**3.4**

(a) Average field over a spherical surface due to charges outside the sphere is the same at the center:

For a charge $q$ a distance $z$ above the center of the sphere, we can use the same geometrical argument from HW 2 Problem 2.7: The average field at over the surface will be in the negative $z$ direction

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\psi(-\hat{\mathbf{z}})$$

where (using law of cosines)

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta \qquad \cos\phi = \frac{z - R\cos\theta}{\mathscr{r}}$$

The surface element is $\mathrm{d}a = R^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$, so the average field is

$$\begin{aligned}
\mathbf{E}_{\text{avg}} &= \frac{1}{4\pi R^2}\frac{1}{4\pi\epsilon_0}(-qR^2)\hat{\mathbf{z}}\int_0^{2\pi}\int_0^{\pi}\frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{4\pi}\frac{1}{4\pi\epsilon_0}(-q)\hat{\mathbf{z}}(2\pi)\int_0^{\pi}\frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{-q}{2}\hat{\mathbf{z}}f
\end{aligned}$$

The integral evaluates to (from Problem 2.7): Using the substitution $u = \cos\theta$: $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$, and the limits of integration are $[\cos 0, \cos\pi]$. So,

$$\begin{aligned}
f(\theta) &= -\int_1^{-1}\frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}}\,\mathrm{d}u \\
&= \int_{-1}^{1}\frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}}\,\mathrm{d}u
\end{aligned}$$

substituting again with $v = \sqrt{z^2 + R^2 - 2zRu}$; $\mathrm{d}v = \frac{zR}{v}\,\mathrm{d}u$; and $u = \frac{1}{2zR}(z^2 + R^2 - v^2)$:

$$\begin{aligned}
f(v) &= -\frac{1}{zR}\int\frac{z - \frac{1}{2z}(z^2 + R^2 - v^2)}{v^3}v\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{2z^2 - (z^2 + R^2 - v^2)}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{v^2 + z^2 - R^2}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\left(1 + \frac{z^2 - R^2}{v^2}\right)\mathrm{d}v \\
&= -\frac{1}{2z^2R}\left(v - \frac{z^2 - R^2}{v}\right)
\end{aligned}$$

substituting back in $v = \sqrt{z^2 + R^2 - 2zRu}$,

$$\begin{aligned}
f(u) &= -\frac{1}{2z^2R}\left(\frac{z^2+R^2-2zRu}{\sqrt{z^2+R^2-2zRu}} - \frac{z^2-R^2}{\sqrt{z^2+R^2-2zRu}}\right)\Bigg|_{-1}^{1} \\
&= -\frac{1}{2z^2R}\left(\frac{2R^2-2zRu}{\sqrt{z^2+R^2-2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{zu-R}{\sqrt{z^2+R^2-2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{z-R}{\sqrt{z^2+R^2-2zR}} - \frac{-z-R}{\sqrt{z^2+R^2+2zR}}\right) \\
&= \frac{1}{z^2}\left(\frac{z-R}{\sqrt{z^2+R^2-2zR}} + \frac{z+R}{z+R}\right) \\
&= \frac{1}{z^2}\left(\frac{z-R}{\sqrt{z^2+R^2-2zR}} + 1\right)
\end{aligned}$$

where the positive root $\sqrt{z^2+R^2-2zR} = (z-R)$ for $z > R$, so

$$\mathbf{E}_{\text{avg}} = \frac{1}{4\pi\epsilon_0}\left(-\frac{q}{2z^2}\right)\left(\frac{z-R}{z-R} + \frac{z+R}{z+R}\right)\hat{\mathbf{z}}$$

Simplfying to

$$\boxed{\mathbf{E}_{\text{avg}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}}$$

which is the same as the field at the center of the sphere. For a collection of particles, we can use superposition and find the net field as the sum of the fields at the center from each charge.

(b) For charges inside the sphere we can use the result from before: for one charge

$$\mathbf{E}_{\text{avg}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left(\frac{z-R}{\sqrt{z^2+R^2-2zR}} + \frac{z+R}{z+R}\right)\hat{\mathbf{z}}$$

but now the positive root is $\sqrt{z^2+R^2-2zR} = (R-z)$ for $z < R$, so

$$\begin{aligned}
\mathbf{E}_{\text{avg}} &= -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left(\frac{z-R}{R-z} + \frac{z+R}{z+R}\right)\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}(-1+1)\hat{\mathbf{z}} \\
\mathbf{E}_{\text{avg}} &= 0
\end{aligned}$$

And we can superimpose the fields from a collection of charges

$$\boxed{\mathbf{E}_{\text{avg}} = 0 + 0 + \cdots = 0}$$

**3.7** Charges $+q$ & $-2q$ are respectively $z = 3d$ & $z = d$ above the $xy$ plane (grounded conductor). Find the force of the charge $+q$:

We can use the method of images and replace the grounded conductor with two charges $-q$ at $z = -3d$ and $+2q$ at $z = -d$. Thus the force on $+q$ is the superposition of the forces from the three charges: The seperation vectors are

$$\begin{aligned}
\mathscr{r}_{-2q} &= 2d\hat{\mathbf{z}} \\
\mathscr{r}_{+2q} &= 4d\hat{\mathbf{z}} \\
\mathscr{r}_{-q} &= 6d\hat{\mathbf{z}}
\end{aligned}$$

Finally, the force on charge $+q$ is

$$\begin{aligned}
\mathbf{F} &= \mathbf{F}_{-2q} + \mathbf{F}_{+2q} + \mathbf{F}_{-q} \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{-2q^2}{(2d)^2} + \frac{2q^2}{(4d)^2} + \frac{-q^2}{(6d)^2}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{36}\right)\hat{\mathbf{z}}
\end{aligned}$$

which simplifies to

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{29q^2}{72d^2}\hat{\mathbf{z}}}$$

**3.8** From Griffiths, where the configuration has another point charge

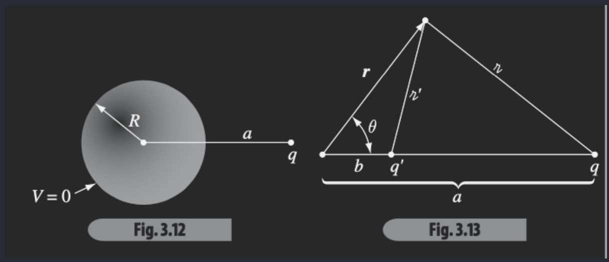


Figure 3.1: From Griffiths Example 3.2

$$q' = -\frac{R}{a}q \tag{3.15}$$

placed a distance

$$b = \frac{R^2}{a} \tag{3.16}$$

thus the potential of the config

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) \tag{3.17}$$

(a) Using law of cosines, show that Eq. (3.17) can be written as

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right]$$

From Fig. 3.1, we can see that

$$\mathscr{r} = \sqrt{r^2+a^2-2ra\cos\theta} \qquad \mathscr{r}' = \sqrt{r^2+b^2-2rb\cos\theta}$$

so using Eq. (3.15) and Eq. (3.16) we can rewrite

$$\begin{aligned}
\frac{q'}{\mathscr{r}'} &= \frac{-R}{a}\frac{q}{\sqrt{r^2+(R^2/a)^2-2r(R^2/a)\cos\theta}} \\
&= -\frac{1}{\sqrt{\frac{a^2}{R^2}}}\frac{q}{\sqrt{r^2+(R^2/a)^2-2r(R^2/a)\cos\theta}} \\
&= -\frac{q}{\sqrt{r^2a/R^2+(R^2/a)^2a^2/R^2-2r(R^2/a)\cos\theta a^2/R^2}} \\
\frac{q'}{\mathscr{r}'} &= -\frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}
\end{aligned}$$

Now we can rewrite Eq. (3.17) as

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right]$$

(b) Finding the induced charge on the sphere & integrating to get total induced charge: The normal component of the potential is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so

$$\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} \\
&= -\epsilon_0 \frac{1}{4\pi\epsilon_0} q \left(-\frac{1}{2}\right) \left[\frac{2r - 2a\cos\theta}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}} - \frac{2ra^2/R^2 - 2a\cos\theta}{(R^2 + (ra/R)^2 - 2ra\cos\theta)^{3/2}}\right]\bigg|_{r=R} \\
&= \frac{q}{4\pi}\left[\frac{R - a\cos\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} - \frac{a^2/R - a\cos\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}\right] \\
&= \frac{q}{4\pi}\left[\frac{R - a^2/R}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}\right]
\end{aligned}$$

which simplifies to

$$\boxed{\sigma(\theta) = \frac{q}{4\pi R}\frac{R^2 - a^2}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}}$$

Integrating to get the total induced charge using the surface element $\mathrm{d}a = R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$:

$$\begin{aligned}
Q &= \int \sigma \mathrm{d}a \\
Q &= \frac{q}{4\pi R}(R^2 - a^2)(2\pi R^2)\int_0^\pi \frac{\sin\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}\mathrm{d}\theta \\
\text{using} \quad u &= R^2 + a^2 - 2Ra\cos\theta; \quad \mathrm{d}u = 2Ra\sin\theta\,\mathrm{d}\theta \\
&= \frac{qR}{2}(R^2 - a^2)\frac{1}{2Ra}\int \frac{1}{u^{3/2}}\,\mathrm{d}u \\
&= \frac{qR}{2}(R^2 - a^2)\frac{1}{2Ra}\frac{-2}{\sqrt{u}} \\
&= -\frac{q}{2a}(R^2 - a^2)\frac{1}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}}\bigg|_0^\pi \\
&= -\frac{q}{2a}(R^2 - a^2)\left[\frac{1}{\sqrt{R^2 + a^2 + 2Ra}} - \frac{1}{\sqrt{R^2 + a^2 - 2Ra}}\right]
\end{aligned}$$

From Fig. 3.1, we can see that $R < a$ so the positive root is $\sqrt{R^2 + a^2 - 2Ra} = (a - R)$. Now the total induced charge is

$$\begin{aligned}
Q &= -\frac{q}{2a}(R^2 - a^2)\left[\frac{1}{a + R} - \frac{1}{a - R}\right] \\
&= -\frac{q}{2a}(R^2 - a^2)\left[\frac{a - R}{(a + R)(a - R)} - \frac{a + R}{(a - R)(a + R)}\right] \\
&= -\frac{q}{2a}(R^2 - a^2)\left[\frac{a - R - (a + R)}{a^2 - R^2}\right] \\
&= -\frac{q}{2a}(R^2 - a^2)\left[\frac{-2R}{-(R^2 - a^2)}\right]
\end{aligned}$$

Thus the total induced charge is

$$\boxed{Q = -\frac{R}{a}q = q'}$$

(c) The energy of the config:

First we find the force on $q$ due the induced charge $q'$ which are separated by a distance $a - b$:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a - b)^2}\hat{\mathbf{n}}$$

Using Eq. (3.15) and Eq. (3.16)

$$\begin{aligned}
\mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{R}{a}\frac{q^2}{(a-R^2/a)^2}\hat{\mathbf{a}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{R}{a}\frac{q^2}{\frac{1}{a^2}(a^2-R^2)^2}\hat{\mathbf{a}} \\
\mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{Raq^2}{(a^2-R^2)^2}\hat{\mathbf{a}}
\end{aligned}$$

Now we can determine the energy by calculating the work it takes to bring $q$ from infinity: The line element is $\mathrm{d}\ell = \mathrm{d}a\hat{\mathbf{a}}$ since the force is in the negative $a$ direction; so the work required to *oppose* the force is

$$\begin{aligned}
W &= -\int_{\infty}^{a} \mathbf{F}\cdot \mathrm{d}\ell \\
&= -\frac{1}{4\pi\epsilon_0}Rq^2\int_{\infty}^{a}\frac{a'}{(a'^2-R^2)^2}(-\,\mathrm{d}a') \\
\text{using}\quad & u = a'^2 - R^2;\quad \mathrm{d}u = 2a'\,\mathrm{d}a' \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\int\frac{1}{2u^2}\,\mathrm{d}u \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2u}\right] \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2(a^2-R^2)}\right]\Bigg|_{\infty}^{a} \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2(a^2-R^2)}-0\right]
\end{aligned}$$

which simplifies to

$$\boxed{W = -\frac{1}{4\pi\epsilon_0}\frac{Rq^2}{2(a^2-R^2)}}$$

**3.10** For a second image charge $q''$ inside the center of the sphere (it must not be outside the sphere) with potential

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) + V_0$$

where

$$V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{\mathscr{r}''} = \frac{1}{4\pi\epsilon_0}\frac{q''}{R} \implies q'' = 4\pi\epsilon_0 R V_0$$

So for a neutral conducting sphere the potential should be zero at the surface, i.e. the magnitude of the image charges $q'$ and $q''$ are equal and opposite:

$$q' = -q''$$

The distance from the second image charge and $q$ is $a$, so

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{qq''}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0} q\left[\frac{q' a^2}{(a^2-R^2)^2} - \frac{q'}{a^2}\frac{(a^2-R^2)^2}{(a^2-R^2)^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a^2-R^2)^2}\left[a^2 - a^2 + 2R^2 - R^4/a^2\right] \end{aligned}$$

Using Eq. (3.15) $q' = -\frac{R}{a}q$

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{(a^2-R^2)^2}\left(\frac{-R}{a}\right)\left[2R^2 - R^4/a^2\right] \\ &= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(a^2-R^2)^2}\left(\frac{R^3}{a^3}\right)\left[2a^2 - R^2\right] \end{aligned}$$

So the force of attraction has magnitude

$$\boxed{F_{\text{att}} = \frac{1}{4\pi\epsilon_0}\frac{q^2 R^3}{a^3(a^2-R^2)^2}\left[2a^2 - R^2\right]}$$

**3.11** Force between point charge $q$ and spherical conductor of total charge $q$:
We can use a second image charge (at the center of the sphere) where

$$q'' + q' = q$$

So the force between $q$ and the conductor is

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{qq''}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}q\left[\frac{q'a^2}{(a^2-R^2)^2} + \frac{q-q'}{a^2}\right] \end{aligned}$$

Using Eq. (3.15) $q' = -\frac{R}{a}q$:

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}q\left[\frac{-Rqa}{(a^2-R^2)^2} + \frac{q+Rq/a}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}q^2\left[-\frac{Ra}{(a^2-R^2)^2} + \frac{a+R}{a^3}\right] \end{aligned}$$

So the force is attractive when $[\,] < 0$, or if we define the critical value $a_c$ where

$$\begin{aligned} \frac{Ra_c}{(a_c^2-R^2)^2} &= \frac{a_c+R}{a_c^3} \\ Ra_c^4 &= (a_c^2-R^2)^2(a_c+R) \\ Ra_c^4 &= (a_c^4 - 2a_c^2R^2 + R^4)(a_c+R) \\ Ra_c^4 &= a_c^5 - 2a_c^3R^2 + R^4a_c + Ra_c^4 - 2a_c^2R^3 + R^5 \\ 0 &= a_c^5 - 2a_c^3R^2 - 2a_c^2R^3 + R^4a_c + R^5 \end{aligned}$$

From the hint, the solution must be in the form

$$a_c = R\frac{1+\sqrt{5}}{2}$$

which is the golden ratio i.e. in quadratic form

$$\phi^2 - \phi - 1 = 0 \implies \phi = \frac{1+\sqrt{5}}{2}$$

So

$$a_c = R\phi \implies \phi = \frac{a_c}{R}$$

With this intuition, we can divide our quintic equation by $R^5$:

$$\begin{aligned} 0 &= \frac{a_c^5}{R^5} - 2\frac{a_c^3}{R^3} - 2\frac{a_c^2}{R^2} + \frac{a_c^4}{R^4} + 1 \\ &= \phi^5 - 2\phi^3 - 2\phi^2 + \phi + 1 \end{aligned}$$

then we can factor it by dividing by the golden ratio equation $\phi^2 - \phi - 1 = 0$ (using polynomial long division):

$$\begin{array}{r|l} & x^3 + x^2 - 1 \\ x^2 - x - 1 & x^5 - 2x^3 - 2x^2 + x + 1 \\ & -x^5 + x^4 + x^3 \\ \hline & x^4 - x^3 - 2x^2 \\ & -x^4 + x^3 + x^2 \\ \hline & -x^2 + x + 1 \\ & x^2 - x - 1 \\ \hline & 0 \end{array}$$

Thus

$$0 = (\phi^2 - \phi - 1)(\phi^3 + \phi^2 - 1)$$

Where the quadratic equation factors to the golden ratio (positive root),

$$\phi^2 - \phi - 1 = 0$$
$$\implies \phi = \frac{1+\sqrt{5}}{2} = \frac{a_c}{R}$$
$$\implies a_c = R\frac{1+\sqrt{5}}{2}$$

So when $a < a_c$, $[\,] < 0$ i.e.

$$F = \frac{1}{4\pi\epsilon_0} q^2 [\,]$$

thus the magnitude of the force $F < 0$ which implies that the force is attractive.

Furthermore, in the cubic equation, there is a real root at approximately $\phi = 0.75488$ (using desmos/root-finding calculator)

$$\implies a_{c2} \approx 0.75488R$$

So, at $a < 0.75488R$ (pretending $\phi = a/R$)

$$(\phi^2 - \phi - 1)(\phi^3 + \phi^2 - 1) = (-C_1)(-C_2) = +C_3 = [\,] > 0$$

where $C_1, C_2, C_3$ are constants. So the force becomes repulsive again when $a < 0.75488R$.

**3.13** Two semi-infinite grounded conducting planes meeting as shown in Fig. 3.2 To set up the problem

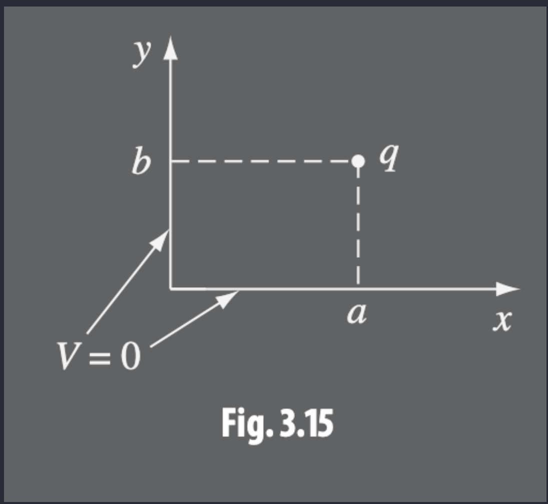


Figure 3.2: From Griffiths

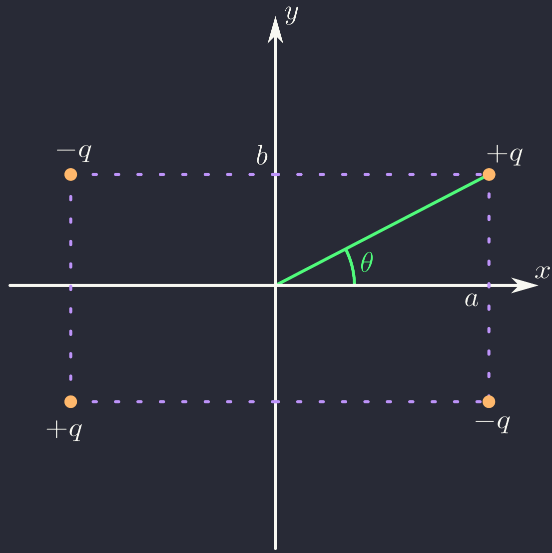


Figure 3.3: Three image charges

so we can have a potential of zero at the planes, we can place image charges $-q$ at $(a, -b)$ and $(-a, b)$, and place an image charge $+q$ at $(-a, -b)$ to balance the potentials at the axes.

The potential in the region $x > 0, y > 0$ is:

$$V = \frac{1}{4\pi\epsilon_0} q \left[ \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2 + z^2}} \right. \\ \left. - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} \right]$$

The force on $q$ is (using Fig. 3.3):

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} q^2 \left[ -\frac{1}{(2a)^2}\hat{\mathbf{x}} - \frac{1}{(2b)^2}\hat{\mathbf{y}} \right. \\ \left. + \frac{1}{(2\sqrt{a^2+b^2})^2}(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}) \right]$$

where

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

So

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left[ -\frac{1}{a^2}\hat{\mathbf{x}} - \frac{1}{b^2}\hat{\mathbf{y}} + \left(\frac{a}{(a^2+b^2)^{3/2}}\hat{\mathbf{x}} + \frac{b}{(a^2+b^2)^{3/2}}\hat{\mathbf{y}}\right)\right]$$

$$\boxed{\mathbf{F} = \frac{q^2}{16\pi\epsilon_0}\left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\hat{\mathbf{y}}\right]}$$

The work done to bring $q$ from infinity to the origin:
Integrating the opposing force from $\infty \to (a, b)$

$$
\begin{aligned}
W &= -\int_{\infty}^{(a,b)} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} \\
&= -\left[\int_{(\infty,\infty)}^{(a,\infty)} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell}_a + \int_{(a,\infty)}^{(a,b)} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell}_b\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\ell}_a = \mathrm{d}x\hat{\mathbf{x}} + (0)\hat{\mathbf{y}} = \mathrm{d}x\hat{\mathbf{x}}, &\quad \mathrm{d}\boldsymbol{\ell}_b = (0)\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}} = \mathrm{d}y\hat{\mathbf{y}} \\
\implies \int_{\infty,\infty}^{(a,\infty)} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell}_a &= \frac{q^2}{16\pi\epsilon_0} \int_{\infty}^{a} \left[\left(\frac{x}{(x^2+b^2)^{3/2}} - \frac{1}{x^2}\right)\mathrm{d}x\right]\Bigg|_{b=\infty} \\
&= \frac{q^2}{16\pi\epsilon_0} \int_{\infty}^{a} \left[-\frac{1}{x^2}\mathrm{d}x\right] \\
\text{and} \quad \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell}_b &= \frac{q^2}{16\pi\epsilon_0} \int_{\infty}^{b} \left[\left(\frac{y}{(a^2+y^2)^{3/2}} - \frac{1}{y^2}\right)\mathrm{d}y\right]
\end{aligned}
$$

So the work done is

$$
W = -\frac{q^2}{16\pi\epsilon_0}\left[\int_{\infty}^{a}\left(-\frac{1}{a^2}\right)\mathrm{d}x + \int_{\infty}^{b}\left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\mathrm{d}y\right]
$$

Evaluating the two integrals using

$$
\begin{aligned}
\int -\frac{1}{x^2}\mathrm{d}x &= \frac{1}{x} \\
\int \frac{y}{(a^2+y^2)^{3/2}}\mathrm{d}y &= -\frac{1}{\sqrt{a^2+y^2}}
\end{aligned}
$$

gives the work done is

$$
W = -\frac{q^2}{16\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{\sqrt{a^2+b^2}} + \frac{1}{b}\right]
$$

or

$$
\boxed{W = \frac{q^2}{16\pi\epsilon_0}\left[\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right]}
$$

We can solve the problem with the method of images, as long as the angle $\phi$ divides $180°$ into an integer, e.g., $\phi = 180, 90, 60, 45, 36, 30, 20, 18, 15, 12, 10, 9, 6, 5, 4, 3, 2, 1, 0.5, \ldots$

We would place a 'mirror' at each $\phi$ division and place an image charge $-q$ that mirrors the point charge $q$ and repeat the process with the next image charge $q$ (making sure to flip charges each time) until we have a symmetric configuration of charges as shown in Fig. 3.4.

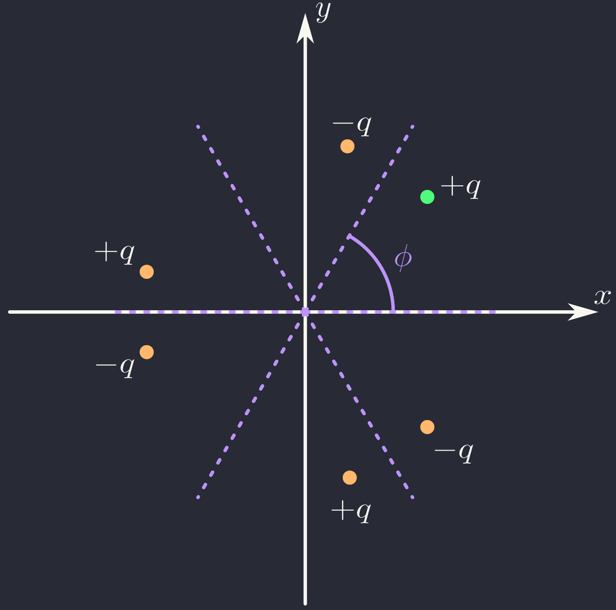


Figure 3.4: Method of images for $\phi = 60°$