Computational Fluid Dynamics

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Introduction

In this project, we implemented a 2D fluid solver using the finite volume method to solve the Euler equations (reduced from the governing Navier-Stokes equation) for the time evolution of fluids. We solve the Euler equations in a 2D box $L_x \times L_y = 1 \times 1$ discretized into $N_x \times N_y = 514 \times 514$ cells with ghost cells at the boundary (therefore 512×512 physical cells) faciliated by periodic boundary conditions, so that the flow of the fluid will wrap around at the boundaries. The `.h` header files contain the computational method for solving the fluid equations, and output the data to a `.csv` file at each time step. The `.cpp` files contain the main function to run the simulation for a simple sound wave or the Kelvin-Helmholtz instability which can help us understand phenomena visible in atmospheres of Jupyter near the Giant Red Spot.

Sound Waves in a Fluid

For a simple test case to verify that our fluid solver works as intended and to test the stability of the simulation, we simulated a 2D soundwave in a fluid with a sinusoidal density perturbation. For sound soundwave we chose to simulate an ideal monoatomic gas—i.e. an adiabatic index $\gamma = 5.0/3.0$—initial density $\rho_0 = 1$, initial pressure $P_0 = 1$, and sinusoidal density perturbation $\rho_1 = 10^{-3}\sin(6\pi x)$. From the equation of states, we can define the speed at which the density fluctuations travel by

$$c = \sqrt{\gamma P_0/\rho_0} = \sqrt{5.0/3.0} \approx 1.29.$$

To estimate this velocity using the simulation data, we can track the position of one of the wave crests and measure the time it takes to travel back to its initial position. After this time t, the wave will have traveled a across 512 physical units, or the full length of the domain $L_x = 1.0$, so we can use the simple velocity formula $v = \Delta x/\Delta t$ to estimate the velocity of the wave. From the simulation data, the wave travels back to its initial position at $t \approx 0.77$, so the estimated velocity is

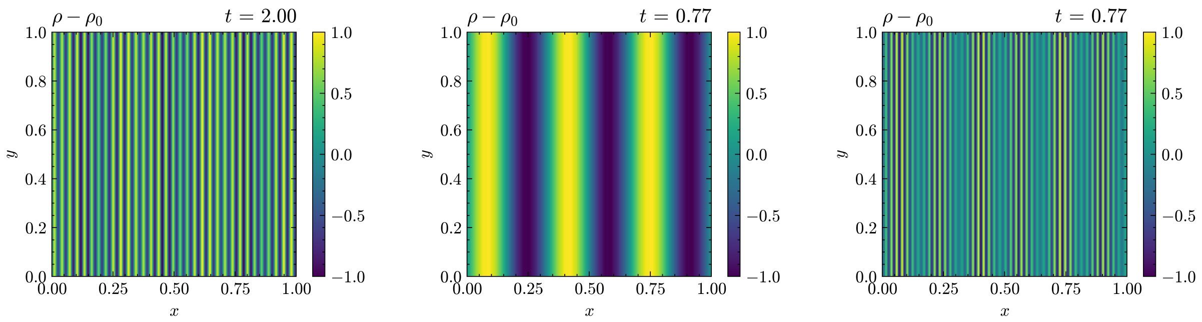
$$c_{\text{est}} = \frac{1.0}{0.77} \approx 1.30$$

This estimate has a perecent error of $\frac{1.30-1.29}{1.29} \times 100\% \approx 1\%$ compared to the theoretical value, so our simulation agrees with the theory even for a low resolution of 512x512 physical unit cells.

CFL Condition

The Courant-Friedrichs-Lewy (CFL) condition has a dimensionless factor C_{CFL} that determines the stability of a given numerical scheme. While running the soundwave simulation for different time intervals and CFL factors, we found that even for small values of C_{CFL}, the simulation would become unstable if the time interval was too large. For example, at $C_{\text{CFL}} = 0.4$, the simulation became visibly unstable at around $t = 2.0$ (Fig. 1a). Furthermore, `sound_wave_c2.mp4` shows the simulation at $C_{\text{CFL}} = 0.2$ ran for longer time, and the shape of the soundwave breaks down by $t = 7.0$.

So with this in mind, we confined the time interval to $t = 0.77$ which is the approximate time interval it takes for a wave crest to travel and return to its initial position. After testing several values of the CFL factor C_{CFL}, we found that the simulation was stable until we reached a maximum value of $C_{\max} \approx 0.5$. Moreover, when we set $C_{\text{CFL}} = 0.5$, the simulation was stable, but when we increased it to $C_{\text{CFL}} = 0.6$, the output at $t = 0.77$ becomes unstable,and the waves look like they break up into smaller waves and the structure of the initial soundwave becomes unrecognizable (Fig. 1c) by the end of the simulation.



(a) $C_{\mathrm{CFL}} = 0.4$ (unstable) (b) $C_{\mathrm{CFL}} = 0.5$ (stable) (c) $C_{\mathrm{CFL}} = 0.6$ (unstable)

Figure 1: 512x512 resolution density plot of the soundwave at time t for different CFL factors.

Kelvin-Helmholtz Instability

Kelvin-Helmholtz (KH) instability develops when 2 fluids of different densities meet each other. To simulate the fluid trajectories, we initialized a screen with x, y dimensions set to 1.0 x 1.0 using 512x512 resolution (defined by N_x and N_y which each include 2 ghost cells) similar to the soundwave simulation. Each position within the matrix corresponds to the mass density at that point. We used the following variables:

$$a = 0.25; b = 0.75;\ \rho_d = 4;\ \rho_0 = 1;\ v_0 = 0.5;\ P_0 = 2.5;\ k = 6.0\pi;\ v_{\mathrm{small}} = 0.01;\ \sigma = 0.05$$

Importantly, a and b describe the respective lower and upper boundaries where the 2 different fluids meet. To generate the initial condition for the KH wave, we used the following initial conditions:

$$\begin{aligned} a \le y \le b: &\quad \rho = \rho_d, \quad v_x = v_0 \\ y < a \text{ or } y > b: &\quad \rho = \rho_0, \quad v_x = -v_0 \end{aligned}$$

These initial conditions generate a slab of density ρ_d surrounded by regions of density ρ_0 with the fluids moving in opposite conditions. We then introduced a sine-wave velocity profile in the y direction to trigger instability at the boundary between both fluids:

$$v_y = v_{\mathrm{small}} \sin(kx)\left(e^{-(y-a)^2/\sigma^2} + e^{-(y-b)^2/\sigma^2}\right) \tag{1}$$

To parallelize the finite volume method, we added the `#pragma omp parallel for collapse(2)` directive at the start of each for-loop in the finite volume solver. However, we were surprised to only see a ~2x improvement in speed, whereas our speed improved significantly when we parallelized the code for the black hole project. This was probably bottlenecked by the csv `output()` function which writes the density data to a new file at each output time step `output_dt = 0.01`.

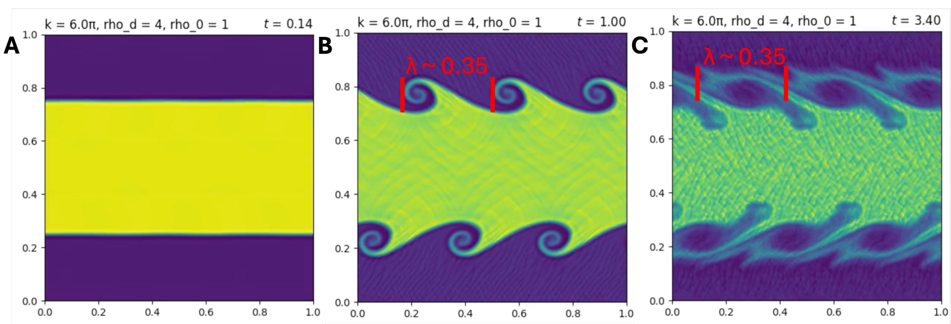


Figure 2: KH waves at different time stamps where increasing color brightness represents increasing density.

At the start the simulation, we see a high-density slab surrounded by low density regions **(Figure 2A)**. By time $t = 1.00$, an ordered pattern of vortices with $\lambda \sim 0.35$ **(Figure 2B)**. This is very close (maybe even exact) to the predicted wavelength of 1/3 which is calculated from $\lambda = 2\pi/k$ where k is the wavenumber that we specified. As time progresses, the vortices become larger and wider, but the wavelength remains the same **(Figure 2C)**. It should be noted that the high density and low-density regions seem to mix.

As we can see in Figure 2, the number wavelengths, or the number of swirling vortices, is is equal to the wavenumber divided by 2π, $k/2\pi = 3$. Furthermore, in the `kh_k12.mp4` video where we change the wavenumber to $k = 12\pi$, we can clearly see 6 swirling patterns form at the interfaces. When we take $k = 100\pi$ (`kh_k100.mp4`), the swirling patterns disappear and the movie is quite boring to watch. However, this happens because the wavelength $\lambda = 2\pi/k = 0.02$ is too short to be captured by the simulation resolution of 512x512 physical cells, so we would have to increase the resolution to capture the instability at this wavelength.

Random noise

To check the effect of the initial v_y perturbation, we changed the sine-wave instability condition to a random number generator `rand` that assigns a random wave number between 0 and 12π to each v_y, so (1) becomes

$$v_y = v_{\text{small}} \sin(\texttt{rand} \times x)\left(e^{-(y-a)^2/\sigma^2} + e^{-(y-b)^2/\sigma^2}\right)$$

This leads to the formation of a disordered pattern of vortices with undefined wavelengths **(Figure 3A, 3B)**. This makes sense since we are introducing a perturbation at many different wavelengths rather than a fixed one.

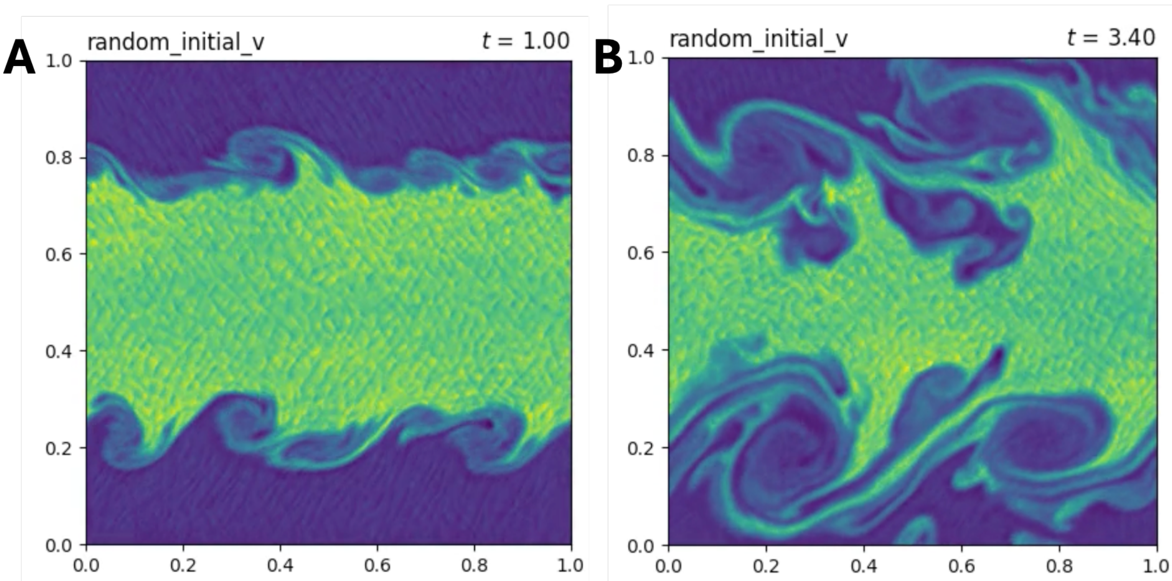
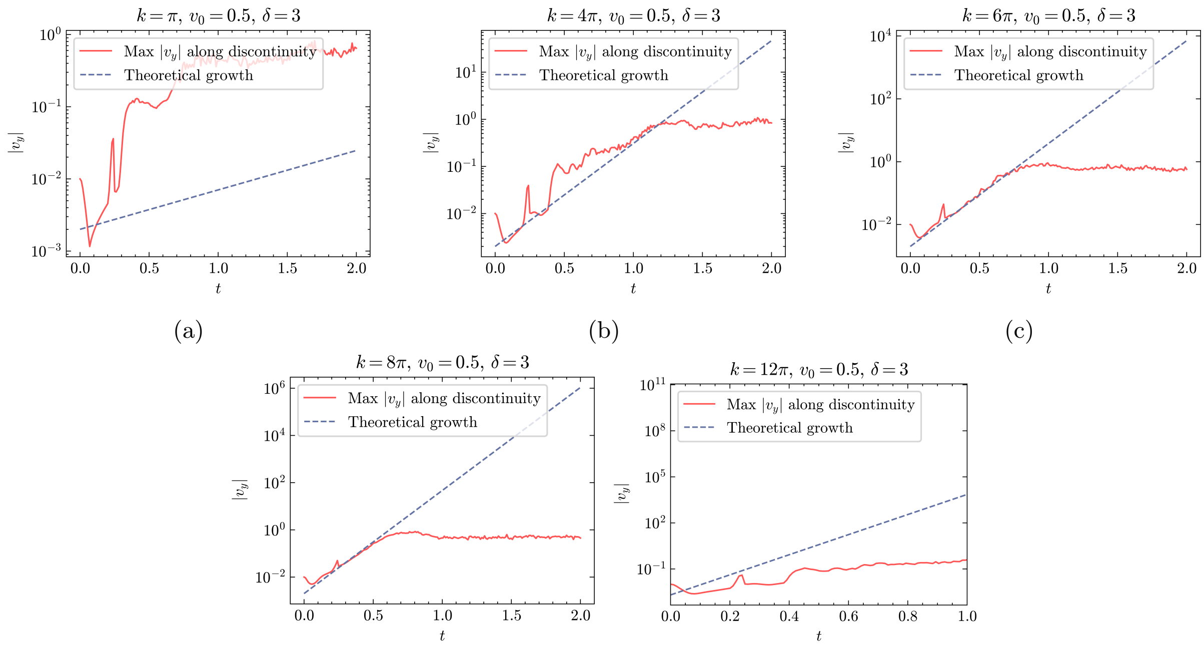


Figure 3: KH waves using $k = 6\pi$, $\rho_d = 4$, $\rho_0 = 1$ and a initial randomized wavenumber



(a) (b) (c) (d) (e)

Figure 4: Growth of Kelvin-Helmholtz instability for different k at 512x512 resolution. $\rho_0 = 1$, $\rho_d = 4$ (therefore $\delta = 3$), and theoretical growth curve $v_y = 0.002e^{\sigma t}$ for all simulations.

Growth rate and wave number

The theoretical growth rate of the Kelvin-Helmholtz instability for the initial v_y perturbation at the interfaces a and b is proportional to an exponential function

$$v_y \propto e^{\sigma t}, \quad \sigma = \frac{2\sqrt{1+\delta}}{2+\delta} k v_0$$

where $\rho_d = (1+\delta)\rho_0$ is the higher density side. In order to align the growth rate curve to the simulation results, we started with the $k = 6\pi$ cases and added a constant of of proportionality $D = 0.002$ to the theoretical growth rate $v_y = De^{\sigma t}$. This constant D was mostly determined by trial and error, but we found that $D = 0.002$ was a good fit for the $k = 4\pi$ and $k = 8\pi$ cases as well, so we used this value for all the theoretical growth rate curves.

Figure 4 shows that at $k = 6\pi, 8\pi$, the growth rate follows the theoretical prediction closely until around $t = 1.0$ where the $|v_y|$ begins to plateau and the instability becomes nonlinear. This nonlinear growth is expected as the vortices eventually merge and the fluid becomes more mixed and turbulent. For $k = \pi$, the growth rate does not follow the theoretical prediction which may be due to the long wavelength $\lambda = 2\pi/k = 2$ not being able to capture the sinusoidal perturbation at the interfaces a and b very well. For $k = 12\pi$, the max $|v_y|$ quickly falls away from the theoretical growth after $t \sim 0.2$, and here, the wavelengths may be getting too short for the simulation resolution to capture the the growth of the instability for an extended period of time.

Density contrasts

Next, we checked the influence of the 2 different fluid densities on the KH wave. Swapping ρ_d and ρ_0 only inverts the direction of the vortices and the density of each fluid region **(Figure 5A, 5B, 5D, 5E)**. When we set $\rho_d = \rho_0$, the vortices appear much quicker and is recognizable by t = 0.31 **(Figure 5G)**. As time evolves, we see homogenous mixing of both fluids and the vortices are spherical shaped rather than elongated **(Figure 5C, 5F)**. Interestingly, changing the $\rho_d : \rho_0$ ratio does not or barely affect the shape of the simulated growth rate curve **(Figure 5H)**. However, for our $\rho_d = \rho_0$ and swapped conditions, the theoretical growth rate deviates more than the original condition from $t = 0.10$ to 1.00 **(Figure 5H).** These results demonstrate that fluid density does not significantly affect the KH instability growth rate.

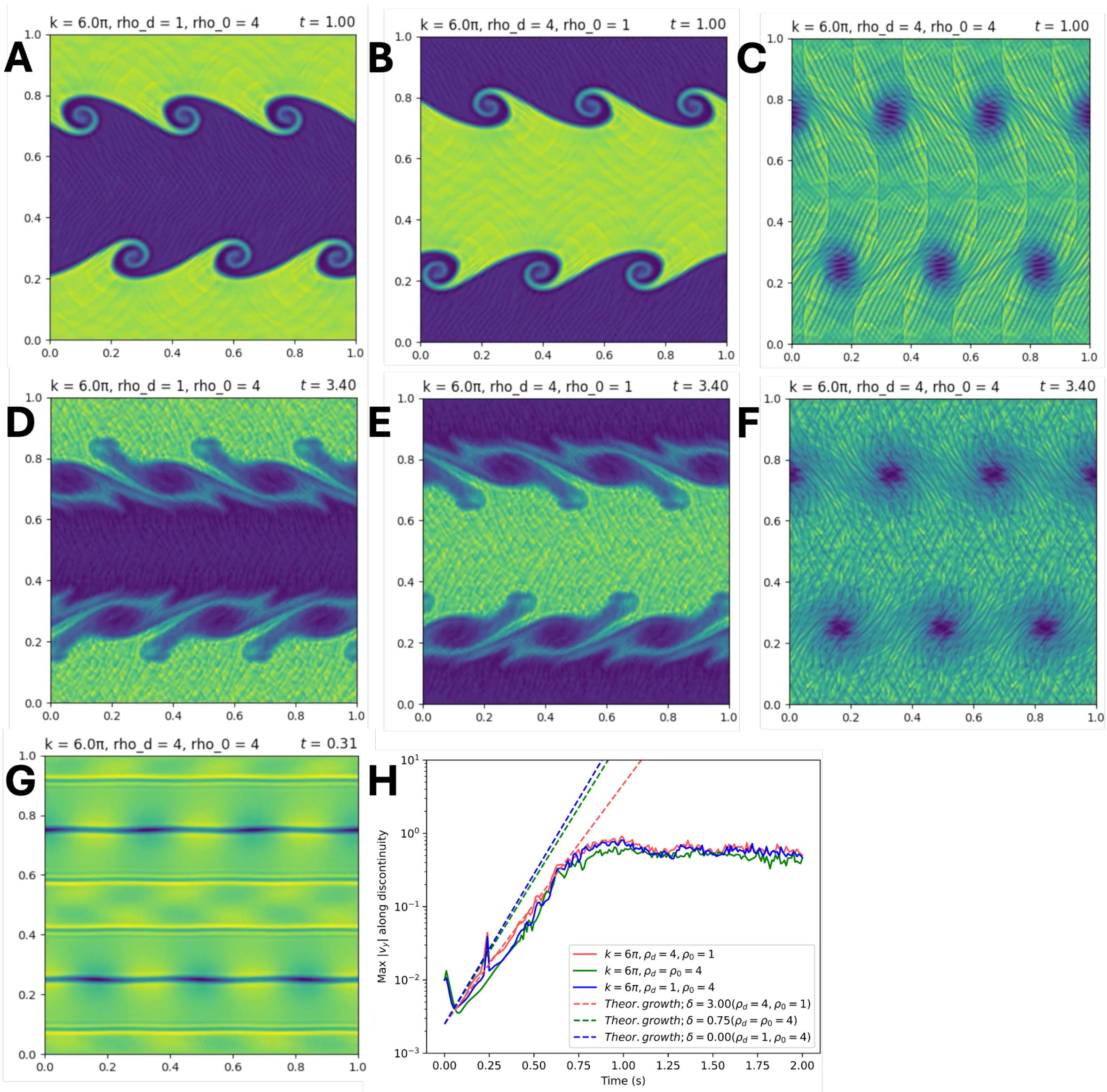


Figure 5: KH waves with different density regions