1 Bessel functions

1.1 Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k} \tag{17.1.2a}$$

$$\mathrm{j}_n(z) = \sqrt{\frac{\pi}{2z}} \left(\frac{z}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma\left(\left(n+\frac{1}{2}\right)+k+1\right)} \left(\frac{z}{2}\right)^{2k} \tag{17.1.12a}$$

$$J_\nu(z) = \frac{e^{-iz}\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\left(\nu+\frac{1}{2}\right)_k (2i)^k z^k}{(2\nu+1)_k k!} \tag{17.1.22}$$

$$\mathrm{j}_n(z) = \frac{\sqrt{\pi}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{1}{\left(n+\frac{3}{2}\right)_k}\left(-\frac{z^2}{4}\right)^k}{k!} \tag{17.1.25}$$

$$\mathrm{j}_n(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{(n+1)_k}{(2n+2)_k}(2iz)^k}{k!} \tag{17.1.26}$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k(\nu,2k)}{(2z)^{2k}} \cos\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right) - \frac{(-1)^k(\nu,2k+1)}{(2z)^{2k+1}} \sin\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right)\right) \tag{17.1.28}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k(\nu,2k)}{(2z)^{2k}} \sin\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right) + \frac{(-1)^k(\nu,2k+1)}{(2z)^{2k+1}} \cos\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right)\right) \tag{17.1.29}$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{\frac{z}{2\nu+2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{(iz)^2}{4(\nu+m-1)(\nu+m)}}{1} \right) \tag{17.1.38}$$

$$\frac{\mathrm{j}_{n+1}(z)}{\mathrm{j}_n(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{(iz)^2}{4\left(n+\frac{1}{2}+m-1\right)\left(n+\frac{1}{2}+m\right)}}{1} \right) \tag{17.1.39}$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = -\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-1}{\frac{2(\nu+m)}{z}} \right) \tag{17.1.40}$$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_\nu(z)} = \frac{-1}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{m-3-2\nu}{-2iz}}{1} \right) \tag{17.1.44}$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{z}{2\nu+2-iz} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{(2\nu+2m-1)\,iz}{2\nu+m+1+(-2i)\,z}\right) \tag{17.1.48}$$

$$\frac{\mathsf{j}_{n+1}(z)}{\mathsf{j}_n(z)} = \frac{z}{2n+3-iz} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{2\,(n+m)\,iz}{2n+m+2+(-2i)\,z}\right) \tag{17.1.49}$$

$$\frac{H^{(1)}_{\nu+1}(z)}{H^{(1)}_\nu(z)} = \frac{2\nu+1-2iz}{2z} - \frac{1}{z}\mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{\nu^2-\frac{(2m-1)^2}{4}}{2\,(iz-m)}\right) \tag{17.1.51}$$

1.2 Modified Bessel functions

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!\,\Gamma(\nu+k+1)} \tag{17.2.20}$$

$$I_\nu(z) = \frac{e^{-z}\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\left(\nu+\frac{1}{2}\right)_k 2^k z^k}{(2\nu+1)_k k!} \tag{17.2.21}$$

$$\mathsf{i}^{(1)}_n(z) = \frac{\sqrt{\pi}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\left(n+\frac{3}{2}\right)_k} \tag{17.2.22}$$

$$\mathsf{i}^{(1)}_n(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(n+1)_k\,(2z)^k}{k!(2n+2)_k} \tag{17.2.23}$$

$$I_\nu(z) = \sum_{k=0}^{\infty}\left(\frac{\left((-1)^k e^z + e^{-z+\frac{(2\nu+1)i\pi}{2}}\right)(\nu,k)}{\sqrt{2\pi z}\,(2z)^k}\right) \tag{17.2.24}$$

$$I_\nu(z) = \sum_{k=0}^{\infty}\left(\frac{\left((-1)^k e^z + e^{-z-\frac{(2\nu+1)i\pi}{2}}\right)(\nu,k)}{\sqrt{2\pi z}\,(2z)^k}\right) \tag{17.2.25}$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{\infty}\left(\frac{(\nu,k)}{(-2z)^k}\right) \tag{17.2.27}$$

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} = \frac{\frac{z}{2(\nu+1)}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{1}{4(\nu+m-1)(\nu+m)}z^2}{1}\right) \tag{17.2.32}$$

$$\frac{\mathsf{i}^{(1)}_{n+1}(z)}{\mathsf{i}^{(1)}_n(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty}\left(\frac{\frac{1}{4\left(\left(n+\frac{1}{2}\right)+m-1\right)\left(\left(n+\frac{1}{2}\right)+m\right)}z^2}{1}\right) \tag{17.2.33}$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{\nu}{z} + \frac{1}{1} + \frac{\frac{-2\nu-1}{2z}}{1} - \mathop{\mathrm{K}}_{m=3}^{\infty}\left(\frac{\frac{\frac{m}{2}+\nu}{2z}}{1}\right) \tag{17.2.34}$$

$$\frac{I_{\nu+1}(z)}{I_\nu(z)} = \frac{z}{2\nu+2+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-(2\nu+2m-1)\,z}{2\nu+m+1+2z} \right) \tag{17.2.38}$$

$$\frac{\mathsf{i}^{(1)}_{n+1}(z)}{\mathsf{i}^{(1)}_{n}(z)} = \frac{z}{2n+3+z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-2\,(n+m)\,z}{2n+m+2+2z} \right) \tag{17.2.39}$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_\nu(z)} = \frac{\nu}{z} - \frac{2\nu+1+2z}{2z} - \frac{1}{z}\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{\nu^2 - \frac{(2m-1)^2}{4}}{2\,(z+m)} \right) \tag{17.2.40}$$

2 Confluent hypergeometric functions

2.1 Confluent hypergeometric series ${}_2F_0$

$$\frac{{}_2F_0(a,b;;z)}{{}_2F_0(a,b+1;;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-\left(b+\frac{m}{2}\right)z}{1} \right) \tag{16.2.4}$$

2.2 Confluent hypergeometric limit function

$${}_0F_1(;b;z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!} \tag{16.3.1}$$

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{\frac{1}{(b+m-1)(b+m)}z}{1} \right) \tag{16.3.4}$$

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \frac{\sqrt{z}}{b} + \frac{1}{2b}\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{-2\,(2b+2m-1)\sqrt{z}}{2b+m+4\sqrt{z}} \right) \tag{16.3.6}$$

2.3 Kummer functions

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{\frac{(a)_k}{(b)_k} z^k}{k!} \tag{16.1.2}$$

$${}_2F_0(a,b;;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{k!} \tag{16.1.12}$$

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a+1;b+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{\frac{z\left(a+\frac{m}{2}\right)}{(b+m-1)(b+m)}}{1} \right) \tag{16.1.13}$$

$$z\,{}_1F_1(1;b+1;z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\frac{z\left(b+\frac{m}{2}-1\right)}{(b+m-2)(b+m-1)}}{1} \right) \tag{16.1.14}$$

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a+1;b+1;z)} = \frac{b-z}{b} + \frac{1}{b} \operatorname*{K}_{m=1}^{\infty} \left(\frac{(a+m)\,z}{b+m-z} \right) \tag{16.1.16}$$

$$ {}_1F_1(1;b+1;z) = \frac{b}{b-z} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)\,z}{b+m-1-z} \right) \tag{16.1.17}$$

$$\frac{U(a,b,z)}{U(a+1,b,z)} = 2a-b+2+z- \operatorname*{K}_{m=1}^{\infty} \left(\frac{(a+m)\,(b-a-m-1)}{b-2a-2m-2-z} \right) \tag{16.1.20}$$

2.4 Parabolic cylinder functions

$$\frac{U(a,x)}{U(a-1,x)} = \frac{1}{x} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{a+(m-1)-\frac{1}{2}}{x} \right) \tag{16.5.7}$$

2.5 Whittaker functions

$$W_{\kappa,\mu}(z) = e^{-z^2} z^{\kappa} \sum_{k=0}^{\infty} \left(\frac{\left(-\kappa-\mu+\frac{1}{2}\right)_k \left(-\kappa+\mu+\frac{1}{2}\right)_k (-z)^{-k}}{k!} \right) \tag{16.4.7}$$

$$\Psi_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \left(\frac{\left(\alpha+\frac{1}{2}\right)_k \left(\beta+\frac{1}{2},k\right) z^{-k-1}}{k!} \right) \tag{16.4.12}$$

3 Mathematical constants

3.1 Apéry's constant, $\zeta(3)$

$$\zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^z} \tag{10.11.1}$$

$$\zeta(3) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{(k!)^{10} \left(205k^2+250k+77\right)}{64\left((2k+1)!\right)^5} \right) \tag{10.11.3}$$

$$\zeta(3) = \frac{6}{5} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(m-1)^6}{34\,(m-1)^3+51\,(m-1)^2+27\,(m-1)+5} \right) \tag{10.11.5}$$

$$\zeta(3) = 1 + \frac{1}{22} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\left(\frac{m}{2}\right)^3}{1} \right) \quad \text{(No label)}$$

$$\zeta(3) = \sum_{n=1}^{k} \left(\frac{1}{n^3} \right) + \frac{1}{2k^2+2k+1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(m-1)^6}{(m-1)^3+m^3+(2m-1)\,(2k^2+2k)} \right) \quad \text{(No label)}$$

$$\zeta(3) = 1 + \frac{1}{5} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(m-1)^6}{2\,(m-1)^3+3\,(m-1)^2+11\,(m-1)+5} \right) \quad \text{(No label)}$$

$$\zeta(3) = 1 + \frac{1}{4} + \frac{1}{1} + \frac{1}{18} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{9} + \frac{1}{9} + \cdots \tag{10.11.4}$$

3.2 Archimedes' constant, symbol π

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tag{10.2.1}$$

$$\pi = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \underset{m=11}{\overset{\infty}{\mathrm{K}}} \left(\frac{0}{1}\right) \tag{10.2.4}$$

$$\pi = \frac{4}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{(m-1)^2}{2m-1}\right) \tag{10.2.5}$$

$$\pi = 3 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{(2m-1)^2}{6}\right) \tag{10.2.6}$$

$$P = 2 + \frac{1}{1} + 2\underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{(m-1)\,m}{1}\right) \qquad \text{(No label)}$$

$$\pi = \frac{4}{4k+1} + 2^{4k}\left(\frac{(k!)^2}{2k}!\right)^2 \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{(2m-3)^2}{8k+2}\right) \qquad \text{(No label)}$$

$$\pi = \frac{4}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{(2m-3)^2}{2}\right) \qquad \text{(No label)}$$

$$\pi = \frac{4}{4+1} + 4\underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{(2m-3)^2}{8+2}\right) \qquad \text{(No label)}$$

3.3 Catalan's constant, symbol G

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{10.12.1}$$

$$G = 0 + \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{88} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{7} + \cdots \tag{10.12.2}$$

$$G = \frac{1}{2} + \frac{1}{\frac{1}{2}} + \frac{1}{2}\underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\left(\frac{m}{2}\right)^2}{\frac{1}{2}}\right) \tag{10.12.3}$$

$$G = \frac{\frac{13}{2}}{7} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{(2m-3)^4\,(2m-2)^4\left(20\,(m-2)^2 - 8\,(m-2) + 1\right)\left(20\,(m)^2 - 8\,(m) + 1\right)}{3520\,(m-1)^6 + 5632\,(m-1)^5 + 2064\,(m-1)^4 - 384\,(m-1)^3 - 156\,(m-1)^2 + 16\,(m-1) + 7}\right) \tag{10.12.5}$$

$$G = 1 + \frac{-1}{3} + \frac{1}{2} \operatorname*{K}_{m=2}^{\infty} \left(\frac{m^2}{1} \right) \tag{No label}$$

3.4 Euler's constant, symbol γ

$$\gamma = -\log(n) + \sum_{k=0}^{\infty} \frac{1}{k} \tag{10.8.1}$$

$$\gamma = 0 + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{4} + \frac{1}{3} + \frac{1}{13} + \frac{1}{5} + \cdots \tag{No label}$$

3.5 Euler's number, base of the natural logarithm

$$e = \sum_{k=0}^{\infty} \frac{1}{k}! \tag{10.3.1b}$$

$$e = 2 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{1} \right) \tag{10.3.5}$$

$$\frac{e-1}{e+1} = \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{(4m-2)} \right) \tag{10.3.6}$$

$$e = 2 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{m+1}{m+1} \right) \tag{No label}$$

$$e = 1 + \frac{2}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{1}{6 + (m-2)\,4} \right) \tag{No label}$$

$$e = \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-1}{m-1} \right) \tag{No label}$$

3.6 Golden ratio, symbol ϕ

$$\phi = 1 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{1} \right) \tag{10.9.4}$$

3.7 Gompertz' constant, symbol G

$$G = \frac{1}{2} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{-(m-1)^2}{2m} \right) \tag{10.13.1}$$

$$G = \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{m}{2}}{1} \right) \tag{No label}$$

3.8 The natural logarithm, $\ln(2)$

$$\ln(2) = \sum_{k=0}^{\infty} \frac{(-1)^k \, 1^{k+1}}{k+1} \tag{10.5.2}$$

$$\ln(2) = \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)^2}{1} \right) \tag{10.5.3}$$

$$\ln(2) = \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\frac{m}{4m-4}}{1} \right) \tag{10.5.4}$$

3.9 Regular continued fractions

$$\sqrt{e} = 1 + \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{1}{1} \right) \tag{10.4.1}$$

$$e^{\frac{1}{\alpha}} = 1 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{\left(2\left(\frac{m+2}{3}\right) - 1\right)\alpha - 1} \right) \tag{10.4.2}$$

$$e^{\frac{1}{\alpha}} = \frac{\alpha+1}{\alpha} + \frac{1}{\alpha} \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{2\alpha - 1} \right) \tag{10.4.3}$$

$$e^{\frac{1}{\alpha}} = \frac{1}{\alpha - 1} + \frac{1}{2\alpha} + \alpha \operatorname*{K}_{m=3}^{\infty} \left(\frac{1}{1} \right) \tag{10.4.4}$$

$$\sqrt{\pi} = 1 + \frac{1}{1} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{28} + \frac{1}{13} + \frac{1}{1} + \cdots \qquad \text{(No label)}$$

$$ee = 7 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{\frac{3(m+4)}{5} - 1} \right) \tag{10.4.5}$$

$$\frac{\pi^2}{12} = \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{(m-1)^4}{2m-1} \right) \tag{10.4.6}$$

$$\frac{e^{\frac{2\alpha}{\beta}} - 1}{e^{\frac{2\alpha}{\beta}} + 1} = \frac{\alpha}{\beta} + \operatorname*{K}_{m=2}^{\infty} \left(\frac{\alpha^2}{(2m-1)\beta} \right) \tag{10.4.7}$$

3.10 Pythagoras' constant, the square root of two

$$1 + \sqrt{2} = 2 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{2} \right) \tag{10.6.3}$$

$$\left(1 + \sqrt{2}\right)^2 = 5 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{1} \right) \qquad \text{(No label)}$$

$$\left(1+\sqrt{2}\right)^3 = 14 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{1}{14}\right) \quad \text{(No label)}$$

$$\left(1+\sqrt{2}\right)^4 = 33 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{1}{1}\right) \quad \text{(No label)}$$

$$\left(1+\sqrt{2}\right)^5 = 82 + \operatorname*{K}_{m=1}^{\infty}\left(\frac{1}{82}\right) \quad \text{(No label)}$$

3.11 The rabbit constant, symbol ρ

$$\rho = \sum_{k=0}^{\infty} 2^{-\left\lfloor (k+1)\left(\frac{\sqrt{5}+1}{2}\right)\right\rfloor} \quad (10.10.2)$$

$$\rho = \operatorname*{K}_{m=1}^{\infty}\left(\frac{1}{2^{F_{m-1}}}\right) \quad (10.10.5)$$

3.12 zeta2, $\zeta(2)$

$$\zeta(2) = \frac{2}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)^4}{2m-1}\right) \quad \text{(No label)}$$

$$\zeta(2) = \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-\frac{\left(\left((m-2)\left(\frac{1}{2}\right)+1\right)\left((m-2)\left(\frac{1}{2}\right)+1\right)\right)\left((m-2)\left(\frac{1}{2}\right)+1\right)}{(m-1)(1+(m-1))}}{1}\right) \quad \text{(No label)}$$

$$\zeta(2) = 1 + \frac{1}{1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{\left(\frac{m}{2}\right)^2}{1}\right) \quad \text{(No label)}$$

$$\zeta(2) = \sum_{n=1}^{k}\left(\frac{1}{n^2}\right) + \frac{2}{2k+1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)^4}{(2m-1)(2k+1)}\right) \quad \text{(No label)}$$

$$\zeta(2) = 2\sum_{n=1}^{k}\left(\frac{(-1)^{n-1}}{n^2}\right) + \frac{(-1)^k}{k^2+k+1} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{-(m-1)^4}{(m-1)^2+m^2+k^2+k}\right) \quad \text{(No label)}$$

$$\zeta(2) = \frac{5}{3} + \operatorname*{K}_{m=2}^{\infty}\left(\frac{(m-1)^4}{11(m-1)^2+11(m-1)+3}\right) \quad \text{(No label)}$$

3.13 zeta4, $\zeta(4)$

$$\zeta(4) = \frac{13}{12} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(m-1)^7 (3m-2)(3m-3)(3m-4)}{3(2(m-1)+1)\left(3(m-1)^2+3(m-1)+1\right)\left(15(m-1)^2+15(m-1)+4\right)} \right) \tag{No label}$$

$$\zeta(4) = 1 + \frac{1}{12\,+}\,\frac{1}{6\,+}\,\frac{1}{1\,+}\,\frac{1}{3\,+}\,\frac{1}{1\,+}\,\frac{1}{4\,+}\,\frac{1}{183\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{2\,+}\cdots \tag{No label}$$

4 Elementary functions

4.1 Inverse trigonometric functions

$$\mathrm{Arccos}\,(z) = \frac{\pi}{2} - z + \sum_{k=0}^{\infty} - \left(\frac{(2k-1)!!}{(2k)!!\,(2k+1)} \right) z^{2k+1} \tag{11.4.2}$$

$$\mathrm{Arccos}\,(z) = \frac{\frac{\sqrt{1-z^2}}{z}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\frac{\left(\frac{(m-1)^2}{(2m-3)(2m-1)}\right)(1-z^2)}{z^2}}{1} \right) \tag{11.4.6}$$

$$\mathrm{Arccos}\,(z) = \frac{z\sqrt{1-z^2}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\left(-\frac{m(m-1)}{(2m-1)(2m-3)}\right)(1-z^2)}{1} \right) \tag{11.4.7}$$

4.2 Inverse hyperbolic functions

$$\mathrm{Arccosh}\left(\frac{1}{z}\right) = \ln\left(\frac{2}{z}\right) + \sum_{k=0}^{\infty} - \left(\frac{(2k-1)!!}{(2k)!!\,(2k)} \right) z^{2k} \tag{11.6.2}$$

$$\mathrm{Arccosh}\,(z) = \frac{z\sqrt{z^2-1}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\left(\frac{m(m-1)}{(2m-3)(2m-1)}\right)(z^2-1)}{1} \right) \tag{11.6.6}$$

$$\mathrm{Arccosh}\,(z) = \frac{\frac{\sqrt{z^2-1}}{z}}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{-\frac{\left(\frac{(m-1)^2}{(2m-3)(2m-1)}\right)(z^2-1)}{z^2}}{1} \right) \tag{11.6.7}$$

4.3 Inverse trigonometric functions

$$\mathrm{Arcsin}\,(z) = z + \sum_{k=0}^{\infty} \left(\frac{(2k-1)!!}{(2k)!!\,(2k+1)} \right) z^{2k+1} \tag{11.4.1}$$

$$\mathrm{Arcsin}\left(z\right)=\frac{\frac{z}{\sqrt{1-z^2}}}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{\frac{\left(\frac{(m-1)^2}{(2m-3)(2m-1)}\right)z^2}{1-z^2}}{1}\right) \tag{11.4.4}$$

$$\mathrm{Arcsin}\left(z\right)=\frac{z\sqrt{1-z^2}}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{\left(-\frac{m(m-1)}{(2m-1)(2m-3)}\right)z^2}{1}\right) \tag{11.4.5}$$

4.4 Inverse hyperbolic functions

$$\mathrm{Arcsinh}\left(z\right)=z+\sum_{k=0}^{\infty}\left(\frac{\left(-1\right)^k\left(2k-1\right)!!}{\left(2k\right)!!\left(2k+1\right)}\right)z^{2k+1} \tag{11.6.1}$$

$$\mathrm{Arcsinh}\left(z\right)=\frac{z\sqrt{1+z^2}}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{\left(\frac{m(m-1)}{(2m-3)(2m-1)}\right)z^2}{1}\right) \tag{11.6.4}$$

$$\mathrm{Arcsinh}\left(z\right)=\frac{\frac{z}{\sqrt{1+z^2}}}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{-\frac{\left(\frac{(m-1)^2}{(2m-3)(2m-1)}\right)z^2}{1+z^2}}{1}\right) \tag{11.6.5}$$

4.5 Inverse trigonometric functions

$$\mathrm{Arctan}\left(z\right)=\sum_{k=0}^{\infty}\left(\frac{\left(-1\right)^k}{2k+1}\right)z^{2k+1} \tag{11.4.3}$$

$$\mathrm{Arctan}\left(z\right)=\frac{z}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{\left(m-1\right)^2z^2}{2m-1}\right) \tag{11.4.8}$$

$$\mathrm{Arctan}\left(z\right)=\frac{\frac{z}{1+z^2}}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{-\left(\frac{m(m-1)}{(2m-3)(2m-1)}\right)\left(\frac{z^2}{1+z^2}\right)}{1}\right) \tag{11.4.9}$$

4.6 Inverse hyperbolic functions

$$\mathrm{Arctanh}\left(z\right)=\sum_{k=0}^{\infty}\left(\frac{1}{2k+1}\right)z^{2k+1} \tag{11.6.3}$$

$$\mathrm{Arctanh}\left(z\right)=\frac{\frac{z}{1-z^2}}{1}+\underset{m=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{\frac{\left(\frac{m(m-1)}{(2m-3)(2m-1)}\right)z^2}{1-z^2}}{1}\right) \tag{11.6.8}$$

$$\operatorname{Arctanh}(z) = \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\left(-\frac{(m-1)^2 z^2}{4(m-1)^2-1}\right)}{1} \right) \tag{11.6.9}$$

4.7 Trigonometric functions

$$\cos(z) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k}! \right) z^{2k} \tag{11.3.2}$$

4.8 Hyperbolic functions

$$\cosh(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2k}! \right) z^{2k} \tag{11.5.2}$$

4.9 Hyperbolic functions

$$\coth(z) = \sum_{k=0}^{\infty} \left(\frac{4^k B_{2k}}{2k}! \right) z^{2k-1} \tag{11.5.4}$$

$$\coth(z) = \frac{1}{z} + \frac{4\pi^{-2} z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(m-1)^2 \left((m-1)^2 + 4\pi^{-2} z^2\right)}{2m-1} \right) \tag{11.5.6}$$

4.10 The exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{11.1.1}$$

$$e^z = 1 + \frac{2z}{2-z} + \frac{\frac{z^2}{6}}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty} \left(\frac{\left(\frac{1}{4(2m-3)(2m-1)}\right) z^2}{1} \right) \tag{11.1.2}$$

$$e^z = 1 + \frac{z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{\left(-\frac{1}{2(m-1)}\right) z}{1} \right) \tag{11.1.3}$$

$$e^z = \frac{1}{1} + \frac{-z}{1} + \mathop{\mathrm{K}}_{m=3}^{\infty} \left(\frac{\left(-\frac{1}{2(m-1)}\right) z}{1} \right) \tag{No label}$$

$$e^z = 1 + \frac{z}{1-z} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{(m-1) z}{m-z} \right) \tag{11.1.4}$$

4.11 The natural logarithm

$$\ln(1+z) = \sum_{k=0}^{\infty} \left(\frac{(-1)^{k+2}}{k+1} \right) z^{k+1} \tag{11.2.1}$$

$$\ln(1+z) = \frac{z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\left(\frac{m}{4(m-1)}\right) z}{1} \right) \tag{11.2.2}$$

$$\ln(1+z) = \frac{2z}{2+z} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-(m-1)^2 z^2}{(2m-1)(2+z)} \right) \tag{11.2.3}$$

$$\ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\left(-\left(\frac{(m-1)^2}{(2m-3)(2m-1)}\right) z^2\right)}{1} \right) \tag{11.2.4}$$

$$\ln(1+z) = z + \frac{-\frac{z^2}{2}}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\left(\left(\frac{m}{2}\right)+1\right)^2 z}{m(m+1)}}{1} \right) \tag{6.8.8}$$

4.12 The power function

$$(1+z)^{\alpha} = 1 + \frac{\alpha z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\left(\left(\frac{m}{2}\right)-\alpha\right)z}{2(m-1)}}{1} \right) \tag{11.7.1}$$

$$(1+z)^{\alpha} = \frac{1}{1} + \frac{-\alpha z}{1} + \underset{m=3}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\left(\frac{m-1}{2}+\alpha\right)z}{2(m-2)}}{1} \right) \tag{11.7.2}$$

$$(1+z)^{\alpha} = \frac{1}{1} + \frac{-\alpha z}{1+z} + \frac{\frac{(\alpha-1)z}{2}}{1} + \underset{m=4}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\left(-\alpha-\left(\frac{m-2}{2}\right)\right)z}{2(m-1)(1+z)}}{1} \right) \tag{11.7.3}$$

$$\left(\frac{z+1}{z-1}\right)^{\alpha} = 1 + \frac{\frac{2\alpha}{z}}{1-\frac{\alpha}{z}} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\frac{\alpha^2-(m-1)^2}{(2(m-1)-1)(2(m-1)+1)z^2}}{1} \right) \tag{11.7.4}$$

4.13 Trigonometric functions

$$\sin(z) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1}! \right) z^{2k+1} \tag{11.3.1}$$

4.14 Hyperbolic functions

$$\sinh(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2k+1}! \right) z^{2k+1} \tag{11.5.1}$$

4.15 Trigonometric functions

$$\tan(z) = \sum_{k=0}^{\infty} \left(\frac{4^{k+1}\left(4^{k+1}-1\right)\left|B_{2(k+1)}\right|}{2\,(k+1)}! \right) z^{2(k+1)-1} \tag{11.3.3}$$

$$\tan(z) = \frac{z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-\frac{z^2}{(2m-1)(2m-3)}}{1} \right) \tag{11.3.7}$$

$$\tan\left(\frac{\pi z}{4}\right) = \frac{z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{(2m-3)^2 - z^2}{2} \right) \tag{11.3.8}$$

$$\tan(z) = \frac{z}{1} + \frac{-4\pi^{-2}z^2}{1} + \underset{m=3}{\overset{\infty}{\mathrm{K}}} \left(\frac{(m-2)^4 - 4\pi^{-2}(m-2)^2 z^2}{2m-3} \right) \tag{11.3.9}$$

4.16 Hyperbolic functions

$$\tanh(z) = \sum_{k=0}^{\infty} \left(4^{k+1}\left(4^{k+1}-1\right)\left(\frac{B_{2(k+1)}}{2\,(k+1)}!\right) \right) z^{2(k+1)-1} \tag{11.5.3}$$

$$\tanh(z) = \frac{z}{1} + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{\left(\frac{1}{(2m-3)(2m-1)}\right) z^2}{1} \right) \tag{11.5.5}$$

5 Error function and related integrals

6 Exponential integrals and related functions

7 Gamma function and related functions

8 Hypergeometric functions

9 q-Hypergeometric function

10 Probability functions