14.04 Recitation 6: Risk-sharing and Portfolio Choice

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1 Insurance in Village India: Townsend (1994)

Let's now work through a simplified version of the model presented in "Risk and Insurance in Village India" (Townsend 1994). This model analyzes the Pareto optimal allocation of state-contingent commodities in risky environments.

Let ε_t be the contemporary realization of all the underlying random variables in the economy, assumed to be observed, for simplicity, at the beginning of date t. Let $h_t = (\varepsilon_1, \varepsilon_2, ...,)$ be the history of realized random variables.

Let $c_t^k(h_t)$ be the consumption of individual k in period t after history h_t. Let $\bar{c}_t(h_t)$ be the total endowment of the village in period t after history h_t. Note that these consumption levels are state-contingent!

The utility of inidividual k is

$$\sum_{t=1}^{T} \beta^t \sum_{h_t} P(h_t) u\left(c_t^k(h_t)\right)$$

where β is the discount factor, $P(h_t)$ is the probability that history h_t is realized.

To find the Pareto optimal allocations, we maximize the linear welfare functional:

$$\sum_{k=1}^{M} \lambda_k \sum_{t=1}^{T} \beta^t \sum_{h_t} P(h_t) u_k\left(c_t^k(h_t)\right)$$

subject to the total endowment of the village in each period t:

$$\sum_{k=1}^{M} c_t^k(h_t) \leq \bar{c}_t(h_t).$$

For each h_t, we solve for the maximum using the using Lagrangian technique, with $\mu(h_t)$ as the multiplier on the feasibility constraint:

$$\sum_{k=1}^{M} \lambda_k \sum_{t=1}^{T} \beta^t \sum_{h_t} P(h_t) u_k\left(c_t^k(h_t)\right) + \mu(h_t)\left(\bar{c}_t(h_t) - \sum_{k=1}^{M} c_t^k(h_t)\right)$$

Differentiating with respect to $c_t^k(h_t)$, we obtain the first order condition:

$$\lambda_k \beta^t P(h_t) u_k'\left(c_t^k(h_t)\right) - \mu(h_t) = 0.$$

Rearranging, we get that

$$\lambda_k u_k'\left(c_t^k(h_t)\right) = \frac{\mu(h_t)}{\beta^t P(h_t)} \equiv \tilde{\mu}(h_t).$$

This says that λ-weighted marginal utilities are equalized across individuals.

Suppose that individuals have constant absolute risk aversion utility functions.

$$u_k(c) = -\frac{1}{\sigma_k}\exp(-\sigma_k c)$$

Note that marginal utility take the following form

$$u_k'(c) = \exp(-\sigma_k c)$$

Substituting this into the first order condition, we get

$$\lambda_k \exp\left(-\sigma_k c_t^k(h_t)\right) = \tilde{\mu}(h_t).$$

Taking logs on both sides, we get

$$\ln\lambda_k - \sigma_k c_t^k(h_t) = \ln(\tilde{\mu}(h_t)).$$

Rearranging, we obtain the following relationship:

$$c_t^k(h_t) = \frac{1}{\sigma_k}\ln\lambda_k - \frac{1}{\sigma_k}\ln(\tilde{\mu}(h_t)).$$

Summing over the first-order conditions, we have that

$$\sum_{k=1}^{M} c_t^k(h_t) = \sum_{k=1}^{M}\frac{1}{\sigma_k}\ln\lambda_k - \sum_{k=1}^{M}\frac{1}{\sigma_k}\ln(\tilde{\mu}(h_t)).$$

Note that rearranging we have

$$\ln(\tilde{\mu}(h_t)) = \frac{\sum_{k=1}^{M}\frac{1}{\sigma_k}\ln\lambda_k}{\sum_{k=1}^{M}1/\sigma_k} - \frac{1}{\sum_{k=1}^{M}1/\sigma_k}\sum_{k=1}^{M}c_t^k(h_t).$$

Plugging it back in, we have

$$c_t^k(h_t) \quad = \quad \left[\frac{1}{\sigma_k}\ln\lambda_k - \frac{\sum_{k=1}^{M}\frac{1}{\sigma_k}\ln\lambda_k}{\sum_{k=1}^{M}1/\sigma_k}\right] + \frac{1/\sigma_k}{\sum_{k=1}^{M}1/\sigma_k}\sum_{k=1}^{M}c_t^k(h_t)$$

or

$$c_t^k(h_t) = \alpha_k + \frac{1/\sigma_k}{\sum_{k=1}^{M}1/\sigma_k}\bar{c}_t(h_t)$$

Townsend (1994) allows for heterogeneous risk aversion and demographics, by letting $u_k(c;A) = -\frac{1}{\sigma_k}\exp\left(-\sigma_k\frac{c}{A}\right)$, where A_t^k are demographic features of k in period t that may affect k's marginal utility of consumption. Letting $\tilde{c}_t^k(h_t) = \frac{c_t^k(h_t)}{A_t^k}$ denote the age-weighted consumption, they obtain in the same manner as above:

$$\tilde{c}_t^k(h_t) = \frac{1}{\sigma_k}\left[\ln\lambda_k - \frac{\sum_{k=1}^M \frac{1}{\sigma_k}\ln\lambda_k}{\sum_{k=1}^M \frac{1}{\sigma_k}}\right] - \frac{1}{\sigma_k}\left[\ln A_t^k - \frac{\sum_{k=1}^M \frac{1}{\sigma}\ln A_t^k}{\sum_{k=1}^M \frac{1}{\sigma_k}}\right] + \frac{\frac{1}{\sigma_k}}{\sum_{k=1}^M \frac{1}{\sigma_k}}\sum_{k=1}^M \tilde{c}_t^k(h_t).$$

Townsend (1994) then tests the following relationship in the data using linear regression:

$$\tilde{c}_t^k = \alpha^k + \beta^k \bar{\bar{c}}_t + \delta^k \bar{A}_t^k + \zeta^k X_t^k + u_t^j$$

where

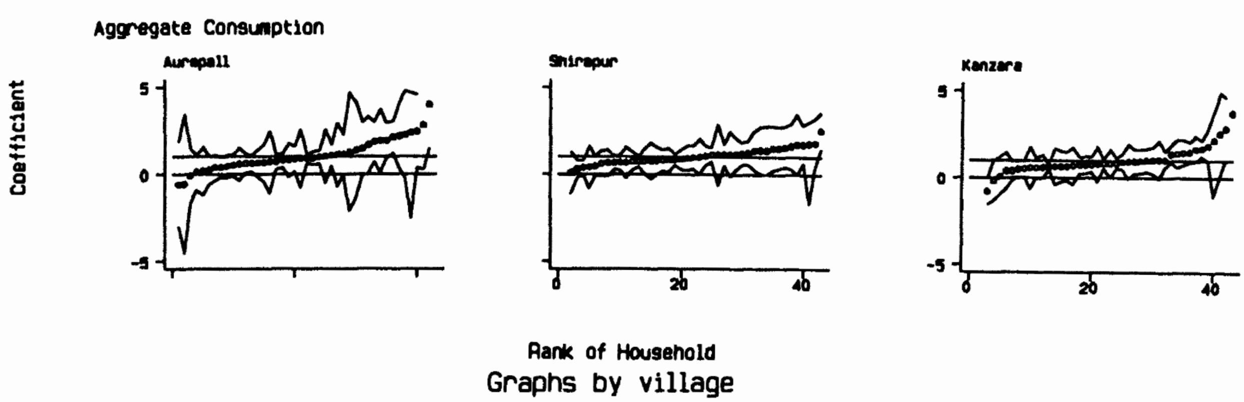
$$\bar{\bar{c}}_t = \frac{1}{M}\sum_{k=1}^M \tilde{c}_t^k.$$

According to the theory, we have that

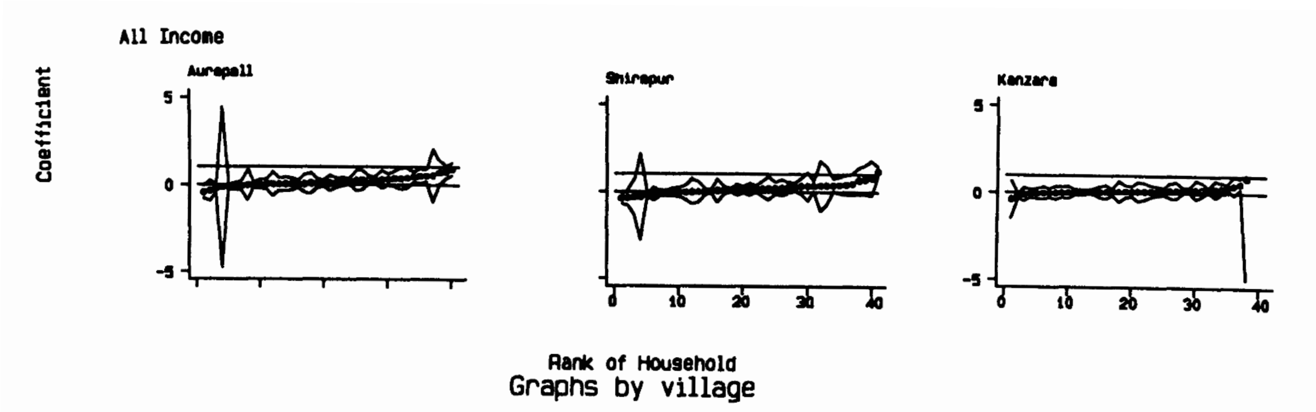
$$\beta^k = \frac{M/\sigma_k}{\sum_{k=1}^M 1/\sigma_k}$$

in a Pareto efficient risk-sharing scheme. Note that if $\sigma_k = \sigma$, then $\beta^k = 1$. More generally, we have that $\frac{1}{M}\sum_{k=1}^M \beta^k = 1$. Furthermore, in a Pareto efficient allocation, there is no relationship between individual income and consumption after controlling for aggregate consumption.

1.1 Empirical Results: Townsend (1994)



(A)



2 Investment and Portfolio Choice

Now let's consider an single-agent economy where there are two investment possibilities $i = 1,2$. When we allocate k_{t+1}^i units of capital to technology i at the end of period t, the agent has $f_i\left(k_{t+1}^i\right) + k_{t+1}^i$ in the beginning of the period $t+1$. The function f_i is a production function whose output is ex ante uncertain in period t. (Here we assume that depreciation $\delta = 1$.) This is a simplified version of "Risk and Return in Village Economies" by Samphantharak & Townsend (2015).

Let W_t be a variable summarizing the initial resources of the economy in a given period.

$$W_t = \sum_{i=1,2} \left(f_i\left(k_t^i\right) + k_t^i\right).$$

In any given period, consumption is equal to the difference between total resources at the start of the period and the amount invested for the next period:

$$c_t = W_t - \sum_{i=1,2} k_{t+1}^i.$$

Then we can write down a value function in the following recursive way:

$$V\left(W_t\right) = \max_{\left\{k_{t+1}^i\right\}} \left[u\left(W_t - \sum_{i=1,2} k_{t+1}^i\right) + \beta \operatorname{E}\left[V\left(\sum_{i=1,2}\left(f_i\left(k_{t+1}^i\right) + k_{t+1}^i\right)\right)\right]\right]$$

subject to non-negativity constraints

$$k_{t+1}^i \geq 0$$

and resource constraint

$$\sum_{i=1,2} k_{t+1}^i \leq W_t$$

By the same logic as in the previous section, we now have the following first order condition (w.r.t. k_{t+1}^i):

$$-u'\left(c_t\right) + \beta \operatorname{E}\left[V'\left(W_{t+1}\right)\left(1 + f_i'\left(k_{t+1}^i\right)\right)\right] \leq 0$$

If this FOC binds (which means the non-negativity and resource constraints don't bind, why?), then we have

$$\mathrm{E}\left[m_{t+1}R^i_{t+1}\right] = 1,$$

where R^i_{t+1} is a random variable denoting the return on investment i given by

$$R^i_{t+1} = 1 + f'_i\left(k^i_{t+1}\right),$$

and m_{t+1} is the stochastic discount factor

$$m_{t+1} = \frac{\beta V'\left(W_{t+1}\right)}{u'\left(c_t\right)}.$$

A covariance decomposition yields

$$\mathrm{E}\left[m_{t+1}R^i_{t+1}\right] = \mathrm{E}\left[m_{t+1}\right]\mathrm{E}\left[R^i_{t+1}\right] + \mathrm{Cov}\left[m_{t+1}, R^i_{t+1}\right] = 1$$

We can rearrange this equation into

$$\mathrm{E}\left[R^i_{t+1}\right] = \underbrace{\frac{1}{\mathrm{E}\left[m_{t+1}\right]}}_{=\gamma} - \underbrace{\frac{\mathrm{Cov}\left[m_{t+1}, R^i_{t+1}\right]}{\mathrm{Var}\left[m_{t+1}\right]}}_{=\beta_i}\underbrace{\frac{\mathrm{Var}\left[m_{t+1}\right]}{\mathrm{E}\left[m_{t+1}\right]}}_{=\lambda},$$

which is an empirical prediction about the relationship between the expected return from an investment ($\mathrm{E}\left[R^i_{t+1}\right]$) and a coefficient β_i, which measures the extent to which the returns from investment i covaries with the stochastic discount factor. This model is sometimes called the consumption-based capital asset pricing model (CCAPM).