Analysis of the Fourier Transformation and its Applications

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The Fourier Transform converts a signal—e.g. AC and radio waves—initially presented in the time domain into the frequency domain. Decomposing a signal into its constituent frequencies allows more information content to be extracted from a signal thus the Fourier Transform became a powerful tool in signal processing and analysis. Modern technology, such as the Stanford Research Systems SR770, utilizes the Fast Fourier Transform (FFT); a numerical algorithm that computes the Discrete Fourier Transform (DFT) of a sequence (or continuous signal) in rapid time— i.e. reducing the computational complexity of a time series from $O(n^2)$ to $O(n \log n)$[1]. Although FFT allows for extremely fast and efficient frequency domain representation, there are limitations regarding the resolution of the frequency domain, its sensitivity to noise, and spectral leakage. Here we show how FFT can be used for finding signals amongst noise, tuning into AM radio, and identifying properties of the fluxgate magnetometer. Fourier Transformations present a new set of tools for understanding how a signal behaves in the frequency domain. We deciphered information in a particular signal by utilizing spectral density, harmonic analysis, and a rf (radio frequency) receiver module. From the three experiments, the analysis paradigm was rendered impossible in the time domain but is now possible in the frequency domain. We prospect that these experimental explorations will capture the utility and importance of Fourier methods in signal processing and analysis. Furthermore, an intuitive understanding of Fourier Transformations will reveal the hidden yet visible applications of Fourier methods in the modern world.

Famous mathematician and educator Gilbert Strang stated that the FFT is "the most important numerical algorithm of our lifetime"[2]. This was in 1994 before the invention of the iPhone, and only a year after the World Wide Web was released into public domain[3]. Now every handheld device utilizes a Fourier Transform in some fashion, from the cellular networks & WiFi signals that connect us to the internet to the software that processes the images we take. But first, we should understand what a Fourier Transform is before we can understand its applications:

The Fourier Transform, shown below, is an integral transform that takes the time series of a function/signal and outputs a function of frequency.

$$f(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} \tag{1}$$

From this, it becomes clear that for frequencies with associated harmonics, the integral product will be greater than zero, whereas those without will collapse to zero. Even more importantly in our new function, the product determined for each frequency is proportional to the strength of its harmonic within the Fourier Series. This can be best visualized using an oscilloscope (TDS1012) which displays data in the time domain and a Stanford Research spectrum analyzer 770 (SR770) which displays the frequency domain. Also, note that the SR770 uses FFT which splices the time series into discrete segments and efficiently computes the DCT at blazing speeds. These tools as well as a function generator and electronic modules provided by Teach Spin are the main tools used in our exploration of the Fourier Transform.

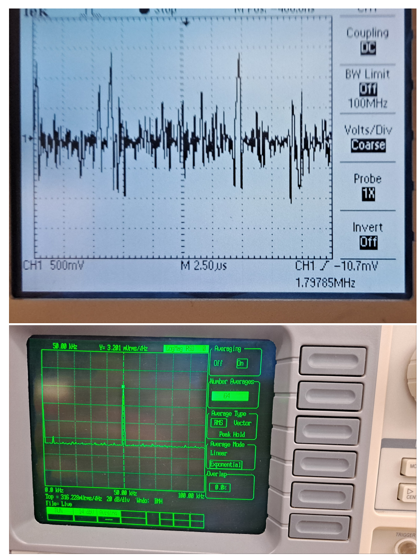

FIG. 1: Oscilloscope (top) and Spectrum (bottom) view for a 50 kHz, 5 V sine wave superimposed with noise

For our first experiment, we analyzed an unknown signal hidden under noise. Initially, as a proof of concept, we had a small amount of noise applied to a distinct strong signal. As shown in Fig. 1, using the time domain approach (scope) demonstrates an interesting but indecipherable signal. In the partnering image is the SR770 display in which it is almost impossible to miss the pure sinusoidal. We can also read the power of that individual signal itself. Due to the product of $f(x)$ and $e^{-i2\pi\xi x}$, each frequency in the spectrum is displayed in a way that is proportional to its power.

One could, theoretically and with great difficulty, examine the time domain signal and derive the pure sinusoidal, but becomes exceptionally difficult for a signal under much heavier noise as shown in Fig. 2.

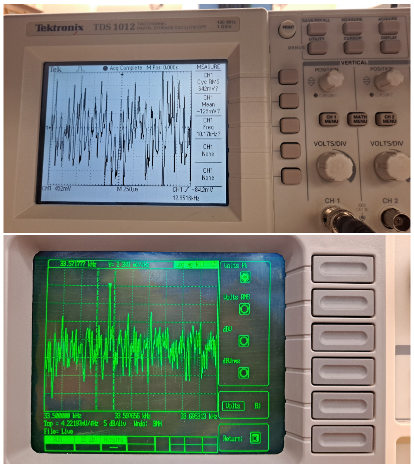

FIG. 2: Weak Signal under noise displayed in both the time and frequency domains.

In this time display it is much harder to understand the nature of the underlying pure waveform. A much weaker waveform has been superimposed upon stronger noise. Despite this, in the frequency domain, it is recoverable. So, how would one measure signal under noise?

Noise randomly sends power to each frequency with equal chance. Therefore if one takes an average of noise over enough time, the averages should resolve to a flat uniform distribution in the frequency domain.

FIG. 3: Spectral Density of Noise

Any interesting signal should be easily visible over the line as its average will vary from this. Practically speaking, the SSR770 takes its averages over a finite amount of time referred to as acquisition time. Thus the process involved in identifying a weak signal among noise is presented as translating the signal to the frequency domain, taking sufficient averages, and using a fine enough span to visually spot it. The act of taking those averages, however, is a process in itself. For a given noisy waveform, the amplitudes could be averages across a finite timespan, our SRR770 uses 4ms as the default, but since noise is statistically random, this would result in a 0 average, which is not useful. A better method would be to square the amplitudes first and then take the average, yielding the average power of the signal known as V_{rms}. We can then, in the frequency domain, divide by each frequency and get $V_{rms}/\sqrt{Hz}$. This new term we've derived, the voltage spectral density (VSD), is the actual value we read on the SSR770 and displays each frequency by its respective power contribution. Correspondingly the square of this value is the (PSD) or power spectral density and is the term used in calculations.

The power of noise is continuously distributed across the spectrum when we measure spectral density, so we can now compare the power (or mean-square voltage) in noise to the power of a sinusoidal signal. Furthermore, averaging the measurements will decrease the magnitude of the spectral density—i.e. lowering the noise floor—while the magnitude of the spectral density of the signal will be unaffected.

With the measure of spectral density, we were able to recover the signal under noise as long as we chose a small enough frequency span and swept through the full span of 100 kHz until we found a visible signal peak. By default, this method of signal recovery is limited by the acquisition time required for a given frequency resolution on the SR770[4] i.e. the frequency-duration "uncertainty principle"

$$f_{\text{resolution}} \cdot T_{acq} \geq \textbf{constant} \tag{2}$$

This dimensionless magic number attaches a frequency-resolution condition on the SR770. For example, the SR770 defines its magic number (with a default acquisition time of 4 ms for the full 100 kHz span) to be 400, so for a frequency span of 195 Hz shown in Fig 2 the acquisition time for a single measurement is 400/195 = 2 s. In addition, averaging 64 measurements to decrease the spectral density of the noise floor adds computational time as well. So we had to repeat this for roughly 170 measurements from 0 to 33.5 kHz as we swept across the full span until we found our small signal at 33.572 kHz shown in Fig 2.

Beyond just detection, the Fourier Transform can be useful in analyzing the properties of waveforms as well. In AM radio, program content consists of a variable signal, and the product of that and a pure carrier frequency creates an amplitude-modulated signal containing program data. This can be formulated as the equation:

$$V(t) = A[cos(2\pi f_c t) + 0.5cos(2\pi(f_c \pm f_p)t)] \tag{3}$$

where A is the fixed amplitude of the carrier frequency. This shows what an amplitude-modulated wave is, a central carrier frequency with a varying program frequency superimposed on it, and the variations in the 2nd term aka the program frequency, contain the data and content of the signal. This allows a higher frequency wave which has a better time propagating across space to "carry" a small program wave. Then at the receiver one simply has to use a local oscillator to "tune" to a specific carrier wave and then demodulate by filtering out that carrier, leaving only the program content. The first image in Fig 4 shows an example we generated, with the full signal above and the carrier sinusoidal below. This is a fairly interesting signal, however, just like in the previous case, we can only gain so much information from this. Moving to the frequency domain, we can better see the properties of the signal, The central peak is the carrier wave and the side bands demonstrate program content. This allows us to better view the entire AM spectrum. If, for instance as there are in real applications, numerous radio stations all broadcast in space, how would our devices tune to the one we desire without this methodology? Using Fourier Transformations and the frequency domain we can see each station and the program content is represented by the sidebands on either side.

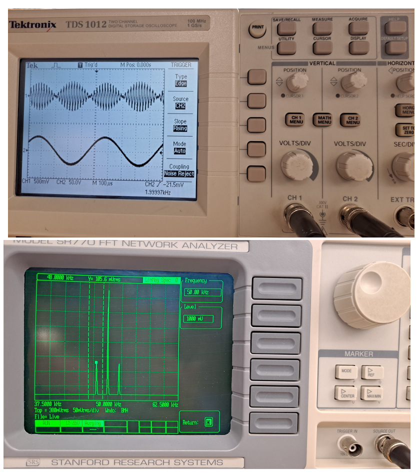

FIG. 4: Amplitude modulated signal in both domains.

Our third attempt at utilizing the Fourier methodology involves magnetic fields. In our next experiment, we examined how the Fourier Transform could be used with a magnetometer to ascertain the information content of magnetic fields. The fluxgate magnetometer is a sensor made up of a ferromagnetic core—specifically ferrite material to reduce eddy current loss—wrapped by a primary driving coil and secondary pickup coil. The driving coil sends an AC to the solenoid which generates an AC magnetic field. As the fluxgate sensor is subject to an external magnetic field, the generated emf is picked up by the secondary coil as an AC signal. The Fourier decomposition of this AC signal contains a 2nd harmonic component in response to the magnetization of the ferromagnetic core which is highly sensitive to an external magnetic field. In the experiment, the fluxgate magnetometer was able to detect external magnetic fields of the Earth and isolate multiple AC magnetic fields being detected by the fluxgate sensor using the SR770. Moreover, the fluxgate sensor becomes a real-time AC-field spectrometer that is highly sensitive to magnetic fields and low AC frequencies. Later on, we noticed that the fluxgate sensor was also affected by the magnetic fields produced by the nearby pulsed NMR machine which contains a 1.4 T or 2.1 T permanent magnet whereas the Earth's magnetic field on the surface is in the order of μT [5].

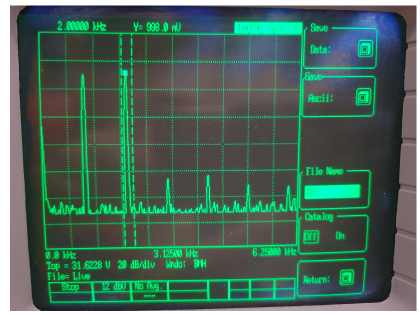

FIG. 5: The harmonics of the Magnetometer

Hopefully, the varied nature of our application and analyses will motivate the reader that the Fourier Transformation is a useful lens for any signal analysis. Viewing signals by their frequency domain should be, in the minds of these authors, a fundamental aspect of any data collection. From radio telescope data to improved sound recognition in audio processing, Fourier demonstrates often unknown properties. It can be applied to sensor data like in the magnetometer, but as we showed in the case of amplitude-modulated waveforms, it can also reveal emergent properties in cases that may not have called for its use.

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