Introduction to Algorithms

Date: 3/12 (Thursday)

Instructor: 유준수

Assignment

- Read 2.3
- Problems:
  - 2.3절 – 1, 4, 6

- Incremental Approach
  - Incrementing i=2 to n
- Correctness
  - State Loop Invariant
  - Check Initialization/Maintenance/Termination

- Assumption: RAM Model
  1. One by one execution of instruction
  2. Instruction takes constant time (not dependent on input size n)
- Analyzing Insertion Sort

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n-1$
// Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n-1$
$j = i - 1$	c_4	$n-1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j+1] = A[j]$	c_6	$\sum_{i=2}^{n} (t_i - 1)$
$j = j - 1$	c_7	$\sum_{i=2}^{n} (t_i - 1)$
$A[j+1] = key$	c_8	$n-1$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n} (t_i - 1) + c_7 \sum_{i=2}^{n} (t_i - 1) + c_8(n-1)$$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{i=2}^{n} t_i + c_6 \sum_{i=2}^{n} (t_i - 1) + c_7 \sum_{i=2}^{n} (t_i - 1) + c_8(n-1)$$

- t_i : number of while loop at i –th iteration
- Best-case time complexity
  - $t_i = 1 \; for \; all \; i$

$$T(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

- Worst-case time complexity
  - $t_i = i \; for \; all \; i$

$$T(n) = \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right) n^2 + \left(\frac{c_1 + c_2 + c_4 + c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right) n - (c_2 + c_4 + c_5 + c_8)$$

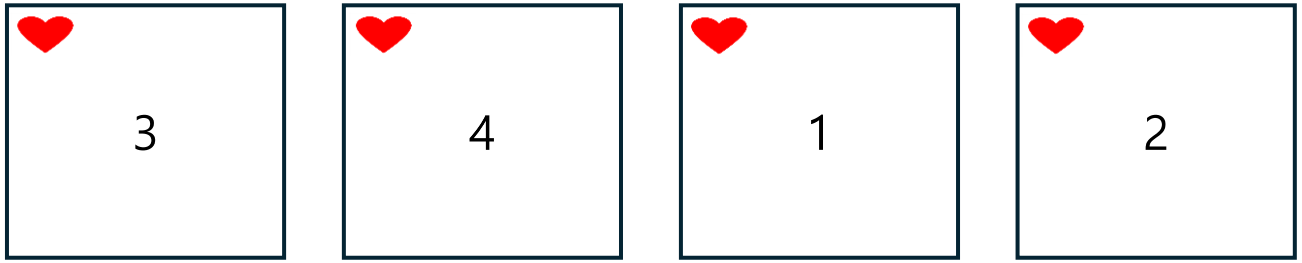
- Order of growth
  - Think of θ as "roughly proportional to" for now
  - Consider only the highest term
  - Ignore the constant term
  - E.g., $\theta(9n^3 + 2n^2 + 100) = \theta(n^3)$

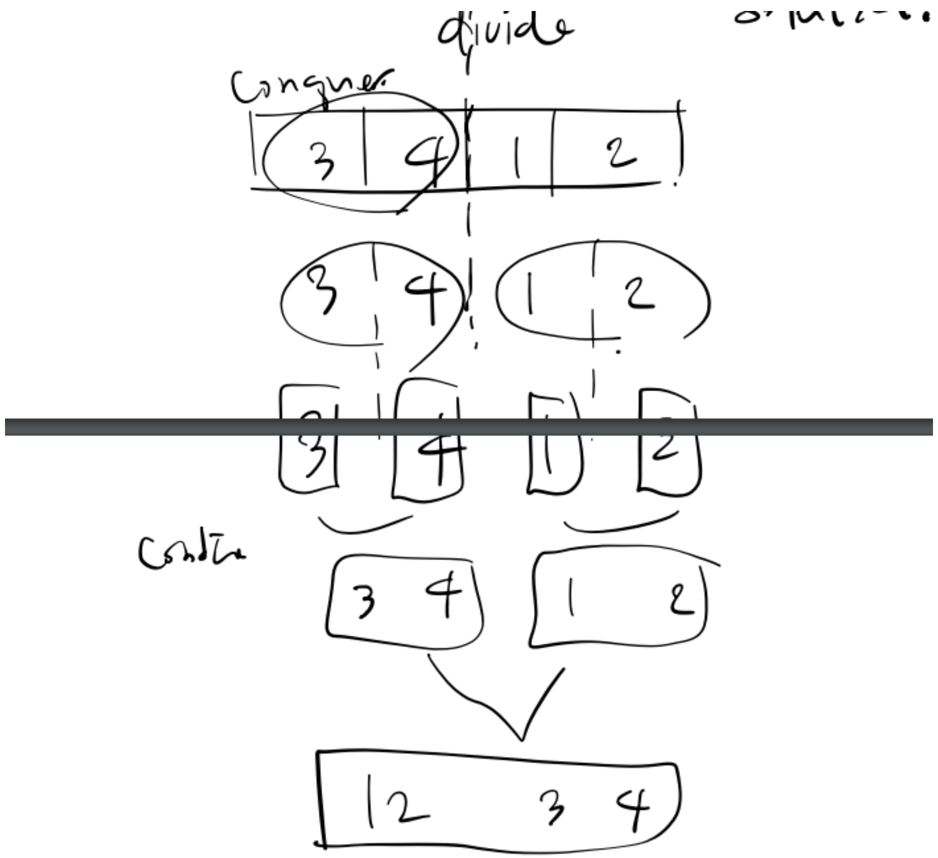
Chapter 2. The Role of Algorithms in Computing

- 2.1 Insertion sort
- 2.2 Analyzing algorithms
- 2.3 Designing algorithms

Divide & Conquer Method

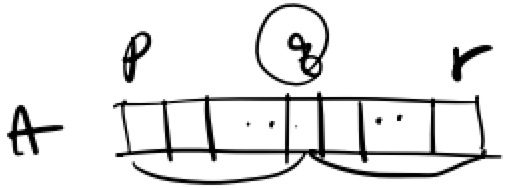
- Divide
  - Divide the problem into sub-problem of the same kind
- Conquer
  - Solve the problem, recursively.
- Combine
  - Combine solutions to form original solution


3
4
1
2


divide
Conquer
Combine

Main Idea of Merge Sort Algorithm

$MS(A, p, r)$ $\qquad \left\lfloor \frac{p+r}{2} \right\rfloor = q$



$q = \left\lfloor \frac{p+r}{2} \right\rfloor$

$MS(A, p, q)$

$MS(A, q+1, r)$

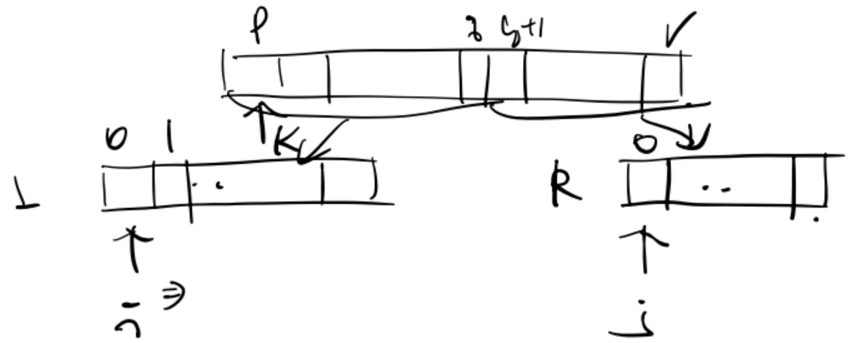
$Merge(A, p, q, r)$

Solve subproblem of its own kind.

Merge

- Combine two sorted arrays to output a sorted combined array

Merge (A, p, q, r)



while $i < n_L$, $j < n_R$

    if $L[i] \leq R[j]$

        $A[k] = L[i]$
        $i = i+1$

    else

        $A[k] = R[j]$

        $j = j+1$

    $k = k+1$

while $i < n_L$

$A[k] = L[i]$

$i = i+1$

$k = k+1$

while $j < n_R$

$A[k] = R[i]$

$j = j+1$

$k = k+1$

Back to Merge Sort Algorithm

$MS(A, p, r)$

If $p = r$
return

$q = \left\lfloor \frac{p+r}{2} \right\rfloor$

$MS(A, p, q)$
$MS(A, q+1, r)$ ⇒ divide & conquer

$Merge(A, p, q, r)$ ⇒ combin

How it works:

① MS(A, 1, n=4)

If $1 \neq 4$

$q = 2$

② MS(A, 1, 2)

MS(A, 3, 4)

Merge(A, 1, 2, 4)

$\underline{MS(A,1,2)}$

If $1 \neq 2$

$q = 1$

$MS(A,1,1)$

$MS(A,2,2)$

$Merge(A,1,1,2)$

③ $MS(A,1,1)$

If $1 = 1$

return

④ $MS(A,2,2)$

If $2=2$

return

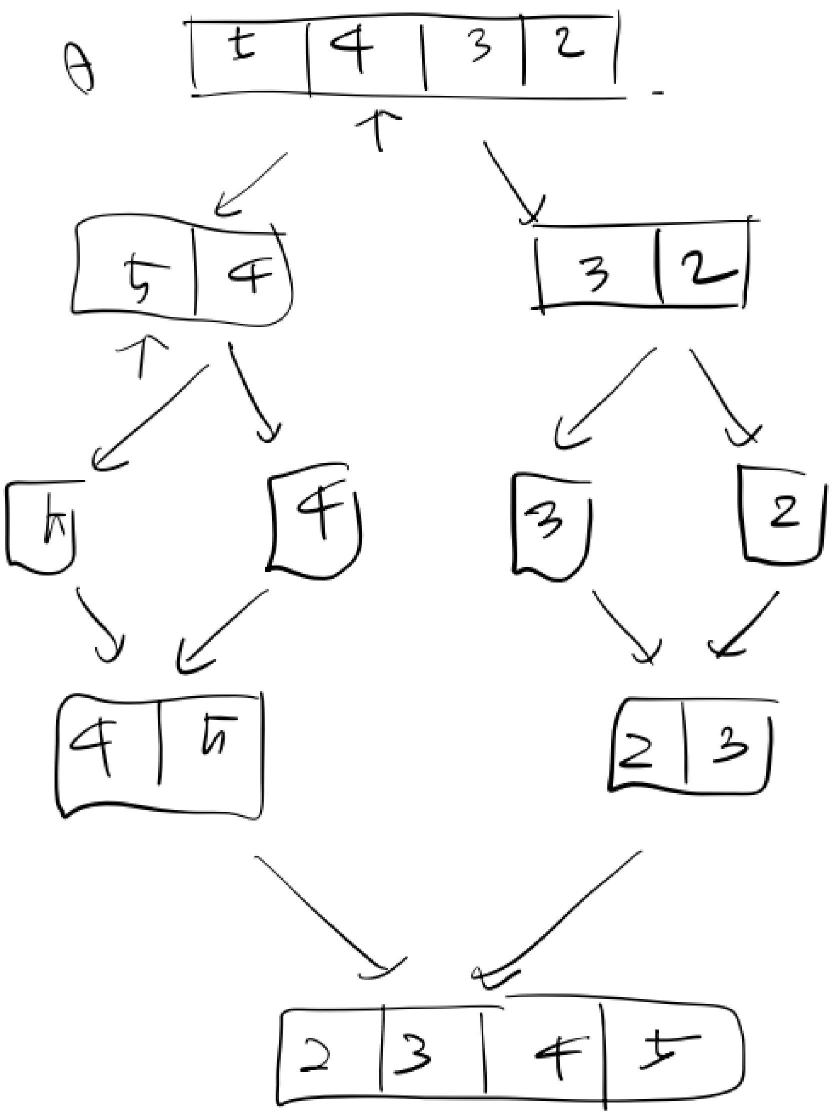
5 Merge (A, 1, 1, 2)

	1	2
A	5	1

$\downarrow$

A	1	5

Example of n=4 and how it works



Complexity of Merge Process

```
MERGE(A, p, q, r)
1  n_L = q − p + 1          // length of A[p : q]
2  n_R = r − q              // length of A[q + 1 : r]
3  let L[0 : n_L − 1] and R[0 : n_R − 1] be new arrays
4  for i = 0 to n_L − 1     // copy A[p : q] into L[0 : n_L − 1]
5      L[i] = A[p + i]
6  for j = 0 to n_R − 1     // copy A[q + 1 : r] into R[0 : n_R − 1]
7      R[j] = A[q + j + 1]
8  i = 0                    // i indexes the smallest remaining element in L
9  j = 0                    // j indexes the smallest remaining element in R
10 k = p                    // k indexes the location in A to fill
11 // As long as each of the arrays L and R contains an unmerged element,
   //       copy the smallest unmerged element back into A[p : r].
12 while i < n_L and j < n_R
13     if L[i] ≤ R[j]
14         A[k] = L[i]
15         i = i + 1
16     else A[k] = R[j]
17         j = j + 1
18     k = k + 1
19 // Having gone through one of L and R entirely, copy the
   //       remainder of the other to the end of A[p : r].
20 while i < n_L
21     A[k] = L[i]
22     i = i + 1
23     k = k + 1
24 while j < n_R
25     A[k] = R[j]
26     j = j + 1
27     k = k + 1
```

size n complexity

4-11: $\Theta(n_L + n_R)$

$= \Theta(n)$

12-27:

comp: $n/2$ at least

$\Theta(n)$ n at most

copy: n

$\Theta(n)$

Therefore, it takes $\theta(n)$

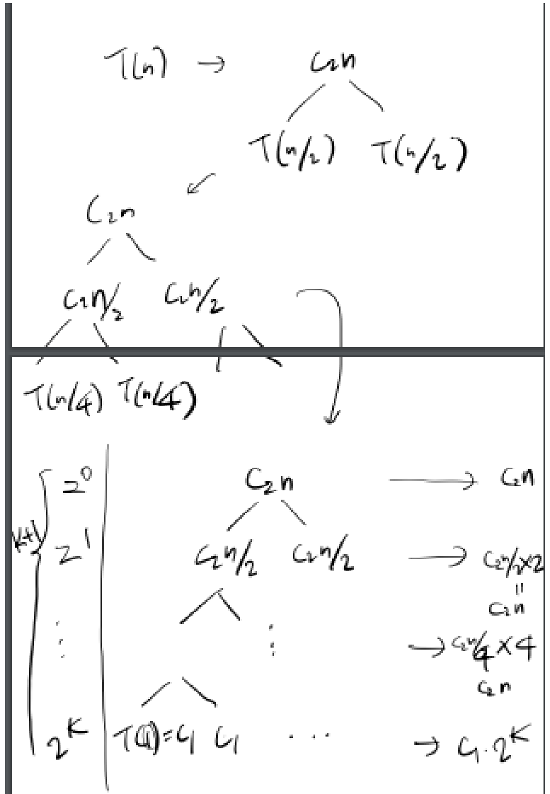
Complexity of Merge Sort: Recurrence Relation

```
MERGE-SORT(A, p, r)
  if p ≥ r                              // zero or one element?
      return
  q = ⌊(p + r)/2⌋                       // midpoint of A[p : r]
  MERGE-SORT(A, p, q)                   // recursively sort A[p : q]
  MERGE-SORT(A, q + 1, r)               // recursively sort A[q + 1 : r]
  // Merge A[p : q] and A[q + 1 : r] into A[p : r].
  MERGE(A, p, q, r)
```

$$T(n) = \begin{cases} T(1) = c_1 : & n = 1 \\ 2T(n/2) + \Theta(n) & \end{cases}$$

$$\Theta(n) = c_2 n$$

Complexity of Merge Sort: Time Complexity Calculation



$$n/2^K = 1 \Rightarrow K = \log_2 n$$

$$T(n) = c_2 n \times K + c_1 \cdot 2^K$$

$$= c_2 n \log n + c_1 n$$

$$= \Theta(n \log n)$$

Question?