Part I

Chapter 6. Heapsort

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Assignment

- Read §6.1, §6.2, §6.3, §6.4
- Problems: Next Class

Part II: Sorting and Order Statistics

- Chapter 6: Heapsort
- Chapter 7: Quicksort
- Chapter 8: Sorting in linear time
- Chapter 9: Medians and order statistics

max

min

Part II: Sorting and Order Statistics

What's in this part?

$O(n^2)$ $O(n \lg n)$

- Fundamental sorting algorithms: insertion sort, merge sort, heapsort, quicksort
- Comparison-based sorting and its limitations $\Omega(n \lg n)$
- Non-comparison sorting: counting sort, radix sort, bucket sort
- Selecting the ith smallest element — order statistics

Big Picture: Sorting is essential in both theory and practice — efficient, optimal, and widely applicable.

Sorting — The Problem and Structure

$\{a_1, \ldots, a_n\}$

$\{a_1', \ldots, a_n'\}$

Goal: Rearrange n input elements so that:

$$a_1' \leq a_2' \leq \cdots \leq a_n'$$

- We often sort **records** (e.g., students, orders) by a **key** (e.g., score).
- Satellite data (e.g., name, address) must move with the key.
- Efficient sorting may use **pointers** instead of physically moving records.

Four Core Sorting Algorithms

- **Insertion Sort:** Simple, quadratic time, but fast for small inputs. In-place.
- **Merge Sort:** $O(n \log n)$ time, divide-and-conquer, not in-place.
- **Heapsort:** $O(n \log n)$, in-place, uses heaps (priority queues).
- **Quicksort:** Fast in practice, $O(n \log n)$ expected time, $O(n^2)$ worst-case.

All four are **comparison sorts**.

Lower Bound for Comparison Sorting

- **Key Result:**
  Any comparison-based sort requires at least $\Omega(n \log n)$ comparisons (worst-case).
- Merge sort and heapsort are **asymptotically optimal**.

$n \log n$

Non-Comparison Sorts (Faster When Possible)

To go faster than $\Omega(n \log n)$, we must **avoid comparisons**.

- **Counting Sort:** When keys are integers in $[0, k]$. Time: $\Theta(n + k)$
- **Radix Sort:** For digit-based numbers. Time: $\Theta(d(n + k))$
- **Bucket Sort:** For uniform real numbers in $[0, 1)$. Time: $\Theta(n)$ average-case

Assumption: Input keys must satisfy special structure or distribution.

Order Statistics (Chapter 9)

- ***i*th order statistic:** The ith smallest element in an array
- Can solve with sorting: $\boxed{O(n \log n)}$
- **Randomized Select:** $O(n)$ expected time
- **Deterministic Select:** $O(n)$ worst-case time (but more complex)

Use case: Find the median, minimum, maximum, etc. without full sort.

Chapter 6: Heapsort

- **Chapter 6.1: Heaps**
- Chapter 6.2: Maintaining the heap property Max-heapify
- Chapter 6.3: Building a heap Build Max Heap
- Chapter 6.4: The heapsort algorithm Heapsort
- Chapter 6.5: Priority queues

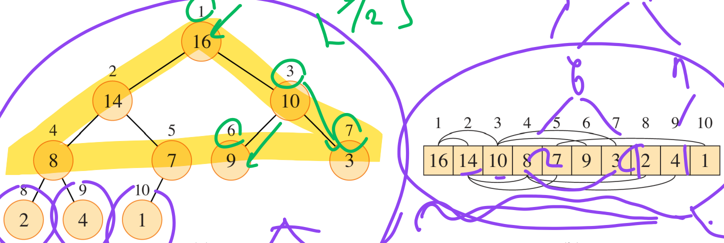
Heapsort: Overview

- Heapsort is a sorting algorithm introduced in this chapter.
- Like **Merge Sort**:
  - Runs in $O(n \log n)$ time
- Like **Insertion Sort**:
  - Sorts **in place**, using only constant extra space
- Combines the best features of both: fast and space-efficient
- Introduces a new algorithmic idea: using a **data structure** called a **heap** to manage information

What is a Binary Heap?

- A (binary) heap is a **nearly complete binary tree**
- Represented as an array $A[1:n]$
- `A.heap_size` denotes the number of elements currently in the heap
- Tree is filled top to bottom, left to right

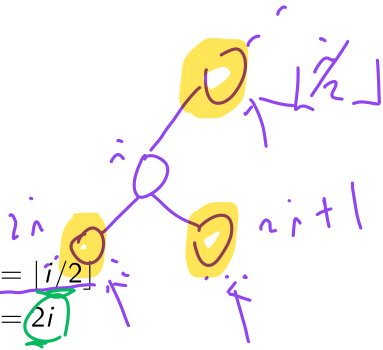
$\lfloor 3/2 \rfloor$



(a) (b)

left

Heap Index Formulas



Given a node at index i:

$$\begin{aligned}\texttt{PARENT}(i) &= \lfloor i/2 \rfloor \\ \texttt{LEFT}(i) &= 2i \\ \texttt{RIGHT}(i) &= 2i + 1\end{aligned}$$

$\lfloor i/2 \rfloor$

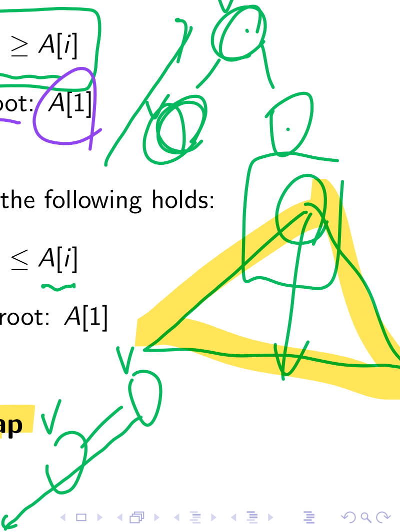
- These are computed efficiently using bit shifts
- Commonly implemented as inline functions or macros

Heap Properties

- **Max-Heap:**
  - For all nodes i (except the root), the following holds:

$$\boxed{A[\text{parent}(i)] \geq A[i]}$$

  - So the largest element is at the root: $A[1]$



- **Min-Heap:**
  - For all nodes i (except the root), the following holds:

$$A[\text{parent}(i)] \leq A[i]$$

  - So the smallest element is at the root: $A[1]$

- Heapsort uses a **max-heap**
- Priority queues often use a **min-heap**

Heap Height and Runtime

Coming up in this chapter:

- `MAX-HEAPIFY` – maintain heap property ($O(\log n)$)
- `BUILD-MAX-HEAP` – make a heap from unordered array ($\Theta(n)$)
- `HEAPSORT` – sorting in-place using a heap ($O(n \log n)$)

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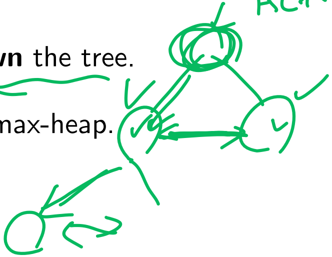
Maintaining the Heap Property

- Core idea: `MAX-HEAPIFY` ensures the **max-heap property** is preserved.
- Input: Array `A` with `A.heap-size`, and index `i`.

  $A[i] \le A[\text{parent}(i)]$
- Assumes:
  - Subtrees rooted at `LEFT(i)` and `RIGHT(i)` are already max-heaps.
  - Node `i` may violate the heap property.
- Action: Let the value at `A[i]` **float down** the tree.

A[i]



Goal: Make the subtree rooted at `i` a valid max-heap.

MAX-HEAPIFY in Action (Figure 6.2)

largest $\leftarrow$ i

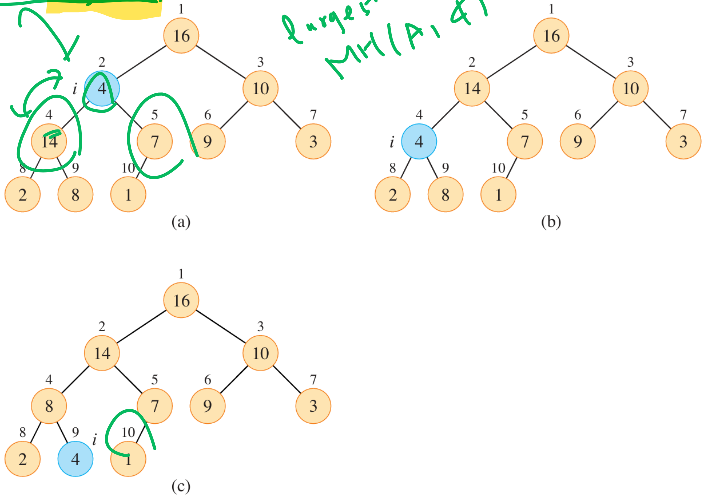
- Step 1: Compare `A[i]`, `A[LEFT(i)]`, and `A[RIGHT(i)]`
- Step 2: Find the largest and set index in `largest`
- Step 3: If largest $\neq$ i:
  - Swap `A[i]` and `A[largest]`
  - Recursively call `MAX-HEAPIFY` on index `largest`

Fixing violations from top-down.

MAX-HEAPIFY in Action (Figure 6.2)

MH(A,2)

largest ← 4
MH(A,4)



Fixing violations from top-down.

MAX-HEAPIFY Algorithm

(A, n) T(n)

Algorithm 1 MAX-HEAPIFY(A, i)

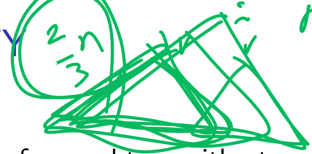
```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i] then
    largest ← l
else
    largest ← i
end if
if r ≤ A.heap-size and A[r] > A[largest] then
    largest ← r
end if
if largest ≠ i then
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest)
end if
```

RAM

O(1)

Note: `LEFT(i) = 2i,   RIGHT(i) = 2i + 1`

Running Time of MAX-HEAPIFY



n

n

- Let $T(n)$ be the worst-case time for a subtree with at most n nodes.
- Time to compare and swap at each step is $O(1)$.
- Recursive call occurs on a subtree of size at most $\frac{2n}{3}$.

T(n)

$$T(n) \leq T\left(\frac{2n}{3}\right) + \Theta(1)$$

MH(A, 2i)

- Using the Master Theorem (Case 2):

$$T(n) = O(\log n)$$

Alternatively: Run time is $O(h)$ where h is the height of node i.

Think About It

$T(n) \leq$

Question 1:
Why does each child subtree in MAX-HEAPIFY have size at most $2n/3$?

Hint: Consider the structure of a complete binary tree.

Question 2:
Given the recurrence:

$$T(n) \leq T(2n/3) + \Theta(1)$$

use the **Master Theorem** to find the time complexity of `MAX-HEAPIFY`.

Answer: Why Subtree Size Is at Most $2n/3$

To analyze the worst-case size of a child subtree in a binary heap:

- Assume a heap with n nodes, rooted at index i.
- We want to find the maximum number of nodes in either the left or right subtree.
- The heap is a complete binary tree: all levels full, except possibly the last, which is filled left-to-right.
- To maximize the size of the left subtree, it should include as many nodes as possible in the last level.

i

k

2^{k-1}

Let k be the height of the last level, with:

$$\text{Left subtree size} = 2^k, \quad \text{Right subtree size} = 2^{k-1}$$

$1+2+\cdots+2^{k-1} = 2^k - 1$

$1+2+\cdots+2^{k-2} = 2^{k-1} - 1$

So:

$$2^k + 2^{k-1} = 3 \cdot 2^{k-1} \le n \Rightarrow 2^{k-1} \le \frac{n}{3} \Rightarrow \text{Max subtree size} \le \frac{2n}{3}$$

Conclusion: Each recursive call to `MAX-HEAPIFY` acts on a subtree of size at most $\boxed{2n/3}$.

Answer: Time Complexity via Master Theorem

We solve the recurrence:

$a=1$ $\quad b=\frac{3}{2}$ $\quad = f(n)$

$$T(n) \leq T(2n/3) + \Theta(1)$$

$n^{\log_b a} = 1$

$O(\log(n))$

Step 1: Match to the Master Theorem form:

$$T(n) = T(n/b) + f(n)$$

Here, $a = 1$, $b = 3/2$ (since $n/b = 2n/3$), and $f(n) = \Theta(1)$

Step 2: Compare $f(n)$ to $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$

Conclusion: This is Case 2 of the Master Theorem:

$$f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(\log n)$$

$T(n) = \Theta\left(\cancel{n^{\log_b a}} \times \log n\right)$

Final Result: `MAX-HEAPIFY` runs in $\boxed{O(\log n)}$ time.

Chapter 6: Heapsort

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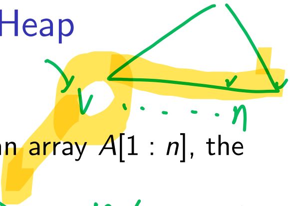
Exercise 6.1-8: Identifying Leaves in a Heap



Statement: In an n-element heap stored as an array $A[1:n]$, the leaves are the nodes with indices:

$$\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor + 2, \ldots, n$$

$$\left\lfloor \frac{n}{2} \right\rfloor + 1$$

Proof:

- In a heap, the children of a node i are at $2i$ and $2i+1$.
- If $i > \left\lfloor \frac{n}{2} \right\rfloor$, then $2i > n \rightarrow i$ has no children $\rightarrow$ it is a leaf.
- If $i \leq \left\lfloor \frac{n}{2} \right\rfloor$, then $2i \leq n \rightarrow$ it has at least a left child $\rightarrow$ it is internal.

Conclusion: Leaves are exactly those with indices from $\left\lfloor \frac{n}{2} \right\rfloor + 1$ to n.

BUILD-MAX-HEAP Algorithm

Goal: Convert an array $A[1:n]$ into a max-heap.

Procedure:

procedure BUILD-MAX-HEAP(A, n)
    A.heap-size $\leftarrow n$
    for $i \leftarrow \lfloor n/2 \rfloor$ **downto** 1 **do**
        MAX-HEAPIFY(A, i)
    end for
end procedure

$\lfloor n/2 \rfloor + 1$ leaf $\leftarrow$ max-heap

Key Idea: Internal nodes are heapified from bottom-up. Leaves are already heaps.

internal node.

Why BUILD-MAX-HEAP Works (Loop Invariant)

Loop Invariant: At the start of each iteration, nodes $i+1$ to n are all roots of valid max-heaps.

Initialization:

- $i = \lfloor n/2 \rfloor$
- All nodes from $i+1$ to n are leaves $\rightarrow$ trivially max-heaps.

Maintenance:

- Children of node i are greater than i
- `MAX-HEAPIFY(A, i)` ensures subtree rooted at i becomes a max-heap

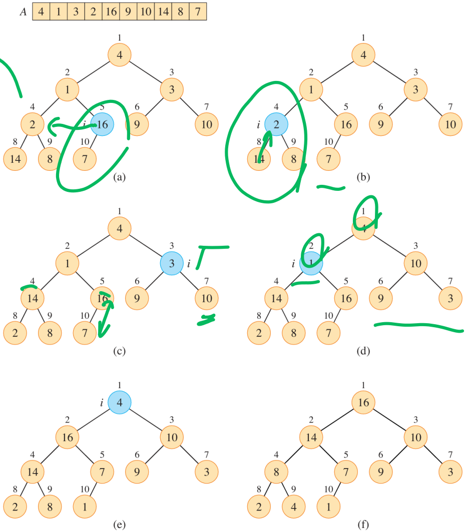
Termination:

- After loop, all nodes 1 through n are max-heap roots
- Especially $A[1]$ — the root of the final heap

Figure: BUILD-MAX-HEAP in Action

BM(A, 10)

$\lfloor 10/2 \rfloor = 5$



h

Naïve Runtime Analysis of BUILD-MAX-HEAP

- The procedure BUILD-MAX-HEAP(A) runs a loop:

$$\texttt{for } i = \lfloor n/2 \rfloor \texttt{ downto } 1 : \quad \texttt{MAX-HEAPIFY(A, i)}$$

- Number of iterations: $\lfloor n/2 \rfloor \approx O(n)$
- Each call to MAX-HEAPIFY can take up to $O(\log n)$ time in the worst case (if the node is near the root)
- Therefore, total worst-case time is:

$$O(n) \cdot O(\log n) = O(n \log n)$$

- This is a correct upper bound — but not the tightest possible.

Tighter Runtime Analysis of `BUILD-MAX-HEAP`

- The simple bound $O(n \log n)$ assumes all calls to `MAX-HEAPIFY` take $O(\log n)$ time
- In reality, many nodes are near the bottom of the heap and have small height
- `MAX-HEAPIFY` runs in $O(h)$ where h is the height of the node

- Idea: sum the work done at each height
- Using known bounds on the number of nodes at each height:

$$\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot O(h) = O(n)$$

- Tighter bound: $O(n)$

Key Questions for the Tighter Bound

To justify the tight $O(n)$ analysis, we ask:

1. Why does an n-element heap have height $\lfloor \log n \rfloor$?
2. Why are there at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of height h?

These structural properties of binary heaps allow us to bound the total work done by `BUILD-MAX-HEAP`.

Why is Heap Height $\lfloor \log_2 n \rfloor$?

Goal: Show that a binary heap with n elements has height $\lfloor \log_2 n \rfloor$

- In a perfect binary tree of height h, number of nodes:

$$1 + 2 + 4 + \cdots + 2^h = 2^{h+1} - 1$$

- In a complete binary tree of height h, minimum number of nodes:

$$2^h \quad \text{(only one node at bottom level)}$$

So the number of nodes n in a binary heap satisfies:

$$2^h \leq n < 2^{h+1}$$

Taking $\log_2$:

$$h \leq \log_2 n < h + 1 \Rightarrow h = \lfloor \log_2 n \rfloor$$

Why Do We Divide n by 2^{h+1}?

Goal: Bound the number of nodes at height h in a binary heap of size n

- A node at height h must be the root of a subtree of height h
- A full binary subtree of height h has:

$$\text{Total nodes} = 1 + 2 + 4 + \cdots + 2^h = 2^{h+1} - 1$$

- So each such node **uses up** at least $2^{h+1} - 1$ positions in the heap
- Total number of such nodes is at most:

$$\left\lfloor \frac{n}{2^{h+1} - 1} \right\rfloor \leq \left\lceil \frac{n}{2^{h+1}} \right\rceil$$

Tighter Runtime of `BUILD-MAX-HEAP`: Setup

Let:

- c: constant factor in the $O(h)$ cost of `MAX-HEAPIFY`
- Nodes at height h: at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$

Total cost:

$$T(n) \le \sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot ch$$

Apply inequality: $\lceil x \rceil \le 2x$ for all $x \ge \frac{1}{2}$

$$\Rightarrow \left\lceil \frac{n}{2^{h+1}} \right\rceil \le \frac{n}{2^h} \quad \text{(CLRS Exercise 6.3-2)}$$

So:

$$T(n) \le cn \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}$$

Transforming the Summation

We now simplify:

$$T(n) \le cn \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}$$

This summation is known to converge:

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = 2$$

From CLRS Equation A.11 (p.1142):

$$\sum_{h=0}^{\infty} h x^h = \frac{x}{(1-x)^2} \quad \text{for } 0 < x < 1$$

$$\Rightarrow \sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = \frac{1/2}{1/4} = 2$$

So we are safe to approximate:

$$T(n) \le cn \cdot 2$$

Final Result: BUILD-MAX-HEAP is Linear Time

Final bound:

$$T(n) \leq cn \cdot \sum_{h=0}^{\infty} \frac{h}{2^h} = cn \cdot 2 = O(n)$$

Interpretation:

- While a single MAX-HEAPIFY may cost $O(\log n)$, most nodes lie near the bottom of the heap
- Their small height makes their cost low
- So the total cost of building a heap is dominated by these shallow calls
- **Hence, BUILD-MAX-HEAP runs in linear time.**

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Overview of HEAPSORT

HEAPSORT Steps:

1. Call `BUILD-MAX-HEAP` to organize the input array into a max-heap.
2. The largest element is now at the root $A[1]$.
3. Swap $A[1]$ with $A[n]$ to move the max to its final sorted position.
4. Reduce the heap size by 1.
5. Call `MAX-HEAPIFY` on $A[1]$ to restore the max-heap.
6. Repeat until heap size is 1.

Each iteration places the next-largest element in its final position.

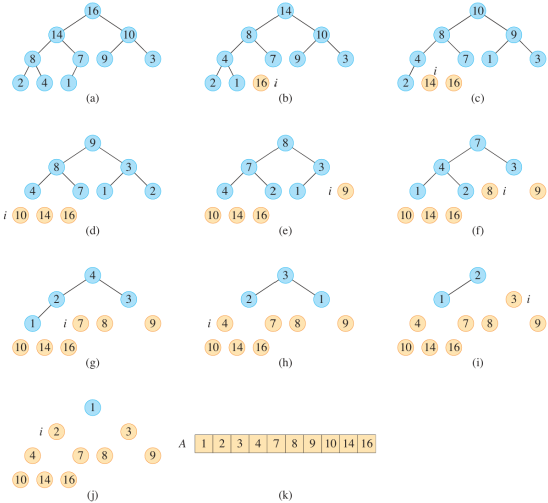
HEAPSORT Pseudocode

```
HEAPSORT(A, n):
```

1. `BUILD-MAX-HEAP(A, n)`
2. `for i = n downto 2:`
   - `exchange A[1] with A[i]`
   - `A.heap-size = A.heap-size - 1`
   - `MAX-HEAPIFY(A, 1)`

Key Idea: Each `MAX-HEAPIFY` fixes the heap with one fewer element.

Visual Example of HEAPSORT



Time Complexity of HEAPSORT

Step-by-step costs:

- `BUILD-MAX-HEAP` takes $O(n)$ time
- The loop runs $n - 1$ times
- Each `MAX-HEAPIFY` call takes $O(\log n)$

$$\Rightarrow T(n) = O(n) + (n-1) \cdot O(\log n) = O(n \log n)$$

HEAPSORT is an in-place, comparison-based sort with worst-case $O(n \log n)$ **time.**

Question?