Stochastics

1. Measure-Theoretic Probability

In this chapter, measure-theoretic probability theory is presented simply.

1.1 Events and Probability

Definition 1.1.1 ($\sigma-algebra$)

Let Ω be an non-empty set. A $\sigma-algebra$ $\mathcal{F}$ on Ω is a family of subsets of Ω such that:

1. The empty set $\emptyset$ belongs to $\mathcal{F}$.
2. if $A \in \mathcal{F}$, then $\Omega \setminus A \in \mathcal{F}$.
3. if $A_1,\ A_2,\ \ldots$ is a sequence of sets in $\mathcal{F}$, then their union $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathcal{F}$.

Definition 1.1.2 (probability measure)

Let $(\Omega, \mathcal{F})$ be a measurable space. Then a *probability measure* P on $(\Omega, \mathcal{F})$ is a function

$$P : \mathcal{F} \to [0,1]$$

such that:

1. $P(\Omega) = 1$;
2. if $A_1,\ A_2, \ldots$ is a pairwise disjoint sets belonging to $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*.

Theorem 1.1.3 (increasing and decreasing sequence of sets)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let

$$A_1 \subset A_2 \subset A_3\ \ldots$$

be an increasing sequence of sets that belongs to $\mathcal{F}$. Then,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(A_n\right)$$

Similary, let

$$A_1 \supset A_2 \supset \ldots$$

be an decreasing sequnce of sets that belongs to $\mathcal{F}$. Then,

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(A_n\right)$$

[1]

Lemma 1.1.4 (Borel-Cantelli Lemma)

Let $A_1,\ A_2,\ \ldots$ be a sequence of events of such that $\sum_{i=1}^{\infty} P(A_i) < \infty$ and let $B_n = \bigcup_{i=n}^{\infty} A_i$. Then $P\left(\bigcap_{i=1}^{\infty} B_i\right) = 0$.
[2]

1.2 Random Variables

Definition 1.2.1 (random variable)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then a *$\mathcal{F}$–measurable* function $\xi : \Omega \to \mathbb{R}$ is called a random variable.

Definition 1.2.2 ($\sigma-algebra$ generated by a random variable)

- Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\xi : \Omega \to \mathbb{R}$ be a random variable. Then, a *$\sigma-algebra$ $\sigma(\xi)$ generated by a random variable ξ* is defined to be a familiy of sets containing all sets of the form $\xi^{-1}(B)$, where B is a Borel set in $\mathbb{R}$.
- Furthermore, let $\{\xi_i : i \in I\}$ be a familiy of random variables. Then a *σ–algebra $\sigma\left\{\xi : i \in I\right\}$ generated by* $\{\xi_i : i \in I\}$ is defined to be a smallest *σ–algebra* that contains all sets of the form $\xi_i^{-1}(B)$, where $i \in I$ and B is a Borel set in $\mathbb{R}$.

Lemma 1.2.3 (Doob-Dynkin)

Let ξ be a random variable. Then each *$\sigma(\xi)-measurable$* random variable η can be written as

$$\eta = f(\xi)$$

for some Borel function $f : \mathbb{R} \to \mathbb{R}$.
[3]

Definition 1.2.4 (distribution and distribution function)

Every random variable $\xi : \Omega \to \mathbb{R}$ gives rise to a probability measure

$$P_\xi(B) = P(\xi^{-1}(B))$$

on $\mathbb{R}$ defined on the *$\sigma-algebra$* of Borel sets $B \in \mathcal{B}(\mathbb{R})$. We call P_ξ the *distribution* of ξ [4]. The function $F_\xi : \mathbb{R} \to [0, 1]$[5] defined by

$$F_\xi(x) = P_\xi((-\infty,\, x))$$

is called the *distribution function* of ξ.

Theorem 1.2.5 (property of the distirbution function)

Let F_ξ be the distribution function of a random variable ξ. Then, F_ξ satisfies the following conditions:

1. F_ξ is non-decreasing;

2. F_ξ is right-continuous;

3. $$\lim_{x\to-\infty} F_\xi(x) = 0, \quad \lim_{x\to\infty} F_\xi(x) = 1.$$

[6]

Definition 1.2.6 (absolutely continuous distribution, discrete distribution)

- If there is a Borel function $f_\xi : \mathbb{R} \to \mathbb{R}$ such that for any Borel set $B \subset \mathbb{R}$

$$P\{\xi \in B\} = \int_B f_\xi(x)\, dx,$$

  then ξ is said to be a random variable with *absolutely continuous distribution* and f_ξ is called the *density* of ξ.
- If there is a (finite or infinite) sequence of pairwise distinct real numbers $x_1,\ x_2,\ \ldots$ such that for any Borel set $B \in \mathbb{R}$

$$P\{\xi \in B\} = \sum_{x_i \in B} P\{\xi = x_i\},$$

  then ξ is said to have *discrete distribution* with values $x_1,\ x_2,\ \ldots$ and *mass* $P\{\xi = x_i\}$ at x_i.

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1. pf) try to use the definition of measure. ↩
2. pf) use the Theorem 1.1 above. ↩
3. pf) not in this book... study it later. ↩
4. Note that $P(a \le X < b) = P_X((a,\, b)),\ P(X = x) = P_X(x)$,... try to compare with the classic probability theory. ↩
5. In the classic probability theory, we learned that its called as CDF. ↩
6. pf) We can use the property of measure and probability measure. ↩