

1 Bessel functions

1.1 Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad (17.1.2a)$$

$$j_n(z) = \sqrt{\frac{\pi}{2z}} \left(\frac{z}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma((n + \frac{1}{2}) + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad (17.1.12a)$$

$$J_\nu(z) = \frac{e^{-iz} \left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{(\nu + \frac{1}{2})_k (2i)^k z^k}{(2\nu + 1)_k k!} \quad (17.1.22)$$

$$j_n(z) = \frac{\sqrt{\pi}}{(2n + 1) \Gamma(n + \frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{1}{(n + \frac{3}{2})_k} \left(-\frac{z^2}{4}\right)^k}{k!} \quad (17.1.25)$$

$$j_n(z) = \frac{\sqrt{\pi} e^{-iz}}{(2n + 1) \Gamma(n + \frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{(n+1)_k}{(2n+2)_k} (2iz)^k}{k!} \quad (17.1.26)$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \cos \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) - \frac{(-1)^k (\nu, 2k + 1)}{(2z)^{2k+1}} \sin \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) \right) \quad (17.1.28)$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \sin \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) + \frac{(-1)^k (\nu, 2k + 1)}{(2z)^{2k+1}} \cos \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) \right) \quad (17.1.29)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{z}{2\nu+2} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{(iz)^2}{4(\nu+m-1)(\nu+m)}}{1} \quad (17.1.38)$$

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{z}{2n+3} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{(iz)^2}{4(n+\frac{1}{2}+m-1)(n+\frac{1}{2}+m)}}{1} \quad (17.1.39)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = - \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{-1}{\frac{2(\nu+m)}{z}} \quad (17.1.40)$$

$$\frac{H_{\nu+1}^{(1)}(z)}{H_\nu^{(1)}(z)} = \frac{-1}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{m-3-2\nu}{-2iz}}{1} \quad (17.1.44)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{z}{2\nu+2-iz} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(2\nu+2m-1)iz}{2\nu+m+1+(-2i)z} \quad (17.1.48)$$

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{z}{2n+3-iz} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{2(n+m)iz}{2n+m+2+(-2i)z} \quad (17.1.49)$$

$$\frac{H_{\nu+1}^{(1)}(z)}{H_{\nu}^{(1)}(z)} = \frac{2\nu+1-2iz}{2z} + -\frac{1}{z} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(iz-m)} \quad (17.1.51)$$

1.2 Modified Bessel functions

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \Gamma(\nu+k+1)} \quad (17.2.20)$$

$$I_{\nu}(z) = \frac{e^{-z} \left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(\nu+\frac{1}{2})_k 2^k z^k}{(2\nu+1)_k k!} \quad (17.2.21)$$

$$i_n^{(1)}(z) = \frac{\sqrt{\pi}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!(n+\frac{3}{2})_k} \quad (17.2.22)$$

$$i_n^{(1)}(z) = \frac{\sqrt{\pi} e^{-iz}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(n+1)_k (2z)^k}{k!(2n+2)_k} \quad (17.2.23)$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z+\frac{(2\nu+1)i\pi}{2}}\right) (\nu, k)}{\sqrt{2\pi z} (2z)^k} \right) \quad (17.2.24)$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z-\frac{(2\nu+1)i\pi}{2}}\right) (\nu, k)}{\sqrt{2\pi z} (2z)^k} \right) \quad (17.2.25)$$

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{\infty} \left(\frac{(\nu, k)}{(-2z)^k} \right) \quad (17.2.27)$$

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{\frac{z}{2(\nu+1)}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{1}{4(\nu+m-1)(\nu+m)} z^2}{1} \quad (17.2.32)$$

$$\frac{i_{n+1}^{(1)}(z)}{i_n^{(1)}(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{1}{4((n+\frac{1}{2})+m-1)((n+\frac{1}{2})+m)} z^2}{1} \quad (17.2.33)$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{1}{1} + \frac{\frac{-2\nu-1}{2z}}{1} + -\mathop{\mathrm{K}}\limits_{m=3}^{\infty} \frac{\frac{\frac{m}{2}+\nu}{2z}}{1} \quad (17.2.34)$$

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{z}{2\nu+2+z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{-(2\nu+2m-1)z}{2\nu+m+1+2z} \quad (17.2.38)$$

$$\frac{i_{n+1}^{(1)}(z)}{i_n^{(1)}(z)} = \frac{z}{2n+3+z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{-2(n+m)z}{2n+m+2+2z} \quad (17.2.39)$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{\nu}{z} - \frac{2\nu+1+2z}{2z} + \frac{1}{z} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(z+m)} \quad (17.2.40)$$

2 Confluent hypergeometric functions

2.1 Confluent hypergeometric series ${}_2F_0$

$$\frac{{}_2F_0(a, b; ; z)}{{}_2F_0(a, b+1; ; z)} = 1 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{-(b+\frac{m}{2})z}{1} \quad (16.2.4)$$

2.2 Confluent hypergeometric limit function

$${}_2F_0(; b; z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!} \quad (16.3.1)$$

$$\frac{{}_2F_0(; b; z)}{{}_2F_0(; b+1; z)} = 1 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{\frac{1}{(b+m-1)(b+m)}z}{1} \quad (16.3.4)$$

2.3 Kummer functions

$${}_2F_0(a; b; z) = \sum_{k=0}^{\infty} \frac{\frac{(a)_k}{(b)_k} z^k}{k!} \quad (17.1.2)$$

$${}_2F_0(a, b; ; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{k!} \quad (17.1.12)$$

$$\frac{{}_2F_0(a; b; z)}{{}_2F_0(a+1; b+1; z)} = 1 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{\frac{z(a+\frac{m}{2})}{(b+m-1)(b+m)}}{1} \quad (17.1.13)$$

$$z {}_2F_0(1; b+1; z) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{\frac{z(b+\frac{m}{2}-1)}{(b+m-2)(b+m-1)}}{1} \quad (17.1.14)$$

$$\frac{{}_2F_0(a; b; z)}{{}_2F_0(a+1; b+1; z)} = \frac{b-z}{b} + \frac{1}{b} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{(a+m)z}{b+m-z} \quad (17.1.16)$$

$${}_2F_0(1; b+1; z) = \frac{b}{b-z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(m-1)z}{b+m-1-z} \quad (17.1.17)$$

$$\frac{U(a, b, z)}{U(a+1, b, z)} = 2a - b + 2 + z + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{(a+m)(b-a-m-1)}{b-2a-2m-2-z} \quad (17.1.20)$$

2.4 Parabolic cylinder functions

$$\frac{U(a, x)}{U(a-1, x)} = \frac{1}{x} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{a + (m-1) - \frac{1}{2}}{x} \quad (16.5.7)$$

2.5 Whittaker functions

$$W_{\kappa, \mu}(z) = e^{-z^2} z^{\kappa} \sum_{k=0}^{\infty} \left(\frac{(-\kappa - \mu + \frac{1}{2})_k (-\kappa + \mu + \frac{1}{2})_k (-z)^{-k}}{k!} \right) \quad (16.4.7)$$

$$functions : WhittakerPsi(\alpha, \beta, z) = \sum_{k=0}^{\infty} \left(\frac{(\alpha + \frac{1}{2})_k (\beta + \frac{1}{2}, k) z^{-k-1}}{k!} \right) \quad (16.4.12)$$

3 Mathematical constants

3.1 Apéry's constant, $\zeta(3)$

$$\zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^z} \quad (10.11.1)$$

$$\zeta(3) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{(k!)^{10} (205k^2 + 250k + 77)}{64 ((2k+1)!)^5} \right) \quad (10.11.3)$$

$$\zeta(3) = \frac{6}{5} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{-(m-1)^6}{34(m-1)^3 + 51(m-1)^2 + 27(m-1) + 5} \quad (10.11.5)$$

3.2 Archimedes' constant, symbol π

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad (10.2.1)$$

$$\pi = \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{14} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + \frac{1}{84} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{15} + \frac{1}{3} + \frac{1}{13} + \frac{1}{1} + \frac{1}{4} + \frac{1}{2} \quad (10.2.4)$$

$$\pi = \frac{4}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \frac{(m-1)^2}{2m-1} \quad (10.2.5)$$

$$\pi = 3 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \frac{(2m-1)^2}{6} \quad (10.2.6)$$

3.3 Catalan's constant, symbol C

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \quad (10.12.1)$$

$$G = \frac{1}{\frac{1}{2}} + \frac{1}{2} \prod_{m=2}^{\infty} \frac{\left(\frac{m}{2}\right)^2}{\frac{1}{2}} \quad (10.12.3)$$

$$G = \frac{\frac{13}{2}}{7} + \prod_{m=2}^{\infty} \frac{(2m-3)^4 (2m-2)^4 \left(20(m-2)^2 - 8(m-2) + 1\right) \left(20(m)^2 - 8(m) + 1\right)}{3520(m-1)^6 + 5632(m-1)^5 + 2064(m-1)^4 - 384(m-1)^3 - 156(m-1)^2 + 16(m-1) + 7} \quad (10.12.5)$$

3.4 Euler's constant, symbol γ

$$\gamma = \sum_{k=0}^{\infty} \frac{1}{k} \quad (10.8.1)$$

3.5 Euler's number, base of the natural logarithm

$$e^1 = \sum_{k=0}^{\infty} \frac{1}{k!} \quad (10.3.1b)$$

$$e^1 = 2 + \prod_{m=1}^{\infty} \frac{1}{1} \quad (10.3.5)$$

$$\frac{e^1 - 1}{e^1 + 1} = \prod_{m=1}^{\infty} \frac{1}{(4m-2)} \quad (10.3.6)$$

4 Elementary functions

5 Error function and related integrals

6 Exponential integrals and related functions

7 Gamma function and related functions

8 Hypergeometric functions

9 q-Hypergeometric function

10 Probability functions