CARLETON UNIVERSITY
Department of Systems and Computer Engineering

SYSC 3600
Lab #1: Simulation of a 1^{st}-Order System

Before coming to Lab #1:

- Complete SIMULINK tutorial in Lab #0.
- Read and understand the below pre-lab for Lab #1.
- Read and understand Appendix A which contains the solution of a 1^{st}-order system with a unit step function as input.

Instructions:
Read through and complete the entire lab. Your report should address the points made in the **Report** suggestion boxes and demonstrate your understanding of the material.

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1 Pre-lab

In this lab, you will simulate a system described by a 1^{st}-order differential equation (DE). This pre-lab gives the needed background on developing a simulation diagram to solve a 1^{st}-order DE. Be sure to read and understand the pre-lab before your scheduled lab section.

1.1 Simple 1^{st}-order systems

First-order systems are some of the simplest systems we will be studying. Understanding how 1^{st}-order systems respond to inputs is the first step towards understanding how higher-order systems will respond to inputs.

Let's start by considering the RC circuit in Fig. 1 and the mass-damper system in Fig. 2. Both of these systems are 1^{st}-order, which should be evident since each only has one internal element that can store the state of the system. The RC circuit can be expressed by the following equation

$$\frac{de_o(t)}{dt} + \frac{1}{RC}e_o(t) = \frac{1}{RC}e_i(t) \tag{1}$$

while the mass-damper system is described by

$$M\frac{dv(t)}{dt} + bv(t) = f(t). \tag{2}$$

In this lab we will investigate how 1^{st}-order systems respond to a step input. For the RC circuit, a step input is like having the input source $e_i(t)$ initially off and then at some point in time switch the input on to a constant voltage (like throwing on the power switch). For the mass-damper system, a step input is like applying no force to the mass initially and then at some point in time applying a constant force $f(t)$ as indicated. Intuitively, it is reasonably easy to imagine how the outputs $e_o(t)$ and $v(t)$ for each system will change under these scenarios. The intent of this lab is for you to obtain a fuller understanding of how the step response occurs along with how the R and C parameters for the RC circuit as well as the M and b parameters for the mass-damper system control the step response.

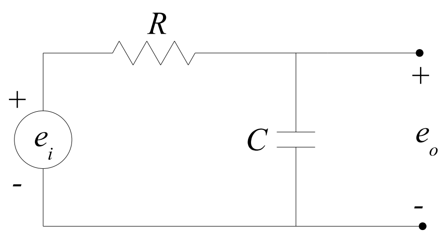

Figure 1: RC circuit 1^{st}-order system.

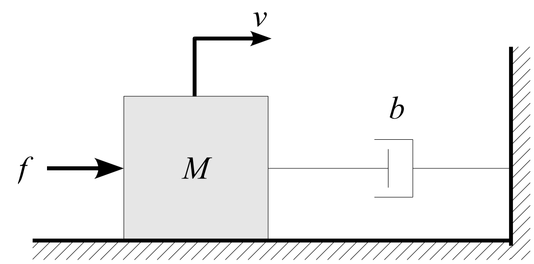

Figure 2: Mass-damper 1^{st}-order system.

1.2 Simulation diagram for a 1^{st}-order differential equation

Let's start by considering a causal, linear time-invariant (LTI) system described by the following 1^{st}-order differential equation

$$\frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) \tag{3}$$

where $x(t)$ is the input to the system and $y(t)$ is the output from the system. Note that $x(t)$ and $y(t)$ are functions over time so you will need to build a simulation that can input the values for $x(t)$ over time as well as monitor the output $y(t)$ over time.

From a simulation standpoint, implementing Eq. 3 directly is not appropriate since differentiation is sensitive to small fluctuations in the signal that may be due to input noise or numerical imprecision during simulation. If we can solve Eq. 3 such that integration is used instead of differentiation, then the simulation is less prone to misleading results due to any input noise or numerical imprecision. We can put Eq. 3 into a more robust form for simulation by first solving for $\frac{dy(t)}{dt}$ to obtain

$$\frac{dy(t)}{dt} = -a_0 y(t) + b_0 x(t) \tag{4}$$

and then performing a running integration on both sides of the equation to give

$$y(t) = \int_{-\infty}^{t} \left[-a_0 y(\lambda) + b_0 x(\lambda)\right] d\lambda \tag{5}$$

where λ is a dummy variable that disappears in the integration. Note that the running integral has limits from $-\infty$ to t and not from $-\infty$ to $+\infty$ since the output $y(t)$ for this causal system is dependent only on inputs from the past up to and including the present time t.

With the 1^{st}-order system written as in Eq. 5, the simulation diagram in Fig. 3 can easily be developed to realize this system. Verify that Fig. 3 does indeed realize Eq. 5. For convenience, each directed arrow in Fig. 3 is labelled.

As a side note, notice that Eq. 4 is embedded Fig. 3 right before the running integrator. Since $y(t)$ is the output of the integrator, it should be clear that the input to the integrator must be $\frac{dy(t)}{dt}$. Looking at the rest of the structure of the simulation diagram, it should be clear that the input to the integrator is $-a_0 y(t) + b_0 x(t)$. Putting these two observations together means we must have $\frac{dy(t)}{dt} = -a_0 y(t) + b_0 x(t)$, which is simply Eq. 4.

1.3 Using initial conditions

For a practical simulation of a 1^{st}-order differential equation, we realistically need to change the $-\infty$ limit of the running integral to a finite value. With Eq. 5, this can be done by splitting the running integral into

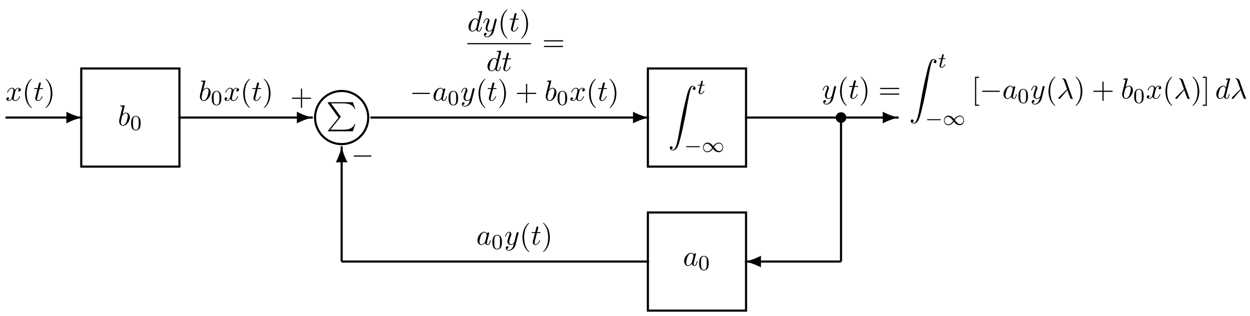

Figure 3: Simulation diagram that realizes Eq. 5.

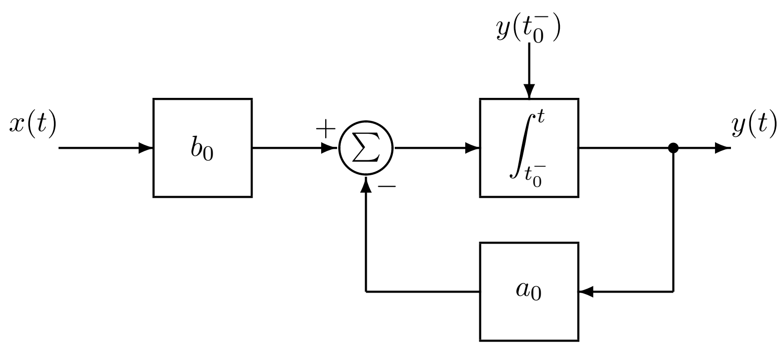

Figure 4: Simulation diagram that realizes Eq. 6 with initial condition set for the integrator.

two portions to give

$$y(t) = y(t_0^-) + \int_{t_0^-}^{t} [-a_0y(t) + b_0x(t)]\, dt \qquad \text{for } t \geq t_0 \tag{6}$$

where t_0 is the start time for the simulation and $y(t_0^-)$ is the initial condition for the output at the start of the simulation. A simulation diagram representing Eq. 6 is given in Fig. 4.

Recall that initial conditions for an integrator can be set, such as the `Integrator` block in SIMULINK. If we set the initial condition of the integrator to some value C, then notice that at the start of the simulation in Fig. 4 that the output $y(t)$ is simply

$$y(t)\Big|_{t=\text{start time}} = \text{C}. \tag{7}$$

Therefore, for a 1^{st}-order system we should set the initial condition of the integrator to $y(t_0^-)$ to satisfy Eq. 6.

2 Lab: Step Response of a 1^{st}-Order System

The primary goal of this lab is to learn and understand how a 1^{st}-order system responds to a step input. This step input could be in the form of the standard unit step function $u(t)$ defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \tag{8}$$

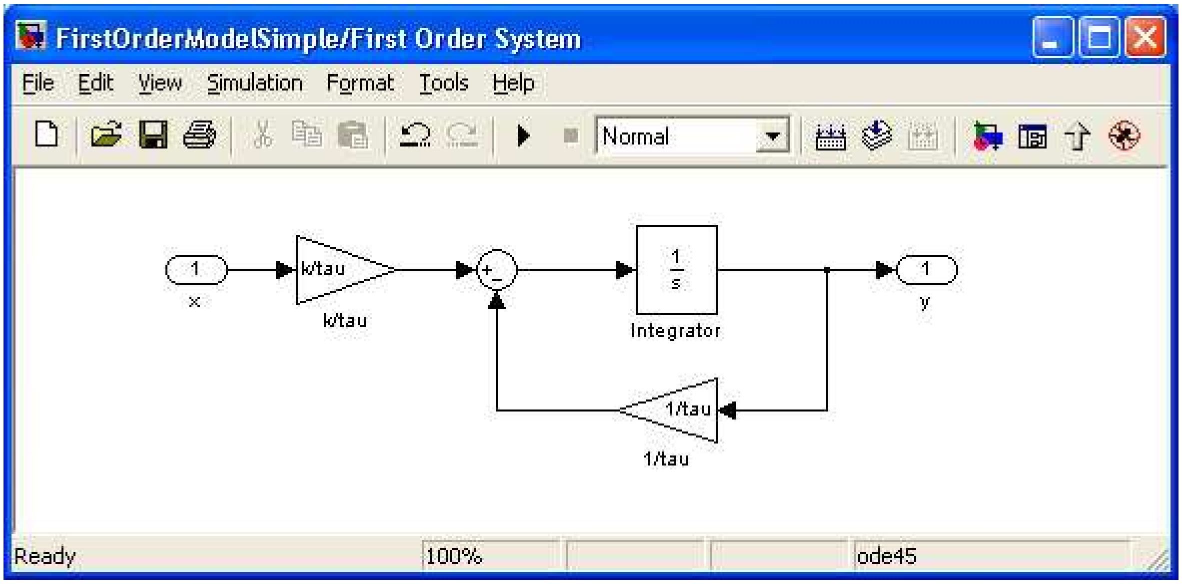

Figure 5: SIMULINK simulation diagram (as a subsystem) for the 1^{st}-order differential equation.

Similarly, the step input could be

- a scaled unit step function, such as $Au(t)$, where A is some scaling factor,
- a delayed unit step function, such as $u(t - t_d)$, where t_d is the delay in seconds, or
- a composite signal containing a superposition of scaled/delayed step functions.

Along with learning the step response of a 1^{st}-order system, this lab also looks out how initial conditions affect the step response for a 1^{st}-order system. For the first part of this lab we will assume that the initial conditions are zero, but in Sec. 2.5 non-zero initial conditions will be explored.

2.1 1^{st}-order differential equations

In this lab we will simulate a system described by the following 1^{st}-order differential equation (DE)

$$\frac{dy(t)}{dt} = \frac{-y(t) + kx(t)}{\tau} \tag{9}$$

where $x(t)$ is the system input and $y(t)$ is the system output. We have used a different form for Eq. 9 compared to Eq. 3 since k and τ have special meaning, particularly when investigating the step response. In this case, k is the static gain of the 1^{st}-order system, and τ is the time constant of the system. It should be clear that Eq. 9 is in the same form as Eq. 3 if we set

$$a_0 = \frac{1}{\tau} \qquad \text{and} \qquad b_0 = \frac{k}{\tau}. \tag{10}$$

A simulation model for this 1^{st}-order DE can be implemented as shown in Fig. 5. Note that Fig. 5 shows only the simulation model for a subsystem and that we can use this subsystem as shown in Fig. 6. Alternative realizations for the simulation model using SIMULINK are shown in Appendix B. For instance, Fig. 9 shows how to realize the simulation model with a step input without using a subsystem.

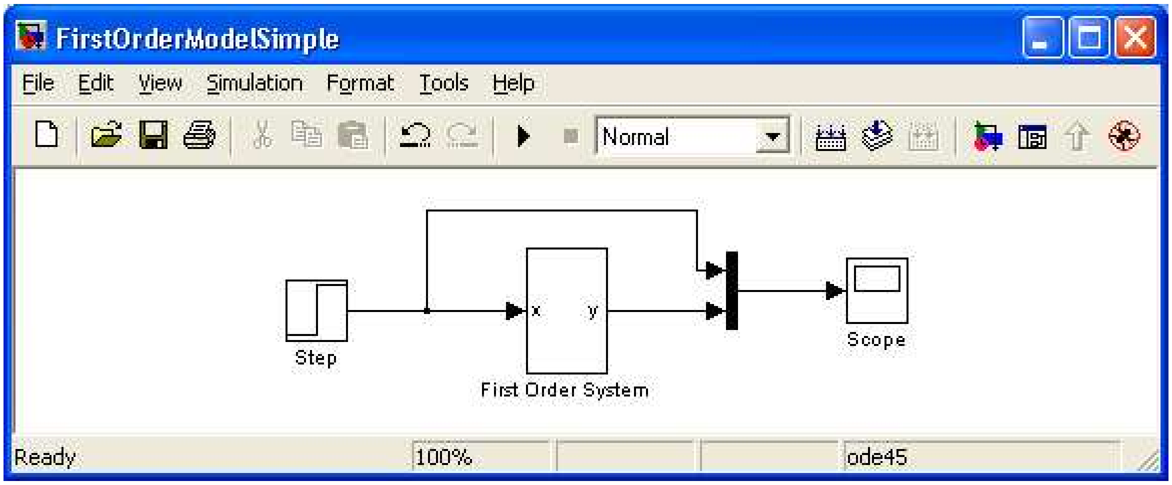

Figure 6: SIMULINK simulation model that determines the step response of the 1^{st}-order subsystem.

2.2 Step response, static gain, and the "time constant"

We will first explore how the 1^{st}-order system given by Eq. 9 responds to an input of a step function. Implement the simulation diagram for Eq. 9 in SIMULINK with an input of $x(t) = u(t)$, zero initial conditions (*i.e.*, $y(t_0^-) = 0$), and for different values of k and τ. In particular, try the following cases:

1. $x(t) = u(t)$, $k = 3$, $\tau = 10$ seconds, and $y(t_0^-) = 0$,
2. $x(t) = u(t)$, $k = 3$, $\tau = 5$ seconds, and $y(t_0^-) = 0$,
3. $x(t) = u(t)$, $k = 5$, $\tau = 10$ seconds, and $y(t_0^-) = 0$.

Make observations as to what k controls and what τ controls. For τ, you should observe what happens at time $t = 4\tau$, which is commonly referred to as the 2% settling time.

Report: Your report should include one representative plot. Discuss how k and τ affect the step response and what is meant by the 2% settling time at $t = 4\tau$.

2.3 Response to a delayed input

For a linear time-invariant system, such as Eq. 9, delaying the input by some time t_d will cause the response to be delayed by the same time delay t_d. Simulate the 1^{st}-order DE using the following settings to see the effect of delaying the input.

1. $x(t) = u(t - 30)$, $k = 1$, $\tau = 10$ seconds, and $y(t_0^-) = 0$
2. $x(t) = u(t - 10)$, $k = 1$, $\tau = 10$ seconds, and $y(t_0^-) = 0$
3. $x(t) = u(t + 10)$, $k = 1$, $\tau = 10$ seconds, and $y(t_0^-) = 0$

What observations can you make as to how the system responds when the input is delayed?

Report: Your report should include one representative plot. Discuss how setting a time delay t_d in the input affects the output.

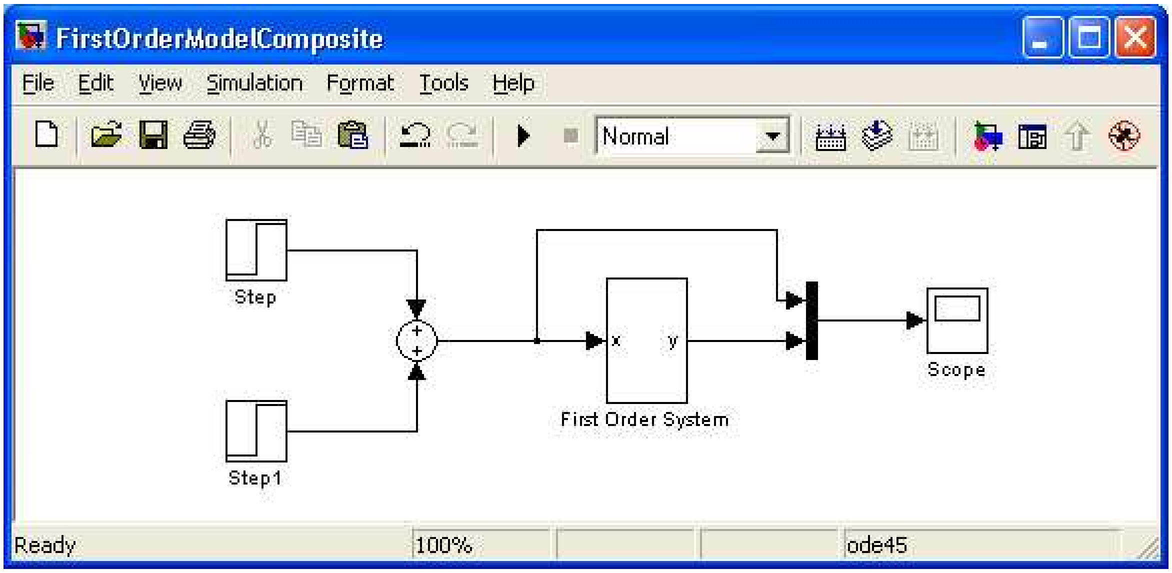

Figure 7: Forming composite inputs with two step functions in SIMULINK.

2.4 Response to a composite input signal

Another feature of linear time-invariant systems is that the response of a system to a composite input signal (input consisting of a number of parts/functions) can be determined through superposition of the responses of each of the parts.

Consider the following cases for the 1^{st}-order system. Try to predict the response of the system first, and then afterwards simulate the system and see if you predicted correctly.

1. $x(t) = u(t) + u(t-10)$, $k = 1$, $\tau = 1$ seconds, and $y(t_0^-) = 0$
2. $x(t) = u(t-2) - u(t-6)$, $k = 1$, $\tau = 1$ seconds, and $y(t_0^-) = 0$
3. $x(t) = u(t-5) - u(t-6)$, $k = 1$, $\tau = 1$ seconds, and $y(t_0^-) = 0$

These cases can be implemented by summing two step functions similar to that shown in Fig. 7.

> **Report:** Your report should include one representative plot. Make observations about the response when the input is a composite signal. What would happen if each component of the input was passed through the system separately and then the responses combined afterwards?

2.5 Response when non-zero initial conditions exist

The previous sections have all assumed a zero-state system (*i.e.*, initial conditions are set to zero). We will now explore how non-zero initial conditions affect the step response of a 1^{st}-order system.

The initial conditions control the state of the system at the *start* of the simulation, which we will designate as time t_0. As seen in Sec. 1.3 for the 1^{st}-order system considered, the `Integrator` block can be set with the initial condition $y(t_0^-)$ to satisfy Eq. 6.

Figure 8 shows one possible realization of a subsystem for implementing the 1^{st}-order system described by Eq. 9 and using initial conditions. In this realization, the initial condition is set through the MATLAB

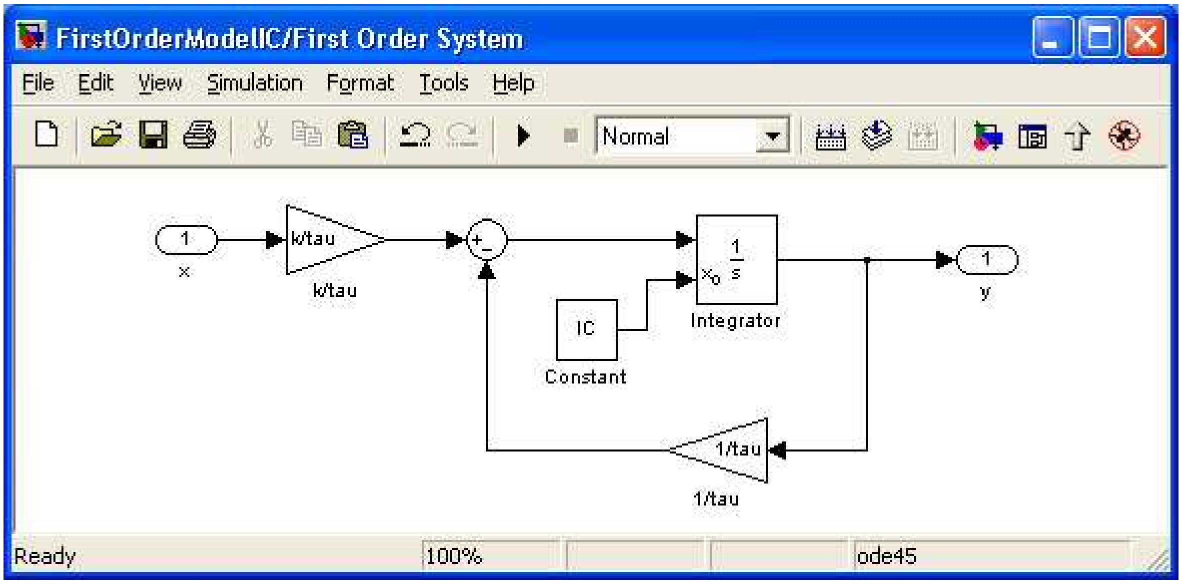

Figure 8: SIMULINK subsystem model for a 1^{st}-order system with non-zero initial conditions.

variable `IC` which you must manually set. Other possible realizations are given in Appendix B, where the initial condition can be set in the `Integrator` block either internally or externally.

Experiment with different initial conditions to see how the initial condition affect the step response. In particular, observe the response of the 1^{st}-order system in the following scenarios.

1. $x(t) = u(t)$, $k = 3$, $\tau = 10$ seconds, $y(t_0^-) = 2$, and $t_0 = 0$
2. $x(t) = u(t)$, $k = 3$, $\tau = 10$ seconds, $y(t_0^-) = 6$, and $t_0 = 0$
3. $x(t) = u(t-5)$, $k = 3$, $\tau = 10$ seconds, $y(t_0^-) = 6$, and $t_0 = 0$

Report: In your report, include a plot for all three scenarios. Discuss how the initial condition affects the response.

2.6 Step response for the RC circuit and the mass-damper system

Now that you hopefully have a good understanding of how a 1^{st}-order system responds to a step input, answer the following questions:

1. Give equations for the static gain k and the time constant τ for the mass-damper system given in Fig. 2.
2. If the input force to the mass-damper system is $f(t) = u(t)$ Newtons, what values for M and b are necessary for $v(t)$ to have settled to within 2% of $v(t) = 4$ m/s within 2 seconds?
3. What is the static gain k and the time constant τ for the RC circuit given in Fig. 1?
4. If the input to the RC circuit is $e_i(t) = u(t)$ and R and C are unknown, determine the output $e_o(t)$ at time $t = \infty$. What do R and C control?

Report: Answer these questions for your report.

A Solution of a 1^{st}-Order System With Unit Step Input

A.1 General solution for a 1^{st}-order differential equation

Consider that you are given a system described by the 1^{st}-order differential equation given in Eq. 3 which is repeated here for convenience.

$$\frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

If we wish to solve for the response of the system (*i.e.*, the output) we must solve for $y(t)$, but the combination of the $\frac{dy(t)}{dt}$ and $y(t)$ terms makes this difficult.

One approach of solving for $y(t)$ is by multiplying both sides of the equation by the integrating factor $e^{a_0 t}$ to give

$$e^{a_0 t}\frac{dy(t)}{dt} + a_0 e^{a_0 t} y(t) = b_0 e^{a_0 t} x(t). \tag{11}$$

We recognize that using the product rule in calculus, given as $(fg)'(t) = f(t)g'(t) + f'(t)g(t)$, that the left-hand side of the equation can be simplified to

$$\frac{d}{dt}\left[e^{a_0 t} y(t)\right] = b_0 e^{a_0 t} x(t). \tag{12}$$

This simplification justifies the choice of this particular integration factor $e^{a_0 t}$ since $y(t)$ now occurs only once in Eq. 12. To now solve for $y(t)$, perform a running integral on both sides of Eq. 12 to obtain

$$e^{a_0 t} y(t) = \int_{-\infty}^{t} b_0 e^{a_0 \lambda} x(\lambda) d\lambda \tag{13}$$

Solving for $y(t)$ gives

$$y(t) = e^{-a_0 t} \int_{-\infty}^{t} b_0 e^{a_0 \lambda} x(\lambda) d\lambda \tag{14}$$

or

$$y(t) = \int_{-\infty}^{t} b_0 e^{-a_0 (t-\lambda)} x(\lambda) d\lambda \tag{15}$$

Eq. 15 is the general solution for a 1^{st}-order differential equation with any input $x(t)$.

A.2 Step response for a 1^{st}-order differential equation

Given a specific input $x(t)$, Eq. 15 can often be simplified. For the step response of the system, we have the input $x(t) = u(t)$ where

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \tag{16}$$

Plugging $x(t) = u(t)$ into Eq. 15 gives

$$y(t) = \begin{cases} 0, & t < 0 \\ \displaystyle\int_{0}^{t} b_0 e^{-a_0 (t-\lambda)} d\lambda, & t \geq 0 \end{cases} \tag{17}$$

Evaluating the definite integral in Eq. 17 gives

$$y(t) = b_0 e^{-a_0 t} \int_0^t e^{a_0 \lambda} d\lambda \tag{18}$$

$$= b_0 e^{-a_0 t} \left[\frac{1}{a_0} e^{a_0 \lambda} \right]_{\lambda=0}^{\lambda=t} \tag{19}$$

$$= \frac{b_0}{a_0} e^{-a_0 t} \left(e^{a_0 t} - 1 \right) \tag{20}$$

$$= \frac{b_0}{a_0} \left(1 - e^{-a_0 t} \right) \qquad t \geq 0 \tag{21}$$

In the lab we will use a 1^{st}-order differential equation in the form

$$\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) = \frac{k}{\tau} x(t) \tag{22}$$

where effectively

$$a_0 = \frac{1}{\tau} \qquad \text{and} \qquad b_0 = \frac{k}{\tau}. \tag{23}$$

Using these values for a_0 and b_0, Eq. 21 can be written as

$$y(t) = k \left(1 - e^{-t/\tau} \right) \qquad t \geq 0 \tag{24}$$

where k scales the output response and τ is a time constant controlling the speed of the step response.

Both Eq. 21 and Eq. 24 can be plotted quickly with most graphing software to verify the simulation results for the step response of these 1^{st}-order differential equations.

B Alternative SIMULINK Realizations

B.1 Straight forward SIMULINK model

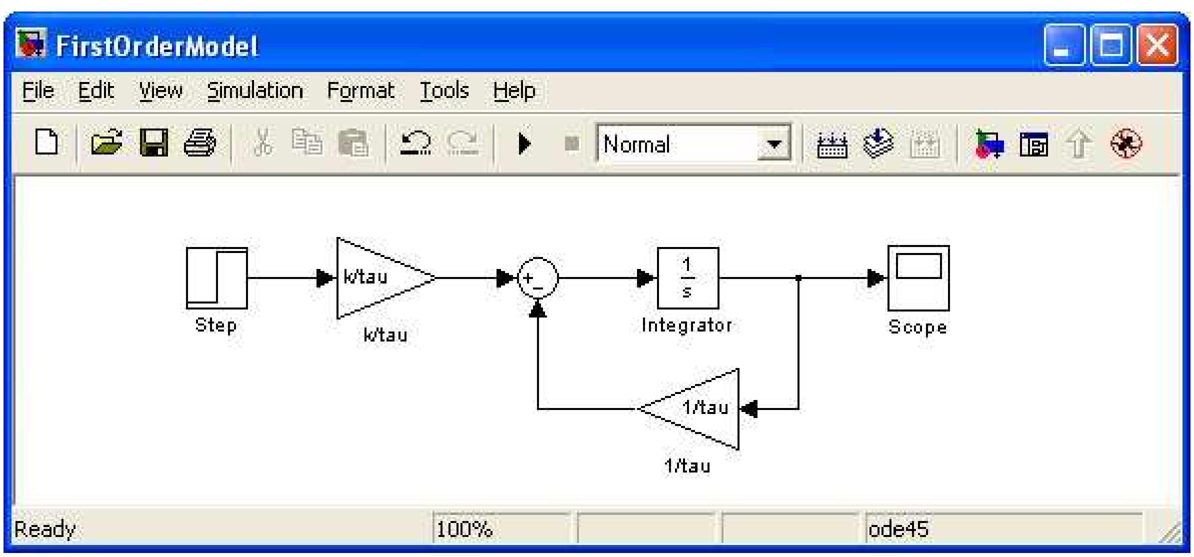

Figure 9: SIMULINK simulation diagram for the 1^{st}-order differential equation with a step function as input.

B.2 Implementation with constant external inputs for the subsystem

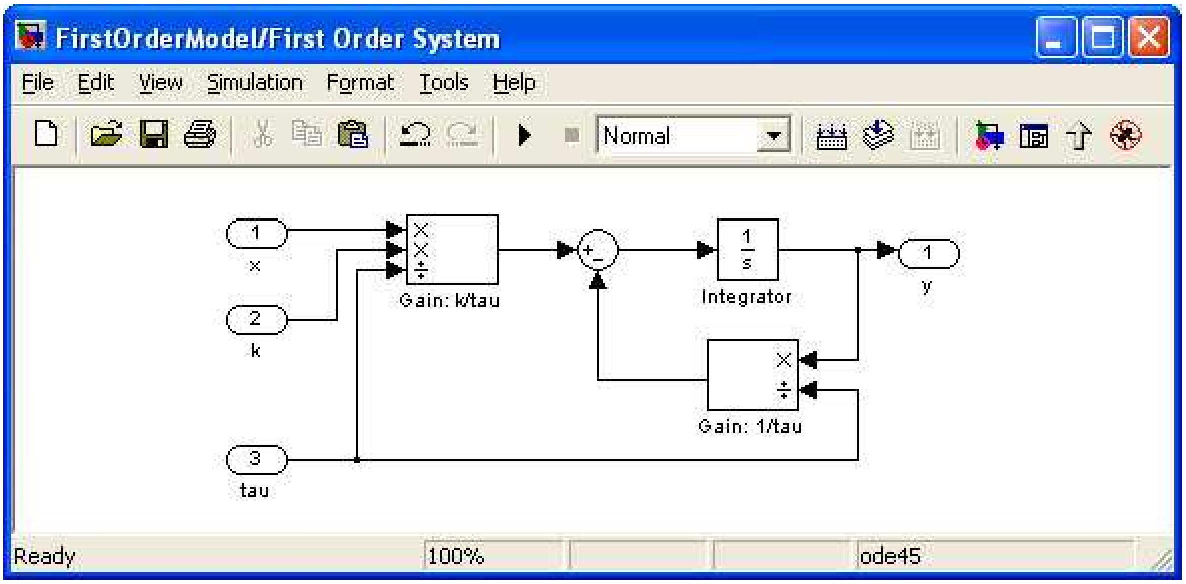

Figure 10: SIMULINK subsystem for a 1^{st}-order system where static gain k and time constant τ are external inputs.

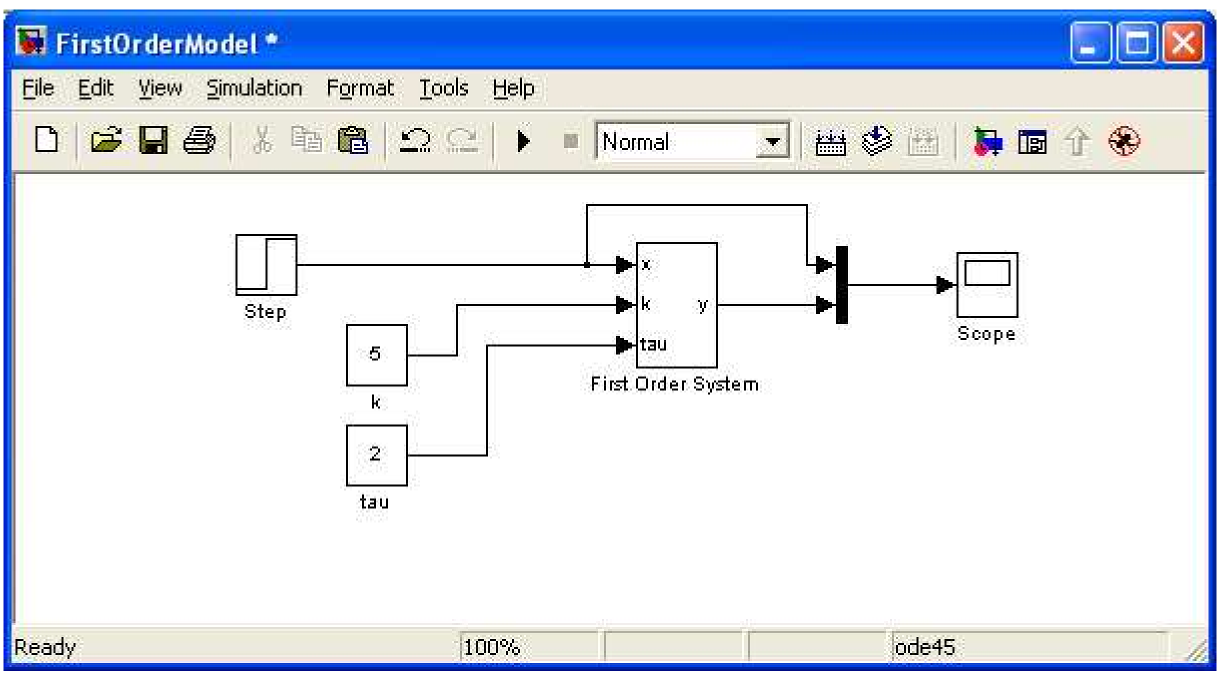

Figure 11: Example use of the subsystem from Fig. 10 to find the step response.

B.3 Implementation with external inputs and external initial condition for the subsystem

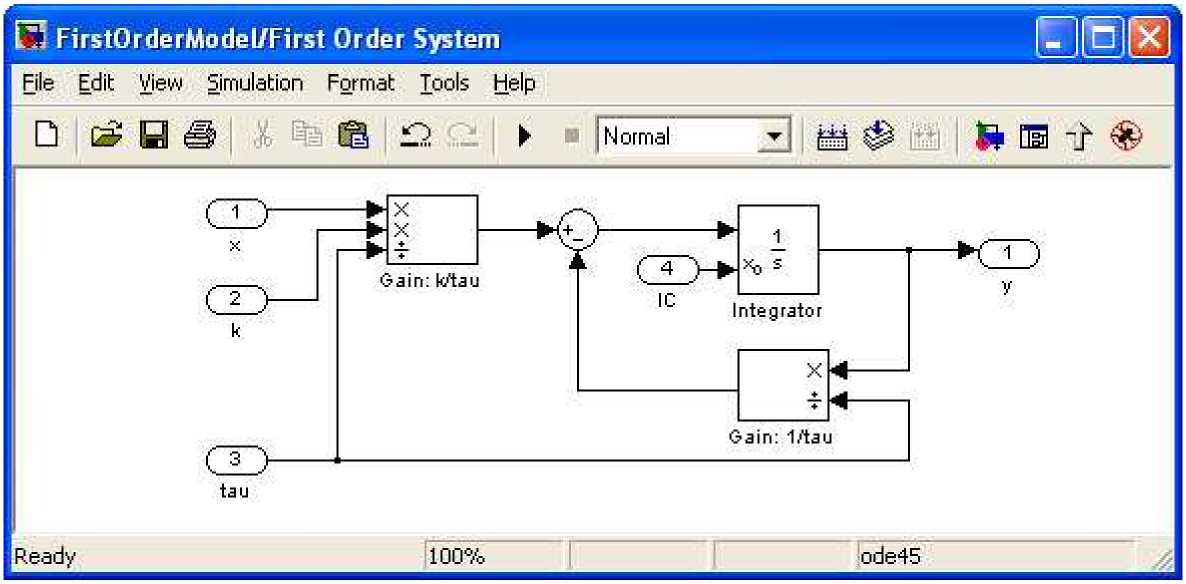

Figure 12: SIMULINK subsystem for a 1^{st}-order system where static gain k, time constant τ, and initial conditions are external inputs.

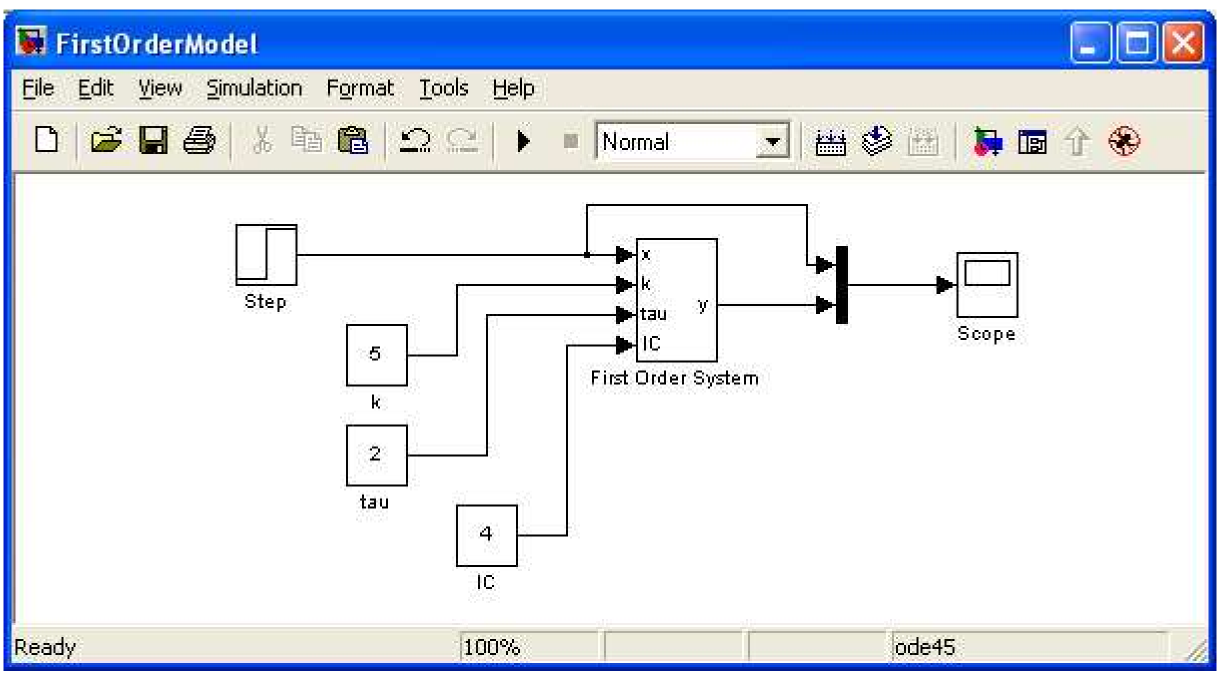

Figure 13: Example use of the subsystem from Fig. 12 to find the step response.