14.04 Recitation 5: A Deep Dive into Pareto Optimality

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Fall 2020

1 Consumption Bundles, Endowments, Feasible Allocations

Today, we begin by developing notation for describing how many of each commodity is assigned to whom.

Suppose there are K goods and I consumers. Each consumer i consumes some bundle $\vec{z}_i \in Z \equiv \mathbb{R}_+^K$. This means each consumption bundle has a nonnegative quantity of each commodity. (Familiarizing yourself with some very basic set notation will make things easier in this class. For instance, $z_i \in Z$ means z_i is an element of the set Z. The set of K-tuples of nonnegative real numbers is representeded as $\mathbb{R}_+^K$.)

To make this concrete, let us imagine that there are two consumers (we will call them 1 and 2) and two goods (apples and bananas). Consumer 1 consumes two apples and no bananas; consumer 2 consumes no apples and five bananas. Formally, we write these two consumption bundles as $\vec{z}_1 = (2,0)$ and $\vec{z}_2 = (0,5)$.

The *social outcome* or *allocation* is denoted by the vector $(\vec{z}_1, ..., \vec{z}_I)$. In our example, the allocation is $((2,0),(0,5))$. Of course, this is not the only possible allocation! In total, there are two apples and five bananas in this society. We say, therefore, that the *social endowment* of this economy is $\vec{e} = (2,5)$. The set of *feasible allocations* (of society's resources) is

$$X = \left\{ (\vec{z}_1, ..., \vec{z}_I) \in Z^I \mid \vec{z}_1 + ... + \vec{z}_I \leq \vec{e} \right\}.$$

This is the set of allocations such that the sum of consumption bundles is less than or equal to the social endowment.

2 Pareto Optimality

We will now formulate a notion of a socially optimal allocation and explore its properties. The concept we will use is Pareto optimality (or Pareto efficiency). Pareto optimality is a very intuitive notion: For any Pareto optimal allocation, there is no alternative allocation that makes everyone better off without making someone worse off.

Definition. A feasible allocation $x = (z_1, ..., z_I)$ is *Pareto optimal* if there is no other feasible allocation $x' = (z'_1, ..., z'_I)$ such that $z'_i \succsim_i z_i$ for all $i \in I$ and $z'_i \succ_i z_i$ for some $i \in I$.

Importantly, the Pareto efficiency concept does not concern itself with distributional issues. For example, an allocation which gives all of society's endowments to a single consumer can be Pareto efficient.

Example. In an Edgeworth box representing an endowment economy with two individuals, the set of Pareto allocations is equivalently the set of points where the two individual's indifference curves are tangent. (Why?) This set is known as the Pareto frontier or contract curve. Look at the picture!

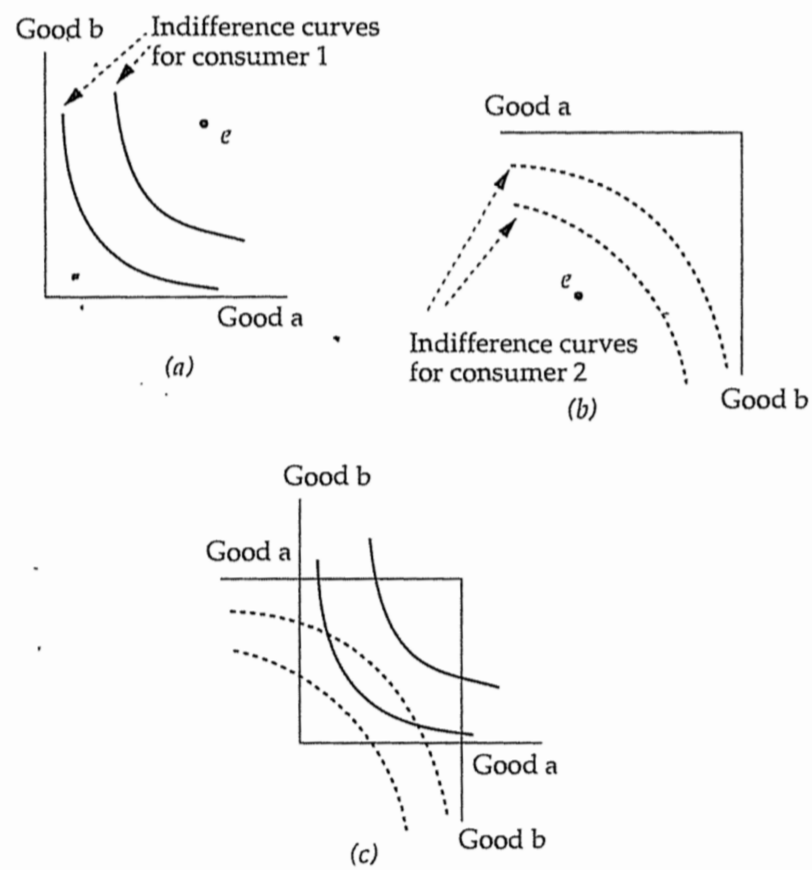
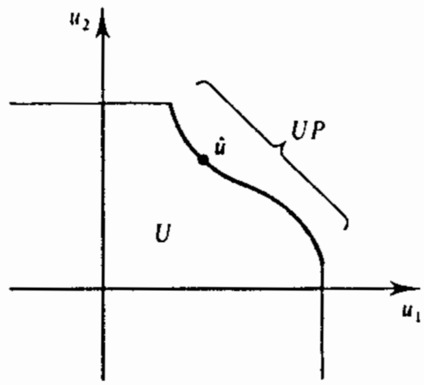


Figure 5.2. Edgeworth boxes.
Figures 5.2(a) and (b) give the two sets of data that make up an *Edgeworth box*: the social endowment and the indifference curves of the two parties, in (a) for consumer 1 and in (b) for 2. In figure 5.2(c) the indifference curves are superimposed.

Each allocation corresponds to a utility level for the individuals. Given a set of feasible allocations, we can define a *utility possibility set U* as follows.

$$U = \left\{ (u_1, ...u_I) \in \mathbb{R}^I : \text{ there exists a feasible allocation } x = (z_1, ..., z_I) \text{ such that } u_i \leq u_i(z_i) \text{ for } i = 1, ..., I \right\}.$$

The set of Pareto allocations therefore corresponds to the boundary of the utility possibility set, and is marked as UP in the picture below.



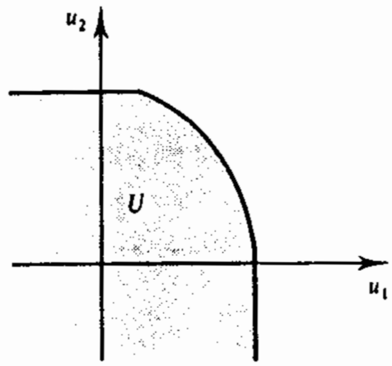


Figure 16.E.1 (left)
The utility possibility set.

Figure 16.E.2 (right)
A convex utility possibility set.

3 Social Welfare Optimality

Let's now suppose that society's distributional principles can be summarized in a social welfare function $W(u_1,...,u_I)$ assigning social utility values to the various possible vectors of utilities for the I consumers. We will focus on a particularly simple class of social welfare functions: those that take the linear form.

$$W(u_1,...,u_I) = \sum_{i=1}^{I} \lambda_i u_i = \lambda \cdot u$$

The linear social welfare maximum for a set of "Pareto weights" $\{\lambda_i\}$ is

$$x^* = \arg\max_{x \in \tilde{X}} \sum_{i=1}^{I} \lambda_i u_i(z_i).$$

For economies with convex utility possibility set, there is a close relation between linear social welfare optima and Pareto optima.[1]

Theorem. *Take an economy with a convex utility possibility set. Every linear social welfare optimum with weights $\lambda_i > 0$ for all i is Pareto optimal. Every Pareto optimal allocation is a linear social welfare optimum for some welfare weights $\lambda_i \geq 0$.*

The proof of this theorem relies on an important mathematical result called the supporting hyperplane theorem. The following figure provides some intuition: Here we have a convex two-person utility possibility set. Recall that the Pareto frontier is the boundary of the utility possibility set. u^* marks the linear social welfare maximum for a given λ vector. The social welfare maximum is achieved at the point where the tangent line is perpendicular to the λ vector. This tangent line more generally called the supporting hyperplane. As you can see, u^* is on the Pareto frontier. Furthermore, for any u on the Pareto frontier, we can find a λ vector such that u is a linear social welfare maximum for those weights.[2]

[1]Recall that the definition of a convex set: Given any two points x and y in a convex set, the line joining them lies entirely within that set.

[2]Homework: Why is convexity of the utility possibility set important? What happens if the utility possibility set is not convex?

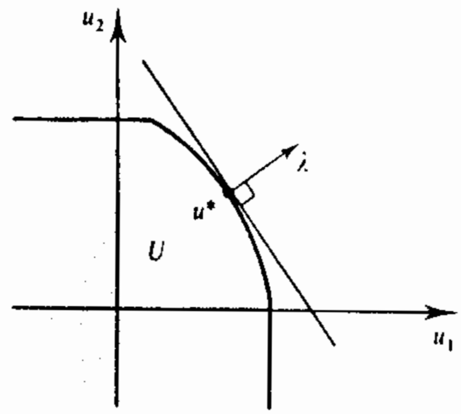


Figure 16.E.3
Maximizing a linear social welfare function.

This theorem is super useful. It tells us that by simply maximizing the linear social welfare function, we are able to find the entire set of Pareto optimal allocations of an economy. We can do this with standard constrained optimization techniques! All we need is to make sure that the utility possibility set is convex. For example, if the feasible set $\tilde{X}$ is convex and the utility functions are concave, then the utility possibility set is convex.

4 Characterizing Pareto Optima: Example from Kreps

Let's work through an example of a Pareto optimal allocation, following Kreps chapter 5.4.

Imagine an economy with K commodities and I individuals. We write $Z = \mathbb{R}_+^K$ and $x = (\vec{z}_1, \vec{z}_2, ..., \vec{z}_I) \in \mathbb{R}_+^{K\times I}$ for a social outcome – an allocation of commodities to individuals. Each z_i is an element of Z and is a K-dimensional vector. We write z_{ik} for the amount of good k allocation to individual i.

Now imagine that society is endowed at the outset with a supply of some $K+1$ good that no one wishes to consume but which can be transfered into various bundles of the K goods for consumption. The production of these K goods is characterized by a function ϕ with the folllowing interpretation: For each bundle of $\vec{z} \in Z$, the amount of the the $K+1$ good needed to produce $\vec{z}$ is $\phi(\vec{z})$.

Therefore the set of feasible allocation is

$$\tilde{X} = \left\{ x \in \mathbb{R}_+^{K\times I} : \phi\left(\sum_{i=1}^{I} z_{i1}, ..., \sum_{i=1}^{I} z_{iK}\right) \leq e_0 \right\}$$

Here, $\sum_{i=1}^{I} z_i$ is the vector sum of the commodity bundles allocated to the individuals. So the constraint above states that it must be feasible to produce from e_0 what is to be allocated. We assume ϕ is strictly increasing, continuously differentiable, and quasi-convex. Quasi-convexity says that if z and z' can be produced from e_0, then so can any combination inbetween z and z'. Quasi-convexity is important for allowing us to use the theorem above.

We now use the theorem to characterize all the Pareto efficient social outcomes as follows:

$$\max_{z_1,\ldots,z_I} \sum_{i=1}^{I} \lambda_i u_i(z_i) \text{ subject to } \phi\left(\sum_{i=1}^{I} z_{i1},\ldots,\sum_{i=1}^{I} z_{iK}\right) \leq e_0$$

We will assume that the weights λ_i are all strictly positive, that the solutions are characterized by the first-order conditions, that all relevant multipliers and partial derivatives are strictly positive, and that the only constraint that binds is the social feasibility constraint. (Please see recitation notes on constrained optimization for a detailed treatment of when we can do this!) Letting μ be the multiplier on the social feasibility constraint, we can write down the Lagrangian as

$$\sum_{i=1}^{I} \lambda_i u_i(z_i) + \mu\left(e_0 - \phi\left(\sum_{i=1}^{I} z_{i1},\ldots,\sum_{i=1}^{I} z_{iK}\right)\right)$$

Differentiating, we obtain the first-order condition for each z_{ik}:

$$\lambda_i \frac{\partial u_i}{\partial z_{ik}} = \mu \frac{\partial \phi}{\partial z_k},$$

where the partial derivatives are evaluated at the optimal solution.

Take any two goods k and k'. We can divide the two first-order conditions to obtain

$$\frac{\partial u_i}{\partial z_{ik}} \Big/ \frac{\partial u_i}{\partial z_{ik'}} = \frac{\partial \phi}{\partial z_k} \Big/ \frac{\partial \phi}{\partial z_{k'}}.$$

Does this look familiar to you? The left-hand side is the *marginal rate of substitution for individual i of good k for k'* (remember?). The right-hand side is called the *marginal rate of technical substitution* of k for k'. For any Pareto optimal allocation, these two should be equal. In fact, they are equal for all individuals!

These are *testable* empirical predictions about substitution patterns in consumption and production. So we can take this relationship, collect data in the real world, and test whether observed allocations are Pareto optimal. Next recitation, we will walk through an example.