Simulation of a 1st Order System

Ghassan Arnouk

SYSC 3600A
Fall 2019
Lab 1 Report

Instructor: Dr. Yegui Cai
Lab Section: L1 – Tuesday: 2.35-5.25 pm

Date Submitted: October 31, 2019

Table of Contents

List of Figures

1 Pre-lab

The purpose of this lab is to develop a simulation diagram to solve a system described by a 1st order differential equation (DE).

1.1 Simple 1st Order Systems

Both of RC circuit and mass-damper systems are 1st order since each only has one internal element that can store the state of the system. The differential equation for both systems has the form:

$$\frac{dy(t)}{dt} + a0y(t) = b0x(t) \tag{1}$$

1.2 Simulation Diagram for a 1st Order Differential Equation

From a simulation standpoint, implementing Eq. 1 directly is not appropriate since differentiation is sensitive to small fluctuations in the signal. Regarding that, integration is used instead of differentiation. Eq. 1 is can be put in the form below:

$$\frac{dy(t)}{dt} = -a0y(t) + b0x(t) \tag{2}$$

and then performing a running integration on both sides of the equation to give

$$y(t) = \int_{-\infty}^{t} -a0y(\lambda) + b0x(\lambda) \tag{3}$$

where λ is a dummy variable that disappears in the integration. Note that the running integral has limits from -∞ to *t* since the output *y(t)* for this casual system is dependent only on the inputs from the past up to and including the present time *t*. With the 1st order system written as in Eq. 3, a simulation diagram can easily be developed to realize the system.

1.3 Using Initial Conditions

For a practical simulation of a 1st order differential equation, we realistically need to change the -∞ limit of the running integral to a finite value. With Eq. 3, this can be done by splitting the running integral into two portions to give

$$y(t) = y(t0^-) + \int_{-\infty}^{t} -a0y(\lambda) + b0x(\lambda) \tag{4}$$

where t_0 is the start time for the simulation and $y(t_0^-)$ is the initial condition for the output at the start of the simulation. Therefore, for a 1st order system we should set the initial condition of the integrator to $y(t_0^-)$ to satisfy Eq. 4.

2 Lab: Step Response of a 1st Order System

2.1 1st order differential equations

In this lab, we will simulate a system described by the following 1st order differential equation (DE)

$$\frac{dy(t)}{dt} = \frac{-y(t) + kx(t)}{tau} \tag{5}$$

where $x(t)$ is the system input and $y(t)$ is the system output. In this case, k is the static gain of the 1st order system, and τ is the time constant of the system. It should be clear that Eq. 5 is in the same form as Eq. 1 if we set

$$a_0 = \frac{1}{tau} \quad \text{and} \quad b_0 = \frac{k}{tau} \tag{6}$$

2.2 Step Response, Static Gain, and the “Time Constant”

Figure 1 below shows the output response when a unit step function was provided as an input, with parameters k=3 & τ=10 to a 1st order system with no initial conditions, $y(t_0^-) = 0$.

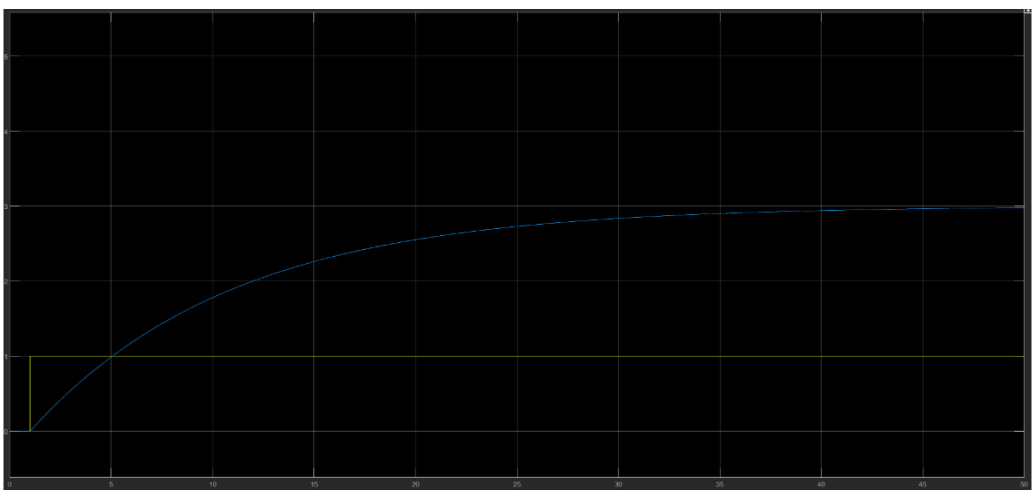

Figure 1 - Response with k=3 & τ=10

Figure 2 below shows the output response when a unit step function was provided as an input, with parameters k=3 & τ=5 to a 1st order system with no initial conditions, $y(t_0^-) = 0$.

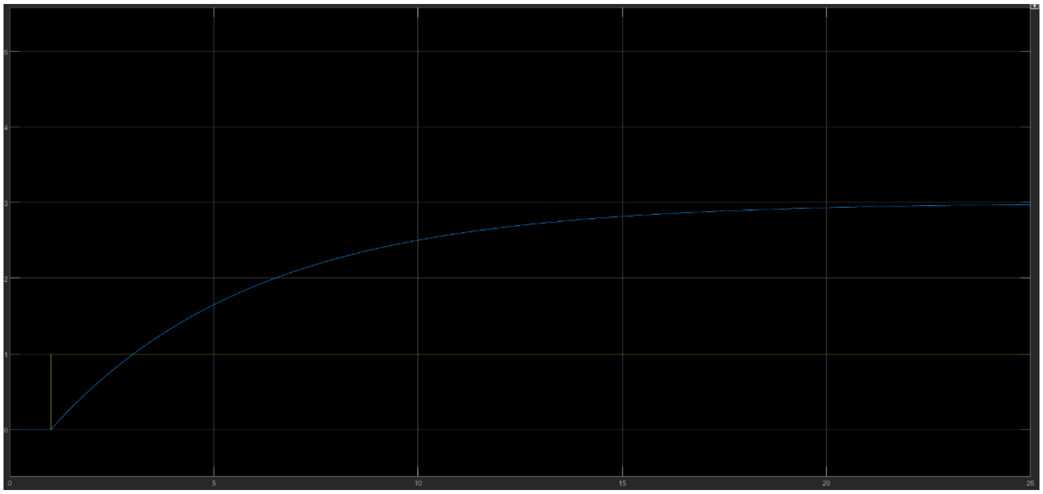

Figure 2 - Response with k=3 & τ=5

Figure 3 below shows the output response when a unit step function was provided as an input, with parameters k=5 & τ=10 to a 1st order system with no initial conditions, $y(t_0^-) = 0$.

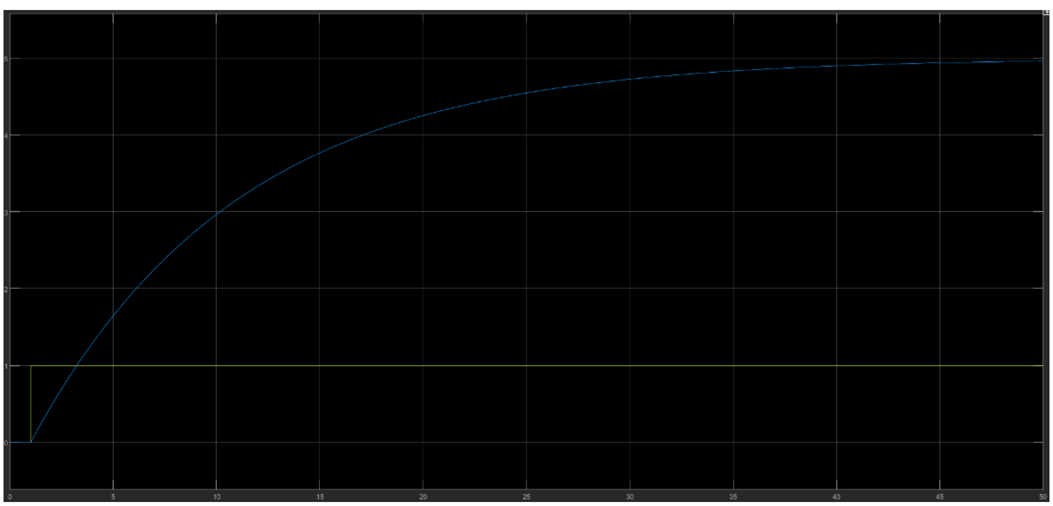

Figure 3 - Response with k=5 & τ=10

τ is defined to be the time constant, which controls the rate of settling of the output response. More specifically, it represents the time needed for the process variable to reach 63.2% of its total and final change. A smaller time constant τ requires less time for the output to reach the steady-state value of the output as shown in Fig. 2. In comparison, the output response of Fig. 1 requires more time to reach the steady-state value of the output since its time constant τ is larger.

k is the gain of the output response relative to the input of unit step function. The steady-state value of the output response is determined using the constant value *k*. Since the value of the unit step is 1, the gain of the output response can be determined as follows

$$y = k \times 1 = k \tag{7}$$

In this case, the gain is the value of the parameter *k*. The output response of Fig. 1 tends to reach the steady-state value, a gain, of 3 while the output response of Fig. 3 tends to reach the steady-state value, a gain, of 5.

Settling time is the time required for an output to reach and settle within a certain percentage of its steady-state value. It happens at time $t = 4\tau$, which is commonly referred to as the 2% settling time.

The 2% settling time for

- Fig. 1 is t = 4 x 10 = 40 seconds.
- Fig. 2 is t = 4 x 5 = 25 seconds.
- Fig. 3 is t = 4 x 10 = 40 seconds.

2.3 Response to a Delayed Input

Figure 4 below shows three output responses when three unit-step responses were provided as an input, with parameters k=1 & τ=1 as well as a time delay of t_d=-10, t_d=10, and t_d=30 seconds respectively, to a 1st order system with no initial conditions, $y(t_0^-) = 0$.

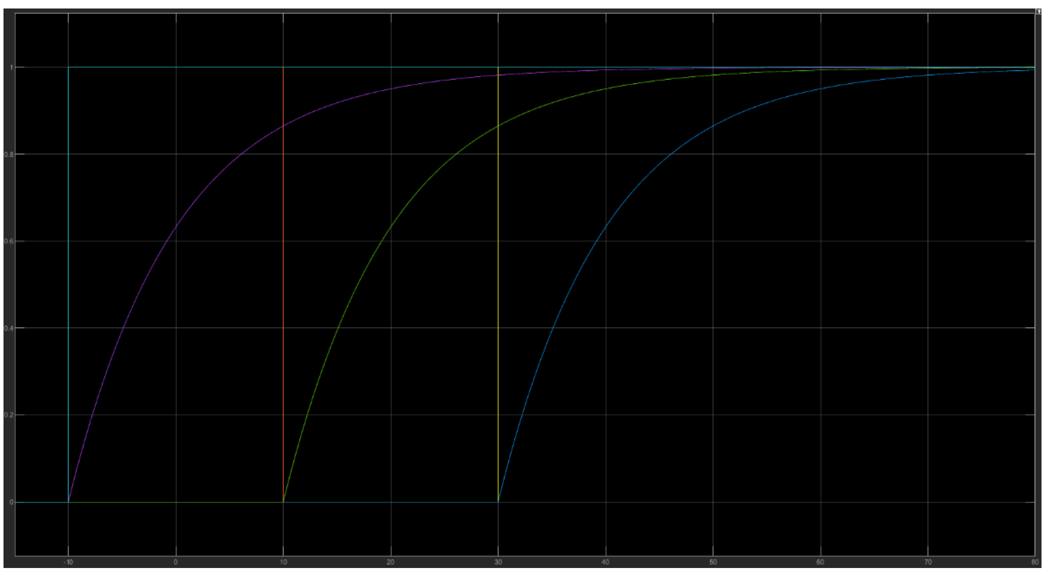

Figure 4 - Three Output Responses with k=1, τ=10, & t_d=-10, t_d=10, t_d=30 Respectively

The input of the unit step function of the light blue line is delayed by -10 seconds which means the input signal is shifted 10 seconds in the negative x-axis direction with reference to zero. In results, the output response of the purple line is time delayed in the same manner as the input.

The input of the unit step function of the red line is delayed by 10 seconds which means the input signal is shifted 10 seconds in the positive x-axis direction with reference to zero. In results, the output response of the green line is time delayed in the same manner as the input.

The input of the unit step function of the yellow line is delayed by 30 seconds which means the input signal is shifted 30 seconds in the positive x-axis direction with reference to zero. In results, the output response of the dark blue line is time delayed in the same manner as the input.

All the output responses of Fig. 4 tend to reach a gain value of 1, the value of *k*, which provides that the delayed time has no impact on the steady-state value. However, the 2% settling time is delayed based on the value of t_d and can be determined using the equation below:

$$t = 4\tau + t_d \tag{8}$$

The 2% settling time of

the purple line output is	t = (4 x 10) - 10 = 30 seconds.
the green line output is	t = (4 x 10) + 10 = 50 seconds.
the dark blue line output is	t = (4 x 10) + 30 = 70 seconds.

2.4 Response to a Composite Input Signal

Figure 5 below shows three output responses when three unit-step responses were provided as an input, with parameters k=1 & τ=1 as well as a time delay of t_d=0 & t_d=10, t_d=2 & t_d=6, and t_d=5 & t_d=6 seconds respectively, to a 1st order system with no initial conditions, $y(t_0^-) = 0$.

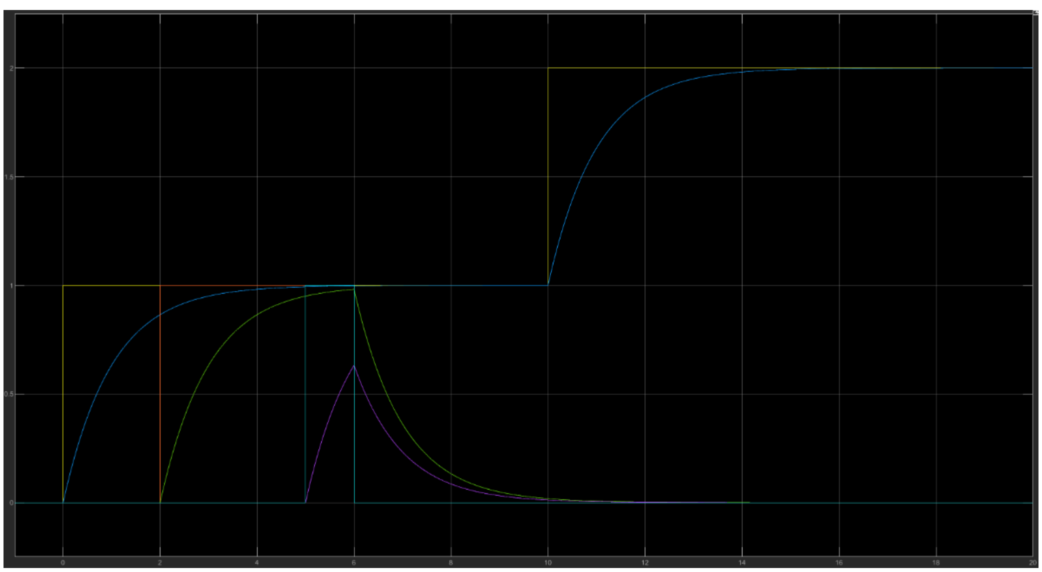

Figure 5 - Three Output Responses with a Composite Input Signal

As observed in Fig. 5, the given input to the system is a composite of two delayed unit step functions, the final value is seen to converge to the sum of both signals. Furthermore, one input starting at t_1 and the other at $t_d = t_2 > t_1$, it is shown that the output response is the superposition of both signals after time $t = t_2$.

In comparison with the case where each input signal is provided separately, the final output is the sum of the two separate outputs of each input signal. This is because the system is a linear time-invariant (LTI) system.

2.5 Response When Non-Zero Initial Conditions Exist

Figure 6 below shows the output response when a unit step function was provided as an input, with parameters k=3, τ=10 to a 1st order system with an initial condition, $y(t_0^-) = 2$.

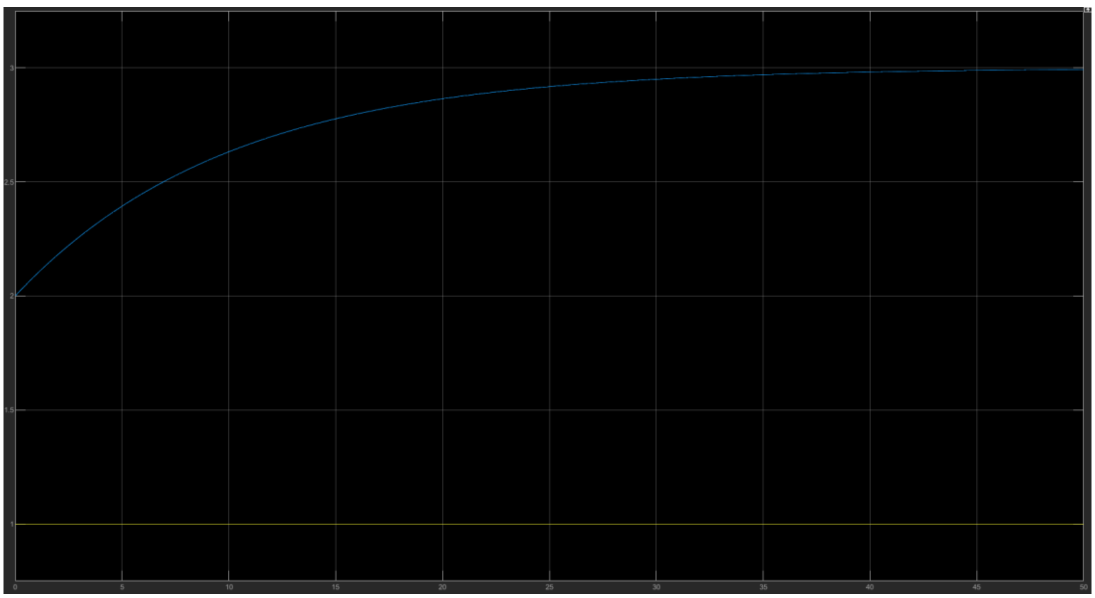

Figure 6 - Response with k=3, τ=10 & $y(t_0^-) = 2$

As shown in Fig. 6 above, the output response of the system is starting at the initial condition value and tending to reach the steady-state value. In this case, the initial condition value is less than the steady-state value which makes the output response increase exponentially in attempt to reach the stead-state value.

Figure 7 below shows the output response when a unit step function was provided as an input, with parameters k=3, τ=10 to a 1st order system with an initial condition, $y(t_0^-) = 6$.

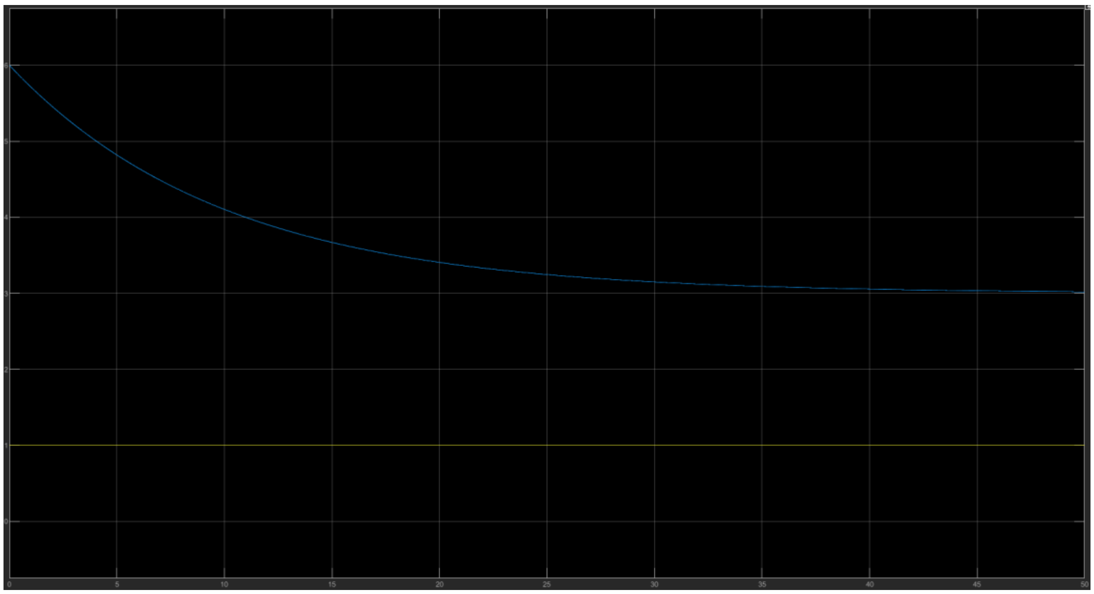

Figure 7 - Response with $k=3$, $\tau=10$, $t_d=5$ & $y(t_0^-) = 6$

As shown in Fig. 7 above, the output response of the system is starting at the initial condition value and tending to reach the steady-state value. In this case, the initial condition value is greater than the steady-state value which makes the output response decrease exponentially in attempt to reach the stead-state value.

Figure 8 below shows the output response when a unit step function was provided as an input, with parameters k=3, τ=10 as well as a time delay of t_d=-10 to a 1st order system with an initial condition, $y(t_0^-) = 6$.

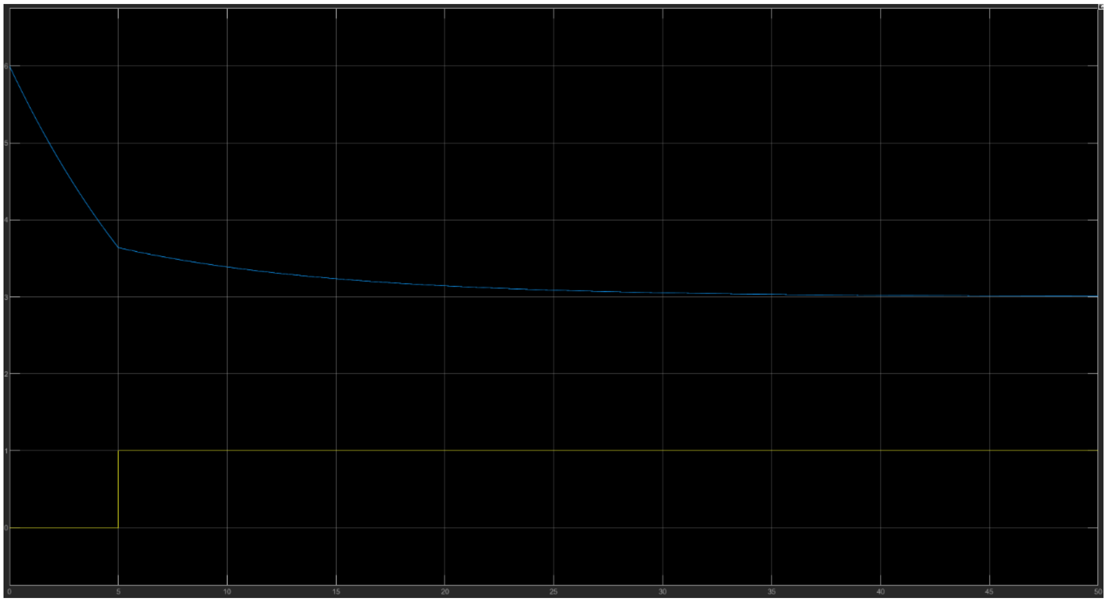

Figure 8 - Response with k=1, τ=10, & t_d =5, & $y(t_0^-)$ = 6

As shown in Fig. 8 above, the output response of the system is starting at the initial condition value and has an initial decay towards zero on the interval t = [0, t_d]. The initial condition value is greater than the steady-state value. As the point of the step in the input, there is a continuous change that makes the output decay to the steady-state value on the interval t = [t_d, ∞] making the output response a continuous piecewise function.

2.6 Step Response for the RC Circuit and the Mass-Damper System

Question #1

Comparing the general DE

$$\frac{dy(t)}{dt} + \frac{1}{tau} y(t) = \frac{k}{tau} x(t)$$

Where,

$$a_0 = \frac{1}{tau} \quad \text{and} \quad b_0 = \frac{k}{tau}$$

to that of the mass-damper system

$$M \frac{dv(t)}{dt} + bv(t) = f(t)$$

It is seen that

$$\frac{k}{tau} = \frac{1}{M} \quad \text{and} \quad \frac{1}{tau} = \frac{b}{M}$$

Solving these equations will give

$$k = \frac{1}{b} \quad \text{and} \quad tau = \frac{M}{b}$$

Question #2

k = 4 m/sec → b = 0.25 sec/m

4τ = 2 sec → τ = 0.5 sec

M * $u(t)$ = [0.125 kg/N] x [1N] = 0.125 kg

Therefore, b=0.25 sec/m and M = 0.125 kg

Question #3

Comparing the general DE

$$\frac{dy(t)}{dt} + \frac{1}{tau} y(t) = \frac{k}{tau} x(t)$$

Where,

$$a_0 = \frac{1}{tau} \quad \text{and} \quad b_0 = \frac{k}{tau}$$

to that of the RC circuit system

$$\frac{de0(t)}{dt} + \frac{1}{RC} e0(t) = \frac{1}{RC} ei(t)$$

It is seen that

$$k = 1 \quad \text{and} \quad \tau = RC$$

Question #4

The output at time t = ∞ is equal to the stead-state value which can be calculated using Eq. 7

$$y = k \text{ x } 1 = 1$$

The values of R & C control the time constant, how quick the output response in reaching steady state.

References

[1] “Lab #1: Simulation of a 1st-Order System,” Carleton Univeristy, Ottawa, 2019.