

1 Bessel functions

1.1 Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad (17.1.2a)$$

$$j_n(z) = \sqrt{\frac{\pi}{2z}} \left(\frac{z}{2}\right)^{n+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma((n + \frac{1}{2}) + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad (17.1.12a)$$

$$J_\nu(z) = \frac{e^{-iz} \left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{(\nu + \frac{1}{2})_k (2i)^k z^k}{(2\nu + 1)_k k!} \quad (17.1.22)$$

$$j_n(z) = \frac{\sqrt{\pi}}{(2n + 1) \Gamma(n + \frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{1}{(n+\frac{3}{2})_k} \left(-\frac{z^2}{4}\right)^k}{k!} \quad (17.1.25)$$

$$j_n(z) = \frac{\sqrt{\pi} e^{-iz}}{(2n + 1) \Gamma(n + \frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\frac{(n+1)_k}{(2n+2)_k} (2iz)^k}{k!} \quad (17.1.26)$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \cos \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) - \frac{(-1)^k (\nu, 2k + 1)}{(2z)^{2k+1}} \sin \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) \right) \quad (17.1.28)$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu, 2k)}{(2z)^{2k}} \sin \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) + \frac{(-1)^k (\nu, 2k + 1)}{(2z)^{2k+1}} \cos \left(z - \left(\frac{\nu}{2} + \frac{1}{4} \right) \pi \right) \right) \quad (17.1.29)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \frac{\frac{z}{2\nu+2}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{(iz)^2}{4(\nu+m-1)(\nu+m)}}{1} \right) \quad (17.1.38)$$

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{(iz)^2}{4(n+\frac{1}{2}+m-1)(n+\frac{1}{2}+m)}}{1} \right) \quad (17.1.39)$$

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = - \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{-1}{\frac{2(\nu+m)}{z}} \right) \quad (17.1.40)$$

$$\frac{H_{\nu+1}^{(1)}(z)}{H_\nu^{(1)}(z)} = \frac{-1}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{m-3-2\nu}{-2iz}}{1} \right) \quad (17.1.44)$$

$$\frac{J_{\nu+1}(z)}{J_{\nu}(z)} = \frac{z}{2\nu+2-iz} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(2\nu+2m-1)iz}{2\nu+m+1+(-2i)z} \right) \quad (17.1.48)$$

$$\frac{j_{n+1}(z)}{j_n(z)} = \frac{z}{2n+3-iz} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{2(n+m)iz}{2n+m+2+(-2i)z} \right) \quad (17.1.49)$$

$$\frac{H_{\nu+1}^{(1)}(z)}{H_{\nu}^{(1)}(z)} = \frac{2\nu+1-2iz}{2z} - \frac{1}{z} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(iz-m)} \right) \quad (17.1.51)$$

1.2 Modified Bessel functions

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \Gamma(\nu+k+1)} \quad (17.2.20)$$

$$I_{\nu}(z) = \frac{e^{-z} \left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(\nu+\frac{1}{2})_k 2^k z^k}{(2\nu+1)_k k!} \quad (17.2.21)$$

$$i_n^{(1)}(z) = \frac{\sqrt{\pi}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!(n+\frac{3}{2})_k} \quad (17.2.22)$$

$$i_n^{(1)}(z) = \frac{\sqrt{\pi}e^{-iz}}{(2n+1)\Gamma(n+\frac{1}{2})} \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(n+1)_k (2z)^k}{k!(2n+2)_k} \quad (17.2.23)$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z+\frac{(2\nu+1)i\pi}{2}}\right)(\nu, k)}{\sqrt{2\pi z} (2z)^k} \right) \quad (17.2.24)$$

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \left(\frac{\left((-1)^k e^z + e^{-z-\frac{(2\nu+1)i\pi}{2}}\right)(\nu, k)}{\sqrt{2\pi z} (2z)^k} \right) \quad (17.2.25)$$

$$K_{\nu}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{\infty} \left(\frac{(\nu, k)}{(-2z)^k} \right) \quad (17.2.27)$$

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{\frac{z}{2(\nu+1)}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{1}{4(\nu+m-1)(\nu+m)}}{1} z^2 \right) \quad (17.2.32)$$

$$\frac{i_{n+1}^{(1)}(z)}{i_n^{(1)}(z)} = \frac{\frac{z}{2n+3}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{1}{4((n+\frac{1}{2})+m-1)((n+\frac{1}{2})+m)}}{1} z^2 \right) \quad (17.2.33)$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{\nu}{z} + \frac{1}{1} + \frac{\frac{-2\nu-1}{2z}}{1} + - \mathop{\mathrm{K}}\limits_{m=3}^{\infty} \left(\frac{\frac{\frac{m}{2}+\nu}{2z}}{1} \right) \quad (17.2.34)$$

$$\frac{I_{\nu+1}(z)}{I_{\nu}(z)} = \frac{z}{2\nu+2+z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-(2\nu+2m-1)z}{2\nu+m+1+2z} \right) \quad (17.2.38)$$

$$\frac{i_{n+1}^{(1)}(z)}{i_n^{(1)}(z)} = \frac{z}{2n+3+z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-2(n+m)z}{2n+m+2+2z} \right) \quad (17.2.39)$$

$$\frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)} = \frac{\nu}{z} - \frac{2\nu+1+2z}{2z} - \frac{1}{z} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{\nu^2 - \frac{(2m-1)^2}{4}}{2(z+m)} \right) \quad (17.2.40)$$

2 Confluent hypergeometric functions

2.1 Confluent hypergeometric series ${}_2F_0$

$$\frac{{}_1F_0(a;;z)}{{}_1F_0(a;;z)} = 1 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{-(b+\frac{m}{2})z}{1} \right) \quad (16.2.4)$$

2.2 Confluent hypergeometric limit function

$${}_0F_1(;b;z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k!} \quad (16.3.1)$$

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{\frac{1}{(b+m-1)(b+m)}z}{1} \right) \quad (16.3.4)$$

$$\frac{{}_0F_1(;b;z)}{{}_0F_1(;b+1;z)} = 1 + \frac{\sqrt{z}}{b} + \frac{1}{2b} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{-2(2b+2m-1)\sqrt{z}}{2b+m+4\sqrt{z}} \right) \quad (16.3.6)$$

2.3 Kummer functions

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{\frac{(a)_k}{(b)_k} z^k}{k!} \quad (16.1.2)$$

$${}_1F_0(a;;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{k!} \quad (16.1.12)$$

$$\frac{{}_1F_1(a;b;z)}{{}_1F_1(a+1;b+1;z)} = 1 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{\frac{z(a+\frac{m}{2})}{(b+m-1)(b+m)}}{1} \right) \quad (16.1.13)$$

$$z {}_1F_1(1;b+1;z) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-\frac{z(b+\frac{m}{2}-1)}{(b+m-2)(b+m-1)}}{1} \right) \quad (16.1.14)$$

$$\frac{{}_1F_1(a; b; z)}{{}_1F_1(a+1; b+1; z)} = \frac{b-z}{b} + \frac{1}{b} \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{(a+m)z}{b+m-z} \right) \quad (16.1.16)$$

$${}_1F_1(1; b+1; z) = \frac{b}{b-z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)z}{b+m-1-z} \right) \quad (16.1.17)$$

$$\frac{U(a, b, z)}{U(a+1, b, z)} = 2a - b + 2 + z - \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{(a+m)(b-a-m-1)}{b-2a-2m-2-z} \right) \quad (16.1.20)$$

2.4 Parabolic cylinder functions

$$\frac{U(a, x)}{U(a-1, x)} = \frac{1}{x} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{a + (m-1) - \frac{1}{2}}{x} \right) \quad (16.5.7)$$

2.5 Whittaker functions

$$W_{\kappa, \mu}(z) = e^{-z^2} z^{\kappa} \sum_{k=0}^{\infty} \left(\frac{(-\kappa - \mu + \frac{1}{2})_k (-\kappa + \mu + \frac{1}{2})_k (-z)^{-k}}{k!} \right) \quad (16.4.7)$$

$$\Psi_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \left(\frac{(\alpha + \frac{1}{2})_k (\beta + \frac{1}{2}, k) z^{-k-1}}{k!} \right) \quad (16.4.12)$$

3 Mathematical constants

3.1 Apéry's constant, $\zeta(3)$

$$\zeta(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^z} \quad (10.11.1)$$

$$\zeta(3) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{(k!)^{10} (205k^2 + 250k + 77)}{64 ((2k+1)!)^5} \right) \quad (10.11.3)$$

$$\zeta(3) = \frac{6}{5} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-(m-1)^6}{34(m-1)^3 + 51(m-1)^2 + 27(m-1) + 5} \right) \quad (10.11.5)$$

$$\zeta(3) = 1 + \frac{1}{22} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{m}{2}\right)^3}{1} \right) \quad (\text{No label})$$

$$\zeta(3) = \sum_{n=1}^k \left(\frac{1}{n^3} \right) + \frac{1}{2k^2 + 2k + 1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-(m-1)^6}{(m-1)^3 + m^3 + (2m-1)(2k^2 + 2k)} \right) \quad (\text{No label})$$

$$\zeta(3) = 1 + \frac{1}{5} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-(m-1)^6}{2(m-1)^3 + 3(m-1)^2 + 11(m-1) + 5} \right) \quad (\text{No label})$$

$$\zeta(3) = 1 + \frac{1}{4} + \frac{1}{1} + \frac{1}{18} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{9} + \frac{1}{9} + \dots \quad (10.11.4)$$

3.2 Archimedes' constant, symbol π

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad (10.2.1)$$

$$\pi = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \dots \quad (10.2.4)$$

$$\pi = \frac{4}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)^2}{2m-1} \right) \quad (10.2.5)$$

$$\pi = 3 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{(2m-1)^2}{6} \right) \quad (10.2.6)$$

$$P = 2 + \frac{1}{1} + {}_2\mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)m}{1} \right) \quad (\text{No label})$$

$$\pi = \frac{4}{4k+1} + {}_2^{4k} \left(\frac{(k!)^2}{(2k)!} \right) \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(2m-3)^2}{8k+2} \right) \quad (\text{No label})$$

$$\pi = \frac{4}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(2m-3)^2}{2} \right) \quad (\text{No label})$$

$$\pi = \frac{4}{4+1} + {}_4\mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(2m-3)^2}{8+2} \right) \quad (\text{No label})$$

3.3 Catalan's constant, symbol G

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \quad (10.12.1)$$

$$G = 0 + \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{88} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{7} + \dots \quad (10.12.2)$$

$$G = \frac{1}{2} + \frac{1}{\frac{1}{2}} + \frac{1}{2} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{m}{2}\right)^2}{\frac{1}{2}} \right) \quad (10.12.3)$$

$$G = \frac{\frac{13}{2}}{7} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(2m-3)^4 (2m-2)^4 \left(20(m-2)^2 - 8(m-2) + 1 \right) \left(20(m)^2 - 8(m) + 1 \right)}{3520(m-1)^6 + 5632(m-1)^5 + 2064(m-1)^4 - 384(m-1)^3 - 156(m-1)^2 + 16(m-1) + 7} \right) \quad (10.12.5)$$

$$G = 1 + \frac{-1}{3} + \frac{1}{2} \tilde{K} \left(\frac{m^2}{1} \right) \quad (\text{No label})$$

3.4 Euler's constant, symbol γ

$$\gamma = -\log(n) + \sum_{k=0}^{\infty} \frac{1}{k} \quad (10.8.1)$$

$$\gamma = 0 + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{4} + \frac{1}{3} + \frac{1}{13} + \frac{1}{5} + \dots \quad (\text{No label})$$

3.5 Euler's number, base of the natural logarithm

$$e = \sum_{k=0}^{\infty} \frac{1}{(k)!} \quad (10.3.1b)$$

$$e = 2 + \tilde{K} \left(\frac{1}{1} \right) \quad (10.3.5)$$

$$\frac{e-1}{e+1} = \tilde{K} \left(\frac{1}{(4m-2)} \right) \quad (10.3.6)$$

$$e = 2 + \tilde{K} \left(\frac{m+1}{m+1} \right) \quad (\text{No label})$$

$$e = 1 + \frac{2}{1} + \tilde{K} \left(\frac{1}{6 + (m-2)4} \right) \quad (\text{No label})$$

$$e = \frac{1}{1} + \tilde{K} \left(\frac{-1}{m-1} \right) \quad (\text{No label})$$

3.6 Golden ratio, symbol ϕ

$$\phi = 1 + \tilde{K} \left(\frac{1}{1} \right) \quad (10.9.4)$$

3.7 Gompertz' constant, symbol G

$$G = \frac{1}{2} + \tilde{K} \left(\frac{-(m-1)^2}{2m} \right) \quad (10.13.1)$$

$$G = \frac{1}{1} + \tilde{K} \left(\frac{\frac{m}{2}}{1} \right) \quad (\text{No label})$$

3.8 The natural logarithm, $\ln(2)$

$$\ln(2) = \sum_{k=0}^{\infty} \frac{(-1)^k 1^{k+1}}{k+1} \quad (10.5.2)$$

$$\ln(2) = \frac{1}{1} + \mathring{K}_{m=2}^{\infty} \left(\frac{(m-1)^2}{1} \right) \quad (10.5.3)$$

$$\ln(2) = \frac{1}{1} + \mathring{K}_{m=2}^{\infty} \left(\frac{m}{1} \right) \quad (10.5.4)$$

3.9 Regular continued fractions

$$\sqrt{e} = 1 + \frac{1}{1} + \mathring{K}_{m=2}^{\infty} \left(\frac{1}{1} \right) \quad (10.4.1)$$

$$e^{\frac{1}{\alpha}} = 1 + \mathring{K}_{m=1}^{\infty} \left(\frac{1}{(2^{\frac{m+2}{3}} - 1)\alpha - 1} \right) \quad (10.4.2)$$

$$e^{\frac{1}{\alpha}} = \frac{\alpha+1}{\alpha} + \frac{1}{\alpha} \mathring{K}_{m=1}^{\infty} \left(\frac{1}{2\alpha-1} \right) \quad (10.4.3)$$

$$e^{\frac{1}{\alpha}} = \frac{1}{\alpha-1} + \frac{1}{2\alpha} + \alpha \mathring{K}_{m=3}^{\infty} \left(\frac{1}{1} \right) \quad (10.4.4)$$

$$\sqrt{\pi} = 1 + \frac{1}{1} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{28} + \frac{1}{13} + \frac{1}{1} + \dots \quad (\text{No label})$$

$$ee = 7 + \mathring{K}_{m=1}^{\infty} \left(\frac{1}{\frac{3(m+4)}{5} - 1} \right) \quad (10.4.5)$$

$$\frac{\pi^2}{12} = \frac{1}{1} + \mathring{K}_{m=2}^{\infty} \left(\frac{(m-1)^4}{2m-1} \right) \quad (10.4.6)$$

$$\frac{e^{\frac{2\alpha}{\beta}} - 1}{e^{\frac{2\alpha}{\beta}} + 1} = \frac{\alpha}{\beta} + \mathring{K}_{m=2}^{\infty} \left(\frac{\alpha^2}{(2m-1)\beta} \right) \quad (10.4.7)$$

3.10 Pythagoras' constant, the square root of two

$$1 + \sqrt{2} = 2 + \mathring{K}_{m=1}^{\infty} \left(\frac{1}{2} \right) \quad (10.6.3)$$

$$(1 + \sqrt{2})^2 = 5 + \mathring{K}_{m=1}^{\infty} \left(\frac{1}{1} \right) \quad (\text{No label})$$

$$(1 + \sqrt{2})^3 = 14 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{1}{14} \right) \quad (\text{No label})$$

$$(1 + \sqrt{2})^4 = 33 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{1}{1} \right) \quad (\text{No label})$$

$$(1 + \sqrt{2})^5 = 82 + \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{1}{82} \right) \quad (\text{No label})$$

3.11 The rabbit constant, symbol ρ

$$\rho = \sum_{k=0}^{\infty} 2^{-\lfloor (k+1) \left(\frac{\sqrt{5}+1}{2} \right) \rfloor} \quad (10.10.2)$$

$$\rho = \mathop{\mathrm{K}}\limits_{m=1}^{\infty} \left(\frac{1}{2^{F_{m-1}}} \right) \quad (10.10.5)$$

3.12 zeta2, $\zeta(2)$

$$\zeta(2) = \frac{2}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)^4}{2m-1} \right) \quad (\text{No label})$$

$$\zeta(2) = \frac{1}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-\frac{((m-2)(\frac{1}{2})+1)((m-2)(\frac{1}{2})+1)((m-2)(\frac{1}{2})+1)}{(m-1)(1+(m-1))}}{1} \right) \quad (\text{No label})$$

$$\zeta(2) = 1 + \frac{1}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{m}{2} \right)^2}{1} \right) \quad (\text{No label})$$

$$\zeta(2) = \sum_{n=1}^k \left(\frac{1}{n^2} \right) + \frac{2}{2k+1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)^4}{(2m-1)(2k+1)} \right) \quad (\text{No label})$$

$$\zeta(2) = 2 \sum_{n=1}^k \left(\frac{(-1)^{n-1}}{n^2} \right) + \frac{(-1)^k}{k^2 + k + 1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-(m-1)^4}{(m-1)^2 + m^2 + k^2 + k} \right) \quad (\text{No label})$$

$$\zeta(2) = \frac{5}{3} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)^4}{11(m-1)^2 + 11(m-1) + 3} \right) \quad (\text{No label})$$

3.13 zeta4, $\zeta(4)$

$$\zeta(4) = \frac{13}{12} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)^7 (3m-2) (3m-3) (3m-4)}{3 (2(m-1)+1) \left(3(m-1)^2 + 3(m-1)+1 \right) \left(15(m-1)^2 + 15(m-1)+4 \right)} \right) \quad (\text{No label})$$

$$\zeta(4) = 1 + \frac{1}{12} + \frac{1}{6} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} + \frac{1}{183} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \dots \quad (\text{No label})$$

4 Elementary functions

4.1 Inverse trigonometric functions

$$\operatorname{Arccos}(z) = \frac{\pi}{2} - z + \sum_{k=0}^{\infty} - \left(\frac{(2k-1)!!}{(2k)!! (2k+1)} \right) z^{2k+1} \quad (11.4.2)$$

$$\operatorname{Arccos}(z) = \frac{\frac{\sqrt{1-z^2}}{z}}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{(m-1)^2}{(2m-3)(2m-1)} \right) (1-z^2)}{z^2} \right) \quad (11.4.6)$$

$$\operatorname{Arccos}(z) = \frac{z\sqrt{1-z^2}}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \left(\frac{\left(-\frac{m(m-1)}{(2m-1)(2m-3)} \right) (1-z^2)}{1} \right) \quad (11.4.7)$$

4.2 Inverse hyperbolic functions

$$\operatorname{Arccosh}\left(\frac{1}{z}\right) = \ln\left(\frac{2}{z}\right) + \sum_{k=0}^{\infty} - \left(\frac{(2k-1)!!}{(2k)!! (2k)} \right) z^{2k} \quad (11.6.2)$$

$$\operatorname{Arccosh}(z) = \frac{z\sqrt{z^2-1}}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{m(m-1)}{(2m-3)(2m-1)} \right) (z^2-1)}{1} \right) \quad (11.6.6)$$

$$\operatorname{Arccosh}(z) = \frac{\frac{\sqrt{z^2-1}}{z}}{1} + \mathop{\mathbb{K}}\limits_{m=2}^{\infty} \left(\frac{-\left(\frac{(m-1)^2}{(2m-3)(2m-1)} \right) (z^2-1)}{z^2} \right) \quad (11.6.7)$$

4.3 Inverse trigonometric functions

$$\operatorname{Arcsin}(z) = z + \sum_{k=0}^{\infty} \left(\frac{(2k-1)!!}{(2k)!! (2k+1)} \right) z^{2k+1} \quad (11.4.1)$$

$$\operatorname{Arcsin}(z) = \frac{\frac{z}{\sqrt{1-z^2}}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{(m-1)^2}{(2m-3)(2m-1)} \right) z^2}{1} \right) \quad (11.4.4)$$

$$\operatorname{Arcsin}(z) = \frac{z\sqrt{1-z^2}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(-\frac{m(m-1)}{(2m-1)(2m-3)} \right) z^2}{1} \right) \quad (11.4.5)$$

4.4 Inverse hyperbolic functions

$$\operatorname{Arsinh}(z) = z + \sum_{k=0}^{\infty} \left(\frac{(-1)^k (2k-1)!!}{(2k)!! (2k+1)} \right) z^{2k+1} \quad (11.6.1)$$

$$\operatorname{Arsinh}(z) = \frac{z\sqrt{1+z^2}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{m(m-1)}{(2m-3)(2m-1)} \right) z^2}{1} \right) \quad (11.6.4)$$

$$\operatorname{Arsinh}(z) = \frac{\frac{z}{\sqrt{1+z^2}}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(-\frac{(m-1)^2}{(2m-3)(2m-1)} \right) z^2}{1} \right) \quad (11.6.5)$$

4.5 Inverse trigonometric functions

$$\operatorname{Arctan}(z) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} \right) z^{2k+1} \quad (11.4.3)$$

$$\operatorname{Arctan}(z) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)^2 z^2}{2m-1} \right) \quad (11.4.8)$$

$$\operatorname{Arctan}(z) = \frac{\frac{z}{1+z^2}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(-\frac{m(m-1)}{(2m-3)(2m-1)} \right) \left(\frac{z^2}{1+z^2} \right)}{1} \right) \quad (11.4.9)$$

4.6 Inverse hyperbolic functions

$$\operatorname{Arctanh}(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} \right) z^{2k+1} \quad (11.6.3)$$

$$\operatorname{Arctanh}(z) = \frac{\frac{z}{1-z^2}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{m(m-1)}{(2m-3)(2m-1)} \right) z^2}{1} \right) \quad (11.6.8)$$

$$\operatorname{Arctanh}(z) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(-\frac{(m-1)^2 z^2}{4(m-1)^2 - 1} \right)}{1} \right) \quad (11.6.9)$$

4.7 Trigonometric functions

$$\cos(z) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k)!} \right) z^{2k} \quad (11.3.2)$$

4.8 Hyperbolic functions

$$\cosh(z) = \sum_{k=0}^{\infty} \left(\frac{1}{(2k)!} \right) z^{2k} \quad (11.5.2)$$

4.9 Hyperbolic functions

$$\coth(z) = \sum_{k=0}^{\infty} \left(\frac{4^k B_{2k}}{(2k)!} \right) z^{2k-1} \quad (11.5.4)$$

$$\coth(z) = \frac{1}{z} + \frac{4\pi^{-2}z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)^2 \left((m-1)^2 + 4\pi^{-2}z^2 \right)}{2m-1} \right) \quad (11.5.6)$$

4.10 The exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad (11.1.1)$$

$$e^z = 1 + \frac{2z}{2-z} + \frac{\frac{z^2}{6}}{1} + \mathop{\mathrm{K}}\limits_{m=3}^{\infty} \left(\frac{\left(\frac{1}{4(2m-3)(2m-1)} \right) z^2}{1} \right) \quad (11.1.2)$$

$$e^z = 1 + \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{-1}{2(m-1)} \right) z}{1} \right) \quad (11.1.3)$$

$$e^z = \frac{1}{1} + \frac{-z}{1} + \mathop{\mathrm{K}}\limits_{m=3}^{\infty} \left(\frac{\left(\frac{-1}{2(m-1)} \right) z}{1} \right) \quad (\text{No label})$$

$$e^z = 1 + \frac{z}{1-z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(m-1)z}{m-z} \right) \quad (11.1.4)$$

4.11 The natural logarithm

$$\ln(1+z) = \sum_{k=0}^{\infty} \left(\frac{(-1)^{k+2}}{k+1} \right) z^{k+1} \quad (11.2.1)$$

$$\ln(1+z) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{m}{4(m-1)} \right) z}{1} \right) \quad (11.2.2)$$

$$\ln(1+z) = \frac{2z}{2+z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-(m-1)^2 z^2}{(2m-1)(2+z)} \right) \quad (11.2.3)$$

$$\ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(-\left(\frac{(m-1)^2}{(2m-3)(2m-1)} \right) z^2 \right)}{1} \right) \quad (11.2.4)$$

$$\ln(1+z) = z + \frac{-\frac{z^2}{2}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{\left(\frac{m}{2} \right) + 1}{m(m+1)} \right)^2 z}{1} \right) \quad (6.8.8)$$

4.12 The power function

$$(1+z)^{\alpha} = 1 + \frac{\alpha z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{\left(\frac{m}{2} \right) - \alpha}{2(m-1)} \right) z}{1} \right) \quad (11.7.1)$$

$$(1+z)^{\alpha} = \frac{1}{1} + \frac{-\alpha z}{1} + \mathop{\mathrm{K}}\limits_{m=3}^{\infty} \left(\frac{\left(\frac{\frac{m-1}{2} + \alpha}{2(m-2)} \right) z}{1} \right) \quad (11.7.2)$$

$$(1+z)^{\alpha} = \frac{1}{1} + \frac{-\alpha z}{1+z} + \frac{\frac{(\alpha-1)z}{2}}{1} + \mathop{\mathrm{K}}\limits_{m=4}^{\infty} \left(\frac{\left(\frac{-\alpha - \left(\frac{m-2}{2} \right)}{2(m-1)(1+z)} \right) z}{1} \right) \quad (11.7.3)$$

$$\left(\frac{z+1}{z-1} \right)^{\alpha} = 1 + \frac{\frac{2\alpha}{z}}{1 - \frac{\alpha}{z}} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{\alpha^2 - (m-1)^2}{(2(m-1)-1)(2(m-1)+1)z^2}}{1} \right) \quad (11.7.4)$$

4.13 Trigonometric functions

$$\sin(z) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k+1)!} \right) z^{2k+1} \quad (11.3.1)$$

4.14 Hyperbolic functions

$$\sinh(z) = \sum_{k=0}^{\infty} \left(\frac{1}{(2k+1)!} \right) z^{2k+1} \quad (11.5.1)$$

4.15 Trigonometric functions

$$\tan(z) = \sum_{k=0}^{\infty} \left(\frac{4^{k+1} (4^{k+1} - 1) |B_{2(k+1)}|}{(2(k+1))!} \right) z^{2(k+1)-1} \quad (11.3.3)$$

$$\tan(z) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-\frac{z^2}{(2m-1)(2m-3)}}{1} \right) \quad (11.3.7)$$

$$\tan\left(\frac{\pi z}{4}\right) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(2m-3)^2 - z^2}{2} \right) \quad (11.3.8)$$

$$\tan(z) = \frac{z}{1} + \frac{-4\pi^{-2}z^2}{1} + \mathop{\mathrm{K}}\limits_{m=3}^{\infty} \left(\frac{(m-2)^4 - 4\pi^{-2}(m-2)^2 z^2}{2m-3} \right) \quad (11.3.9)$$

4.16 Hyperbolic functions

$$\tanh(z) = \sum_{k=0}^{\infty} \left(4^{k+1} (4^{k+1} - 1) \left(\frac{B_{2(k+1)}}{(2(k+1))!} \right) \right) z^{2(k+1)-1} \quad (11.5.3)$$

$$\tanh(z) = \frac{z}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\left(\frac{1}{(2m-3)(2m-1)} \right) z^2}{1} \right) \quad (11.5.5)$$

5 Error function and related integrals

5.1 Complementary and complex error function

$$\operatorname{erfc}(z) = e^{-z^2} \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma\left(\frac{k}{2} + 1\right)} \quad (13.2.9)$$

$$w(z) = \sum_{k=0}^{\infty} \frac{(iz)^k}{\Gamma\left(\frac{k}{2} + 1\right)} \quad (13.2.10)$$

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi} z e^{z^2}} \sum_{k=0}^{\infty} \left((-1)^k \left(\frac{1}{2} \right)_k z^{-2k} \right) \quad (13.2.11)$$

$$w(z) = \frac{i}{\pi z} \sum_{k=0}^{\infty} \left(\left(\frac{1}{2} \right)_k z^{-2k} \right) \quad (13.2.12)$$

$$\operatorname{erfc}(z) = \frac{1}{z^2} + \frac{z}{\sqrt{\pi}} e^{-z^2} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{m-1}{2}}{1} \right) \quad (13.2.20a)$$

$$w(z) = \frac{1}{-z^2} + -\frac{iz}{\sqrt{\pi}} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{m-1}{2}}{1} \right) \quad (13.2.20b)$$

$$\operatorname{erfc}(z) = \frac{2z}{2z^2 + 1} + \frac{e^{-z^2}}{\sqrt{\pi}} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-(2m-3)(2m-2)}{4m-3+2z^2} \right) \quad (13.2.23a)$$

$$w(z) = \frac{-\frac{iz}{\sqrt{\pi}}}{\frac{1}{2} - z^2} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-\left(-\frac{3}{2} + m\right)(m-1)}{2m - \frac{3}{2} - z} \right) \quad (13.2.23b)$$

5.2 Error function and Dawson's integral

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)k!} \quad (13.1.7)$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{\left(\frac{3}{2}\right)_k} \quad (13.1.8)$$

$$F(z) = e^{-z^2} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)k!} \quad (13.1.9)$$

$$F(z) = 1 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\left(\frac{3}{2}\right)_k} \quad (13.1.10)$$

$$\operatorname{erf}(z) = \frac{2z^2}{1} + \frac{1}{\sqrt{\pi}ze^{z^2}} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{-2(m-1)}{(2m-3)(2m-1)}z^2}{1} \right) \quad (13.1.11a)$$

$$F(z) = \frac{-2z^2}{1} + -\frac{1}{2z} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{2(m-1)}{(2m-3)(2m-1)}z^2}{1} \right) \quad (13.1.11b)$$

$$\operatorname{erf}(z) = \frac{2z^2}{1-2z^2} + \frac{1}{\sqrt{\pi}ze^{z^2}} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{4(m-1)}{(2m-3)(2m-1)}z^2}{1 - \frac{2}{2m-1}z^2} \right) \quad (13.1.13a)$$

$$F(z) = \frac{-2z^2}{1+2z^2} + -\frac{1}{2z} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{4(m-1)}{(2m-3)(2m-1)}z^2}{1 + \frac{2}{2m-1}z^2} \right) \quad (13.1.13b)$$

5.3 Fresnel integrals

$$C(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\pi}{2}\right)^{2k}}{(2k)!(4k+1)} z^{4k+1} \quad (13.4.6a)$$

$$S(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!(4k+3)} z^{4k+3} \quad (13.4.6b)$$

$$C(z) + iS(z) = \frac{z^2}{1} + \frac{e^{\frac{i\pi z^2}{2}}}{z} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{i\pi(m-1)}{(2m-3)(2m-1)} z^2}{1} \right) \quad (13.4.9)$$

$$C(z) + iS(z) = \frac{z^2}{1 + i\pi z^2} + \frac{e^{\frac{i\pi z^2}{2}}}{z} \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{-\frac{2i\pi(m-1)}{(2m-3)(2m-1)} z^2}{1 + \frac{i\pi}{2m-1} z^2} \right) \quad (13.4.10)$$

5.4 Repeated integrals

$$I^\alpha \operatorname{erfc}(z) = \frac{\frac{2}{\sqrt{\pi}} e^{-z^2}}{(2z)^{\alpha+1}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (2k + \alpha)!}{\alpha! k! (2z)^{2k}} \right) \quad (13.3.2)$$

$$\frac{I^\alpha \operatorname{erfc}(z)}{I^{\alpha-1} \operatorname{erfc}(z)} = \frac{\frac{1}{2}}{z} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{\alpha+m-1}{2}}{z} \right) \quad (13.3.5)$$

6 Exponential integrals and related functions

6.1 Exponential integrals

$$E_1(z) = -1 \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k k!} \quad (14.1.10)$$

$$E_n(z) = \left(\frac{(-z)^{n-1}}{(n-1)!} \right) \left(-\gamma - \ln(z) + \sum_{i=1}^{n-1} \left(\frac{1}{i} \right) \right) - \sum_{i=0}^{n-2} \left(\frac{(-z)^i}{(i-n+1)! i!} \right) + \sum_{k=0}^{\infty} - \left(\frac{(-z)^{k+n-1}}{(k)(k+n-1)!} \right) \quad (14.1.11)$$

$$E_\nu(z) = \Gamma(1-\nu) z^{\nu-1} - \frac{1}{1-\nu} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^k}{(k!) (k+1-\nu)} \quad (14.1.12)$$

$$E_\nu(z) = e^{-z} \sum_{k=0}^{\infty} \left(\frac{(-1)^k (\nu)_k}{z^{k+1}} \right) \quad (14.1.13)$$

$$E_n(z) = \frac{1}{z} + \frac{n}{1} + e^{-z} \mathop{\mathrm{K}}\limits_{m=3}^{\infty} \left(\frac{n + \frac{m-2}{2}}{1} \right) \quad (14.1.16)$$

$$E_\nu(z) = \frac{\frac{e^{-z}}{z}}{1} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{\frac{\frac{m}{2} + \nu - 1}{z}}{1} \right) \quad (14.1.19)$$

$$E_\nu(z) = z^{\nu-1} \Gamma(1-\nu) + \frac{-e^{-z}}{1-\nu} + \mathop{\mathrm{K}}\limits_{m=2}^{\infty} \left(\frac{(\nu - \frac{m}{2}) z}{m - \nu} \right) \quad (14.1.20)$$

$$E_\nu(z) = z^{\nu-1} \Gamma(1-\nu) + \frac{1}{1-\nu-z} + -e^{-z} \mathring{K}_{m=2}^{\infty} \left(\frac{(m-1)z}{m-\nu-z} \right) \quad (14.1.22)$$

$$E_\nu(z) = \frac{1}{\nu+z} + e^{-z} \mathring{K}_{m=2}^{\infty} \left(\frac{-(m-1)(\nu+m-2)}{\nu+(m-1)2+z} \right) \quad (14.1.23)$$

$$E_\nu(z) = \frac{e^{-z}}{z} + \frac{-e^{-z}\nu}{z(1+z)+\nu z} + \mathring{K}_{m=2}^{\infty} \left(\frac{-(m-1)(\nu+m-1)z^2}{z(m+z)+(\nu+m-1)z} \right) \quad (14.1.24)$$

6.2 Related functions

$$E_1(x) = \gamma + \ln(x) + \sum_{k=0}^{\infty} \frac{x^k}{kk!} \quad (14.2.14)$$

$$\text{Ein}(z) = -1 \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{k+1}}{(k+1)(k+1)!} \quad (14.2.16)$$

$$E_1(x) = \frac{e^x}{x} \sum_{k=0}^{\infty} (k! x^{-k}) \quad (14.2.19)$$

$$E_1(z) = 2 \sum_{i=0}^{\infty} \left(\frac{z^{2i+1}}{(2i+1)(2i+1)!} \right) + \frac{1}{z} + -e^{-z} \mathring{K}_{m=2}^{\infty} \left(\frac{\lfloor \frac{m}{2} \rfloor}{\frac{z}{1}} \right) \quad (14.2.21)$$

$$\text{Ein}(z) = \gamma + \ln(z) + \frac{1}{z} + e^{-z} \mathring{K}_{m=2}^{\infty} \left(\frac{\lfloor \frac{m}{2} \rfloor}{\frac{z}{1}} \right) \quad (14.2.23)$$

$$E_1(x) = \frac{e^x}{1} + \mathring{K}_{m=2}^{\infty} \left(\frac{-\lfloor \frac{m}{2} \rfloor}{\frac{x}{1}} \right) \quad (14.2.24)$$

7 Gamma function and related functions

8 Hypergeometric functions

9 q-Hypergeometric function

10 Probability functions