14.04 Recitation 10: Fundamental Welfare Theorems

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1 Definitions

Today we will talk about two important mathematical results that shape the thinking of many economists and their love for markets – the fundamental welfare theorems. To start we need to define a few things.

Private Ownership Economy First, we need the concept of a private ownership economy. This is an slight elaboration on the economy we defined earlier in this class. As before, the private economy consists of a finite number L of commodities, I consumers each with a consumption set $X_i \subseteq \mathbb{R}^L_+$ and preferences $\succeq_i$ over bundles on X_i, J firms with production set $Y_j \subseteq \mathbb{R}^L_+$, and an endowment vector $\omega_i \subseteq \mathbb{R}^L_+$ for each consumer i. In the private ownership economy, each consumer additionally has an ownership share θ_{ij} of each of the J firms, such that $\sum_{i=1} \theta_{ij} = 1$. This means each consumer will receive share θ_{ij} of firm j's profits. We can summarize the private ownership economy as follows:

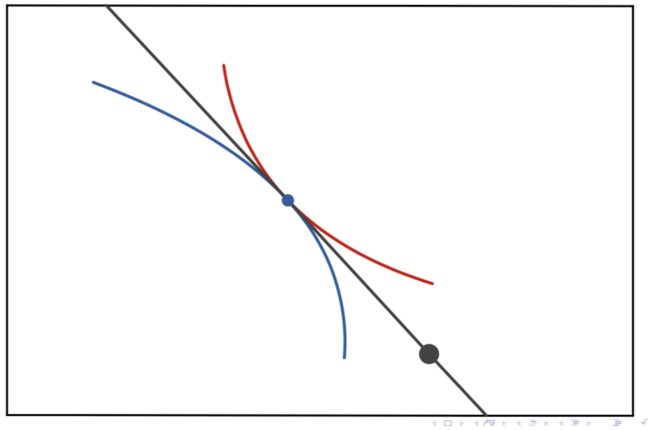
$$\mathscr{E} \equiv \left\{ \{X_i, \succeq_i, \omega_i\}_{i=1}^{I}, \left\{Y_j\right\}_{j=1}^{J}, \left\{\theta_{ij}\right\}_{i=1,j=1}^{I,J} \right\}$$

Walrasian Equilibrium Next, we will introduce the concept of a price equilibrium, also known as a Walrasian or competitive equilibrium.

Definition. A Walrasian equilibrium is an allocation (x^*, y^*) and a price vector $p^* \in \mathbb{R}^L_+$ such that:

1. For each j, y_j^* maximizes profits given prices p^*. (That is, $p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_j$.)
2. For each i, x_i^* maximizes preferences given her budget. (That is, $x_i^* \succeq_i x_i$ for all $x_i \in X_i$ such that $p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_j \theta_{ij} p^* \cdot y_j^*$.)
3. The allocation is feasible. (That is, $\sum_i x_i \leq \sum_i \omega_i + \sum_j y_j^*$)

The Walrasian equilibrium can be visualized in the Edgeworth box as follows:



In this graph, the endowment level and the price vector defines the budget sets of the two consumers. Each consumer maximizes: The first-order condition implies that the individual consumption bundle in the Walrasian equilibrium is where the indifference curve is tangent to the budget line. (Why do we call this an "equilibrium"?)

Price Equilibrium with transfers A slight variant of the Walrasian equilibrium is to allow any allocation of wealth that may not be equal to the consumer's initial endowment.

Definition. An allocation (x^*, y^*) and a price vector $p \in \mathbb{R}^L_+$ constitute a price equilibrium with transfers if there exists an *assignment* of wealth levels $w = (w_1, w_2, ..., w_I)$ such that

1. For each j, y_j^* maximizes profits given prices. (That is, $p \cdot y_j^* \geq p \cdot y_j$ for all $y_j \in Y_j$.)
2. For each i, x_i^* maximizes preferences given her budget. (That is, $x_i^* \succeq_i x_i$ for all $x_i \in X_i$ such that $px_i \leq w_i$.)
3. The allocation is feasible. (That is, $\sum_i x_i \leq \bar{\omega} + \sum_j y_j^*$)
4. The assignment of wealth levels is feasible: $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$.

Note that any Walrasian equilibrium is a price equilibrium with transfers, where $w_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$.

2 First Welfare Theorem

The First Welfare theorem shows that price equilibria are Pareto optimal.

Theorem 1. *Let (x^*, y^*, p, w) be a price equilibrium with transfers. Then, if preferences are rational and locally non-satiated, then (x^*, y^*) is a Pareto Optimal allocation.*

The intuition for this First Welfare theorem is best explained using the Edgeworth box. Recall from above that the wealth level and the price vector defines the budget sets of the two consumers. Individual consumer maximization implies that the individual consumption bundle in the price equilibrium is where the indifference curve is tangent to the budget line. This point is exactly the intersection of the budget line with the contract curve. Since this point lies on the contract curve, it is a Pareto Optimal allocation!

Do you hear choirs of angles and choruses of trumpets? The “Invisible Hand” of the price mechanism produces equilibria that cannot be improved upon. This theorem requires very few assumptions: only local-nonsatiation of preferences, and can be proven by contradiction using a simple manipulation of the definitions. (See lecture slides.) But there is a problem. Does a competitive equilibrium even exist? We’ll come back to this question next recitation.

3 Second Welfare Theorem

The Second Welfare theorem is a statement about the (almost) converse of the First Welfare theorem: Any Pareto optimal allocation can be implemented by a price equilibrium *with transfers*. To show this, we need a few more conditions.

Theorem 2. *Take any economy* $\mathscr{E} \equiv \left\{ \{X_i, \succeq_i, \omega_i\}_{i=1}^{I}, \left\{Y_j\right\}_{j=1}^{J}, \left\{\theta_{ij}\right\}_{i=1,j=1}^{I,J} \right\}$ *such that*

1. $X_i \subseteq \mathbb{R}_+^L$ *are convex, open sets*
2. $Y_j \subseteq \mathbb{R}_+^L$ *are convex and closed sets, and admit a concave transformation function* $F_j(y)$ *such that*
$$Y_j = \left\{ y \in \mathbb{R}^L : F_j(y) \geq 0 \right\}$$
3. *Preferences are given by concave and locally non-satiated utility functions* $u_i : X_i \to \mathbb{R}$ *(equivalent to preferences being rational, convex, continuous and locally non-satiated)*
4. *There exist* $(\tilde{x}, \tilde{y})$ *such that* $\tilde{x}_i \in X_i$, $F_j(y_j) > 0$, *and* $\sum_i \tilde{x}_i \ll \sum_i \omega_i + \sum_j \tilde{y}_j$

Then, for any Pareto Optimal allocation (x^*, y^*), *there exist a price vector* $p \in \mathbb{R}_+^L$ *and a vector of wealth levels* $w^* \in \mathbb{R}_+^I$ *such that* (x^*, y^*, p, w^*) *is a Walrasian equilibrium with transfers of* $\mathscr{E}$.

What does this theorem say? Suppose you are a benevolent social dictator. You want to implement a Pareto optimal allocation. The Second Welfare Theorem states that you can do so simply by choosing initial endowments and then letting the market take over.

3.1 Proof Sketch

The proof is lengthy, but instructive, and consists of 3 parts:

1. Characterize the solution to the Pareto Problem for a given vector λ.
2. Characterize the conditions for an equilibrium with transfers.
3. Make the mapping between the Pareto optimal allocation and the Walrasian Equilibrium with transfers that implements it, by identifying that under some conditions, the conditions for both allocations are identical.

It is instructive to walk through the proof idea together, so let’s do that.

Step 1: characterize the Pareto Problem. We should all be familiar with how to do this. The Pareto Problem is the following program.

$$\max_{x_1,\dots,x_I} \sum_{i=1}^{I} \lambda_i u_i(x_i) \text{ subject to } \sum_i x_{i,l} \leq \sum_i \omega_{i,l} + \sum_j y_{j,l} \forall l \text{ and } F_j(y_j) \geq 0 \forall j$$

To solve it, this we invoke Kuhn-Tucker. This yields first order conditions:

$$\lambda_i \frac{\partial u_i}{\partial x_l}(x_i^*) - \gamma_l^* = 0, \qquad \gamma_l^* + \phi_j^* \frac{\partial F_j}{\partial y_l}(y_j^*) = 0,$$

where γ_l^* and ϕ_j^* are respective Lagrange multipliers, along with along with the binding resouce constraint: $\sum_i x_{i,l} \leq \sum_i \omega_{i,l} + \sum_j y_{j,l}$.

Step 2: characterize an equilibrium with transfers. There are four conditions:

1. For each i, x_i^* maximizes preferences given her budget:

$$x_i^* \in \arg\max_{x_i} u_i(x_i) \text{ subject to } px \leq w^*$$

or

$$\frac{\partial u_i}{\partial x_l}(x_i^*) - \mu_i p_l = 0$$

where μ_i is the Lagrange multiplier.

2. For each j, y_j^* maximizes profits given prices:

$$y_j^* \in \arg\max_{y_j} p y_j \text{ subject to } F_j(y_j) \geq 0$$

or

$$p_l + \delta_j \frac{\partial F_j}{\partial y_l}(y_j^*) = 0$$

where δ_j is the Lagrange multiplier.

3. The allocation is feasible:

$$\sum_i x_i \leq \sum_i \omega_i + \sum_j y_j^*$$

4. The assignment of wealth levels is feasible:

$$\sum_i w_i^* = \sum_i p\omega_i + \sum_j p y_j^*.$$

Step 3: make a mapping between the two. To do so, we show that, for any Pareto optimal allocation (x^*, y^*), there exists $(p, w^*, \mu_i, \delta_j)$ such that (x^*, y^*, p, w^*) satisfies the four conditions of a price equilibrium.

This can be done by letting

$$p_l \equiv \gamma_l^*, \qquad \mu_i \equiv \frac{1}{\lambda_i}, \qquad \delta_j \equiv \phi_j^*, \qquad w_i^* = \sum_l \gamma_l^* x_{i,l}^*$$

We can then carefully check that each of these four conditions of a price equilibrium from step 2 is satisfied. (See lecture slides for more detail.)