1 Bessel functions

1.1 Bessel functions

$$J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, \Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}$$
 (17.1.2a)

$$j_n(z) = \sqrt{\frac{\pi}{2z}} \left(\frac{z}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma((n+\frac{1}{2})+k+1)} \left(\frac{z}{2}\right)^{2k}$$
(17.1.12a)

$$J_{\nu}(z) = \frac{e^{-iz} \left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\left(\nu + \frac{1}{2}\right)_{k} (2i)^{k} z^{k}}{(2\nu+1)_{k} k!}$$
(17.1.22)

$$j_n(z) = \frac{\sqrt{\pi}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{1}{(n+\frac{3}{2})_k} \left(-\frac{z^2}{4}\right)^k}{k!}$$
(17.1.25)

$$j_n(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{(n+1)_k}{(2n+2)_k} (2iz)^k}{k!}$$
(17.1.26)

$$J_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \cos\left(z - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi\right) - \frac{(-1)^k (\nu, 2k+1)}{(2z)^{2k+1}} \sin\left(z - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi\right) \right)$$
(17.1.28)

$$Y_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \sin\left(z - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi\right) + \frac{(-1)^k (\nu, 2k+1)}{(2z)^{2k+1}} \cos\left(z - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi\right) \right)$$

$$(17.1.29)$$

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \frac{\frac{z}{2\nu+2}}{1} + \prod_{m=2}^{\infty} \frac{\frac{(iz)^2}{4(\nu+m-1)(\nu+m)}}{1}$$
(17.1.38)

$$\frac{\mathbf{j}_{n+1}(z)}{\mathbf{j}_n(z)} = \frac{\frac{z}{2n+3}}{1} + K \sum_{m=2}^{\infty} \frac{\frac{(iz)^2}{4(n+\frac{1}{2}+m-1)(n+\frac{1}{2}+m)}}{1}$$
(17.1.39)

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = -\prod_{m=1}^{\infty} \frac{-1}{\frac{2(\nu+m)}{z}}$$
 (17.1.40)

$$\frac{H_{\nu+1}^{(1)}(z)}{H_{\nu}^{(1)}(z)} = \frac{-1}{1} + \prod_{m=2}^{\infty} \frac{\frac{m-3-2\nu}{-2iz}}{1}$$
(17.1.44)

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \frac{z}{2\nu + 2 - iz} + \prod_{m=2}^{\infty} \frac{(2\nu + 2m - 1)iz}{2\nu + m + 1 + (-2i)z}$$
(17.1.48)

$$\frac{\mathbf{j}_{n+1}(z)}{\mathbf{j}_n(z)} = \frac{z}{2n+3-iz} + \prod_{m=2}^{\infty} \frac{2(n+m)iz}{2n+m+2+(-2i)z}$$
(17.1.49)

$$\frac{H_{\nu+1}^{(1)}(z)}{H_{\nu}^{(1)}(z)} = \frac{2\nu + 1 - 2iz}{2z} - \frac{1}{z} \prod_{m=1}^{\infty} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(iz - m)}$$
(17.1.51)

1.2 Modified Bessel functions

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \, \Gamma(\nu+k+1)}$$
 (17.2.20)

$$I_{\nu}(z) = \frac{e^{-z} \left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\left(\nu+\frac{1}{2}\right)_{k} 2^{k} z^{k}}{(2\nu+1)_{k} k!}$$
(17.2.21)

$$i_n^{(1)}(z) = \frac{\sqrt{\pi}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!(n+\frac{3}{2})_k}$$
(17.2.22)

$$\mathbf{i}_{n}^{(1)}(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^{n} \sum_{k=0}^{\infty} \frac{(n+1)_{k}(2z)^{k}}{k!(2n+2)_{k}}$$
(17.2.23)

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z + \frac{(2\nu + 1)i\pi}{2}} \right) (\nu, k)}{\sqrt{2\pi z} (2z)^k} \right)$$
(17.2.24)

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z - \frac{(2\nu+1)i\pi}{2}} \right) (\nu, k)}{\sqrt{2\pi z} (2z)^k} \right)$$
(17.2.25)

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{\infty} \left(\frac{(\nu, k)}{(-2z)^k} \right)$$
 (17.2.27)

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{\frac{z}{2(\nu+1)}}{1} + K \sum_{m=2}^{\infty} \frac{\frac{1}{4(\nu+m-1)(\nu+m)}z^2}{1}$$
(17.2.32)

$$\frac{\mathbf{i}_{n+1}^{(1)}(z)}{\mathbf{i}_{n}^{(1)}(z)} = \frac{\frac{z}{2n+3}}{1} + K \sum_{m=2}^{\infty} \frac{\frac{1}{4((n+\frac{1}{2})+m-1)((n+\frac{1}{2})+m)}z^{2}}{1}$$
(17.2.33)

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{1}{1} + \frac{\frac{-2\nu - 1}{2z}}{1} - \prod_{m=3}^{\infty} \frac{\frac{\frac{m}{2} + \nu}{2z}}{1}$$
(17.2.34)

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{z}{2\nu+2+z} + \prod_{m=2}^{\infty} \frac{-(2\nu+2m-1)z}{2\nu+m+1+2z}$$
(17.2.38)

$$\frac{\mathbf{i}_{n+1}^{(1)}(z)}{\mathbf{i}_{n}^{(1)}(z)} = \frac{z}{2n+3+z} + \prod_{m=2}^{\infty} \frac{-2(n+m)z}{2n+m+2+2z}$$
(17.2.39)

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{\nu}{z} - \frac{2\nu + 1 + 2z}{2z} - \frac{1}{z} \prod_{m=1}^{\infty} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(z+m)}$$
(17.2.40)

2 Confluent hypergeometric functions

2.1 Confluent hypergeometric series $_2F_0$

$$\frac{{}_{2}F_{0}(a,b;;z)}{{}_{2}F_{0}(a,b+1;;z)} = 1 + \mathop{K}\limits_{m=1}^{\infty} \frac{-\left(b + \frac{m}{2}\right)z}{1}$$
(16.2.4)

2.2 Confluent hypergeometric limit function

$$_{0}F_{1}(;b;z) = \sum_{k=0}^{\infty} \frac{z^{k}}{(b)_{k}k!}$$
 (16.3.1)

$$\frac{{}_{0}F_{1}(;b;z)}{{}_{0}F_{1}(;b+1;z)} = 1 + \prod_{m=1}^{\infty} \frac{\frac{1}{(b+m-1)(b+m)}z}{1}$$
(16.3.4)

$$\frac{{}_{0}F_{1}(;b;z)}{{}_{0}F_{1}(;b+1;z)} = 1 + \frac{\sqrt{z}}{b} + \frac{1}{2b} \prod_{m=1}^{\infty} \frac{-2(2b+2m-1)\sqrt{z}}{2b+m+4\sqrt{z}}$$
(16.3.6)

2.3 Kummer functions

$$_{1}F_{1}(a;b;z) = \sum_{k=0}^{\infty} \frac{\frac{(a)_{k}}{(b)_{k}} z^{k}}{k!}$$
 (16.1.2)

$$_{2}F_{0}(a,b;;z) = \sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}z^{k}}{k!}$$
 (16.1.12)

$$\frac{{}_{1}F_{1}(a;b;z)}{{}_{1}F_{1}(a+1;b+1;z)} = 1 + \prod_{m=1}^{\infty} \frac{\frac{z(a+\frac{m}{2})}{(b+m-1)(b+m)}}{1}$$
(16.1.13)

$$z_{1}F_{1}(1;b+1;z) = \frac{z}{1} + K \frac{\sum_{m=2}^{\infty} \frac{-\frac{z\left(b + \frac{m}{2} - 1\right)}{(b+m-2)(b+m-1)}}{1}}{1}$$
(16.1.14)

$$\frac{{}_{1}F_{1}(a;b;z)}{{}_{1}F_{1}(a+1;b+1;z)} = \frac{b-z}{b} + \frac{1}{b} \prod_{m=1}^{\infty} \frac{(a+m)z}{b+m-z}$$
(16.1.16)

$$_{1}F_{1}(1;b+1;z) = \frac{b}{b-z} + \prod_{m=2}^{\infty} \frac{(m-1)z}{b+m-1-z}$$
 (16.1.17)

$$\frac{U(a,b,z)}{U(a+1,b,z)} = 2a - b + 2 + z - \prod_{m=1}^{\infty} \frac{(a+m)(b-a-m-1)}{b-2a-2m-2-z}$$
(16.1.20)

2.4 Parabolic cylinder functions

$$\frac{U(a,x)}{U(a-1,x)} = \frac{1}{x} + \prod_{m=2}^{\infty} \frac{a + (m-1) - \frac{1}{2}}{x}$$
 (16.5.7)

2.5 Whittaker functions

$$W_{\kappa,\mu}(z) = e^{-z^2} z^{\kappa} \sum_{k=0}^{\infty} \left(\frac{\left(-\kappa - \mu + \frac{1}{2}\right)_k \left(-\kappa + \mu + \frac{1}{2}\right)_k \left(-z\right)^{-k}}{k!} \right)$$
(16.4.7)

$$functions: WhittakerPsi\left(\alpha,\beta,z\right) = \sum_{k=0}^{\infty} \left(\frac{\left(\alpha + \frac{1}{2}\right)_k \left(\beta + \frac{1}{2},k\right) z^{-k-1}}{k!}\right) \tag{16.4.12}$$

3 Mathematical constants

3.1 Apéry's constant, $\zeta(3)$

$$\zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^z}$$
 (10.11.1)

$$\zeta(3) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{(k!)^{10} \left(205k^2 + 250k + 77 \right)}{64 \left((2k+1)! \right)^5} \right)$$
 (10.11.3)

$$\zeta(3) = \frac{6}{5} + \prod_{m=2}^{\infty} \frac{-(m-1)^6}{34(m-1)^3 + 51(m-1)^2 + 27(m-1) + 5}$$
(10.11.5)

$$\zeta(3) = \frac{1}{22} + \sum_{m=2}^{\infty} \frac{\left(\frac{m}{2}\right)^3}{1} \tag{}$$

$$\zeta(3) = \frac{1}{2k^2 + 2k + 1} + \prod_{m=2}^{\infty} \frac{-(m-1)^6}{(m-1)^3 + m^3 + (2m-1)(2k^2 + 2k)} \tag{}$$

$$\zeta(3) = \frac{1}{5} + K \sum_{m=2}^{\infty} \frac{-(m-1)^6}{2(m-1)^3 + 3(m-1)^2 + 11(m-1) + 5}$$
 ()

$$\zeta(3) = \frac{1}{4} + \frac{1}{1} + \frac{1}{18} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1$$

3.2 Archimedes' constant, symbol π

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tag{10.2.1}$$

$$\pi = \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{14} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{84} + \frac{1}{2} + \frac{1}{1} + \frac{1}{15} + \frac{1}{3} + \frac{1}{15} + \frac{1}{$$

$$\pi = \frac{4}{1} + \prod_{m=2}^{\infty} \frac{(m-1)^2}{2m-1}$$
 (10.2.5)

$$\pi = 3 + \prod_{m=1}^{\infty} \frac{(2m-1)^2}{6}$$
 (10.2.6)

$$P = \frac{1}{1} + 2 \sum_{m=2}^{\infty} \frac{(m-1)m}{1} \tag{}$$

$$\pi = \frac{4}{4k+1} + 2^{4k} \left(\frac{(k!)^2}{2k!} \right)^2 \prod_{m=2}^{\infty} \frac{(2m-3)^2}{8k+2}$$
 ()

$$\pi = \frac{4}{1} + \prod_{m=2}^{\infty} \frac{(2m-3)^2}{2} \tag{}$$

$$\pi = \frac{4}{4+1} + 4 \int_{-\infty}^{\infty} \frac{(2m-3)^2}{8+2} \tag{}$$

3.3 Catalan's constant, symbol C

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$
 (10.12.1)

$$G = \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{88} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{7} + \frac{1}{22} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{26} + \frac{1}{1} + \frac{1}{11} + \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \frac{1}{9} + \frac{1}{3} + \frac{1}{1} + \frac{1}$$

$$G = \frac{1}{\frac{1}{2}} + \frac{1}{2} \prod_{m=2}^{\infty} \frac{\left(\frac{m}{2}\right)^2}{\frac{1}{2}}$$
 (10.12.3)

$$G = \frac{\frac{13}{2}}{7} + K \frac{(2m-3)^4 (2m-2)^4 (20 (m-2)^2 - 8 (m-2) + 1) (20 (m)^2 - 8 (m) + 1)}{3520 (m-1)^6 + 5632 (m-1)^5 + 2064 (m-1)^4 - 384 (m-1)^3 - 156 (m-1)^2 + 16 (m-1) + 7}$$
(10.12.5)

$$G = \frac{-1}{3} + \frac{1}{2} \sum_{m=2}^{\infty} \frac{m^2}{1} \tag{}$$

3.4 Euler's constant, symbol γ

$$\gamma = \sum_{k=0}^{\infty} \frac{1}{k} \tag{10.8.1}$$

$$\gamma = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{4} + \frac{1}{3} + \frac{1}{13} + \frac{1}{5} + \frac{1}{1} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{14} + \frac{1}{11} + \frac{1}{3} + \frac{1}{7} + \frac{1}{1} + \frac{1}{7} + \frac{1}{1} + \frac{1}{1}$$

3.5 Euler's number, base of the natural logarithm

$$e^1 = \sum_{k=0}^{\infty} \frac{1}{k!}$$
 (10.3.1b)

$$e^1 = 2 + \prod_{m=1}^{\infty} \frac{1}{1} \tag{10.3.5}$$

$$\frac{e^1 - 1}{e^1 + 1} = \prod_{m=1}^{\infty} \frac{1}{(4m - 2)}$$
 (10.3.6)

$$e^{1} = 2 + \prod_{m=1}^{\infty} \frac{m+1}{m+1} \tag{}$$

$$e^{1} = \frac{2}{1} + K \int_{m=2}^{\infty} \frac{1}{6 + (m-2)4}$$
 ()

$$e^{1} = \frac{1}{1} + \sum_{m=2}^{\infty} \frac{-1}{m-1} \tag{}$$

4 Elementary functions

- 5 Error function and related integrals
- 6 Exponential integrals and related functions
- 7 Gamma function and related functions
- 8 Hypergeometric functions
- 9 q-Hypergeometric function
- 10 Probability functions