1 Electrodynamics

1.1 Electromotive Force

1.1.1 Ohm's Law

$$\mathbf{J} = \sigma \mathbf{f}$$

$\mathbf{J}$ is the current density, σ is the constant of proportionality (conductivity), and $\mathbf{F}$ is the force/charge i.e. the electric field $\mathbf{f}/q \equiv \mathbf{F} = \mathbf{E}$. So the electromagnetic force is

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where the velocity of the charge $\mathbf{v}$ is small enough which leaves us with "Ohm's Law"

$$\mathbf{J} = \sigma \mathbf{E}$$

Example A 'cylindrical' resistor with resistivity $\rho = 1/\sigma$, cross-sectional area A, and length L, under a potential difference V.

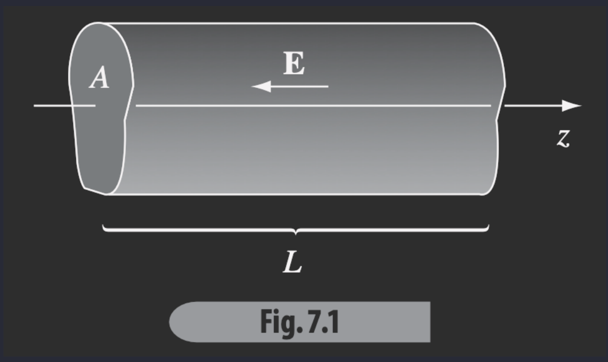

Figure 1.1: Cylindrical Resistor

The current I flowing through the resistor is given by

$$I = JA = \sigma EA = \sigma \frac{V}{L} A = \frac{\sigma A}{L} V$$

Example Coaxial cable with inner radius a, outer radius b, separated by a material with conductivity σ:

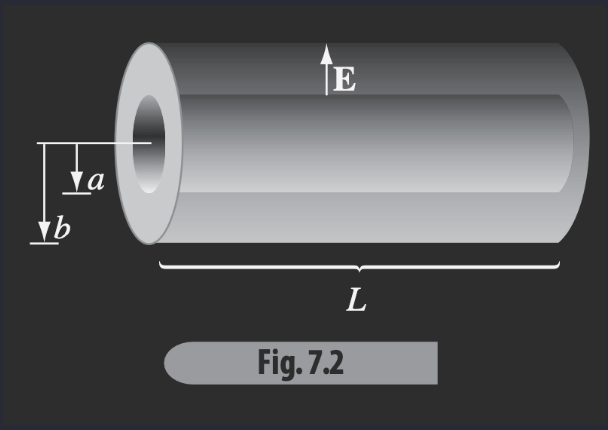

Figure 1.2: Coaxial Cable

The electric field will point radially outward in the region between the two cylinders:

$$E = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

where $\lambda \sim$ charge/length. We get the current

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma \int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}\lambda L$$

and the potential

$$V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$$

So we get

$$\boxed{I = \frac{2\pi\sigma L}{\ln(b/a)}V} \qquad \boxed{V = IR}$$

which is the more common form of Ohm's Law.

For simple steady currents, we know that

$$\boldsymbol{\nabla}\cdot\mathbf{J} = 0$$

$\rightarrow$ no source or sink of charge. Furthermore, from Ohm's law,

$$= \boldsymbol{\nabla}\cdot(\sigma\mathbf{E}) = \sigma\boldsymbol{\nabla}\cdot\mathbf{E} = 0$$

$\Longrightarrow$ uniform $\mathbf{E}$ in conductors carrying steady currents.

So $\mathbf{J} = \sigma\mathbf{E}$ tells us a steady current flows due to an electric field, *but* $\mathbf{F} = q\mathbf{E} = m\mathbf{a}$ tells us that the electrons should accelerate... Think of it like this: Electrons are like cars stopping at a stop sign; they accelerate but then stop when they reach the next stop sign, so they have a constant average velocity as they accelerate and stop. This average is roughly

$$v_{\text{avg}} = \frac{1}{2}a\tau = \frac{1}{2}\frac{a\lambda}{v_{\text{thermal}}} = \frac{1}{2}\frac{qE\lambda}{mv_{\text{thermal}}}$$

so for n moles per unit volume of charge and f electrons per molecule

$$\mathbf{J} = nfqv_{\text{avg}} = \left(\frac{nfq^2\lambda}{2mv_{\text{thermal}}}\right)\mathbf{E}$$

How much energy/time is transferred to conductor?

$$V \equiv \frac{W}{q}, \quad I \equiv \frac{q}{s} \implies P = IV = I^2R$$

aka the "Joule heating law".

1.1.2 Electromotive Force

$\mathcal{E}$ or EMF

Current can experience forces

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

where $\mathbf{f}$ is the total force/charge, and $\mathbf{f}_s$ can be chemical (from battery), mechanical (piezoelectric), etc.

The EMF can be the line integral

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} + \cancel{\oint \mathbf{E} \cdot d\mathbf{l}}$$

where the second terms cancels from Stokes's $\boldsymbol{\nabla} \times \mathbf{E} = 0$. This EMF has units of volts, but it is not a potential difference....

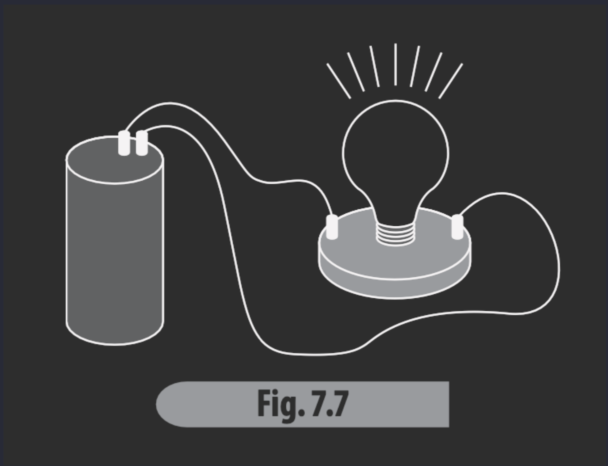

Figure 1.3: EMF

In the current wire connected to the battery (EMF), there is a $\sigma \sim 1/\rho \to R$ such that $V = IR$ or in terms of the EMF $\mathcal{E} = IR$. In the battery we idealize $\sigma \to \infty$ thus $R \to 0$.

1.1.3 Motional EMF

"(E)motional"

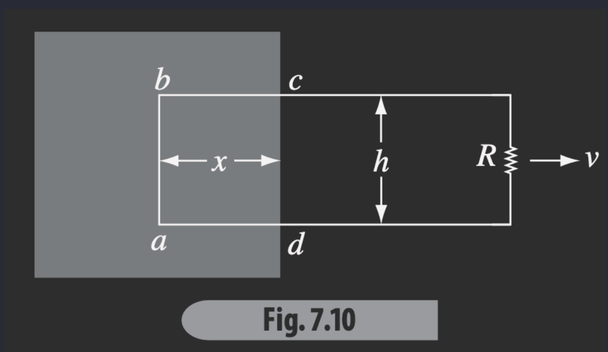

Figure 1.4: Motional EMF

For a B-field into the page, the current moves clockwise (RHR), so the EMF is providing a magnetic force and not just a potential difference:

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh = IR_{\text{wire}}$$

We can also think of the area in the magnetic field as a flux

$$\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{a} = Bhx = B\text{area}$$

so the time rate of change (as we pull the wire with velocity v) is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = Bh\frac{\mathrm{d}x}{\mathrm{d}t} = -Bhv$$

or generally

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

AKA “Faraday’s Law”.

1.2 Electromagnetic Induction

1.2.1 Faraday's Law

$\sim$ 1831ish Faraday observed three important experiments:

1. Moving a loop of wire rightward through a magnetic field induced a current in the wire.
2. Moving a magnet to the left also induced a current.
3. Increasing the magnetic field strength of a stationary magnet also induced a current on the stationary loop of wire.

(1.) Describes motional EMF, (2.) changes at rest! Empirically

$$\mathcal{E} = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \quad \Phi = BA$$

where $\Phi = \int_S \mathbf{B}(\mathbf{r}) \cdot \mathrm{d}\mathbf{a}$ so

$$\implies \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{a} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{a}$$

which gives us Faraday's Law

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

1.2.2 The Induced Electric Field

$$\boldsymbol{\nabla} \times E = 0 \rightarrow = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \rightarrow = 0$$

Where we can get the magnetostatics equations by changing the terms

$$\boldsymbol{\nabla} \times B = \mu_0 J, \qquad \boldsymbol{\nabla} \cdot B = 0$$

From the Biot-Savart Law,

$$\mathbf{E} = -\frac{1}{4\pi} \int \frac{\frac{\partial \mathbf{B}}{\partial t} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \mathrm{d}\tau = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B}(\mathbf{r}) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \mathrm{d}\tau$$

and using the Ampere's law integral form $\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{enc}}$, we have this integral form of Faraday's law

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

Example A uniform B-field $\mathbf{B} = B(t)\hat{\mathbf{z}}$ filling a circular region of radius s. When B changes, what is E?

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = E 2\pi s$$

so

$$= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}[\pi s^2 B(t)] = -\pi s^2 \frac{\partial B}{\partial t}$$

Thus we have a circular electric field

$$\mathbf{E} = -\frac{s}{2}\frac{\partial B}{\partial t}\hat{\boldsymbol{\phi}}$$

Example Ring of radius b with line charge λ concentric with a region with magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, but constrained to a region smaller than the ring a.

Again, an amperian loop of radius s so

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = E2\pi s = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\pi s^2 \frac{\partial B}{\partial t} \implies \mathbf{E} = -\frac{s}{2}\frac{\partial B}{\partial t}\hat{\boldsymbol{\phi}}$$

inside the magnetic field region. Outside the magnetic field region, we have

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = E2\pi s = -\pi a^2 \dot{\mathbf{B}} \implies \mathbf{E} = -\frac{a^2}{2s}\frac{\partial B}{\partial t}\hat{\boldsymbol{\phi}}$$

So at the ring or $s = b$,

$$\mathbf{E} = -\frac{a^2}{2b}\frac{\partial B}{\partial t}\hat{\boldsymbol{\phi}}$$

The torque on the segment of the ring is

$$F = dqE$$

so

$$\mathrm{d}\mathbf{N} = \mathbf{r} \times \mathbf{F} = b\mathrm{d}qE = b\lambda E\,\mathrm{d}l$$

The total toque is

$$\mathbf{N} = b\lambda\left(-\frac{a^2}{2b}\dot{B}\oint \mathrm{d}l = -b\lambda\pi a^2\dot{B}\hat{\boldsymbol{\phi}}\right)$$

And the angular momentum is

$$\int N dt = -\lambda\pi a^2 b\int_{B_0}^{0} \mathrm{d}B$$

so

$$\implies \Delta\text{ang momentum} = \lambda\pi a^2 bB_0$$