14.04 Recitation 4: Storage and Investment

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1 Allocation over Time

So far, we talked about the allocation of goods. Now we do something more – let's allocate goods over time. How do we talk about time? We just switch to indexing the commodity space using time periods. For example, let's say that there are two consumers (1 and 2), one good (apples), and two periods (0 and 1). Consumer 1 consumes two apples in period 0 and none in period 1. We can represent this as $z_1 = (2,0)$. Simiarly, consumer 2 consumes zero apples in period 0 and five in period 1, so $z_2 = (0,5)$. This social outcome is $((2,0),(0,5))$. The notation is the same before, but the interpretation is now different. If there are many individuals and many periods, then for easier reading, we can index the consumption bundle z_{it} using both i (for the individual) and t (for the time period).

What about preferences over time? Economists often imagine that intemporal utility functions take the following mathematical form:

$$U_i(c_{i0,}c_{i1},...) = \sum_{t=0}^{\infty} \beta^t u_i(c_{it})$$

where $\beta \in (0,1)$ is the discount factor, c_{it} is agent i's consumption level in period t, and $u_i(\cdot)$ is a twice differentiable, increasing, and concave with-in-period utility function.

This intertemporal utility function implies that agents prefer to have "smooth" consumption over time if possible. Preference for smooth consumption seems sensable since most humans prefer to eat three times a day, not once a year! (Can you see that preference for smooth consumption is a consequence of diminishing marginal rates of substitution, exactly like a preference for balance?) This idea of "consumption smoothing" will come up again and again because it generates lots of empirical predictions about economic behavior. Below are some examples.

Storage and Saving Suppose that there is only one person – a biblical farmer named Joseph – and one good – grain. Joseph can perfectly predict the future, and he knows that there will be seven years of plenty followed by seven years of famine. He can either eat the grain now or store it for next year. Preference for smooth consumption means that Joseph will want to store the grain! We might call this saving.

Credit and Gift Exchange Now suppose that there are two famers – Joseph and Job. Joseph currently has a year of plenty but expects famine next year. Job has the opposite situation – famine now and plenty later.

If both prefer smooth consumption, then the “optimal” allocation is that Joseph gives Job some grain now and Job gives Joseph some grain later. I.e. Joseph lends to Job! We might call this credit or gift exchange.

Risk-sharing and Social Insurance Finally, suppose there is just a single future period, and that either Job will have plenty while Joseph experiences famine or Job will have famine while Joseph has plenty. So there is *risk* of famine for each individual. If both prefer to “smooth” consumption across potential states of the world, then it may be best for Job and Joseph to “share risk” by agreeing beforehand that whoever has plenty gives a little to whoever has famine. We might call this insurance!

2 Storage and Saving

Now we discuss “dynamic optimization.” This just means that we are optimizing over time and there is some technology that allows us to move resources between time periods. Most of the intuition for a typical dynamic optimization problem can be understood from considering just two periods. So let’s start with that.

Let’s imagine a single farmer economy. There are two period $t = 0, 1$. In period 0, the farmer has endowment e_0. In period 1, he gets additional endowment e_1. There is a storage technology. If the farmer saves s_0 in period 0, then in period 1 he has an additional $k_1 = \delta s_0$ units of resources, where δ represents depreciation (if $\delta < 1$) or interest (if $\delta > 1$).

We can set up the programming problem as follows:

$$\begin{aligned}
\max_{(c_0,c_1,s_0)} \quad & u(c_0) + \beta u(c_1) \\
\text{s.t.} \quad & c_0 \geq 0 \\
& c_1 \geq 0 \\
& s_0 \geq 0 \\
& c_0 + s_0 \leq e_0 \\
& c_1 \leq e_1 + k_1 \\
& k_1 = \delta s_0
\end{aligned}$$

Since the farmer prefers to consume more if possible, we can reason that the first two constraints do not bind, i.e $c_0 > 0$ and $c_1 > 0$. For the same reason, we can also reason that there is nothing left after consumption and saving in the first period, so $c_0 + s_0 = e_0$. Similarly, in the later period, the farmer will consume everything that’s available, $c_1 = e_1 + k_1$. We can therefore rewrite the above problem as:

$$\begin{aligned}
\max_{(c_0,c_1)} \quad & u(c_0) + \beta u(c_1) \\
& c_0 \leq e_0 \\
& c_0 + \frac{c_1}{\delta} = e_0 + \frac{e_1}{\delta}
\end{aligned}$$

The first constraint here might be called a liquidity constraint. (Why?) The second is the resource constraint. The constraints give two possibilities: (1) $s_0 = 0$, or $c_0 = e_0$, i.e. the farmer doesn’t store at all, or (2) $s_0 > 0$, or $c_0 < e_0$, i.e. the farmer stores some positive amount. We can consider these two possibilities separately.

In case 1, we simply have $c_0 = e_0$ and $c_1 = e_1$.

In case 2, we have Lagrangian

$$\mathscr{L} = u(c_0) + \beta u(c_1) + \mu \left[e_0 + \frac{e_1}{\delta} - c_0 - \frac{c_1}{\delta} \right]$$

which yield FOCs

$$u'(c_0) = \mu$$
$$\beta u'(c_1) = \frac{\mu}{\delta}$$
$$c_0 + \frac{c_1}{\delta} = e_0 + \frac{e_1}{\delta}.$$

Suppose $u(c) = \ln c$. Then consumption in the periods are related by

$$c_1 = \beta \delta c_0.$$

(What's the intuition for this equation?) Plugging this into the resource constraint yields the levels.

$$c_0 = \frac{1}{1+\beta} \left(e_0 + \frac{e_1}{\delta} \right)$$
$$c_1 = \frac{\beta \delta}{1+\beta} \left(e_0 + \frac{e_1}{\delta} \right)$$

(When is the economy in case 1 vs case 2? What's the intuition?)

3 A Recursive Fomulation

Now let's imagine that there are a large number of periods $t = 0, 1, \ldots, \infty$ and still just one farmer. Let's imagine that the economy starts out with abundant resources, i.e. $e_0 \gg 0$, but that $e_t = 0$ for all $t > 0$, so the liquidity constraints never bind. Let k_t be the total amount of resources available at the end of period t, with $k_0 = e_0$. We will think of period 1 as the first consumption period. The programming problem is

$$\max_{(c_1, \ldots)} \quad \sum_{t=1}^{\infty} \beta^t u(c_t)$$

subject to storage technology

$$k_t = \delta k_{t-1} - c_t,$$

and initial condition

$$k_0 = e_0.$$

Plugging the storage technology into the objective function, this programming problem can be rewritten as

$$\max_{(k_1,\ldots)} \quad \sum_{t=0}^{\infty} \beta^t u\left(\delta k_t - k_{t+1}\right)$$

$$k_0 = e_0$$

Instead of optimizing over consumption flows c_t, we are now optimizing over the path of capital stock k_t. This stock variable k_t summarizes the state of the economy at k_t, so we can write this problem recursively with *no* constraints.

$$V\left(k_0\right) = \max_{k_1}\left[u\left(\delta k_0 - k_1\right) + \beta V\left(k_1\right)\right]$$

The first order condition for k_1 is

$$-u'\left(\delta k_0 - k_1\right) + \beta V'\left(k_1\right) = 0.$$

Being loose for just a minute, notice that this FOC can be expanded into

$$-u'\left(c_1\right) + \beta \frac{\partial}{\partial k_1}\left\{\max_{k_1}\left[u\left(\delta k_1 - k_2\right) + \beta V\left(k_2\right)\right]\right\} = 0.$$

Ignoring some technical details we need to deal with the awkward "max" in this FOC, then this is the same FOC as we went over in the previous section.

$$u'\left(c_0\right) = \beta \delta u'\left(c_1\right).$$

Indeed this FOC holds for all t due to the recursive formulation.

$$u'\left(c_t\right) = \beta \delta u'\left(c_{t+1}\right) = 0 \quad \forall t = 1, 2, \ldots$$

(When there are only a finite number of periods, we can also write down a recursive formulation, but we need both an initial condition and also a termination condition. You'll get to think about this in the problem set!)