Notes to accompany `tovh.pro`

The TOV equations are written using the function h as the independent variable, where

$$dh = \frac{dp}{\rho + p}, \qquad h\,(p = 0) = 0,$$

and the dependent variables are chosen to be $r^2, m, \mathcal{N}$ and ϕ. $\mathcal{N}$ is the enclosed baryon number times the baryon rest mass and ϕ is related to the rotational drag function ω (see below). All units are in km ($c = G = 1$). Note that h does not correspond to the enthalpy or any other usual thermodynamic quantity. The TOV equations are

$$\begin{aligned}
\frac{dr^2}{dh} &= -2r^2 \frac{r - 2m}{m + 4\pi p r^3}\,, \\
\frac{dm}{dh} &= -4\pi \rho r^3 \frac{r - 2m}{m + 4\pi p r^3}\,, \\
\frac{d\mathcal{N}}{dh} &= -4\pi \frac{m_b}{m_b + u_0} \rho r^{7/2} \frac{\sqrt{r - 2m}}{m + 4\pi p r^3}\,, \\
\frac{d\phi}{dh} &= \frac{\phi\,(\phi + 3)\,(r - 2m) - 4\pi\,(\phi + 4)\, r^3\,(\rho + p)}{m + 4\pi p r^3}.
\end{aligned}$$

The boundary conditions at $r = 0$ are

$$h = h_c, \qquad r^2\,(h_c) = m\,(h_c) = \mathcal{N}\,(h_c) = \phi\,(h_c) = 0.$$

The TOV equations, expressed in this way, do not become singular at either boundary, and very close to the origin the following limits apply:

$$\frac{dr^2}{dh} \to -\frac{3}{2\pi\,(\rho + 3p)}, \qquad \frac{dm}{dh} \to -\frac{3\rho r}{\rho + 3p} \to 0,$$

$$\frac{d\mathcal{N}}{dh} \to -\frac{3\rho r}{\rho + 3p} \frac{m_b}{m_b + u_0} \to 0, \qquad \frac{d\phi}{dh} = -\frac{24}{5} \frac{\rho + p}{\rho + 3p}.$$

The range of integration is $h_c \geq h \geq 0$ so h_c must be varied to determine a given mass or baryon mass.

The internal energy per baryon u, energy density ρ, baryon number density n and pressure p are related by

$$\rho = n\,(m_b + u)\,, \qquad p = n^2 \frac{du}{dn}.$$

u_0 is the internal energy at zero density, usually taken to be the binding energy of iron. The binding energy of the star is $\mathrm{BE} = \mathcal{N}(R) - M$ where $M = m(R)$ and R is the stellar radius where $p(R) = 0$.

Moment of Inertia

In the slow-motion approximation, the moment of inertia is given by

$$I = \frac{8\pi}{3} \int_0^R r^4 \left(\rho + p\right) e^{(\lambda - \nu)/2} \omega dr,$$

where $\lambda = -\ln(1 - 2m/r)$ and ν are the metric coefficients and ω is the rotational drag function. In terms of the function $j = e^{-(\lambda+\nu)/2}$, the rotational drag satisfies

$$\frac{d}{dr}\left(r^4 j \frac{d\omega}{dr}\right) = -4r^3 \omega \frac{dj}{dr},$$

with the boundary conditions

$$\omega_R = 1 - \frac{2I}{R^3}, \qquad \left(\frac{d\omega}{dr}\right)_0 = 0.$$

Therefore, the moment of inertia can be written as

$$I = -\frac{2}{3} \int_0^R r^3 \omega \frac{dj}{dr} dr = \frac{1}{6} \int_0^R d\left(r^4 \omega \frac{dj}{dr}\right) = \frac{R^4}{6} \left(d\omega/dr\right)_R .$$

We note that the second order differential equation ω satisfies can be instead written as a first order differential equation in terms of the function $\phi = d\ln\omega/d\ln r$,

$$\frac{d\phi}{dr} = -\frac{\phi}{r}\left(\phi + 3\right) - \left(4 + \phi\right) \frac{d\ln j}{dr},$$

where

$$\frac{d\ln j}{dr} = -\frac{4\pi r^2}{r - 2m}\left(\rho + p\right),$$

with the boundary condition $\phi(0) = 0$. The moment of inertia becomes

$$I = \frac{R^3}{6}\phi_R \omega_R = \frac{\phi_R}{6}\left(R^3 - 2I\right),$$

using the boundary condition for ω. This simplifies to

$$I = \frac{R^3 \phi_R}{6 + 2\phi_R}.$$

Quark-Hadron Equation of State

We use an approximation to an equation of state representing at low densities a hadronic phase and at high densities a deconfined quark phase. Intermediate between the densities ρ_h and ρ_q is a mixed phase of hadrons and quarks. In this section, we use $\rho = m_b n$ for the baryon density and ϵ for the energy density. For hadrons a polytropic EOS with index n_h is employed; in the mixed phase a polytropic EOS with index n_m is employed; in the quark phase an MIT bag model is employed. The equation of state is

$$\begin{aligned}
p &= p_h \left(\frac{\rho}{\rho_h}\right)^{1+1/n_h}, \quad \epsilon = \rho + n_h p && \rho < \rho_h \\
p &= p_h \left(\frac{\rho}{\rho_h}\right)^{1+1/n_m}, \quad \epsilon = \rho\left(1 + (n_h + n_m)\frac{p_h}{\rho_h}\right) + n_m p && \rho_h < \rho < \rho_q \\
p &= 3\epsilon - B = (B + p_q)\left(\frac{\rho}{\rho_q}\right)^{4/3} - B && \rho > \rho_q
\end{aligned}$$

The parameters of the EOS are p_h, ρ_h, n_h, n_m and ρ_q. The auxiliary quantities needed are

$$p_q = p_h \left(\frac{\rho_q}{\rho_h}\right)^{1+1/n_m}, \qquad B = \frac{1}{4}\left[\rho_q\left(1 + (n_h - n_m)\frac{p_h}{\rho_h}\right) + (n_m - 3)\, p_q\right],$$

which assure continuity of pressure and energy density at the densities ρ_h and ρ_q.

In order to implement an equation of state into `tovh.pro`, it is necessary to express thermodynamic quantities in terms of the variable h. For the case at hand, one finds

$$\begin{aligned}
\rho &= \rho_h \left(\frac{e^h - 1}{e^{h_h} - 1}\right)^{n_h} && h < h_h \\
\rho &= \rho_h \left[1 + \frac{\rho_h\left(e^h - e^{h_h}\right)}{p_h\,(1 + n_m)}\right]^{n_m} && h_h < h < h_q \\
\rho &= \rho_q e^{3(h - h_q)} && h > h_q
\end{aligned}$$

where

$$\begin{aligned}
h_h &= \ln\left[1 + \frac{p_h}{\rho_h}(n_h + 1)\right], \\
h_q &= h_h + \ln\left[\left(1 + \frac{p_h}{\rho_h} n_h + \frac{p_q}{\rho_q}(1 + n_m)\right)\right] - \ln\left[1 + \frac{p_h}{\rho_h}(1 + n_h + n_m)\right].
\end{aligned}$$