1 Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k} \tag{17.1.2a}$$

$$\mathrm{j}_n(z) = \sqrt{\frac{\pi}{2z}} \left(\frac{z}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma\left(\left(n+\frac{1}{2}\right)+k+1\right)} \left(\frac{z}{2}\right)^{2k} \tag{17.1.12a}$$

$$J_\nu(z) = \frac{e^{-iz}\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\left(\nu+\frac{1}{2}\right)_k (2i)^k z^k}{(2\nu+1)_k k!} \tag{17.1.22}$$

$$\mathrm{j}_n(z) = \frac{\sqrt{\pi}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{1}{\left(n+\frac{3}{2}\right)_k}\left(-\frac{z^2}{4}\right)^k}{k!} \tag{17.1.25}$$

$$\mathrm{j}_n(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{(n+1)_k}{(2n+2)_k}(2iz)^k}{k!} \tag{17.1.26}$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu,2k)}{(2z)^{2k}} \cos\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right) - \frac{(-1)^k (\nu,2k+1)}{(2z)^{2k+1}} \sin\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right)\right) \tag{17.1.28}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu,2k)}{(2z)^{2k}} \sin\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right) + \frac{(-1)^k (\nu,2k+1)}{(2z)^{2k+1}} \cos\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right)\right) \tag{17.1.29}$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{\frac{z}{2\nu+2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{\frac{(iz)^2}{4(\nu+m-1)(\nu+m)}}{1} \tag{17.1.38}$$

$$\frac{\mathrm{j}_{n+1}(z)}{\mathrm{j}_n(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{\frac{(iz)^2}{4\left(n+\frac{1}{2}+m-1\right)\left(n+\frac{1}{2}+m\right)}}{1} \tag{17.1.39}$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = -\mathop{\mathrm{K}}_{m=1}^{\infty} \frac{-1}{\frac{2(\nu+m)}{z}} \tag{17.1.40}$$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{-1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{\frac{m-3-2\nu}{-2iz}}{1} \tag{17.1.44}$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{z}{2\nu+2-iz} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{(2\nu+2m-1)\,iz}{2\nu+m+1-(2i)\,z} \tag{17.1.48}$$

$$\frac{\mathsf{j}_{n+1}(z)}{\mathsf{j}_n(z)} = \frac{z}{2n+3-iz} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \frac{2\,(n+m)\,iz}{2n+m+2-(2i)\,z} \tag{17.1.49}$$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_{\nu}(z)} = \frac{2\nu+1-2iz}{2z} - \frac{1}{z}\underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2\,(iz-m)} \tag{17.1.51}$$

2 Modified Bessel functions

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(\nu+k+1)} \tag{17.2.20}$$

$$I_\nu(z) = \frac{e^{-z}\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\left(\nu+\frac{1}{2}\right)_k 2^k z^k}{(2\nu+1)_k k!} \tag{17.2.21}$$

$$\mathsf{i}^{(1)}_n(z) = \frac{\sqrt{\pi}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\left(n+\frac{3}{2}\right)_k} \tag{17.2.22}$$

$$\mathsf{i}^{(1)}_n(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(n+1)_k\,(2z)^k}{k!(2n+2)_k} \tag{17.2.23}$$

$$I_\nu(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z+\frac{(2\nu+1)i\pi}{2}}\right)(\nu,k)}{\sqrt{2\pi z}\,(2z)^k} \right) \tag{17.2.24}$$

$$I_\nu(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z-\frac{(2\nu+1)i\pi}{2}}\right)(\nu,k)}{\sqrt{2\pi z}\,(2z)^k} \right) \tag{17.2.25}$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{\infty} \left(\frac{(\nu,k)}{(-2z)^k} \right) \tag{17.2.27}$$

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} = \frac{\frac{z}{2(\nu+1)}}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \frac{\frac{1}{4(\nu+m-1)(\nu+m)} z^2}{1} \tag{17.2.32}$$

$$\frac{\mathsf{i}^{(1)}_{n+1}(z)}{\mathsf{i}^{(1)}_n(z)} = \frac{\frac{z}{2n+3}}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \frac{\frac{1}{4\left(\left(n+\frac{1}{2}\right)+m-1\right)\left(\left(n+\frac{1}{2}\right)+m\right)} z^2}{1} \tag{17.2.33}$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{1}{1} + \frac{\frac{-2\nu-1}{2z}}{1} - \underset{m=3}{\overset{\infty}{\mathrm{K}}} \frac{\frac{\frac{m}{2}+\nu}{2z}}{1} \tag{17.2.34}$$

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} = \frac{z}{2\nu+2+z} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \frac{-\,(2\nu+2m-1)\,z}{2\nu+m+1+2z} \tag{17.2.38}$$

$$\frac{\mathsf{i}_{n+1}^{(1)}(z)}{\mathsf{i}_n^{(1)}(z)} = \frac{z}{2n+3+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{-2\,(n+m)\,z}{2n+m+2+2z} \tag{17.2.39}$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{\nu}{z} - \frac{2\nu+1+2z}{2z} - \frac{1}{z} \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{\nu^2 - \frac{(2m-1)^2}{4}}{2\,(z+m)} \tag{17.2.40}$$

3 Confluent hypergeometric series ${}_2F_0$

$$\frac{{}_2F_0(a,b;;z)}{{}_2F_0(a,b+1;;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{-\left(b+\frac{m}{2}\right)z}{1} \tag{16.2.4}$$

4 Confluent hypergeometric series ${}_2F_0$

$$\frac{{}_2F_0(a,b;;z)}{{}_2F_0(a,b+1;;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{-\left(b+\frac{m}{2}\right)z}{1} \tag{16.2.4}$$

5 Confluent hypergeometric limit function

$${}_2F_0(;b;z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!} \tag{16.3.1}$$

$$\frac{{}_2F_0(;b;z)}{{}_2F_0(;b+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{\frac{1}{(b+m-1)(b+m)}z}{1} \tag{16.3.4}$$

6 Kummer functions

$${}_2F_0(a;b;z) = \sum_{k=0}^{\infty} \frac{\frac{(a)_k}{(b)_k} z^k}{k!} \tag{17.1.2}$$

$${}_2F_0(a,b;;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{k!} \tag{17.1.12}$$

$$\frac{{}_2F_0(a;b;z)}{{}_2F_0(a+1;b+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{\frac{z\left(a+\frac{m}{2}\right)}{(b+m-1)(b+m)}}{1} \tag{17.1.13}$$

$$z\,{}_2F_0(1;b+1;z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{-\frac{z\left(b+\frac{m}{2}-1\right)}{(b+m-2)(b+m-1)}}{1} \tag{17.1.14}$$

$$\frac{{}_2F_0(a;b;z)}{{}_2F_0(a+1;b+1;z)} = \frac{b-z}{b} + \frac{1}{b} \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{(a+m)\,z}{b+m-z} \tag{17.1.16}$$

$$ {}_2F_0(1;b+1;z) = \frac{b}{b-z} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \frac{(m-1)\,z}{b+m-1-z} \tag{17.1.17} $$

$$ \frac{U(a,b,z)}{U(a+1,b,z)} = 2a-b+2+z - \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{(a+m)\,(b-a-m-1)}{b-2a-2m-2-z} \tag{17.1.20} $$

7 Parabolic cylinder functions

$$ \frac{U(a,x)}{U(a-1,x)} = \frac{1}{x} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \frac{a+(m-1)-\frac{1}{2}}{x} \tag{16.5.7} $$

8 Whittaker functions

$$ W_{\kappa,\mu}(z) = e^{-z^2} z^{\kappa} \sum_{k=0}^{\infty} \left(\frac{\left(-\kappa-\mu+\frac{1}{2}\right)_k \left(-\kappa+\mu+\frac{1}{2}\right)_k (-z)^{-k}}{k!} \right) \tag{16.4.7} $$

$$ functions : WhittakerPsi\,(\alpha,\beta,z) = \sum_{k=0}^{\infty} \left(\frac{\left(\alpha+\frac{1}{2}\right)_k \left(\beta+\frac{1}{2},k\right) z^{-k-1}}{k!} \right) \tag{16.4.12} $$

9 Apéry's constant, $\zeta(3)$

$$ \zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^z} \tag{10.11.1} $$

$$ \zeta(3) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{(k!)^{10} \left(205k^2+250k+77\right)}{64\left((2k+1)!\right)^5} \right) \tag{10.11.3} $$

$$ \zeta(3) = \frac{6}{5} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \frac{-(m-1)^6}{34\,(m-1)^3+51\,(m-1)^2+27\,(m-1)+5} \tag{10.11.5} $$