

Homework 5

3.16 From Griffiths:

$$V(x, y) = \frac{2V_0}{\pi} \arctan\left(\frac{\sin \pi y/a}{\sinh \pi x/a}\right)$$

where the surface charge density is given by a Gaussian pillbox

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

since normal of the surface is $\hat{\mathbf{n}} = \hat{\mathbf{x}}$,

$$\begin{aligned} \sigma &= -\epsilon_0 \frac{\partial}{\partial x} \left(\frac{2V_0}{\pi} \arctan(b) \right) \\ &= -\frac{2V_0\epsilon_0}{\pi} \frac{1}{1+b^2} \left(\frac{-\sin(\pi y/a)}{\sinh^2(\pi x/a)} \right) \cosh(\pi x/a) \frac{\pi}{a} \\ &= \frac{2V_0\epsilon_0}{a} \frac{\sin(\pi y/a)}{\sinh^2(\pi x/a) + \sin^2(\pi y/a)} \cosh(\pi x/a) \end{aligned}$$

where at the boundary $x = 0$,

$$\sinh(0) = 0, \cosh(0) = 1$$

so

$$\sigma = \frac{2V_0\epsilon_0}{a} \frac{\sin(\pi y/a)}{\sin^2(\pi y/a)} = \boxed{\frac{2V_0\epsilon_0}{a} \sin(\pi y/a)}$$

3.18 Since only the top plate has a potential V_0 the boundary conditions are

(i) $V = V_0$ at $z = a$

(ii) $V = 0$ at $x = 0, y = 0, z = 0, x = a,$ and $y = a$

We look for solutions

$$V(x, y, z) = X(x)Y(y)Z(z)$$

and solving for Laplace's equation

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0$$

where

$$\frac{1}{X} \frac{d^2 X}{dx^2} = C_1, \frac{1}{Y} \frac{d^2 Y}{dy^2} = C_2, \frac{1}{Z} \frac{d^2 Z}{dz^2} = C_3 \quad \text{with} \quad C_1 + C_2 + C_3 = 0$$

The textbook suggests that $C_1 = -k^2$, $C_2 = -l^2$, and $C_3 = k^2 + l^2$ where k, l are constants. We can solve for X, Y, Z separately:

$$X(x) = A \sin(kx) + B \cos(kx)$$

$$Y(y) = C \sin(ly) + D \cos(ly)$$

$$Z(z) = E e^{\sqrt{k^2 + l^2} z} + F e^{-\sqrt{k^2 + l^2} z}$$

So for the boundary conditions (ii)

$$\begin{aligned} \left. \frac{d^2 X}{dx^2} \right|_{x=0} &= -A^2 k^2 \sin^2(0) - B^2 k^2 \cos^2(0) = 0 \implies B = 0 \\ \left. \frac{d^2 X}{dx^2} \right|_{x=a} &= -A^2 k^2 \sin^2(ka) = 0 \implies k = \frac{n\pi}{a} \quad \text{for } n = 1, 2, 3, \dots \\ \left. \frac{d^2 Y}{dy^2} \right|_{y=0} &= -C^2 l^2 \sin^2(0) - D^2 l^2 \cos^2(0) = 0 \implies D = 0 \\ \left. \frac{d^2 Y}{dy^2} \right|_{y=a} &= -C^2 l^2 \sin^2(la) = 0 \implies l = \frac{m\pi}{a} \quad \text{for } m = 1, 2, 3, \dots \\ \left. \frac{d^2 Z}{dz^2} \right|_{z=0} &= E(k^2 + l^2) + F(k^2 + l^2) = 0 \implies F = -E \end{aligned}$$

thus

$$\begin{aligned} Z(z) &= E e^{\sqrt{k^2 + l^2} z} - E e^{-\sqrt{k^2 + l^2} z} \\ &= 2E \sinh\left(\sqrt{k^2 + l^2} z\right) \\ &= 2E \sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2} z\right) \\ &= 2E \sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} z\right) \end{aligned}$$

The potential is now

$$V(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} z\right)$$

where we can solve for the Constants C_{nm} using the boundary condition (i) $V = V_0$ at $z = a$:

$$V_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} a\right)$$

Doing the Fourier series by multiplying by $\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)$ and integrating over $x, y = [0, a]$:

$$\begin{aligned} \int_0^a \int_0^a V_0 \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dx dy \\ = \sum \sum C_{nm} \sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} a\right) \int_0^a \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) \sin^2\left(\frac{m\pi y}{a}\right) dx dy \end{aligned}$$

where the integral

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx = \begin{cases} \frac{a}{2} & \text{if } n = n' \\ 0 & \text{if } n \neq n' \end{cases}$$

So

$$V_0 \int_0^a \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dx dy = C_{nm} \sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} a\right) \frac{a^2}{4}$$

where

$$C_{nm} \sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} a\right) = \begin{cases} 0 & \text{if } n \text{ or } m \text{ is even} \\ \frac{16V_0}{\pi^2 nm} & \text{if } n \text{ and } m \text{ are odd} \end{cases}$$

Thus the potential is

$$V(x, y, z) = \frac{16V_0}{\pi^2} \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \frac{\sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} z\right)}{\sinh(\pi \sqrt{n^2 + m^2})}$$

The potential at the center is

$$V(a/2, a/2, a/2) = \frac{16V_0}{\pi^2} \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1}{nm} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{m\pi}{2}\right) \frac{\sinh\left(\frac{\pi}{a} \sqrt{n^2 + m^2} a/2\right)}{\sinh(\pi \sqrt{n^2 + m^2})}$$

where a calculator gives us

$$V(a/2, a/2, a/2) \approx 0.167V_0$$

Which roughly $V_0/6$ or a sixth of the potential in the center of a cube with V_0 on each face.

3.25 The potential inside and outside the sphere is given by (Griffiths 3.78 & 3.79)

$$V(r, \theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) & (r \leq R) \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) & (r \geq R) \end{cases}$$

where (3.81 & 3.84)

$$B_l = A_l R^{2l+1} \quad (3.81)$$

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^\pi \sigma_0(\theta) P_l(\cos \theta) \sin \theta d\theta \quad (3.84)$$

Since the charge density is constant $\sigma_0(\theta) = \sigma_0$ and using $u = \cos \theta$; $\cos(0) = 1, \cos(\pi) = -1$ and $du = -\sin \theta d\theta$ we get

$$A_l = -\frac{\sigma_0}{2\epsilon_0 R^{l-1}} \int_1^{-1} P_l(u) du$$

where $P_l(u)$ is odd for odd l and even for even l so the integral is zero for odd l and non-zero for even l :

$$\int_1^{-1} P_l(u) du = \begin{cases} 0 & \text{if } l \text{ is even} \\ -2 \int_0^1 P_l(u) du & \text{if } l \text{ is odd} \end{cases}$$

Thus

$$A_l = -\frac{\sigma_0}{2\epsilon_0 R^{l-1}} (-2) \int_0^1 P_l(u) du = \begin{cases} 0 & \text{if } l \text{ is odd} \\ \frac{\sigma_0}{\epsilon_0 R^{l-1}} \int_0^1 P_l(u) du & \text{if } l \text{ is even} \end{cases}$$

So for the odd P_l

$$\begin{aligned} \int_0^1 P_1(u) du &= \int_0^1 u du = \frac{1}{2} \\ \int_0^1 P_3(u) du &= \frac{1}{2} \int_0^1 (5u^3 - 3u) du = -\frac{1}{8} \\ \int_0^1 P_5(u) du &= \frac{1}{8} \int_0^1 (63u^5 - 70u^3 + 15u) du = \frac{1}{16} \end{aligned}$$

and for the even P_l the integral is zero; therefore

$$\begin{aligned} A_0 &= A_2 = A_4 = A_6 = 0 \\ A_1 &= \frac{\sigma_0}{\epsilon_0} \frac{1}{2}, \quad A_3 = \frac{\sigma_0}{\epsilon_0 R^2} \left(\frac{-1}{8} \right), \quad A_5 = \frac{\sigma_0}{\epsilon_0 R^4} \frac{1}{16} \end{aligned}$$

and

$$\begin{aligned} B_0 &= B_2 = B_4 = B_6 = 0 \\ B_1 &= \frac{\sigma_0}{\epsilon_0} R^3 \frac{1}{2}, \quad B_3 = \frac{\sigma_0}{\epsilon_0} R^5 \left(\frac{-1}{8} \right), \quad B_5 = \frac{\sigma_0}{\epsilon_0} R^7 \frac{1}{16} \end{aligned}$$

So finally the potential is

$$V(r, \theta) = \frac{\sigma_0}{\epsilon_0} \left[\frac{r}{2} P_1(\cos \theta) - \frac{r^3}{8R^2} P_3(\cos \theta) + \frac{r^5}{16R^4} P_5(\cos \theta) \right] \quad (r \leq R)$$

and

$$V(r, \theta) = \frac{\sigma_0}{\epsilon_0} \left[\frac{R^3}{2r^2} P_1(\cos \theta) - \frac{R^5}{8r^4} P_3(\cos \theta) + \frac{R^7}{16r^6} P_5(\cos \theta) \right] \quad (r \geq R)$$

3.29 Given charge density

$$\rho(r, \theta) = k \frac{R}{r^2} (R - 2r) \sin \theta$$

for a sphere of radius R and constant k . The monopole term is

$$\int \rho d\tau = kR \int \left(\frac{1}{r^2} R - 2r \sin \theta \right) r^2 \sin \theta dr d\theta d\phi$$

Where the radial integral is

$$\int_0^R (R - 2r) dr = 0$$

Moving on to the dipole term

$$\int r \cos \theta \rho d\tau = kR \int (r \cos \theta) (R - 2r) \sin \theta \sin \theta dr d\theta d\phi$$

where the polar integral is

$$\int_0^\pi \sin \theta \cos \theta d\theta = 0$$

Maybe the quadrupole term will be non-zero:

$$\begin{aligned} \int r^2 \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \rho d\tau &= \frac{1}{2} kR \int_0^{2\pi} \int_0^\pi \int_0^R r^2 ((3 \cos^2 \theta - 1)(R - 2r) \sin \theta) dr d\theta d\phi \\ &= \frac{1}{2} kR (2\pi) \left(-\frac{\pi}{8} \right) \left(-\frac{R^4}{6} \right) \\ &= \frac{\pi^2 k R^5}{48} \end{aligned}$$

So far from the sphere for points on the z -axis the potential is

$$V(z) = \frac{1}{4\pi\epsilon_0} \frac{\pi^2 k R^5}{48 z^3}$$

3.38 Given

$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \quad (3.103)$$

and

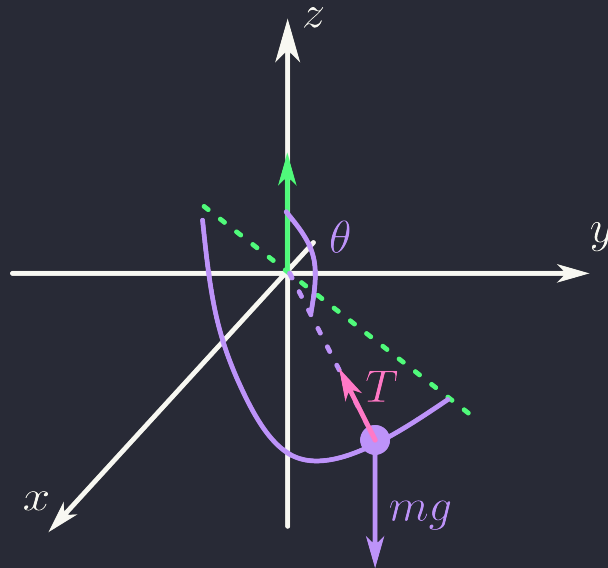
$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}] \quad (3.104)$$

The dipole moment has vector components

$$\mathbf{p} = p \cos \theta \hat{\mathbf{r}} - p \sin \theta \hat{\theta}$$

so now we can sub this into (3.104):

$$\begin{aligned} \mathbf{E}_{\text{dip}}(r, \theta) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}] \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} 3(p \cos \theta) \hat{\mathbf{r}} - (p \cos \theta \hat{\mathbf{r}} - p \sin \theta \hat{\theta}) \\ \text{Eq. 3.103} &= \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \end{aligned}$$

Figure 0.1: The electric field of a dipole at a distance r from the origin.

3.61 Given the potential from (3.103) and the force

$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta})$$

Pretending this is a pendulum as shown in Fig. 0.1 the force is

$$\mathbf{F} = -T\hat{\mathbf{r}} - mg\hat{\mathbf{z}} = -T\hat{\mathbf{r}} - mg(\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta})$$

T is the tension given by the centripetal acceleration

$$\begin{aligned} ma_c &= T - mg \cos(\pi - \theta) \\ m \frac{v^2}{r} &= T + mg \cos \theta \end{aligned}$$

where v is the velocity given by energy conservation

$$\begin{aligned} KE &= \Delta PE \\ \frac{1}{2}mv^2 &= -mgr \cos \theta \\ \implies v^2 &= -2gr \cos \theta \end{aligned}$$

So

$$T = m \frac{v^2}{r} - mg \cos \theta = -2mg \cos \theta - mg \cos \theta = -3mg \cos \theta$$

Thus we get the force

$$\begin{aligned} \mathbf{F} &= 3mg \cos \theta \hat{\mathbf{r}} - mg(\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta}) \\ &= mg(2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \end{aligned}$$

Which is telling us that the electric charge will swing in an arc like a pendulum i.e.

$$mg \equiv \frac{qp}{4\pi\epsilon_0 r^3}$$

or we can think of the charge q as an inertial mass and group the other terms as the gravitational constant.