MACHINE LEARNING (XAI501), SPRING 2024

ASSIGNMENT #1: BAYESIAN LINEAR REGRESSION

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[Problem #1] With the likelihood function and the prior over the model parameter $\mathbf{w}$ defined as below, please derive the posterior probability distribution of $\mathbf{w}$ and the predictive distribution of t for a new sample x.

- Likelihood function $p(\mathbf{t}|\mathbf{w})$: exponential of a quadratic function of $\mathbf{w}$

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{n=1}^{N} \mathcal{N}\left(t_n | \mathbf{w}^\top \phi(\mathbf{x}_n), \beta^{-1}\right)$$

- (Conjugate prior) Gaussian distribution

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

- Posterior probability distribution of $\mathbf{w}$

$$\begin{aligned} p(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta) &\propto p(\mathbf{t}|\mathbf{w}, \mathbf{x}, \beta)\, p(\mathbf{w}) \\ &= \mathcal{N}\left(\mathbf{t}|\Phi^\top \mathbf{w}, \beta^{-1}\mathbf{I}\right) \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0) \\ p(\mathbf{w}|\mathbf{x}, \mathbf{t}, \alpha, \beta) &= \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N) \\ \text{where} &\left\{ \begin{array}{l} \mathbf{m}_N = \mathbf{S}_N\left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\Phi^\top \mathbf{t}\right) \\ \mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \beta\Phi^\top\Phi \end{array} \right. \end{aligned}$$

- Predictive distribution of t for a new sample x

$$\begin{aligned} p(t|x, \mathbf{x}, \mathbf{t}) &= \int p(t|x, \mathbf{w})\, p(\mathbf{w}|\mathbf{x}, \mathbf{t})\, d\mathbf{w} \\ &= \int \mathcal{N}\left(t \,\middle|\, \phi(x)^\top \mathbf{w}, \beta^{-1}\right) \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N) \\ &= \mathcal{N}\left(t \,\middle|\, \underbrace{\phi(x)^\top \mathbf{m}_N}_{m(x)}, \underbrace{\beta^{-1} + \phi(x)^\top \mathbf{S}_N \phi(x)}_{s^2(x)}\right) \\ \text{where} &\left\{ \begin{array}{l} \mathbf{m}_N = \mathbf{S}_N\left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\Phi^\top \mathbf{t}\right) \\ \mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \beta\Phi^\top\Phi \end{array} \right. \end{aligned}$$

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- (Conjugate prior) Gaussian distribution

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

① posterior distribution of w

$$p(w|X,t,\alpha,\beta) \propto p(t|w,X,\beta)\, p(w) = N(t|\Phi^T w, \beta^{-1} I)\, N(w|m_0, S_0)$$

To get $p(w|x,t,\alpha,\beta)$ into the form of a Gaussian, complete the square on w.

$$-\frac{\beta}{2}(t-\Phi^T w)^T(t-\Phi^T w) - \frac{1}{2}(w-m_0)^T S_0^{-1}(w-m_0)$$

$$= -\frac{\beta}{2}(w^T\Phi\Phi^T w - 2w^T\Phi t) - \frac{1}{2}(w^T S_0^{-1} w - 2w^T S_0^{-1} m_0) + \text{const.}$$

$$= -\frac{1}{2}(w^T(\beta\Phi\Phi^T + S_0^{-1})w - 2w^T(\beta\Phi t + S_0^{-1} m_0)) + \text{Const.}$$

$$= -\frac{1}{2}(w-m_N)^T S_N^{-1}(w-m_N) + \text{const.}$$

where $m_N = S_N(S_0^{-1}m_0 + \beta\Phi^T t)$

$S_N^{-1} = S_0^{-1} + \beta\Phi^T\Phi$

therefore, the posterior is

$$p(w|x,t,\alpha,\beta) = N(w|m_N, S_N)$$

② predictive distribution

Integrate out w to get the predictive distribution.

$$p(t|x,X,t) = \int \underbrace{p(t|x,w)}_{\text{Likelihood}}\ \underbrace{p(w|X,t)}_{\text{posterior}}\, dw$$

$$= \int N(t|\phi(x)^T w, \beta^{-1})\, N(w|m_N, S_N)\, dw$$ → product of Gaussian dist. is another Gaussian distribution

$$= N(t|\phi(x)^T m_N, \beta^{-1} + \phi(x)^T S_N \phi(x))$$

$$= N(t|m(x), \beta^{-1} + s^2(x))$$

where

$m(x) = \phi(x)^T m_N$ mean of predictive dist. for a new input x

$s^2(x) = \phi(x)^T S_N \phi(x)$ model's uncertainty.

$m_N = S_N(S_0^{-1}m_0 + \beta\Phi^T t)$

$S_N^{-1} = S_0^{-1} + \beta\Phi^T\Phi$