Posets and Dilworth's Theorem

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You will receive full credit if you solve 3 problems. You may receive partial credit even if you didn't fully solve a problem.

Exercise 1. Prove the weighted Mirsky's theorem and the weighted Dilworth's theorem. (Hint: for the weighted Mirsky's theorem, replace each element $a \in P$ with a chain of size $w(a)$.)

Exercise 2. Recall that in Gallai-Milgram theorem, a path cover need not be disjoint. If there is disjoint condition, does the theorem hold? In other words, prove or disprove the following statement.

- Let $G = (V, E)$ be a DAG. Then the minimum size of a collection of *disjoint* paths covering V is equal to the maximum size of a path-independent set of vertices.

Exercise 3. For the given set X of positive integers, find a maximum size set $Y \subseteq X$ with the following property: for any two distinct elements in Y, one does not divide the other.

1. $X = \{1, 2, \cdots, 10\}$
2. $X = \{1, 2, \cdots, 100\}$

Exercise 4. Devise a polynomial time algorithm finding a maximum weight antichain. An algorithm that has a runtime linear on the weights is not a polynomial time algorithm. (Hint: one of the algorithmic proofs of Dilworth's theorem generalizes well to the weighted version.)

Exercise 5. Let $G = (V, E)$ be a DAG.

1. Prove that the minimum size of a collection of directed cuts covering E is equal to the length of the longest path.
2. Prove that the minimum size of a collection of paths covering E is equal to the maximum size of a directed cut.

Exercise 6. Let $(P, \leq)$ be a poset. Let ω be the maximum size of a chain and let α be the maximum size of an antichain. Prove that $\omega \cdot \alpha \geq |P|$.

Exercise 7. This exercise will guide you through the proof of Dilworth's theorem using Mirsky's theorem and the perfect graph theorem. This partially accounts for the relationship between Dilworth's theorem and Mirsky's theorem. Necessary definitions and theorems are stated in the Appendix and you are free to use them without proof. Let $(P, \leq)$ be a poset.

1. Let $G = (V, E)$ be an (undirected) graph. Prove that the chromatic number $\chi(G)$ is equal to the minimum size of a collection of disjoint stable sets covering V.
2. Let $G = (P, E)$ be a graph where $\{u, v\} \in E$ if and only if $u < v$ or $v < u$. This graph is called the **comparability graph** of $(P, \leq)$. Using Mirsky's theorem, prove that G is a perfect graph.
3. Using the perfect graph theorem, prove Dilworth's theorem.

Appendix: the perfect graph theorem

Definition 1. Let $G = (V, E)$ be an (undirected) graph. The **complement** of G is a graph $\bar{G} = (V, \bar{E})$ where $\{u, v\} \in \bar{E}$ if and only if $\{u, v\} \notin E$.

Definition 2. Let $G = (V, E)$ be an (undirected) graph. A **clique** is a vertex set $C \subseteq V$ such that every pair of vertices in C are adjacent. An **independent set** or a **stable set** is a vertex set $S \subseteq V$ such that every pair of vertices in S are not adjacent.

Clearly, it follows that a clique in a graph is a stable set in its complement. The maximum size of a clique is denoted by $\omega(G)$. In a general graph, finding $\omega(G)$ is a well known NP-hard problem. Now we define chromatic number.

Definition 3. For a color set X, a **proper coloring** of G is a map $c : V \to X$ such that $c(u) \neq c(v)$ for every $\{u, v\} \in E$. The minimum possible $|c(V)|$ of a proper coloring is called the **chromatic number**, and is denoted by $\chi(G)$.

Finally, we can define a perfect graph. It is obvious that $\chi(G) \geq \omega(G)$ for any graph G. The graph is perfect when this inequality is tight for every induced subgraph.

Definition 4. A graph G is called a **perfect graph** if for every induced subgraph H of G we have $\chi(G) = \omega(G)$.

The notion of perfect graph was first introduced by Claude Berge, and it contains many interesting classes of graphs, such as trees, interval graphs, bipartite graphs, line graphs, and so on. The following is the perfect graph theorem which was conjectured by Berge, and was proved by László Lovász.

Theorem 1 (Perfect graph theorem)**.** *Let G be an (undirected) graph. Then G is a perfect graph if and only if its complement $\bar{G}$ is a perfect graph.*