for approximate policy evaluation  $m \geq 1$ , number of iterations  $n \geq 1$ .

```

for  $k = 1, \dots, n$  do
  for  $\ell = 1, \dots, m$  do
     $v' \leftarrow v$ 
    for  $s \in \mathcal{S}$  do
       $v(s) \leftarrow \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s, \pi(a))(r + \gamma v'(s'))$ 
    for  $s \in \mathcal{S}$  do
      for  $a \in \mathcal{A}$  do
         $q(s, a) \leftarrow \sum_{(r,s') \in \mathcal{R} \times \mathcal{S}} p(r, s' | s, a) (r + \gamma v(s'))$ 
       $\pi(s) \leftarrow \arg \min_{a \in \mathcal{A}} q(s, a)$ 
  return  $\pi$ 

```

**Proposition 3.2.5** (Linear convergence of policy iteration). *Let  $(\pi_k)_{k \geq 0}$  be a policy iteration. Then for all  $k \geq 0$ ,*

$$\begin{aligned} \|v_{\pi_k} - v_*\|_{\infty} &\leq \gamma^k \|v_{\pi_0} - v_*\|_{\infty}, \\ \|q_{\pi_k} - q_*\|_{\infty} &\leq \gamma^k \|q_{\pi_0} - q_*\|_{\infty}. \end{aligned}$$

*Proof.* Denote  $v_k = v_{\pi_k}$  for  $k \geq 0$ .

$$\begin{aligned} v_* - v_{k+1} &= B_* v_* - B_* v_k + (B_* - B_{\pi_{k+1}}) v_k + B_{\pi_{k+1}} (v_k - v_{k+1}) \\ &\leq B_* v_* - B_* v_k, \end{aligned}$$

where the inequality holds because the second term is zero:

$$B_{\pi_{k+1}} v_k = B_{\pi_g[v_k]} v_k = B_* v_k$$

and the last term is nonpositive because  $B_{\pi_{k+1}}$  is monotone according to Proposition 2.1.6, and  $v_k \leq v_{k+1}$  by property of greedy policy improvement from Proposition 3.2.1. Moreover, by definition of  $v_*$ ,  $v_* \geq v_{\pi_{k+1}} = v_{k+1}$ . Therefore,

$$0 \leq v_* - v_{k+1} \leq B_* v_* - B_* v_k$$

and using the Lipschitz continuity of  $B_*$  from Proposition 2.2.5,

$$\|v_* - v_{k+1}\|_{\infty} \leq \|B_* v_* - B_* v_k\|_{\infty} \leq \gamma \|v_* - v_k\|.$$

The result for action-value functions is proved similarly.  $\square$

*Remark 3.2.6* (Generalized iteration). It is possible to define a family of iterations, which generalizes both value iteration and policy iteration. It is sometimes called *generalized policy iteration* or *optimistic policy iteration*. A sequence  $(v_k)_{k \geq 0}$  in  $\mathbb{R}^S$  (resp.  $(q_k)_{k \geq 0}$  in  $\mathbb{R}^{S \times \mathcal{A}}$ ) is a *generalized iteration* for state-value functions (resp. action-value functions) if there exists a sequence  $(m_k)_{k \geq 0}$  in  $\{1, 2, \dots\} \cup \{\infty\}$  such that for all  $k \geq 0$ ,

$$\begin{aligned} \pi_k &\in \Pi_g[v_k], & (\text{resp. } \pi_k &\in \Pi_g[q_k]) \\ v_{k+1} &= B_{\pi_k}^{m_k} v_k, & (\text{resp. } q_{k+1} &= B_{\pi_k}^{m_k} q_k), \end{aligned}$$

where by convention,  $B_{\pi}^{\infty} v = v_{\pi}$  (for all  $\pi \in \Pi_0$  and  $v \in \mathbb{R}^{S \times \mathcal{A}}$ ). Then, value iteration corresponds to  $m_k = 1$  (for all  $k \geq 0$ ) and policy iteration corresponds to  $m_k = \infty$  (for all  $k \geq 0$ ). A practical implementation of policy iteration where  $m_k$  may be large but not infinite then belongs to this family.

### 3.3 Asynchronous fixed point iterations

**Theorem 3.3.1** (A generalized fixed point theorem). *Let  $(X, d)$  a metric space,  $(\gamma_k)_{k \geq 0}$  nonnegative sequence in  $(0, 1)$  and  $(F_k)_{k \geq 0}$  a sequence of operators in  $X$  that share a common fixed point  $x_* \in X$  and so that  $F_k$  is  $\gamma_k$ -Lipschitz continuous. If  $(x_k)_{k \geq 0}$  satisfies  $x_{k+1} = F_k x_k$  for all  $k \geq 0$ , then*

$$d(x_k, x_*) \leq d(x_0, x_*) \left( \prod_{\ell=0}^{k-1} \gamma_{\ell} \right).$$

*If the above product converges to zero, then  $x_k \rightarrow x_*$  as  $k \rightarrow +\infty$ .*

*Proof.* Let  $k \geq 0$ .

$$d(x_{k+1}, x_*) = d(F_k x_k, F_k x_*) \leq \gamma_k d(x_k, x_*),$$

hence the result.  $\square$

For the remaining of this section,  $d \geq 1$  will be a given integer.

**Definition 3.3.2.** Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . For  $J \subset \{1, \dots, d\}$  and denote  $F^{|J|} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  the operator defined as

$$(F^{|J|} x)_j = (F x)_j$$

for  $j \in J$  and  $(F^{|J|} x)_{j'} = x_{j'}$  for  $j' \notin J$ . If  $J = \{j\}$  for some  $j \in \{1, \dots, d\}$ , we denote  $F^{|j|} = F^{\{j\}}$ .

*Remark 3.3.3.*  $F^{|J|}$  can be written as

$$F^{|J|} = I + \mathbb{1}_J \otimes (F - I) = (1 - \mathbb{1}_J) \otimes I + \mathbb{1}_J \otimes F$$

where  $\mathbb{1}_J$  denotes the vector with value 1 for components in  $J$  and value 0 for the other components, and  $\otimes$  denotes component-wise multiplication. This expression will be easier to generalize.

**Proposition 3.3.4.** *Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a 1-Lipschitz continuous map for  $\|\cdot\|_\infty$ . Then for all  $J \subset \{1, \dots, d\}$ ,  $F^{|J|}$  is 1-Lipschitz continuous for  $\|\cdot\|_\infty$ .*

*Proof.* For  $x, x' \in \mathbb{R}^d$ ,

$$\begin{aligned} \|F^{|J|}x - F^{|J|}x'\|_\infty &= \max_{1 \leq j \leq d} |(F^{|J|}x)_j - (F^{|J|}x')_j| \\ &= \max \left\{ \max_{j \in J} |(F^{|J|}x)_j - (F^{|J|}x')_j|, \max_{j \notin J} |(F^{|J|}x)_j - (F^{|J|}x')_j| \right\} \\ &\leq \max \left\{ \max_{j \in J} |(Fx)_j - (Fx')_j|, \max_{j \notin J} |x_j - x'_j| \right\} \\ &\leq \max \{ \|Fx - Fx'\|_\infty, \|x - x'\|_\infty \} \\ &\leq \|x - x'\|_\infty. \end{aligned}$$

□

**Proposition 3.3.5.** *Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $x \in \mathbb{R}^d$ . The following propositions are equivalent:*

- (i)  $x$  a fixed point of  $F$ ,
- (ii)  $x$  a is fixed point of  $F^{|j|}$  for all  $j \in \{1, \dots, d\}$ ,
- (iii)  $x$  a is fixed point of  $F^{|J|}$  for all  $J \subset \{1, \dots, d\}$ .

*Proof.* Immediate

□

**Proposition 3.3.6.** *Let  $\gamma \in [0, 1]$  and  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  a  $\gamma$ -Lipschitz continuous map for  $\|\cdot\|_\infty$ . Let  $J_1, \dots, J_M$  be such that  $\bigcup_{m=1}^M J_m = \{1, \dots, d\}$ . Then,  $F^{|J_M|} \circ \dots \circ F^{|J_1|}$  is  $\gamma$ -Lipschitz continuous for  $\|\cdot\|_\infty$ .*

*Proof.* For each  $1 \leq m \leq M$ , denote  $F^{1:m} = F^{|J_m|} \circ F^{|J_{m-1}|} \circ \dots \circ F^{|J_1|}$ .

Now fix  $1 \leq j \leq d$  and let  $m$  be an integer such that  $j \in J_m$ . Then it follows that,

$$(F^{1:M}x)_j = (F^{|J_M|}(F^{1:M-1}x))_j = (F^{1:M-1}x)_j = \dots = (F^{1:m}x)_j = (F(F^{1:m-1}x))_j.$$

Similarly,  $(F^{1:M}x')_j = (F(F^{1:m-1}x'))_j$ . Then using the above,

$$\begin{aligned} |(F^{1:M}x)_j - (F^{1:M}x')_j| &= |(F(F^{1:m-1}x))_j - (F(F^{1:m-1}x'))_j| \\ &\leq \|F(F^{1:m-1}x) - F(F^{1:m-1}x')\|_\infty \\ &\leq \gamma \|F^{1:m-1}x - F^{1:m-1}x'\|_\infty \\ &\leq \gamma \|x - x'\|_\infty, \end{aligned}$$

where we used the  $\gamma$ -Lipschitz property of  $F$  and for the last inequality the 1-Lipschitz continuity of each map  $F_1, F_2, \dots, F_{m-1}$  from Proposition 3.3.4. Taking the maximum over  $j$  yields

$$\|F^{1:M}x - F^{1:M}x'\|_\infty \leq \gamma \|x - x'\|_\infty.$$

□

**Theorem 3.3.7.** *Let  $\gamma \in (0, 1)$ ,  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  a  $\gamma$ -Lipschitz continuous map for  $\|\cdot\|_\infty$ , and  $(J_k)_{k \geq 0}$  a sequence of sets so that each  $j \in \{1, \dots, d\}$  belongs to infinitely many sets. If  $(x_k)_{k \geq 0}$  satisfies*

$$x_{k+1} = F^{|J_k|} x_k,$$

*then it converges to the unique fixed point of  $F$ .*

*Proof.* Recursively define an increasing sequence of integers  $(k_\ell)_{\ell \geq 0}$  as follows. Let  $k_0 = 0$  and  $k_1$  be the smallest integer such that

$$\bigcup_{k=0}^{k_1-1} J_k = \{1, \dots, d\},$$

which exists by assumption. Similarly for  $\ell \geq 2$ , let  $k_\ell$  the smallest integer larger than  $k_{\ell-1}$  such that

$$\bigcup_{k=k_{\ell-1}}^{k_\ell-1} J_k = \{1, \dots, d\}.$$

Denote  $F_k = F^{|J_k|}$  for all  $k \geq 0$  and  $G_\ell = F_{k_{\ell+1}-1} \circ \dots \circ F_{k_\ell+1} \circ F_{k_\ell}$  for all  $\ell \geq 0$ . Then we can apply Proposition 3.3.6 which gives that each map  $G_\ell$  is  $\gamma$ -Lipschitz continuous for  $\|\cdot\|_\infty$ . Because  $x_{k_{\ell+1}} = G_\ell x_{k_\ell}$  for all  $\ell \geq 0$ , by Theorem 3.3.1, we can write

$$\|x_{k_\ell} - x_*\| \leq \gamma^\ell \|x_0 - x_*\|, \quad \ell \geq 0,$$

where  $x_*$  is the unique fixed point of  $F$ . Moreover, using the fact that each map  $F_k$  ( $k \geq 0$ ) is 1-Lipschitz continuous for  $\|\cdot\|_\infty$  and has  $x_*$  as fixed point thanks to Propositions 3.3.4 and 3.3.5, we can write for  $k > k_\ell$ ,

$$\begin{aligned} \|x_k - x_*\|_\infty &= \|(F_{k-1} \circ \dots \circ F_{k_\ell})x_{k_\ell} - (F_{k-1} \circ \dots \circ F_{k_\ell})x_*\|_\infty \\ &\leq \|x_{k_\ell} - x_*\|_\infty \leq \gamma^\ell \|x_0 - x_*\|_\infty. \end{aligned}$$

Hence the convergence of  $x_k$  to  $x_*$  as  $k \rightarrow +\infty$ .  $\square$

### 3.4 Asynchronous value iterations

**Definition 3.4.1** (Asynchronous value iterations). Let  $\pi$  be a stationary policy.

- (i) A sequence  $(v_k)_{k \geq 0}$  in  $\mathbb{R}^{\mathcal{S}}$  is an asynchronous state-value iteration for the evaluation of policy  $\pi$  (resp. for control) if there exists a sequence  $(\mathcal{S}_k)_{k \geq 0}$  of subsets of  $\mathcal{S}$  such that

$$v_{k+1} = B_\pi^{\mathcal{S}_k} v_k, \quad \left( \text{resp. } v_{k+1} = B_*^{\mathcal{S}_k} v_k \right).$$

$(\mathcal{S}_k)_{k \geq 0}$  is then called the sequence of updated states.



- (ii) A sequence  $(q_k)_{k \geq 0}$  in  $\mathbb{R}^{\mathcal{S} \times \mathcal{A}}$  is an asynchronous state-value iteration for the evaluation of policy  $\pi$  (resp. for control) if there exists a sequence  $(\mathcal{Q}_k)_{k \geq 0}$  of subsets of  $\mathcal{S} \times \mathcal{A}$  such that

$$q_{k+1} = B_\pi^{\mathcal{Q}_k} q_k, \quad \left( \text{resp. } q_{k+1} = B_*^{\mathcal{Q}_k} q_k \right).$$

$(\mathcal{Q}_k)_{k \geq 0}$  is then called the sequence of updated state-action pairs.

**Proposition 3.4.2** (Convergence of asynchronous value iterations). *Let  $\pi$  be a stationary policy.*

- (i) *Let  $(v_k)_{k \geq 0}$  be a state-value iteration for the evaluation of policy  $\pi$  (resp. for control) where each state is updated infinitely. Then,  $v_k$  converges to  $v_\pi$  (resp.  $v_*$ ) as  $k \rightarrow +\infty$ .*
- (ii) *Let  $(q_k)_{k \geq 0}$  be a action-value iteration for the evaluation of policy  $\pi$  (resp. for control) where each state-action pair is updated infinitely. Then,  $q_k$  converges to  $q_\pi$  (resp.  $q_*$ ) as  $k \rightarrow +\infty$ .*

*Proof.* Follows from Theorem 3.3.7. □

*Remark 3.4.3* (Single-component updates).