

1 Bessel functions

1.1 Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad (17.1.2a)$$

$$j_n(z) = \sqrt{\frac{\pi}{2z}} \left(\frac{z}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma((n + \frac{1}{2}) + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad (17.1.12a)$$

$$J_\nu(z) = \frac{e^{-iz} \left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{(\nu + \frac{1}{2})_k (2i)^k z^k}{(2\nu + 1)_k k!} \quad (17.1.22)$$

$$j_n(z) = \frac{\sqrt{\pi}}{(2n + 1) \Gamma(n + \frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{1}{(n + \frac{3}{2})_k} \left(-\frac{z^2}{4}\right)^k}{k!} \quad (17.1.25)$$

$$j_n(z) = \frac{\sqrt{\pi} e^{-iz}}{(2n + 1) \Gamma(n + \frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{(n+1)_k}{(2n+2)_k} (2iz)^k}{k!} \quad (17.1.26)$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \cos \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) - \frac{(-1)^k (\nu, 2k + 1)}{(2z)^{2k+1}} \sin \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) \right) \quad (17.1.28)$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \sin \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) + \frac{(-1)^k (\nu, 2k + 1)}{(2z)^{2k+1}} \cos \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) \right) \quad (17.1.29)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{\frac{z}{2\nu+2}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{(iz)^2}{4(\nu+m-1)(\nu+m)}}{1} \quad (17.1.38)$$

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{(iz)^2}{4(n+\frac{1}{2}+m-1)(n+\frac{1}{2}+m)}}{1} \quad (17.1.39)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = - \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{-1}{\frac{2(\nu+m)}{z}} \quad (17.1.40)$$

$$\frac{H_{\nu+1}^{(1)}(z)}{H_\nu^{(1)}(z)} = \frac{-1}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{m-3-2\nu}{-2iz}}{1} \quad (17.1.44)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{z}{2\nu + 2 - iz} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(2\nu + 2m - 1) iz}{2\nu + m + 1 + (-2i) z} \quad (17.1.48)$$

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{z}{2n+3-iz} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{2(n+m)iz}{2n+m+2+(-2i)z} \quad (17.1.49)$$

$$\frac{H_{\nu+1}^{(1)}(z)}{H_{\nu}^{(1)}(z)} = \frac{2\nu+1-2iz}{2z} - \frac{1}{z} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(iz-m)} \quad (17.1.51)$$

1.2 Modified Bessel functions

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \Gamma(\nu+k+1)} \quad (17.2.20)$$

$$I_{\nu}(z) = \frac{e^{-z} \left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(\nu+\frac{1}{2})_k 2^k z^k}{(2\nu+1)_k k!} \quad (17.2.21)$$

$$i_n^{(1)}(z) = \frac{\sqrt{\pi}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!(n+\frac{3}{2})_k} \quad (17.2.22)$$

$$i_n^{(1)}(z) = \frac{\sqrt{\pi} e^{-iz}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(n+1)_k (2z)^k}{k!(2n+2)_k} \quad (17.2.23)$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z+\frac{(2\nu+1)i\pi}{2}}\right) (\nu, k)}{\sqrt{2\pi z} (2z)^k} \right) \quad (17.2.24)$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z-\frac{(2\nu+1)i\pi}{2}}\right) (\nu, k)}{\sqrt{2\pi z} (2z)^k} \right) \quad (17.2.25)$$

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{\infty} \left(\frac{(\nu, k)}{(-2z)^k} \right) \quad (17.2.27)$$

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{\frac{z}{2(\nu+1)}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{1}{4(\nu+m-1)(\nu+m)} z^2}{1} \quad (17.2.32)$$

$$\frac{i_{n+1}^{(1)}(z)}{i_n^{(1)}(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{1}{4((n+\frac{1}{2})+m-1)((n+\frac{1}{2})+m)} z^2}{1} \quad (17.2.33)$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{1}{1} + \frac{\frac{-2\nu-1}{2z}}{1} - \mathop{\mathrm{K}}\limits_{m=3}^{\infty} \frac{\frac{\frac{m}{2}+\nu}{2z}}{1} \quad (17.2.34)$$

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{z}{2\nu+2+z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{-(2\nu+2m-1)z}{2\nu+m+1+2z} \quad (17.2.38)$$

$$\frac{i_{n+1}^{(1)}(z)}{i_n^{(1)}(z)} = \frac{z}{2n+3+z} + \prod_{m=2}^{\infty} \frac{-2(n+m)z}{2n+m+2+2z} \quad (17.2.39)$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{\nu}{z} - \frac{2\nu+1+2z}{2z} - \frac{1}{z} \prod_{m=1}^{\infty} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(z+m)} \quad (17.2.40)$$

2 Confluent hypergeometric functions

2.1 Confluent hypergeometric series ${}_2F_0$

$$\frac{{}_2F_0(a, b; ; z)}{{}_2F_0(a, b+1; ; z)} = 1 + \prod_{m=1}^{\infty} \frac{-(b + \frac{m}{2})z}{1} \quad (16.2.4)$$

2.2 Confluent hypergeometric limit function

$${}_0F_1(; b; z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!} \quad (16.3.1)$$

$$\frac{{}_0F_1(; b; z)}{{}_0F_1(; b+1; z)} = 1 + \prod_{m=1}^{\infty} \frac{\frac{1}{(b+m-1)(b+m)}z}{1} \quad (16.3.4)$$

$$\frac{{}_0F_1(; b; z)}{{}_0F_1(; b+1; z)} = 1 + \frac{\sqrt{z}}{b} + \frac{1}{2b} \prod_{m=1}^{\infty} \frac{-2(2b+2m-1)\sqrt{z}}{2b+m+4\sqrt{z}} \quad (16.3.6)$$

2.3 Kummer functions

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{\frac{(a)_k}{(b)_k} z^k}{k!} \quad (16.1.2)$$

$${}_2F_0(a, b; ; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{k!} \quad (16.1.12)$$

$$\frac{{}_1F_1(a; b; z)}{{}_1F_1(a+1; b+1; z)} = 1 + \prod_{m=1}^{\infty} \frac{\frac{z(a + \frac{m}{2})}{(b+m-1)(b+m)}}{1} \quad (16.1.13)$$

$$z {}_1F_1(1; b+1; z) = \frac{z}{1} + \prod_{m=2}^{\infty} \frac{\frac{z(b + \frac{m}{2} - 1)}{-(b+m-2)(b+m-1)}}{1} \quad (16.1.14)$$

$$\frac{{}_1F_1(a; b; z)}{{}_1F_1(a+1; b+1; z)} = \frac{b-z}{b} + \frac{1}{b} \prod_{m=1}^{\infty} \frac{(a+m)z}{b+m-z} \quad (16.1.16)$$

$${}_1F_1(1; b+1; z) = \frac{b}{b-z} + \prod_{m=2}^{\infty} \frac{(m-1)z}{b+m-1-z} \quad (16.1.17)$$

$$\frac{U(a, b, z)}{U(a+1, b, z)} = 2a - b + 2 + z - \prod_{m=1}^{\infty} \frac{(a+m)(b-a-m-1)}{b-2a-2m-2-z} \quad (16.1.20)$$

2.4 Parabolic cylinder functions

$$\frac{U(a, x)}{U(a-1, x)} = \frac{1}{x} + \prod_{m=2}^{\infty} \frac{a + (m-1) - \frac{1}{2}}{x} \quad (16.5.7)$$

2.5 Whittaker functions

$$W_{\kappa, \mu}(z) = e^{-z^2} z^{\kappa} \sum_{k=0}^{\infty} \left(\frac{(-\kappa - \mu + \frac{1}{2})_k (-\kappa + \mu + \frac{1}{2})_k (-z)^{-k}}{k!} \right) \quad (16.4.7)$$

$$functions : WhittakerPsi(\alpha, \beta, z) = \sum_{k=0}^{\infty} \left(\frac{(\alpha + \frac{1}{2})_k (\beta + \frac{1}{2}, k) z^{-k-1}}{k!} \right) \quad (16.4.12)$$

3 Mathematical constants

3.1 Apéry's constant, $\zeta(3)$

$$\zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^z} \quad (10.11.1)$$

$$\zeta(3) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{(k!)^{10} (205k^2 + 250k + 77)}{64 ((2k+1)!)^5} \right) \quad (10.11.3)$$

$$\zeta(3) = \frac{6}{5} + \prod_{m=2}^{\infty} \frac{-(m-1)^6}{34(m-1)^3 + 51(m-1)^2 + 27(m-1) + 5} \quad (10.11.5)$$

$$\zeta(3) = \frac{1}{22} + \prod_{m=2}^{\infty} \frac{\left(\frac{m}{2}\right)^3}{1} \quad ()$$

$$\zeta(3) = \frac{1}{2k^2 + 2k + 1} + \prod_{m=2}^{\infty} \frac{-(m-1)^6}{(m-1)^3 + m^3 + (2m-1)(2k^2 + 2k)} \quad ()$$

$$\zeta(3) = \frac{1}{5} + \prod_{m=2}^{\infty} \frac{-(m-1)^6}{2(m-1)^3 + 3(m-1)^2 + 11(m-1) + 5} \quad ()$$

$$\zeta(3) = \frac{1}{4} + \frac{1}{1} + \frac{1}{18} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{9} + \frac{1}{9} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{7} + \frac{1}{1} + \frac{1}{1} + \frac{1}{7} + \frac{1}{11} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} \quad (10.11.4)$$

3.2 Archimedes' constant, symbol π

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad (10.2.1)$$

$$\pi = \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{14} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{84} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{15} + \frac{1}{3} + \frac{1}{1} \quad (10.2.4)$$

$$\pi = \frac{4}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(m-1)^2}{2m-1} \quad (10.2.5)$$

$$\pi = 3 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{(2m-1)^2}{6} \quad (10.2.6)$$

$$P = \frac{1}{1} + 2 \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(m-1)m}{1} \quad ()$$

$$\pi = \frac{4}{4k+1} + 2^{4k} \left(\frac{(k!)^2}{2k!} \right)^2 \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(2m-3)^2}{8k+2} \quad ()$$

$$\pi = \frac{4}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(2m-3)^2}{2} \quad ()$$

$$\pi = \frac{4}{4+1} + 4 \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(2m-3)^2}{8+2} \quad ()$$

3.3 Catalan's constant, symbol C

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \quad (10.12.1)$$

$$G = \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{88} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{7} + \frac{1}{22} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{26} + \frac{1}{1} + \frac{1}{11} + \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \frac{1}{9} + \frac{1}{3} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} \quad (10.12.2)$$

$$G = \frac{1}{\frac{1}{2}} + \frac{1}{2} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\left(\frac{m}{2}\right)^2}{\frac{1}{2}} \quad (10.12.3)$$

$$G = \frac{\frac{13}{2}}{7} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(2m-3)^4 (2m-2)^4 \left(20(m-2)^2 - 8(m-2) + 1\right) \left(20(m)^2 - 8(m) + 1\right)}{3520(m-1)^6 + 5632(m-1)^5 + 2064(m-1)^4 - 384(m-1)^3 - 156(m-1)^2 + 16(m-1) + 7} \quad (10.12.5)$$

$$G = \frac{-1}{3} + \frac{1}{2} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{m^2}{1} \quad ()$$

3.4 Euler's constant, symbol γ

$$\gamma = -\log n + \sum_{k=0}^{\infty} \frac{1}{k} \quad (10.8.1)$$

$$\gamma = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{4} + \frac{1}{3} + \frac{1}{13} + \frac{1}{5} + \frac{1}{1} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{40} + \frac{1}{1} + \frac{1}{11} + \frac{1}{3} + \frac{1}{7} + \frac{1}{1} + \frac{1}{7} + \frac{1}{1} + \frac{1}{1} +$$

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3.5 Euler's number, base of the natural logarithm

$$e^1 = \sum_{k=0}^{\infty} \frac{1}{k}! \quad (10.3.1b)$$

$$e^1 = 2 + \prod_{m=1}^{\infty} \frac{1}{1} \quad (10.3.5)$$

$$\frac{e^1 - 1}{e^1 + 1} = \prod_{m=1}^{\infty} \frac{1}{(4m-2)} \quad (10.3.6)$$

$$e^1 = 2 + \prod_{m=1}^{\infty} \frac{m+1}{m+1} \quad (1)$$

$$e^1 = \frac{2}{1} + \sum_{m=2}^{\infty} \frac{1}{6 + (m-2)4} \quad ()$$

$$e^1 = \frac{1}{1} + \sum_{m=2}^{\infty} \frac{-1}{m-1} \quad ()$$

3.6 Golden ratio, symbol ϕ

$$\phi = 1 + \prod_{m=1}^{\infty} \frac{1}{1} \quad (10.9.4)$$

3.7 Gompertz' constant, symbol G

$$G = \frac{1}{2} + \sum_{m=2}^{\infty} \frac{-(m-1)^2}{2m} \quad (10.13.1)$$

$$G = \frac{1}{1} + \prod_{m=2}^{\infty} K_{\frac{m}{1}} \quad (1)$$

3.8 The natural logarithm, $\ln(2)$

$$\ln(2) = \sum_{k=0}^{\infty} \frac{(-1)^k 1^{k+1}}{k+1} \quad (10.5.2)$$

$$\ln(2) = \frac{1}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \frac{(m-1)^2}{1} \quad (10.5.3)$$

$$\ln(2) = \frac{1}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \frac{\frac{m}{4m-4}}{1} \quad (10.5.4)$$

3.9 Regular continued fractions

$$\sqrt{e^1} = \frac{1}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \frac{1}{1]} \quad (10.4.1)$$

$$e^{\frac{1}{\alpha}} = 1 + \mathop{\mathbb{K}}\limits_{m=1}^{\infty} \frac{1}{(2(\frac{m+2}{3}) - 1)\alpha - 1} \quad (10.4.2)$$

$$e^{\frac{1}{\alpha}} = \frac{\alpha+1}{\alpha} + \frac{1}{\alpha} \mathop{\mathbb{K}}\limits_{m=1}^{\infty} \frac{1}{2\alpha - 1} \quad (10.4.3)$$

$$e^{\frac{1}{\alpha}} = \frac{1}{\alpha - 1} + \frac{1}{2\alpha} + \alpha \mathop{\mathbb{K}}\limits_{m=3}^{\infty} \frac{1}{1} \quad (10.4.4)$$

$$\sqrt{\pi} = \frac{1}{1} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{28} + \frac{1}{13} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{18} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{83} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{1} + \frac{1}{288} + \frac{1}{1} + \frac{1}{90} \quad (10.4.5)$$

$$e^1 e^1 = 7 + \mathop{\mathbb{K}}\limits_{m=1}^{\infty} \frac{1}{\frac{3(m+4)}{5} - 1} \quad (10.4.6)$$

$$\frac{\pi^2}{12} = \frac{1}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \frac{(m-1)^4}{2m-1} \quad (10.4.7)$$

$$\frac{e^{\frac{2\alpha}{\beta}} - 1}{e^{\frac{2\alpha}{\beta}} + 1} = \frac{\alpha}{\beta} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \frac{\alpha^2}{(2m-1)\beta} \quad (10.4.8)$$

3.10 Pythagoras' constant, the square root of two

$$1 + \sqrt{2} = 2 + \prod_{m=1}^{\infty} \frac{1}{2} \quad (10.6.3)$$

$$\left(1 + \sqrt{2}\right)^2 = 5 + \prod_{m=1}^{\infty} \frac{1}{1} \quad ()$$

$$\left(1 + \sqrt{2}\right)^3 = 14 + \prod_{m=1}^{\infty} \frac{1}{14} \quad ()$$

$$\left(1 + \sqrt{2}\right)^4 = 33 + \prod_{m=1}^{\infty} \frac{1}{1} \quad ()$$

$$\left(1 + \sqrt{2}\right)^5 = 82 + \prod_{m=1}^{\infty} \frac{1}{82} \quad ()$$

3.11 The rabbit constant, symbol ρ

$$\rho = \sum_{k=0}^{\infty} 2^{-\left\lfloor (k+1)\left(\frac{\sqrt{5}+1}{2}\right) \right\rfloor} \quad (10.10.2)$$

$$\rho = \prod_{m=1}^{\infty} \frac{1}{2^{F(m-1)}} \quad (10.10.5)$$

4 Elementary functions

5 Error function and related integrals

6 Exponential integrals and related functions

7 Gamma function and related functions

8 Hypergeometric functions

9 q-Hypergeometric function

10 Probability functions