14.04 Recitation 11: Existence of Walrasian Equilibria

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Fall 2020

1 Existence of Walrasian Equilibrium

The First Welfare Theorem suggests that Walrasian equilibria (also known as competitive equilibria) are wonderful because they are guaranteed to be Pareto optimal. Furthermore, competitive equilibria does not require an all-knowing "social planner" to decide on optimal allocations. Instead it relies only on "decentralized" optimizing behavior by consumers and firms facing market prices. This leads economists to love markets. But there is a deep problem. Do Walrasian equilibria even exist? Put different, is there a price vector that clears markets given any initial assignment of endowments and private ownership of production technology?

It turns out that existence is a pretty hard mathematical issue that did not get properly resolved until the 1960s. The treatment in Krep's textbook is very transparent, so let's go over his discussion here. We will then come back and talk about some of the more technical arguments that Prof. Townsend discussed in class.

2 Example

Let's consider a simple endowment economy with two goods. Note that that if (p_1, p_2) is an equilibrium price vector, so is $(\lambda p_1, \lambda p_2)$ for any positive constant λ. So without loss of generality, let's normalize the prices so that $p_1 + p_2 = 1$. This means $p_2 = 1 - p_1$.

Optimizing behavior by consumer i facing prices (p_1, p_2) gives demand x_1^i and x_2^i. And since the consumer's budget constraint must bind, we then have that

$$p_1 x_1^i + p_2 x_2^i = p_1 e_1^i + p_2 e_2^i$$

where e_1^i denotes i's endowment of good 1 and so on. Note that in our two-good setting, given p_1, we know $p_2 = 1 - p_1$, and given x_1^i, we know x_2^i. So we can just keep track of x_1^i and p_1.

To guarantee that the allocations $\left(x_1^i, x_2^i\right)_{i=1}^I$ and prices (p_1, p_2) define a competitive equilibrium, we just need to make sure that the markets clear; that is, we need that

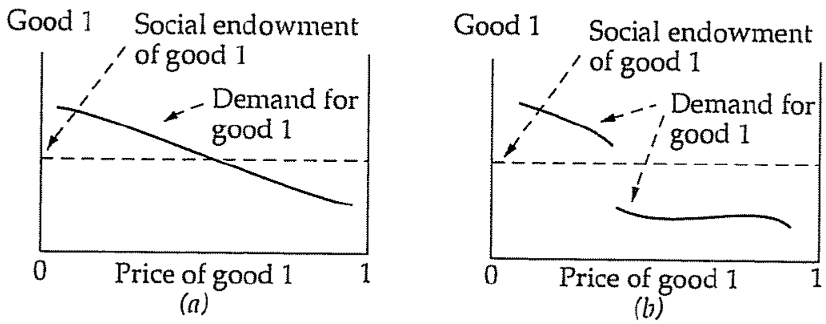
$$\sum_i x_l^i = \sum_i e_l^i, \qquad \forall l = 1, 2.$$

Let's assume that given any pair of strictly positive prices p_1 and p_2 and for consumer i, each consumer's utility maximization problem has a unique solution. With this assumption, we can write $x_1^i(p_1)$. Then we can define the aggregate demand $X_1(p_1) = \sum_i x_1^i(p_1)$. The market clearing condition is therefore

$$X_1(p_1) = e_1.$$

In other words, at any price p_1 such that $X_1(p_1) = e_1$, we have a competitive equilibrium.

Let's plot some examples of $X_1(p_1)$ to get an intuition for when this happens... The basic idea is that you need to find a solution a non-linear equation.



3 Fix Point Theorems

In this section we will study the existence of solutions to general systems of nonlinear equations. The first thing to notice is that solving equations can be also thought of solving fixed point problems. The following theorem states a very general set of conditions under which we can guarantee the existence of a fix point of a function.

Theorem 1. *(Brower's Fixed Point Theorem) Suppose $A \subset \mathbb{R}^L$ is non-empty, convex, compact, and $f : A \to A$ is continuous. Then f has a fixed point; i.e. there exists $x^* \in A$ such that $f(x^*) = x^*$.*

The following theorem is an extension of Brower's theorem to the case of correspondences. Correspondences are like functions except the function output may not be single-valued.

Theorem 2. *(Kakutani's Fixed Point Theorem) Suppose $A \subset \mathbb{R}^L$ is non-empty, convex, compact, and $f : A \rightrightarrows A$ is a correspondence such that*

1. *f is non-empty, i.e. $f(x) \neq \emptyset$ for all $x \in A$*
2. *f is convex valued, i.e. $f(x) \subseteq A$ is a convex set for all $x \in A$*
3. *f is upper hemicontinuous.*

Then f has a fixed point; i.e. there exists $x^* \in A$ such that $f(x^*) = x^*$.

Uh oh. Lots of complicated words we haven't defined. Don't worry! You don't need to know the details of these fix point theorems. You just need to have a rough intuition about what they say and why we need them to prove the existence of Walrasian equilibria.

4 Negishi's Theorem

Now that we have fix point theorem under our belt, we can come back to do Negishi's classic proof of the existence of competitive equilibria. This proof is pretty neat because it leverages results we have from the Welfare Theorems.

The First Welfare Theorem implies that the set of Walrasian Equilibrium allocations is included in the set of Pareto Optimal Allocations. Therefore, for each Walrasian Equilibrium allocation (x^*, y^*) there must exist a vector of Pareto weights λ such that (x^*, y^*) maximizes the λ-weighted Welfare function (as we saw in the Second welfare theorem notes).

Negishi's method relies on this fact, and instead of trying to compute the equilibrium prices for the economy, it tries to find the corresponding vector of Pareto weights (using the fix point methods briefly discussed above). Once the right vector of Pareto Weights is found, one can find the Walrasian equilibrium allocation, and we can use the technique developed in the Second Welfare theorem to derive the equilibrium prices from the Pareto Optimal allocation.

For simplicity we will assume there is only one firm in the economy

Theorem 3. *(Negishi 1960) Suppose we have a private ownership economy*

$$\mathscr{E} \equiv \left\{ \{X_i, \succeq_i, \omega_i\}_{i=1}^I, Y, \{\theta_i\}_{i=1}^I \right\}$$

that satisfy assumptions (A.1) to (A.4).[1] *Then, there exists a Walrasian equilibrium (x^*, y^*, p) for this economy.*

The proof starts with solving the Pareto problem with arbitrary λ weights to define the corresponding Pareto optimal allocations $(x(\lambda), y(\lambda))$. Note that $(x(\lambda), y(\lambda))$ is an equilibrium with transfers allocation, by virtue of the second welfare theorem. Therefore, we know there exist prices $\hat{p}(\lambda) = \hat{\gamma}(\lambda)$ (i.e. the Lagrange multipliers of the resource constraints) and wealth levels $w_i(\lambda) = \sum_{l=1}^L \hat{p}_l(\lambda)\hat{x}_{il}(\lambda)$ such that $(x(\lambda), y(\lambda), \hat{p}(\lambda), w(\lambda))$ is a price equilibrium with transfers.

We now want to show that there exists λ such that $(x(\lambda), y(\lambda), \hat{p}(\lambda))$ is a Walrasian equilibrium for $\mathscr{E}$. This means that we must have

$$w_i(\lambda) = \sum_{l=1}^L \hat{p}_l(\lambda)\hat{x}_{il}(\lambda) = \sum_{l=1}^L \hat{p}_l(\lambda)(\omega_{il} + \theta_i \hat{y}_l(\lambda)).$$

This amounts to finding λ^* such that the above equation is satisfied. The proof will rely on the Kakutani's fixed point theorem, and we will need to define a mapping f and checking the conditions where there exists λ^* such that $f(\lambda^*) = \lambda^*$. I'll omit details here, but you can refer to the lecture slides for details.

[1] See lecture notes for these conditions.