10.1 Curves Defined by Parametric Equations

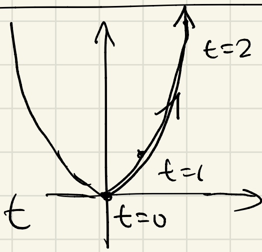
$\begin{cases} x = f(t) \\ y = g(t), \end{cases} \quad t_0 \le t \le t_{end} \Rightarrow$ Parametric Equation

---

ex1) Sketch $\begin{cases} x = t \\ y = t^2 \end{cases}$

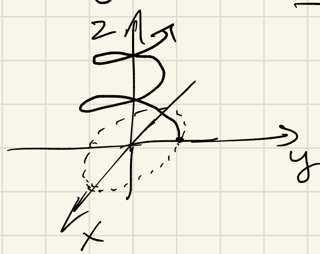
$\rightarrow y = (\overset{=t}{x})^2 = \boxed{x^2}$

↪ eliminated parameter t



ex2) $\begin{cases} x = 2\cos t \\ y = 2\sin t, \end{cases} \quad 0 \le t \le 2\pi \rightarrow x^2 + y^2 = 4 \rightarrow$ circle of $r=2$

ex3) $\begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = t \end{cases} \rightarrow$ Spiral

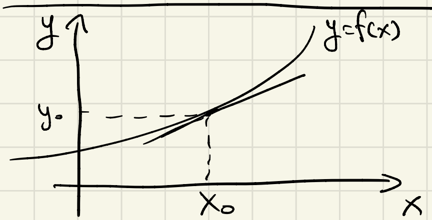


ex4) $\begin{cases} x = \sin 2t \\ y = \cos 2t, \end{cases} \quad 0 \le t \le 2\pi$

traverses twice!

Two different parametric equations can describe the same curve but not the same parametric curve!

10.2 Calculus with Parametric Curves



$$y - y_0 = f'(x)(x - x_0)$$

↳ tangent line formula

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \Rightarrow$$ if there is a function $F(x)$ that satisfies $y = F(x)$,

$$\frac{dy}{dt} = F'(x) = F'(f(t)) \cdot f'(t) = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow \text{if } \frac{dx}{dt} \neq 0, \quad \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$ ☆

---

ex1) $\left.\frac{dy}{dx}\right|_{t=2}$, $\begin{cases} x = t \\ y = t^2 \end{cases}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1} = 2t \rightarrow \left.\frac{dy}{dx}\right|_{t=2} = 2 \cdot 2 = \underline{4}$$

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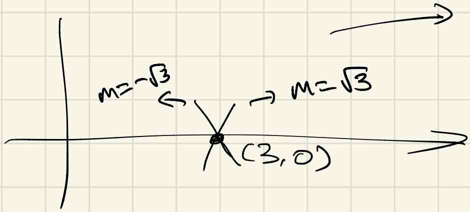
ex2) $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases} \rightarrow \left.\frac{dy}{dx}\right|_{(3,0)} = \left.\frac{3t^2 - 3}{2t}\right|_{t=\pm\sqrt{3}} = \frac{6}{2\sqrt{3}} = \pm\sqrt{3}$

$$\begin{cases} 3 = t^2 \longrightarrow \boxed{t = \pm\sqrt{3}} \\ 0 = t^3 - 3t \rightarrow t(t^2 - 3) = 0 \rightarrow t = 0, \boxed{\pm\sqrt{3}} \end{cases} \Rightarrow \pm\sqrt{3}$$

⟶ two tangent lines



ex2) cont. horizontal tangent? $\left.\frac{dy}{dx}\right|_{p=?}=0$ ☆

since $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$, $\boxed{\text{if } \left.\frac{dy}{dx}\right|_p=0, \text{ then } \left.\frac{dy}{dt}\right|_p=0 \text{ and } \left.\frac{dx}{dt}\right|_p\neq 0.}$

$$\begin{cases} x=t^2 \\ y=t^3-3t \end{cases} \rightarrow \frac{dy}{dt}=3t^2-3=0 \rightarrow t=\pm 1,$$

$$\left.\frac{dx}{dt}\right|_{t=\pm 1}\neq 0 \Rightarrow \text{horizontal tangent at } t=\pm 1$$

$\Rightarrow$ h.t. at $(1,2)$, $(1,-2)$
$\hookrightarrow t=-1$ $\hookrightarrow t=1$

Vertical tangent? $\left.\frac{dx}{dy}\right|_{p=?}=0$

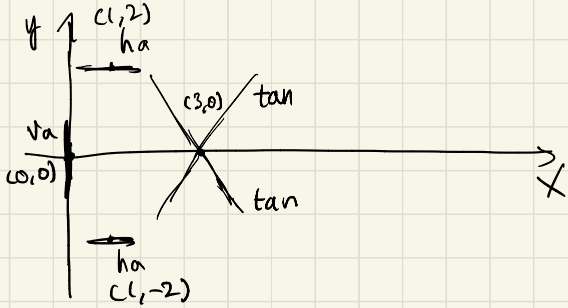
$$\frac{dx}{dy}=\frac{dx/dt}{dy/dt} \Rightarrow \text{if } \left.\frac{dx}{dt}\right|_p=0 \text{ and } \left.\frac{dy}{dt}\right|_p\neq 0, \text{ then } \left.\frac{dx}{dy}\right|_p=0$$

$\frac{dx}{dt}=2t=0 \rightarrow t=0$

$\left.\frac{dy}{dt}\right|_{t=0}=\left.3t^2-3\right|_{t=0}=-3\neq 0$

$\Rightarrow$ vertical tangent at $t=0$

$\Rightarrow$ v.a. at $(0,0)$



Calculating $\frac{d^2y}{dx^2}$ for parametric equation?

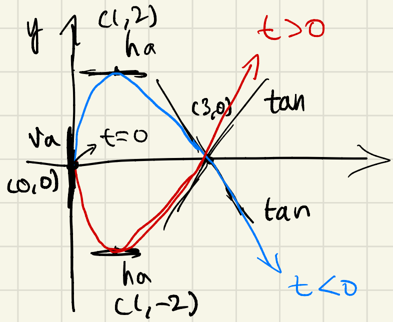
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\boxed{\frac{dy}{dx}}\right) = \frac{d\,\square/dt}{dx/dt} = \boxed{\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}}$$

ex) $\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}, \quad \frac{d^2y}{dx^2} = ?$ $\quad \frac{dy}{dx} = \frac{3t^2-3}{2t}, \quad \frac{d^2y}{dx^2} = \frac{\frac{3}{2} + \frac{3}{2t^2}}{2t}$

$$= \frac{3(t^2+1)}{4t^3}$$

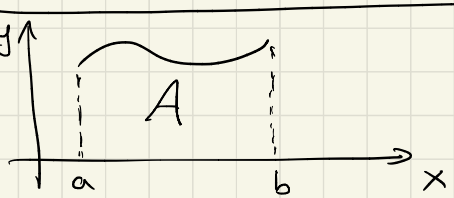
$\frac{d^2y}{dx^2} > 0$ when $t > 0$ →

$\frac{d^2y}{dx^2} < 0$ when $t < 0$ →



Area under a Parametrized Curve

$y = f(x)$

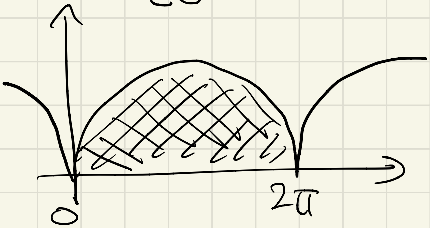


$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

let $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$, $\alpha \le t \le \beta$, area? , let α s.t. $f(\alpha) = a$, β s.t. $f(\beta) = b$.

$$\boxed{A = \int_a^b y\,dx = \int_\alpha^\beta \underbrace{g(t)}_{y} \cdot \underbrace{f'(t)\,dt}_{dx}}$$

ex) $\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1-\cos\theta) \end{cases}$ for $0 \le \theta \le 2\pi$. A?



$$A = \int_0^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)\,d\theta$$

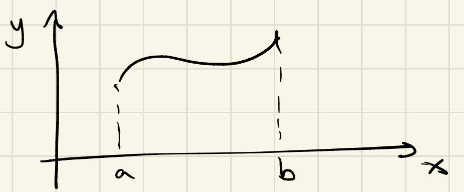
$$= r^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

$$= r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

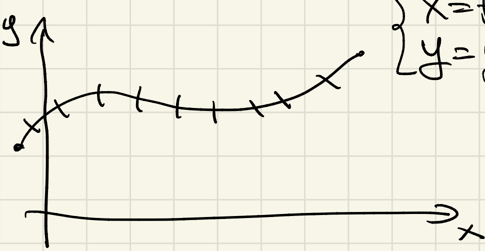
$$= r^2\int_0^{2\pi}\left(1-2\cos\theta+\frac{1+\cos2\theta}{2}\right)d\theta = r^2\left[\theta-2\sin\theta+\frac{1}{2}\theta+\frac{\sin2\theta}{4}\right]_0^{2\pi}$$

$$= r^2\left(\frac{3}{2}\cdot2\pi\right) = \boxed{3\pi r^2}$$

Arclength



$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



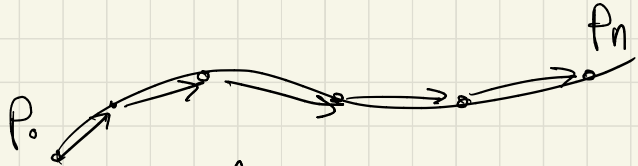
$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ $\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$$L = \int_a^b \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\,dx$$

$$= \int_a^b \sqrt{1+\frac{(dy/dt)^2}{(dx/dt)^2}}\,dx = \int_a^b \sqrt{\frac{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\,dx$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot\cancel{\frac{1}{dx/dt}}\cdot\cancel{\frac{dx}{dt}}\cdot dt = \boxed{\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

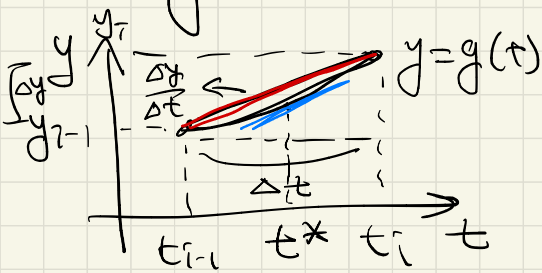
Geometric Interpretation:



$$L \simeq \sum_{i=1}^{n} |P_{i-1} P_i| = \sum_{i=1}^{n} |(x_{i-1}, y_{i-1})(x_i, y_i)|$$

$$= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$\rightarrow$ Δy in terms of t? $y = g(t)$



by MVT, there exists t^* s.t. $g'(t^*) = \frac{\Delta y}{\Delta t}$

$$\rightarrow \Delta y = g'(t^*) \cdot \Delta t$$

similarly, $\Delta x = f'(t^{**}) \cdot \Delta t$

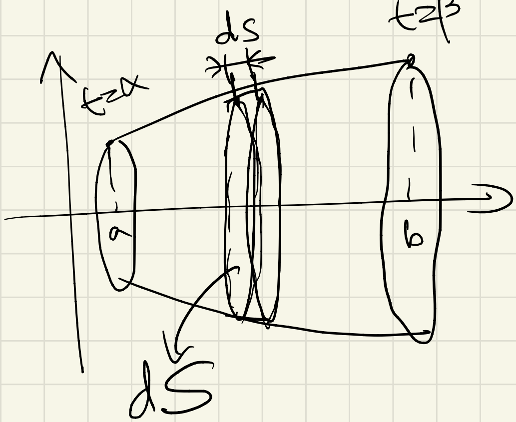
$$\rightarrow L \simeq \sum_{i=1}^{n} \sqrt{(f'(t^{**})\Delta t)^2 + (g'(t^*)\Delta t)^2}$$

$$= \sum_{i=1}^{n} \sqrt{[f'(t_i^{**})]^2 + [g'(t_i^*)]^2}\, \Delta t$$

$$\Rightarrow L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^*)]^2}\, \Delta t = \int_\alpha^\beta \underbrace{\sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt}_{ds}$$

$$= \int_\alpha^\beta ds$$

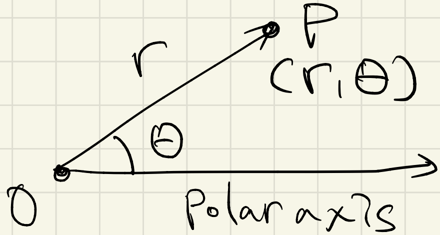
Surface area



$$dS \simeq 2\pi \cdot y \cdot ds$$

$$\boxed{S = \int_{\alpha}^{\beta} 2\pi y \, ds}$$ ☆

10.3 Polar Coordinates



r = distance from O to P

θ = angle between $\overline{OP}$ and polar axis

$(-r, \theta) = (r, \theta + \pi)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \end{cases}$$

---

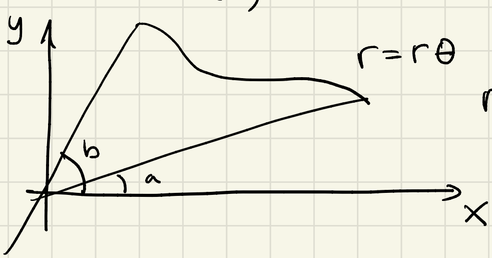
ex) $r = 2\cos\theta$ to cartesian

$$r^2 = 2r\cos\theta \rightarrow x^2+y^2 = 2x \rightarrow \boxed{(x-1)^2+y^2=1}$$

Tangents to Polar Curves

$r = r(\theta)$, $a \le \theta \le b$

recall: $\begin{cases} x = f(t) \\ y = g(t) \end{cases} \Rightarrow \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$



now: $\begin{cases} x = r\cos\theta = r(\theta)\sin\theta \\ y = r\sin\theta = r(\theta)\cos\theta \end{cases}, \quad a \le \theta \le b$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \boxed{\frac{r(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}}$$

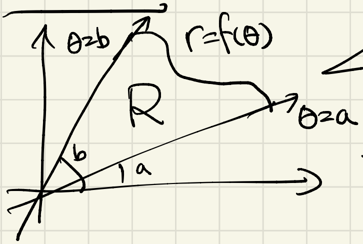
---

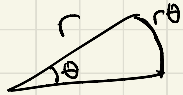
ex) $r = 1+\sin\theta$. slope at $\theta = \pi/3$?

$$\frac{dy}{dx} = \left.\frac{(1+\sin\theta)'\sin\theta + (1+\sin\theta)\cos\theta}{(1+\sin\theta)'\cos\theta - (1+\sin\theta)\sin\theta}\right|_{\pi/3} = -1$$

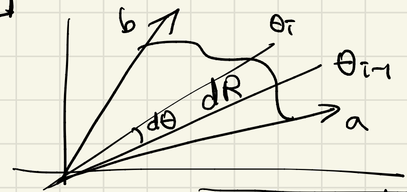
10.4 Areas and Length in Polar

Areas





sector of circle: $A = \frac{1}{2} r^2 \theta$

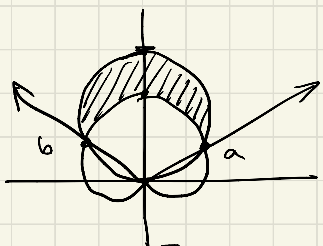


$\theta_{i-1} < \theta_i^* < \theta_i$

$$dR \simeq \frac{1}{2}[f(\theta_i^*)]^2 \cdot d\theta \rightarrow \boxed{R = \frac{1}{2}\int_a^b r^2 \, d\theta}$$

---

ex) A of region inside $r = 3\sin\theta$, outside of $r = 1+\sin\theta$.



① Intersection a, b:

$3\sin\theta = 1+\sin\theta \rightarrow \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \rightarrow a = \frac{\pi}{6}, b = \frac{5\pi}{6}$

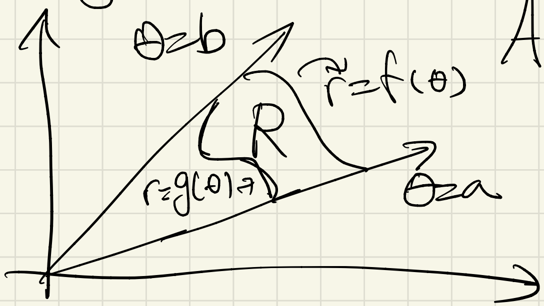
$$R = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left[9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right] d\theta$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 2\sin\theta - 1)\, d\theta = \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta$$

$$= 3\theta - 2\sin 2\theta + 2\cos\theta \Big|_{\pi/6}^{\pi/2} = 3\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \boxed{\pi}$$

In general:



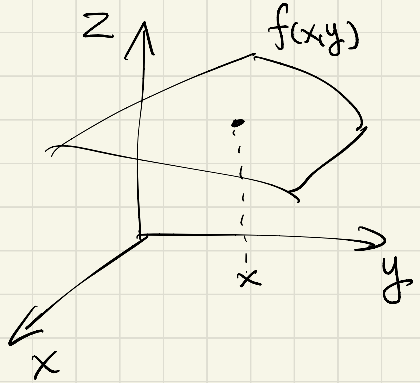
$$A=\frac{1}{2}\int_a^b\left[f^2(\theta)-g^2(\theta)\right]d\theta$$

Arc length

$$\begin{cases} x=f(\theta)\cos\theta \\ y=f(\theta)\sin\theta \end{cases} \Rightarrow L=\int_a^b\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$=\int_a^b\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

12.1 3D Coordinate System



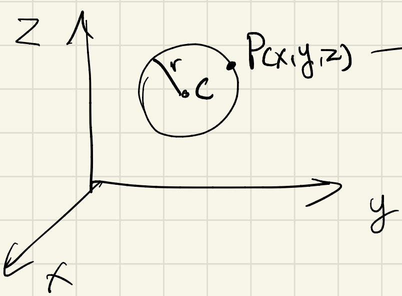
Surface: $z = f(x,y)$

↳ defining z interms of x and y in $\mathbb{R}^3$ space

$$\text{Distance in } \mathbb{R}^3 = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

---

ex) Equation of a sphere, $r = r$, $c = (h, k, l)$



$\rightarrow |\overline{PC}| = r$ (def. of sphere)

$\rightarrow \sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r$

$\Rightarrow (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

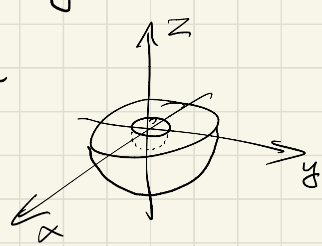
---

ex) Region represented by $\begin{cases} 1 \le x^2 + y^2 + z^2 \le 9 & ① \\ z \le 0 & ② \end{cases}$

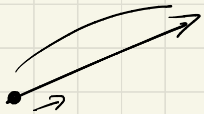
① $x^2+y^2+z^2 \le 9 \rightarrow$ solid sphere, $r=3$, origin-centered

$1 \le x^2+y^2+z^2 \rightarrow$ exterior of sphere, $r=1$, origin-centered

② $z \le 0 \rightarrow$ all points below the xy-plane



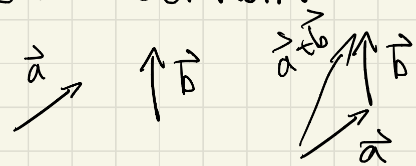
12.2 Vectors



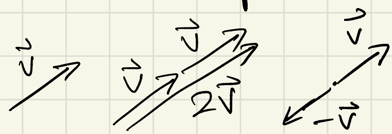
Vector — Magnitude (Length)
— Direction

Equivalent Vectors – if vectors have the same magnitude and direction, they are equivalent.

Vector Addition:



Scalar Multiplication:



$\vec{v} \cdot 0 = \vec{0}$

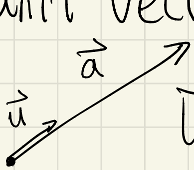
Length: $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ in $\mathbb{R}^3$

"Special" Vectors: $\hat{i} = \langle 1,0,0 \rangle$, $\hat{j} = \langle 0,1,0 \rangle$, $\hat{k} = \langle 0,0,1 \rangle$

↳ any $\mathbb{R}^3$ vector can be represented as:

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

Unit Vector: vector of length 1



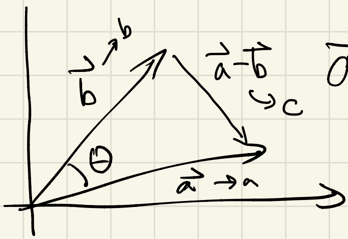
$$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{|\vec{a}|} \cdot \vec{a} = \vec{a}_0$$

12.3&4 Multiplication of Vectors

Dot Product $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ $(\mathbb{R}^3)$

$\vec{a}\cdot\vec{a} = |\vec{a}|^2 \Leftrightarrow |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$

Geometric Interpretation of $\vec{a}\cdot\vec{b}$



$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\theta}$ $(0 \le \theta \le \pi)$

Proof: Law of Cosines

$c^2 = a^2 + b^2 - 2ab\cos\theta$

$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |b|^2 - 2|\vec{a}||\vec{b}|\cos\theta \quad \cdots ①$

$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = (\vec{a}-\vec{b})\cdot\vec{a} - (\vec{a}-\vec{b})\vec{b}$

$= |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \quad \cdots ②$

$\to \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}||\vec{b}|\cos\theta = \cancel{|\vec{a}|^2} - 2\vec{a}\cdot\vec{b} + \cancel{|\vec{b}|^2}$ ①=②

$\to |\vec{a}||\vec{b}|\cos\theta = \vec{a}\cdot\vec{b}$ //

$\Rightarrow \cos\theta = \frac{\vec{a}}{|\vec{a}|}\cdot\frac{\vec{b}}{|\vec{b}|} = \vec{a}_0\cdot\vec{b}_0$

---

ex) angle between $\vec{a} = \hat{i}$, $\vec{b} = \hat{i} + \hat{j}$?

$\cos\theta = \frac{1}{1\cdot\sqrt{2}}\cdot\langle 1,0\rangle\cdot\langle 1,1\rangle = \frac{1}{\sqrt{2}}$

$\to \theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\pi/4}$

Orthogonal Vectors : $\theta = \frac{\pi}{2}$, $\vec{a} \perp \vec{b}$

$\rightarrow$ Thus, $\vec{a} \cdot \vec{b} = 0$! ($\vec{0}$ is perpendicular to all!)

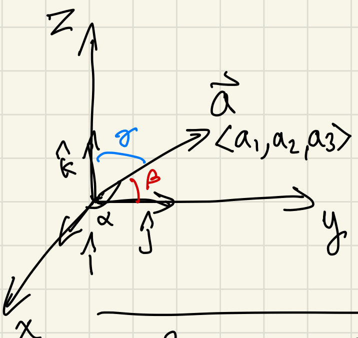
$$\boxed{\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b}}$$ *

$\vec{a} \cdot \vec{b} > 0$ when $0 \leq \theta \leq \frac{\pi}{2}$

$\vec{a} \cdot \vec{b} = 0$ when $\theta = 0$

$\vec{a} \cdot \vec{b} < 0$ when $\frac{\pi}{2} \leq \theta \leq \pi$

Direction Angles & Direction Cosines



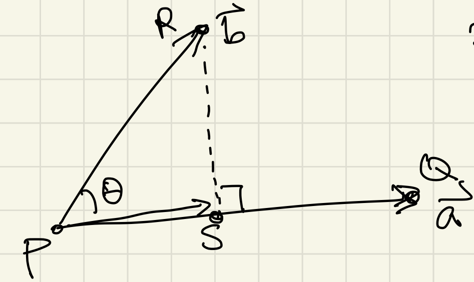
$$\left.\begin{aligned} \cos\alpha &= \frac{\vec{a} \cdot \hat{i}}{|\vec{a}|} = \frac{a_1}{|\vec{a}|} \\ \cos\beta &= \frac{\vec{a} \cdot \hat{j}}{|\vec{a}|} = \frac{a_2}{|\vec{a}|} \\ \cos\gamma &= \frac{\vec{a} \cdot \hat{k}}{|\vec{a}|} = \frac{a_3}{|\vec{a}|} \end{aligned}\right] \text{direction cosines}$$ *

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

$$\vec{a} = |\vec{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

$$\rightarrow \vec{u}_{\vec{a}} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Projections



$\overrightarrow{PS} = \mathrm{proj}_{\vec{a}}\vec{b}$

(projection of $\vec{b}$ onto $\vec{a}$)

$|\overrightarrow{PS}| = \mathrm{comp}_{\vec{a}}\vec{b}$

(component of $\vec{b}$ along $\vec{a}$)

$\cos\theta = \dfrac{|\overrightarrow{PS}|}{|\vec{b}|} \Rightarrow |\overrightarrow{PS}| = |\vec{b}|\cos\theta \to |\vec{a}||\vec{b}|\cos\theta = |\overrightarrow{PS}|\cdot|\vec{a}|$

$\to \vec{a}\cdot\vec{b} = |\overrightarrow{PS}||\vec{a}| \to |\overrightarrow{PS}| = \boxed{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \mathrm{comp}_{\vec{a}}\vec{b}}$ ☆

then, $\mathrm{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\dfrac{\vec{a}}{|\vec{a}|} = \boxed{\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}}$

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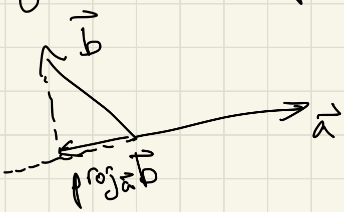
ex) $\mathrm{proj}_{\vec{a}}\vec{b}$, $\mathrm{comp}_{\vec{a}}\vec{b}$, $\vec{a} = \langle 3, -3, 1\rangle$, $\vec{b} = \langle 2, 4, -1\rangle$

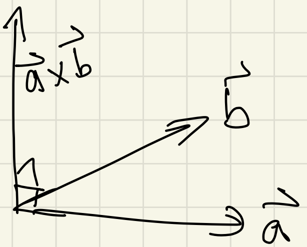
$\mathrm{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{6-12-1}{\sqrt{9+9+1}} = -\dfrac{7}{\sqrt{19}}$

$\mathrm{proj}_{\vec{a}}\vec{b} = \mathrm{comp}_{\vec{a}}\vec{b}\cdot\dfrac{\vec{a}}{|\vec{a}|} = -\dfrac{7}{19}\langle 3, -3, 1\rangle = \boxed{\left\langle -\dfrac{21}{19}, \dfrac{21}{19}, -\dfrac{7}{19}\right\rangle}$

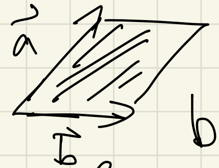
Negative $\mathrm{comp}_{\vec{a}}\vec{b}$? $\to \vec{a}\cdot\vec{b} < 0 \to \frac{\pi}{2} < \theta \leq \pi$



Cross Product $\vec{a} \times \vec{b}$



$|\vec{a} \times \vec{b}|$ = area of parallelogram,



perpendicular to both $\vec{a}$ and $\vec{b}$
(Curl using right hand)

Determinants

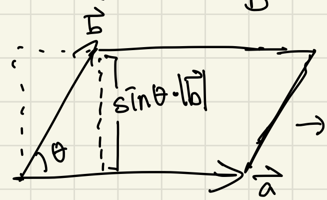
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \times \vec{b}$?

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boxed{(a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}}$$

Is $(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp b$? $(\vec{a} \times \vec{b}) \cdot \vec{a} \text{ or } \vec{b} = 0$!

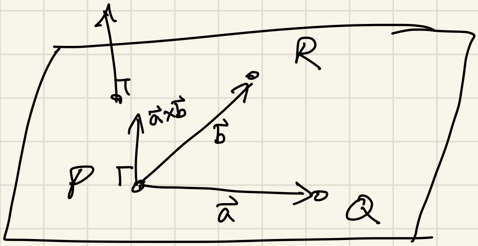
$$\boxed{|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta}$$



$\rightarrow A = |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta$

$$\boxed{|\vec{a} \times \vec{b}| = 0 \iff \vec{a} \parallel \vec{b}}$$

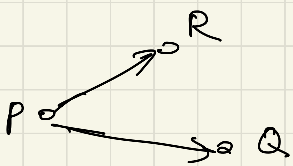
ex) Find a vector $\perp$ to the plane passing through $P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$



$\vec{a} = \overrightarrow{PQ} = \langle -3,1,-7\rangle$, $\vec{b} = \overrightarrow{PR} = \langle 0,-5,-5\rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \langle -5-35, -15, 15\rangle = \boxed{\langle -40,-15,15\rangle}$$

ex) Area of $\triangle PQR$?



$$|\vec{a}\times\vec{b}| = 2\triangle PRQ$$

$$\underbrace{64+\underbrace{9+9}_{18}}_{82}$$

$$\rightarrow \triangle PRQ = \frac{|5\langle -8,-3,3\rangle|}{2} = \boxed{\frac{5\sqrt{82}}{2}}$$

Cross Product is <u>NOT</u> commutative!

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

Triple Product $\vec{a}\cdot(\vec{b}\times\vec{c})$

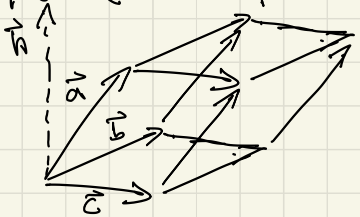
$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot\left[(b_2c_3 - b_3c_2)\hat{i} - \cdots\right]$$

→ same as replacing unit vectors with components of $\vec{a}$

$$= \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \text{ (scalar triple product)}$$

$|\vec{a}\cdot(\vec{b}\times\vec{c})| = V$ of the parallelepiped formed



$$V = |\vec{b}\times\vec{c}|\cdot\overbrace{\text{comp}_{(\vec{b}\times\vec{c})}\vec{a}}^{\vec{h}}$$

$$= |\vec{b}\times\vec{c}|\cdot\frac{|\vec{a}\cdot(\vec{b}\times\vec{c})|}{|\vec{b}\times\vec{c}|} = \vec{a}\cdot(\vec{b}\times\vec{c})$$

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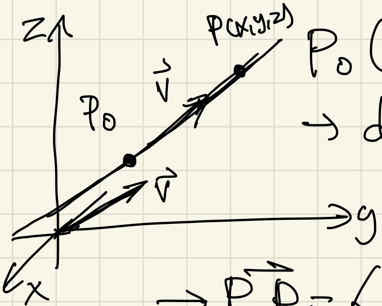
ex) Show that $\vec{a} = \langle 1, 4, -7\rangle$, $\vec{b} = \langle 2, -1, 4\rangle$, $\vec{c} = \langle 0, -9, 18\rangle$ are coplanar. → $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix}1 & 4 & -7\\ 2 & -1 & 4\\ 0 & -9 & 18\end{vmatrix} = 1(-18+36) - 4(36) - 7(-18)$$

$$= 18 - 8\cdot 18 + 7\cdot 18 = 0$$

12.5 Equations of Lines and Planes



$P_0(x_0, y_0, z_0)$, $\vec{V} = \langle a, b, c \rangle$

→ describe all points P on the line

$$\overrightarrow{P_0P} = t\vec{V}$$

$\rightarrow \overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$, $t\vec{v} = \langle ta, tb, tc \rangle$

$$\Rightarrow \begin{cases} x - x_0 = ta \\ y - y_0 = tb \\ z - z_0 = tc \end{cases} \rightarrow \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}, \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\vec{r} = \langle x, y, z \rangle, \vec{r_0} = \langle x_0, y_0, z_0 \rangle \rightarrow \vec{r} = \vec{r_0} + t\vec{V}$$

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ex) passes $(5, 1, 3)$, // to $\langle 1, 4, -2 \rangle$

$$\underbrace{\langle x, y, z \rangle}_{=\vec{r}} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle = \langle 5+t, 1+4t, 3-2t \rangle$$

↳ $t = 1$: $\langle x, y, z \rangle = \langle 5+1, 1+4, 3-2 \rangle = \langle 6, 5, 1 \rangle$

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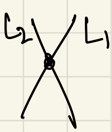
ex) show that L_1 and L_2 are skew lines (do not intersect, are not parallel)

L_1: $x = 1 + t$, $y = -2 + 3t$, $z = 4 - t$, $t \in \mathbb{R}$

L_2: $x = 2s$, $y = 3 + s$, $z = -3 + 4s$, $s \in \mathbb{R}$

① $\vec{v_1} = \langle 1, 3, -1 \rangle$, $\vec{v_2} = \langle 2, 1, 4 \rangle \rightarrow \vec{v_1} \neq \alpha \vec{v_2} \rightarrow$ not $\vec{v_1} \parallel \vec{v_2}$ → not $L_1 \parallel L_2$

②



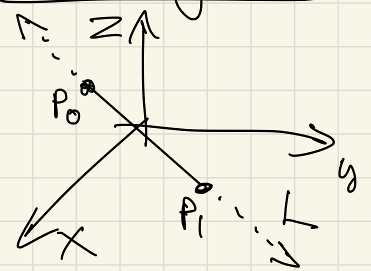
$$\begin{cases} 1 + t = 2s \\ -2 + 3t = 3 + s \\ 4 - t = -3 + 4s \end{cases}$$

$1 + t = 2s \rightarrow t = 2s - 1 \leftarrow t = \frac{11}{5}$

$-2 + 3t = 3 + s \rightarrow -2 + 6s - 3 = 3 + s \rightarrow 5s = 8 \rightarrow s = \frac{8}{5}$

$4 - t = -3 + 4s \rightarrow 4 - \frac{11}{5} = -3 + \frac{32}{5} \rightarrow 1 = \frac{43}{5} \rightarrow$ False, $\boxed{\text{no intersection}}$

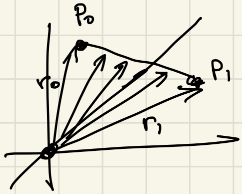
Line Segment $\overline{P_0 P_1}$



$$\vec{r} = \vec{r_0} + t\overbrace{(\vec{r_1} - \vec{r_0})}^{\vec{v}}$$

$$= (1-t)\vec{r_0} + t\vec{r_1}$$

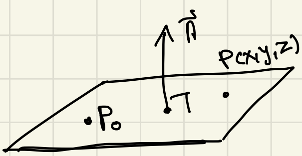
$\rightarrow$ for $t=0$, $\vec{r} = \vec{r_0}$; for $t=1$, $\vec{r} = \vec{r_1}$

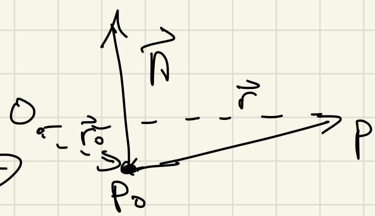


$\Rightarrow$ for $0 \le t \le 1$, $\vec{r}$ describes the points in the segment $\overline{P_0 P_1}$.

$$\boxed{\vec{r} = (1-t)\vec{r_0} + t\vec{r_1},\ 0 \le t \le 1}$$

Planes





$\overrightarrow{P_0 P} \perp \vec{n}$

$\hookrightarrow \boxed{(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0}$

if $\vec{n} = \langle a, b, c \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, $\vec{r} = \langle x, y, z \rangle$

$\rightarrow \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

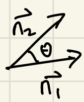
$\Leftrightarrow ax + by + cz + d = 0 \quad (d = -(ax_0 + by_0 + cz_0))$

ex) Find the intersection point $\begin{cases} x = 2+3t \\ y = -4t \\ z = 5+t \end{cases}$, $4x+5y-2z=18$

$\Rightarrow 4(2+3t)+5(-4t)-2(5+t)=18 \rightarrow t=-2$

$\rightarrow \boxed{x=-4,\ y=8,\ z=3} \rightarrow P(-4,8,3)$

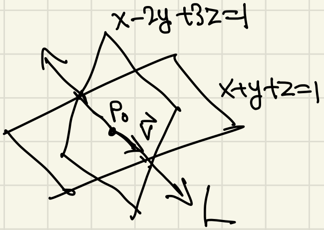
$\vec{n}_1 \parallel \vec{n}_2 \rightarrow$ parallel planes



$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

angle between planes = angle between $\vec{n}$

Line of intersection: ① P_0 ② direction $\vec{v}$



① Find (x, y, z) that satisfies both

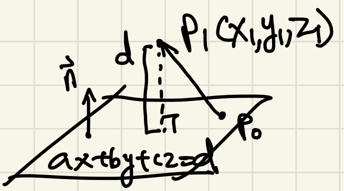
$\begin{cases} x-2y+3z=1 \\ x+y+z=1 \end{cases} \rightarrow$ set $y=0$ (at x-z plane) $\begin{cases} x+3z=1 \\ x+z=1 \end{cases} \rightarrow \begin{matrix} x=1 \\ z=0 \end{matrix}$

$\rightarrow P_0(1,0,0)$

② Find $\vec{v}$ ($\vec{v} \perp \vec{n}_1$ and $\vec{v} \perp \vec{n}_2 \Rightarrow \vec{v} = \vec{n}_1 \times \vec{n}_2$)

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, 3 \rangle \Rightarrow \boxed{\dfrac{x-1}{5} = \dfrac{y}{-2} = \dfrac{z}{3}}$

Distance between Point P_1 and plane

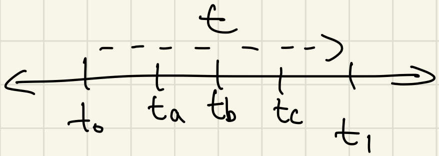


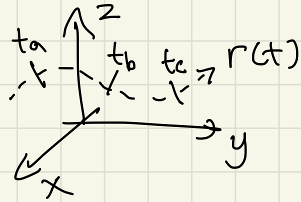
$d = \text{comp}_{\vec{n}}\vec{b} = \dfrac{\vec{n}\cdot\vec{b}}{|\vec{n}|} = \boxed{\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$

13.1 Vector Functions

$\vec{v}(t) = \langle \cos t, \sin t, t \rangle \rightarrow$ vector function

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad t = [t_0, t_1]$





$\vec{r}(t) = \langle t, 1+t, 3t \rangle = \begin{cases} x = t \\ y = 1+t \\ z = 3t \end{cases} \rightarrow$ line!

Limits of $\vec{r}(t)$: $\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$

$\vec{r}(t)$ is continuous if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

13.2 Derivatives and Integrals of $\vec{r}(t)$

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \quad \text{(tangent line)}$$

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

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ex) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, tangent line at $(0, 1, \pi/2)$

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \to \vec{r}'(\pi/2) = \langle -1, 0, 1 \rangle = \vec{v}$

$\to \{ x = -t, \ y = 1, \ z = t + \pi/2 \}$

$$\frac{d}{dt}[f(t) \cdot \vec{v}(t)] = f'(t)\vec{v}(t) + f(t)\vec{v}'(t)$$

$$\frac{d}{dt}[\vec{v}(t) \times \vec{u}(t)] = \vec{v}'(t) \times \vec{u}(t) + \vec{v}(t) \times \vec{u}'(t)$$

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ex) if $|\vec{r}(t)| = C$, then $\vec{r}'(t) \perp \vec{r}(t)$. Prove.

$\underline{\vec{r}'(t) \cdot \vec{r}(t) = 0}$, $|\vec{r}(t)| = \sqrt{(\vec{r}(t))^2} = C \to (\vec{r}(t))^2 = C^2$

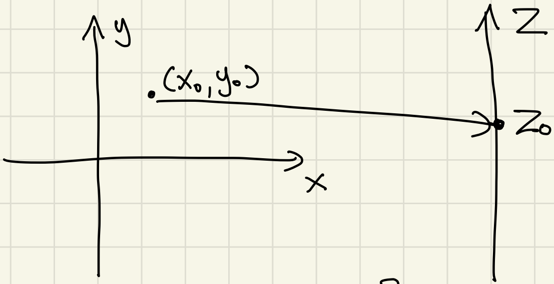
$\to \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0$

$\to 2\vec{r}'(t) \cdot \vec{r}(t) = 0 \to \vec{r}'(t) \cdot \vec{r}(t) = 0$ //

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

$$\int \vec{r}(t)\,dt = \langle F(t), G(t), H(t) \rangle + \vec{C}$$

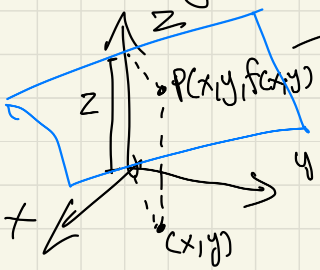
14.1 Functions of Multiple Variables



$\mathbb{R}^2 \to \mathbb{R}$

↳ A set of two inputs correspond to a single output

Visualizing in $\mathbb{R}^3$



→ set of all points for $(x, y) \in D$

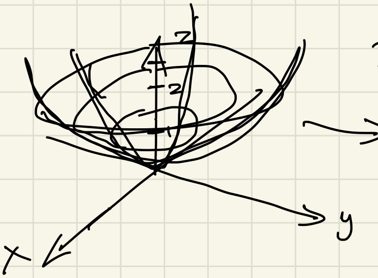
→ $\underline{z = f(x, y)}$ (surface)

what about $w = f(x, y, z)$?

---

ex) find D, R, and sketch $f(x,y) = x^2 + y^2$

$z = x^2 + y^2 \to D = \{(x,y) \in \mathbb{R}^2\}, R = \{z \in \mathbb{R} \mid z \geq 0\}$



for a fixed $z \to x^2 + y = C$

↳ circle of radius $\sqrt{C}$

→ Paraboloid in $\mathbb{R}^3$

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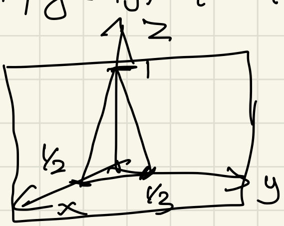
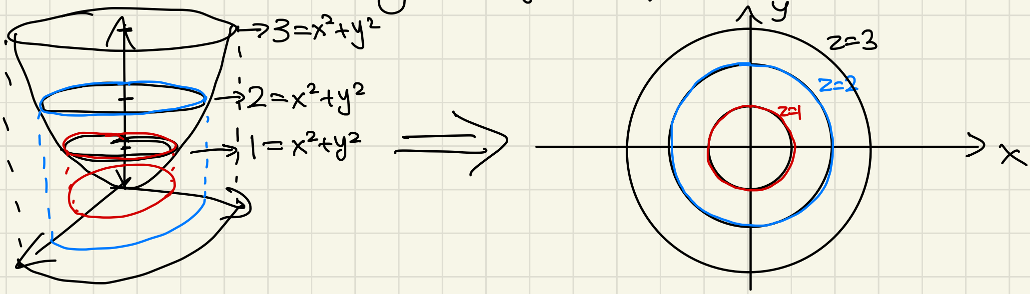
ex) $g(x,y) = 1 - 2x - 3y \to 2x + 3y + z = 1$ (plane)

↳ $f(x,y) = ax + by + c \to$ always a plane
$(a, b \neq 0)$



Level Curves

Curves defined by $f(x,y)=k$, where k is a constant → for $\mathbb{R}^3$



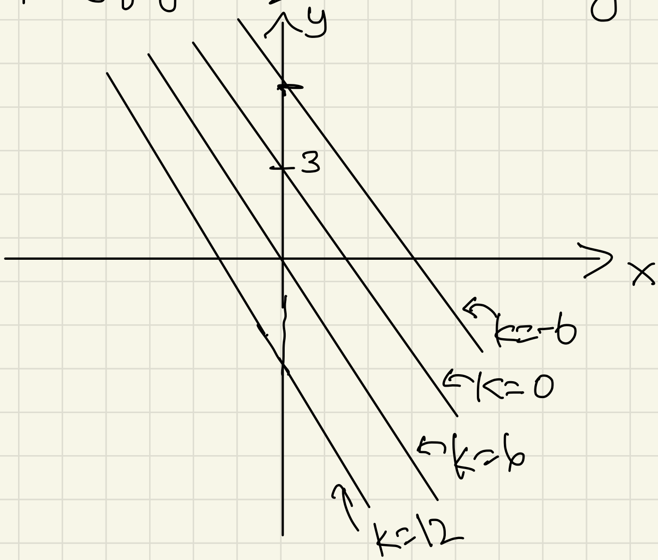
ex) $f(x,y)=6-3x-2y$, $k=-6, 0, 6, 12$

$\rightarrow 3x-2y+(k-6)=0$ (straight lines)

$k=-6$: $3x+2y-12=0 \rightarrow y=-\frac{3}{2}x+6$

$k=0$: $3x+2y-6=0 \rightarrow y=-\frac{3}{2}x+3$

$k=6$: $y=-\frac{3}{2}x$ $\quad$ $k=12$: $y=-\frac{3}{2}x-3$

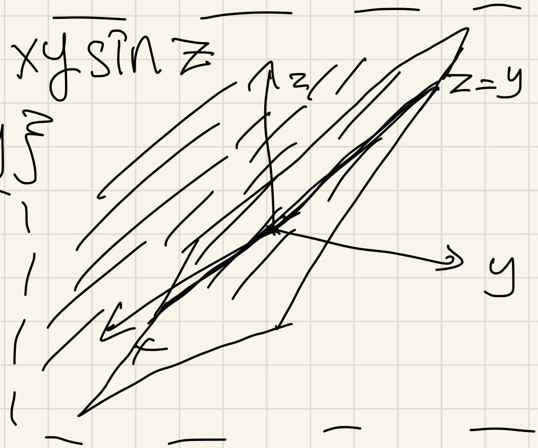


Function of 3 Variables

Assign a surface to a real number w

ex) $f(x,y,z) = \ln(z-y) + xy\sin z$

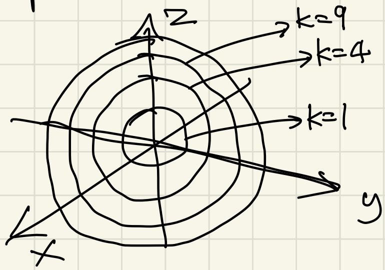
$D = \{(x,y,z) \in \mathbb{R}^3 \mid z > y\}$



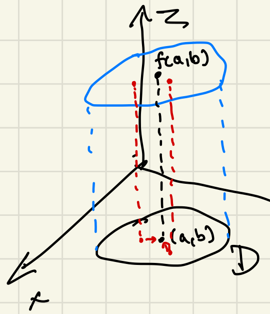
ex) Level surfaces of

$f(x,y,z) = x^2+y^2+z^2$

$\rightarrow$ spheres of $r = \sqrt{k}$



14.2 Limits and Continuity for Multivariable



$\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, if for every $\varepsilon > 0$, there is a corresponding $\delta > 0$ st. if $(x,y)\in D$ and $\text{dist}((x,y),(a,b)) < \delta$ then $|f(x,y) - L| < \varepsilon$ (δ-ε limit def.)

If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along path C_1
and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along path C_2 $\Big]$ $\lim = \text{DNE}$

---

ex) Prove $\lim\limits_{(x,y)\to(0,0)}\left(\dfrac{x^2-y^2}{x^2+y^2}\right) = \text{DNE}$

Proof: Let C_1 be path: $y=0$, then

$$\lim_{(x,0)\to(0,0)} \frac{x^2-y^2}{x^2-y^2} = \lim_{x\to 0}\frac{x^2}{x^2} = \boxed{1} = L_1$$

Let C_2 be path: $x=0$, then

$$\lim_{(0,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0}\frac{-y^2}{y^2} = \boxed{-1} = L_2$$

$L_1 \neq L_2$
$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \text{DNE}$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = ?$ $C_1: y = mx \to L_1 = 0$, $C_2: x = y^2 \to L_2 = \frac{1}{2}$ $\to$ DNE

---

ex) Prove $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy^2}{x^2+y^2} = 0$.

For any $\varepsilon>0$, find $\delta>0$ st. $0<\sqrt{x^2+y^2}<\delta$ then $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$.

$\to$ can we express $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| < f(\delta) < \varepsilon$

ex. cont.) $\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} = \frac{3x^2\sqrt{y^2}}{x^2+y^2} \leq \frac{3x^2\sqrt{x^2+y^2}}{x^2+y^2}$

$= \frac{3x^2}{x^2+y^2}\cdot\delta \leq \frac{3(x^2+y^2)}{x^2+y^2}\delta = 3\delta < \varepsilon$

$\rightarrow \left|\frac{3x^2y}{x^2+y^2} - 0\right| \leq 3\delta < \varepsilon$

$\Rightarrow$ for any $\varepsilon > 0$, $\exists\, \delta > 0$, e.g. $\delta = \frac{\varepsilon}{3}$ s.t.

$$\sqrt{x^2+y^2} < \delta \rightarrow \left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$$

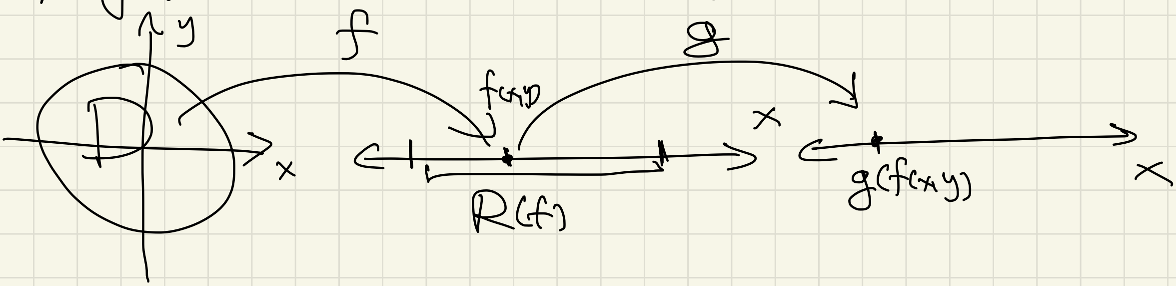
---

f is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. ☆

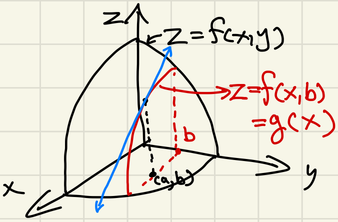
f is continuous over D if all points on D are cont. ☆

Arithmetic of continuous functions $(+,-,\times,\div)$ results in a continuous function. ☆

If f is cont. on D & g is cont. on $R(f)$, then $h = g\circ f$ is cont. on $D(f)$.



14.3 Partial Derivatives



keep y constant at b, vary x

→ make into single variable function

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{f(x+h,b)-f(x,b)}{h}$$

$$\Rightarrow g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$f_x(a,b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} \quad \text{[Partial Derivative]}$$

Tangent Line: $z-c = f_x(a,b)(x-a)$ *

---

ex) $f(x,y) = 4-2x^2-y^2$. $f_x(1,1)$?

$f(x,1) = 4-2x^2-1 = -2x^2+3 \to f_x(x) = \boxed{-4x}$ $f_x(1) = \boxed{-4}$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{(treat } y \text{ as constant)}$$

---

ex) $f(x,y) = x^3+3x^2y-2xy^2+xy-3$

$f_x(x,y) = 3x^2+6xy-2y^2+y$, $f_y(x,y) = 3x^2-4xy+x$

Implicitly Defined Function $\boxed{F(x,y,z)=0}$

$$x^3+y^3+z^3+6xyz=1 \Rightarrow x^3+y^3+[f(x,y)]^3+6xyf(x,y)=1$$

$$\frac{\partial z}{\partial x}? \to 3x^2+3z^2\cdot\frac{\partial z}{\partial x}+6y\left(z+x\frac{\partial z}{\partial x}\right)=0$$

$$\to (3z^2+6xy)\frac{\partial z}{\partial x}=-3x^2-6yz \to \frac{\partial z}{\partial x}=\boxed{\frac{-3x^2-6yz}{3z^2+6xy}}$$

Higher Order Derivatives

$$f_{xx}=(f_x)_x=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2} \quad f_{yy}=\frac{\partial^2 z}{\partial y^2}$$

$$f_{xy}=(f_x)_y=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\,\partial x} \quad f_{yx}=\frac{\partial^2 z}{\partial x\,\partial y}$$

---

ex) Show that $u(x,t)=f(x+at)+g(x-at)$ solves $u_{tt}=a^2u_{xx}$

$$u_x=f'(x+at)+g'(x-at),\ u_{xx}=f''(x+at)+g''(x-at)$$

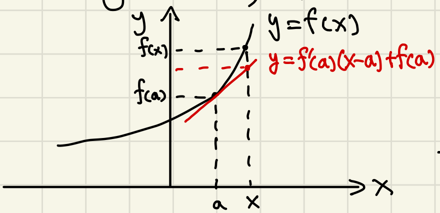
$$u_t=af'(x+at)-ag'(x-at),\ u_{tt}=a^2f''(x+at)+a^2g''(x+at)$$

$$\Rightarrow u_{tt}=a^2\left(f''(x+at)+g''(x+at)\right)=a^2u_{xx}$$

If f_{xy} and f_{yx} are both continuous, $f_{xy}(x,y)=f_{yx}(x,y)$

14.4 Tangent Planes & Linear Approximation

for $y=f(x)$, $f(x) \simeq f(a)+f'(a)(x-a)$

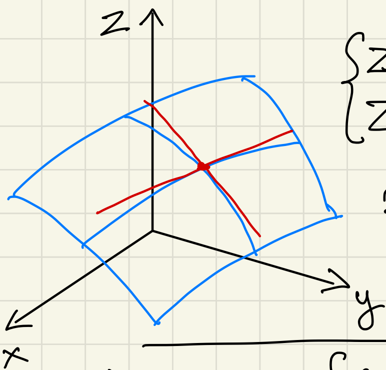


for points near $(a, f(a))$

$\Rightarrow$ In 3D, a plane to a surface

Tangent Plane through $P(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$

$\Rightarrow A(x-x_0)+B(y-y_0)+C(z-z_0)=0$



$$\begin{cases} z-z_0 = f_x(x_0,y_0)(x-x_0) \;\ldots\; y=y_0 \\ z-z_0 = f_y(x_0,y_0)(y-y_0) \;\ldots\; x=x_0 \end{cases}$$

Plane: $\frac{A}{C}(x-x_0)+\frac{B}{C}(y-y_0)+(z-z_0)=0$

$\rightarrow f_x(x_0,y_0) = -\frac{A}{C},\ f_y(x_0,y_0) = -\frac{B}{C}$

$$\Rightarrow \boxed{z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)} \star$$

---

ex) Tangent Plane of $z=2x^2+y^2$ at $P(1,1,3)$?

$z-3 = (4x)\big|_{x=1}(x-1) + (2y)\big|_{y=1}(y-1)$

$\rightarrow z-3 = 4(x-1)+2(y-1) \rightarrow 4x+2y-z-3=0$

$\star$

$z_T = f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \simeq f(x,y)$

$\hookrightarrow$ good approximation when (x,y) is close to (x_0,y_0)

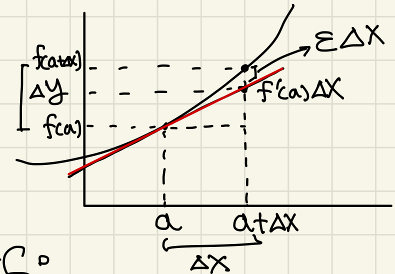
Differentiability of $f(x,y)$

for $y=f(x)$, $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, $h = \Delta x$

$\rightarrow \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} \Rightarrow f'(a)$ exists if $\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)$ as $\Delta x \to 0$

$\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a) = \varepsilon \to 0$ as $\Delta x \to 0$

$\Rightarrow \underbrace{f(a+\Delta x)-f(a)}_{:=\Delta y} = f'(a)\Delta x + \varepsilon \Delta x$



$\Rightarrow \Delta y = f'(a)\Delta x + \varepsilon \Delta x$

$y=f(x)$ is differentiable at $x=a$ if:

$\Delta y = f'(a)\Delta x + \varepsilon\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$

For $z=f(x,y)$, differenitability is:

$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 = 0$ as $(\Delta x, \Delta y) \to (0,0)$.

... or use a simpler theorem:

if f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable.

ex) $f(x,y) = xe^{xy}$, $f(1.1, -0.1) \simeq ?$

$(f_x(x,y) = e^{xy} + xye^{xy}, \ f_x(1,0) = 1)$

$(f_y(x,y) = x^2 e^{xy}, \ f_y(1,0) = 1)$

$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = x+y$

$\rightarrow f(1.1, -0.1) \simeq L(1.1, -0.1) = 1.1 - 0.1 = \boxed{1}$

Differentials

if dx is an independent small change of x,

$dy := f'(x)\,dx \quad (\underbrace{y - f(a)}_{dy} = f'(a)\underbrace{(x-a)}_{dx})$

$\rightarrow$ for 3D,

$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

14.5 Chain Rule

$$z=f(x,y),\ x=g(t),\ y=h(t) \rightarrow z=f(g(t),h(t))$$

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dg}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dh}{dt}$$

$$z=f(\overset{x}{\overset{\shortparallel}{g(s,t)}},\overset{y}{\overset{\shortparallel}{h(s,t)}}) \rightarrow \frac{\partial z}{\partial s},\ \frac{\partial z}{\partial t}$$

$$\frac{\partial z}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial h}{\partial s},\quad \frac{\partial z}{\partial t}=\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial h}{\partial t}$$

If $u=f(x_1,x_2,\dots,x_n)$ and $x_j=x_j(t_1,t_2,\dots,t_m)$

$$\frac{\partial u}{\partial t_i}=\frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i}+\frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_i}+\dots+\frac{\partial f}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}$$

14.6 Directional Derivatives & Gradient

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} = \lim_{h\to 0} \frac{f(\langle x_0,y_0\rangle + h\langle 1,0\rangle) - f(\langle x_0,y_0\rangle)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h} = \lim_{h\to 0} \frac{f(\langle x_0,y_0\rangle + h\langle 0,1\rangle) - f(\langle x_0,y_0\rangle)}{h}$$

--- then $\frac{\partial f}{\partial \vec{u}} = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$ $(\vec{u} = \langle a,b\rangle)$ unit vector

let $g(h) = f(x_0+ha, y_0+hb)$, then $g(0) = f(x_0,y_0)$

$\rightarrow \frac{\partial f}{\partial \vec{u}} = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = g'(0)$

$\rightarrow \frac{dg}{dh} = \frac{d}{dh}\left(f(x_0+ha, y_0+hb)\right) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial h} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$

$\Rightarrow \frac{\partial f}{\partial \vec{u}} = f_x\cdot a + f_y\cdot b = D_{\vec{u}}f = \langle f_x, f_y\rangle \cdot \langle a,b\rangle$ ☆

---

ex) $f(x,y) = x^3 - 4xy^2 + y^2$, $\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$, $D_{\vec{u}}f(1,0)$?

$D_{\vec{u}}f = \langle 3x^2-4y^2, -8xy+2y\rangle\cdot\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

$D_{\vec{u}}f(1,0) = \langle 3,0\rangle\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle = \boxed{\frac{3\sqrt{3}}{2}}$

Gradient Vector $\nabla f := \langle f_x, f_y\rangle$

$\rightarrow D_{\vec{u}}f = \nabla f \cdot \vec{u}$ ☆

Maximizing the directional derivative $\to \vec{u} \ni D_{\vec{u}} f$ max

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cos\theta \to$ max when $\theta = 0$

$\to$ max when $\vec{u}$ points in direction of ∇f, $\max(D_{\vec{u}} f) = |\nabla f|$

Let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be vector of level curves

$\to f(x(t), y(t)) = k \to \frac{d}{dt} f(x(t), y(t)) = 0$

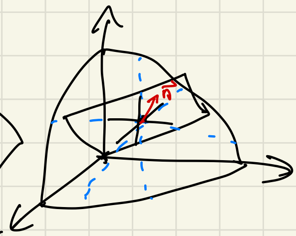
$\to \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$

$\Rightarrow \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle = 0$ ☆

Thus, ∇f is orthogonal to the tangent of the level curve at that point. $(\nabla f \cdot \vec{r}'(t) = 0 \Leftrightarrow \nabla f \perp \vec{r}'(t))$

Tangent planes for $w = F(x, y, z)$

$F(x, y, z) = k$, $\vec{n} = \nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $F(x(t), y(t), z(t)) = k$

$\frac{d}{dt} F(x(t), y(t), z(t)) = 0 \to \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$

$\to \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \langle x'(t), y'(t), z'(t) \rangle = 0$ ☆

$\Rightarrow \nabla F \cdot \vec{r}'(t) = 0$, $\nabla F \perp \vec{r}'(t) = 0 \Rightarrow \underline{\underline{\vec{n} = \nabla F}}$

$\Rightarrow$ Tangent plane: $\frac{\partial F}{\partial x} \cdot (x - x_0) + \frac{\partial F}{\partial y} \cdot (y - y_0) + \frac{\partial F}{\partial z} \cdot (z - z_0) = 0$ ☆

14.7 Maximum & Minimum Values

For $f(x,y)$,

A local maximum (a,b): $f(a,b) \geq f(x,y)$ for all (x,y) in some disk centered at (a,b)

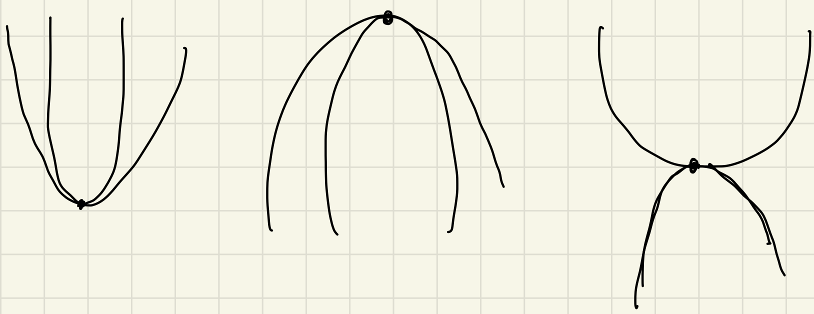
A local minimum (a,b): $f(a,b) \leq f(x,y)$ for all (x,y) in some disk centered at (a,b)

Global min/max: $f(a,b) \geq f(x,y)$ or $f(a,b) \leq f(x,y)$ for all (x,y) in domain D of f

If f has a local min/max at (a,b), and f_x and f_y exist, then $f_x(a,b) = f_y(a,b) = 0$. *

Critical point of f: $\nabla f = \langle 0,0 \rangle$ or DNE *

(not all critical points are extrema → saddle points)



2^{nd} Derivative Test: suppose (a,b) is a critical point of f.

$$D(x,y) = f_{xx}(a,b)\cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

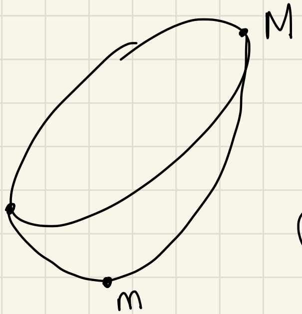
- if $D>0$ and $f_{xx}>0 \rightarrow f(a,b)$ is a local minimum
- if $D>0$ and $f_{xx}<0 \rightarrow f(a,b)$ is a local maximum
- if $D<0 \rightarrow f(a,b)$ is a saddle point

Absolute Extrema

$\mathbb{R}^2$

Extreme Value Theorem: f has a global min&max on $[a,b]$!

for $z=f(x,y)$: if f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f has absolute max $M=f(x_M,y_M)$ and absolute min $m=f(x_m,y_m)$ s.t. $(x_M,y_M),(x_m,y_m)\in D$.



① Find values of critical points in D.

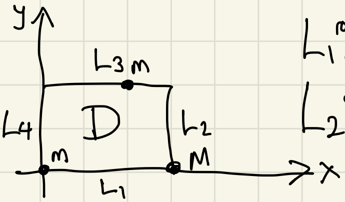
② Find extreme values of boundaries of D.

③ Compare extreme values for min, max

ex) $f(x,y)=x^2-2xy+2y$, $D=\{(x,y)\mid 0\le x\le 3,\ 0\le y\le 2\}$

$\nabla f=\langle 2x-2y,\,-2x+2\rangle=\vec{0} \to (1,1)$ is a critical point

$\to f(1,1)=\boxed{1}$



$L_1:\ y=0 \to f(x)=x^2,\ 0\le x\le 3 \to \overset{(3,0)}{M=9},\ \overset{(0,0)}{m=0}$

$L_2:\ x=3 \to f(y)=9-6y+2y,\ 0\le y\le 2 \to \overset{(3,0)}{M=9},\ \overset{(3,2)}{m=1}$

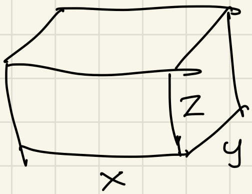
$L_3:\ y=2 \to f(x)=x^2-4x+4,\ 0\le x\le 3$

$\to \overset{(0,2)}{M=4},\ \overset{(2,2)}{m=0}$ $\quad L_4:\ f(y)=2y \to \overset{(0,2)}{M=4},\ \overset{(0,0)}{m=0}$

$\Rightarrow \boxed{M=f(3,0)=9,\ m=f(0,0)=f(2,2)=0}$

14.8 Lagrange Multipliers

ex) open top box with $12m^2$ surface area, max V?



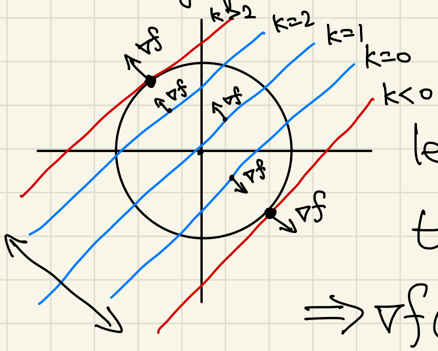
$$V = xyz,\quad A = 2xz + 2yz + xy = 12$$

$$\{\underbrace{f(x,y,z)}_{=V} \to \max,\quad \underbrace{g(x,y,z)}_{=A} = \underbrace{k}_{=12}\}$$

In 2D: $\{f(x,y) \to \text{max or min},\ g(x,y) = k\}$

ex) min/max of $z = f(x,y) = 1 + y - x$, $\underbrace{x^2+y^2=1}_{g(x,y)=k}$

in x-y plane: $f(x,y) = k \to y = k - 1 + x = x + (k-1)$



At the point of extrema, the level curve of f must be tangent to the constraint!

$$\Rightarrow \nabla f(x_0,y_0) \parallel \nabla g(x_0,y_0) \Rightarrow \nabla f \overset{(x_0,y_0)}{=} \lambda \nabla g$$

$$\to \left\{\begin{array}{l} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = k \end{array}\right\} \text{ for } (x,y), \lambda$$

$\hookrightarrow$ evaluate f at (x,y) found

for $\mathbb{R}^3$: $\left\{\begin{array}{l} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{array}\right.$

ex) $\{f(x,y) = -x+y \to \max,\ x^2+y^2=1\}$

$\to \underbrace{\langle -1, 1 \rangle}_{\nabla f} = \lambda \underbrace{\langle 2x, 2y \rangle}_{\nabla g} \to (x,y) = \left(-\frac{1}{2\lambda}, \frac{1}{2\lambda}\right)$

$\left(\frac{-1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1 \to \lambda = \frac{\pm 1}{\sqrt{2}} \to (x,y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$\to f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \sqrt{2},\ f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

Two constraints:

$f(x,y,z) \to$ extremum

$g(x,y,z) = k,\ h(x,y,z) = c$

$$\Rightarrow \begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z), \\ g(x,y,z) = k,\ h(x,y,z) = c \end{cases}$$

15.1 Double Integrals over Rectangles

for $y=f(x)$, $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ (Riemann Sum)

$\rightarrow$ for $z=f(x,y)$, $\iint_A f(x,y)\,dxdy \simeq \sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\,\Delta x\,\Delta y$

Iterated Integrals

$$\int_a^b\int_c^d f(x,y)\,dxdy = \int_a^b\left[\int_c^d f(x,y)\,dx\right]dy$$

$\rightarrow$ ex) $\int_0^2\int_0^3 x^2y\,dxdy = \int_0^2\left[\frac{x^3y}{3}\right]_0^3 dy = \int_0^2 9y\,dy = \frac{9}{2}y^2\Big|_0^2 = \boxed{18}$

Fubini's Theorem: if $[\overset{x}{a,b}]\times[\overset{y}{c,d}]$ is well-defined for f,

then $\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dydx = \int_c^d\int_a^b f(x,y)\,dxdy$

ex) $[1,2]\times[0,\pi]$, $f(x,y) = y\sin(xy)$

$\rightarrow \int_0^\pi\int_1^2 y\sin(yx)\,dxdy = \int_0^\pi(-\cos(yx))\Big|_1^2 dy = \int_0^\pi(\cos y - \cos 2y)\,dy$

$= \left[-\sin y + \frac{\sin 2y}{2}\right]_0^\pi = -\sin\pi + \frac{\sin 2\pi}{2} = \boxed{0}$

$\rightarrow \int_1^2\int_0^\pi y\sin(yx)\,dydx$ (=)

$z = f(x,y) = 16-x^2-2y^2$

ex) V bounded $[0,2]\times[0,2]$, $x^2+2y^2+z=16$

$\frac{96}{3}-\frac{8}{3}=\frac{88}{3}$

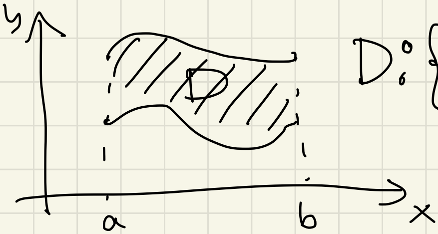
$\int_0^2\int_0^2\left[16-x^2-2y^2\right]dxdy = \int_0^2\left[-\frac{x^3}{3}+16x-2xy^2\right]_0^2 dy = \int_0^2\left(-\frac{8}{3}+32-4y^2\right)dy$

$= \frac{88}{3}y - \frac{4}{3}y^3\Big|_0^2 = \frac{176}{3}-\frac{32}{3} = \frac{144}{3} = \boxed{48}$

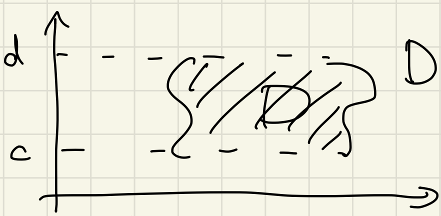
if $f(x,y)=g(x)\cdot h(y) \to \iint_R f(x,y)\,dx = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$

$$f_{avg} = \frac{1}{\text{Area } R}\iint_R f(x,y)\,dA$$

15.2 Double Integral over General Regions



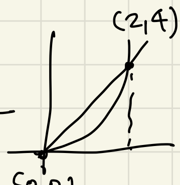
$D: \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



$D: \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(}^{g_2(} f(x,y)\,dy\,dx \text{ or } \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

ex) $z = x^2+y^2$, D is bounded by $y=2x$, $y=x^2$



$$\int_0^2\int_{x^2}^{2x} z\,dy\,dx = \int_0^2 \left[x^2y + \frac{y^3}{3}\right]_{x^2}^{2x} dx = \int_0^2 \left(x^2(2x-x^2) + \frac{(8x^3-x^6)}{3}\right)dx$$

$$= \int_0^2 \left(\overbrace{2x^3 - x^4 + \frac{8x^3}{3}}^{\frac{14x^3}{3}} - \frac{x^6}{3}\right)dx = \frac{7x^4}{6} - \frac{x^5}{5} - \frac{x^7}{21}\Bigg|_0^2 = \boxed{\frac{216}{35}}$$

ex) $\iint_D xy\,dA$, $y=x-1$, $y^2=2x+6 \to x=-1,\ 5$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow \quad\ \downarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad y=-2 \quad 4$

$$\int_{-2}^{4}\int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{\frac{y^2-6}{2}}^{y+1} dy$$

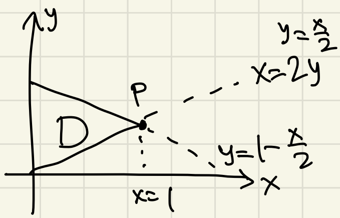
$$=\int_{-2}^{4}\frac{y}{2}\left(y^2+2y+1-\frac{1}{4}\left(y^4-12y^2+36\right)\right)dy$$

$$=\int_{-2}^{4}\left(\frac{y^3}{2}+\overbrace{y^2+\frac{y}{2}-\frac{y^4}{4}+3y^2}^{4y^2}-9\right)dy$$

$$=-\frac{y^5}{20}+\frac{y^4}{8}+\frac{4}{3}y^3+\frac{y^2}{4}-9y\Big|_{-2}^{4}=\boxed{36}$$

---

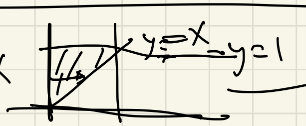
ex) $x+2y+z=2$, $x=2y$, $x=0$, $z=0$



$V=\iint_D (2-x-2y)\,dA \qquad P: \frac{x}{2}=1-\frac{x}{2} \to x=1$

$$V=\int_0^1\int_{x/2}^{1-x/2}(2-x-2y)\,dy\,dx \ \cdots = \frac{1}{3}$$

$\int_0^1\int_x^1 \sin(y^2)\,dy\,dx$

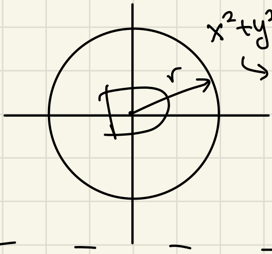


$\longrightarrow \int_0^1\int_0^y \sin y^2\,dx\,dy$

$$=\int_0^1 y\sin y^2\,dy = \frac{1}{2}\int_0^1 2y\cdot\sin y^2\,dy = \frac{1}{2}\left(-\cos y^2\right)\Big|_0^1$$

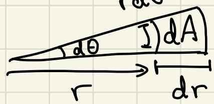
$$=\frac{1}{2}(1-\cos 1)$$

15.3 Double Integrals in Polar Coordinates



$$I = \iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,dA$$

$$I = \int_\alpha^\beta \int_a^b f(r,\theta)\,\underbrace{r\,dr\,d\theta}_{dA}$$



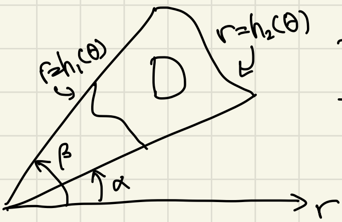
$dA \simeq r\,d\theta \cdot dr$
$(d\theta, dr \lll 1)$

---

ex) $\iint_R (3x+4y^2)\,dA$, $1 \le x^2+y^2 \le 4 \to 1 \le r \le 2$, $0 \le \theta \le \pi$

$$\int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta = \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_1^2 d\theta = \int_0^\pi (7\cos\theta + 15\overbrace{\sin^2\theta}^{\frac{1}{2}-\frac{\cos 2\theta}{2}})\,d\theta$$

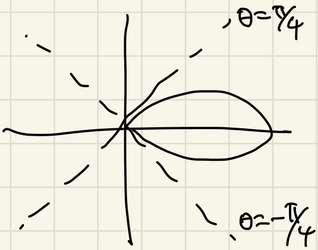
$$= \left[7\sin\theta + 15\left(\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right)\right]_0^\pi = \boxed{\frac{15}{2}\pi}$$



$$I = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

---

ex) Area of one loop of $r=\cos 2\theta$



$$\int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta = \int_0^{\pi/4} \overbrace{\cos^2 2\theta}^{\frac{1}{2}+\frac{\cos 4\theta}{2}}\,d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi/4} = \boxed{\pi/8}$$

ex) under $z=x^2+y^2$, above x-y plane, inside $x^2+y^2=2x$

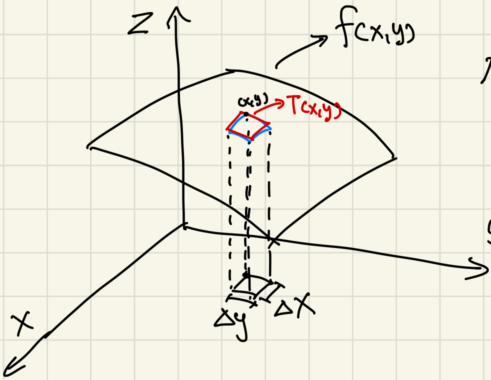
$x^2-2x+y^2=0 \rightarrow (x-1)^2+y^2=0 \rightarrow r=2\cos\theta,\ -\pi/2 \le \theta \le \pi/2$

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 \cdot r\,dr\,d\theta = 2\int_{0}^{\pi/2}\left[\frac{1}{4}r^4\right]_{0}^{2\cos\theta} d\theta = \int_{0}^{\pi/2} 8\cos^4\theta\, d\theta = 8 \cdot \frac{\pi}{2} \cdot \frac{1\cdot 3}{2\cdot 4}$$

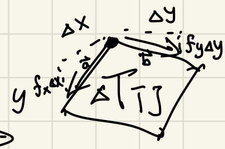
$$= \boxed{\frac{3\pi}{2}}$$

15.5 Surface Area

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$



$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$



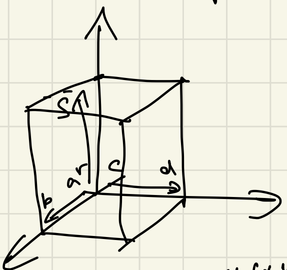
$$\Delta T = |\vec{a} \times \vec{b}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix}$$

$$= |-f_x \Delta x \Delta y\, \hat{i} - f_y \Delta x \Delta y\, \hat{j} + \Delta x \Delta y\, \hat{k}|$$

$$\Rightarrow \Delta T = \Delta x \Delta y \sqrt{f_x^2 + f_y^2 + 1} \xrightarrow{\lim} dT = dA \sqrt{f_x^2 + f_y^2 + 1}$$

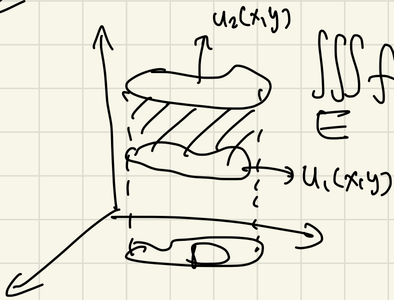
$$\Rightarrow T = \int dT = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

15.6 Triple Integrals



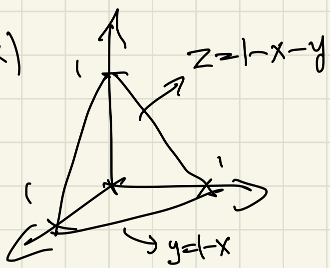
for domain box $B=[a,b]\times[c,d]\times[r,s]$:

$$\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

ex)



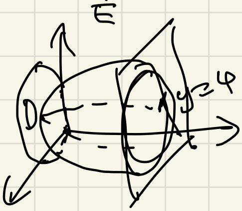
$$\iint_D \int_0^{1-x-y} z\,dz\,dA = \iint_D \frac{1}{2}(1-x-y)^2\,dA$$

$$=\frac{1}{2}\int_0^1\int_0^{1-x}(1-x-y)^2\,dy\,dx = \frac{1}{2}\int_0^1\left[-\frac{1}{3}(1-x-y)^3\right]_0^{1-x}dx$$

$$=\frac{1}{6}\int_0^1(1-x)^3\,dx = \frac{1}{6}\left[-\frac{1}{4}(1-x)^4\right]_0^1 = \boxed{\frac{1}{24}}$$

Also applies to x and y being the innermost integral.

ex) $\iiint_E \sqrt{x^2+z^2}\,dV$, E: bounded by $y=x^2+z^2$ and $y=4$

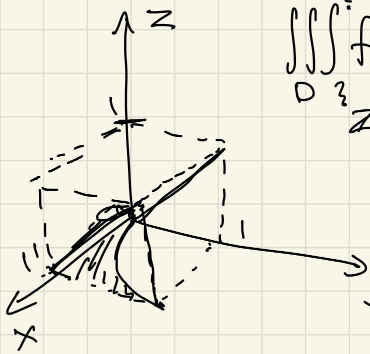


$E: x^2+z^2\le y\le 4$, $D: x^2+z^2=4$

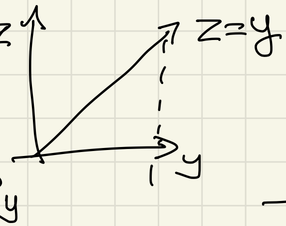
$$\iint_D\int_{x^2+z^2}^4 \sqrt{x^2+z^2}\,dy\,dA = \iint_D \sqrt{x^2+z^2}\left[4-(x^2+z^2)\right]dA$$

$$=\int_0^{2\pi}\int_0^2(4r^2-r^4)\,dr\,d\theta = \boxed{\frac{128\pi}{15}}$$

ex) $I=\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx \to E: 0\le x\le 1,\ 0\le y\le x^2,\ 0\le z\le y$



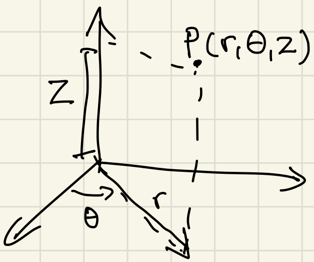
$\iint_D \int_{?}^{?} f(x,y,z)\,dx\,dA \qquad x=\sqrt{y},\ x=1$



$\to \iint_D \int_{\sqrt{y}}^{1} f(x,y,z)\,dx\,dz\,dy$

$\to \int_0^1\int_0^y\int_{\sqrt{y}}^{1} f(x,y,z)\,dx\,dz\,dy$

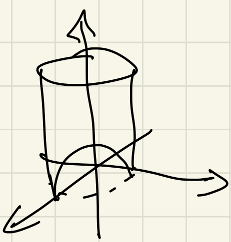
15.7 Triple Integrals in Cylindrical Coordinates



$\to \iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$

$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(r\cos\theta, r\sin\theta, z)\,dz\; r\,dr\,d\theta$

ex) $x^2+y=1,\ z=4,\ z=1-x^2-y^2,\ \rho=k\sqrt{x^2+y^2}$

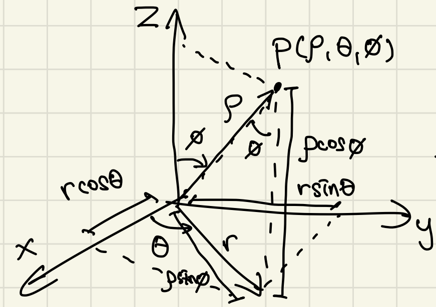


$m=\iiint_E k\sqrt{x^2+y^2}\,dV = k\iint_D \int_{1-x^2-y^2}^{4} \sqrt{x^2+y^2}\,dz\,dA$

$= k\iint_D r[3+r^2]\,dA = k\int_0^{2\pi}\int_0^1 (3r^2+r^4)\,dr\,d\theta$

$= \boxed{\frac{128\pi k}{5}}$

15.8 Triple Integrals in Spherical Coordinates

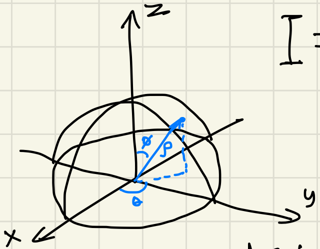


$\rho \geq 0,\ \varnothing \in [0, \pi]$

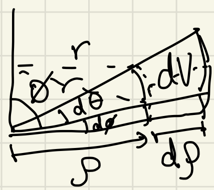
$$\begin{cases} x = \rho\cos\theta\sin\varnothing \\ y = \rho\sin\theta\sin\varnothing \\ z = \rho\cos\varnothing \end{cases}$$

$\rho^2 = x^2 + y^2 + z^2$

$r = \rho\sin\varnothing$
$= \sqrt{x^2+y^2}$



$$I = \iiint_E f(\rho\cos\theta\sin\varnothing, \rho\sin\theta\sin\varnothing, \rho\cos\varnothing)\, dV$$

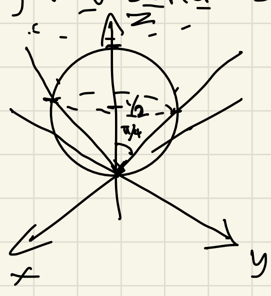


$dV \simeq (r\, d\theta\, d\rho) \cdot \rho\, d\varnothing$
$= \underline{\rho^2 \sin\varnothing\, d\rho\, d\theta\, d\varnothing}$

$$\Rightarrow I = \int_c^d \int_\alpha^\beta \int_a^b f \cdot \underbrace{\rho^2 \sin\varnothing\, dr\, d\theta\, d\varnothing}_{dV}$$

---

ex) V of solid above $z = \sqrt{x^2+y^2}$, below $x^2+y^2+z^2 = z$



$z = r \quad x^2+y^2+x^2+y^2 = \sqrt{x^2+y^2}$

$2\alpha^2 - \alpha = 0$

$\alpha = 0, \frac{1}{2} \rightarrow x^2+y^2 = \frac{1}{2}$

$\rho^2 = x^2+y^2+z^2 \rightarrow \rho^2 = z \rightarrow \rho^2 - \rho\cos\varnothing = 0$

$\rightarrow \rho = \cos\varnothing,\ \rho = 0$

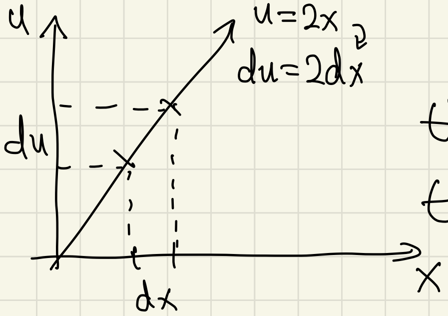
$$V = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varnothing} \rho^2 \sin\varnothing\, d\rho\, d\varnothing\, d\theta = 2\pi \int_0^{\pi/4} \sin\varnothing \left[\frac{\rho^3}{3}\right]_0^{\cos\varnothing} d\varnothing = \frac{2}{3}\pi \int_0^{\pi/4} \sin\varnothing\cos^3\varnothing\, d\varnothing$$

$$= \frac{2}{3}\pi\left[-\frac{\cos^4\varnothing}{4}\right]_0^{\pi/4} = \frac{1}{6}\pi\left(1 - \frac{1}{4}\right) = \frac{3}{4}\cdot\frac{1}{6}\pi = \boxed{\frac{\pi}{8}}$$

15.9 Change of Variables in Multiple Integrals

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du,\quad a=g(c),\ b=g(d),\ x=g(u)$$

$$\rightarrow \int_a^b f(x)\,dx = \int_c^d f(x(u))\cdot\frac{dx}{du}\cdot du,\quad a=x(c),\ b=x(d)$$



when there is a change of variables, there is a "factor" that relates the relative ratio of the two differentials.

$$\begin{cases} x = r\cos\theta = x(r,\theta) \\ y = r\sin\theta = y(r,\theta) \end{cases} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

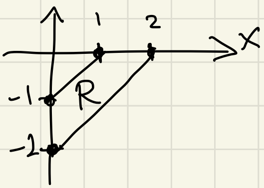
In general, in 2D, if $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \rightarrow T(u,v) = (x,y)$ and T is one-to-one, there exists T^{-1} and G and H such that $u = G(x,y)$, $v = H(x,y)$.

$$I = \iint_R f(x,y)\,dA \rightarrow \left| \begin{matrix} x = x(u,v) \\ y = y(u,v) \end{matrix} \right| \rightarrow \iint_S f(x(u,v),y(u,v)) \overbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}^{dA}$$

where $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \dfrac{\partial x}{\partial u}\cdot\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial y}{\partial u}$ (Jacobian)

Also applies to 3D! $\rightarrow \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

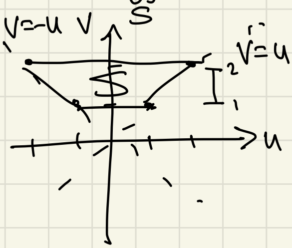
ex) $I=\iint_R e^{\frac{x+y}{x-y}}\,dA$, R: $(1,0),(2,0),(0,-2),(0,-1)$ trapazoid



set $x+y=u$, $x-y=v$ $\rightarrow$ $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=\left|-\frac{1}{4}-\frac{1}{4}\right|=\frac{1}{2}$$

$\rightarrow \iint_S e^{\frac{u}{v}}\left(\frac{1}{2}\right)du\,dv$, S: $(1,1),(2,2),(-2,2),(-1,1)$

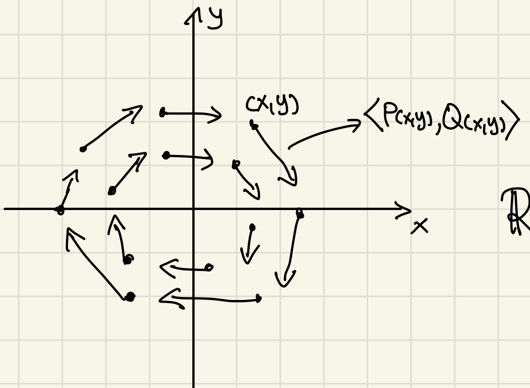


$$\int_1^2\int_{-v}^{v}\frac{1}{2}e^{\frac{u}{v}}\,du\,dv=\frac{1}{2}\int_1^2\left[ve^{\frac{u}{v}}\right]_{-v}^{v}dv=\frac{1}{2}\int_1^2 v\left(e-\frac{1}{e}\right)dv$$

$$=\frac{\left(e-\frac{1}{e}\right)}{4}v^2\Big|_1^2=\boxed{\frac{3}{4}\left(e-\frac{1}{e}\right)}=\frac{3\sinh(1)}{2}$$

16.1 Vector Fields

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$



$\vec{F}$ assigns every point on its domain a 2-D vector ($\mathbb{R}^2$)

$\mathbb{R}^3$: $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

Gradient Fields

$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right\rangle$

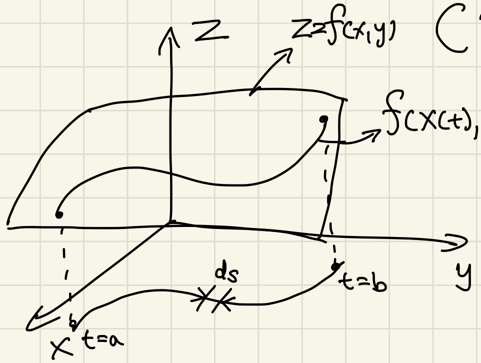
ex) ∇f of $f = x^2y - y^3 \rightarrow \nabla f = \langle 2xy, x^2-3y^2, 0 \rangle$

$\hookrightarrow \vec{F} = \langle 2xy, x^2-3y^2, 0 \rangle$ is conservative.

$\vec{F}$ is conservative if $\exists$ a scalar function f s.t. $\nabla f = \vec{F}$.

Then, f is a potential function of $\vec{F}$.

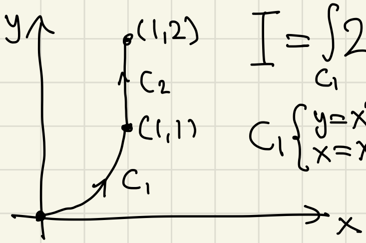
16.2 Line Integrals



C: curve in $\mathbb{R}^2$, parametric $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $(a \le t \le b)$

$$I = \int_C f(x,y)\cdot ds = \int_a^b f(x(t),y(t))\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

ex) $\int_C 2x\,ds$, where C: $(0,0)\to(1,1)$ over $y=x^2$, $(1,1)\to(1,2)$ over $x=1$
(C1: the first part, C2: the second part)



$$I = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \int_{x=0}^{1} 2x\sqrt{(2x)^2+1}\,dx + \int_{y=1}^{2} \underbrace{2(1)\sqrt{1+0}}_{2}\,dy$$

$C_1 \begin{cases} y = x^2 \\ x = x \end{cases}$ $C_2 \begin{cases} y = y \\ x = 1 \end{cases}$

$$= \frac{1}{4}\int_0^1 8x\sqrt{4x^2+1}\,dx + 2y\Big|_1^2$$

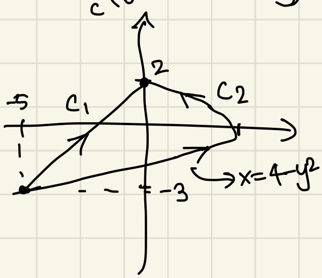
$$= \frac{1}{4}\left(\frac{2}{3}(4x^2+1)^{3/2}\right)\Big|_0^1 + 2 = \boxed{\frac{1}{6}\left(5^{3/2}-1\right)+2}$$

$\int_C f(x,y)\,dx$, $\int_C f(x,y)\,dy$? $dx = x'(t)\,dt$, $dy = y'(t)\,dt$

$$\Rightarrow \int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\cdot x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\cdot y'(t)\,dt$$

ex) $\int_C (y^2dx + x\,dy)$? $C_1 \begin{cases} x = 5t-5 \\ y = 5t-3 \end{cases} (0 \le t \le 1)$ $C_2 \begin{cases} x = 4-t^2 \\ y = t \end{cases} (-3 \le t \le 2)$



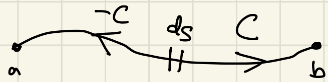
$$\int_{C_1}(y^2dx + x\,dy) = \int_0^1 \left[(5t-3)^2\cdot 5\,dt + (5t-5)\cdot 5\,dt\right] = -\frac{5}{6}$$

$$\int_{C_2}(y^2dx + x\,dy) = \int_{-3}^{2}\left[t^2(-2t)\,dt + (4-t^2)(1)\,dt\right] = 40\frac{5}{6}$$

Curve C has orientation according to parameter t!

$\Rightarrow \int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx$, $\int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy$

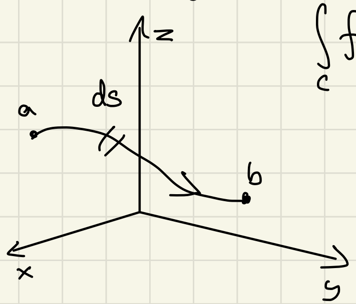
however, $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$



$\therefore$ if C is backwards, $dx = x'(t)\,dt$, and $x'(t)$ and dt has opposite signs $\rightarrow dx < 0 \rightarrow \int_C f\,dx = \int_{-C} f(-dx)$

In ds, both $x'(t)$ and $y'(t)$ are squared $\rightarrow ds > 0$

Line Integrals in Space



$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

or in compact notation where

$$\vec{r} = \langle x(t), y(t), z(t) \rangle$$

$$\rightarrow \int_a^b f(\vec{r}(t))\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

$$\int_C \left(P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz\right)$$

... why would we make such an integral?

Line Integrals of Vector Space

Let $\vec{F} = \langle P, Q, R \rangle (x, y, z)$

Let $\vec{r} = \langle x, y, z \rangle (t)$, $t \in [a, b]$

The "Work" done by the force field in moving a particle along path C is: $\int_C \vec{F} \cdot d\vec{r}$ wher $d\vec{r} = \vec{r}'(t)dt$

Then: $W = \int_C \vec{F} \cdot d\vec{r} = \int_C Px'(t)dt + Qy'(t)dt + Rz'(t)dt$

$$= \int_C Pdx + Qdy + Rdz$$

ex) $\vec{F}(x,y) = x^2 \hat{i} - xy \hat{j}$, $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j}$, $0 \le t \le \pi/2$.

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \langle x^2, -xy \rangle \cdot \langle -\sin t, \cos t \rangle dt =$$

$$= \int_0^{\pi/2} \langle \cos^2 t, -\cos t \cdot \sin t \rangle \cdot \langle -\sin t, \cos t \rangle dt$$

$$= \int_0^{\pi/2} (-\cos^2 t \sin t - \cos^2 t \sin t) dt = -2 \int_0^{\pi/2} \cos^2 t \sin t \, dt$$

$$= +2 \left[\frac{\cos^3 t}{3} \right]_0^{\pi/2} = +\frac{2}{3} [0 - 1] = \boxed{-2/3}$$

16.3 Fundamental Theorem for Line Integrals

For single variable scalar integrals:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \text{ where } F' \text{ is cont. on } [a,b].$$

Vector Calculus version:

Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$.

Let f be a differentiable function, ∇f cont. on C.

Then, $\int_a^b \nabla f\, d\vec{r} = \int_a^b \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

(if $\vec{F}$ is a conservative field, $\int_C \vec{F}\,d\vec{r} = \Delta f\,[a,b]$)

ex) $\vec{F} = \frac{mMG}{|\vec{x}|^3}\vec{x}$, find work from $(3,4,12)$ to $(2,2,0)$

$$f = -\frac{mMG}{\sqrt{x^2+y^2+z^2}} = -\frac{mMG}{|\vec{x}|}$$

$$\nabla f = -mMG\left\langle -\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}^{\,3}},\ -\frac{1}{2}\frac{2y}{|\vec{x}|^3},\ -\frac{1}{2}\frac{2z}{|\vec{x}|^3}\right\rangle = \frac{mMG}{|\vec{x}|^3}\langle x, y, z\rangle = \frac{mMG}{|\vec{x}|^3}\vec{x}$$

$$\rightarrow \nabla f = \vec{F} \rightarrow \int_C \vec{F}\cdot dr = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$W = f(2,2,0) - f(3,4,12) = \boxed{mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)}$$

Path Dependence

if C_1 and C_2 are two separate paths with same endpoints,

in general, $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$, but for conservative fields,

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot dr$ (Independent of Path!)

$\Rightarrow$ Assume a closed curve, where $P_0 = P_1$.

If $\vec{F}$ is path independent, $\oint_C \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0) = 0$.

$$\boxed{\oint_C \vec{F} \cdot dr = 0 \text{ for } \forall \underset{\text{(closed)}}{\tilde{C}} \iff \vec{F} \text{ is path independent}}$$

If: D is open, D is connected, and $\vec{F}$ is continuous,

$$\boxed{\vec{F} \text{ is path independent} \iff \vec{F} \text{ is conservative } (\exists \nabla f = \vec{F})}$$

... but how to determine $\vec{F}$ is conservative?

If $\vec{F}$ is conservative: $\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle P(x,y), Q(x,y) \right\rangle$

$$\begin{cases} P(x,y) = \frac{\partial f}{\partial x} & \partial/\partial y \\ Q(x,y) = \frac{\partial f}{\partial y} & \partial/\partial x \end{cases} \Rightarrow \begin{cases} \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} \end{cases}$$

if $\frac{\partial^2 f}{\partial y \partial x}$ & $\frac{\partial^2 f}{\partial x \partial y}$ $\left(\frac{\partial P}{\partial y} \text{ \& } \frac{\partial Q}{\partial x}\right)$ are continuous, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

If D is a simply connected region, then $\vec{F}$ cons. $\iff \left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\right)$

16.5 Curl and Divergence (16.4 comes later!)

f: scalar function, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \to \nabla$: gradient operator

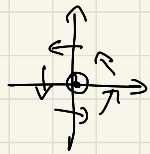
For vector field $\vec{F} = \langle P, Q, R \rangle$, 2 ways of differentiation:

* Curl: $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \underline{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}}$

* Divergence: $\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \underline{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$

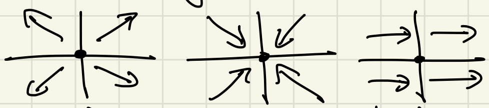
* Physical Meaning of curl $\vec{F}$ and div $\vec{F}$ *

Curl: measures the "rotation of fluids"



- Direction of curl $\vec{F}$ represents axis of rotation
- Magnitude of curl $\vec{F}$ represents how fast particles rotate

Divergence: measures the "rate of change of mass flowing out" on a point



$\text{div}\vec{F}(P) > 0$ $\quad$ $\text{div}\vec{F}(P) < 0$ $\quad$ $\text{div}\vec{F}(P) = 0$

$\vec{F} = \langle P(x,y), Q(x,y) \rangle$ is a conservative vector field. curl $\vec{F}$?

$$\text{curl}\,\vec{F} = \left\langle \underbrace{\frac{\partial Q}{\partial z}}_{0}, \underbrace{\frac{\partial P}{\partial z}}_{0}, \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0} \right\rangle = \vec{0} \quad \left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\right)$$

$\Rightarrow$ Conservative vector fields (in 2D) have no curl! *

* $\text{curl}\,\vec{F} = \text{curl}(\nabla f) = \vec{0}$! (for all dimensions)

ex) $\vec{F} = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$

(a) show that $\vec{F}$ is conservative.

$$\operatorname{curl}\vec{F} = \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle = \vec{0}$$

$\operatorname{curl}\vec{F} = \vec{0} \Longleftrightarrow \vec{F}$ is conservative. ($\vec{F}$ has no holes)

(b) find f s.t. $\nabla f = \vec{F}$.

$$f = xy^2z^3 + \cancelto{0}{C(y,z)} + \cancelto{0}{D(y)} + \cancelto{0}{G(z)} \rightarrow f = \boxed{xy^2z^3 + K}$$

Theorem: $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$. $(\nabla\cdot(\nabla\times\vec{F}) = 0)$ if $\vec{F}$ has continuous 2^{nd} order partial derivatives. ☆

Proof: $\nabla\cdot(\nabla\times\vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$= 0$ (By Clairaut's Theorem, order of ∂ can be switched)

$$\operatorname{div}(\operatorname{grad}\vec{F}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$\rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplace Operator, Δ.

$$\operatorname{div}(\operatorname{grad}\vec{F}) = \nabla\cdot(\nabla f) = \Delta f$$

$\Delta f = 0 \Rightarrow$ Laplace's Equation

16.4 Green's Theorem

A line integral over a simple, closed curve: $\oint_C P\,dx + Q\,dy$

In terms of a double integral over a region D

bounded by C: $\pm\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$ (sign by orientation)

$$\Rightarrow \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy \quad \star$$

In vector notation: $d\vec{r} = \langle dx, dy \rangle$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \text{curl}\,\vec{F}\cdot\hat{k}$

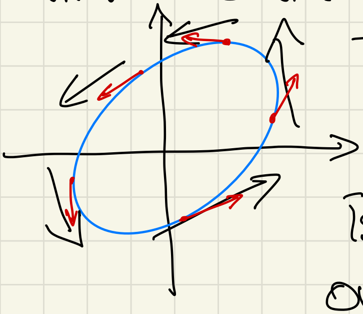
$$\Rightarrow \underline{\oint_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F}\cdot\hat{k}\,dA} \quad \star\star \text{ (Green's Theorem I)}$$

$$d\vec{r} = \langle dx, dy \rangle = \langle x'(t)dt, y'(t)dt \rangle = \langle x'(t), y'(t) \rangle dt$$

$$d\vec{r} = \vec{r}\,'(t)dt = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\cdot|\vec{r}\,'(t)|\,dt = \vec{t}\,ds$$

$$\rightarrow \oint_C \vec{F}\cdot d\vec{r} = \underline{\oint_C \vec{F}\cdot\vec{t}\,ds = \iint_D \text{curl}\,\vec{F}\cdot\hat{k}\,dA} \quad \star \text{ (Green's Theorem II)}$$

What does this show?



The total contribution of $\vec{F}$ to the rotation in region D: $\iint_D \text{curl}\,\vec{F}\cdot\hat{k}\,dA$

is equal to the tangential components over the boundary of D: $\oint_C \vec{F}\cdot\vec{t}\,ds$

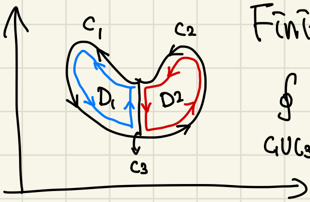
Second Vector Form of Green's Theorem

$$\oint_C \vec{F}\cdot\vec{n}\cdot ds \quad \left(\vec{n}(t) = \langle y'(t), -x'(t)\rangle \cdot \frac{1}{|\vec{r}'(t)|},\ \text{normal vector}\right)$$

$$= \oint_C \langle P, Q\rangle \cdot \frac{\langle y'(t), -x'(t)\rangle}{|\vec{r}'(t)|} \cdot |\vec{r}'(t)|\, dt = \oint_C (P y'(t)dt - Q x'(t) dt)$$

$$= \oint_C (-Q dx + P dy) = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \underline{\iint_D \operatorname{div}(\vec{F})\, dA}$$ ☆☆

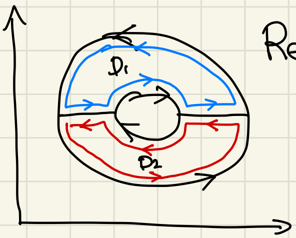
Extension to Non-simple Regions



Finite Union of Simple Regions:

$$\oint_{C_1 \cup C_3} \vec{F}\cdot d\vec{r} = \iint_{D_1} \operatorname{curl}\vec{F}\cdot\vec{k}\, dA, \quad \oint_{C_1 \cup (-C_3)} \vec{F}\cdot dr = \iint_{D_2} \operatorname{curl}\vec{F}\cdot\vec{k}\cdot dA$$

$$\rightarrow \oint_{C_1 \cup C_2} \vec{F}\cdot dr = \oint_C \vec{F}\cdot d\vec{r} = \iint_{D_1 \cup D_2} \operatorname{Curl}\vec{F}\cdot\vec{k}\cdot dA$$



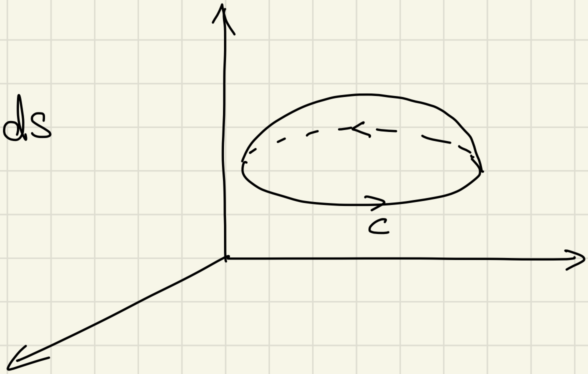
Region with Holes:

$$\int_{\partial D_1} \vec{F}\cdot d\vec{r} + \int_{\partial D_2} \vec{F}\cdot d\vec{r} = \iint_D \operatorname{curl}\vec{F}\cdot\vec{k}\, dA$$

16.8 Stokes' Theorem

in 3D: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$

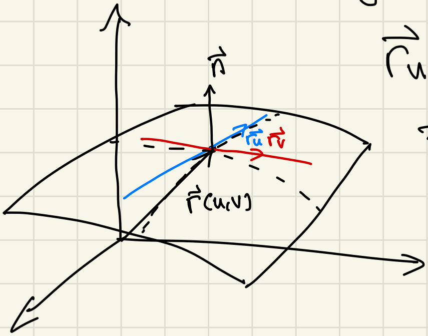
but first - - -



16.6 Parametric Surfaces

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ (parametric curves)

$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$ (parametric surfaces) ☆



$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$

$\vec{n} = \vec{r}_u \times \vec{r}_v$ ☆

$\Rightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

where $\langle a,b,c\rangle = \vec{n} = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$

Surface Area: $\Delta S \simeq |\vec{r}_u \times \vec{r}_v|\,\Delta u\,\Delta v \rightarrow A = \iint_D |\vec{r}_u \times \vec{r}_v|\,du\,dv$ ☆

16.7 Surface Integrals

Surface Integrals of Scalar Functions f

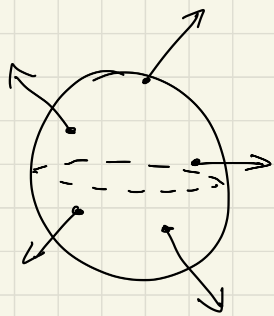
$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\underbrace{|\vec{r}_u \times \vec{r}_v|\,du\,dv}_{dS} \quad \text{(parametric)}$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\underbrace{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy}_{dS} \quad (z=z(x,y))$$

$$\left(\vec{r}(x,y) = x\hat{i}+y\hat{j}+z(x,y)\hat{k} \rightarrow \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle\right.$$

$$\left.\Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\right)$$

Surface Integrals of Vector Functions $\vec{F}$



$\iint_S \vec{F}$? → The unit normal vector denotes the "positive" orientation of the surface ($\vec{n}$)

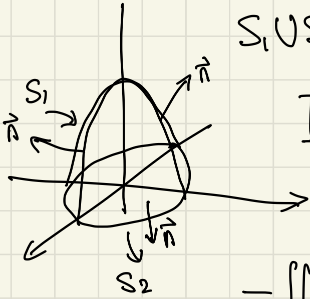
An infinitesimally small surface area $d\vec{S} = \vec{n}\,dS$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\cdot dS$$

$$\vec{n} = \begin{cases} \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} & \text{(parametric)} \\[2ex] \dfrac{\left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}} & (z=z(x,y)) \end{cases} \quad \Bigg| \quad dS = \begin{cases} |\vec{r}_u \times \vec{r}_v| & \text{(parametric)} \\[2ex] \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} & (z=z(x,y)) \end{cases}$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\overbrace{dA}^{du\,dv} = \iint_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right)\overbrace{dA}^{dx\,dy}$$

ex) $\iint_S \vec{F}\cdot d\vec{S}$, $\vec{F}=\langle y,x,z\rangle$, $S: 0\le z\le 1-x^2-y^2$



$S_1 \cup S_2 \to$ closed surface $\to \vec{n}$ pointing outwards

$$I=\iint_S \vec{F}\cdot d\vec{S}=\iint_{S_1}\vec{F}\cdot d\vec{S}+\iint_{S_2}\vec{F}\cdot d\vec{S}=I_1+I_2$$

$$S_1: z=1-x^2-y^2 \to I_1=\iint_{D_1}\vec{F}\cdot\left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle dA$$

$$=\iint_{D_1}\langle y,x,z\rangle\cdot\langle +2x,+2y,1\rangle\, dA,\quad D_1=\{(x,y)\mid x^2+y^2\le 1\}$$

$$I_1=\iint_{D_1}(4xy+1-x^2-y^2)\,dA \to \begin{pmatrix}x=r\cos\theta\\ y=r\sin\theta\end{pmatrix} \to \int_0^{2\pi}\int_0^1(4r^2\sin\theta\cos\theta+1-r^2)\,r\,dr\,d\theta$$

$$\cdots=\boxed{\pi/2}$$

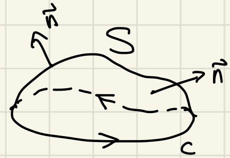
$$I_2=\iint_{D_2}\langle y,x,z\rangle\cdot\langle 0,0,-1\rangle\, d\vec{S}=\iint_{D_2}\cancel{\langle y,x,0\rangle\cdot\langle 0,0,-1\rangle}\, d\vec{S}=\boxed{0}$$

$$\Rightarrow I=I_1+I_2=\boxed{\pi/2}$$

16.8 & 16.9 Stokes' & Divergence Theorem

(Green's Theorem: $\oint_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F}\cdot\hat{k}\,ds$, $\oint_C \vec{F}\cdot\vec{n}\cdot ds = \iint_D \text{div}\,\vec{F}\,dA$)

Stokes' Theorem



S: oriented piecewise smooth surface

→ $\hat{k}$ turns into a normal vector to the surface, $\vec{n}$.

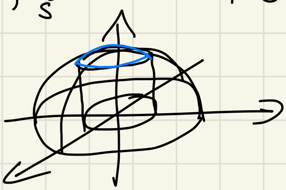
$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\cdot dS = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$ ★★★ (S: any surface w/boundary C)

ex) $\oint_C \vec{F}\cdot d\vec{r}$, $\vec{F} = \langle -y^2, x, z^2\rangle$, C: $(y+z=2)\cap(x^2+y^2\le 1)$, counterclockwise ($\to z=2-y$)

$\to \text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y\rangle$ → stokes' theorem works simpler!

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \langle 0,0,1+2y\rangle\cdot\langle 0,1,1\rangle\,dA = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta = \boxed{\pi}$$

ex) $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$, S: $x^2+y^2+z^2=4$, $x^2+y^2=1$, $z>0$, $\vec{F} = \langle xz, yz, xy\rangle$

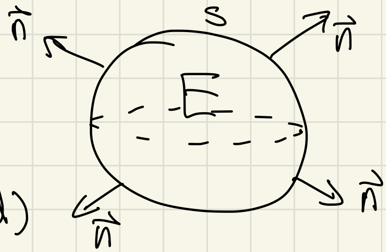


$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$, $z=\sqrt{3}$

$\to \int_0^{2\pi} \langle \sqrt{3}x, \sqrt{3}y, xy\rangle\cdot\langle dx, dy, 0\rangle$ → $\begin{cases} x=\cos\theta & dx=-\sin\theta\,d\theta \\ y=\sin\theta & dy=\cos\theta\,d\theta \end{cases}$

$$\int_0^{2\pi} (-\sqrt{3}\cos\theta\sin\theta + \sqrt{3}\cos\theta\sin\theta)\,d\theta = \boxed{0}$$

Divergence Theorem



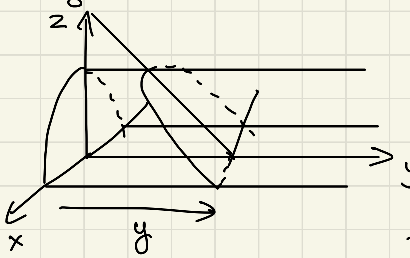
$$\iint_S \vec F\cdot d\vec S = \iiint_E \operatorname{div}\vec F\cdot dV$$

$$= \iint_{\partial E} \vec F\cdot\vec n\cdot dS \quad (\vec n\text{: unit outward normal})$$

ex) Flux of $\vec F = \langle z, y, x\rangle$ across unit sphere

$$\iint_S \vec F\cdot d\vec S = \iiint_E \operatorname{div}\vec F\, dV \to \iiint_E 1\cdot dV = \boxed{\frac{4}{3}\pi}$$

ex) $\iint_S \vec F\cdot d\vec S$, $\vec F = \langle xy,\ y^2+e^{xz^2},\ \sin(xy)\rangle$, S: surface of $(z=1-x^2,\ z=0,\ y=0,\ y+z=2)$

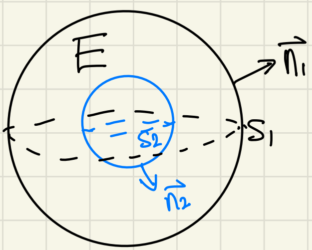


$$\operatorname{div}\vec F = y+2y = 3y$$

$$\iiint_E 3y\,dV = \iint_{D_{xz}}\left(\int_0^{2-z} 3y\,dy\right)dA = \iint_{D_{xz}}\left(\frac{3}{2}(2-z)^2\right)dA$$

$$= \int_{-1}^{1}\int_0^{1-x^2} \frac{3}{2}(2-z)^2\,dz\,dx = -\frac{1}{2}\int_{-1}^{1}\left[(2-z)^3\right]_0^{1-x^2} dx$$

$$= -\frac{1}{2}\int_{-1}^{1}\left((1+x^2)^3-8\right)dx \rightsquigarrow \boxed{\frac{184}{35}}$$



Boundary of E is given by

$$\begin{cases} \vec n_1 \text{ on } S_1 \\ -\vec n_2 \text{ on } S_2 \end{cases} \to \iiint_E \operatorname{div}\vec F\cdot dV = \iint_{S_1}\vec F\cdot\vec n_1\cdot dS + \iint_{S_2}\vec F\cdot(-\vec n_2)\,dS$$

$$= \iint_{\substack{S_1\\(\text{ext})}} \vec F\cdot d\vec S - \iint_{\substack{S_2\\(\text{int})}} \vec F\cdot d\vec S$$

Q.11 Practice B

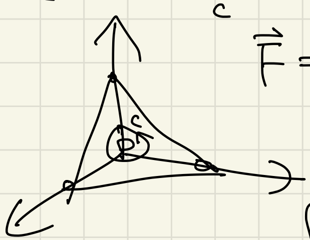
Use Stokes' Theorem to show that

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \oint_C \vec{F}\cdot d\vec{r}$$

where C is a simple closed curve on plane $x+y+z=1$

depends only on its area and not its shape or location

(Stokes': $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot d\vec{S}$)



$$\vec{F} = \langle z, -2x, 3y\rangle \rightarrow \text{curl}\vec{F} = \langle 3, 1, -2\rangle$$

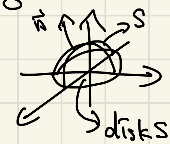
$$d\vec{S} = \vec{n}\cdot dS = \frac{\langle 1,1,1\rangle}{\sqrt{3}}\cdot dS$$

$$\int_C \langle 3,1,-2\rangle\cdot\frac{\langle 1,1,1\rangle}{\sqrt{3}}\,dS = \iint_S \left(\frac{2}{\sqrt{3}}\right) dS = \frac{2}{\sqrt{3}}\cdot\iint_S dS$$

$$= \frac{2}{\sqrt{3}}\cdot \text{Area of } D$$

Q.12 Practice A

$\iint_S \langle z^2x, \frac{1}{3}y^3+\tan z, x^2z+y^2\rangle\cdot d\vec{S}$, S: top half of unit sphere, outward $\vec{n}$



$$\iint_{S\cup S_1}\vec{F}\cdot d\vec{S} = \iiint_{\partial E}\text{div}\vec{F}\cdot dV = \iiint_{\partial E}(z^2+y^2+x^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div}\vec{F}\cdot dV - \underbrace{\iint_{S_1}\vec{F}\cdot d\vec{S}} \rightarrow \iint_{\partial S_1}\langle 0, \tfrac{1}{3}y^3, y^2\rangle\cdot\langle 0,0,-1\rangle\,dS$$

$$= -\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\; r\,dr\,d\theta$$

Q12 cont.

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^1 \rho^4 \, d\rho = \frac{2\pi}{5}$$

$$-\int_0^{2\pi}\int_0^1 r^2 \sin^2\theta \, r \, dr \, d\theta = -\int_0^{2\pi} \sin^2\theta \, d\theta \cdot \frac{1}{4} = -\pi \cdot \frac{1}{4} = -\frac{\pi}{4}$$

$$\frac{1}{2} - \frac{\cos 2\theta}{2}$$

$$\frac{1}{2}\theta - \cancel{\sin}$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right) = \frac{8\pi + 5\pi}{20} = \boxed{\frac{13\pi}{20}}$$