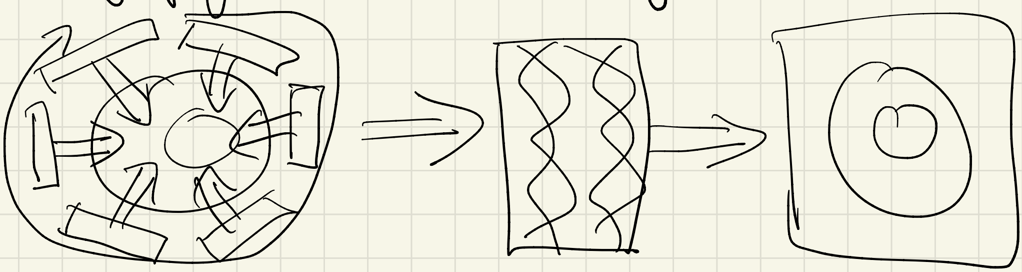
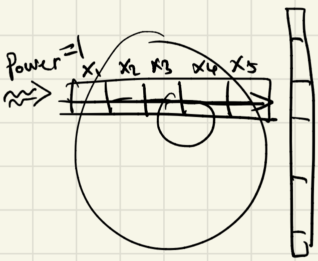
Imaging

Energy Source → Body → Energy Detection

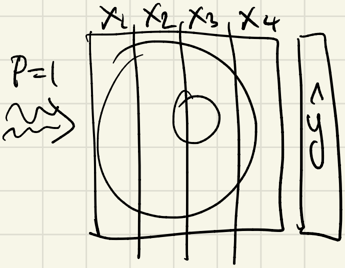
tomography → reconstructing images from projections





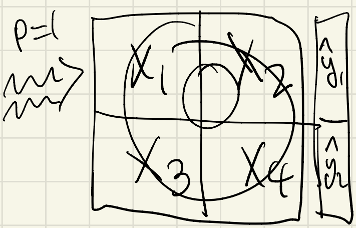
$$\to 1\cdot e^{-(x_1+x_2+x_3+\cdots+x_n)} = \hat{y}$$

$$\to x_1+x_2+\cdots+x_n = y \quad (y=-\log(\hat{y}))$$



$$\to y = x_1+x_2+x_3+x_4$$

↳ unsolvable

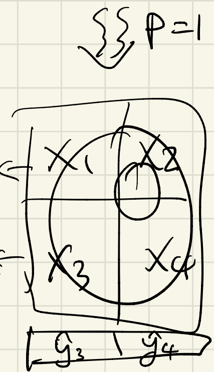


$$\to y_1 = x_1+x_2$$

$$\to y_2 = x_3+x_4$$

$$y_3 = x_1+x_3$$

$$y_4 = x_2+x_4$$



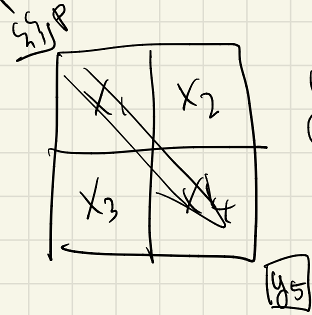
$$\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_3 + x_4 \\ y_3 = x_1 + x_3 \\ y_4 = x_2 + x_4 \end{cases}$$

$y_1 + y_2 = x_1 + x_2 + x_3 + x_4$

$\underbrace{y_1 + y_2 - y_3}_{} = x_2 + x_4$

↪ basically y_4

$\Rightarrow$ y_4 is not a new measurement!



$y_5 \simeq \sqrt{2}\, x_1 + \sqrt{2}\, x_4$

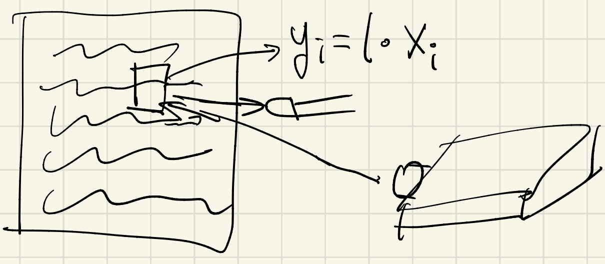
or $y_5 = x_1 + \frac{1}{4} x_2 + \frac{1}{4} x_3 + x_4$

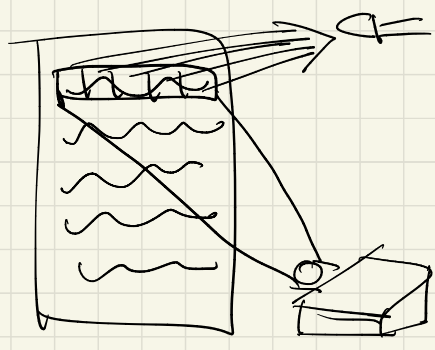
All of the measurements are linear

↪ each variable is multiplied by a scalar

Single Pixel Scanner

$y = x_1 + x_2 + \cdots + x_n$





Linear Algebra

Linear Equations: consider $f(x_1, x_2, \cdots, x_n) : \mathbb{R}^n \to \mathbb{R}$.

f is linear if following identity holds:

1) homogeneity: $f(ax_1, ax_2, \cdots, ax_n) = af(x_1, x_2, \cdots x_n)$

2) superposition: if $x_i = y_i + z_i$, then

$$f(y_1+z_1, y_2+z_2, \cdots, y_n+z_n) = f(y_1, y_2, \cdots, y_n) + f(z_1, z_2, \cdots z_n)$$

$\rightarrow$ Claim: linear functions can always be expressed as

$$f(x_1, x_2, \cdots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Proof for $\mathbb{R}^2$

$f(x_1, x_2)$ is linear; need to prove $f(x_1, x_2) = c_1x_1 + c_2x_2$

trick: $x_1 = \underbrace{1}_{y_1} \cdot x_1 + \underbrace{0}_{z_1} \cdot x_2$, $x_2 = \underbrace{0}_{y_2} \cdot x_1 + \underbrace{1}_{z_2} \cdot x_2$

$$\Rightarrow x_1 = y_1x_1 + z_1x_2, \quad x_2 = y_2x_1 + z_2x_2$$

So, $f(x_1, x_2) = f(x_1y_1 + x_2z_1, x_1y_2 + x_2z_2)$

$= x_1 f(y_1, y_2) + x_2 f(z_1, z_2)$

$= c_1x_1 + c_2x_2$

let $\underbrace{f(y_1, y_2)}_{=f(1,0)} = c_1$

$\underbrace{f(z_1, z_2)}_{f(0,1)} = c_2$

Linear set of Equations

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & b_M \end{array}\right]$$

ex) tomography

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

Algorithm for solving linear equations

1) Multiply an equation with a nonzero scalar

$$2x+3y=4 \iff 4x+6y=8$$

2) Adding a scalar constant multiple of one equation to another equation

3) Swapping equations → trivial

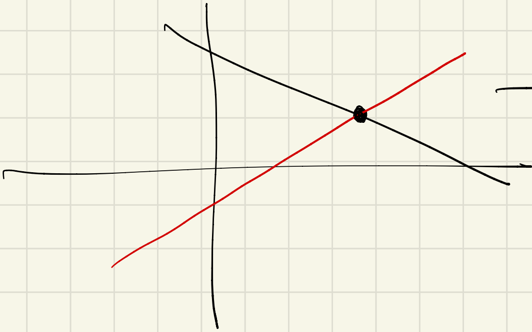
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{infinite solution!}$$

$$\hookrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] \rightarrow \text{no solution!}$$

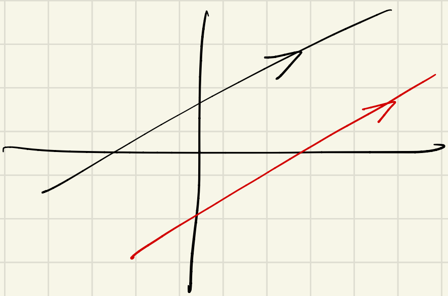
$$\hookrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

Geometric Interpretation



→ one unique solution

→ infinite solution



→ no solution

Vectors

Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \vec{x} \in \mathbb{R}^N$$

$$\vec{e}_N = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \text{N-th element}$$

Addition: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_4 \end{bmatrix}$

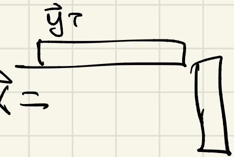
Matrix

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{M1} & x_{M2} & \cdots & x_{MN} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_N \end{bmatrix}$$

Matrix Addition → if same size, by element

Matrix Multiplication → width of 1st = length of 2nd (inner product)

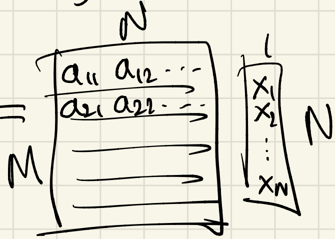
$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$ $\quad \vec{y}^T \cdot \vec{x} =$



$\rightarrow \underbrace{y_1 x_1 + y_2 x_2 \cdots y_N x_N}_{1 \times 1 \text{ (scalar)}}$

$A \in \mathbb{R}^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$

$A\vec{x} =$



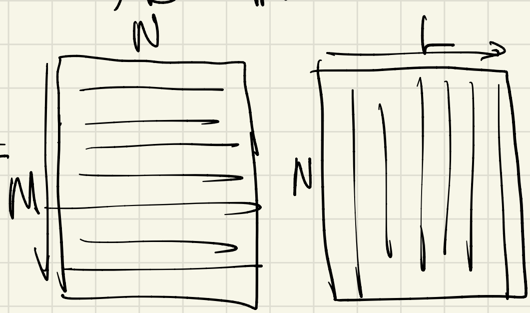
$$= \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1N} \cdot x_N \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2N} \cdot x_N \\ \vdots \\ a_{M1} \cdot x_1 \cdots \end{bmatrix}$$

$\rightarrow$

$M \times 1$ (Vector)

$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$

$AB =$



$$= \begin{array}{cc} & L \\ M & \begin{bmatrix} \underbrace{a_{11}b_{11} + a_{12}b_{21} \cdots} & \underbrace{a_{11}b_{12} \cdots} \\ \vdots & \vdots \end{bmatrix} \end{array}$$

result in location $\alpha, \beta = a_{\alpha 1} b_{1\beta} + a_{\alpha 2} b_{2\beta} + \cdots + a_{\alpha N} b_{N\beta}$

$$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$$

(outer product)

$$\vec{x}\,\vec{y}^T = \underbrace{\boxed{}}_{N\times 1}\;\overbrace{\boxed{}}^{1\times N} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & \cdots & \cdots & x_2y_N \\ \vdots & & \ddots & \vdots \\ x_Ny_1 & \cdots & \cdots & x_Ny_N \end{bmatrix}$$

Matrix Form of LSE

$$A\vec{x} = \vec{b}$$

$$\boxed{A}\;\boxed{\vec{x}} = \begin{bmatrix} a_{11}x_1 & \cdots & a_{1N}x_N \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & a_{MN}x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

$$\Longleftrightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & & \ddots & & \vdots \\ a_{M1} & \cdots & \cdots & a_{MN} & b_M \end{array}\right]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

Row → how each variable affects the measurement

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Column → how a particular variable affects all measurements

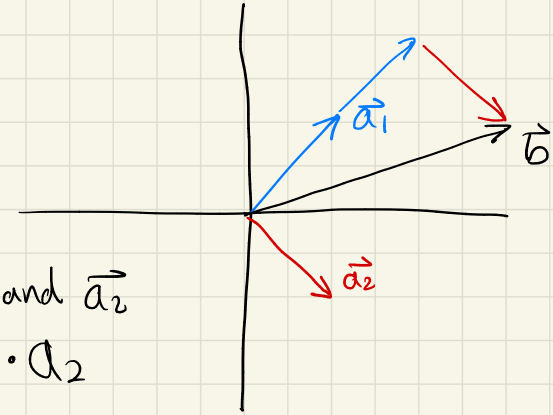
for $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_m\} \in \mathbb{R}^n$, and $\{a_1, a_2, \cdots, a_m\} \in \mathbb{R}$
a linear combination of vectors is

$$\vec{b} \triangleq a_1\vec{a}_1 + a_2\vec{a}_2 + \cdots + a_m\vec{a}_m$$

↳ (defined as)

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}3\\ 1\end{bmatrix}$$

$\vec{a_1}$ $\vec{a_2}$ $\vec{b}$



what linear comb. of $\vec{a_1}$ and $\vec{a_2}$ gives $\vec{b}$? → $2\cdot a_1 + 1\cdot a_2$

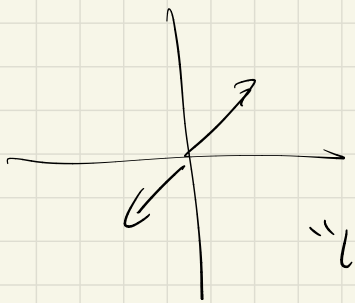
$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{b}$$

$\vec{a_1}$ $\vec{a_2}$

can a linear comb. of $\vec{a_1}$ and $\vec{a_2}$ give any $\vec{b}$? → yes

$$\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{b}$$

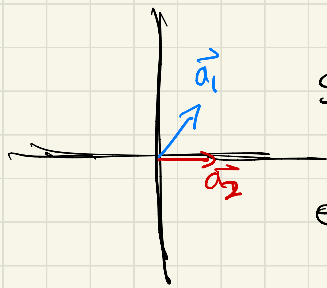
$\vec{a_1}$ $\vec{a_2}$



only if $\vec{b}$ is on the "line"

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{b}$$

$\vec{a_1}$ $\vec{a_2}$



still can "sweep" the entire surface!

Span

Span of the columns of A is the set of all vectors $\vec{b}$ such that $A\vec{x}=\vec{b}$ has a solution

↳ set of all vectors that can be reached by all possible linear comb. of the columns of A

$(\mathbb{R}^2)$

$$\text{span}(A) = \left\{ \vec{v} \mid \vec{v} = \alpha \left[\vec{a}_1 \right] + \beta \left[\vec{a}_2 \right] \quad \alpha, \beta \in \mathbb{R} \right\}$$

↳ if $\vec{a}_1$ and $\vec{a}_2$ are scalar multiples, β can be ignored.

there exists

If $\exists\, \vec{x}$ s.t. $A\vec{x}=\vec{b}$, then $\vec{b} \in \text{span}(A)$

↳ there is a solution for $A\vec{x}=\vec{b}$

if $\vec{b} \notin \text{span}(A)$, then there are no $\vec{x}$ that solves $A\vec{x}=\vec{b}$

Proofs

1) Operations that do not change the solution

1) Multiply with non-zero scalar

proof for $N=2$:

let $ax+by=c$, solution $x_0, y_0 \rightarrow ax_0+by_0=c$.

Show that $\beta ax+\beta by=\beta c$ has the same solution.

substitue x_0, y_0 for x, y:

$\beta ax_0+\beta by_0=\beta c \rightarrow \beta(ax_0+by_0)=\beta c \rightarrow \beta c=\beta c$ //

$\beta ax+\beta by=\beta c$, solution $x_1, y_1 \rightarrow \beta ax_1+\beta by_1=\beta c$

Show that $\beta ax+\beta by=\beta c$ has the same solution.

$\frac{\beta ax_1}{\beta}+\frac{\beta by_1}{\beta}=\frac{\beta c}{\beta} \rightarrow ax_1+by_1=c$

~~$\beta ax_0+\beta by_0=\beta c \rightarrow \beta(ax_0+by_0)=\beta c \rightarrow \beta c=\beta c$ //~~

2) Adding scalar multipled equations

$\rightarrow$ Show that the altered equations have the same solution as the original

Theorem: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

2) Adding multiple of equations

Know: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \left\{\vec{v} \mid \vec{v} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix}, \alpha,\beta \in \mathbb{R}\right\} = S$

Need to show:

pick a specific vector $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \in \mathbb{R}^2$,

show that it belongs to S

Need to solve: $\alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$

$\rightarrow \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \rightarrow$ Gaussian Elimination

$\rightarrow \left[\begin{array}{cc|c}1 & 1 & b_1\\0 & 2 & b_1-b_2\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 1 & b_1\\0 & 1 & (b_1-b_2)/2\end{array}\right] \rightarrow \begin{array}{l}\alpha = \frac{b_1+b_2}{2}\\ \beta = \frac{b_1-b_2}{2}\end{array}$

$\Rightarrow$ every $\vec{b} \in \mathbb{R}$ can be written as linear combinations. Also, $\vec{b} \in S$

Linear Dependence

$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \vec{a}_1 = -\vec{a}_2$

$\vec{a}_1 \quad \vec{a}_2$

Definition 1: A set of vectors $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_N\}$ are linearly dependent if $\exists \{\alpha_1, \alpha_2, \cdots, \alpha_N\} \in \mathbb{R}$ s.t.

$$\vec{a}_i = \sum_{j \neq i} \alpha_j \vec{a}_j \quad 1 \leq i, j \leq M$$

ex) $\vec{a}_2 = 3\vec{a}_1 - 2\vec{a}_5 + 6\vec{a}_7$

($\vec{a}_i$ in span of all $\vec{a}_j$)

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \rightarrow$ linearly dependent?

Need to solve: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} \alpha = 2 \\ \beta = 1 \end{matrix}$

Definition 2: when $\sum_{i=1}^{N} \alpha_i \vec{a}_i = 0$ (as long as all $\alpha = 0$)

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} \right\} \rightarrow$ still linearly dependent

---

Theorem: If columns of matrix A are linearly dependent, then $A\vec{x} = \vec{b}$ does not have a unique solution.

Let $A\vec{x^*} = \vec{b}$, $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3] \rightarrow [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3] \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\vec{\alpha}} = 0 \rightarrow A\vec{\alpha} = 0$

set $\vec{x^*} = \vec{x^*} + \vec{\alpha}$, then $A\vec{x^*} = A(\vec{x^*} + \vec{\alpha}) = \vec{b} + 0 = \vec{b}$

$\rightarrow \vec{x^*}$ is not a unique solution for $A\vec{x} = \vec{b}$ //

Matrix Transformation

$A\vec{x} = \vec{b}$ → A transforms $\vec{x}$ to $\vec{b}$

ex) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$ → reflection!

ex) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \sin\theta\, x_1 + \cos\theta\, x_2 \end{bmatrix}$

↳ rotates $\vec{x}$ by θ cclw.

Linear Transformation of Vectors

$f(ax) = af(x)$, $a \in \mathbb{R}$

$f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$

↳ Matrix-vector multiplication satisfies linear transformation.

$A \cdot (\alpha\vec{x}) = \alpha A\vec{x}$

$A \cdot (\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

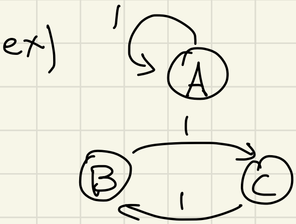
Vectors as States, Matrices as State Transitions

ex)

$$\vec{S}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right] \text{position} \\ -\text{velocity} \end{array}$$

→ add $\theta(t)$ to know where the car will be in time t,

ex)



$$\vec{X}(t) = \begin{bmatrix} X_A(t) \\ X_B(t) \\ X_C(t) \end{bmatrix} \quad \begin{cases} X_A(t+1) = X_A(t) \\ X_B(t+1) = X_C(t) \\ X_C(t+1) = X_B(t) \end{cases}$$

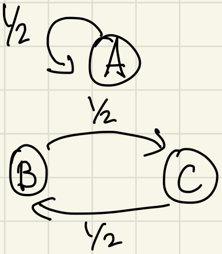
$$\rightarrow \begin{bmatrix} X_A(t+1) \\ X_B(t+1) \\ X_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_A(t) \\ X_B(t) \\ X_C(t) \end{bmatrix} \rightarrow \vec{X}(t+1) = Q\vec{X}(t)$$

$$\vec{X}(t+2) = Q \cdot \vec{X}(t+1) = Q\,Q\,\vec{X}(t) = Q^2\,\vec{X}(t)$$

$$\vec{X}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \vec{X}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

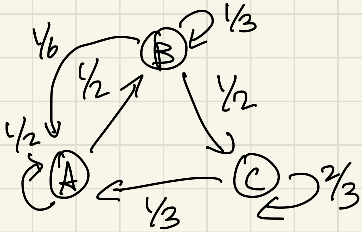
$$\vec{X}(2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow Q^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\vec{X}(0) = \vec{X}(2)$



$$\begin{cases} X_A(t+1) = \frac{1}{2} X_A(t) \\ X_B(t+1) = \frac{1}{2} X_C(t) \\ X_C(t+1) = \frac{1}{2} X_B(t) \end{cases} \rightarrow \vec{X}(t+1) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \vec{X}(t)$$

$$Q^2 = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$ → water 1/4 thed every 2days



$$\begin{bmatrix} X_A(t+1) \\ X_B(t+1) \\ X_C(t+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} X_A(t) \\ X_B(t) \\ X_C(t) \end{bmatrix}$$

$$X_A(t+1) = \frac{1}{2} X_A(t) + \frac{1}{6} X_B(t) + \frac{1}{3} X_C(t)$$

Reversing the arrow transposes the matrix, but it does not "turn back time" (inverse)

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix}$$

$$A \in \mathbb{R}^{N \times M} \qquad A^T \in \mathbb{R}^{M \times N}$$

Matrix Inversion

$\vec{x}(t+1) = Q\vec{x}(t)$, is there a matrix P s.t.

$\vec{x}(t) = P\vec{x}(t+1)$? $\rightarrow$ only if $P \cdot Q = I$

$$P\vec{x}(t+1) = PQ\vec{x}(t) \Rightarrow \vec{x}(t) \cdot \cancel{I} = P\vec{x}(t+1)$$

$$(Q \cdot P = I)$$

P is inverse of Q if: $PQ = QP = I$

$P = Q^{-1}$, $Q = P^{-1}$

$$QP = I$$

$$\overset{Q}{\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}} \overset{P}{\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{array}{ccc} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \end{array} \qquad \begin{array}{ccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array}$$

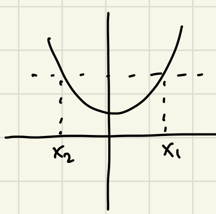
$$\to \left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{Q} & \multicolumn{3}{c}{I} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{I} & \multicolumn{3}{c}{P} \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\Rightarrow P = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Can we always invert a function?



$y = f(x)$

$f^{-1}(y) = x_1 ? x_2 ?$

↳ not invertible

$f(\vec{x}) = A\vec{x}$? → yes, as long as A^{-1} exists.

Invertibility of Linear Transformations

A is invertible iff the columns of A are linearly independent.

$\rightarrow$ if columns of A are lin. dep. $\rightarrow A^{-1}$ d.n.e.

$\rightarrow$ if A^{-1} exists $\rightarrow$ columns of A are lin. independent.

Proof: Assume lin. dep. & invertability $\rightarrow$ contradiction

$\exists \vec{a} \neq 0$ s.t. $A\vec{\alpha} = 0 \rightarrow A^{-1} \cdot A\vec{\alpha} = A^{-1} \cdot 0$

$\rightarrow I\vec{\alpha} = 0 \Rightarrow$ but $\vec{\alpha} \neq 0$, thus A^{-1} DNE.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $(\det(A) = ad - bc)$

Matrix A is invertible

$= A\vec{x} = \vec{b}$ has a unique solution

$=$ A is full rank (linearly independent columns)

$=$ A has a trivial nullspace

$= \det(A) \neq 0$

Vector Space: set of vectors ($V \in \mathbb{R}^n$) and scalars ($F \in \mathbb{R}$), $\cdot$ and $+$ s.t.

1) $\alpha\vec{x} \in V$ 2) $\vec{x}+\vec{y} \in V$ } Axiom of Closure

3) $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$

4) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$

5) $\exists \vec{0} \in V$ s.t. $\vec{x}+\vec{0} = \vec{x}$

6) $\exists (-\vec{x}) \in V$ s.t. $\vec{x}+(-\vec{x}) = \vec{0}$

} Axiom of Addition

7) $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$

8) $\alpha\cdot(\beta\vec{x}) = (\alpha\beta)\vec{x}$

9) $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$

10) $1\cdot\vec{x} = \vec{x}$

} Axiom of Scaling

$\mathbb{R}^2 \to$ vector space, $\mathbb{R}^{2\times 2} \to$ vector space

$\alpha \in \mathbb{R}, a \geq 0 \to$ X vector space

$\text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\} \to$ vector space $\begin{bmatrix}0\\1\end{bmatrix} \to$ X vector space

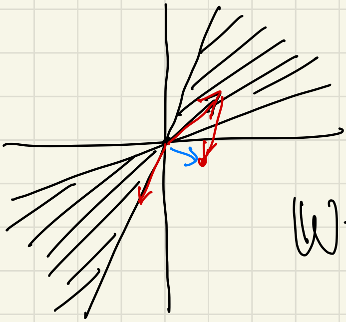
$0 \to$ vector space

Subspace: $\mathbb{U} \subset \mathbb{V}$ has 3 properties

1) $\vec{0} \in \mathbb{U}$ 2) $\vec{v}_1 + \vec{v}_2 \in \mathbb{U}$ 3) $\alpha \vec{v} \in \mathbb{U}$

$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} \to$ subspace!

2D planes in $\mathbb{R}^3$
↳ subspace!
(as long as passing orig in (0,0,0))



→ not a subspace!

$\mathbb{W} = \left\{\begin{bmatrix}a & b\\0 & d\end{bmatrix} \middle| \begin{matrix}a b\\d\end{matrix} \in \mathbb{R}\right\}, \mathbb{V} = \mathbb{R}^{2\times 2}$ $\mathbb{W} \subset \mathbb{V}$?
↳ yes

Basis: Minimum set of vectors that spans a vector space

In $\mathbb{V}$, $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a basis of $\mathbb{V}$ if:

1) $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ are linearly independent

2) $\forall \vec{v} \in \mathbb{V}, \exists \alpha_1, \alpha_2, \cdots, \alpha_N \in \mathbb{R}^N$ s.t. $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_1 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$
(for any)

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \to$ basis for $\mathbb{R}^3$ (identity basis)

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} \to$ not a basis for $\mathbb{R}^3$

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \to$ not a basis for $\mathbb{R}^3$

Column Space = span of column vectors = range

$$\text{span}\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} \triangleq \left\{\sum_{m=1}^{M} \alpha_m \vec{a}_m \,\middle|\, \alpha_1, \alpha_2, \cdots, \alpha_M \in \mathbb{R}\right\}$$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \rightarrow \text{not a basis, but maybe a basis for a subspace?}$$

$$\vec{v}_1 = A\vec{u}_1, \vec{v}_2 = A\vec{v}_2 \quad (\vec{u} \in \mathbb{R}^{2\times 1}, \vec{v} \in \mathbb{R}^{3\times 1})$$

$$A\vec{0} = \vec{0}, \quad \vec{v}_1 + \vec{v}_2 = A\vec{u}_1 + A\vec{u}_2 = A(\vec{u}_1 + \vec{u}_2), \quad \overset{\alpha\vec{v}_1}{\alpha(A\vec{u}_1)} = A(\alpha\vec{u}_1)$$

↳ columnspace of A is a subspace!

Rank: $\text{Rank}\{A\} = \dim\{\text{span}\{A\}\} \leq \min(M, N) \quad (A \in \mathbb{R}^{M\times N})$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \rightarrow \text{Rank}\{A\} = 2$$

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Rank}\{A\} = 2$$

$$A = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \rightarrow \text{Rank}\{A\} = 1$$

Nullspace: $A \in \mathbb{R}^{M\times N}$, set of all vectors $\vec{x} \in \mathbb{R}^M$ s.t. $A\vec{x} = 0$

$A\vec{x} = 0 \rightarrow$ How many solutions?

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{only } \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ (trivial nullspace)}$$

↳ lin. indep.

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 2x_2 \rightarrow \vec{x} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} \rightarrow \vec{x} = \alpha\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\rightarrow$ non-trivial nullspace, $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$$\text{Rank}\{A\} + \dim\{\text{Null}\{A\}\} = \min(M, N)$$

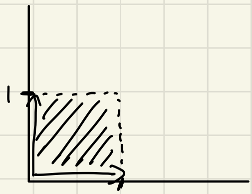
$A\vec{x}=\vec{b}$, $\vec{v}_0 \in Null\{A\} \rightarrow A\vec{v}_0=0$, $A\vec{x}_0=\vec{b}$

then, $\vec{x}_0+\alpha\vec{v}_0$ is also a solution!

$\because A(\vec{x}_0+\alpha\vec{v}_0) = A\vec{x}_0 + A(\alpha\vec{v}_0) = \vec{b}+\cancel{\alpha(A\vec{v}_0)} = \vec{b}$

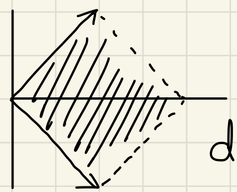
Determinant

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad-bc$$



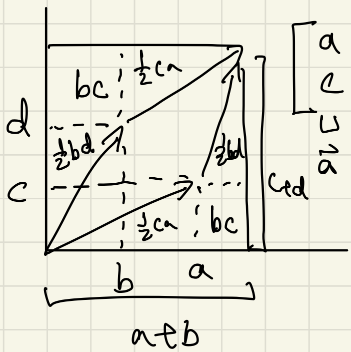
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0$



$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = -2 \neq 0$



$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\vec{a}$ $\vec{b}$

$A =$

$(a+b)(c+d)$

$-(bc)\times 2 - (\frac{1}{2}ca)\times 2 - (\frac{1}{2}bd)\times 2$

$= \underline{ac}+bc+ad+\underline{bd}-2bc-\underline{ac}-\underline{bd}$

$= \boxed{ad-bc} = \det(A)$

PageRank

$$\text{To}\;\begin{array}{c}\text{From}\\ \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix}\end{array} \rightarrow \vec{x}(t+1) = A \cdot \vec{x}(t)$$

$$\vec{x}(0) = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} \quad \vec{x}(1) = \begin{bmatrix} 0.125 \\ 0.208 \\ 0.208 \\ 0.458 \end{bmatrix} \cdots \vec{x}(100) = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} \rightarrow \text{steady state}$$

$$\vec{X}_{ss} = Q \cdot \vec{X}_{ss} \quad (Q \in \mathbb{R}^{N \times N}, \vec{X} \in \mathbb{R}^{N})$$

$$\rightarrow Q \cdot \vec{X}_{ss} - \vec{X}_{ss} = \vec{0} \rightarrow \boxed{(Q - I)\vec{X}_{ss} = \vec{0}}$$

↳ $\vec{X}_{ss}$ lies in the $\mathrm{Null}\{Q - I\}$

Eigen Values

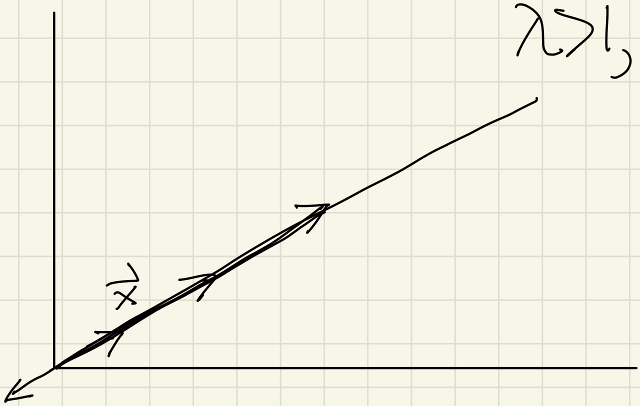
$$Q \cdot \vec{X}_{ss} = 1 \cdot \vec{X}_{ss}, \quad Q \cdot \vec{x} = \lambda \cdot \vec{x}\ ?$$

↳ $\vec{x}$ is an Eigenvector of Q with Eigenvalue λ

$\mathrm{span}\{\vec{x}\}$ is associated with Eigen-space

$Q \cdot \vec{x} = \lambda \cdot \vec{x}$ if $\lambda = 1$, $\vec{x}$ maps to itself.

$\lambda > 1$, $\lambda < 1$, $\lambda < 0$?



Eigen Definitions

Let $Q \in \mathbb{R}^{N \times N}$ (square matrix), and $\lambda \in \mathbb{R}$

$\exists \vec{x} \neq \vec{0}$ s.t. $Q\vec{x} = \lambda \vec{x}$,

the λ is an eigen value of Q, $\vec{x}$ is an eigen vector,

and $\mathrm{Null}(Q - \lambda I)$ is its eigenspace.

$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$, find λ and $\vec{x}$ s.t. $Q\vec{x} = \lambda \vec{x} \rightarrow (Q - \lambda I)\vec{x} = \vec{0}$

$\hookrightarrow \vec{x} \in \mathrm{Null}(Q - \lambda I)$

$Q - \lambda I = \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1 - \lambda \end{bmatrix} \rightarrow$ nontrivial nullspace?

$\Longleftrightarrow \det = 0$!

$\det(Q - \lambda I) = \underbrace{(\tfrac{1}{2} - \lambda)(1 - \lambda) - \tfrac{1}{2} \cdot 0}_{\text{characteristic polynomial}} = 0 \rightarrow \lambda_1 = \tfrac{1}{2}, \lambda_2 = 1$

$$Q-\lambda I = \begin{bmatrix} \frac{1}{2}-\lambda & 0 \\ \frac{1}{2} & 1-\lambda \end{bmatrix},\ \lambda_1 = \frac{1}{2},\ \lambda_2 = 1$$

$$\rightarrow \lambda = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x} = 0 \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{x} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$\rightarrow \lambda = 1 \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \vec{x} = 0 \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{x} = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

eigen ⋯ $\left(\lambda = \frac{1}{2}, \vec{x} \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}\right), \left(\lambda = 1, \vec{x} \in \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$

$\hookrightarrow Q\vec{x} = \frac{1}{2}\vec{x}$ $\hookrightarrow Q\vec{x} = 1 \cdot \vec{x}$

Circuits

Quantity	Symbol	Unit
Current	I	Amperes [A]
Voltage	V	Volts [V]
Resistance	R	Ohms [Ω]

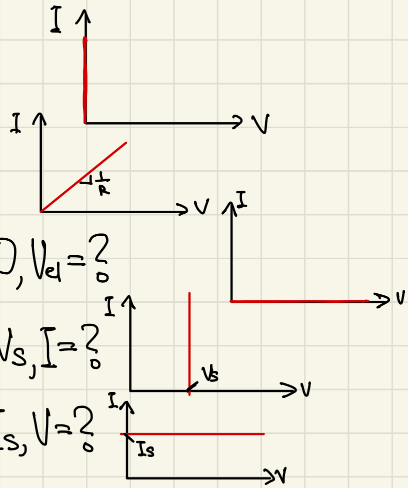
Wire: $+ \xrightarrow{I_{el}} -$, V_{el} $V_{el}=0,\ I_{el}=?$

Resistor: $+ \xrightarrow{I_{el}} -$, V_{el} $V_{el}=I_{el}\cdot R$

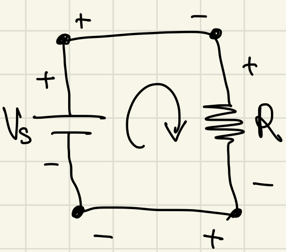
Open Circuit: $\xrightarrow{I_{el}}$ $+$ $-$, V_{el} $I_{el}=0,\ V_{el}=?$

Voltage Source: $\xrightarrow{I_{el}}$ $+$ $-$, V_S $V_{el}=V_S,\ I=?$

Current Source: $\xrightarrow{I_{el}}$ $+$ $-$, I_S $I_{el}=I_S,\ V=?$

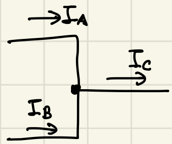


Kirchoff's Rule: $\sum V$ of a loop $=0$, $\sum I_{in}=\sum I_{out}$ for junctions



$$V_S - V_R = 0$$

$$\rightarrow V_S - IR = 0$$



$$I_A + I_B - I_C = 0$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc = 0$

$\rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$

Theorem: Let $A \in \mathbb{R}^{N \times N}$, with M distinct eigenvalues & eigenvectors $\lambda_i, \vec{v}_i \,|\, 1 \leq i \leq M$. Then, all $\vec{v}_i$ are linearly independent.

$\Rightarrow$ If $A \in \mathbb{R}^{2 \times 2}$ have $\vec{v}_1, \vec{v}_2$ as eigenvectors, then $\text{span}\{\vec{v}_1, \vec{v}_2\}$ forms a basis!

$\therefore$ Proof by contradiction: assume linear dependence, then either $\lambda_1 = \lambda_2$ or $\vec{v}_2 = \vec{0}$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow (\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}), (\lambda_2 = 2, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix})$

$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$? $\rightarrow \vec{v}_3 = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2$

$\Rightarrow A\vec{v}_3 = A(1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2) = \lambda_1 \cdot 1 \cdot \vec{v}_1 + \lambda_2 \cdot 1 \cdot \vec{v}_2 = \vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \rightarrow (\lambda_1 = \frac{1}{2}, \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}), (\lambda_2 = 1, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix})$

$\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? $\rightarrow \vec{v}_3 = 2 \cdot \vec{v}_1 + 4 \cdot \vec{v}_2$ (use gaussian for more complex ones)

$\Rightarrow A\vec{v}_3 = A(2\vec{v}_1 + 4\vec{v}_2) = 2 \cdot (\frac{1}{2}\vec{v}_1) + 4(1 \cdot \vec{v}_2) = \vec{v}_1 + 4\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$A=\begin{bmatrix}2&0\\0&2\end{bmatrix} \rightarrow \lambda_{1,2}=2$, $\vec{v} \in \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right\}$ (eigenspace is 2D!)

In general, multiplicity in λ results in multidimensional eigenspace... except when the matrix is <u>defective</u>

<u>Defective Matrix</u>

$A=\begin{bmatrix}1&1/4\\0&1\end{bmatrix} \rightarrow (1-\lambda)^2=0 \quad \lambda_{1,2}=1 \rightarrow \text{null}\left\{\begin{bmatrix}0&1/4\\0&0\end{bmatrix}\right\}=\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

$\vec{x}(t+1)=A\vec{x}(t)$

Assume $\lambda_i \mid 1\le i\le N$ are distinct $\Rightarrow \text{Span}\{v_i \mid 1\le i\le N\} \in \mathbb{R}^N$

$$\vec{x}(1)=A\vec{x}(0)=A(\alpha_1\vec{v}_1+\alpha_2\vec{v}_2 --- +\alpha_N\vec{v}_N)$$

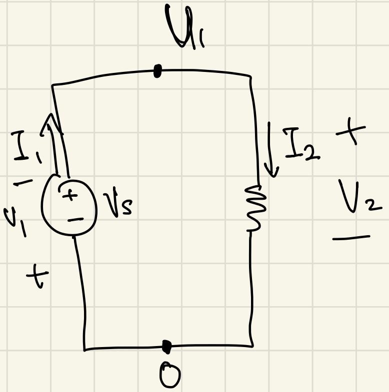
$$=\alpha_1A\vec{v}_1+\alpha_2A\vec{v}_2+---+\alpha_NA\vec{v}_N$$

$$=\alpha_1\lambda_1\vec{v}_1+\alpha_2\lambda_2\vec{v}_2+---+\alpha_N\lambda_N\vec{v}_N$$

$$\vec{x}(2)=A\vec{x}(1)=\alpha_1\lambda_1^2\vec{v}_1+\alpha_2\lambda_2^2\vec{v}_1+---+\alpha_N\lambda_N^2\vec{v}_N$$

$$\vec{x}(t)=\alpha_1\lambda_1^t\vec{v}_1+\alpha_2\lambda_2^t\vec{v}_1+\cdots+\alpha_N\lambda_N^t\vec{v}_N$$

$$\lim_{t\to\infty}\vec{x}(t)\ ? \quad \begin{cases}|\lambda|<1 \rightarrow 0\\ \lambda=1 \rightarrow 1\\ \lambda=-1 \rightarrow \text{oscillates}\\ |\lambda|>1 \rightarrow \infty\end{cases}$$



$$\vec{x} = \begin{bmatrix} I_1 \\ I_2 \\ U_1 \end{bmatrix}$$

$I_1 = I_2 \to \boxed{I_1 - I_2 = 0}$ ①

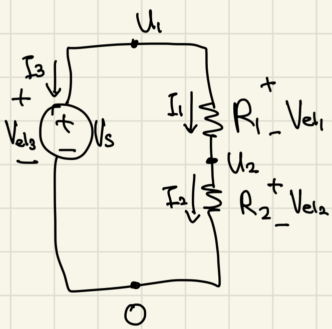
$V_1 = -V_s = 0 - U_1 = -U_1$

$\to \boxed{U_1 = V_s} = -V_1$ ②

$V_2 = I_2 R = U_1 - 0$

$\to \boxed{R \cdot I_2 - U_1 = 0}$ ③

$$A\vec{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & R & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0 \\ V_s \\ 0 \end{bmatrix} \to \begin{aligned} U_1 &= V_s \\ I_1 &= V_s/R \\ I_2 &= V_s/R \end{aligned}$$



$$\vec{x} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ U_1 \\ U_2 \end{bmatrix}$$

$\boxed{I_1 + I_3 = 0}$ ①

$\boxed{-I_1 + I_2 = 0}$ ②

$V_{el_1} = I_1 \cdot R_1 = U_1 - U_2 \to \boxed{U_1 - U_2 - R_1 I_1 = 0}$ ③

$V_{el_2} = I_2 \cdot R_2 = U_2 - 0 \to \boxed{U_2 - R_2 I_2 = 0}$ ④

$V_{el_3} = V_s = U_1 - 0 \quad \boxed{U_1 = V_s}$ ⑤

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -R_1 & 0 & 0 & 1 & -1 \\ 0 & -R_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_s \end{bmatrix} \Rightarrow$$

$I_1 = \frac{V_s}{R_1 + R_2} \quad I_2 = \frac{V_s}{R_1 + R_2} \quad I_3 = -\frac{V_s}{R_1 + R_2}$

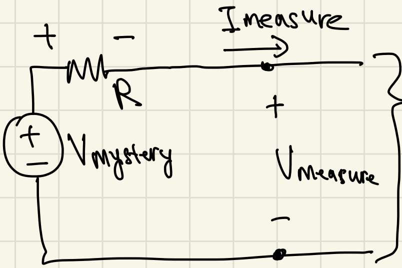
$U_1 = V_s \quad U_2 = \frac{R_2}{R_1 + R_2} V_s$

Energy & Power

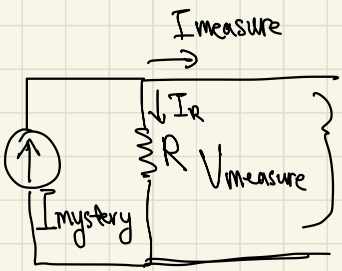
$$\left.\begin{aligned} \text{Current } I &= \frac{dQ}{dt} \\ \text{Voltage } V &= \frac{dE}{dq} \end{aligned}\right\} \rightarrow \text{Power } P = \frac{dE}{dt} = VI$$

$P_{el} = V_{el} \cdot I_{el}$; if E_{el} is a resistor $\rightarrow P = I^2 R = \frac{V^2}{R} = VI$

$P > 0 \rightarrow$ Power dissipated



opencircit $(I=0)$ $\rightarrow V_{mystery} = V_{measure}$

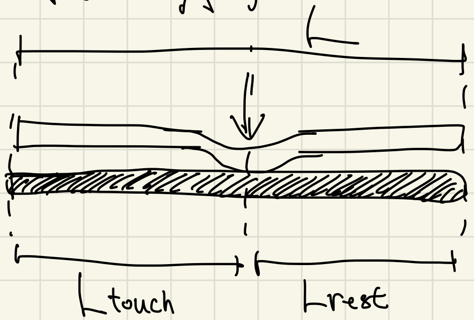


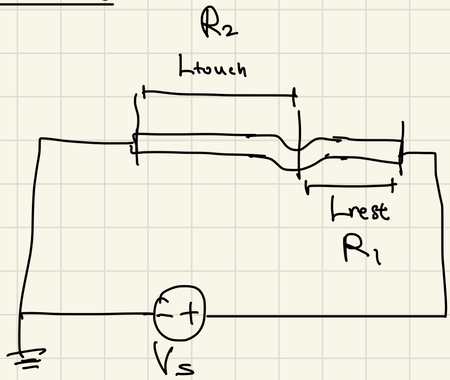
$$I_{myst} = I_{meas} + \cancelto{0}{I_R} = I_{meas} + \cancelto{0}{\frac{V}{R}}$$

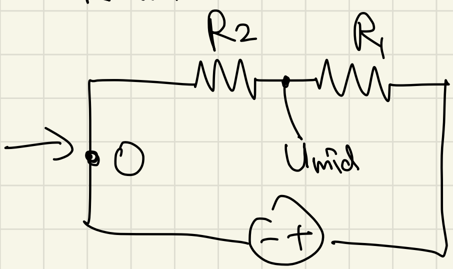
Wire $(V=0)$ $\rightarrow I_{myst} = I_{meas}$

Resistance & Resistivity

$$R = \frac{L}{A}\rho$$



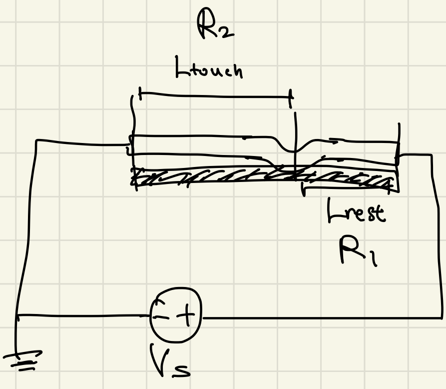


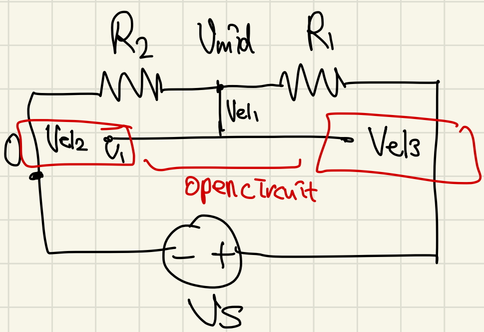


$$R_1 = \rho \cdot \frac{L_{rest}}{A}$$

$$R_2 = \rho \cdot \frac{L_{touch}}{A}$$

$$U_{mid} = \frac{R_2}{R_1 + R_2} V_s = \frac{L_{touch}}{L} V_s$$



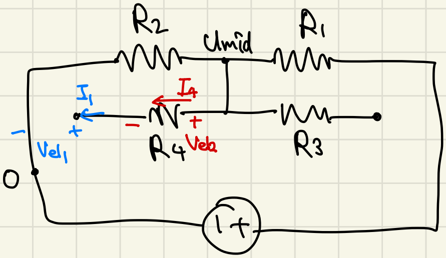


$$Vel_2 = U_1 - 0 \rightarrow U_1 = Vel_2$$

$$Vel_1 = U_{mid} - U_1 \rightarrow U_{mid} = \cancelto{0}{Vel_1} + Vel_2$$

$$U_{mid} - 0 = Vel_2 + Vel_1 \Rightarrow \text{measure } V_{out} \text{ with } Vel_2$$

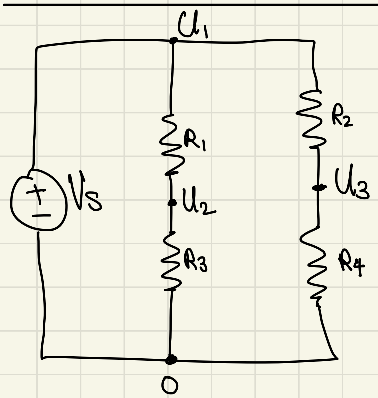
Model 2: more realistic model (imperfect conductor)



$I_4 = I_1,\ I_1 = 0$ (KCL)

$U_{mid} - 0 = V_{el_1} + \cancel{R_4 I_4} \to 0$

$\to U_{mid} = V_{el_1}$

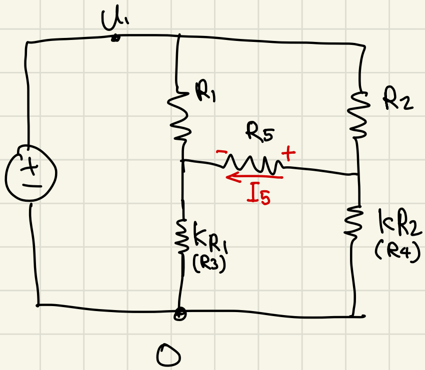


$U_2 = \frac{R_3}{R_1+R_3}\cdot V_s$

$U_3 = \frac{R_4}{R_2+R_4}\cdot V_s$

if $R_3 = KR_1$, $R_4 = KR_2$

$\to U_2 = \frac{k}{k+1} V_s$, $U_3 = \frac{k}{k+1} V_s$

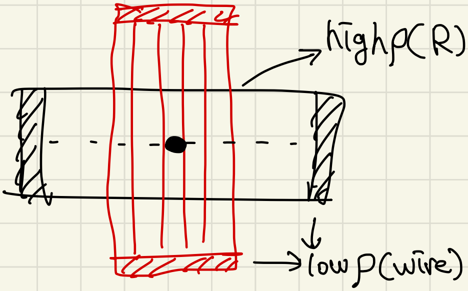


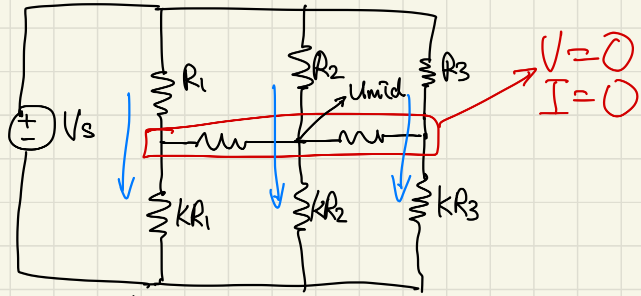
$U_3 - U_2 = I_5 R_5 = V_{el5}$

If $V_{el5} = 0$, $I_5 = 0$

$\to R_5$ is open circuit!

2D Touchscreen

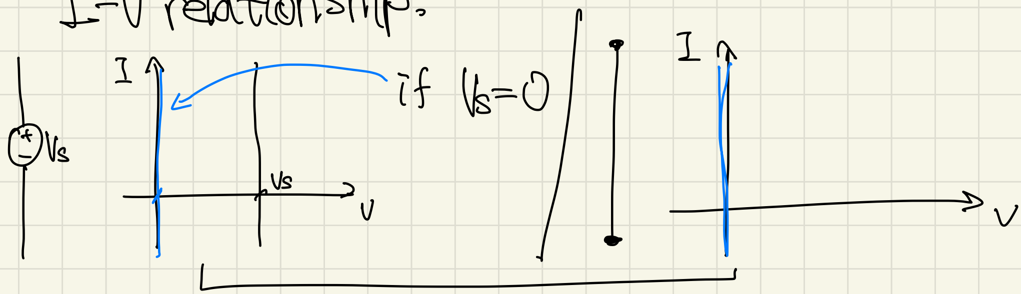




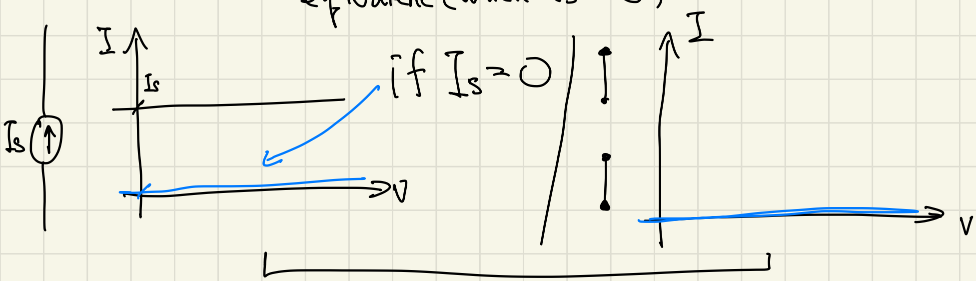
$$U_{mid} = \frac{L_{touch}}{L_V} V_s$$

Equivalence

Two circuits are equivalent if they have the same ☆ I-V relationship.

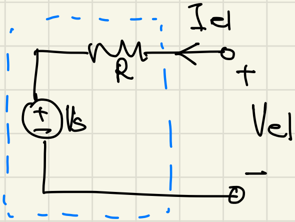


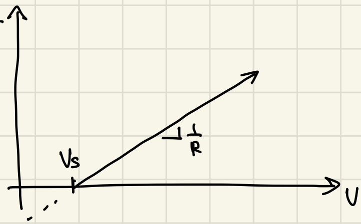
equivalent (when Vs = 0)



equivalent (when Is = 0)

As long as I-V relations are same, circuits are equivalent



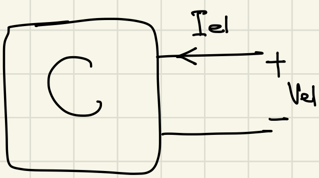


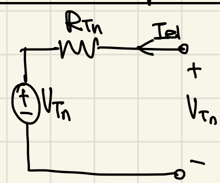
$$V_{el} = V_s + V_R$$

$$V_{el} = V_s + I_{el} \cdot R$$

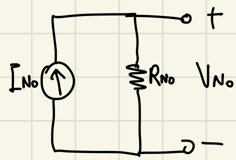
$$\underbrace{I_{el} = \frac{1}{R} V_{el} - \frac{V_s}{R}}$$

Thevenin and Norton Equivalent

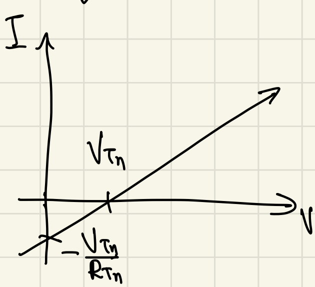




Thevenin Eq.

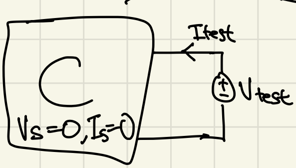


Norton Eq.

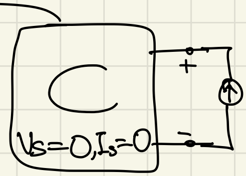


1) connect to open circuit → find V_{Th}, $I = 0$

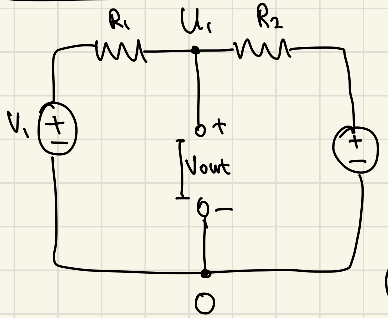
2) find slope (R_{Th})



$$R_{Th} = \frac{V_{test}}{I_{test}} = R_{N_0}$$

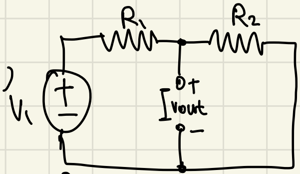


$$\underbrace{A\vec{x}}_{I_i,\,U_i} = \underbrace{\vec{b}}_{I_s,\,V_s} \Rightarrow \vec{x} = A^{-1}\vec{b} \Rightarrow \begin{cases} I_i = \alpha_1 I_{s_1} + \cdots + \alpha_n V_{s_n} \\ \vdots \\ U_i = \beta_1 I_{s_1} + \cdots + \beta_n V_{s_n} \end{cases}$$



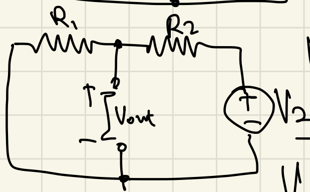
$U_1 - 0 = V_{out} \rightarrow U_1 = V_{out}$

① if $V_2 = 0$,



$$V_{out_1} = \frac{R_2}{R_1+R_2} V_1$$

② if $V_1 = 0$,



$$V_{out_2} = \frac{R_1}{R_1+R_2} V_2$$

$$\boxed{V_{out} = \frac{R_1 V_2 + R_2 V_1}{R_1 + R_2}}$$

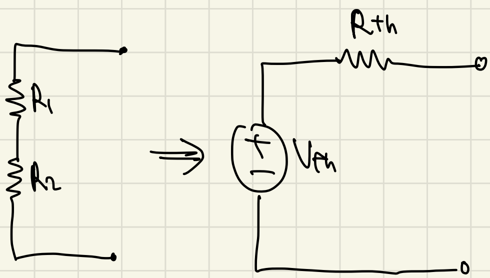
Superposition

Set all other V_s or I_s to 0

→ V_s to wire, I_s to open circuit (equivalence!)

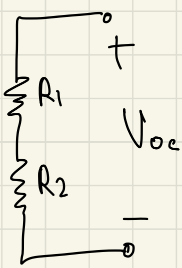
$V_{out} = \sum V_{out_i}$

ex)



$$R_{th} = \frac{V_{test}}{I_{test}}$$

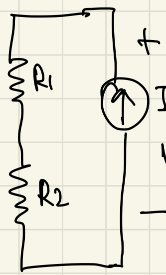
①



$V_{oc} = 0$

$V_{th} = 0$

②



$$V_{test} - I_{test}(R_1 + R_2) = 0$$

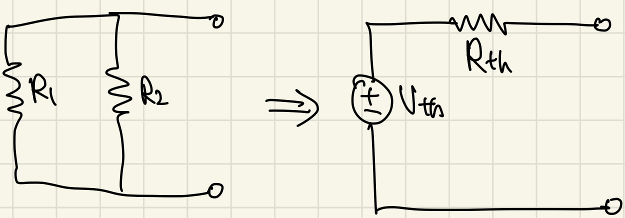
$\downarrow$

V_{test} $V_{test} = (R_1 + R_2) I_{test}$

$$\rightarrow R = R_{th} = \frac{V_{test}}{I_{test}} = R_1 + R_2$$

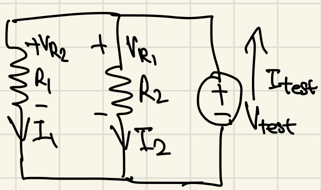
---

ex)



① $V_{th} = 0$

②

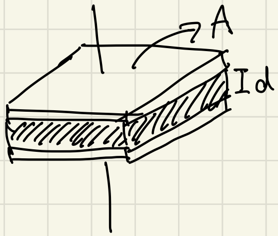


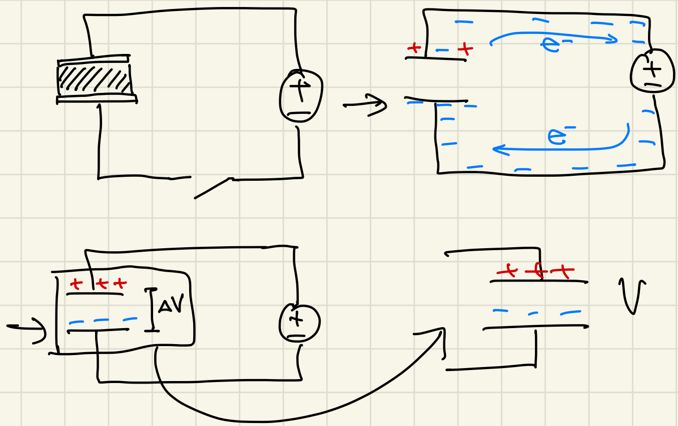
$$I_{test} = I_1 + I_2$$

$$V_{test} = V_{R_1} = V_{R_2} = I_1 R_1 = I_2 R_2$$

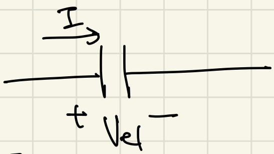
$$I_{test} = \frac{V_{test}}{R_1} + \frac{V_{test}}{R_2} \rightarrow R_{th} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

Capacitors





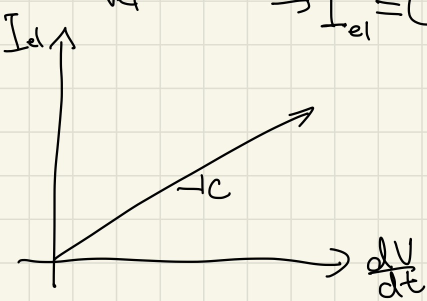
IV-relationship

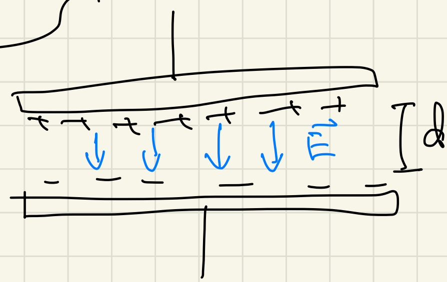


const.

$Q_{el} = C \cdot V_{el}$, $I = \frac{dQ}{dt}$, $I_{el} = \frac{d}{dt}(C \cdot V_{el})$

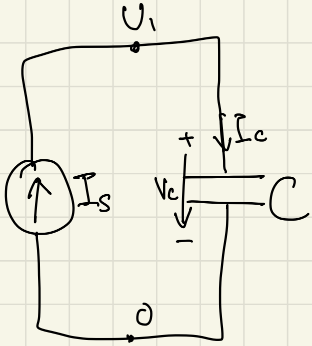
$\rightarrow I_{el} = C\frac{dV_{el}}{dt}$ → $C\,[F]$





$C = \varepsilon \frac{A}{d}$ → permitivity, $\varepsilon = \varepsilon_0 \cdot \varepsilon_r$

→ thickness of dielectric

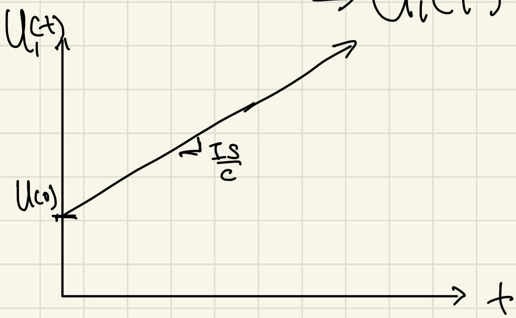


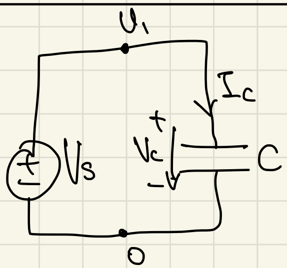
$$I_s = I_c,\ I_c = C\frac{dV_c}{dt},\ V_c = U_1 - 0$$

$$\rightarrow I_s = C\cdot\frac{dU_1}{dt} \rightarrow \int_0^t I_s\cdot dt' = C\int_{U_1(0)}^{U_1(t)} dU_1'$$

$$\rightarrow I_s\cdot t = C\left(U_1(t) - U_1(0)\right)$$

$$\Rightarrow U_1(t) = \frac{I_s\cdot t}{C} + U(0)$$

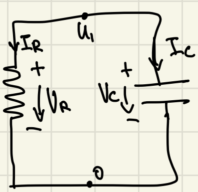




$$U_1 = V_s,\ V_c = U_1 \rightarrow \underline{\underline{V_s = V_c}}$$

$$I_c = C\frac{dV_c}{dt},\ V_c = \text{const} \rightarrow \frac{dV_c}{dt} = 0$$

$$\rightarrow \underline{\underline{I_c = 0}}$$



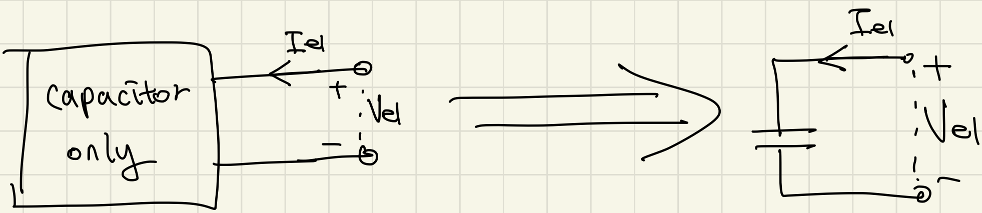
for steady-state, V_c = constant

$$\rightarrow I_c = C\cdot\cancelto{0}{\frac{dV_c}{dt}} = 0 \rightarrow I_R = 0,\ V_R = 0$$

$$\rightarrow U_1 - 0 = V_R = 0 \rightarrow \underline{\underline{U_1 = 0}}$$

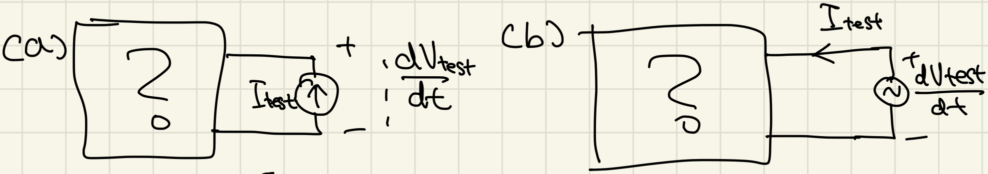
$\Rightarrow$ behaves like an open circuit!

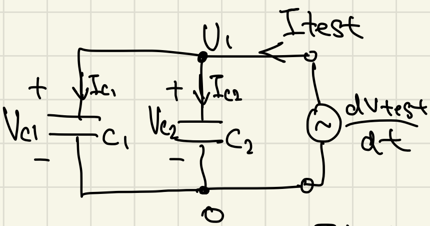
Equivalent Circuits with Capacitors



a) apply I_{test} and measure $\frac{dV_{test}}{dt}$.
b) apply $\frac{dV_{test}}{dt}$ and measure I_{test}.

$$C_{eq} = \frac{I_{test}}{\frac{dV_{test}}{dt}}$$

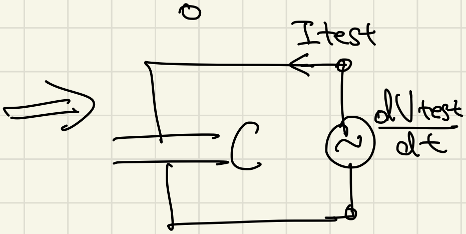




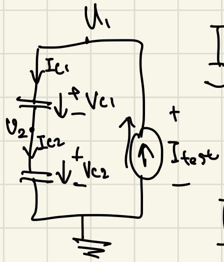
$$I_{C_1} = C_1 \cdot \frac{dU_1}{dt}, \quad I_{C_2} = C_2 \cdot \frac{dU_1}{dt}$$

$$I_{test} = I_{C_1} + I_{C_2} = \underbrace{\frac{dU_1}{dt}}_{\frac{dV_{test}}{dt}} (C_1 + C_2)$$

$\Rightarrow$



$$\boxed{C_{eq} = (C_1 + C_2)}, \quad I_{test} = \frac{dV_{test}}{dt}(C_1 + C_2)$$



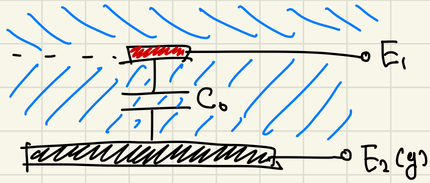
$$I_{test} = I_{C_1} = I_{C_2}, \quad I_{C_1} = C_1 \cdot \frac{dV_{C_1}}{dt}, \quad I_{C_2} = C_2 \cdot \frac{dV_2}{dt}, \quad V_{C_1} = U_1 - U_2, \; V_{C_2} = U_2$$

$$I_{test} = C_2 \frac{dU_2}{dt} \rightarrow \frac{dU_2}{dt} = \frac{I_{test}}{C_2}$$

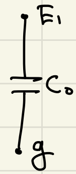
$$I_{test} = C_1\left(\frac{dU_1}{dt} - \frac{dU_2}{dt}\right) \rightarrow \frac{dU_1}{dt} = \frac{dU_2}{dt} + \frac{I_{test}}{C_1} = \frac{I_{test}}{C_2} + \frac{I_{test}}{C_1}$$

$$\frac{dU_1}{dt} = \frac{dV_{test}}{dt} = \frac{I_{test}}{C_2} + \frac{I_{test}}{C_1} \rightarrow \boxed{C_{eq} = \left(\frac{1}{C_2} + \frac{1}{C_1}\right)^{-1}}$$

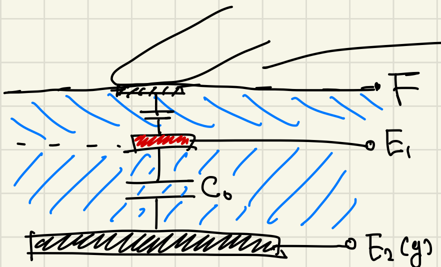
Capacitive Touchscreen



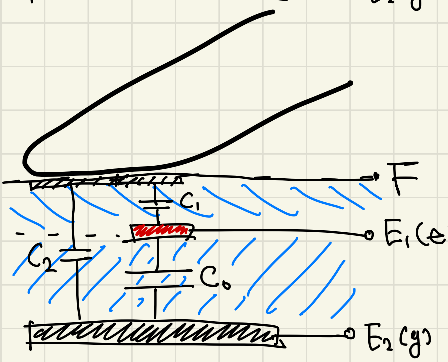
$\rightarrow$

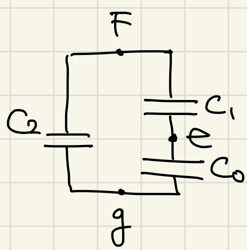


$\rightarrow C_0 = \varepsilon \frac{A}{d}$

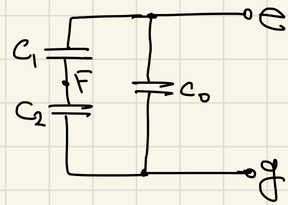


touch forms a capacitor

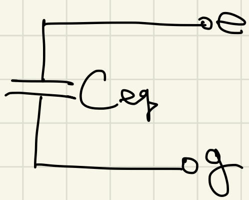




$\Rightarrow$

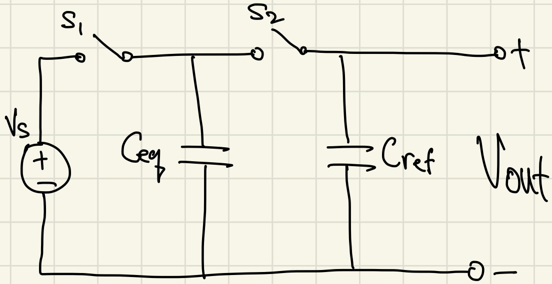


$\Rightarrow$



$$C_{eq} = C_0 + (C_1 // C_2)$$

$$C_\Delta (C_{eq} - C_0)$$

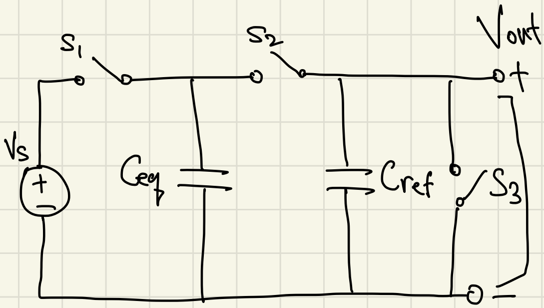


Phase 1) Close S_1, Open S_2

→ Charge C_{eq}, $q = C_{eq} \cdot V_s$

Phase 2) Close S_2, open S_1

→ charge sharing



Phase 1) Close S_1, S_3, open S_2

Phase 2) Close S_2, open S_1, S_3

$\Rightarrow$ ensures $q_{ref}|_{t=0} = 0$!

$q_{eq}|_{t=0} = C_{eq} \cdot V_s$

After $t \to \infty$ for phase 2, $V_{eq} = V_{ref} = V_{out}$! conservation of q

$q_{eq} = C_{eq} \cdot V_{out}$, $q_{ref} = C_{ref} \cdot V_{out}$, $q_{eq} + q_{ref} = C_{eq} \cdot V_s$

$\rightarrow C_{eq} \cdot V_{out} + C_{ref} \cdot V_{out} = C_{eq} \cdot V_s$

$\Rightarrow V_{out} = \dfrac{C_{eq} \cdot V_s}{C_{eq} + C_{ref}}$ → "capacitance dividers"

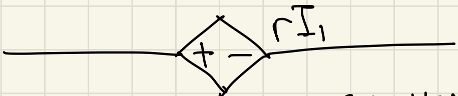
Vout changes when C_{eq} changes!

With touch, $V_{out} = \dfrac{(C_0 + \Delta C)}{(C_0 + \Delta C + C_{ref})} V_s \quad (C_{eq} = C_0 + \Delta C)$ ☆

But how to measure touch as Yes / No?

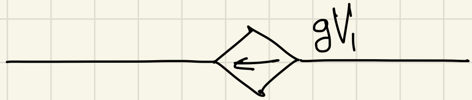
- Threshold Voltage (V_{th}): $V_{out_{notouch}} \leq V_{th} \leq V_{out_{touch}}$
- $V > V_{th} \rightarrow$ Touch(1), $V < V_{th} \rightarrow$ No Touch(0)

New Circuit Elements



Current-controlled (CCVS) Voltage source

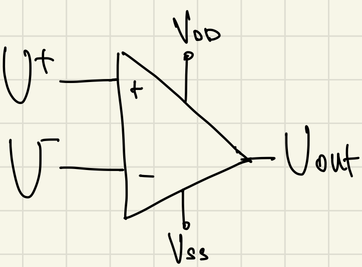
[Op-amp]

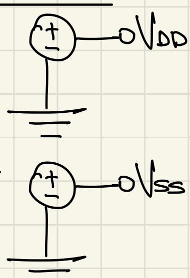


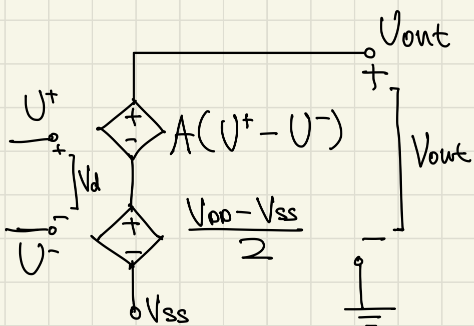
Voltage-controlled (VCCS) current source

[Transistor]

Operational Amplifiers







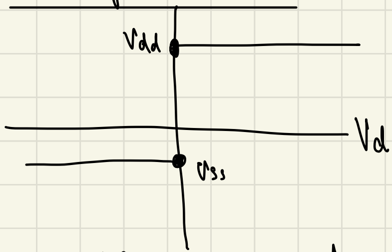
A: gain, ideally $\rightarrow \infty$

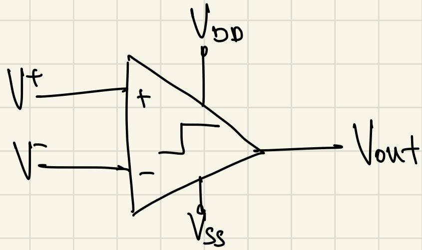
$$V_{out} = V_{ss} + \frac{V_{DD} - V_{ss}}{2} + A\overset{(V^+ - V^-)}{V_d} = \frac{V_{DD} + V_{ss}}{2} + AV_d$$

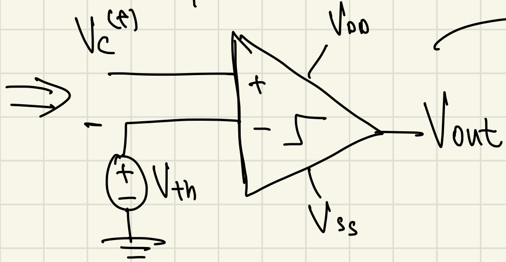
when $V_{ss} \leq \underbrace{\frac{V_{DD} + V_{ss}}{2} + AV_d}_{V^*} \leq V_{DD}$ (railing)

$$\begin{cases} V_{out} = V_{DD} \text{ if } V^* > V_{DD} \\ V_{out} = V_{ss} \text{ if } V^* < V_{ss} \end{cases} \rightarrow \text{used to compare voltage}$$

Comparator



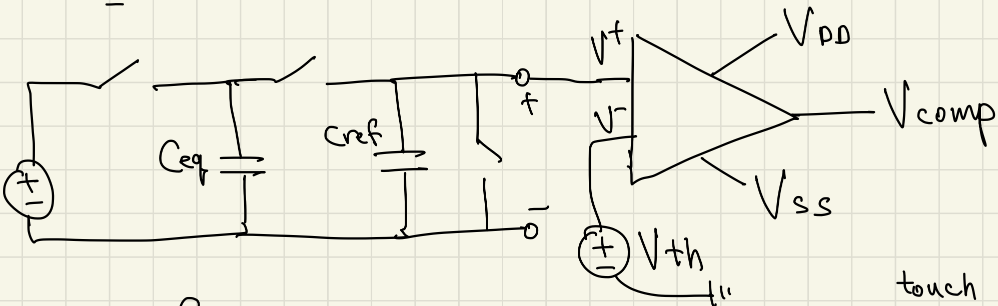




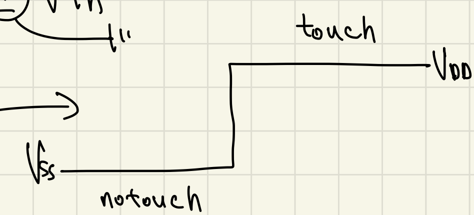
if $V_c > V_{th} \rightarrow V_{out} = V_{DD}$

if $V_c \leq V_{th} \rightarrow V_{out} = V_{SS}$

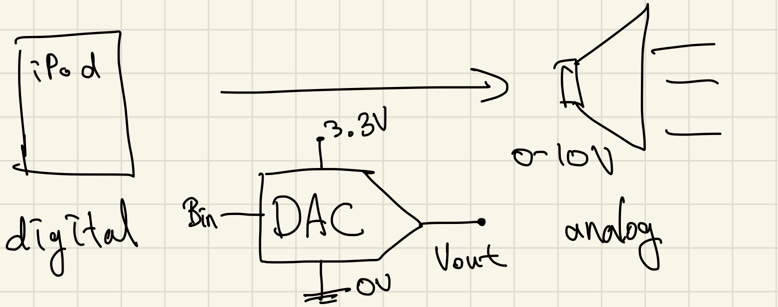
$\Rightarrow$ gives a binary comparison!

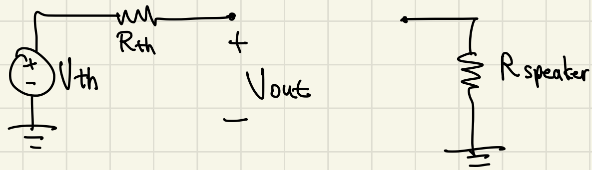


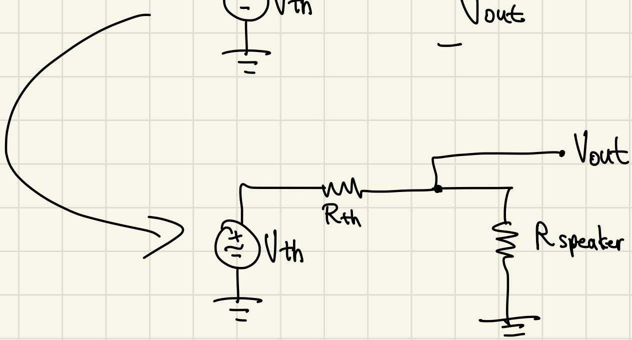
$$\Rightarrow C_{eq} = \begin{cases} C_0 + \Delta C & \rightarrow \text{touch} \\ C_0 & \rightarrow \text{no touch} \end{cases}$$



Playing Music



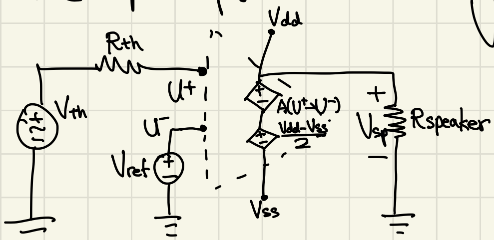




$$V_{out} = V_{th} \cdot \frac{R_{speaker}}{R_{th} + R_{speaker}}$$

↳ too small!

→ Use op-amp!



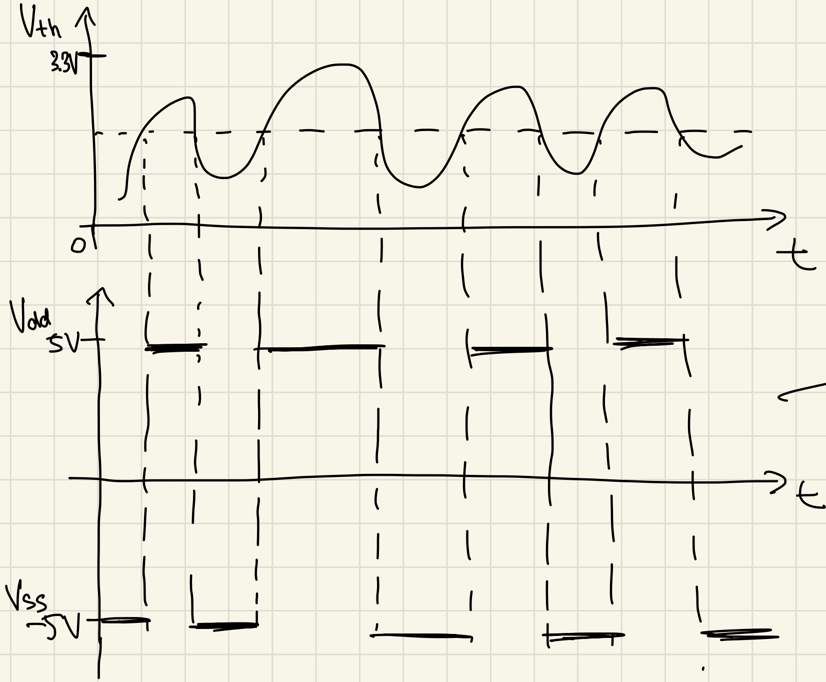
$V_{dd} = -V_{ss} = 5V \rightarrow 10V$ output

$U^+ - U^- = V_d$

ideally, $A \rightarrow \infty$

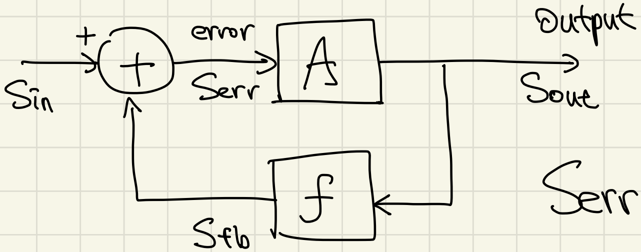
$$\text{KVL: } V_{speaker} = V_{ss} + \frac{V_{dd} - V_{ss}}{2} + AV_d = \cancel{\frac{V_{dd} + V_{ss}}{2}}^{\,0} + AV_d$$

$$\rightarrow V_{speaker} = AV_d, \quad V_{ss} < AV_d < V_{dd}, \quad V_d = V_{th} - V_{ref}$$



bad sound!

Negative Feedback



if $S_{in}\nearrow, S_{out}\nearrow, S_{fb}\nearrow$

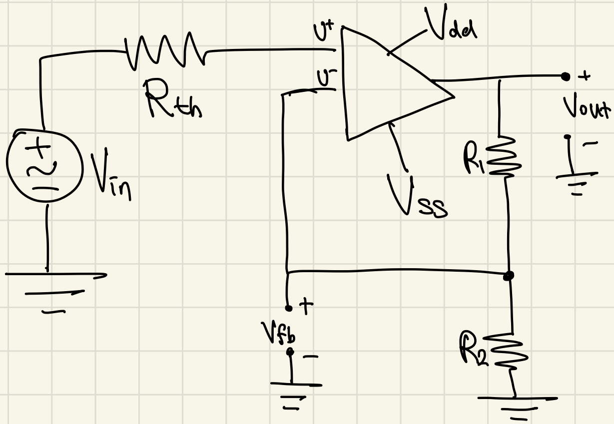
$\Rightarrow S_{in}-S_{fb} < S_{in} \rightarrow S_{err}\searrow, S_{out}\searrow$

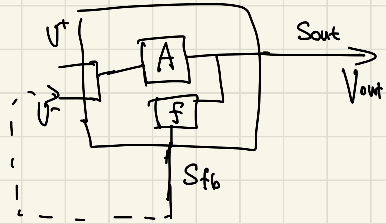
$$S_{err} = S_{in}-S_{fb}$$

$$S_{out} = A\cdot S_{err},\ S_{fb} = f\cdot S_{out} \rightarrow S_{out} = A(S_{in}-S_{fb})$$

$$\rightarrow S_{out}\left(\frac{1}{A}+f\right) = S_{in} \Rightarrow \frac{S_{out}}{S_{in}} = \frac{1}{\frac{1}{A}+f} = \frac{A}{1+fA}$$

$$\frac{S_{out}}{S_{in}} = \frac{A}{1+fA},\ A\rightarrow\infty,\ \frac{S_{out}}{S_{in}} \rightarrow \frac{1}{f} \Rightarrow \underline{\underline{S_{out} = \frac{1}{f}S_{in}}}$$





$$V_d = U^+ + U^- = V_{in} - V_{fb}$$

$$V_{out} = A V_d$$

$$V_{fb} = V_{out} \cdot \boxed{\frac{R_2}{R_1+R_2}} \rightarrow f$$

$$V_{out} = A \cdot \left(V_{in} - \overbrace{V_{out}\frac{R_2}{R_1+R_2}}^{V_{fb}}\right)$$

$$V_{out}(1+Af) = AV_{in}$$

$$\frac{V_{out}}{V_{in}} = \text{gain} = \frac{A}{1+Af} \xrightarrow{A\to\infty} \frac{1}{f}$$

$$\Rightarrow \text{gain} = \frac{1}{f} = \frac{R_1+R_2}{R_2} = \boxed{1+\frac{R_1}{R_2}}$$

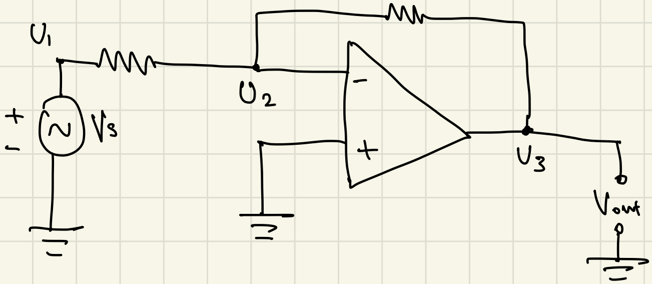
gain of 3 $(3.3V \rightarrow 10V) \rightarrow \underline{\underline{\frac{R_1}{R_2} = 2}}$

$$V_d = \frac{V_{out}}{A}, \text{ if } A\to\infty, \; V_d = \frac{1}{A}\cdot\frac{A}{1+Af}V_{in} \rightarrow 0$$

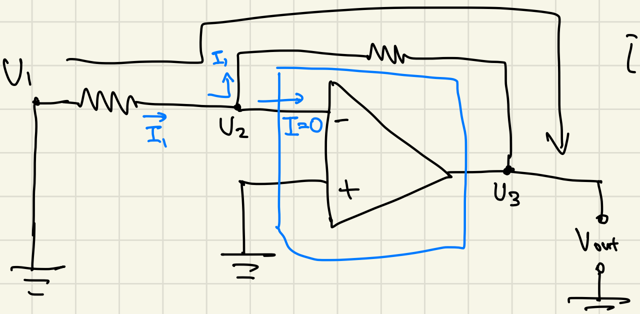
only if: $U^+ \simeq U^-$ when $A \to \infty$
neg. feedback

$\underline{I^+ = I^- = 0}$ → always

Op-amp Circuit Analysis



→ zero the V source



Analysis:

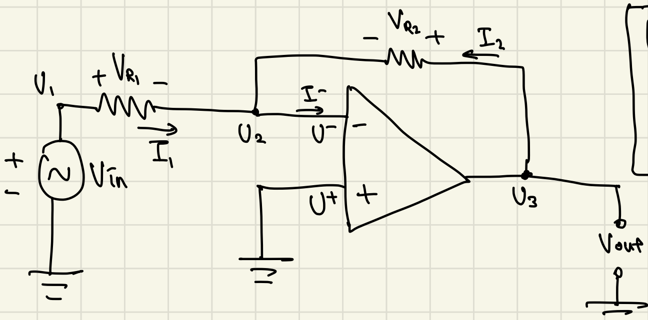
(voltage divider)

if $U_3 \nearrow$, $U_2 \nearrow$,

(op-amp)

but if $U_2 \nearrow$, $U_3 \searrow$

⇒ Neg. feedback



$$\boxed{\begin{aligned} &U_1 = V_{in}, U_3 = V_{out} \\ &U_2 = U^- = U^+ = 0 \end{aligned}}$$

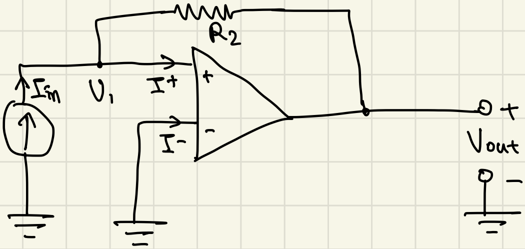
$$\boxed{\begin{aligned} &V_{R_1} = I_1 R_1, \quad V_{R_2} = I_2 R_2 \\ &= V_{in} \qquad\qquad = V_{out} \end{aligned}}$$

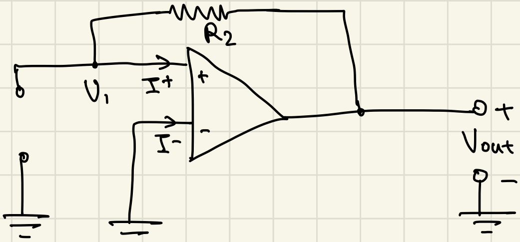
$$I_1 + I_2 = I^- = 0 \rightarrow I_1 = -I_2$$

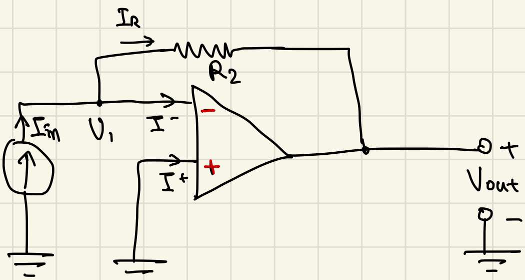
$$V_{in} = U_1 = I_1 R_1, \quad V_{out} = U_3 = I_2 R_2 \Rightarrow \frac{V_{in}}{R_1} + \frac{V_{out}}{R_2} = 0$$

$$\rightarrow V_{out} = -\frac{R_2}{R_1} V_{in}, \quad gain = -\frac{R_2}{R_1}$$

Inverting op-amp







If $V_{out} \nearrow$, $U_1 \nearrow$

if $U_1 \nearrow$, $V_{out} \nearrow$

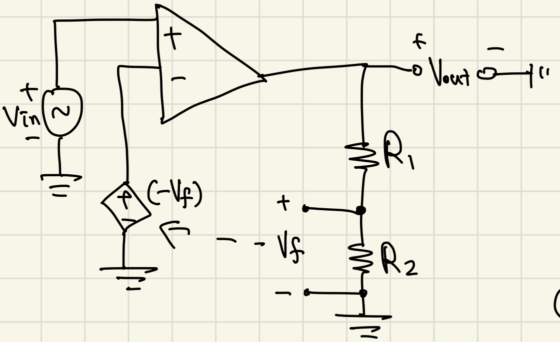
$\Rightarrow$ Pos. feedback

↪ switch polarity!

$U_1 = 0$, $I_{in} = I_R$

$0 - V_{out} = I_{in} \cdot R_2$

$\rightarrow \dfrac{V_{out}}{I_{in}} = -R$

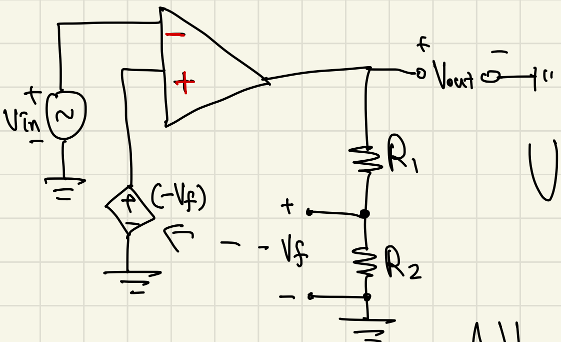


if $V_{out}\nearrow$, $V_f\nearrow$, $V_d\nearrow$

if $V_d\nearrow$, $V_{out}\nearrow$

$\Rightarrow$ Pos. feedback

↪ switch polarity!



$$V_f = \frac{R_2}{R_2+R_1} V_{out}$$

$$V_{out} = A(V_{in} - (-V_f))$$

$$= A\left(V_{in} + \frac{R_2}{R_2+R_1} V_{out}\right)$$

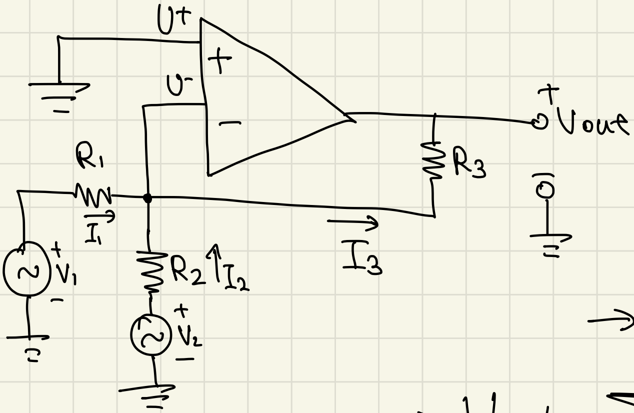
$$AV_{in} = \left(1 - A\frac{R_2}{R_2+R_1}\right) V_{out}$$

$V_{in} = \frac{1}{A} - \frac{R_2}{R_2+R_1} V_{out}$, $\lim_{A\to\infty} \to V_{in} = -\frac{R_2}{R_1+R_2} V_{out}$

$\Rightarrow V_{out} = -\frac{R_1+R_2}{R_2} V_{in}$, gain $= -\frac{R_1+R_2}{R_2}$

$$V_{in} = -V_f = -\frac{R_2}{R_2+R_1} V_{out}$$

Artificial Neuron



$$U^+ = U^- = 0$$

$$I_1 + I_2 = I_3$$

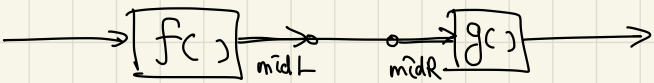
$$\frac{-V_1}{R_1} + \frac{(-V_2)}{R_2} = \frac{V_{out}}{R_3}$$

$$\rightarrow V_{out} = R_3\left(-\frac{V_1}{R_1} - \frac{V_2}{R_2}\right)$$

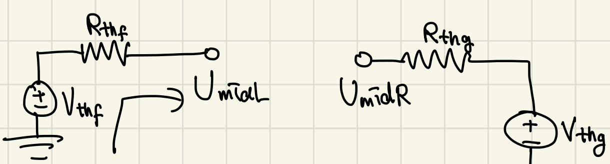
$$\Rightarrow V_{out} = \sum_{i=1}^{n} -\frac{R_3}{R_i} V_i$$

add an inverting amplifier of gain=1 $\rightarrow V_{out} = \sum_{i=1}^{n} \frac{R_3}{R_i} V_i$ ☆

Cascading Blocks

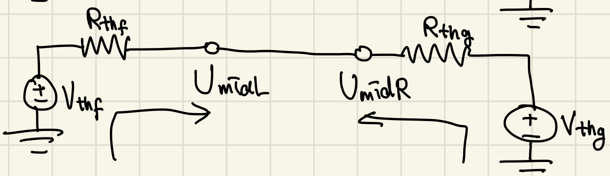


how to maintain functionality when connected to other blocks?



Before connection:

$$U_{midL} = V_{thf}$$



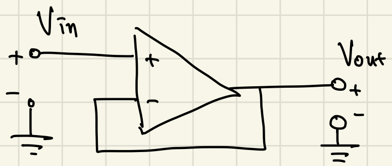
After connection:

$$U_{midL} = V_{thf} \cdot \frac{R_{thg}}{R_{thf} + R_{thg}} + V_{thg} \cdot \frac{R_{thf}}{R_{thf} + R_{thg}}$$

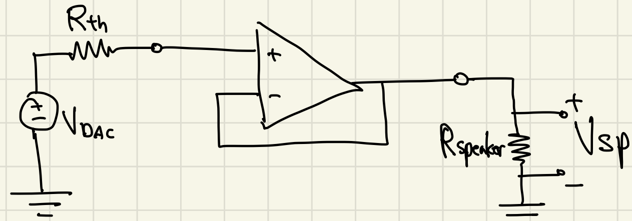
1) If $R_{thf} = 0$ (wire) OR 2) If $R_{thg} \rightarrow \infty$ (open circuit)

↳ block g ↳ block f

Unity Gain Buffer

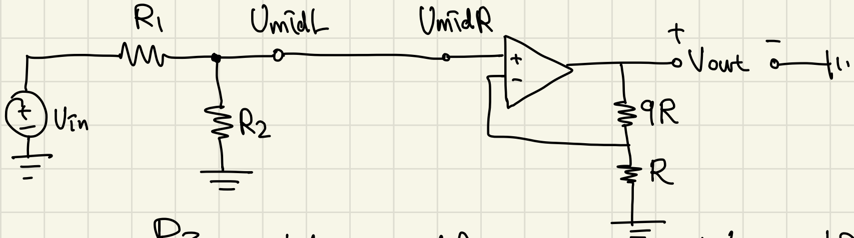


$U^+ = U^- \rightarrow \underline{\underline{V_{in} = V_{out}}}$



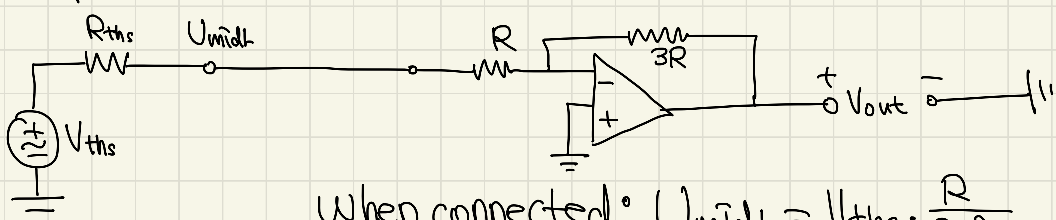
$\Rightarrow V_{DAC} = V_{sp}$ (isolated)

Example 1: $V_{in} \longrightarrow \boxed{\dfrac{R_2}{R_1+R_2}} \longrightarrow \boxed{A_v = 10} \longrightarrow V_{out}$



$U_{midL} = \dfrac{R_2}{R_1+R_2}$ $\quad U_{midR} = U_{midL}$ $\quad \underline{V_{out} = 10\left(\dfrac{R_2}{R_1+R_2} V_{in}\right)}$

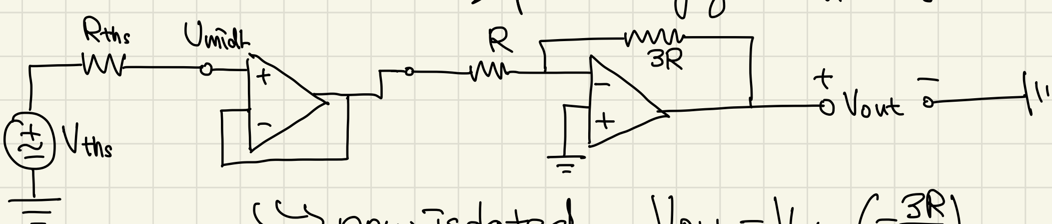
Example 2: $V_{in} \longrightarrow \boxed{\text{sensor}} \xrightarrow{U_{mid}} \boxed{-3} \xrightarrow{V_{out}}$



When connected: $U_{midL} = V_{ths} \cdot \dfrac{R}{R+R_{ths}}$

$U_{midL} = V_{ths}$

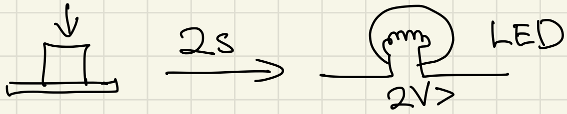
$\Rightarrow$ put a unity gain buffer!



↪ now isolated

$\underline{V_{out} = V_{ths}\left(-\dfrac{3R}{R}\right)}$

Example 3:



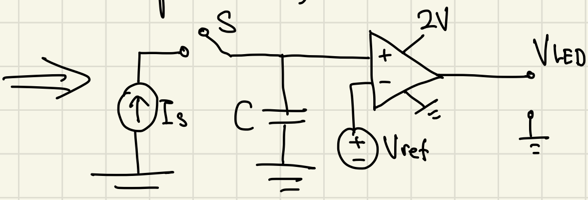
1) Specification: measures 2 seconds after button is pressed and apply 2V across the LED

2) strategy: $\boxed{\text{Push}} \rightarrow \boxed{\text{Timer}} \xrightarrow{2s} \boxed{\geq} \rightarrow \boxed{\uparrow 2V}$

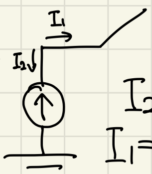
$\left(V_c(t) = \frac{I_s}{C} \cdot t + V_c(0)\right)$

3) Implementation: Push → Switch, Timer → Capacitor + I_s

$\geq$: Comparator, rail = 2V



Problem:

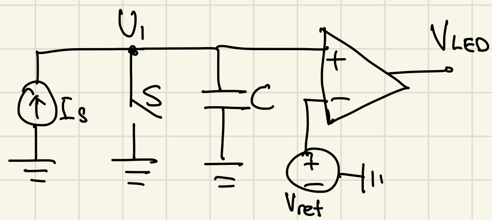


$I_1 = 0$

$I_2 = -I_s$

$I_1 = I_2 \rightarrow 0 = -I_s$?

Revise:



Verify: when S is closed → $U_1 = 0 \rightarrow I_s = I_{sw}$

when S is open → $V_{time} = \frac{I_s}{C} t$, make $V_{time}(t_0 + 2) \geq V_{ref}$

Inner Product

$\Rightarrow$ measures similarity between vectors

For a real-valued vector space $\mathbb{V}$, the mapping

$$\vec{u}, \vec{v} \in \mathbb{V} \Rightarrow \langle \vec{u}, \vec{v} \rangle \in \mathbb{R}$$

is called an <u>inner product</u> if it satisfies:

1) Symmetry $(\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle)$
2) Linearity $(\langle \alpha\vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle, \langle \vec{u}+\vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle)$
3) Positive-definiteness $(\langle \vec{v}, \vec{v} \rangle \geq 0, \langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = \vec{0})$

ex1) Euclidean Inner Product (Dot product)

$$\vec{x}, \vec{y} \in \mathbb{R}^N, \langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \sum_{i=1}^{N} x_i y_i = \vec{y}^T \vec{x} = \langle \vec{y}, \vec{x} \rangle \quad (1)$$

$$\langle \alpha\vec{x}, \vec{y} \rangle = (\alpha\vec{x})^T \vec{y} = \alpha(\vec{x})^T \vec{y} = \alpha(\vec{x}^T \vec{y}) = \alpha \langle \vec{x}, \vec{y} \rangle \quad (2)$$

$$\langle \vec{x}+\vec{z}, \vec{y} \rangle = (\vec{x}+\vec{z})^T \vec{y} = \vec{x}^T \vec{y} + \vec{z}^T \vec{y} = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$$

$$\langle \vec{x}, \vec{x} \rangle = \sum_{i=0}^{N} x_i^2 \geq 0 \quad (3)$$

ex2) Weighted Inner Product

$\vec{x}, \vec{y} \in \mathbb{R}^N$, $Q \in \mathbb{R}^{N\times N}$ symmetric with positive eigenvalues

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y} \quad \left(Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right)$$

$$[x_1 \ x_2] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2 \qquad (1)$$

$$\|$$

$$[y_1 \ y_2] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 x_1 + 3 y_2 x_2$$

Linearity is trivial (matrices & vectors) (2)

$$\vec{x}^T Q \vec{x} = [x_1 \ x_2] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 3x_2^2 \geq 0 \quad (3)$$

Norms

$\Rightarrow$ A measure of <u>length</u> of elements in the vector space

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

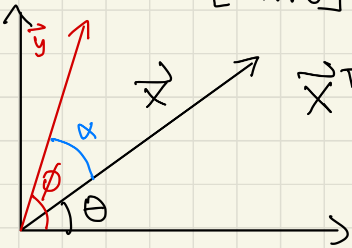
1) Homogeneity ($\|\alpha \vec{v}\| = |\alpha| \|\vec{v}\|$, $\alpha \in \mathbb{R}$)
2) Non-negativity ($\|\vec{v}\| \geq 0$)
3) Triangle Inequality ($\|\vec{v} + \vec{u}\| \leq \|\vec{v}\| + \|\vec{u}\|$)

Euclidean Norm

$\vec{x} \in \mathbb{R}^N$, $\langle \vec{x}, \vec{x} \rangle = \vec{x}^T \vec{x}$, $\|\vec{x}\| = \sqrt{\vec{x}^T \vec{x}}$

$\vec{x} \in \mathbb{R}^2 \Rightarrow \|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$

$\vec{x} = \|\vec{x}\| \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, $\vec{y} = \|\vec{y}\| \begin{bmatrix}\cos\phi \\ \sin\phi\end{bmatrix}$



$$\vec{x}^T\vec{y} = \|\vec{x}\|\cdot\|\vec{y}\|\cdot(\cos\theta\cos\phi + \sin\theta\sin\phi)$$
$$= \|\vec{x}\|\cdot\|\vec{y}\|\cdot\cos(\phi-\theta) = \boxed{\|\vec{x}\|\|\vec{y}\|\cos\alpha}$$

Orthogonality

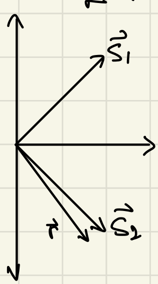
⇒ For an inner product, two vectors $\vec{x}, \vec{y}$ are orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$.

Cauchy-Schwartz Inequality

Consider: $|\langle \vec{x}, \vec{y}\rangle| = \|\vec{x}\|\|\vec{y}\||\cos\alpha| \Rightarrow |\langle \vec{x}, \vec{y}\rangle| \leq \|\vec{x}\|\|\vec{y}\|$

Classification

$\vec{r} = \begin{bmatrix}0.93 \\ -1.1\end{bmatrix}$, is it closer to $\vec{s}_1 = \begin{bmatrix}1 \\ 1\end{bmatrix}$ or $\vec{s}_2 = \begin{bmatrix}1 \\ -1\end{bmatrix}$?



A: use $\|\vec{r} - \vec{s}_i\|$, $i^* = \operatorname{argmin}\|\vec{r} - \vec{s}_i\|$, $i \in \{1,2\}$

observe: $\operatorname{argmin}\|\vec{r} - \vec{s}_i\| = \operatorname{argmin}\|\vec{r} - \vec{s}_i\|^2$

$$\|\vec{r} - \vec{s}_i\| = \langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle = \langle \vec{r}, \vec{r} - \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} - \vec{s}_i \rangle$$
$$= \langle \vec{r}, \vec{r} \rangle - \langle \vec{r}, \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} \rangle + \langle \vec{s}_i, \vec{s}_i \rangle$$
$$= \underbrace{\|\vec{r}\|}_{\text{fixed}} + \overbrace{\|\vec{s}_i\|}^{=2,\ \text{fixed}} - 2\langle \vec{r}, \vec{s}_i \rangle \Rightarrow \text{maximize } \langle \vec{r}, \vec{s}_i \rangle!$$

Interference

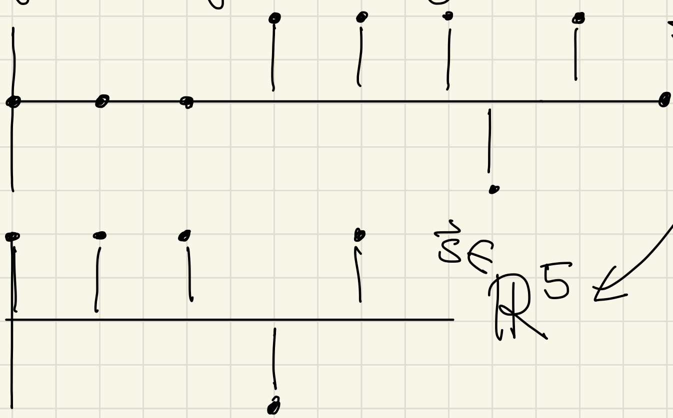
$$\vec{r} = \vec{s}_1 + \vec{s}_2 + \vec{n} \to \langle \vec{r}, \vec{s}_1 \rangle = \underbrace{\langle \vec{s}_1, \vec{s}_1 \rangle}_{\text{desired}} + \underbrace{\langle \vec{s}_2, \vec{s}_1 \rangle}_{\text{interference}} + \cancel{\langle \vec{n}, \vec{s}_1 \rangle} \to \text{small}$$

what makes interference negligible? $\to$ if $\vec{s}_1 \perp \vec{s}_2$, $\langle \vec{s}_1, \vec{s}_2 \rangle = 0$!

Timing

Satellites send a modulated unique code

Signal is digitized by the receiver



$\vec{r} \in \mathbb{R}^9$

$\vec{s} \in \mathbb{R}^5$

how to compare?

define: $\vec{r} = [r_0 \; r_1 \; r_2 \; \cdots \; r_8]^T$

$$\Rightarrow r(n) = \begin{cases} r_n \; ; & 0 \le n \le 8 \\ 0 \; ; & \text{elsewhere} \end{cases}$$

$\vec{s} = [s_0 \; s_1 \cdots s_4]^T$

$$\Rightarrow s(n) = \begin{cases} s_n \; ; & 0 \le n \le 4 \\ 0 \; ; & \text{elsewhere} \end{cases}$$

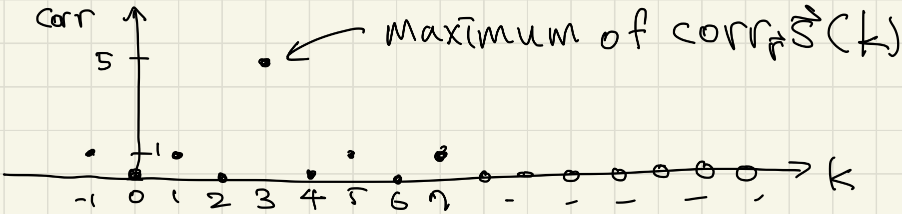
$$\langle r(n), s(n) \rangle = \sum_{n=-\infty}^{\infty} r(n) \cdot s(n) = \sum_{n=0}^{7} r(n) \cdot s(n) = 0$$

Then compute $\langle r(n), s(n-1) \rangle$ → delayed by 1 timestamp

$$\langle r(n), s(n-1) \rangle = \sum_{n=-\infty}^{\infty} r(n) \cdot s(n-1) = 1$$

Now let the "delay" be a variable k. Then,

$$corr_{\vec{r}}\vec{s}(k) = \langle \vec{r}(n), \vec{s}(n-k) \rangle$$



$$k^* = \underset{k}{\operatorname{argmax}}(corr_{\vec{r}}\vec{s}(k)) = 3$$

- If $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^M$, then length of $corr_{\vec{x}}\vec{y}$ is $N+M-1$
- $corr_{\vec{x}}\vec{y} \neq corr_{\vec{y}}\vec{x}$

In reality, signals are periodically repeated

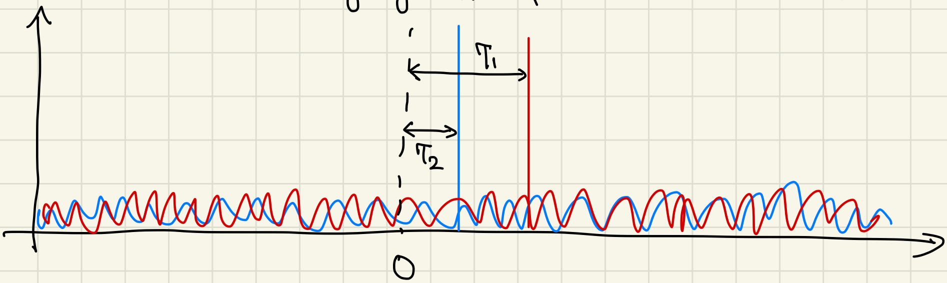
→ cross correlation is "periodically expanded"

Localization

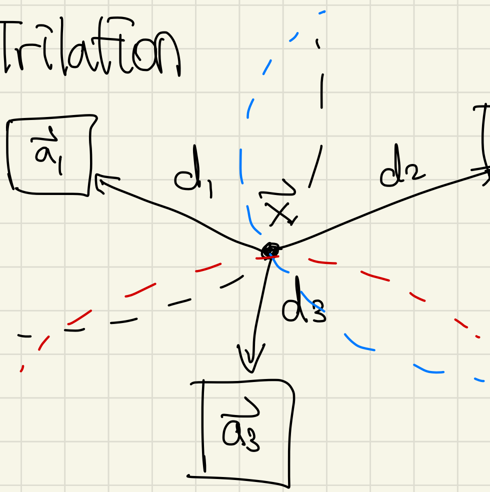
$$\vec{r}(n) = S_1(n-T_1) + S_2(n-T_2) + w(n)$$

$$\rightarrow \mathrm{corr}_{\vec{r}}\vec{S}_1(k) = \langle r(n), S_1(n-k)\rangle$$

$$= \underbrace{\langle S_1(n-T_1), S_1(n-k)\rangle}_{\text{want to be only big at } k=T_1} + \underbrace{\langle S_2(n-T_2), S_1(n-k)\rangle}_{\text{want to be small}} + \underbrace{\langle w(n), S_1\rangle}_{\text{always small}}$$



Trilation



$$\begin{cases} \|\vec{x}-\vec{a}_1\|^2 = d_1^2 \\ \|\vec{x}-\vec{a}_2\| = d_2^2 \\ \|\vec{x}-\vec{a}_3\| = d_3^2 \end{cases}$$

$$\|\vec{x}-\vec{a}_1\| = d_1^2$$

$$\rightarrow (\vec{x}-\vec{a}_1)^T(\vec{x}-\vec{a}_1) = d_1^2$$

$$\rightarrow \vec{x}^T\vec{x} - \vec{a}_1^T\vec{x} - \vec{x}^T\vec{a}_1 + \vec{a}_1^T\vec{a}_1 = d_1^2$$

$$\rightarrow \begin{cases} \|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = (C_1T_1)^2 \\ \|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = (C_2T_2)^2 \\ \|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = (C_3T_3)^2 \end{cases}$$

$$\begin{cases} \|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = (C_1T_1)^2 & (1) \\ \|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = (C_2T_2)^2 & (2) \\ \|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = (C_3T_3)^2 & (3) \end{cases}$$

$(2)-(1)$: $-2\vec{a}_2^T\vec{x} + 2\vec{a}_1^T\vec{x} + \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 = (C_2T_2)^2 - (C_1T_1)^2$

$\rightarrow 2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(T_2^2 - T_1^2)$

$(3)-(1)$: $2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(T_3^2 - T_1^2)$

$$2\begin{bmatrix} a_{11}-a_{21} & a_{12}-a_{22} \\ a_{11}-a_{31} & a_{12}-a_{32} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(T_2^2 - T_1^2) \\ \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(T_3^2 - T_1^2) \end{bmatrix}$$

Problem: T can only be measured in relative times!

$\Delta T_2 = T_2 - T_1$, $\Delta T_3 = T_3 - T_1$

$$\rightarrow C^2(T_2^2 - T_1^2) = C^2\underbrace{(T_2 - T_1)}_{\Delta T_2}(T_2 + T_1)$$

$$= C^2(\Delta T_2)(\underbrace{T_2 - T_1}_{\Delta T_2} + 2T_1) = C^2(\Delta T_2)(\Delta T_2 + 2T_1)$$

$$\Rightarrow 2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta T_2 T_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta T_2)^2$$

$\Rightarrow$ put one more satellite to make 3 equations

$$2\begin{bmatrix} a_{11}-a_{21} & a_{12}-a_{22} & -c^2\Delta T_2 \\ & \ddots & \\ & & \ddots \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ T_1 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + c^2(\Delta T_2)^2 \\ \vdots \\ \vdots \end{bmatrix}$$

$\Rightarrow$ Now we can find <u>time</u> and <u>position</u>!

Multi-Lateration

If we have more measurements than unknowns...

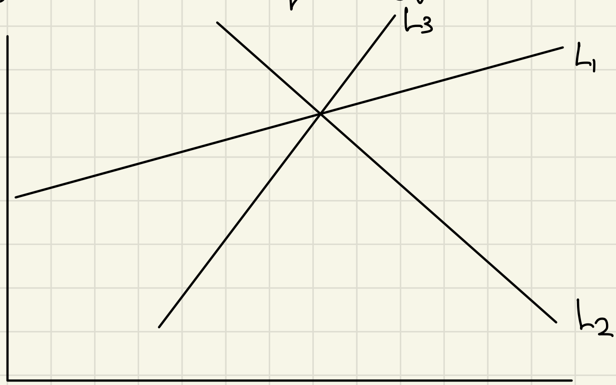
$\rightarrow$ the system might not have a unique solution!

ex) $a_{11}x_1 + a_{12}x_2 = b_1 \cdots L_1$

$a_{21}x_1 + a_{22}x_2 = b_2 \cdots L_2$

$a_{31}x_1 + a_{31}x_2 = b_3 \cdots L_3$

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix}$$



$\rightarrow$ has solutions when $\vec{b} \in \text{Span}(\text{col}(A))$

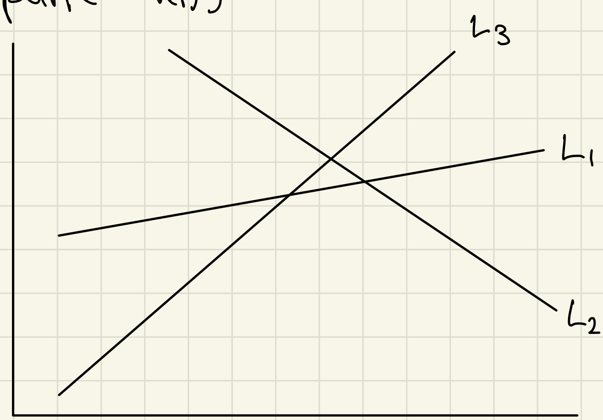
In the real world...

$a_{11}x_1 + a_{12}x_2 = b_1 + e_1 \cdots L_1$

$a_{21}x_1 + a_{22}x_2 = b_2 + e_2 \cdots L_2$

$a_{31}x_1 + a_{31}x_2 = b_3 + e_3 \cdots L_3$

there are errors.



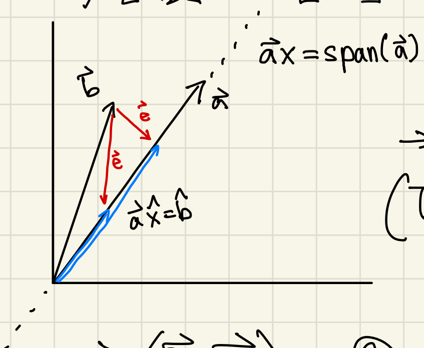
Least Squares Algorithm

we have $\vec{b}$, a model $A\vec{x}=\vec{b}$

Problem: $A\vec{x}=\vec{b}$ does not have a solution!

→ Instead, find $\hat{x}$ s.t. $A\hat{x}$ is closest to $\vec{b}$

ex) $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ (1 unknown, 2 equation)



find $\hat{x}$ s.t. $\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\|$ is minimized

→ that $\vec{e}_{min} \perp \vec{a}$, $\vec{a}\hat{x} = \hat{b}$

(Theorem: minimum distance btwn line and a point is an orthogonal projection)

$$\Rightarrow \langle \vec{e}, \vec{a} \rangle = 0 \rightarrow \langle \vec{b} - \hat{b}, \vec{a} \rangle = 0$$

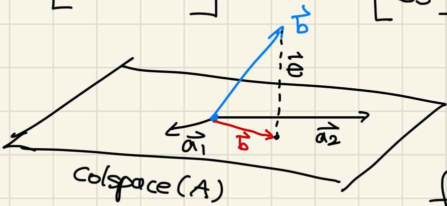
$$\rightarrow \langle \vec{b}, \vec{a} \rangle - \langle \hat{b}, \vec{a} \rangle = 0 \rightarrow \langle \vec{b}, \vec{a} \rangle = \langle \hat{b}, \vec{a} \rangle$$

$$\rightarrow \langle \vec{b}, \vec{a} \rangle = \langle \vec{a}\hat{x}, \vec{a} \rangle \rightarrow \langle \vec{b}, \vec{a} \rangle = \hat{x} \langle \vec{a}, \vec{a} \rangle$$

$$\rightarrow \hat{x} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2} \Rightarrow \hat{b} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2} \vec{a} = \frac{\vec{b}^T \vec{a}}{\vec{a}^T \vec{a}} \vec{a}$$

ex) 3 equations, 2 unknowns

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$



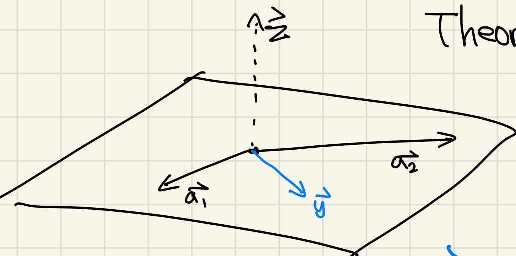
no solution to $A\vec{x} = \vec{b}$

$\Rightarrow \vec{b} \notin$ colspace(A)

find $\hat{x}$ s.t. $\|\vec{e}\| = \|A\vec{x} - \vec{b}\|$ is minimized

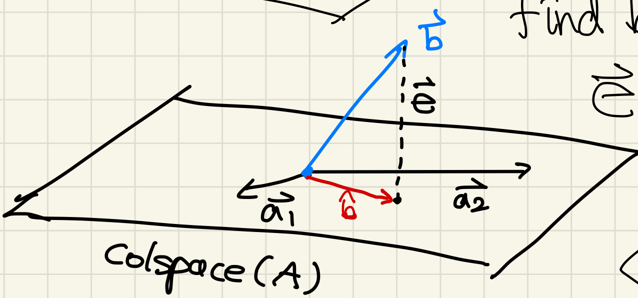
$\Rightarrow$ Orthogonal projection onto colspace(A)!

Consider $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_N]$, $\vec{y} \in$ colspace(A)



Theorem: if $\exists \vec{z}$ s.t. $\langle \vec{z}, \vec{a}_i \rangle = 0 \rightarrow \langle \vec{z}, \vec{y} \rangle = 0$.

Now Least squares ...



find $\hat{b} = A\hat{x}$

$\vec{e} = \vec{b} - \hat{b}$

Since $\vec{e} \perp$ col(A),

$\langle \vec{a}_i, \vec{e} \rangle = 0$

$\rightarrow \langle \vec{a}_i, \vec{b} - \hat{b} \rangle = 0 \rightarrow \vec{a}_i^T(\vec{b} - \hat{b}) = 0$

$$\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_N^T \end{bmatrix}\begin{bmatrix} | \\ \vec{b} - \hat{b} \\ | \end{bmatrix} = \vec{0} \longrightarrow A^T(\vec{b} - A\hat{x}) = 0 \rightarrow A^T\vec{b} = A^TA\hat{x}$$

$\Rightarrow$ if A is full rank, $\underline{\hat{x} = (A^TA)^{-1}A^T\vec{b}}$

$$\hat{X} = (A^TA)^{-1}A^T\vec{b}$$

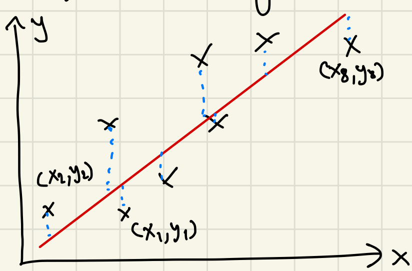
ex1) $\begin{bmatrix}2\\1\end{bmatrix}[x] = \begin{bmatrix}1\\1\end{bmatrix}$ $\quad A^TA = [2\ \ 1]\begin{bmatrix}2\\1\end{bmatrix} = 5$

$\rightarrow (A^TA)^{-1} = \frac{1}{5} \rightarrow \left(\frac{1}{5}\right)\begin{bmatrix}2\\1\end{bmatrix}[1\ \ 1] = \boxed{3/5} = \hat{X}$

ex2) $\begin{bmatrix}1 & 0\\0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $\quad (A^TA)^{-1} = \left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\right)^{-1} = \begin{bmatrix}1 & 0\\0 & 1/2\end{bmatrix}$

$\rightarrow \hat{X} = \begin{bmatrix}1 & 0\\0 & 1/2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\0 & 1\end{bmatrix}^T\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1/2\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}1\\2.5\end{bmatrix}$

ex3) Linear regression $(ax+b)$



$$\begin{array}{ccccc} A & & \vec{p} & & \vec{y} \\ \begin{bmatrix}x_1 & 1\\x_2 & 1\\x_3 & 1\\ \vdots & \\ x_8 & 1\end{bmatrix} & & \begin{bmatrix}a\\b\end{bmatrix} & = & \begin{bmatrix}y_1\\y_2\\y_3\\ \vdots\\ y_8\end{bmatrix}\end{array} \Rightarrow \vec{p} = (A^TA)^{-1}A^T\vec{y}$$

$\vec{e} = A\vec{p} - \vec{y}$ (vertical difference between line and data)

ex4) Regression

Model: $ax^2+by^2+Cxy+dx+ey=1 \rightarrow$ linear to $(a \sim e)$

$$\begin{array}{ccccc} A & & \vec{p} & & \vec{y} \\ \begin{bmatrix}x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1\\x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2\\ & & \vdots & & \\ x_N^2 & y_N^2 & x_Ny_N & x_N & y_N\end{bmatrix} & & \begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix} & = & \begin{bmatrix}1\\1\\ \vdots\\ 1\end{bmatrix}\end{array} \quad \hat{p} = (A^TA)^{-1}A^T\vec{y}$$

ex5) exponential regression $(y = Ce^{ax})$

$$\rightarrow \log(y) = \log(C) + ax = b + ax$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \log(y_1) \\ \log(y_2) \\ \vdots \\ \log(y_N) \end{bmatrix} \rightarrow \hat{c} = e^{\hat{b}} \rightarrow \begin{pmatrix} \hat{a} \\ \hat{c} \end{pmatrix}$$

Matrix Transposes

$$(AB)^T = B^T A^T$$

$\rightarrow$ A^TA is invertible

Theorem: $\mathrm{Null}(A^TA) = \mathrm{Null}(A)$

1) $\vec{w} \in \mathrm{Null}(A) \rightarrow A\vec{w} = \vec{0} \rightarrow A^TA\vec{w} = A^T\vec{0} = \vec{0} \rightarrow \vec{w} \in N(A^TA)$

2) $\vec{v} \in \mathrm{Null}(A^TA) \rightarrow A^TA\vec{v} = \vec{0}$, $\|A\vec{v}\|^2 = (A\vec{v})^T(A\vec{v}) = \vec{v}^T \underbrace{A^TA\vec{v}}_{} = 0$

$\rightarrow \|A\vec{v}\|^2 = 0 \implies A\vec{v} = 0 \rightarrow \vec{v} \in N(A)$

If A^TA is not Invertible

$A^TA\hat{x} = A^T\vec{b} \rightarrow$ infinite solutions with same $\vec{e} = A\hat{x} - \vec{b}$