Problem 1. (a) In the vacuum $\varphi_0(x,z) = A\cos(kx)e^{kz}$, and the electric field is the negative gradient of the electrostatic potential, so

$$\begin{aligned}\mathbf{E}_0 = -\boldsymbol{\nabla}\varphi_0 &= \left(-\frac{\partial\varphi_0}{\partial x}, -\frac{\partial\varphi_0}{\partial z}\right)\\ &= (kA\sin(kx)e^{kz}, kA\cos(kx)e^{kz})\end{aligned}$$

and the tangential component of the electric field $\mathbf{E}_{x0} = Ak\sin(kx)e^{kz}$ satisfies the boundary condition $\mathbf{E}_{xi} = kA\sin(kx)e^{-kz} = \mathbf{E}_{x0}$ for $z < 0$.

(b) The normal (or z) component of the Displacement field at the boundary is given as

$$D_{zi} = \epsilon(\boldsymbol{\omega})E_{zi} = \epsilon(\boldsymbol{\omega})kA\cos(kx)e^0 = \epsilon(\boldsymbol{\omega})kA\cos(kx)$$

for a vacuum we have

$$D_{z0} = E_{z0} = -\frac{\partial\varphi_0}{\partial z} = -kA\cos(kx)e^0 = -kA\cos(kx)$$

So for

$$\epsilon(\boldsymbol{\omega})kA\cos(kx) = -kA\cos(kx) \implies \epsilon(\boldsymbol{\omega}) = -1$$

And the dielectric function for a plasma is

$$\begin{aligned}\epsilon(\boldsymbol{\omega}) &= 1 - \frac{\omega_p^2}{\omega^2}\\ -1 &= 1 - \frac{\omega_p^2}{\omega^2} \implies \omega^2 = \frac{1}{2}\omega_p^2\end{aligned}$$

Problem 2. Metal 1 on the positive side of the interface can be treated as the plasma from Problem 1, and vice versa for Metal 2, so the dielectric functions are

$$\begin{aligned}\epsilon_1(\boldsymbol{\omega}) &= 1 - \frac{\omega_{p1}^2}{\omega^2}\\ \epsilon_2(\boldsymbol{\omega}) &= 1 - \frac{\omega_{p2}^2}{\omega^2}\end{aligned}$$

And the boundary conditions require the Displacement field to be continuous across the interface:

$$\begin{aligned}D_{z01} &= D_{z02}\\ \epsilon_1(\boldsymbol{\omega})\left[-\frac{\partial\varphi_{01}}{\partial z}\right] &= \epsilon_2(\boldsymbol{\omega})\left[-\frac{\partial\varphi_{02}}{\partial z}\right]\\ \epsilon_1(\boldsymbol{\omega})\left[-\frac{\partial}{\partial z}\left(A\cos(kx)e^{-kz}\right)\right] &= \epsilon_2(\boldsymbol{\omega})\left[-\frac{\partial}{\partial z}\left(A\cos(kx)e^{kz}\right)\right]\\ \epsilon_1(\boldsymbol{\omega}) &= -\epsilon_2(\boldsymbol{\omega})\end{aligned}$$

So the frequency associated with the interface is

$$\begin{aligned}1 - \frac{\omega_{p1}^2}{\omega^2} &= -\left(1 - \frac{\omega_{p2}^2}{\omega^2}\right)\\ 2 &= \frac{\omega_{p2}^2 + \omega_{p1}^2}{\omega^2}\\ \implies \omega &= \left[\frac{1}{2}(\omega_{p1}^2 + \omega_{p2}^2)\right]^{1/2}\end{aligned}$$

Problem 3. (a) Starting with the electromagnetic wave equation (53) from Kittel becomes

$$c^2K^2E^2 = \omega^2(E + 4\pi P) \to c^2K^2E^2 = \omega^2(\epsilon(\infty)E + 4\pi P)$$

or

$$E(\omega^2\epsilon(\infty) - c^2K^2) + P(4\pi\omega^2) = 0$$

and (54) remains

$$\begin{aligned} -\omega^2 P + \omega_T^2 P &= (Nq^2/M)E \\ \text{or} \quad E(Nq^2/M) + P(\omega^2 - \omega_T^2) &= 0 \end{aligned}$$

The two equations have a solution when the determinant of the matrix is zero:

$$\begin{vmatrix} \omega^2\epsilon(\infty) - c^2K^2 & 4\pi\omega^2 \\ Nq^2/M & \omega^2 - \omega_T^2 \end{vmatrix} = 0$$

so

$$\begin{aligned} [\omega^2\epsilon(\infty) - c^2K^2][\omega^2 - \omega_T^2] - 4\pi\omega^2\frac{Nq^2}{M} &= 0 \\ \omega^2[\omega^2\epsilon(\infty) - \omega_T^2\epsilon(\infty) - c^2K^2] + c^2K^2\omega_T^2 - 4\pi\omega^2\frac{Nq^2}{M} &= 0 \end{aligned}$$

at $K = 0$ we have a two roots for ω^2:

$$\begin{aligned} \omega^2\left[\omega^2\epsilon(\infty) - \omega_T^2\epsilon(\infty) - 4\pi\frac{Nq^2}{M}\right] &= 0 \\ \implies \omega^2 &= \omega_T^2 + \frac{4\pi Nq^2}{M\epsilon(\infty)} \end{aligned}$$

(b) For low ω we can neglect the ω^4 and $\omega^2c^2k^2$ which leaves us with

$$\begin{aligned} -\omega^2[\omega_T^2\epsilon(\infty) + 4\pi Nq^2/M] + c^2k^2\omega_T^2 = 0 \\ \implies \omega^2 &= \frac{c^2k^2\omega_T^2}{\omega_T^2\epsilon(\infty) + 4\pi Nq^2/M} \\ &= \frac{c^2k^2}{\epsilon(\infty) + 4\pi Nq^2/M\omega_T^2} \end{aligned}$$

where we know the dielectric function at $\omega = 0$ is from Kittel is

$$\epsilon(0) = \epsilon(\infty) + \frac{4\pi Nq^2}{M\omega_T^2} \tag{59}$$

so

$$\omega^2 = \frac{c^2k^2}{\epsilon(0)} \implies \omega = \frac{ck}{\sqrt{\epsilon(0)}}$$