Homework 9

5.14 For a steady current I flowing down a long cylindrical wire of radius a the magnetic field inside and out the wire

(a) Given a uniform surface current

$$\oint B \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}} \implies B(2\pi s) = \mu_0 I \pi$$

where s is the distance from the axis of the wire. Thus

$$\boxed{B = \frac{\mu_0 I}{2s} \quad s > a, \quad \text{and} \quad B = 0 \quad s < a}$$

which is points in the $\hat{\boldsymbol{\phi}}$ direction.

(b) For $\mathbf{J} \propto s$

$$J = ks \implies I = \int \mathbf{J} \cdot \mathrm{d}\mathbf{a} = \int_0^a (ks) 2\pi s \,\mathrm{d}s = \frac{2\pi k a^3}{3}$$

So the constant k is

$$k = \frac{3I}{2\pi a^3}$$

Outside the wire $I_{\text{enc}} = I$ so

$$\boxed{\mathbf{B} = \frac{\mu_0 I_{\text{enc}}}{2\pi s} \hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \quad s > a}$$

But for $s < a$ the current only a fraction of the current is enclosed:

$$I_{\text{enc}} = \int_0^s k s' (2\pi s') \,\mathrm{d}s' = \frac{2\pi k s^3}{3} = \frac{2\pi s^3}{3} \left(\frac{3I}{2\pi a^3} \right) = I \frac{s^3}{a^3}$$

Therefore

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi s} \left(\frac{s^3}{a^3} \right) \hat{\boldsymbol{\phi}} = \frac{\mu_0 I s^2}{2\pi a^3} \hat{\boldsymbol{\phi}} \quad s < a}$$

5.18 A parallel plate capacitor with uniform surface charge $\pm\sigma$ moves at constant speed v.

(a) Between the plates, the magnetic field is (using a rectangular Amperian loop perpendicular to the moving plate from Example 5.8)

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}} \implies B(2l) = \mu_0 K l$$

so at each region of a single plate

$$B = \pm\frac{\mu_0 K}{2} = \pm\frac{\mu_0 \sigma v}{2}$$

Above and below the plate, the magnetic field is zero

$$\boxed{B = 0 \quad \text{above and below the plate}}$$

and between the plates

$$\boxed{B = \mu_0 K = \mu_0 \sigma v \quad \text{between the plates}}$$

(b) The magnetic force per unit area on the upper plate is

$$\begin{aligned}
\frac{F_{\text{upper}}}{A} &= \frac{1}{A}\int K \times B \,\mathrm{d}A \\
&= K \times B \\
&= (\sigma v) \times \frac{\mu_0 \sigma v}{2} \\
&= \boxed{\frac{\mu_0 \sigma^2 v^2}{2}}
\end{aligned}$$

(c) The speed required to balance the electrical force: From electrostatics the force on the upper plate is

$$F = \sigma E = \sigma \frac{\sigma}{2\epsilon_0} = \frac{\sigma^2}{2\epsilon_0}$$

so for the forces to balance

$$\frac{\mu_0 \sigma^2 v^2}{2} = \frac{\sigma^2}{2\epsilon_0} \implies v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

where the values are related to the speed of light c by (From Wikipedia)

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \implies v = c$$

So the force will balance at the speed of light c.

5.26 What current density would produce the vector potential $A = k\hat{\boldsymbol{\phi}}$ in cylindrical coordinates? Finding $\mathbf{B} \to \mathbf{J}$:

$$\mathbf{B} = \boldsymbol{\nabla} \times A = \frac{1}{s}\begin{vmatrix} \hat{\boldsymbol{s}} & s\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & s(k) & 0 \end{vmatrix} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\hat{\mathbf{z}} = \frac{k}{s}\hat{\boldsymbol{z}}$$

So using Ampère's Law

$$\mathbf{J} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) = \frac{1}{\mu_0 s}\begin{vmatrix} \hat{\boldsymbol{s}} & s\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & k/s \end{vmatrix} = -\frac{1}{\mu_0 s} s\left(\frac{\partial}{\partial z}\frac{k}{s}\right)\hat{\phi} = \boxed{\frac{k}{\mu_0 s^2}\hat{\phi}}$$

5.32 To find the magnetic field inside a solid sphere of uniform charge density ρ, radius R, and angular velocity ω:

From Example 5.11 we just change $\sigma \to \rho$ so

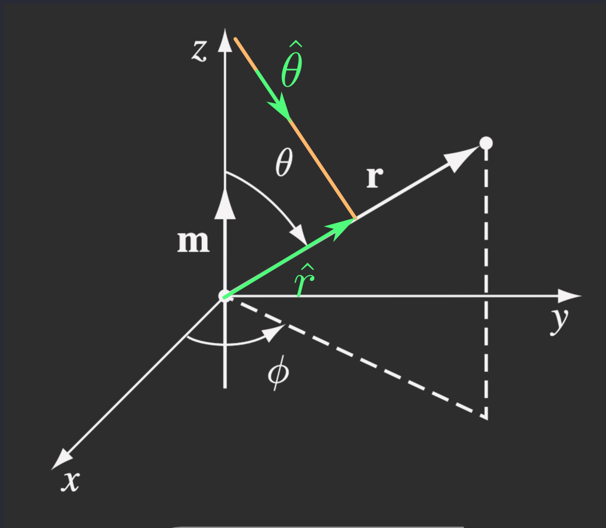
$$\mathbf{A}(r,\theta,\phi) = \begin{cases} \frac{1}{3}\mu_0 R' \rho\omega r \sin\theta\hat{\phi} & \text{Inside sphere} \\ \frac{1}{3r^2}\mu_0 R'^4 \rho\omega \sin\theta\hat{\phi} & \text{Outside sphere} \end{cases}$$

where we integrate R' from $0 \to r$ to find the potential $A_{\phi_{\text{out}}}$ and $r \to R$ to find $A_{\phi_{\text{in}}}$: The potential is then the sum of the two

$$\begin{aligned} \mathbf{A} &= \frac{1}{3}\hat{\boldsymbol{\phi}}\mu_0\rho\omega r\sin\theta\int_r^R R'\,\mathrm{d}R' + \frac{1}{3r^2}\mu_0\rho\omega\sin\theta\int_0^r R'^4\,\mathrm{d}R' \\ &= \frac{1}{3}\mu_0\rho\omega\sin\theta\left[\frac{1}{2}(R^2-r^2) + \frac{1}{r^3}\frac{r^5}{5}\right]\hat{\boldsymbol{\phi}} \\ &= \mu_0\rho\omega r\sin\theta\left[\frac{R^2}{6} - \frac{r^2}{6} + \frac{r^2}{15}\right]\hat{\boldsymbol{\phi}} \\ &= \mu_0\rho\omega r\sin\theta\left[\frac{R^2}{6} - \frac{r^2}{10}\right]\hat{\boldsymbol{\phi}} \end{aligned}$$

And using the curl in spherical coordinates

$$\begin{aligned} \boldsymbol{\nabla}\times A &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & (r\sin\theta)A_\phi \end{vmatrix} \\ &= \frac{1}{r^2\sin\theta}\left[\hat{\mathbf{r}}\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) - r\hat{\boldsymbol{\theta}}\frac{\partial}{\partial r}(r\sin\theta A_\phi)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\hat{\mathbf{r}}\frac{\partial}{\partial\theta}\left(\mu_0\rho\omega r^2\sin^2\theta\left[\frac{R^2}{6} - \frac{r^2}{10}\right]\right) - r\hat{\boldsymbol{\theta}}\frac{\partial}{\partial r}\left(\mu_0\rho\omega\sin^2\theta\left[\frac{r^2R^2}{6} - \frac{r^4}{10}\right]\right)\right] \\ &= \frac{1}{r^2\sin\theta}\left[\hat{\mathbf{r}}\mu_0\rho\omega r^2(2\sin\theta\cos\theta)\left[\frac{R^2}{6} - \frac{r^2}{10}\right] - r\hat{\boldsymbol{\theta}}\mu_0\rho\omega\sin^2\theta\left[\frac{2rR^2}{6} - \frac{4r^3}{10}\right]\right] \\ \mathbf{B} &= \mu_0\rho\omega\left[\hat{\mathbf{r}}\cos\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right) - \hat{\boldsymbol{\theta}}\sin\theta\left(\frac{R^2}{3} - \frac{2r^2}{5}\right)\right] \end{aligned}$$



5.36 Show that thet magnetic dipole can be written in coordinate free form:

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \tag{5.89}$$

From the Figure we can see that

$$\mathbf{m}\cdot\hat{\mathbf{r}} = m\cos\theta, \quad \mathbf{m}\cdot\hat{\boldsymbol{\theta}} = -m\sin\theta$$

so

$$\mathbf{m} = (\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + (\mathbf{m}\cdot\hat{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}} = m\cos\theta\hat{\mathbf{r}} - m\sin\theta\hat{\boldsymbol{\theta}}$$

Therefore

$$3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m} = 3m\cos\theta\hat{\mathbf{r}} - (m\cos\theta\hat{\mathbf{r}} - m\sin\theta\hat{\boldsymbol{\theta}}) = 2m\cos\theta\hat{\mathbf{r}} + m\sin\theta\hat{\boldsymbol{\theta}}$$

or from the textbook

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) \tag{5.88}$$

5.39

(a) A phonograph of radius R, surface charge σ rotating at ω, find its magnetic dipole moment: For a ring $\mathrm{d}r$ using $v = \omega r$ and $I = \sigma v \,\mathrm{d}r$ we have

$$m \equiv I \int \mathrm{d}\mathbf{A} = \sigma v(\pi r^2)\,\mathrm{d}r = \pi\sigma\omega r^3\,\mathrm{d}r$$

and integrating over the entire disk

$$m = \pi\sigma\omega \int_0^R r^3\,\mathrm{d}r = \frac{\pi\sigma\omega R^4}{4}$$

(b) Find the magnetic dipole moment of a spinning spherical shell: For one ring of width $R\,\mathrm{d}\theta$ around the sphere where $v = C/T$. For circumference $C = 2\pi R\sin\theta$ and time $T = 2\pi/\omega$, the current is

$$I = \sigma v(R\,\mathrm{d}\theta) = \sigma\frac{2\pi R\sin\theta}{2\pi/\omega}\,\mathrm{d}\theta = \sigma\omega R^2\sin\theta\,\mathrm{d}\theta$$

Using the area around the ring $A = \pi(R\sin\theta)^2$, we can integrate over the entire sphere $\theta \in [0, \pi]$:

$$\begin{aligned}
m = IA = \int \pi\sigma\omega R^2\sin\theta(R\sin\theta)^2\,\mathrm{d}\theta &= \pi\sigma\omega R^4\int_0^\pi \sin^3\theta\,\mathrm{d}\theta \\
= \pi\sigma\omega R^4\int(\sin^2\theta)\sin\theta\,\mathrm{d}\theta &= \pi\sigma\omega R^4\int(1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta \\
\text{using}\quad \cos\theta = u &\implies -\sin\theta\,\mathrm{d}u = \mathrm{d}\theta \\
= \pi\sigma\omega R^4\int(u^2-1)\,\mathrm{d}u &= \pi\sigma\omega R^4\left[\frac{u^3}{3} - u\right]_1^{-1} = \boxed{\frac{4}{3}\pi\sigma\omega R^4}
\end{aligned}$$

For $r > R$ the dipole term in the multipole expansion is

$$\mathbf{A}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}} = \frac{\mu_0}{4\pi}\frac{4}{3}\pi\sigma\omega R^4\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}} = \boxed{\frac{1}{3r^2}\mu_0\sigma\omega R^4\sin\theta\hat{\boldsymbol{\phi}}}$$

which is the same term from Problem 5.32 $\mathbf{A} = \mathbf{A}_{\text{dip}}$ thus a perfect dipole for points outside the sphere.