Solid Body Rotation

Last Week: Non-inertial Frames

1. Just linear acceleration $\mathbf{A}$, N2L

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}$$

2. Rotating frame:

$$m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \Omega + m(\Omega \times \mathbf{r}) \times \Omega$$

Solid body

N particles on a continuous distribution

$$\begin{aligned} m_\alpha, \qquad \alpha &= 1, 2, \ldots, N \\ \mathbf{r}_\alpha, \qquad \mathbf{r}_\alpha - \mathbf{r}_\beta &= \text{constant} \end{aligned}$$

With a center of mass (COM/CM)

$$\begin{aligned} \mathbf{R} = \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}_\alpha, &\qquad M = \sum_\alpha m_\alpha \\ \mathbf{P} = \sum_\alpha m_\alpha \mathbf{v}_\alpha &= \sum_\alpha m\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}} \\ \dot{\mathbf{P}} &= M\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}} \end{aligned}$$

Angular Momentum

$$\begin{aligned} \ell_\alpha &= \mathbf{r}_\alpha \times \mathbf{p}_\alpha \\ &= \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha \end{aligned}$$

and the total angular momentum

$$\mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha$$

Defining a position $\mathbf{r}'_\alpha$ relative to the CM

$$\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}, \quad \mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$$

we can rewrite the total angular momentum as

$$\begin{aligned} \mathbf{L} &= \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}'_\alpha) \times (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha) \\ &= \sum_\alpha m_\alpha \mathbf{R} \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{R} \times \dot{\mathbf{r}}'_\alpha + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha \end{aligned}$$

but since we know that

$$\begin{aligned} \mathbf{R} &= \frac{1}{M}\sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}'_\alpha) \\ &= \frac{1}{M}\sum_\alpha m_\alpha \mathbf{R} + \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}'_\alpha \\ \implies \sum_\alpha m_\alpha \mathbf{r}'_\alpha &= 0 \\ \sum_\alpha m_\alpha \dot{\mathbf{r}}'_\alpha &= 0 \end{aligned}$$

so the middle terms of the total angular momentum are zero:

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha$$

which can be re-expressed as

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_{\text{cm}} + \mathbf{L}_{\text{rel}} \\ \mathbf{L}_{\text{cm}} &= M\mathbf{R} \times \dot{\mathbf{R}} \\ \mathbf{L}_{\text{rel}} &= \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha \end{aligned}$$

For example we can consider the earth as a rigid body with angular momentum

$$\mathbf{L}_E = \mathbf{L}_{\text{spin}} + \mathbf{L}_{\text{orb}}$$

Time derivative of angular momentum we have two parts

$$\begin{aligned} \dot{\mathbf{L}}_{\text{cm}} &= M\dot{\mathbf{R}} \times \dot{\mathbf{R}} + M\mathbf{R} \times \ddot{\mathbf{R}} \\ &= M\mathbf{R} \times \mathbf{F}_{\text{ext}} = \mathbf{\Gamma}_{\text{cm}} \end{aligned}$$

and

$$\begin{aligned} \dot{\mathbf{L}}_{\text{rel}} &= \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{r}}'_\alpha, \quad \ddot{\mathbf{r}}'_\alpha = \ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}} \\ &= \mathbf{\Gamma}_{\text{rel}} \end{aligned}$$

Energy The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha)^2 \\ &= \frac{1}{2}\sum_\alpha m_\alpha (\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{r}}'^2_\alpha) \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}'^2_\alpha \end{aligned}$$

and the potential energy is

$$U = U_{\text{ext}} + U_{\text{int}} = U_{\text{ext}}$$

where there is no relative motion between the particles, the internal potential energy is a constant which can be ignored.

Example: Rotating disk We consider a disk rotating about the z-axis with angular velocity

$$\omega = (0, 0, \omega)$$

with a particle with position and velocity

$$\begin{aligned} \mathbf{r}_\alpha &= (x_\alpha, y_\alpha, z_\alpha) \\ \dot{\mathbf{r}}_\alpha &= (\dot{x}_\alpha, \dot{y}_\alpha, \dot{z}_\alpha) \end{aligned}$$

the time derivative of the position vector is

$$\dot{\mathbf{r}}_\alpha = \omega \times \mathbf{r}_\alpha = (-\omega y_\alpha, \omega x_\alpha, 0)$$

and the angular momentum is

$$\begin{aligned}\ell_\alpha &= m_\alpha \mathbf{r}_\alpha \times \dot{\mathbf{r}}_\alpha = m_\alpha \mathbf{r}_\alpha \times (\omega \times \mathbf{r}_\alpha) \\ &= m_\alpha(-\omega x_\alpha z_\alpha, -\omega y_\alpha z_\alpha, \omega(x_\alpha^2 + y_\alpha^2))\end{aligned}$$

thus the z component of total angular momentum is

$$L_z = \sum_\alpha m_\alpha \ell_{\alpha,z} = \sum_\alpha m_\alpha \omega(x_\alpha^2 + y_\alpha^2) = \omega \sum_\alpha m_\alpha \rho_\alpha^2 = \omega I_z$$

where ρ is radius in cylindrical coordinates and I_z is the moment of inertia about the z-axis (parallel axis theorem). The other two components of angular momentum are

$$\begin{aligned}L_x &= -\sum_\alpha m_\alpha \omega x_\alpha z_\alpha \\ L_y &= -\sum_\alpha m_\alpha \omega y_\alpha z_\alpha\end{aligned}$$

and since L_x and L_y can be nonzero, that means that $\mathbf{L}$ can be in any direction! If we define the products of inertia

$$\begin{aligned}I_{xz} &= -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ I_{yz} &= -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ I_{zz} &= \sum_\alpha m_\alpha (x_\alpha + y_\alpha)^2\end{aligned}$$

we define the total angular momentum as

$$\begin{aligned}\mathbf{L} &= I \cdot \omega \\ &= (I_{xz} \cdot \omega_z, I_{yz} \cdot \omega_z, I_{zz} \cdot \omega_z)\end{aligned}$$

Lecture 25: 3/27/24

HW 8 Hint For a puck on a rotating table

$$\ddot{\mathbf{r}} = 2\dot{\mathbf{r}} \times \mathbf{\Omega} + (\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}$$

where the first term points inward, and the second term points outward. Since

$$\dot{\mathbf{r}} = -\mathbf{\Omega} \times \mathbf{r}$$
$$\implies \ddot{\mathbf{r}} = -r\Omega^2$$

or the centripetal acceleration.

Inertia Tensor

For a general rigid body, we define the angular velocity

$$\omega = (\omega_x, \omega_y, \omega_z)$$

the angular momentum

$$\mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \dot{\mathbf{r}}_\alpha$$

where $\dot{\mathbf{r}}_\alpha = \omega \times \mathbf{r}_\alpha$. Using the BAC-CAB rule we can write

$$\mathbf{r} \times (\omega \times \mathbf{r}) = \omega(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \omega)$$

so

$$\begin{aligned} L_x &= \sum_\alpha m_\alpha(\omega_x(x_\alpha^2 + y_\alpha^2 + z_\alpha^2) - x(\cancel{\omega_x x_\alpha} + \omega_y y_\alpha + \omega_z z_\alpha)) \\ &= \sum_\alpha m_\alpha(\omega_x(y^2 + z^2)) - m_\alpha x_\alpha y_\alpha \omega_y - m_\alpha x_\alpha z_\alpha \omega_z \end{aligned}$$

where we define the products of inertia for $\mathbf{L} = I\omega$:

$$L_i = \sum_j^3 I_{ij}\omega_j$$
$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

where

$$I_{xx} = \sum_\alpha m_\alpha(y_\alpha^2 + z_\alpha^2)$$
$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha = I_{yx}$$
$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha = I_{zx}$$

for the the first row, and

$$I_{yy} = \sum_\alpha m_\alpha(x_\alpha^2 + z_\alpha^2)$$
$$I_{yx} = -\sum_\alpha m_\alpha y_\alpha x_\alpha$$
$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = I_{zy}$$

similarly for the second row and third row. These creates a real 3×3 matrix that is symmetric or

$$I = I^T$$

Example: Dumbell in the yz plane The masses are placed at $(0, \pm y_0, z)$, so the products of inertia are

$$I_{zz} = 2my_0^2, \quad I_{xx} = 2m(y_0^2 + z_0^2), \quad I_{yy} = 2mz_0^2$$

and for the nondiagonal terms

$$I_{xy} = 0 = I_{xz}, \quad I_{yz} = 0 \dots$$

are all zero. Thus we create the inertia tensor

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Another example: A disk on the xy plane The disk has radius R and mass M lying on the xy plane at $z = z_0$. The mass is distributed evenly, so

$$\begin{aligned} m_\alpha &= \mathrm{d}m = \rho R \,\mathrm{d}\theta \\ M &= \int \mathrm{d}m = \int_0^{2\pi} \rho R \,\mathrm{d}\theta = 2\pi\rho R^2 \\ \implies \rho &= \frac{M}{2\pi R^2} \end{aligned}$$

The products of inertia are now calcaulable:

$$\begin{aligned} I_{zz} &= \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = 0 \\ &= \int_0^{2\pi} \rho R \,\mathrm{d}\theta\, (x^2 + y^2) \qquad R = x^2 + y^2 \\ &= \int_0^{2\pi} \rho R^3 \,\mathrm{d}\theta = 2\pi\rho R^3 = 2\pi \frac{M}{2\pi R^2} R^3 = MR \end{aligned}$$

and the other products

$$\begin{aligned} I_{xx} &= \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) \\ &= \int_0^{2\pi} \rho R \,\mathrm{d}\theta\, (y^2 + z_0^2) \\ &= \int_0^{2\pi} \rho R \,\mathrm{d}\theta\, (R^2 \cos^2\theta + z_0^2) \\ &= Mz_0^2 + \int_0^{2\pi} \rho R^3 \cos^2\theta \,\mathrm{d}\theta \\ &= Mz_0^2 + \pi\rho R^3 \\ &= Mz_0^2 + \pi \frac{M}{2\pi R^2} R^3 = Mz_0^2 + \frac{M}{2} R^2 \end{aligned}$$

we can see the familar term for the moment of inertia of a disk $I = \frac{1}{2}MR^2$ which is shifted. The cross terms are

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_0 = 0 = I_{xz}$$
$$I_{xy} = 0 \dots$$

where can see that the average of y is zero for the first term, and we see that all the cross terms are zero as well.

Final Example: A cube with corner at the origin Given the side length a we know that the mass is simply

$$M = \rho a^3, \quad \rho = \frac{M}{a^3}$$

and the products of inertia are

$$\begin{aligned} I_{xy} &= -\int_0^a \mathrm{d}x \int_0^a \mathrm{d}y \int_0^a \mathrm{d}z \, \rho xy \\ &= -\frac{1}{2}a^2 \cdot \frac{1}{2}a^2 \cdot a \cdot \rho = -\frac{1}{4}Ma^2 \end{aligned}$$

and the other cross terms are the same as well

$$I_{yz} = I_{xz} = -\frac{1}{4}Ma^2$$

The diagonal terms

$$\begin{aligned} I_{zz} &= \iiint_0^a \rho(x^2 + y^2) \, \mathrm{d}V \\ &= \frac{1}{3}a^3 \cdot a \cdot a \cdot \rho + a \cdot \frac{1}{3}a^3 \cdot a \cdot \rho \\ &= \frac{2}{3}Ma^2 = I_{xx} = I_{yy} \end{aligned}$$

this gives us the inertia tensor

$$I = Ma^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$

The symmetry of the axes tell us how lopsided or asymmetrical the object is.

Inertial Tensor We can define the inertial product

$$I_{ij} = \sum_{\alpha} m_\alpha (r_\alpha^2 \delta_{ij} - r_{\alpha,i} r_{\alpha,j})$$

where δ_{ij} is the kronicker delta. We know that I in 3×3 is real and symmetric. I is diagonalizeable

$$\exists \quad 3 \text{ axes} \quad \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$$

such that I is diagonal i.e.

$$I = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

Where

$$\lambda_1, \quad \lambda_2, \quad \lambda_3$$

are the principle moments of inertia and the principle axes are

$$\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$$

Thus the matrix rows are in the form

$$I\hat{\mathbf{e}}_i = \lambda_i \hat{\mathbf{e}}_i$$

And I is diagonalized by rotation. To solve a diagonization problem we have to solve for λ by using the angular momentum equation

$$\begin{aligned} \mathbf{L} = I\omega &= \lambda\omega \\ I\omega - \lambda\omega &= 0 \\ (I - \lambda\mathbb{1})\omega &= 0 \\ \implies \det(I - \lambda\mathbb{1}) &= 0 \end{aligned}$$

which gives us the matrix

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{vmatrix} = 0$$

so from the cube example we have

$$I = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \qquad \mu = \frac{Ma^2}{12}$$

thus

$$\det(I - \lambda\mathbb{1}) = \begin{vmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{vmatrix}$$

To solve this, we can use some tricks:

- Circular matrix: Since the sum of every row is the same

$$\sum_j I_{ij} = \text{constant} = \lambda_1$$

- Trace of a matrix (sum of diagonal entries)

$$\begin{aligned}\mathrm{Tr}(I) &= I_{xx} + I_{yy} + I_{zz} \\ &= \sum_i \lambda_i\end{aligned}$$

- Determinant of a matrix

$$\det(I) = \lambda_1\lambda_2\lambda_3$$

so

$$\lambda_1 = 2\mu, \quad \mathrm{Tr}(I) = 24\mu, \quad \det(I) = 242\mu^3$$

so form the other two equations

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= 24\mu \\ \lambda_2\lambda_3 &= 22\mu^2\end{aligned}$$

and

$$\begin{aligned}\lambda_1\lambda_2\lambda_3 &= 242\mu^3 \\ \lambda_2\lambda_3 &= 121\mu^2\end{aligned}$$

which gives us

$$\lambda_2 = \lambda_3 = 11\mu$$

And to get the principle axes we can use the eigenvectors of the matrix

$$\begin{aligned}\lambda\hat{\mathbf{e}}_1 &= I\hat{\mathbf{e}}_1 \\ \begin{pmatrix} 8\mu & -3\mu & -3\mu \\ -3\mu & 8\mu & -3\mu \\ -3\mu & -3\mu & 8\mu \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= 2\mu\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ \begin{pmatrix} 6\mu & -3\mu & -3\mu \\ -3\mu & 6\mu & -3\mu \\ -3\mu & -3\mu & 6\mu \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= 0\end{aligned}$$

and we can guess a solution from the first two rows

$$\begin{aligned}6\mu w_1 - 3\mu w_2 - 3\mu w_3 = 0 \\ 6\mu w_2 - 3\mu w_1 - 3\mu w_3 = 0\end{aligned}$$

summing these two equations gives us

$$9\mu w_1 - 9\mu w_2 = 0 \implies w_1 = w_2 = w_3$$

and we can normalize the vector to get

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for the second unit vector

$$\begin{aligned}I\hat{\mathbf{e}}^2 &= \lambda_2\hat{\mathbf{e}}^2 \\ \begin{pmatrix} -3\mu & -3\mu & -3\mu \\ -3\mu & -3\mu & -3\mu \\ -3\mu & -3\mu & -3\mu \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= 0\end{aligned}$$

where we can guess that

$$\frac{1}{\sqrt{3}}(w_1 + w_2 + w_3) = 0 \leftrightarrow \omega \cdot \hat{\mathbf{e}}_1 = 0$$

We call the new basis vectors of the principle axes the *Body Frame*: $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. The *Space Frame* is the original frame of reference: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. And in the body frame

$$\mathbf{L}_{\text{body}} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

Note that this may point in a different direction than the angular velocity vector.

If $\omega = \omega e_3$

$$\mathbf{L} = I\omega = \lambda_3\omega_3\hat{\mathbf{e}}_1$$

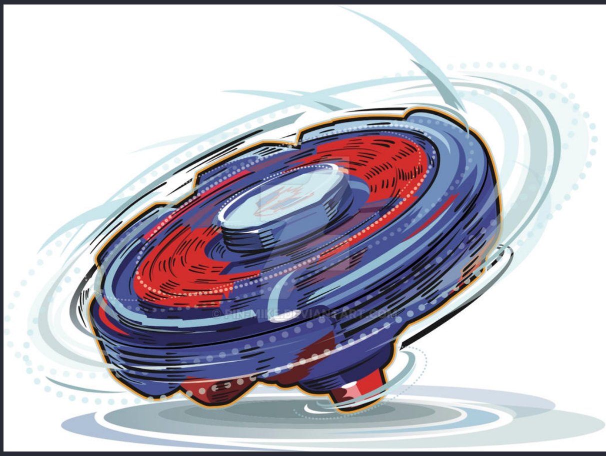

Figure 0.1: The body frame has $\hat{\mathbf{e}}_3$ point in the direction of the spinning beyblade.

Since $\dot{\mathbf{L}} = \mathbf{\Gamma} = \mathbf{R} \times M\mathbf{g}$ where a nonzero torque allows the body to precess. If we look at the torque more carefully given

$$\mathbf{R} = R\hat{\mathbf{e}}_1, \quad \mathbf{g} = -g\hat{\mathbf{z}}$$

$$\begin{aligned} \lambda_3\omega_3\dot{\hat{e}}_1 &= MgR\hat{\mathbf{z}} \times \hat{\mathbf{e}}_3 \\ \dot{\hat{e}}_3 &= \frac{MgR}{\lambda_3\omega_3}\hat{\mathbf{z}} \times \hat{\mathbf{e}}_3 = \Omega \times \hat{\mathbf{e}}_3 \end{aligned}$$

Euler's Equations

- Body Frame $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3 \qquad S$
- Space Frame $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \qquad S_0$

$$\begin{aligned} \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{space}} &= \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{body}} + \omega \times \mathbf{L} \\ &= \dot{\mathbf{L}} + \omega \times \mathbf{L} = \mathbf{\Gamma} \end{aligned}$$

where

$$\begin{aligned} \mathbf{L} &= \lambda_1\omega_1\hat{\mathbf{e}}_1 + \lambda_2\omega_2\hat{\mathbf{e}}_2 + \lambda_3\omega_3\hat{\mathbf{e}}_3 \\ \omega &= \omega_1\hat{\mathbf{e}}_1 + \omega_2\hat{\mathbf{e}}_2 + \omega_3\hat{\mathbf{e}}_3 \end{aligned}$$

Keep in mind that the dot product is not always zero(only if $\mathbf{L} = \lambda\omega$ i.e. a sphere).

$$\begin{aligned} \omega \times \mathbf{L} &= \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1\omega_1 & \lambda_2\omega_2 & \lambda_3\omega_3 \end{pmatrix} \\ &= \hat{\mathbf{e}}_1[\omega_2\omega_3(\lambda_3 - \lambda_2)] \\ &+ \hat{\mathbf{e}}_2[\omega_3\omega_1(\lambda_1 - \lambda_3)] \\ &+ \hat{\mathbf{e}}_3[\omega_1\omega_2(\lambda_2 - \lambda_1)] \end{aligned}$$

so the three components of the torque(or Euler's equations) are

$$\begin{aligned}\Gamma_1 &= \lambda\dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_2\omega_3 \\ \Gamma_2 &= \lambda\dot{\omega}_2 + (\lambda_1 - \lambda_3)\omega_3\omega_1 \\ \Gamma_3 &= \lambda\dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1\omega_2\end{aligned}$$

Zero Torque Case Setting the RHS to zero and moving the lambda terms

$$\begin{aligned}\lambda_1\dot{\omega}_1 &= (\lambda_2 - \lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 &= (\lambda_3 - \lambda_1)\omega_3\omega_1 \\ \lambda_3\dot{\omega}_3 &= (\lambda_1 - \lambda_2)\omega_1\omega_2\end{aligned}$$

Example: Let $\omega = \omega_3\hat{\mathbf{e}}_3, \quad \mathbf{L} = \lambda_3\omega = \lambda_3\omega_3\hat{\mathbf{e}}_3$ thus

$$\omega_1 = \omega_2 = 0$$

and the RHS for all three equations are zero. Thus the body keeps rotating in the same direction. If initially $\omega = \sum_i^3 \omega_i\hat{\mathbf{e}}_i$ we have a lot of motion in any direction.

Small Deviation $\mathbf{L} = \lambda_3\omega_3\hat{\mathbf{e}}_3$ add small ω_1, ω_2: The third equation would be approximately zero, i.e., $\lambda_3\dot{\omega}_3 = 0$ is constant. We are then left with two equations

$$\begin{aligned}\lambda_1\dot{\omega}_1 &= (\lambda_2 - \lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 &= (\lambda_3 - \lambda_1)\omega_3\omega_1\end{aligned}$$

where we have cross terms in the coupled equations. Taking the time derivative of the first equation

$$\begin{aligned}\ddot{\omega}_1 &= -\left[\frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)\omega_3^2}{\lambda_2\lambda_1}\right]\omega_1 \\ \ddot{x} &= -\frac{k}{m}x = -\omega_0^2 x\end{aligned}$$

where this resembles a harmonic oscillator so

$$\omega_0^2 = \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_2\lambda_1}\omega_3^2$$

but the caveat is that the term must be positive or

$$\lambda_3 > \lambda_2, \lambda_1$$

or both negative

$$\lambda_3 < \lambda_2, \lambda_1$$

for $\omega_0^2 > 0$ if

$$\lambda_1 < \lambda_3 < \lambda_2 \qquad \omega_0^2 < 0$$

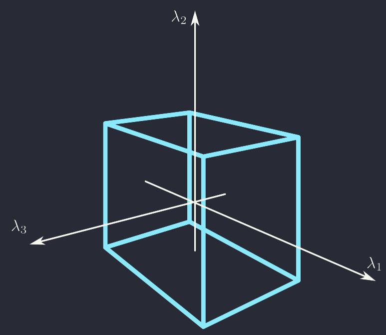

Figure 0.2: Book and its three principal moments of inertia.

Looking at a book, we can see that from the 3 principal moments of inertia, rotating around the largest moment (pointing out of the page) is stable, and rotating around the two smaller moments are unstable.

For a book.

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \quad \text{initially} \quad \omega = \omega_3 \hat{\mathbf{e}}_3$$

Adding a small ω_1, ω_2 where

$$\ddot{\omega}_1 = -\left[\frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2} \omega_3^2\right] \omega_1$$

We can see in the largest moment of inertia, a rotation is stable, but rotation in λ_1 in the figure leads to an unstable rotation, i.e., the book when tossed will rotate around the other axes.

Symmetric top $\lambda_1 = \lambda_2 \neq \lambda_3$ Then from the Euler's equations

$$\lambda_3 \dot{\omega}_3 = 0$$

and using $\lambda_1 = \lambda_2$ the other two equations are

$$\begin{aligned} \lambda_1 \dot{\omega}_1 &= (\lambda_1 - \lambda_3)\omega_3 \omega_2 \\ \lambda_1 \dot{\omega}_2 &= -(\lambda_1 - \lambda_3)\omega_3 \omega_1 \end{aligned}$$

and using

$$\Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3$$

the equations are

$$\begin{aligned} \dot{\omega}_1 &= \Omega_b \omega_2 \\ \dot{\omega}_2 &= -\Omega_b \omega_1 \end{aligned}$$

Which can be solved using the solution

$$\begin{aligned} \eta &= \omega_1 + i\omega_2 \\ \dot{\omega}_1 + i\dot{\omega}_2 &= \Omega_b \omega_2 - i\Omega_b \omega_1 \\ &= \Omega_b(\omega_2 - i\omega_1) \\ &= i\Omega_b(i\omega_2 - \omega_1) \\ \dot{\eta} &= -\Omega_b \eta \implies \eta = \eta_0 e^{-i\Omega_b t} \end{aligned}$$

so

$$\begin{aligned} \omega_1 &= \omega_0 \cos(\Omega_b t) \\ \omega_2 &= \omega_0 \sin(\Omega_b t) \end{aligned}$$

where Ω_b is the free precession frequency (zero torque still results in precession).

Euler Angles

Goal: To find the Lagrangian for a rotating body. We can describe the orientation of the body axes (principal moments) within a space frame: The three angles ϕ, θ, ψ

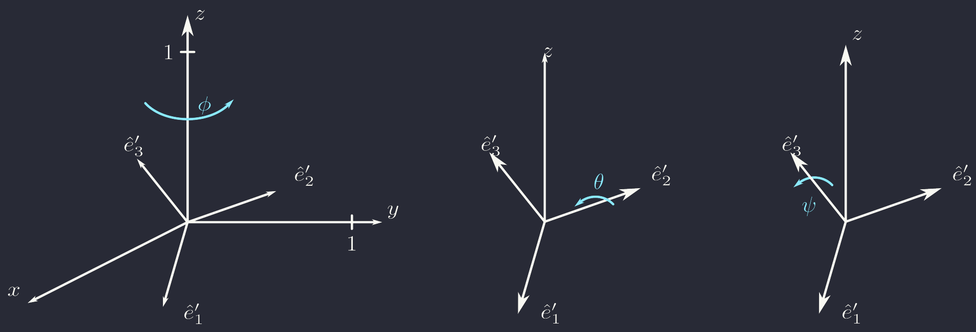

Figure 0.3: Euler angles ϕ, θ, ψ. We can relate $\cos\theta = \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{z}}$

First you rotate around $\hat{\mathbf{z}}$ by ψ, then around $\hat{\mathbf{e}}_2'$ by θ, and finally around $\hat{\mathbf{e}}_3'$ by ϕ.
The three operations can be defined as the angular velocity vector sum

$$
\begin{aligned}
\omega &= \omega_a + \omega_b + \omega_c \\
\omega_a &= \dot{\phi}\hat{\mathbf{z}} \\
\omega_b &= \dot{\theta}\hat{\mathbf{e}}_2' \\
\omega_c &= \dot{\psi}\hat{\mathbf{e}}_3' \\
\omega &= \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{e}}_2' + \dot{\psi}\hat{\mathbf{e}}_3'
\end{aligned}
$$

For the symmetric top we can discount the third step, and if $\lambda_1 = \lambda_2$ then

$$
\begin{aligned}
\hat{\mathbf{e}}_1 &= \hat{\mathbf{e}}_1' \\
\hat{\mathbf{e}}_2 &= \hat{\mathbf{e}}_2'
\end{aligned}
$$

so

$$\hat{\mathbf{z}} = \hat{\mathbf{e}}_3 \cos\theta - \hat{\mathbf{e}}_1' \sin\theta$$

which gives the angular velocity vector

$$
\begin{aligned}
\omega &= -\dot{\phi}\sin\theta\hat{\mathbf{e}}_1' + \dot{\theta}\hat{\mathbf{e}}_1' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{e}}_3' \\
&= \omega_1\hat{\mathbf{e}}_1' + \omega_2\hat{\mathbf{e}}_2' + \omega_3\hat{\mathbf{e}}_3'
\end{aligned}
$$

The angular momentum vector is then

$$
\begin{aligned}
\mathbf{L} &= I\mathbf{o}mega \\
&= \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3
\end{aligned}
$$

and the Kinetic energy is

$$
\begin{aligned}
T &= \frac{1}{2}\omega \cdot \mathbf{L} \\
&= \frac{1}{2}[\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \lambda_3(\dot{\psi} + \phi\cos\theta)^2]
\end{aligned}
$$

To find the potential energy we use the CM R and gravity $\mathbf{g} = -g\hat{\mathbf{z}}$:

$$U = Mgh = MgR\cos\theta$$

where we can find the Lagrangian $\mathcal{L} = T - U$

Euler Angles cont'd For a rotating rigid body where $\lambda_1 = \lambda_2$:

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \phi\cos\theta)^2 - MgR\cos\theta\end{aligned}$$

And the two conserved quantities are ψ and ϕ: Since $\mathcal{L}$ is independent of ψ, ϕ

$$\begin{aligned}p_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \text{constant} = L_z \\ p_\psi &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{constant} = L_3\end{aligned}$$

So the momentum is conserved in the z direction and the 3 direction. Two get the third equation we can use the Euler-Lagrange equations for θ:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) \\ \lambda_1\ddot{\theta} &= \lambda_1\dot{\phi}^2\sin\theta\cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + MgR\sin\theta\end{aligned}$$

or

$$\frac{L_z - L_3\cos\theta}{\lambda_1\sin^2\theta} = \dot{\theta}$$

Assuming that $\theta = \text{constant}, \quad \dot{\phi} = \Omega$. We can also see that $(\dot{\psi} + \dot{\phi}\cos\theta) = \omega_3$, so

$$\begin{aligned}0 &= \lambda_1\Omega^2\sin\theta\cos\theta - \lambda_3\omega_3\Omega\sin\theta + MgR\sin\theta \\ &= \lambda_1\Omega^2\cos\theta - \lambda_3\omega_3\Omega + MgR\end{aligned}$$

and since everything except Ω is constant we can solve using the quadratic formula:

$$\Omega = \frac{\lambda_3\omega_3 \pm \sqrt{(\lambda_3\omega_3)^2 - 4\lambda_1 MgR\cos\theta}}{2\lambda_1\cos\theta}$$

and for $\omega_3 \gg 1$ we can find the free precession frequency Ω_1:

$$\begin{aligned}\Omega_1 &= \frac{\lambda_3\omega_3}{\lambda_1\cos\theta} \\ \Omega_2 &= \frac{MgR}{\lambda_3\omega_3}\end{aligned}$$

where Ω_2 is the precession due to gravity. Looking at the Lagrangian but rewriting as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta \\ &= \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{\text{eff}}\end{aligned}$$

where U_{eff} is the effective potential energy. Theta ranges from $0 \to \pi$ (From the $\sin\theta$ term), and as $\theta \to 0, \pi$ the potential energy goes to infinity!