Stochastics

1. Measure-Theoretic Probability

In this chapter, measure-theoretic probability thoery is presented simply.

1.1 Events and Probability

Definition 1.1 ($\sigma-algebra$)

Let Ω be an non-empty set. A $\sigma-algebra$ $\mathcal{F}$ on Ω is a family of subsets of Ω such that:

1. The empty set $\emptyset$ belongs to $\mathcal{F}$.
2. if $A \in \mathcal{F}$, then $\Omega \setminus A \in \mathcal{F}$.
3. if $A_1,\ A_2,\ \dots$ is a sequence of sets in $\mathcal{F}$, then their union $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathcal{F}$.

Definition 1.2 (probability measure)

Let $(\Omega, \mathcal{F})$ be a measureable space. Then a *probability measure* P on $(\Omega, \mathcal{F})$ is a function

$$P : \mathcal{F} \to [0, 1]$$

such that:

1. $P(\Omega) = 1$;
2. if $A_1,\ A_2, \dots$ is a pairwise disjoint sets belonging to $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

Theorem 1.1 (increasing and decreasing sequence of sets)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let

$$A_1 \subset A_2 \subset A_3\ \dots$$

be an increasing sequence of sets that belongs to $\mathcal{F}$. Then,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(A_n\right)$$

Similary, let

$$A_1 \supset A_2 \supset \dots$$

be an decreasing sequnce of sets that belongs to $\mathcal{F}$. Then,

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(A_n\right)$$

[1]

1.2 Random Variables

Definition 1.3 (random variable)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then a $\mathcal{F}$*–measureable* function $\xi : \Omega \to \mathbb{R}$ is called a random variable.

Definition 1.4 ($\sigma-algebra$ generated by a random variable)

- Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\xi : \Omega \to \mathbb{R}$ be a random variable. Then, a σ*–algebra* generated by a random variable ξ is defined to be a familiy of sets containing all sets of the form $\xi^{-1}(B)$, where B is a Borel set in $\mathbb{R}$.
- Furthermore, let $\{\xi_i : i \in I\}$ be a familiy of random variables. Then a σ*–algebra* generated by $\{\xi_i : i \in I\}$ is defined to be a smallest σ*–algebra* that contains all sets of the form $\xi_i^{-1}(B)$, where $i \in I$ and B is a Borel set in $\mathbb{R}$.

1. pf) try to use the definition of measure. ↩