Math 310: Foundations for Higher Mathematics

Lectures: [insert name]
Notes and Homework: Junseo Shin

Contents

1 The Foundations: Logic and Proofs

Consider the following argument:

> i eat chocolate if i am depressed
> i am not depressed
> therefore i am not eating chocolate

Obviously, the logic is flawed...but how do we write this in a more formal way?

1.1 Propositional Logic

A *statement* is a senetence or mathematical expression that is either *true* or *false*—e.g.

- P : The number 3 is odd
- Q : The number 6 is even
- R : The number 4 is odd

Not a statement

- $x > 2$ (the true value depends on x)
- $x = 2,\ t + 4q = 17$

Combining statements

Given statements P and Q:

- "P and Q" is a statement ($P \wedge Q$)
- "P or Q" is a statement ($P \vee Q$)

We can construct a truth table to represent the truth values of $P \wedge Q$ and $P \vee Q$:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1: Truth tables for conjunction ($\wedge$) and disjunction ($\vee$)

Conditional Statements

The expression:

If P, then Q (or $P \Rightarrow Q$)

is a *conditional statement.*

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2: Truth table for conditional statements

Example:

$P(n)$: The integer n is odd

$Q(n)$: The integer n^2 is odd

$P(n)$ and $Q(n)$ are not statements, but they are *predicates* (statements once n is determined). So the conditional statement is

$P(n) \Rightarrow Q(n)$: If the integer n is odd, then the integer n^2 is odd

Proving a statement of the form $P \Rightarrow Q$

1. Direct proof: Assume P is true and "prove" that Q is also true

Example: Let's construct a truth table for $(P \vee Q) \Rightarrow R$

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$
T	T	T	T	T
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T
F	F	F	F	T

Table 3: Truth table for $(P \vee Q) \Rightarrow R$

Where we want to prove

If n is odd, then n^2 is odd.

The first proposition is symbolically $O(n) : n$ is odd, and the conditional statement is

$$O(n) \Rightarrow O(n^2)$$

Def First we define and integer n odd if $n = 2k + 1$ for some integer k. An integer is even if $n = 2k$ for some integer k.

Remark The set of integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

where k is an integer is denoted as $k \in \mathbb{Z}$.

Proof Suppose n is odd. So by definition, $n = 2k + 1$ for some $k \in \mathbb{Z}$.

$$\implies n^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is an integer, we have that n^2 is in fact odd. $\square$

Another Example (Because students love examples) Suppose x and y are positive numbers. Prove that if $x < y$ then $x^2 < y^2$.

Sol Suppose x and y are positive real numbers and further suppose that $x < y$. A fundamental property of $<$ on the real numbers is that if $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$ since if

$$a < b \implies 0 < b - a$$

and the product of the two positve numbers is positive, i.e.

$$0 < c(b - a) = cb - ca$$

Which now implies $ca < cb$. In this case, if $a = x, b = y, c = x$, then

$$x^2 = x \cdot x < x \cdot y$$

Now if we swap and use $c = y$, we have

$$x \cdot y < y \cdot y = y^2$$

Concatenating the two inequalities, we find that

$$x^2 = x \cdot x < x \cdot y < y \cdot y = y^2$$

Because x and y were arbitrary positive numbers, the conclusion holds. $\square$

1.2 Logical Equivalence

Two statements are *logically equivalent* if they have the same truth value, e.g. x & y are real numbers

$P:\ x \cdot y = 0$

$Q:\ x = 0$ or $y = 0$

are equivalence since they are either both T or both F.

If P and Q are equivalent we say P if and only if Q and we write

$$P \iff Q \quad \text{or} \quad P \equiv Q$$

which is a *biconditional statement*. Note that P & Q are predicates but $P \iff Q$ is a statement.

Example P, Q, and R are statements

$$((P \vee Q) \Rightarrow R) \iff ((P \Rightarrow R)) \wedge (Q \Rightarrow R)$$

P	Q	R	$P \vee Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F

Table 4: Truth table

Contrapositive The *contrapositive* state is

If not Q, then not P

Claim The statement $P \Rightarrow Q$ and its contrapositive $\neg Q \Rightarrow \neg P$ are logically equivalent.

Proof For fun watch the YouTube video Not Knot

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Table 5: Truth table proof

Remark A proof of a condition statement by proving the contrapositive is called a *contrapositive proof.*

Example Let's prove the statement

Suppose x is a real number. If $x^2 + 5x < 0$, then $x < 0$

using a contrapositive proof.

Proof

$$P : x^2 + 5x < 0$$
$$Q : x < 0$$

So $\neg Q \Rightarrow \neg P$ is

If $x \geq 0$, then $x^2 + 5x \geq 0$

Suppose x is a real number satisfying $x \geq 0$. Then $5x \geq 0$ & $x^2 \geq 0$. Thus

$$x^2 + 5x \geq 0$$

Because $x \geq 0$ was arbitrary, we have $\neg Q \Rightarrow \neg P$.

Converse $Q \Rightarrow P$ is called the *converse* of $P \Rightarrow Q$.

Example

P: f is differentiable at $x = 0$

Q: f is continuous at $x = 0$

As an example, $f = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$—so here

$P \Rightarrow Q$ is true, but
$Q \Rightarrow P$ is false

Another example is

P: A is an invertible 2×2 matrix
Q: $\det A \neq 0$

Negation & Quantifiers

Example Let m and n be integers. If 4 divides the product mn (results in an integer), then 4 divides m or 4 divides n.

- Converse: If 4 divides m or 4 divides n, then 4 divides mn
- Contrapositive: If 4 does not divide m and 4 does not divide n, then 4 does not divide mn

This statement is False!

Proof If $m = n = 2$, then 4 divides $mn = 4$. But 4 does *not* divide m or n, thus the statement is F. □

The *negation* of a statement P is the statement whose truth values are opposite for those of P and is denoted as $\neg P$.

Claim Let P and Q be statements.

The negation of the conditional statement $P \Rightarrow Q$ is $P \wedge (\neg Q)$.

Proof We check that $\neg(P \Rightarrow Q)$ and $P \wedge (\neg Q)$ are logically equivalent with a truth table.

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$\neg Q$	$P \wedge (\neg Q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Table 6: Truth table for negation of a conditional statement

Discussion Let P and Q be statements and negate $P \vee Q$, and find what it is equivalent to.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Table 7: Truth table for negation of a disjunction

So the two statements are logically equivalent $\neg(P \vee Q) \iff \neg P \wedge \neg Q$. This is one of De Morgan's Laws:

$$\neg(P \vee Q) \iff \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) \iff \neg P \vee \neg Q$$

Table 8: De Morgan's Laws

Example Every nonempty subset of $\mathbb{N}$ has a smallest element.

Notation $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.

Definition The symbols $\forall$ and $\exists$ are called *quantifiers*.

- $\forall$ stands for "for all" or "for every"
- $\exists$ stands for "there exists" or "there is"

thus we write the above statement as logical mathematical symbols is

$\forall X \subset \mathbb{N}$ with $X \neq \phi$, $\exists x_0 \in X$ such that $x_0 \leq x \quad \forall x \in X$

240 Lecture Notes*

1.3 Propositional Logic

Proposition = statement that has a true value (T or F)

$p =$ "$1 + 1 = 2$": T

$q =$ "St. Louis is the capial of MO": F

Negation NOT $\neg$

$\neg p =$ "not p" or "p is false"

Or in a truth table:

p	$\neg p$
T	F
F	T

e.g. from before:

- $\neg p =$ "$1 + 1 \neq 2$": F
- $\neg q =$ "St. Louis is *not* the capital of MO": T

Conjunction: AND $\wedge$

$p \wedge q =$ "p and q" or "both p and q are true"

Or in a truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

e.g.

$p =$ "Alan Turing was born in England": T

$q =$ "Alan Turing was born in 1912": T

$p \wedge q =$ "Alan Turing was born in England in 1912": T

Disjunction OR $\vee$

$p \vee q =$ "p or q" or "p is true or q is true (or both)" (inclusive)

As a truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

e.g.

$p =$ "2 is a prime number": T

$q =$ "The Blues will win the Stanley Cup this year"

$p \vee q =$ T (since p is true, we can't determine the truth value without knowing q)

To wrap this around our head listen to Conjunction Junction - Schoolhouse Rock.

Exclusive OR XOR $\oplus$

$p \oplus q =$ "p x-or q" or "p or q is true but not both"

As a truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Rather than using T or F we can use bits (1 or 0) to represent the truth values such that

$$1 \oplus 1 = 2 \equiv 0 (\text{mod } 2)$$

Multiple Propositions

$p \wedge q \wedge r =$ "All of p, q, r are true"

$p \vee q \vee r =$ "At least one of p, q, r is true"

For the truth table:

p	q	r	$p \wedge q \wedge r$	$p \vee q \vee r$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	F

which can be generalized to n propositions.

Truth Tables for Compound propositions

$(p \vee q) \wedge (\neg q \vee r)$

So filling out the truth table:

p	q	r	$\neg q$	$p \vee q$	$\neg q \vee r$	$(p \vee q) \wedge (\neg q \vee r)$
T	T	T	F	T	T	T
T	T	F	F	T	F	F
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	T	F	F	T	F	F
F	F	T	T	F	T	F
F	F	F	T	F	T	F

We don't always have to construct truth tables, especially when we are given the truth values of the propositions—e.g.

$$(\neg p \wedge q) \vee (q \wedge \neg r) \qquad p = T \; q = F \; r = T$$
$$(\neg T \wedge F) \vee (F \wedge \neg T) = (F \wedge F) \vee (F \wedge F) = F \vee F = F$$