14.04 Recitation 3: Risk, Uncertainty and Expected Utility

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Fall 2020

1 Uncertainty

In order to talk about uncertainty, we will need to introduce a few concepts which you may know from probability theory and mathematical statistics.

1.1 Random Variables

Suppose that a lottery offers n distinct prizes (some of which may be 0 or even negative), $x_1, x_2, \ldots, x_n$, and that the probabilities of winning these prizes are $\pi_1, \pi_2, \ldots, \pi_n$. If we assume that one and only one prize will be awarded to a player, it must be the case that

$$\sum_{i=1}^{n} \pi_i = 1$$

This equation simply says that our list indicates all possible outcomes of the lottery and that one of these has to occur.

For a lottery (or random variable) X with possible realizations $x_1, x_2, \ldots, x_n$ and probabilities $\pi_1, \pi_2, \ldots, \pi_n$, the expectation (also called mean or expected value) is

$$\mu_X = \mathrm{E}(X) = \sum_{i=1}^{n} \pi_i X_i.$$

The expected value is a weighted sum of the possible outcomes, where the weights are the respective probabilities.

Next we quantify how far a random variable might be spread out from its mean. The variance of a random variable X is the expected value of the squared difference between the outcome and the expectation:

$$\sigma_X^2 = \mathrm{Var}(X) = \mathrm{E}\left[(X - \mu_X)^2\right]$$

Variance is often used as a proxy for risk. The square root of this variance σ_X is called the standard deviation. The coefficient of variance is σ_X / μ_X.

1.2 Covariance and Correlation

Now suppose that there are two lotteries we can play at the same time. There are n possible joint realizations of (X,Y), namely (x_i, y_i) for $i = 1, ..., n$. Each of these joint outcomes has probability π_i.

The covariance between X and Y is the expected value of the difference between the outcome of X and the expectation of X multiplied by the difference between the outcome of Y and the expectation of Y.

$$\sigma_{XY} = \mathrm{Cov}(X,Y) = \mathrm{E}\left[(X - \mu_X)(Y - \mu_Y)\right]$$

The covariance captures the extent to which the lotteries "co-vary." If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the lesser values, the covariance is positive. In the opposite case, when the greater values of one variable mainly correspond to the lesser values of the other, the covariance is negative.

The magnitude of covariances are hard to interpret, so we often talk about correlations instead, since correlations are normalized to between -1 and 1. The correlation coefficient is

$$\rho_{XY} = \mathrm{Corr}(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

If $X = Y$, then $\rho_{XY} = 1$. If $X = -Y$, then $\rho_{XY} = -1$.

1.3 Example: Townsend (1994)

TABLE II

COEFFICIENTS OF VARIATION AND CORRELATION OVER INCOME SOURCES[a]

Village	Profits from Crop Prod.	Livestock Income	Earned Wages	Trade & Handicraft
Aurepalle	0.4227 (0.1101)	−0.0188 [−0.50, 0.50]	0.5800 [0.05, 0.85]	0.6297 [0.05, 0.85]
		0.2136 (0.0499)	0.3607 [−0.25, 0.75]	0.4586 [−0.20, 0.75]
			0.4554 (0.1211)	0.8194 [0.45, 0.95]
				0.4292 (0.1123)
Shirapur	0.2442 (0.0578)	0.5817 [0.05, 0.85]	0.6386 [0.05, 0.85]	0.7913 [0.45, 0.95]
		0.1938 (0.0449)	0.2535 [−0.30, 0.70]	0.6738 [0.05, 0.85]
			1.3068 (0.6140)	0.7352 [0.35, 0.90]
				0.3235 (0.0795)
Kanzara	0.4048 (0.1043)	0.8721 [−0.55, 0.95]	0.8067 [0.45, 0.95]	0.9345 [0.85, 1.00]
		0.3830 (0.0974)	0.7436 [0.35, 0.90]	0.8586 [0.55, 0.95]
			0.5330 (0.1493)	0.8240 [0.45, 0.95]
				0.2973 (0.0721)

[a] Numbers in parentheses are standard deviations, and those in brackets are 95% confidence intervals.

2 Expected Utility and Risk Aversion

In order to reason about utility over uncertain consumption bundles, we introduce a new concept – *von Neumann-Morgenstern expected utility*. The vNM expected utility from a lottery X is

$$U(X) = \mathrm{E}[u(X)] = \sum_i \pi_i u(x_i).$$

When faced with a choice of lotteries X and Y, individuals obeying the von Neumann-Morgenstern axioms of behavior will choose the option that maximizes their vNM expected utility.

We will typically think of $u(\cdot)$ as concave – i.e. exhibiting a diminishing marginal utility of wealth or consumption. In this case, the person will refuse fair bets. A 50–50 bet of winning or losing h dollars, for example, yields less expected utility than does refusing the bet. The reason for this is that winning h dollars means less to this individual than does losing h dollars. An individual who always refuses fair bets is said to be *risk averse*.

The most commonly used measure of risk aversion is the *absolute risk aversion*:

$$A(c) = -\frac{u''(c)}{u'(c)}$$

Another is *relative risk aversion*:

$$R(c) = -\frac{cu''(c)}{u'(c)}$$

A commonly used utility function is the constant absolute risk aversion (CARA) utility function

$$u(c) = -\frac{1}{A}\exp[-Ac].$$

You can check that the absolute risk aversion coefficient for this utility function is a constant.

Another commonly used utility function is the constant relative risk aversion (CRRA) utility function

$$u(c) = \frac{c^{1-R}}{1-R}.$$

Similarly, you can check the relative risk aversion coefficient for this utility function is constant. For an individual with CRRA utility, absolute risk aversion declines as her consumption level c increases. This means potential losses are less serious for high-wealth/consumption individuals.

3 States of the World and Contingent Commodities

Suppose outcomes of any random event can be categorized into a number of *states of the world* which occur with probabilities that sum to one. For example, we might make the very crude approximation of saying that the world will be in only one of two possible states tomorrow: It will be either "good times" or "bad times." One could make a much finer gradation of states of the world which may unfold over multiple periods of time (possibly involving millions of possible states – e.g. hurricane, then famine, then an unexpected death, but then an unexpected windfall, etc.).

Next, suppose there are *state-contingent commodities*. These are goods delivered only if a particular state of the world occurs. "\$1 in good times" is an example of a contingent commodity that promises the individual \$1 in good times but nothing should tomorrow turn out to be bad times. We imagine we can obtain this commodity—that is, someone might promise to give me \$1 if tomorrow turns out to be good times. If someone were also willing to give me the contingent commodity "\$1 in bad times," then I could assure myself of having \$1 tomorrow by obtaining the two contingent commodities "\$1 in good times" and

“$1 in bad times.” A state-contigent commodity is also known as an *Arrow-Debreu security* – a contract that agrees to pay one unit of a numeraire (a currency or a commodity) if a particular state occurs at a particular time in the future and pays zero numeraire in all the other states.