1 Bessel functions

1.1 Bessel functions

$$J_{\nu}(z) = \frac{z}{2}^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \frac{z}{2}^{2k} \tag{17.1.2a}$$

$$\mathrm{j}_n(z) = \sqrt{\frac{\pi}{2z}} \frac{z}{2}^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma\left(\left(n+\frac{1}{2}\right)+k+1\right)} \frac{z}{2}^{2k} \tag{17.1.12a}$$

$$J_{\nu}(z) = \frac{e^{-iz} \frac{z}{2}^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\left(\nu+\frac{1}{2}\right)_k (2i)^k z^k}{(2\nu+1)_k k!} \tag{17.1.22}$$

$$\mathrm{j}_n(z) = \frac{\sqrt{\pi}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \frac{z}{2}^{n} \sum_{k=0}^{\infty} \frac{\frac{1}{\left(n+\frac{3}{2}\right)_k} \left(-\frac{z^2}{4}\right)^k}{k!} \tag{17.1.25}$$

$$\mathrm{j}_n(z) = \frac{\sqrt{\pi} e^{-iz}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \frac{z}{2}^{n} \sum_{k=0}^{\infty} \frac{\frac{(n+1)_k}{(2n+2)_k} (2iz)^k}{k!} \tag{17.1.26}$$

$$J_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \cos\left(z - \frac{\nu}{2} + \frac{1}{4}\pi\right) - \frac{(-1)^k (\nu, 2k+1)}{(2z)^{2k+1}} \sin\left(z - \frac{\nu}{2} + \frac{1}{4}\pi\right) \right) \tag{17.1.28}$$

$$Y_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \sin\left(z - \frac{\nu}{2} + \frac{1}{4}\pi\right) + \frac{(-1)^k (\nu, 2k+1)}{(2z)^{2k+1}} \cos\left(z - \frac{\nu}{2} + \frac{1}{4}\pi\right) \right) \tag{17.1.29}$$

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \frac{\frac{z}{2\nu+2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{(iz)^2}{4(\nu+m-1)(\nu+m)}}{1} \right) \tag{17.1.38}$$

$$\frac{\mathrm{j}_{n+1}(z)}{\mathrm{j}_n(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{(iz)^2}{4\left(n+\frac{1}{2}+m-1\right)\left(n+\frac{1}{2}+m\right)}}{1} \right) \tag{17.1.39}$$

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = -\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-1}{\frac{2(\nu+m)}{z}} \right) \tag{17.1.40}$$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{-1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{m-3-2\nu}{-2iz}}{1} \right) \tag{17.1.44}$$

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \frac{z}{2\nu+2-iz} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(2\nu+2m-1)\,iz}{2\nu+m+1+(-2i)\,z}\right) \tag{17.1.48}$$

$$\frac{\mathsf{j}_{n+1}(z)}{\mathsf{j}_{n}(z)} = \frac{z}{2n+3-iz} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{2\,(n+m)\,iz}{2n+m+2+(-2i)\,z}\right) \tag{17.1.49}$$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{2\nu+1-2iz}{2z} - \frac{1}{z}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{\nu^2-\frac{(2m-1)^2}{4}}{2\,(iz-m)}\right) \tag{17.1.51}$$

1.2 Modified Bessel functions

$$I_{\nu}(z) = \frac{z}{2}^{\nu}\sum_{k=0}^{\infty}\frac{\frac{z}{2}^{2k}}{k!\,\Gamma(\nu+k+1)} \tag{17.2.20}$$

$$I_{\nu}(z) = \frac{e^{-z}\frac{z}{2}^{\nu}}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{\left(\nu+\frac{1}{2}\right)_k 2^k z^k}{(2\nu+1)_k k!} \tag{17.2.21}$$

$$\mathsf{i}^{(1)}_{n}(z) = \frac{\sqrt{\pi}}{(2n+1)\,\Gamma\!\left(n+\frac{1}{2}\right)}\frac{z}{2}^{n}\sum_{k=0}^{\infty}\frac{\frac{z^2}{4}^{k}}{k!\left(n+\frac{3}{2}\right)_k} \tag{17.2.22}$$

$$\mathsf{i}^{(1)}_{n}(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\,\Gamma\!\left(n+\frac{1}{2}\right)}\frac{z}{2}^{n}\sum_{k=0}^{\infty}\frac{(n+1)_k\,(2z)^k}{k!(2n+2)_k} \tag{17.2.23}$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty}\left(\frac{\left((-1)^k e^{z}+e^{-z+\frac{(2\nu+1)i\pi}{2}}\right)(\nu,k)}{\sqrt{2\pi z}\,(2z)^k}\right) \tag{17.2.24}$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty}\left(\frac{\left((-1)^k e^{z}+e^{-z-\frac{(2\nu+1)i\pi}{2}}\right)(\nu,k)}{\sqrt{2\pi z}\,(2z)^k}\right) \tag{17.2.25}$$

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{\infty}\left(\frac{(\nu,k)}{(-2z)^k}\right) \tag{17.2.27}$$

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{\frac{z}{2(\nu+1)}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{1}{4(\nu+m-1)(\nu+m)}z^2}{1}\right) \tag{17.2.32}$$

$$\frac{\mathsf{i}^{(1)}_{n+1}(z)}{\mathsf{i}^{(1)}_{n}(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{1}{4\left(\left(n+\frac{1}{2}\right)+m-1\right)\left(\left(n+\frac{1}{2}\right)+m\right)}z^2}{1}\right) \tag{17.2.33}$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{\nu}{z} + \frac{1}{1} + \frac{\frac{-2\nu-1}{2z}}{1} + -\mathop{\mathrm{K}}_{m=3}^{\infty}\left(\frac{\frac{\frac{m}{2}+\nu}{2z}}{1}\right) \tag{17.2.34}$$

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} = \frac{z}{2\nu+2+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2\nu+2m-1)\,z}{2\nu+m+1+2z} \right) \tag{17.2.38}$$

$$\frac{\mathsf{i}_{n+1}^{(1)}(z)}{\mathsf{i}_n^{(1)}(z)} = \frac{z}{2n+3+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-2\,(n+m)\,z}{2n+m+2+2z} \right) \tag{17.2.39}$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{\nu}{z} - \frac{2\nu+1+2z}{2z} - \frac{1}{z} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{\nu^2 - \frac{(2m-1)^2}{4}}{2\,(z+m)} \right) \tag{17.2.40}$$

2 Confluent hypergeometric functions

2.1 Confluent hypergeometric series ${}_2F_0$

$$\frac{{}_2F_0(a,b;;z)}{{}_2F_0(a,b+1;;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-\left(b+\frac{m}{2}\right)z}{1} \right) \tag{16.2.4}$$

2.2 Confluent hypergeometric limit function

$${}_0F_1(;b;z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!} \tag{16.3.1}$$

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{\frac{1}{(b+m-1)(b+m)}z}{1} \right) \tag{16.3.4}$$

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \frac{\sqrt{z}}{b} + \frac{1}{2b} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-2\,(2b+2m-1)\sqrt{z}}{2b+m+4\sqrt{z}} \right) \tag{16.3.6}$$

2.3 Kummer functions

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{\frac{(a)_k}{(b)_k} z^k}{k!} \tag{16.1.2}$$

$${}_2F_0(a,b;;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{k!} \tag{16.1.12}$$

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a+1;b+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{\frac{z\left(a+\frac{m}{2}\right)}{(b+m-1)(b+m)}}{1} \right) \tag{16.1.13}$$

$$z\,{}_1F_1(1;b+1;z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\frac{z\left(b+\frac{m}{2}-1\right)}{(b+m-2)(b+m-1)}}{1} \right) \tag{16.1.14}$$

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a+1;b+1;z)} = \frac{b-z}{b} + \frac{1}{b} \operatorname*{K}_{m=1}^{\infty} \left(\frac{(a+m)\, z}{b+m-z} \right) \tag{16.1.16}$$

$$ {}_1F_1(1;b+1;z) = \frac{b}{b-z} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)\, z}{b+m-1-z} \right) \tag{16.1.17}$$

$$\frac{U(a,b,z)}{U(a+1,b,z)} = 2a-b+2+z - \operatorname*{K}_{m=1}^{\infty} \left(\frac{(a+m)\,(b-a-m-1)}{b-2a-2m-2-z} \right) \tag{16.1.20}$$

2.4 Parabolic cylinder functions

$$\frac{U(a,x)}{U(a-1,x)} = \frac{1}{x} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{a+(m-1)-\frac{1}{2}}{x} \right) \tag{16.5.7}$$

2.5 Whittaker functions

$$W_{\kappa,\mu}(z) = e^{-z^2} z^{\kappa} \sum_{k=0}^{\infty} \left(\frac{\left(-\kappa-\mu+\frac{1}{2}\right)_k \left(-\kappa+\mu+\frac{1}{2}\right)_k (-z)^{-k}}{k!} \right) \tag{16.4.7}$$

$$\Psi_{n,\beta}(z) = \sum_{k=0}^{\infty} \left(\frac{\left(n+\frac{1}{2}\right)_k \left(\beta+\frac{1}{2},k\right) z^{-k-1}}{k!} \right) \tag{16.4.12}$$

3 Mathematical constants

3.1 Apéry's constant, $\zeta(3)$

$$\zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^z} \tag{10.11.1}$$

$$\zeta(3) = \sum_{k=0}^{\infty} (-1)^k \frac{(k!)^{10} \left(205k^2+250k+77\right)}{64\left((2k+1)!\right)^5} \tag{10.11.3}$$

$$\zeta(3) = \frac{6}{5} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(m-1)^6}{34\,(m-1)^3+51\,(m-1)^2+27\,(m-1)+5} \right) \tag{10.11.5}$$

$$\zeta(3) = 1 + \frac{1}{22} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{m}{2}3}{1} \right) \qquad \text{(No label)}$$

$$\zeta(3) = \sum_{n=1}^{k} \left(\frac{1}{n^3} \right) + \frac{1}{2k^2+2k+1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(m-1)^6}{(m-1)^3+m^3+(2m-1)\left(2k^2+2k\right)} \right) \qquad \text{(No label)}$$

$$\zeta(3) = 1 + \frac{1}{5} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(m-1)^6}{2\,(m-1)^3+3\,(m-1)^2+11\,(m-1)+5} \right) \qquad \text{(No label)}$$

$$\zeta(3) = 1, +\frac{1}{4\,+}\,\frac{1}{1\,+}\,\frac{1}{18\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{4\,+}\,\frac{1}{1\,+}\,\frac{1}{9\,+}\,\frac{1}{9\,+}\,\cdots \tag{10.11.4}$$

3.2 Archimedes' constant, symbol π

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tag{10.2.1}$$

$$\pi = 3 + \frac{1}{7\,+}\,\frac{1}{15\,+}\,\frac{1}{1\,+}\,\frac{1}{292\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{2\,+}\,\frac{1}{1\,+}\,\frac{1}{3\,+}\,\cdots \tag{10.2.4}$$

$$\pi = \frac{4}{1\,+}\,\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(m-1)^2}{2m-1}\right) \tag{10.2.5}$$

$$\pi = 3 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{(2m-1)^2}{6}\right) \tag{10.2.6}$$

$$\pi = 2 + \frac{1}{1\,+}\,2\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(m-1)\,m}{1}\right) \qquad \text{(No label)}$$

$$\pi = \frac{4}{4k+1\,+}\,2^{4k}\,\frac{(k!)^2}{(2k)!}^{2}\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(2m-3)^2}{8k+2}\right) \qquad \text{(No label)}$$

$$\pi = \frac{4}{1\,+}\,\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(2m-3)^2}{2}\right) \qquad \text{(No label)}$$

$$\pi = \frac{4}{4+1\,+}\,4\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(2m-3)^2}{8+2}\right) \qquad \text{(No label)}$$

3.3 Catalan's constant, symbol G

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{10.12.1}$$

$$G = 0, +\frac{1}{1\,+}\,\frac{1}{10\,+}\,\frac{1}{1\,+}\,\frac{1}{8\,+}\,\frac{1}{1\,+}\,\frac{1}{88\,+}\,\frac{1}{4\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{7\,+}\,\cdots \tag{10.12.2}$$

$$G = \frac{1}{2} + \frac{1}{\frac{1}{2}\,+}\,\frac{1}{2}\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{m}{2}^{2}}{\frac{1}{2}}\right) \tag{10.12.3}$$

$$G = \frac{\frac{13}{2}}{7\,+}\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(2m-3)^4\,(2m-2)^4\left(20\,(m-2)^2-8\,(m-2)+1\right)\left(20\,(m)^2-8\,(m)+1\right)}{3520\,(m-1)^6+5632\,(m-1)^5+2064\,(m-1)^4-384\,(m-1)^3-156\,(m-1)^2+16\,(m-1)+7}\right) \tag{10.12.5}$$

$$G=1+\frac{-1}{3}+\frac{1}{2}\operatorname*{K}_{m=2}^{\infty}\left(\frac{m^{2}}{1}\right) \qquad \text{(No label)}$$

3.4 Delian

$$\sqrt[3]{2}=1+\operatorname*{K}_{m=1}^{\infty}\left(\frac{\frac{3(m+1)}{2}-2}{3m}\right) \qquad \text{(No label)}$$

$$\sqrt[3]{2}=1,+\frac{1}{3}+\frac{1}{1}+\frac{1}{5}+\frac{1}{1}+\frac{1}{1}+\frac{1}{4}+\frac{1}{1}+\frac{1}{1}+\frac{1}{8}+\frac{1}{1}+\cdots \qquad (10.7.1.)$$

3.5 Euler's constant, symbol γ

$$\gamma=-\log(n)+\sum_{k=0}^{\infty}\frac{1}{k} \qquad (10.8.1)$$

$$\gamma=0,+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{4}+\frac{1}{3}+\frac{1}{13}+\frac{1}{5}+\cdots \qquad \text{(No label)}$$

3.6 Euler's number, base of the natural logarithm

$$e=\sum_{k=0}^{\infty}\frac{1}{(k)!} \qquad (10.3.1b)$$

$$e=2+\operatorname*{K}_{m=1}^{\infty}\left(\frac{1}{1}\right) \qquad (10.3.5)$$

$$\frac{e-1}{e+1}=\operatorname*{K}_{m=1}^{\infty}\left(\frac{1}{4m-2}\right) \qquad (10.3.6)$$

$$e=2+\operatorname*{K}_{m=1}^{\infty}\left(\frac{m+1}{m+1}\right) \qquad \text{(No label)}$$

$$e=1+\frac{2}{1}+\operatorname*{K}_{m=2}^{\infty}\left(\frac{1}{6+(m-2)\,4}\right) \qquad \text{(No label)}$$

$$e=\frac{1}{1}+\operatorname*{K}_{m=2}^{\infty}\left(\frac{-1}{m-1}\right) \qquad \text{(No label)}$$

3.7 Golden ratio, symbol ϕ

$$\phi=1+\operatorname*{K}_{m=1}^{\infty}\left(\frac{1}{1}\right) \qquad (10.9.4)$$

3.8 Gompertz' constant, symbol G

$$G = \frac{1}{2} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-\left(m-1\right)^2}{2m} \right) \tag{10.13.1}$$

$$G = \frac{1}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{m}{2}}{1} \right) \quad \text{(No label)}$$

3.9 The natural logarithm, $\ln(2)$

$$\ln\left(2\right) = \sum_{k=0}^{\infty} \frac{\left(-1\right)^k 1^{k+1}}{k+1} \tag{10.5.2}$$

$$\ln\left(2\right) = \frac{1}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\left(m-1\right)^2}{1} \right) \tag{10.5.3}$$

$$\ln\left(2\right) = \frac{1}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{m}{4m-4}}{1} \right) \tag{10.5.4}$$

3.10 Regular continued fractions

$$\sqrt{e} = 1 + \frac{1}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{1}{1} \right) \tag{10.4.1}$$

$$e^{\frac{1}{n}} = 1 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{1}{\left(2\frac{m+2}{3} - 1\right) n - 1} \right) \tag{10.4.2}$$

$$e^{\frac{1}{n}} = \frac{n+1}{n} + \frac{1}{n} \underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{1}{2n-1} \right) \tag{10.4.3}$$

$$e^{\frac{1}{n}} = \frac{1}{n-1} + \frac{1}{2n} + n \underset{m=3}{\overset{\infty}{\mathrm{K}}} \left(\frac{1}{1} \right) \tag{10.4.4}$$

$$\sqrt{\pi} = 1, + \frac{1}{1} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{28} + \frac{1}{13} + \frac{1}{1} + \ldots \quad \text{(No label)}$$

$$ee = 7 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{1}{\frac{3(m+4)}{5} - 1} \right) \tag{10.4.5}$$

$$\frac{\pi^2}{12} = \frac{1}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\left(m-1\right)^4}{2m-1} \right) \tag{10.4.6}$$

$$\frac{e^{\frac{2n}{\beta}} - 1}{e^{\frac{2n}{\beta}} + 1} = \frac{n}{\beta} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{n^2}{\left(2m-1\right) \beta} \right) \tag{10.4.7}$$

3.11 Pythagoras' constant, the square root of two

$$1+\sqrt{2} = 2 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{2} \right) \tag{10.6.3}$$

$$\left(1+\sqrt{2}\right)^{2} = 5 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{1} \right) \qquad \text{(No label)}$$

$$\left(1+\sqrt{2}\right)^{3} = 14 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{14} \right) \qquad \text{(No label)}$$

$$\left(1+\sqrt{2}\right)^{4} = 33 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{1} \right) \qquad \text{(No label)}$$

$$\left(1+\sqrt{2}\right)^{5} = 82 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{82} \right) \qquad \text{(No label)}$$

3.12 The rabbit constant, symbol ρ

$$\rho = \sum_{k=0}^{\infty} 2^{-\left\lfloor (k+1)\frac{\sqrt{5}+1}{2} \right\rfloor} \tag{10.10.2}$$

$$\rho = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{2^{F_{m-1}}} \right) \tag{10.10.5}$$

3.13 Theodorus

$$\sqrt{3} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{1}{1} \right) \qquad \text{(No label)}$$

3.14 zeta2, $\zeta(2)$

$$\zeta(2) = \frac{2}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(m-1)^4}{2m-1} \right) \qquad \text{(No label)}$$

$$\zeta(2) = \frac{1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\frac{\left(\left((m-2)\frac{1}{2}+1\right)\left((m-2)\frac{1}{2}+1\right)\right)\left((m-2)\frac{1}{2}+1\right)}{(m-1)(1+(m-1))}}{1} \right) \qquad \text{(No label)}$$

$$\zeta(2) = 1 + \frac{1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{m}{2}^{2}}{1} \right) \qquad \text{(No label)}$$

$$\zeta(2) = \sum_{n=1}^{k} \left(\frac{1}{n^2} \right) + \frac{2}{2k+1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(m-1)^4}{(2m-1)(2k+1)} \right) \qquad \text{(No label)}$$

$$\zeta(2) = 2\sum_{n=1}^{k}\left(\frac{(-1)^{n-1}}{n^2}\right) + \frac{(-1)^k}{k^2+k+1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-(m-1)^4}{(m-1)^2+m^2+k^2+k}\right) \qquad \text{(No label)}$$

$$\zeta(2) = \frac{5}{3} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)^4}{11(m-1)^2+11(m-1)+3}\right) \qquad \text{(No label)}$$

3.15 zeta4, $\zeta(4)$

$$\zeta(4) = \frac{13}{12} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)^7(3m-2)(3m-3)(3m-4)}{3(2(m-1)+1)\left(3(m-1)^2+3(m-1)+1\right)\left(15(m-1)^2+15(m-1)+4\right)}\right) \qquad \text{(No label)}$$

$$\zeta(4) = 1, + \frac{1}{12} + \frac{1}{6} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} + \frac{1}{183} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \ldots \qquad \text{(No label)}$$

4 Elementary functions

4.1 Inverse trigonometric functions

$$\operatorname{Arccos}(z) = \frac{\pi}{2} - z + \sum_{k=0}^{\infty} -\frac{(2k-1)!!}{(2k)!!\,(2k+1)} z^{2k+1} \qquad (11.4.2)$$

$$\operatorname{Arccos}(z) = \frac{\frac{\sqrt{1-z^2}}{z}}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{\frac{\frac{(m-1)^2}{(2m-3)(2m-1)}\left(1-z^2\right)}{z^2}}{1}\right) \qquad (11.4.6)$$

$$\operatorname{Arccos}(z) = \frac{z\sqrt{1-z^2}}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{\left(-\frac{m(m-1)}{(2m-1)(2m-3)}\right)\left(1-z^2\right)}{1}\right) \qquad (11.4.7)$$

4.2 Inverse hyperbolic functions

$$\operatorname{Arccosh}\left(\frac{1}{z}\right) = \ln\left(\frac{2}{z}\right) + \sum_{k=0}^{\infty} -\frac{(2k-1)!!}{(2k)!!\,(2k)} z^{2k} \qquad (11.6.2)$$

$$\operatorname{Arccosh}(z) = \frac{z\sqrt{z^2-1}}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{\frac{m(m-1)}{(2m-3)(2m-1)}\left(z^2-1\right)}{1}\right) \qquad (11.6.6)$$

$$\operatorname{Arccosh}(z) = \frac{\frac{\sqrt{z^2-1}}{z}}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-\frac{\frac{(m-1)^2}{(2m-3)(2m-1)}\left(z^2-1\right)}{z^2}}{1}\right) \qquad (11.6.7)$$

4.3 Inverse trigonometric functions

$$\mathrm{Arcsin}\,(z) = z + \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!\,(2k+1)} z^{2k+1} \tag{11.4.1}$$

$$\mathrm{Arcsin}\,(z) = \frac{\frac{z}{\sqrt{1-z^2}}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{\frac{(m-1)^2}{(2m-3)(2m-1)} z^2}{1-z^2}}{1} \right) \tag{11.4.4}$$

$$\mathrm{Arcsin}\,(z) = \frac{z\sqrt{1-z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\left(-\frac{m(m-1)}{(2m-1)(2m-3)}\right) z^2}{1} \right) \tag{11.4.5}$$

4.4 Inverse hyperbolic functions

$$\mathrm{Arcsinh}\,(z) = z + \sum_{k=0}^{\infty} \frac{(-1)^k\,(2k-1)!!}{(2k)!!\,(2k+1)} z^{2k+1} \tag{11.6.1}$$

$$\mathrm{Arcsinh}\,(z) = \frac{z\sqrt{1+z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{m(m-1)}{(2m-3)(2m-1)} z^2}{1} \right) \tag{11.6.4}$$

$$\mathrm{Arcsinh}\,(z) = \frac{\frac{z}{\sqrt{1+z^2}}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\frac{\frac{(m-1)^2}{(2m-3)(2m-1)} z^2}{1+z^2}}{1} \right) \tag{11.6.5}$$

4.5 Inverse trigonometric functions

$$\mathrm{Arctan}\,(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} \tag{11.4.3}$$

$$\mathrm{Arctan}\,(z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(m-1)^2 z^2}{2m-1} \right) \tag{11.4.8}$$

$$\mathrm{Arctan}\,(z) = \frac{\frac{z}{1+z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\frac{m(m-1)}{(2m-3)(2m-1)} \frac{z^2}{1+z^2}}{1} \right) \tag{11.4.9}$$

4.6 Inverse hyperbolic functions

$$\mathrm{Arctanh}\,(z), = \sum_{k=0}^{\infty} \frac{1}{2k+1} z^{2k+1} \tag{11.6.3}$$

$$\mathrm{Arctanh}\,(z) = \frac{\frac{z}{1-z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{\frac{m(m-1)}{(2m-3)(2m-1)} z^2}{1-z^2}}{1} \right) \tag{11.6.8}$$

$$\operatorname{Arctanh}(z) = \frac{z}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-\frac{(m-1)^2 z^2}{4(m-1)^2-1}}{1} \right) \tag{11.6.9}$$

4.7 Trigonometric functions

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \tag{11.3.2}$$

4.8 Hyperbolic functions

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} z^{2k} \tag{11.5.2}$$

4.9 Hyperbolic functions

$$\coth(z) = \sum_{k=0}^{\infty} \frac{4^k \, B_{2k}}{(2k)!} z^{2k-1} \tag{11.5.4}$$

$$\coth(z) = \frac{1}{z} + \frac{4\pi^{-2} z}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)^2 \left((m-1)^2 + 4\pi^{-2} z^2 \right)}{2m-1} \right) \tag{11.5.6}$$

4.10 The exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{11.1.1}$$

$$e^z = 1 + \frac{2z}{2-z} + \frac{\frac{z^2}{6}}{1} + \operatorname*{K}_{m=3}^{\infty} \left(\frac{\frac{1}{4(2m-3)(2m-1)} z^2}{1} \right) \tag{11.1.2}$$

$$e^z = 1 + \frac{z}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\left(-\frac{1}{2(m-1)} \right) z}{1} \right) \tag{11.1.3}$$

$$e^z = \frac{1}{1} + \frac{-z}{1} + \operatorname*{K}_{m=3}^{\infty} \left(\frac{\left(-\frac{1}{2(m-1)} \right) z}{1} \right) \tag*{(No label)}$$

$$e^z = 1 + \frac{z}{1-z} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)\, z}{m-z} \right) \tag{11.1.4}$$

4.11 The natural logarithm

$$\ln(1+z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+2}}{k+1} z^{k+1} \tag{11.2.1}$$

$$\ln(1+z) = \frac{z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{m}{4(m-1)} z}{1} \right) \tag{11.2.2}$$

$$\ln(1+z) = \frac{2z}{2+z} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-(m-1)^2 z^2}{(2m-1)(2+z)} \right) \tag{11.2.3}$$

$$\ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-\frac{(m-1)^2}{(2m-3)(2m-1)} z^2}{1} \right) \tag{11.2.4}$$

$$\ln(1+z) = z + \frac{-\frac{z^2}{2}}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\frac{m}{2}+1^2 z}{m(m+1)}}{1} \right) \tag{6.8.8}$$

4.12 The power function

$$(1+z)^n = 1 + \frac{nz}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\frac{m}{2}-nz}{2(m-1)}}{1} \right) \tag{11.7.1}$$

$$(1+z)^n = \frac{1}{1} + \frac{-nz}{1} + \underset{m=3}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\frac{m-1}{2}+nz}{2(m-2)}}{1} \right) \tag{11.7.2}$$

$$(1+z)^n = \frac{1}{1} + \frac{-nz}{1+z} + \frac{\frac{(n-1)z}{2}}{1} + \underset{m=4}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\left(-n-\frac{m-2}{2}\right)z}{2(m-1)(1+z)}}{1} \right) \tag{11.7.3}$$

$$\frac{z+1}{z-1}^n = 1 + \frac{\frac{2n}{z}}{1-\frac{n}{z}} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{n^2-(m-1)^2}{(2(m-1)-1)(2(m-1)+1)z^2}}{1} \right) \tag{11.7.4}$$

4.13 Trigonometric functions

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \tag{11.3.1}$$

4.14 Hyperbolic functions

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1} \tag{11.5.1}$$

4.15 Trigonometric functions

$$\tan(z) = \sum_{k=0}^{\infty} \frac{4^{k+1}\left(4^{k+1}-1\right)\left|B_{2(k+1)}\right|}{(2(k+1))!} z^{2(k+1)-1} \tag{11.3.3}$$

$$\tan(z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{-\frac{z^2}{(2m-1)(2m-3)}}{1}\right) \tag{11.3.7}$$

$$\tan\left(\frac{\pi z}{4}\right) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(2m-3)^2 - z^2}{2}\right) \tag{11.3.8}$$

$$\tan(z) = \frac{z}{1} + \frac{-4\pi^{-2}z^2}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty}\left(\frac{(m-2)^4 - 4\pi^{-2}(m-2)^2 z^2}{2m-3}\right) \tag{11.3.9}$$

4.16 Hyperbolic functions

$$\tanh(z) = \sum_{k=0}^{\infty}\left(4^{k+1}\left(4^{k+1}-1\right)\frac{B_{2(k+1)}}{(2(k+1))!}\right) z^{2(k+1)-1} \tag{11.5.3}$$

$$\tanh(z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{1}{(2m-3)(2m-1)}z^2}{1}\right) \tag{11.5.5}$$

5 Error function and related integrals

5.1 Complementary and complex error function

$$\operatorname{erfc}(z) = e^{-z^2}\sum_{k=0}^{\infty}\frac{(-z)^k}{\Gamma\left(\frac{k}{2}+1\right)} \tag{13.2.9}$$

$$w(z) = \sum_{k=0}^{\infty}\frac{(iz)^k}{\Gamma\left(\frac{k}{2}+1\right)} \tag{13.2.10}$$

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi} z e^{z^2}}\sum_{k=0}^{\infty}\left((-1)^k\left(\frac{1}{2}\right)_k z^{-2k}\right) \tag{13.2.11}$$

$$w(z) = \frac{i}{\pi z}\sum_{k=0}^{\infty}\left(\left(\frac{1}{2}\right)_k z^{-2k}\right) \tag{13.2.12}$$

$$\operatorname{erfc}(z) = \frac{1}{z^2} + \frac{z}{\sqrt{\pi}}e^{-z^2}\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{m-1}{2}}{1}\right) \tag{13.2.20a}$$

$$w(z) = \frac{1}{-z^2} + -\frac{iz}{\sqrt{\pi}}\mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{m-1}{2}}{1}\right) \tag{13.2.20b}$$

$$\operatorname{erfc}(z) = \frac{2z}{2z^2+1} + \frac{e^{-z^2}}{\sqrt{\pi}} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \tag{13.2.23a}$$

$$w(z) = \frac{-\frac{iz}{\sqrt{\pi}}}{\frac{1}{2}-z^2} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\left(-\frac{3}{2}+m\right)(m-1)}{2m-\frac{3}{2}-z} \right) \tag{13.2.23b}$$

5.2 Error function and Dawson's integral

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)\,k!} \tag{13.1.7}$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{\left(\frac{3}{2}\right)_k} \tag{13.1.8}$$

$$F(z) = e^{-z^2} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)\,k!} \tag{13.1.9}$$

$$F(z) = 1 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\left(\frac{3}{2}\right)_k} \tag{13.1.10}$$

$$\operatorname{erf}(z) = \frac{2z^2}{1} + \frac{1}{\sqrt{\pi} z e^{z^2}} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{-2(m-1)}{(2m-3)(2m-1)} z^2}{1} \right) \tag{13.1.11a}$$

$$F(z) = \frac{-2z^2}{1} + -\frac{1}{2z} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{2(m-1)}{(2m-3)(2m-1)} z^2}{1} \right) \tag{13.1.11b}$$

$$\operatorname{erf}(z) = \frac{2z^2}{1-2z^2} + \frac{1}{\sqrt{\pi} z e^{z^2}} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{4(m-1)}{(2m-3)(2m-1)} z^2}{1-\frac{2}{2m-1} z^2} \right) \tag{13.1.13a}$$

$$F(z) = \frac{-2z^2}{1+2z^2} + -\frac{1}{2z} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\frac{4(m-1)}{(2m-3)(2m-1)} z^2}{1+\frac{2}{2m-1} z^2} \right) \tag{13.1.13b}$$

5.3 Fresnel integrals

$$C(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \frac{\pi}{2}^{2k}}{(2k)!\,(4k+1)} z^{4k+1} \tag{13.4.6a}$$

$$S(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \frac{\pi}{2}^{2k+1}}{(2k+1)!\,(4k+3)} z^{4k+3} \tag{13.4.6b}$$

$$C(z) + i\,S(z) = \frac{z^2}{1} + \frac{e^{\frac{i\pi z^2}{2}}}{z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{i\pi(m-1)}{(2m-3)(2m-1)} z^2}{1} \right) \tag{13.4.9}$$

$$C(z) + i\,S(z) = \frac{z^2}{1 + i\pi z^2} + \frac{e^{\frac{i\pi z^2}{2}}}{z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{-\frac{2i\pi(m-1)}{(2m-3)(2m-1)} z^2}{1 + \frac{i\pi}{2m-1} z^2} \right) \tag{13.4.10}$$

5.4 Repeated integrals

$$I^n \operatorname{erfc}(z) = \frac{\frac{2}{\sqrt{\pi}} e^{-z^2}}{(2z)^{n+1}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (2k+n)!}{n!k!\,(2z)^{2k}} \right) \tag{13.3.2}$$

$$\frac{I^n \operatorname{erfc}(z)}{I^{n-1} \operatorname{erfc}(z)} = \frac{\frac{1}{2}}{z} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{n+m-1}{2}}{z} \right) \tag{13.3.5}$$

6 Exponential integrals and related functions

6.1 Exponential integrals

$$E_1(z) = -1 \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{kk!} \tag{14.1.10}$$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} \left(-\gamma - \ln(z) + \sum_{k=1}^{n-1} \left(\frac{1}{k} \right) \right) - \sum_{k=0}^{n-2} \left(\frac{(-z)^k}{(k-n+1)\,k!} \right) + \sum_{k=0}^{\infty} - \frac{(-z)^{k+n-1}}{(k)\,(k+n-1)!} \tag{14.1.11}$$

$$E_\nu(z) = \Gamma(1-\nu) z^{\nu-1} - \frac{1}{1-\nu} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^k}{(k!)\,(k+1-\nu)} \tag{14.1.12}$$

$$E_\nu(z) = e^{-z} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu)_k}{z^{k+1}} \right) \tag{14.1.13}$$

$$E_n(z) = \frac{1}{z} + \frac{n}{1} + e^{-z} \operatorname*{K}_{m=3}^{\infty} \left(\frac{n + \frac{m-2}{2}}{1} \right) \tag{14.1.16}$$

$$E_\nu(z) = \frac{\frac{e^{-z}}{z}}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{\frac{m}{2}+\nu-1}{z}}{1} \right) \tag{14.1.19}$$

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) + \frac{-e^{-z}}{1-\nu} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\left(\nu - \frac{m}{2}\right) z}{m-\nu} \right) \tag{14.1.20}$$

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) + \frac{1}{1-\nu-z} + -e^{-z}\operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)\,z}{m-\nu-z}\right) \tag{14.1.22}$$

$$E_\nu(z) = \frac{1}{\nu+z} + e^{-z}\operatorname*{K}_{m=2}^{\infty}\left(\frac{-(m-1)\,(\nu+m-2)}{\nu+(m-1)\,2+z}\right) \tag{14.1.23}$$

$$E_\nu(z) = \frac{e^{-z}}{z} + \frac{-e^{-z}\nu}{z\,(1+z)+\nu z} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-(m-1)\,(\nu+m-1)\,z^2}{z\,(m+z)+(\nu+m-1)\,z}\right) \tag{14.1.24}$$

6.2 Related functions

$$E_1(x) = \gamma + \ln(x) + \sum_{k=0}^{\infty}\frac{x^k}{kk!} \tag{14.2.14}$$

$$\mathrm{Ein}(z) = -1, \sum_{k=0}^{\infty}\frac{(-1)^{k+1}\,z^{k+1}}{(k+1)\,(k+1)!} \tag{14.2.16}$$

$$E_1(x) = \frac{e^x}{x}\sum_{k=0}^{\infty}\left(k!x^{-k}\right) \tag{14.2.19}$$

$$E_1(z) = 2\sum_{k=0}^{\infty}\left(\frac{z^{2k+1}}{(2k+1)\,(2k+1)!}\right) + \frac{\frac{1}{z}}{1} + -e^{-z}\operatorname*{K}_{m=2}^{\infty}\left(\frac{\frac{\lfloor\frac{m}{2}\rfloor}{z}}{1}\right) \tag{14.2.21}$$

$$\mathrm{Ein}(z) = \gamma + \ln(z) + \frac{\frac{1}{z}}{1} + e^{-z}\operatorname*{K}_{m=2}^{\infty}\left(\frac{\frac{\lfloor\frac{m}{2}\rfloor}{z}}{1}\right) \tag{14.2.23}$$

$$E_1(x) = \frac{\frac{e^x}{x}}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-\frac{\lfloor\frac{m}{2}\rfloor}{x}}{1}\right) \tag{14.2.24}$$

7 Gamma function and related functions

7.1 Binet function

$$functions : Binet\,(z) = \sum_{k=0}^{\infty}\left(\frac{B_{2(k+1)}}{(2k+1)\,(2k+2)}z^{-2k-1}\right) \tag{12.2.6}$$

7.2 Incomplete gamma functions

$$\gamma(a,z) = z^a \sum_{k=0}^{\infty} \frac{(-z)^k}{(a+k)\,k!} \tag{12.6.7}$$

$$\gamma(a,z) = \frac{z^a e^{-z}}{a} \sum_{k=0}^{\infty} \frac{z^k}{(1+a)_k} \tag{12.6.8}$$

$$\Gamma(a,z) = z^a e^{-z} \sum_{k=0}^{\infty} \left((-1)^k (1-a)_k z^{-k-1} \right) \tag{12.6.10}$$

$$\Gamma(a,z) = \frac{1}{z +} \; z^a e^{-z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{m}{2} - a}{1} \right) \tag{12.6.15}$$

$$\Gamma(a,z) = \frac{\frac{1}{z}}{1 +} \; z^a e^{-z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{m}{2} - az^{-1}}{1} \right) \tag{12.6.17}$$

$$\gamma(a,z) = \frac{\frac{z}{a}}{1 +} \; z^{a-1} e^{-z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{\left(-\frac{a+\frac{m}{2}-1}{(a+m-2)(a+m-1)} \right) z}{1} \right) \tag{12.6.23}$$

$$\Gamma(a,z) = \Gamma(a) + \frac{\frac{z}{a}}{1 +} - \frac{z^a e^{-z}}{z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{\left(-\frac{a+\frac{m}{2}-1}{(a+m-2)(a+m-1)} \right) z}{1} \right) \tag{12.6.24}$$

$$\gamma(a,z) = \frac{1}{a - z +} \; z^a e^{-z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)\,z}{a + (m-1) - z} \right) \tag{12.6.30}$$

$$\Gamma(a,z) = \Gamma(a) + \frac{-1}{a - z +} \; z^a e^{-z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)\,z}{a + (m-1) - z} \right) \tag{12.6.31}$$

$$\Gamma(a,z) = \frac{1}{1 - a + z +} \; z^a e^{-z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{(1-m)\,(m-1-a)}{((2m)-1) - a + z} \right) \tag{12.6.34}$$

$$\Gamma(a,z) = z^{a-1} e^{-z} + \frac{a-1}{2 + z - a +} \; z^{a-1} e^{-z} \operatorname*{K}_{m=2}^{\infty} \left(\frac{(1-m)\,(m-a)}{2m + z - a} \right) \tag{12.6.35}$$

7.3 Polygamma functions

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty} \frac{1}{k} - \frac{1}{z+k-1} \tag{12.3.2a}$$

$$\psi^{(n)}(z) = (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}} \tag{12.3.2b}$$

$$\psi(z) = \ln(z) - \frac{1}{2z} + \sum_{k=0}^{\infty} \left(-\frac{B_{2k}}{2k} z^{-2k} \right) \tag{12.3.7}$$

$$\psi^{(n)}(z) = (-1)^{n-1} \sum_{k=0}^{\infty} \left(\frac{B_{2k}\,(2k+n-1)!}{(2k)!} z^{-2k-n} \right) \tag{12.3.8}$$

7.4 Tetragamma function

$$\psi^{(2)}(z) = -\frac{1}{z^2} - \frac{1}{z^3} + \frac{\frac{1}{8\pi^2}}{z^2} + -\frac{2\pi}{z}^2 \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{\frac{m}{2}^2 \frac{m}{2}+1}{2(m+1)}}{1} \right) \tag{12.5.1}$$

$$\psi^{(2)}(z) = \frac{\frac{1}{z(z-1)}}{1} + -\mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{\frac{m}{2}^4}{z(z-1)}}{m} \right) \tag{12.5.2}$$

$$\psi^{(2)}(z) = -\frac{1}{z} + \frac{\frac{1}{z}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{\frac{m+2}{4}^2 - 2\frac{m+2}{4} + 2}{2\frac{m+2}{4} - 1} z^{-1}}{1} \right) \tag{12.5.3}$$

7.5 Trigamma function

$$\psi^{(1)}(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{\frac{1}{12\pi}}{z^2} + \frac{2\pi}{z} \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{m^2\left(m^2-1\right)}{4(4m^2-1)}}{1} \right) \tag{12.4.1}$$

$$\psi^{(1)}(z) = \frac{z^{-1}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\left(-\frac{\frac{m}{2}^2}{2m-2} \right) z^{-1}}{1} \right) \tag{12.4.2}$$

$$\psi^{(1)}(z) = \frac{1}{-\frac{1}{2}+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{(m-1)^4}{4(2m-3)(2m-1)}}{-\frac{1}{2}+z} \right) \tag{12.4.3}$$

8 Hypergeometric functions

8.1 Hypergeometric functions

$$\frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} = \frac{c+(b-a+1)\,z}{c} + \frac{1}{c} \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-\,(c-a+m)\,(b+m)\,z}{c+m+(b-a+m+1)\,z} \right) \tag{15.3.8}$$

$${}_2F_1(a,1;c+1;z) = \frac{c}{c+(1-a)\,z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\,(m-1)\,z\,(c-a+m-1)}{c+m-1+(m-a)\,z} \right) \tag{15.3.9a}$$

$$ {}_2F_1\left(1-c,1;2-a;\frac{1}{z}\right) = \frac{c}{c+(1-a)\,z} + \frac{(1-a)\,z}{c}\,\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{-\,(m-1)\,z\,(c-a+m-1)}{c+m-1+(m-a)\,z}\right) \tag{15.3.9b} $$

$$ {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};z\right) = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}z} + \underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{-\,(m-1)\,z\,(m-1)}{\frac{1}{2}+m-1+\left(m-\frac{1}{2}\right)z}\right) \tag{15.3.12} $$

$$ \frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a+1,b+1;c+1;z)} = 1-\frac{a+b+1}{c}z+\underset{m=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{\frac{(a+m)(b+m)}{(c+m-1)(c+m)}\left(z-z^2\right)}{1-\frac{a+b+2m+1}{c+m}z}\right) \tag{15.3.13} $$

$$ {}_2F_1(a+1,1;c+1;z) = \frac{c}{c-(a+1)\,z} + \underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{(a+m-1)\,(m-1)\left(z-z^2\right)}{c+m-1-(a+2\,(m-1)+1)\,z}\right) \tag{15.3.14} $$

$$ {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};z\right) = \frac{1}{1-z} + \underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{\frac{m-1}{m-1+\frac{1}{2}}\left(z-z^2\right)}{1-\frac{2(m-1)+\frac{1}{2}}{m-1+\frac{1}{2}}z}\right) \tag{15.3.17} $$

$$ \frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a+1,b,c;d,e;1)} = 1+\frac{-\frac{bc}{d}}{e-a-1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{\frac{\left(a+\frac{m}{2}\right)\left(d-b+\frac{m}{2}-1\right)\left(d-c+\frac{m}{2}-1\right)}{(d+m-2)(d+m-1)}}{1}\right) \tag{15.6.4} $$

$$ \frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a,b,c;d+1,e;1)} = 1+\underset{m=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{-\frac{\frac{m}{2}+d-a\frac{m}{2}+d-b\frac{m}{2}+d-c}{(d+m-1)(d+m)}}{1}\right) \tag{15.6.5} $$

$$ \frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a+1,b,c;d+1,e;1)} = 1+\underset{m=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{\frac{\left(a+\frac{m}{2}\right)\left(d-b+\frac{m}{2}\right)\left(d-c+\frac{m}{2}\right)}{(d+m-1)(d+m)}}{1}\right) \tag{15.6.6} $$

$$ \frac{{}_3F_2(a,b,c;d,e;1)}{{}_3F_2(a,b+1,c+1;d+1,e+1;1)} = \frac{e-(a-1)-1}{e} + \underset{m=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{-\frac{\left((d-1)-(a-1)+\frac{m}{2}\right)\left(b+\frac{m}{2}\right)\left(c+\frac{m}{2}\right)}{\left(e+\frac{m}{2}\right)((d-1)+m)((d-1)+m+1)}}{\frac{e-(a-1)-1}{e+\frac{m}{2}}}\right) \tag{15.6.7} $$

9 q-Hypergeometric function

9.1 q-Hypergeometric function

$$\frac{{}_2\phi_1(a,b;c;q,z)}{{}_2\phi_1(a,bq;cq;q,z)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{\frac{\left(1-bq^{\frac{m}{2}}\right)\left(cq^{\frac{m}{2}}-a\right)q^{\frac{m}{2}-1}}{(1-cq^{m-1})(1-cq^{m})}z}{1}\right) \tag{19.2.1}$$

$${}_2\phi_1(a,q;cq;q,z) = \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{\frac{\left(1-aq^{\frac{m-2}{2}}\right)\left(cq^{\frac{m-2}{2}}-1\right)q^{\frac{m-2}{2}}}{(1-cq^{m-2})(1-cq^{m-1})}z}{1}\right) \tag{19.2.2}$$

$$\frac{{}_2\phi_1(a,b;c;q,z)}{{}_2\phi_1(a,bq;cq;q,z)} = \frac{q\,(1-c)+(a-bq)\,z}{q\,(1-c)} + \frac{1}{q\,(1-c)}\operatorname*{K}_{m=1}^{\infty}\left(\frac{q\,(1-bq^m)\,(cq^m-a)\,z}{q\,(1-cq^m)+(a-bq^{m+1})\,z}\right) \tag{19.2.5a}$$

$$\frac{{}_2\phi_1(a,b;c;q,z)}{{}_2\phi_1(a,bq;cq;q,z)} = 1 + \frac{q\,(1-c)}{(a-bq)\,z} + \frac{1}{(a-bq)\,z}\operatorname*{K}_{m=1}^{\infty}\left(\frac{q\,(1-bq^m)\,(cq^m-a)\,z}{q\,(1-cq^m)+(a-bq^{m+1})\,z}\right) \tag{19.2.5b}$$

$$\frac{{}_2\phi_1(a,b;c;q,z)}{{}_2\phi_1(aq,bq;cq;q,z)} = \frac{1-c-(a+b-ab-abq)\,z}{1-c} + \frac{1}{1-c}\operatorname*{K}_{m=1}^{\infty}\left(\frac{(1-aq^m)\,(1-bq^m)\left(cz-abq^m z^2\right)q^{m-1}}{(1-cq^m)-(a+b-abq^m-abq^{m+1})\,q^m z}\right) \tag{19.2.7}$$

$$\frac{{}_2\phi_1(a,b;c;q,z)}{{}_2\phi_1(aq,b;c;q,z)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{a}{-1}\right) \tag{19.2.10}$$

$$\frac{{}_2\phi_1(a,b;c;q,qz)}{{}_2\phi_1(a,b;c;q,z)} = \frac{(1-z)}{(1+cq^{-1}-(a+b)\,z)} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-\left(cq^{-1}-abq^{m-2}z\right)\left(1-q^{m-1}z\right)}{1+cq^{-1}-(a+b)\,q^{m-1}z}\right) \tag{19.2.11}$$

$$\frac{{}_2\phi_1(a,b;c;q,z)}{{}_2\phi_1(a,b;c;q,qz)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{\left(1-aq^{\frac{m-1}{2}}\right)\left(1-bq^{\frac{m-1}{2}}\right)z}{1-z}\right) \tag{19.2.12}$$

$$\frac{{}_2\phi_1(aq,b;c;q,z)}{{}_2\phi_1(a,b;c;q,qz)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{\left(1-bq^{\frac{m-1}{2}}\right)z}{(1-a)\,(1-z)}\right) \tag{19.2.13}$$

$$\frac{{}_3\phi_2\Big(a,b,c;e,f;q,\frac{ef}{abc}]\Big)}{{}_3\phi_2\Big(a,b,c;eq,f;q,\frac{efq}{abc}]\Big)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{-\frac{\left(1-\frac{eq^{\frac{m}{2}-1+1}}{a}\right)\left(1-\frac{eq^{\frac{m}{2}-1+1}}{b}\right)\left(1-\frac{eq^{\frac{m}{2}-1+1}}{c}\right)fq^{\frac{m}{2}-1}}{\left(1-eq^{2\frac{m}{2}-1+1}\right)\left(1-eq^{2\frac{m}{2}-1+2}\right)\left(1-fq^{\frac{m}{2}-1}\right)}}{1}\right) \tag{19.3.1}$$

$$\frac{{}_3\phi_2\Big(aq,b,c;e,f;q,\frac{ef}{abcq}]\Big)}{{}_3\phi_2\Big(a,b,c;e,f;q,\frac{ef}{abc}]\Big)} = 1 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{a\left(1-\frac{eq^{\frac{m}{2}-1}}{a}\right)\left(1-\frac{fq^{\frac{m}{2}-1}}{a}\right)}{1}\right) \tag{19.3.2}$$

10 Probability functions