Tolman-Oppenheimer-Volkoff Equation

Application to Neutron Stars

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Table of Contents

Table of Contents

Semi-empirical Mass Formula

- TOV equation

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{m\rho}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi P r^3}{m}\right)\left(1-\frac{2m}{r}\right)^{-1}$$

$$c = G = 1$$

- Macroscopic differential equation for pressure
- Derived assuming spherically symmetric and perfect fluid(Neutron stars, etc)
  - P: isotropic rest frame pressure
  - ρ: rest frame energy density
  - m: mass
- All functions of radial coordinate r.

Derivation of TOV

- Non-vacuum Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Assume spherically symmetric, perfect fluid

$T_{\mu\nu}$: Energy-momentum tensor

$G_{\mu\nu}$: Einstein tensor

$R_{\mu\nu}$: Ricci tensor

R : Curvature scalar(Ricci scalar, trace of Ricci tensor)

$g_{\mu\nu}$: Metric tensor(E.g: δ_{ij}, $\eta_{\mu\nu}$)

- Spherically symmetric metric

$$\mathrm{d}s^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega$$

Table of Contents

Coupled Equation

- The mass and radius are derived by solving the coupled differential equation

$$\begin{cases} \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{m\rho}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi P r^3}{m}\right)\left(1-\frac{2m}{r}\right)^{-1} \\ \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \end{cases}$$

- Mass is defined as the integrated energy density, theoretically when $P = 0$.

$$M = m(R) = 4\pi \int_0^R \rho(r) r^2 \mathrm{d}r$$

- End of integration when pressure is about $10^{-9} \sim 10^{-8}$MeV/fm^3, equivalent to about $1.602 \times 10^{25} \sim 1.602 \times 10^{26}$dyne/cm^3

Runge-Kutta 4th order

- Very famous and frequently used method, used to solve non-linear differential equations.
- Change of metric[1] is needed due to singularities at $r = 0$.

$$\frac{\mathrm{d}P}{\mathrm{d}h} = \rho + P,\ h(p=0) = 0$$
$$\frac{\mathrm{d}r^2}{\mathrm{d}h} = -2r^2 \frac{r - 2m}{m + 4\pi p r^3}$$
$$\frac{\mathrm{d}m}{\mathrm{d}h} = -4\pi\rho r^3 \frac{r - 2m}{m + 4\pi p r^3}$$

[1] tovh.pro, J. M. Lattimer

Table of Contents

Polytropic EoS

- Non-relativistic Polytropic EoS $P(\rho) = K_{nrel}\rho^{5/3}$
- when $k_n << m_n c$

$$P = \frac{1}{3}\frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_\mathrm{n}} d^3k \frac{k^4c^2}{(k^2c^2 + m_n^2c^4)^{1/2}} \approx K_{nrel}\rho^{5/3}$$

$$K_{nrel} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2}{m_n c^2}\right)^{5/3} \approx 6.428 \cdot 10^{-26}\,\frac{\mathrm{cm}^2}{\mathrm{erg}^{2/3}}$$

$$\therefore\ \ \rho = (P/K_{nrel})^{3/5}$$

Polytropic EoS

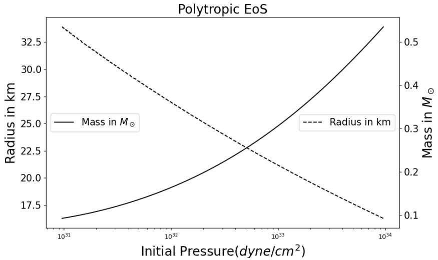

Figure 1: Polytropic EoS, relation between $p_0 - M$, $p_0 - R$

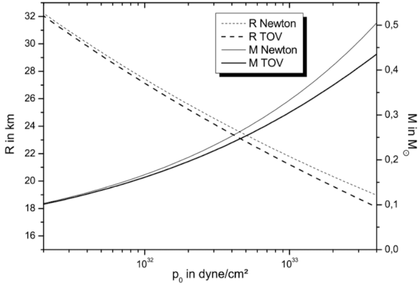

Figure 2: Polytropic EoS, relation between $p_0 - M$, $p_0 - R$, TOV compared to Newtonian limit. Compact Stars for Undergraduates(2004), Sagert

Polytropic EoS M-R

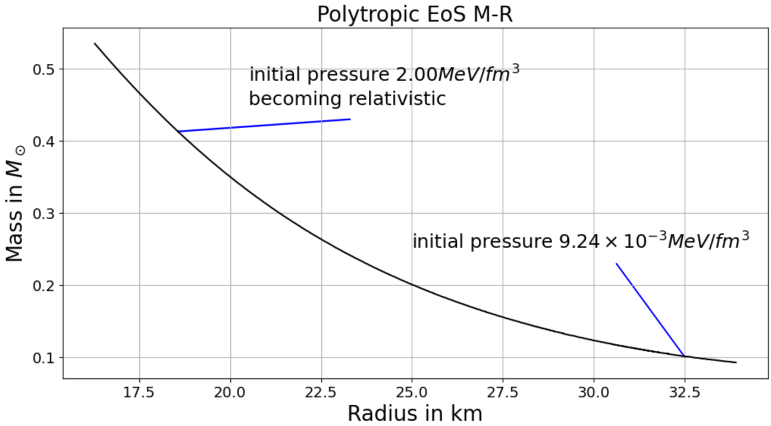

Figure 3: Polytropic EoS, M-R plot.

Relativistic pure neutron gas

- EoS, derived from Fermi distribution, $x = \dfrac{k_n}{m_n c}$

  k_n is Fermi momentum of neutrons.

$$
\begin{aligned}
P &= \frac{1}{3}\frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_\mathrm{f}} d^3k \frac{k^4c^2}{(k^2c^2+m_n^2c^4)^{1/2}} \\
&= \frac{\epsilon_0}{24}\left[(2x^3-3x)(1+x^2)^{1/2}+3\sinh^{-1}x\right]
\end{aligned}
$$

$$
\begin{aligned}
\rho &= \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_\mathrm{f}} d^3k(k^2c^2+m_n^2c^4)^{1/2}k^2 \\
&= \frac{\epsilon_0}{8}\left[(2x^3+x)(1+x^2)^{1/2}-\sinh^{-1}x\right]
\end{aligned}
$$

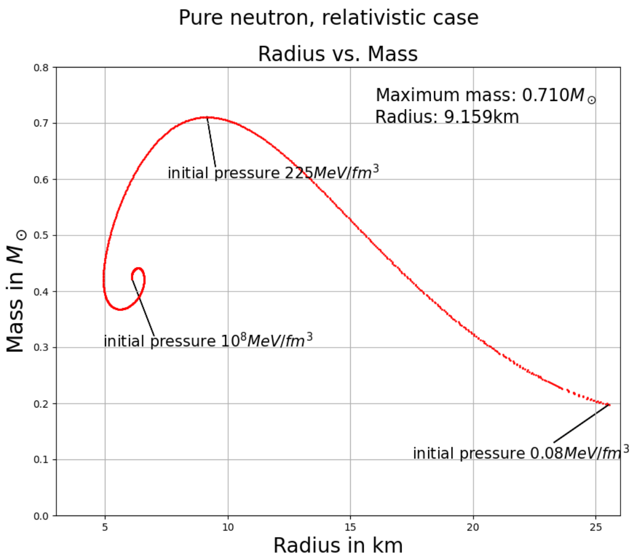

Figure 4: Relativistic pure neutron, M-R plot

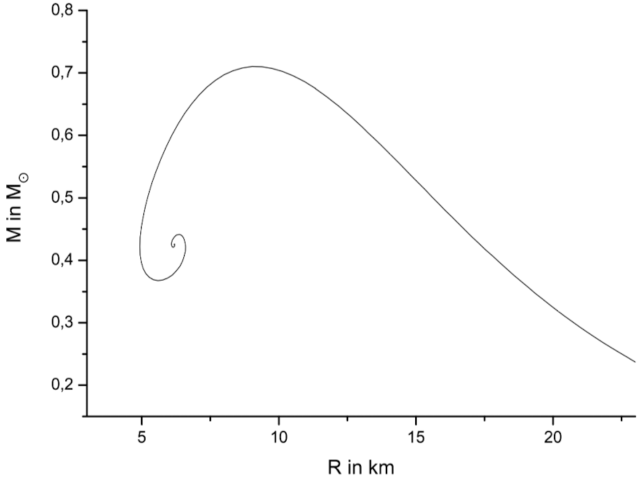

Figure 5: Relativistic pure neutron, M-R plot, *Compact Stars for Undergraduates*(2004), Sagert, maximum mass $M = 0.712 M_\odot$, radius $R = 9.14$km

Empirical Interaction

Empirical interaction for symmetric($n_n = n_p$) matter. [2]

$$\frac{\rho(n)}{n} = m_N + \langle E_0 \rangle u^{2/3} + \frac{1}{2} A u + \frac{B}{\sigma + 1} u^\sigma$$

- All units in MeV and fm
  - $u = n/n_0$
  - $\langle E_0 \rangle = 22.1\text{MeV}$: average kinetic energy per nucleon
  - $\sigma = 2.112,\ A = -118.2\text{MeV},\ B = 65.39\text{MeV}$: fit parameters, constrained by binding energy and nuclear compressibility K_0.
  - $K_0 = 400\text{MeV}$
- Neutron star is assumed to be asymmetric ($n_n > n_p$).

[2] R. R. Silbar and S. Reddy, Am. J. Phys. 72, 892 (2004), nucl-th/0309041.

Considering asymmetry

Function $S(u)$ describes symmetry energy

$$\frac{\rho(n,\delta)}{n} = \frac{\rho(n,0)}{n} + \delta^2 S(u) + O(\delta^3)$$

$$\delta = \frac{n_n - n_p}{n}$$

$S(u)$ is found to be[3]

$$S(u) = (2^{2/3} - 1)\langle E_0 \rangle (u^{2/3} - F(u)) + S_0 F(u)$$

$F(u)$ defines potential contributions, with boundary conditions $F(0) = 0,\ F(1) = 1$. S_0 is chosen to be $30\mathrm{MeV}$ and $F(u)$ is chosen to be u.

[3] M. Prakash, T. L. Ainsworth, and J. M. Lattimer, Phys. Rev. Lett. 61, 2518 (1988).

Empirical Interaction

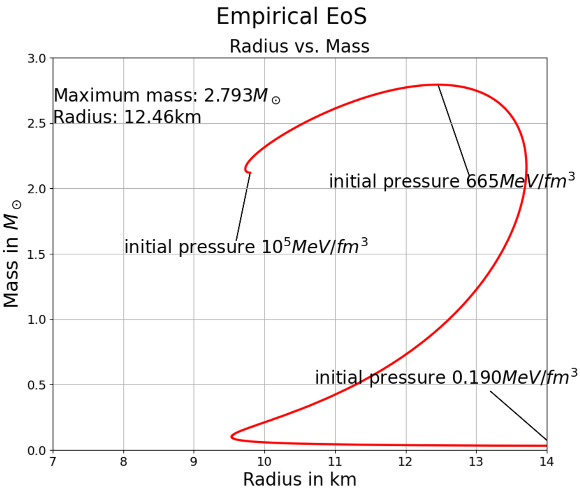

Figure 6: Empirical EoS, M-R plot

Empirical Interaction

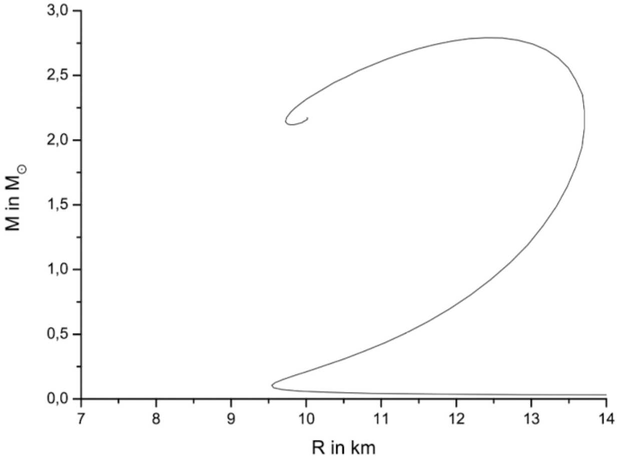

Figure 7: Empirical EoS, M-R plot, *Compact Stars for Undergraduates*(2004), Sagert, maximum mass $M = 2.792 M_\odot$, radius $R = 12.46$km

Table of Contents

Next

1. Study Skyrme model
2. Keep track of numerical error
3. Any suggestions will help!