# Chapter 8. Sorting in Linear Time

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# Assignment

- Read §8
- Problems
  - §8.2 - 3
  - §8.3 - 3, 5
  - §8.4 - 2

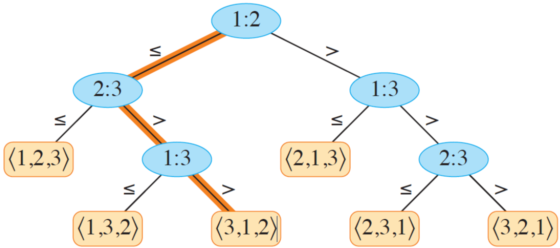
# Chapter 8: Sorting in Linear Time

- **Chapter 8.1: Lower Bounds for Sorting**
- Chapter 8.2: Counting Sort
- Chapter 8.3: Radix Sort
- Chapter 8.4: Bucket Sort

# Why Assume Distinct Elements in Lower Bound Proofs?

- We aim to prove a **lower bound** for comparison-based sorting algorithms.
- Sorting with **duplicates** can be easier:
  - Some comparisons may be skipped.
  - Fewer rearrangements may be required.
- To make the argument general, we assume all elements are **distinct**.
- **Key Insight:** Proving a lower bound for distinct elements means the bound also holds when duplicates are allowed.
- Comparisons like $a_i = a_j$ are useless when all elements are distinct.
- So, we only consider comparisons of the form $a_i < a_j$.

# The Decision-Tree Model




- A **comparison sort** uses only element comparisons like $a_i < a_j$ to determine order.
- We model these sorts using a **decision tree**:
  - A full binary tree where each internal node represents a comparison $a_i : a_j$
  - Each leaf corresponds to a permutation of the input
- The path from root to leaf = sequence of comparisons made for a specific input

# Key Properties of the Decision Tree

- Internal nodes: comparisons $a_i : a_j$
- Leaves: permutations $< \pi(1), \pi(2), \ldots, \pi(n) >$
- The tree must have at least $n!$ **reachable leaves** to represent all permutations
- Execution follows a single root-to-leaf path, depending on comparison outcomes
- The height of the tree $=$ **worst-case number of comparisons**

# Theorem 8.1: Lower Bound for Comparison Sorts

**Statement:** Any comparison sort algorithm requires $\boxed{\Omega(n \log n)}$ comparisons in the worst case.

**Proof:**

- A correct comparison sort must distinguish all $n!$ input permutations.
- Therefore, its decision tree has at least $n!$ **reachable leaves**.
- Any binary tree of height $h$ has at most $2^h$ leaves.

$$n! \leq 2^h \Rightarrow h \geq \log_2(n!)$$

- Apply Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \Rightarrow \quad \log_2(n!) = \Theta(n \log n)$$

- Thus, worst-case comparisons: $\boxed{\Omega(n \log n)}$

# Corollary 8.2: Asymptotic Optimality of Heapsort and Merge Sort

**Corollary:** *Heapsort and merge sort are asymptotically optimal comparison sorts.*

**Justification:**

- By Theorem 8.1, any comparison sort requires $\Omega(n \log n)$ comparisons in the worst case.
- Both heapsort and merge sort achieve a worst-case time of $O(n \log n)$.
- Therefore, they match the lower bound up to constant factors:

$$\boxed{\text{Worst-case time} = \Theta(n \log n)}$$

**Conclusion:** No comparison sort is asymptotically faster than heapsort or merge sort.

# Chapter 8: Sorting in Linear Time

- Chapter 8.1: Lower Bounds for Sorting
- **Chapter 8.2: Counting Sort**
- Chapter 8.3: Radix Sort
- Chapter 8.4: Bucket Sort

# Counting Sort — Introduction

- Counting sort assumes that the input array contains $n$ integers in the range 0 to $k$, for some integer $k$.
- It runs in $\Theta(n + k)$ time.
- **When $k = O(n)$, counting sort runs in $\Theta(n)$ time.**
- Unlike comparison sorts, it uses the **value of the input** to determine position.
- Key idea:
  - For each input $x$, count how many elements are $\leq x$
  - Place $x$ directly into its final position in the output array

**Example:** If 17 elements are $\leq x$, then $x$ goes into position 17.

# Understanding COUNTING-SORT Step-by-Step

1. **Initialize:** Zero out array $C[0 \dots k]$
2. **Count:** For each value $x$ in $A$, increment $C[x]$
3. **Accumulate:** Compute how many values are $\leq i$ by cumulative sum
4. **Place:** For each $A[j]$, set $B[C[A[j]]] = A[j]$, and then decrement $C[A[j]]$

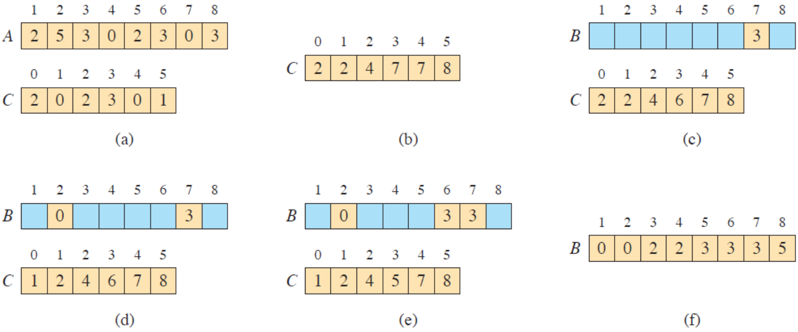
**Key point:** Decrementing $C[A[j]]$ handles duplicates by placing them in stable, ordered positions.

**Note:** We process $A$ in reverse (from $n$ to 1) to ensure **stability**.

# COUNTING-SORT Procedure

```
COUNTING-SORT(A, n, k)
let B[1 : n] and C[0 : k] be new arrays
for i = 0 to k
    C[i] = 0
for j = 1 to n
    C[A[j]] = C[A[j]] + 1
// C[i] now contains the number of elements equal to i.
for i = 1 to k
    C[i] = C[i] + C[i - 1]
// C[i] now contains the number of elements less than or equal to i.
// Copy A to B, starting from the end of A.
for j = n downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1    // to handle duplicate values
return B
```

# COUNTING-SORT Diagram



# Time Complexity of Counting Sort

- Let $n$: number of input elements, $k$: maximum value in the input
- Loop breakdown:
  - Lines 2–3: Initialize $C[0 \dots k] \to \Theta(k)$
  - Lines 4–5: Count frequencies from $A \to \Theta(n)$
  - Lines 7–8: Compute prefix sums $\to \Theta(k)$
  - Lines 11–13: Place elements into $B \to \Theta(n)$
- **Total running time:**

$$\boxed{\Theta(n+k)}$$

- When $k = O(n)$, counting sort runs in $\boxed{\Theta(n)}$ time.

# Why Counting Sort Can Beat $\Omega(n \log n)$

- In Section 8.1, we proved that any comparison sort must take $\Omega(n \log n)$ in the worst case.
- **Counting sort is not a comparison sort:**
  - It performs no comparisons between input elements.
  - Instead, it uses the **actual values** as indices into array $C$.
- Therefore, the lower bound of $\Omega(n \log n)$ does not apply.

# Stability of Counting Sort

- **Definition:** A sorting algorithm is **stable** if it preserves the relative order of equal elements.
- In counting sort, if two elements have the same value, the one appearing first in the input appears first in the output.
- Stability matters when elements carry **satellite data**.
- **Example:**
  - Input: [(5, A), (3, B), (5, C)]
  - Stable output: [(3, B), (5, A), (5, C)]
- **Conclusion:** Counting sort's stability is **required** for radix sort to work correctly.

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# Radix Sort and Card-Sorting Machines

- Radix sort was originally implemented using **mechanical card-sorting machines**.
- Each card had **80 columns**.
- In each column, a hole could be punched in **1 of 12 positions**.
- The sorter could be mechanically programmed to:
  - Examine a specific column
  - Distribute each card into one of 12 bins based on the punched position
- An operator would then **gather cards bin by bin**, so that cards with the first punch appear on top.

# Understanding Punch Card Encoding (IBM Standard)

- The top two unlabeled rows correspond to **zone punches**:
  - Top row = **row 12**
  - Second row = **row 11**
- Digits 0–9 are encoded by a single punch in rows 0–9.
- Letters are encoded by combining for instance, **Row 12 + 1–9** = A–I

# Encoding Digits in Punch Cards

- For decimal digits, each column used only **10 of the 12 punch positions**:
  - Rows 0 through 9 $\rightarrow$ digits 0 through 9
  - Rows 11 and 12 reserved for encoding letters or special characters
- A $d$-digit number occupies a field of **d columns**.
- Since the card sorter can look at only **one column at a time**, sorting $n$ cards by a $d$-digit number requires a sorting algorithm that processes each digit.

# The Need for a Sorting Algorithm

- You might intuitively think to sort numbers by the **most significant digit (MSD)**:
  - Sort by the leftmost digit
  - Recursively sort each bin by the next digit
  - Combine the bins in order to complete the sort
- This seems natural, but in physical systems, it presents a practical challenge...

# Drawback of MSD Sorting

- MSD-based sorting creates **many intermediate piles** of cards.
- After sorting by the first digit, you have 10 bins.
- To recursively sort one bin, the other 9 bins must be **set aside and tracked**.
- This results in increased space and logistical complexity, especially in manual sorting.

**Conclusion:** MSD sorting leads to extra work managing intermediate piles—hence a better algorithmic strategy is needed.

# Radix Sort: LSD-First Strategy and Stability

- Radix sort solves the problem of card sorting **counterintuitively** by sorting on the **least significant digit first**.
- Cards are first sorted by the rightmost digit:
  - Cards go into bins (0–9) based on that digit.
  - Bins are gathered in order: bin 0, then bin 1, ..., bin 9.
- Then the entire deck is sorted again on the next digit to the left, and recombined in the same way.
- This process continues until all $d$ digits have been sorted.
- **Remarkably**, at that point the cards are fully sorted by their entire $d$-digit number.
- Thus, radix sort requires only $d$ passes through the deck.

# Stability in Radix Sort

| | | | |
|---|---|---|---|
| 329 | 720 | 720 | 329 |
| 457 | 355 | 329 | 355 |
| 657 | 436 | 436 | 436 |
| 839 | 457 | 839 | 457 |
| 436 | 657 | 355 | 657 |
| 720 | 329 | 457 | 720 |
| 355 | 839 | 657 | 839 |

- For radix sort to work correctly, the digit-level sorting must be **stable**.
- Stability ensures that the relative order of elements with equal digits is preserved.
- Mechanical card sorters are stable by design.

# Radix Sort for Multi-Field Sorting (e.g., Dates)

- Radix sort can be used to sort **records with multiple fields**.
- Example: sorting dates by **year**, **month**, and **day**.
- One approach:
  - Use a comparison sort with a comparator that checks year first, then month, then day.
- Alternatively, use radix sort:
  - **First pass:** sort by day (least significant)
  - **Second pass:** sort by month
  - **Final pass:** sort by year (most significant)
- Requires each sort to be **stable**.

# RADIX-SORT Pseudocode

**RADIX-SORT assumes:**

- Each element in array $A[1 \dots n]$ has $d$ digits
- Digit 1 is the **least significant digit (LSD)**, and digit $d$ is the most significant

```
RADIX-SORT(A, n, d)
for i = 1 to d
    use a stable sort to sort array A[1 : n] on digit i
```

- The pseudocode does not specify which stable sort to use
- **COUNTING-SORT** is commonly used for digit-level sorting
- Why counting sort?
  - Runs in $\Theta(n + k)$, where $k$ is the digit range (e.g., 0–9)
  - Naturally stable and efficient for small-digit alphabets

# Time Complexity of RADIX-SORT

**Lemma 8.3.**
Given $n$ $d$-digit numbers in which each digit can take on up to $k$ possible values, **RADIX-SORT correctly sorts** these numbers in

$$\Theta(d(n+k))$$

time, assuming the stable sort used runs in $\Theta(n+k)$.

**Proof:**

- Each pass processes $n$ items, each with digit range 0 to $k-1$.
- COUNTING-SORT takes $\Theta(n+k)$ per pass.
- With $d$ passes, total running time is:

$$\boxed{\Theta(d(n+k))}$$

# Corollary: Linear-Time Sorting When $d$ and $k = O(n)$

- If $d$ is constant and $k = O(n)$, then:

$$\Theta(d(n+k)) = \Theta(n)$$

- So **radix sort runs in linear time** in this case.
- We also have flexibility in how we define "digits," especially for binary keys.

# Lemma 8.4: Binary Keys in Radix Sort

**Lemma 8.4.**
Given $n$ binary keys of $b$ bits each, and any positive integer $r \leq b$, radix sort can sort the keys in:

$$\boxed{\Theta\left(\frac{b}{r}(n + 2^r)\right)}$$

time, assuming the stable sort used takes $\Theta(n + k)$ time for inputs in the range $[0, k]$.

# Proof of Lemma 8.4

- A binary key is just a number made of $b$ bits (e.g., 32-bit integers).
- Radix sort requires us to define "digits" — but in binary, we get to choose what a digit means.
- Let $r$ be the number of bits per digit.
- Then:
  - Number of digits $d = \lceil b/r \rceil$
  - Each digit ranges from 0 to $2^r - 1$
  - Counting sort takes $\Theta(n + 2^r)$ per pass
  - Total time: $\Theta(d(n + 2^r)) = \Theta\left(\frac{b}{r}(n + 2^r)\right)$

# Example: Sorting 32-bit Keys with Byte-wise Digits

- Let $b = 32$, $r = 8$ (one digit = 1 byte).
- Then:
  - Number of digits: $d = b/r = 4$
  - Digit range: 0 to $2^8 - 1 = 255$
  - Each counting sort pass: $\Theta(n + 256)$
- Total cost:

$$\Theta(4(n + 256)) = \Theta(n)$$

- **Conclusion:** Efficient for fixed-width binary keys like 32-bit integers.

# Why Is This Useful?

- **Tuning** $r$ gives you flexibility:
  - Larger $r$: fewer passes, but larger bins
  - Smaller $r$: more passes, but smaller bins
- **Summary:** Lemma 8.4 gives a general formula to analyze radix sort's behavior on binary data.

# Chapter 8: Sorting in Linear Time

# Bucket Sort Overview

**Bucket sort** assumes the input consists of $n$ real numbers drawn **uniformly and independently** from the interval $(0, 1)$.

- Like **counting sort**, it achieves **average-case** running time of $\Theta(n)$ by making assumptions about the input.
- **Counting sort** assumes input integers lie in a small range.
- **Bucket sort** assumes inputs are randomly spread over $(0, 1)$ (uniform distribution).

**Key idea:** Partition the interval $(0, 1)$ into $n$ equal-sized buckets, distribute elements into these buckets, sort within each bucket, and concatenate.

# Bucket Sort: Why It Works Well

- Since inputs are **uniformly and independently** distributed:
  - Most buckets will have very few elements
  - This makes sorting inside each bucket fast
- We sort each bucket using a simple algorithm like **insertion sort**
- Finally, we **concatenate** the sorted buckets in order

**Conclusion:** With high probability, the total running time is $\Theta(n)$

# Bucket Sort Assumptions and Structure

**Assumptions:**

- Input array $A[1 \dots n]$ such that each $A[i] \in [0, 1)$

**Algorithm structure:**

- Create an auxiliary array $B[0 \dots n-1]$ of empty linked lists (buckets)
- For each input $A[i]$, place it in bucket:

$$B[\lfloor n \cdot A[i] \rfloor]$$

- Sort each bucket using **insertion sort**
- Scan all buckets in order and concatenate results

# Example Setup: Input Array $A[1 \ldots 10]$

**Input values (uniform in $[0, 1)$)**:

$$A = [0.78,\ 0.17,\ 0.39,\ 0.26,\ 0.72,\ 0.94,\ 0.21,\ 0.12,\ 0.23,\ 0.68]$$

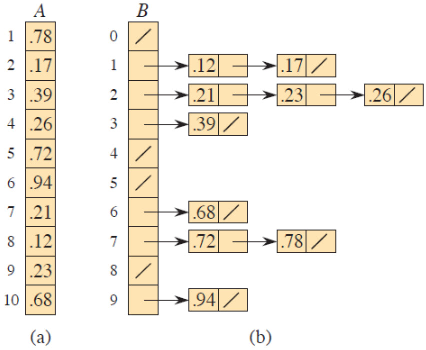
**Buckets** $B[0 \ldots 9]$ will be created. Each value goes into:

$$\text{Bucket } \lfloor 10 \cdot A[i] \rfloor$$

For example:

- $0.78 \to B[7]$
- $0.17 \to B[1]$
- $0.39 \to B[3]$
- $0.94 \to B[9]$

# Example of BUCKET-SORT (n = 10)




- (a) Input array $A[1 \ldots 10]$
- (b) Buckets $B[0 \ldots 9]$ after distributing and sorting
- Each bucket $B[i]$ holds values in the interval:

$$\left[\frac{i}{10}, \frac{i+1}{10}\right)$$

- Slashes indicate the end of each bucket
- Final output is the concatenation of all buckets in order

# BUCKET-SORT Pseudocode and Why It Works

BUCKET-SORT$(A, n)$

1 let $B[0:n-1]$ be a new array
2 **for** $i = 0$ **to** $n-1$
3     make $B[i]$ an empty list
4 **for** $i = 1$ **to** $n$
5     insert $A[i]$ into list $B[\lfloor n \cdot A[i] \rfloor]$
6 **for** $i = 0$ **to** $n-1$
7     sort list $B[i]$ with insertion sort
8 concatenate the lists $B[0], B[1], \ldots, B[n-1]$ together in order
9 **return** the concatenated lists

**Why it works:**

- For any two elements $A[i] \leq A[j]$:
  - If they go into the same bucket $\rightarrow$ insertion sort maintains order (line 7)
  - If they go into different buckets $\rightarrow$ lower-indexed bucket comes first (line 8)
- So final output is fully sorted

# Analyzing the Running Time of Bucket Sort

**Observation:** All lines **except line 7** (insertion sort) run in $O(n)$ time.

**Main cost:** Line 7 — insertion sort in each of the $n$ buckets.

- Let $n_i$ be the number of elements in bucket $B[i]$
- Insertion sort on each bucket takes $O(n_i^2)$

**Total time:**

$$T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2) \tag{8.1}$$

# Average-Case Running Time of BUCKET-SORT

We analyze the expected time based on the input distribution (uniform over $[0, 1)$).

$$\begin{aligned}
\mathbb{E}[T(n)] &= \mathbb{E}\left[\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)\right] \\
&= \Theta(n) + \sum_{i=0}^{n-1} \mathbb{E}\left[O(n_i^2)\right] \quad \text{(by linearity of expectation)} \\
&= \Theta(n) + \sum_{i=0}^{n-1} O\left(\mathbb{E}[n_i^2]\right) \qquad (8.2)
\end{aligned}$$

We now compute $\mathbb{E}[n_i^2]$ for each bucket $i \in \{0, 1, \ldots, n-1\}$

## Claim: $\mathbb{E}[n_i^2] = 2 - \frac{1}{n}$ (Equation 8.3)

Each input element is equally likely to fall into any of the $n$ buckets.

- Let $n_i$ be the number of elements in bucket $B[i]$
- View $n_i$ as the number of **successes** in $n$ Bernoulli trials:
  - Success: input falls into bucket $B[i]$
  - Probability of success: $p = 1/n$, failure $q = 1 - 1/n$
  - So $n_i \sim \text{Binomial}(n, 1/n)$

Using binomial distribution properties:

$$\mathbb{E}[n_i] = np = 1 \quad \text{and} \quad \text{Var}(n_i) = npq = 1 - \frac{1}{n}$$

From identity:

$$\mathbb{E}[n_i^2] = \text{Var}(n_i) + (\mathbb{E}[n_i])^2 = \left(1 - \frac{1}{n}\right) + 1^2 = 2 - \frac{1}{n} \qquad (8.3)$$

# Final Result: Linear Expected Time

Now plug Equation (8.3) into Equation (8.2):

$$\begin{aligned}
\mathbb{E}[T(n)] &= \Theta(n) + \sum_{i=0}^{n-1} O\left(2 - \frac{1}{n}\right) \\
&= \Theta(n) + n \cdot O\left(2 - \frac{1}{n}\right) \\
&= \Theta(n) + O(n) = \boxed{\Theta(n)}
\end{aligned}$$

**Conclusion:** Bucket sort runs in expected linear time under uniform distribution.

# Appendix - Binomial Distribution

The **binomial distribution** models the number of successes in $n$ independent yes/no trials, each with the same probability of success.

**Definition:** Let $X \sim \text{Binomial}(n, p)$, where:

- $n$: number of trials
- $p$: probability of success on each trial
- $q = 1 - p$: probability of failure
- $X$: number of successes in $n$ trials

**Properties:**

$$\mathbb{E}[X] = np, \quad \text{Var}(X) = np(1-p)$$

Binomial distribution is used to model bucket sizes in the analysis of bucket sort.

# Example: Tossing Coins

Imagine tossing a fair coin $n = 10$ times:

- Each toss has:
  - Success = heads, with probability $p = 0.5$
  - Failure = tails, with probability $q = 0.5$
- Let $X$ be the number of heads obtained
- Then $X \sim \text{Binomial}(10, 0.5)$

**Expected number of heads:**

$$\mathbb{E}[X] = 10 \cdot 0.5 = 5$$

# Question?