Coupled Oscillators

Three springs in series and two carts We define equilibrium at $x_1 = 0, x_2 = 0$ and given the Lagrangian

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \\ U &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2 \end{aligned}$$

For the x_1 equation the EL equation gives

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= -k_1x_1 - k_2(x_1 - x_2) \\ \frac{\partial \mathcal{L}}{\partial \dot{x}_1} &= m_1\ddot{x}_1 m_1\ddot{x}_1 & &= -(k_1 + k_2)x_1 + k_2x_2 \end{aligned}$$

and for x_2 we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_2} &= k_2(x_1 - x_2) - k_3x_2 \\ \frac{\partial \mathcal{L}}{\partial \dot{x}_2} &= m_2\ddot{x}_2 \\ m_2\ddot{x}_2 &= k_2x_1 - (k_2 + k_3)x_2 \end{aligned}$$

We can rewrite these equations in matrix form

$$\begin{aligned} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} &= \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ M\ddot{\mathbf{x}} &= -K\mathbf{x} \end{aligned}$$

where we must find $\mathbf{x}(t)$. From oscillators we know that the solution is in the form

$$m\ddot{x} = -kx \implies x = x_0e^{\pm i\omega t}$$

so we can write the solution as

$$\mathbf{x}(t) = \mathbf{a}e^{i\omega t}$$

where we must find $\mathbf{A}$ and ω separately. Since

$$\ddot{\mathbf{x}} = -\omega^2\mathbf{A}e^{i\omega t} = -\omega^2\mathbf{x}$$

then we know that

$$\begin{aligned} -\omega^2M\mathbf{x} &= -K\mathbf{x} \\ \implies (K - \omega^2M)\mathbf{x} &= 0 \end{aligned}$$

so ω^2 is an eigenvalue of KM^{-1}

$$\implies \det\big(K - \omega^2M\big) = 0$$

With the assumption

$$m_1 = m_2 = m, \quad k_1 = k_2 = k_3 = k$$

we have

$$\det\begin{pmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{pmatrix} = 0$$
$$(2k-\omega^2 m)^2 - k^2 = 0$$

which is in the form $a^2 - b^2 = (a+b)(a-b) = 0$. So we have

$$(2k-\omega^2 m + k)(2k - \omega^2 m - k) = 0$$

which gives us the two solutions

$$\omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{3k}{m}$$

Question: Why not 4 solutions, i.e., $\pm\omega_1, \pm\omega_2$?
Plug in ω_1 into $K\mathbf{a} = \omega_1^2 M\mathbf{a}$ to find $\mathbf{a}$:

$$\begin{pmatrix} 2k-\omega_1^2 m & -k \\ -k & 2k-\omega_1^2 m \end{pmatrix} = \begin{pmatrix} k & -k \\ k & -k \end{pmatrix}\mathbf{a} = 0$$

with $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the solution is

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{a}(C_1 e^{i\omega t} + C_2 e^{i\omega t}) \\ &= A\mathbf{a}\cos(\omega_1 t - \delta) \end{aligned}$$

Which gives us the first normal mode

$$\begin{cases} x_1 = A\cos(\omega_1 t - \delta) \\ x_2 = A\cos(\omega_1 t - \delta) \end{cases}$$

This describes when the two carts are moving in phase. For the second normal mode:

$$\begin{aligned} (K - \omega_2 M)\mathbf{a} &= 0 \\ \begin{pmatrix} 2k-\omega_2^2 m & -k \\ -k & 2k-\omega_2^2 m \end{pmatrix}\mathbf{a} &= 0 \\ \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}\mathbf{a} &= 0 \end{aligned}$$

where $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so the solution is

$$\begin{aligned} \mathbf{x} &= A\mathbf{a}\cos(\omega_2 t - \delta) \\ \implies &\begin{cases} x_1 = A\cos(\omega_2 t - \delta) \\ x_2 = -A\cos(\omega_2 t - \delta) \end{cases} \end{aligned}$$

This describes when the two carts are moving in opposite directions. The generalized solution is a linear combination of the normal modes

$$\mathbf{x}(t) = A_1\mathbf{a}_1\cos(\omega_1 t - \delta_1) + A_2\mathbf{a}_2\cos(\omega_2 t - \delta_2)$$

which can describe the complicated motion of the two carts when they are not completely in or out of phase.

Normal Coordinates

$$\xi_1 = \frac{1}{2}(x_1 + x_2)$$
$$\xi_2 = \frac{1}{2}(x_1 - x_2)$$

$\implies \xi_1, \xi_2$ into the EOM:

$$\ddot{\xi}_1 = f(\xi_1)$$
$$\ddot{\xi}_2 = f(\xi_2)$$

decouples the equations.

Last Time:

$$M\ddot{\mathbf{x}} = -K\mathbf{x}$$

For an undiagonalized matrix K we have to solve

$$\ddot{\mathbf{x}} = M^{-1}K\mathbf{x}$$

where

$$\mathbf{x} = \mathbf{a}e^{\pm i\omega t} \implies \omega^2\mathbf{a} = M^{-1}K\mathbf{a}$$

where the general solution is a linear combination of the normal modes

$$\mathbf{x}(t) = A_1\mathbf{a}_1\cos(\omega_1 t - \delta_1) + A_2\mathbf{a}_2\cos(\omega_2 t - \delta_2)$$

Double Pendulum

The Potential energy is made up of two parts

$$\begin{aligned} U_1 &= m_1 g L_1(1-\cos\phi_1) \\ U_2 &= m_2 g L_1(1-\cos\phi_1) + m_2 g L_2(1-\cos\phi_2) \end{aligned}$$

And the two kinetic energies are

$$\begin{aligned} T_1 &= \frac{1}{2}m_1 L_1^2\dot{\phi}_1^2 \\ T_2 &= \frac{1}{2}m_2(L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)) \end{aligned}$$

where we use the Law of Cosines (or dot product), so the Lagrangian is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}(m_1+m_2)L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) \\ &- (m_1+m_2)gL_1(1-\cos\phi_1) - m_2gL_2(1-\cos\phi_2) \end{aligned}$$

Using a small angle approximation where both ϕ_1, ϕ_2 is small:

$$\cos\phi \approx 1 - \frac{\phi^2}{2}$$

we can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(m_1+m_2)L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2 \\ &- (m_1+m_2)gL_1\phi_1^2 - m_2gL_2\phi_2^2 \end{aligned}$$

where we use the second order terms in the potential energy, i.e.

$$T(\dot{\phi}_1, \dot{\phi}_2) \quad U(\phi_1, \phi_2)$$

So for the EL equations:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} &= (m_1+m_2)L_1^2\dot{\phi}_1 + m_2L_1L_2\dot{\phi}_2 \\ \frac{\partial\mathcal{L}}{\partial\phi_1} &= -(m_1+m_2)gL_1\phi_1 \\ \implies (m_1+m_2)L_1^2\ddot{\phi}_1 &+ m_2L_1L_2\ddot{\phi}_2 = -(m_1+m_2)gL_1\phi_1 \end{aligned}$$

and for ϕ_2:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = m_2 L_2^2 \dot{\phi}_2 + m_2 L_1 L_2 \dot{\phi}_1 \\
&\frac{\partial \mathcal{L}}{\partial \phi_2} = -m_2 g L_2 \phi_2 \\
&\implies m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2 = -m_2 g L_2 \phi_2
\end{aligned}$$

This is a matrix in the form

$$M = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{pmatrix} \quad K = \begin{pmatrix} (m_1+m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{pmatrix}$$

here the K matrix is diagonal (opposite from the previous example). But we solve this the same way

$$\det\bigl(K - \omega^2 M\bigr) = 0$$

where ω^2 is the eigenvalue of $M^{-1}K$.

Equal Mass and Length Case Assume $m_1 = m_2 = m, L_1 = L_2 = L$, then

$$M = mL^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad K = mgL \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

From the simple pendulum we know that

$$\omega_0 = \sqrt{\frac{g}{L}}, \quad g = \omega_0^2 L$$

so we can rewrite

$$K = m\omega_0^2 L^2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant of the matrix is

$$\det\bigl(K - \omega^2 M\bigr) = m^2 L^4 \begin{pmatrix} 2\omega_0^2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{pmatrix} = 0$$

which gives us the equation

$$\begin{aligned}
2(\omega_0^2 - \omega^2)^2 - \omega^4 = 0 \\
\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0
\end{aligned}$$

so the two normal frequencies are

$$\omega_1^2 = (2-\sqrt{2})\omega_0^2, \quad \omega_2^2 = (2+\sqrt{2})\omega_0^2$$

Normal Modes To find the normal modes, we substitute ω_1, ω_2 into the equation again:

$$\begin{aligned}
K - \omega_1^2 M &= mL^2\omega_0^2 \begin{pmatrix} 2-(4-2\sqrt{2}) & -(2-\sqrt{2}) \\ -(2-\sqrt{2}) & 1-(2-\sqrt{2}) \end{pmatrix} \\
&= mL^2\omega_0^2 \begin{pmatrix} 2\sqrt{2}-2 & 2-\sqrt{2} \\ 2-\sqrt{2} & \sqrt{2}-1 \end{pmatrix} \\
&= mL^2\omega_0^2(\sqrt{2}-1) \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}
\end{aligned}$$

So

$$(k - \omega_1^2 M)\mathbf{a}_1 = 0, \quad \mathbf{a}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

and the first normal mode is

$$\begin{aligned}\phi(t) &= A_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_1 t - \delta_1) \\ \phi_1(t) &= A_1 \cos(\omega_1 t - \delta_1) \\ \phi_2(t) &= \sqrt{2} A_1 \cos(\omega_1 t - \delta_1)\end{aligned}$$

where the two pendulums are moving exactly in phase (or the 2nd pendulum angle is always $\sqrt{2}$ times the first pendulum). The second normal mode is

$$\phi(t) = A_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_2 t - \delta_2)$$

Lecture 32: **4/15/24**

Review For the General Case of Coupled Oscillators:

$$M\ddot{q} = -K\mathbf{q}$$

where $\mathbf{q} = \mathbf{a}e^{i\omega t}$, and ω^2 is an eigenvalue of $M^{-1}K$.

$$(K - \omega^2 M)\mathbf{a} = 0 \implies \det\big(K - \omega^2 M\big) = 0$$

which gives the normal frequency, and to determine $\mathbf{a}$:

$$\mathbf{q} = \sum_i A_i \mathbf{a}_i \cos(\omega_i t - \delta_i)$$

where we have $2n$ unknowns and $2n$ initial conditions.

Nodes Of a String

For a string of mass M under tension T, we separate the string into small nodes of length ℓ, and the nodes deviate y_i to form a segmented wave. Assuming y_i is small: N2L gives

$$m\ddot{y}_2 = F_y = -T\sin\theta_1 - T\sin\theta_2$$
$$\sin\theta_1 = \frac{y_i - y_{i-1}}{\ell}, \quad \sin\theta_2 = \frac{y_i - y_{i+1}}{\ell}$$

and

$$\begin{aligned} m\ddot{y}_1 &= -T\frac{y_i - y_{i-1}}{l} - T\frac{y_i - y_{i+1}}{\ell} \\ &= \frac{T}{\ell}(y_{i-1} - 2y_i + y_{i+1}) \\ \implies M\ddot{\mathbf{y}} &= -K\mathbf{y} \end{aligned}$$

e.g. For $n = 2$ we have the two equations

$$\begin{aligned} i = 1: &\quad \ddot{y}_1 = \frac{T}{m\ell}(y_2 - 2y_1) \\ i = 2: &\quad \ddot{y}_2 = \frac{T}{m\ell}(y_1 - 2y_2) \end{aligned}$$

which can be written in matrix form

$$M = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad K = \frac{T}{m\ell}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and for n nodes we have a tri-diagonal matrix for K:

$$K = \frac{T}{m\ell}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

Solving for the normal modes where $n = 2$:

$$\det\big(K - \omega^2 M\big) = 0$$
$$\omega_1^2 = \omega_0^2, \quad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\omega_2^2 = 3\omega_0^2, \quad \mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$n \to \infty$? We take the limit of the continuous string...

$$m = \frac{M}{n} \to 0, \quad \ell = \frac{L}{n+1} \to 0$$

since these quanities go to zero, we have to define a nonzero quantity

$$\mu = \frac{M}{L} \approx \frac{m}{\ell}$$

The equation of motion is

$$\ddot{y}_i = \frac{T}{m\ell}(y_{i-1} - 2y_i + y_{i+1})$$

and since $y \to y(x), \qquad x \in [0, L]$, we can Taylor expand

$$y_{i+1} = y_i + y_i'\ell + \frac{1}{2}y_i''\ell^2 y_{i-1} = y_i - y_i'(-\ell) + \frac{1}{2}y_i''\ell^2$$

where the first two terms cancel out, so we have

$$\ddot{y} = \frac{T}{m\ell}y''\ell^2 = \frac{T}{\mu}y'' = c^2 y''$$

A solution to y is exponential in the form

$$y(x) = a(x)e^{i\omega t}$$
$$-\omega^2 a(x) = c^2 a''(x)$$
$$a'' = -\frac{\omega^2}{c^2}a = -k^2 a$$

where k is the wave vector. The general solution is

$$a(x) = C_1 \sin kx + C_2 \cos kx$$
$$a(0) = 0 \implies C_2 = 0$$
$$a(L) = 0 \implies \sin(kL) = 0 = \sin(n\pi) \implies k = \frac{n\pi}{L}$$

so

$$k_n = \frac{n\pi}{L}, \quad \omega_n = \frac{n\pi c}{L}$$

Which gives us

$$a_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$$
$$y(x,t) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)e^{i\omega_n t}$$

The initial conditions tell us

$$y(x,0) = f(x) = \sum_n A_n \sin\left(\frac{m\pi}{L}x\right)$$
$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x$$

this is from the Fourier coefficient:

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = \delta_{nm}$$