14.04 Recitation 1: Preferences and Utility Maximization

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1 Overview

In this course, we formally analyze the problem of resource allocation under scarcity. Many real world issues can be thought of as resource allocation problems. Examples include...

- Division of farm land in Medieval England
- Villager occupation choice in Northern Thailand
- Risk sharing through insurance or gift exchange
- Intertemporal storage of grain seeds
- Investment in stocks and bonds

This course proceeds in the following steps:

1. Develop a language for analyzing resource allocation problems
2. Develop concepts for socially optimal resource allocation given technological or informational constraints (i.e. Pareto optimality)
3. Develop concepts for decentralized resource allocation using the market (e.g. the competitive equilibrium)
4. Compare decentralized allocation with social optimum (i.e. welfare theorems)
5. Analyze individual and aggregate behavior of economic agents facing competitive market prices (i.e. consumer theory)
6. Analyze the role of money in facilitating exchange (i.e. monetary theory)

2 Preferences, utility, and indifference curves

To know how to allocate resources, we need to know what people want or need. Today, we develop mathematical notions for describing the "wants" or "needs" of economic agents. These include preferences, utility functions, and indifference curves.

2.1 Consumption Bundle

An agent consumes some bundle $x \in X \equiv \mathbb{R}_+^K$. X is called the commodity space. To make this concrete, let us imagine that there are two consumers (we will call them 1 and 2) and two goods (apples and bananas). Alice consumes two apples and no bananas; Bob consumes no apples and five bananas. Formally, we write these two consumption bundles as $x^A = (2,0)$ and $x^B = (0,5)$.

2.2 Preferences

We assume each individual has a *preference relation* $\succsim$ on the set of possible consumption bundles Z.

- When we write $x \succsim y$, we mean that "x is at least as good as y" for any $x, y \in Z$
- We write that $x \succ y$ if and only if $x \succsim y$ and $y \not\succsim x$. This is called a "strict preference."
- We write that $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. This is called "indifference."

For example, if i prefers consuming two apples and a banana to one apple and one banana, we can write $(2,1) \succsim_i (1,1)$. For most of this course, we will consider scenarios where agents care only about what they consume themselves and do not care what other people consume. We will relax this assumption when we discuss externalities.

2.3 Utility

A *utility function* $u(z)$ represents a preference relation $\succsim$ if and only if for all $x, y \in Z$,

$$x \succsim y \iff u(x) \geq u(y).$$

For example, we say that $(2,1) \succsim_i (1,1)$ if and only if $u_i(2,1) \geq u_i(1,1)$.

Things to remember:

- Axioms of rational choice: Not every preference relation has a utility function representation! When we choose to represent preferences using utility function, we implicitly make assumptions about the underlying preference relations. These assumptions are:
  1. Completeness: For any $x, y \in Z$, then either $x \succsim y$ or $y \succsim x$.
  2. Transitivity: For any $x, y, z \in Z$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$.
  3. Continuity: For any sequence $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$, $x^n \to x$ and $y^n \to y$ implies $x \succsim y$.

  If a preference relation satisfies the above three properties, then a utility function representation is guaranteed to exist.
- Nonuniqueness: A given preference relation may have *multiple* utility representations! It is not generally possible to compare utility functions across individuals. For example, any $f(u(z))$ where f is increasing represents the same preferences.

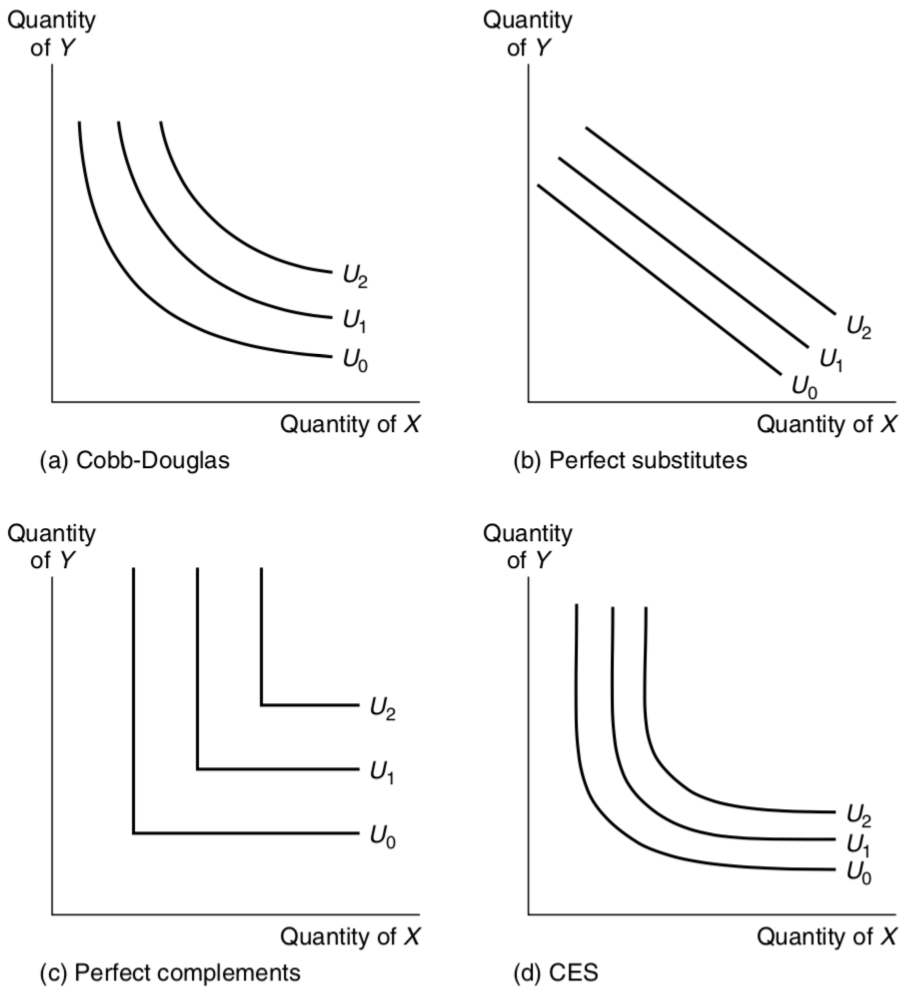

Figure 1: Indifference curves

- Monotonicity: Economic agents typically prefer more of something than less. in fact, the definition of a "good" is that people like more of it. Conversely, utility decreases if we consume more of a "bad." This means utility is monotonically increasing in the quantity of "goods" consumed. The first derivative of the utility function, i.e. the "marginal utility," is therefore positive. In symbols, $\frac{\partial u}{\partial z_i} > 0$.

2.4 Indifference curves

We can also represent preferences using indifference curves. Indifference curves are like contour plots on maps, as shown in Figure 1. Each point on the map represents a consumption bundle. An indifference curve is the set of points that yield the same utility level.

Things to remember:

- The marginal rate of substitution (MRS) is the negative slope of an indifference curve at some point U_0. The MRS is the rate at which the individual is willing to trade x for y, and is related to the marginal

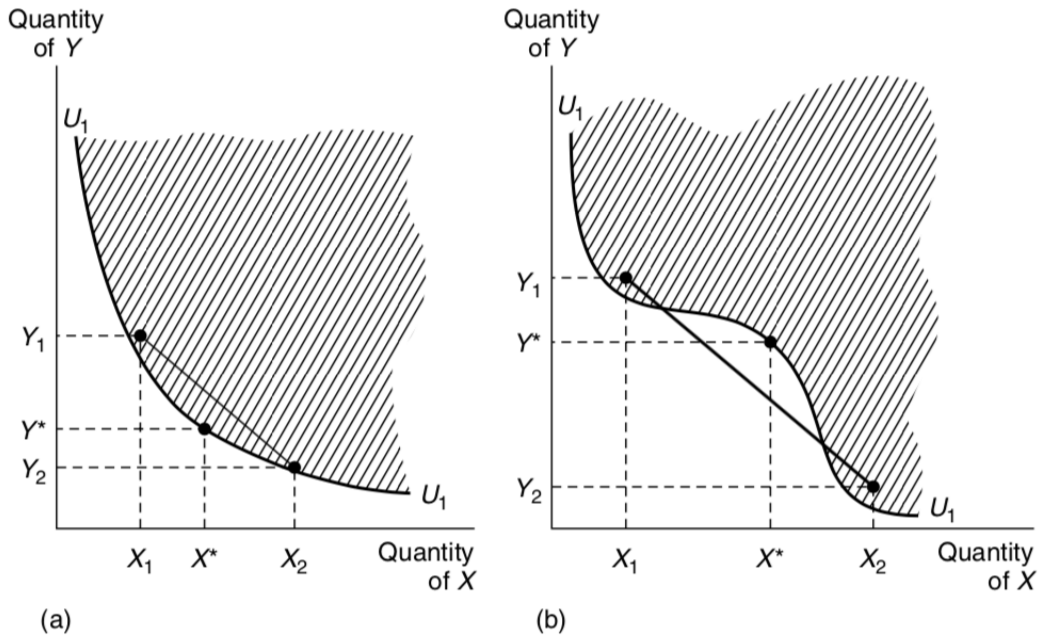

Figure 2: Convex and non-convex indifference curves

utilities as follows.

$$MRS = -\left.\frac{dy}{dx}\right|_{U=U_0} = \frac{\partial U}{\partial x} \Big/ \frac{\partial U}{\partial y}.$$

(Sometimes, you will see the also marginal utility of x written as U_x or MU_x.)

- We typically assume that MRS is decreasing in the quantity of x. That is, if I have more of x, then I'm willing to give up more of x for the same quantity of y. This captures a "preference for balance." Diminishing MRS correspond to convex indifference curves and quasi-concave utility functions. See Figure 2. (Note that a function $f : S \to \mathbb{R}$ defined on a convex subset S of a real vector space is quasiconcave if for all $x, y \in S$ and $\lambda \in [0,1]$ we have $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$.)

3 Utility Maximization subject to a Budget

Suppose a consumer has a utility function u. The consumer maximizes u subject to the budget constraint

$$\begin{aligned} \max_{x_1,x_2,\ldots,x_L} \quad & u(x_1,x_2,\ldots,x_L) \\ \text{subject to} \quad & \sum_l p_l x_l \leq I \\ & x_l \geq 0 \,\forall l \end{aligned}$$

How to solve for the consumer's demand $x^*(p,I)$?

- Step 1: Form the Lagrangian.

$$\mathscr{L} = u(x_1, x_2, \ldots, x_L) + \lambda \left(I - \sum_l p_l x_l \right) + \sum_l \mu_l x_l$$

- Step 2: Write out the first-order conditions for the x_ls.

$$\frac{\partial \mathscr{L}}{\partial x_l} = \frac{\partial u}{\partial x_l} + \lambda p_l + \mu_l = 0 \, \forall l$$

- Step 3: Write the constraint.

$$I - \sum_l p_l x_l \geq 0$$

$$x_l \geq 0$$

- Step 4: Write the inequality contraint for the multipliers.

$$\lambda \geq 0$$

$$\mu_i \geq 0 \;\; \forall l$$

- Step 5: Write the complementary slackeness condition.

$$\lambda \left(I - p_l x_l \right) = 0$$

$$\mu_l x_l = 0 \;\; \forall l$$

- Step 6: Look for a solution for x and λ by combining above equations and inequalities.

The recipe above produces **candidate solutions**, but does not in general guarantee that the candidate solutions are globally (or even locally) optimal. However, if u is quasi-concave and the budget set is convex, then any solution satisfying the above conditions must be a **global maximer**.

3.1 Intuitions

We can look at this graphically for the two good case. There are two possibilities: an interior solution ($x_l > 0$ and $\mu_l = 0 \, \forall l$), as shown in Figure 3, or a corner solution ($x_l = 0$ and $\mu_l > 0$ for some l), as shown in Figure 4.

If we have an interior solution, we have the following intuitions for the tangency condition.

- At the optimum, the utility per dollar on the margin must be equal across all goods, and also equal to the marginal utility of income λ

$$\frac{MU_i}{p_i} = \frac{MU_j}{p_j} = \ldots = \lambda$$

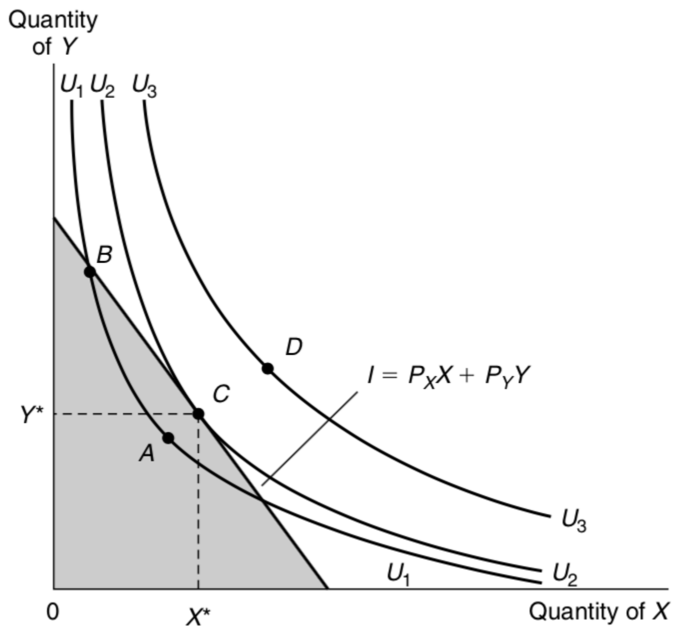

Figure 3: Interior solution

- Another way to write the same equation is as follow

$$MRS = \frac{MU_i}{MU_j} = \frac{p_i}{p_j}$$

What is the right hand side? To get one unit of good i, we must give up p_i/p_j of good j. So p_i/p_j is the cost of good i in units of j. Note that the right hand side is the marginal rate of substitution, i.e. the value of good i in units of good j. So this equation says that the "mariginal benefit must equal the marginal cost."

- Finally, note that at the optimum, prices are equal to marginal utilities.

$$p_i = \frac{MU_i}{\lambda}$$

This means that if we can measure prices, we have an indirect measure of people's marginal utilities.

If a **corner solution** is obtained, i.e. $x_k = 0$ for some good k, then

$$\frac{\partial U}{\partial x_k} - \lambda p_k < 0,$$

This can be rearranged as

$$p_k > \frac{MU_k}{\lambda}$$

Interpretation: Any good whose price (p_k) exceeds its marginal value (MU_k/λ) to the consumer will not be purchased.

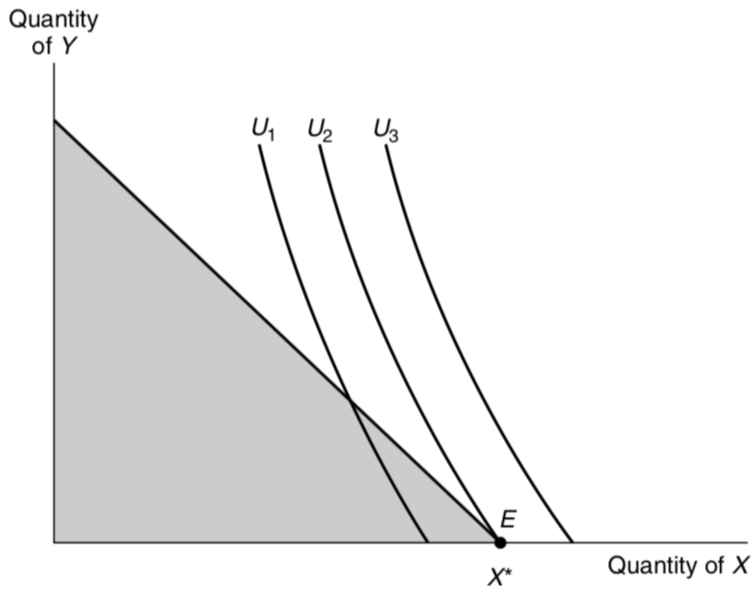

Figure 4: Corner solution

4 Example: Cobb-Douglas Utility

- We solve

$$\max U(X,Y) = X^{\alpha} Y^{1-\alpha}$$

subject to budget constraint:

$$p_X X + p_Y Y = I$$

- We set up Lagrangian:

$$\mathscr{L} = X^{\alpha} Y^{1-\alpha} + \lambda \left(I - p_X X - p_Y Y \right)$$

- The first order conditions are:

$$\begin{aligned} \alpha X^{\alpha-1} Y^{1-\alpha} &= \lambda p_X \\ (1-\alpha) X^{\alpha} Y^{-\alpha} &= \lambda p_Y \end{aligned}$$

- Taking ratio of the two F.O.C.s yields:

$$\frac{\alpha Y}{(1-\alpha) X} = \frac{p_X}{p_Y}$$

- Rearranging, we have

$$p_X X = \frac{\alpha}{(1-\alpha)} p_Y Y$$

- Plugging this back into the budget constraint, we find that

$$\begin{aligned} X^* &= \frac{\alpha I}{p_X} \\ Y^* &= \frac{(1-\alpha)I}{p_Y} \end{aligned}$$