Posets and Dilworth's Theorem

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Exercise 1. Prove the weighted Mirsky's theorem and the weighted Dilworth's theorem. (Hint: for the weighted Mirsky's theorem, replace each element $a \in P$ with a chain of size $w(a)$.)

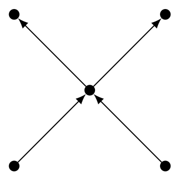
For the weighted Mirsky's theorem, replace each element $a \in P$ with new elements $a_1, \cdots, a_{w(a)}$, and let $a_i \leq b_j$ if and only if $a \leq b$. Note that if an antichain contains one of $a_1, \cdots, a_{w(a)}$, then it can be extended by adding all of them. Then applying Mirsky's theorem immediately gives the result.

Similar argument can be made for the weighted Dilworth's theorem, by replacing each element with a chain.

Exercise 2. Recall that in Gallai-Milgram theorem, a path cover need not be disjoint. If there is disjoint condition, does the theorem hold? In other words, prove or disprove the following statement.

- Let $G = (V, E)$ be a DAG. Then the minimum size of a collection of *disjoint* paths covering V is equal to the maximum size of a path-independent set of vertices.

The following graph needs at least 3 disjoint paths, whereas the size of a path-independent vertex set is at most 2.



Exercise 3. For the given set X of positive integers, find a maximum size set $Y \subseteq X$ with the following property: for any two distinct elements in Y, one does not divide the other.

1. $X = \{1, 2, \cdots, 10\}$
2. $X = \{1, 2, \cdots, 100\}$

Consider the chain cover $\{A_m\}_{2 \nmid m}$ where $A_m = \{2^k \cdot m : k \in \mathbb{Z}\} \cap X$. This chain cover has size $\lceil |X|/2 \rceil$ giving the upper bound of an antichain, which can be achieved by choosing all elements greater than $|X|/2$.

Exercise 4. Devise a polynomial time algorithm finding a maximum weight antichain. An algorithm that has a runtime linear on the weights is not a polynomial time algorithm. (Hint: one of the algorithmic proofs of Dilworth's theorem generalizes well to the weighted version.)

Recall the proof using minimum flow. An algorithm for the weighted version can be obtained by setting the flow lower bound of each edge (a_1, a_2) to the weight of $a \in P$.

Exercise 5. Let $G = (V, E)$ be a DAG.

1. Prove that the minimum size of a collection of directed cuts covering E is equal to the length of the longest path.
2. Prove that the minimum size of a collection of paths covering E is equal to the maximum size of a directed cut.

Let P be the set of edges and let $e_1 \leq e_2$ if and only if there is a path that passes through e_1 and e_2 in this order. To apply Mirsky's theorem and Dilworth's theorem, we should prove that for any antichain A we have a directed cut with size greater than $|A|$. One way is to take X the set of vertices that can reach any edge in A, and then take $\delta^+(X)$.

Exercise 6. Let $(P, \leq)$ be a poset. Let ω be the maximum size of a chain and let α be the maximum size of an antichain. Prove that $\omega \cdot \alpha \geq |P|$.

By Mirsky's theorem, we have an antichain cover of size ω. Since each antichain is of size at most α, it follows that $\omega \cdot \alpha \geq |P|$. We can also apply Dilworth's theorem to obtain the same result.

Exercise 7. This exercise will guide you through the proof of Dilworth's theorem using Mirsky's theorem and the perfect graph theorem. This partially accounts for the relationship between Dilworth's theorem and Mirsky's theorem. Necessary definitions and theorems are stated in the Appendix and you are free to use them without proof. Let $(P, \leq)$ be a poset.

1. Let $G = (V, E)$ be an (undirected) graph. Prove that the chromatic number $\chi(G)$ is equal to the minimum size of a collection of disjoint stable sets covering V.

2. Let $G = (P, E)$ be a graph where $\{u, v\} \in E$ if and only if $u < v$ or $v < u$. This graph is called the **comparability graph** of $(P, \leq)$. Using Mirsky's theorem, prove that G is a perfect graph.

3. Using the perfect graph theorem, prove Dilworth's theorem.

The first statement is immediate if one notices that the set of vertices of same color is a stable set.

For the second statement, note that a chain is a clique in the comparability graph, and an antichain is a stable set in the comparability graph. So the antichain cover corresponds to a proper coloring. Then Mirsky's theorem gives the desired result.

For the last statement, the roles of an antichain and a chain is reversed. First, note that a clique is a stable set in its complement, and vice versa. By the perfect graph theorem, the complement of the comparability graph is perfect. This gives Dilworth's theorem.