1 Decay and Scattering

Decay rate Γ

- Probability per unit time for the decay to happen

For a decay process the change in the number of particles (amount of stuff that decayed)

$$-N(t)\Gamma \mathrm{d}t = \mathrm{d}N$$

we can solve this differential equation to find

$$\begin{aligned} \int \frac{\mathrm{d}N}{N} &= -\int \Gamma \mathrm{d}t \\ \ln N &= -\Gamma t + C \\ \implies N(t) &= N_0 e^{-\Gamma t} \end{aligned}$$

we can find the mean lifetime $\tau = \frac{1}{\Gamma}$ so

$$N(t) = N_0 e^{-t/\tau}$$

Half time and the half-life is when

$$\begin{aligned} N(t_{1/2}) &= \frac{N_0}{2} = N(0)e^{-\Gamma t_{1/2}} \\ \implies e^{\Gamma t_{1/2}} &= 2 \\ \Gamma t_{1/2} &= \ln 2 \end{aligned}$$

or

$$t_{1/2} = \tau \ln 2$$

Example

$$\begin{aligned} \pi^+ \rightarrow & \mu^+ + \nu_\mu \qquad \Gamma_1 \gg \Gamma_2 \\ & e^+ + \nu_e \qquad \Gamma_2 \end{aligned}$$

and

$$\Gamma_{tot} = \sum_i \Gamma_i \qquad \tau_{tot} = \frac{1}{\Gamma_{tot}}$$

we have a branching ratio (or fraction)

$$\mathrm{Br}_i = \frac{\Gamma_i}{\Gamma_{tot}} \qquad [0, 1]$$

and we find the branching ratio of the pion decay is experimentally

$$\begin{aligned} \mathrm{Br}_1 &= 0.999877 \\ \mathrm{Br}_2 &= 0.000123 \end{aligned}$$

Insert Griffiths Figure 6.1 here

Scattering From the impact parameter b and scattering angle θ we can find the cross section, or the probability of scattering. We have an infinitesimal area of

$$\mathrm{d}\sigma = |\mathrm{d}b \cdot b\mathrm{d}\phi|$$

which is like the area of a rectangle made by the differential impact parameter. The solid angle is

$$\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

like the theta and phi part of spherical coordinates. The differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left| \frac{b}{\sin\theta} \cdot \frac{\mathrm{d}b}{\mathrm{d}\theta} \right|$$

Hard Sphere Scattering We have a hard sphere of radius R and we send a particle toward the sphere and it scatters on the surface. Thus the cross section is expected to be

$$\sigma = \pi R^2$$

or the area of a circle that cuts the sphere. From the law of inflection we have an inflection

$$2\alpha + \theta = \pi$$

and the trigonometry shows that the impact parameter is

$$b = R\sin\alpha$$

or

$$\begin{aligned} b &= R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\ &= R\cos\left(\frac{\theta}{2}\right) \end{aligned}$$

so the differential cross section is

$$\begin{aligned} \frac{\mathrm{d}b}{\mathrm{d}\theta} &= -\frac{R}{2}\sin\frac{\theta}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \left| \frac{R\cos\frac{\theta}{2}}{\sin\theta} \cdot \frac{R}{2}\sin\frac{\theta}{2} \right| \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= \frac{R^2}{4} \end{aligned}$$

and

$$\begin{aligned} \int \mathrm{d}\sigma &= \int \frac{R^2}{4}\,\mathrm{d}\Omega \\ \sigma &= \frac{R^2}{4} \cdot 4\pi = \pi R^2 \end{aligned}$$

Rutherford Scattering In the experiment we can find the impact parameter

$$b = \frac{q_1 q_2}{2E}\cot\frac{\theta}{2}$$

so

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{q_1 q_2}{4E}\csc^2\frac{\theta}{2}$$

and

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \left|\frac{b}{\sin\theta}\cdot\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \\ &= \left|\frac{q_1q_2}{2E}\cot\frac{\theta}{2}\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\cdot -\frac{q_1q_2}{4E}\csc^2\frac{\theta}{2}\right| \\ &= \frac{q_1^2q_2^2}{16E^2}\csc^4\frac{\theta}{2}\end{aligned}$$

so the cross section is

$$\begin{aligned}\sigma &= \int\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega \\ &= \frac{q_1^2q_2^2}{16E^2}\int\csc^4\frac{\theta}{2}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\ &= 2\pi\frac{q_1^2q_2^2}{16E^2}\int\frac{\sin\theta}{\sin^4\frac{\theta}{2}}\mathrm{d}\theta \\ &= 2\pi\frac{q_1^2q_2^2}{16E^2}\int\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}\mathrm{d}\theta\end{aligned}$$

and substituting

$$x = \sin\frac{\theta}{2} \implies \mathrm{d}x = \frac{1}{2}\cos\frac{\theta}{2}\mathrm{d}\theta$$

so

$$2\pi\int_0^1\frac{2x}{x^4}\mathrm{d}x = 2\pi\left(\frac{1}{2x^2}\right)\Big|_0^1 \to \infty$$

Fermi Golden Rule For nonrelativistic system

$$\text{Transition probability} = \text{phase space} \times |\text{amplitude}|^2$$

or

$$\rho\cdot|\langle f|\,0\,|i\rangle|^2$$

where ρ is the density of states.

Relativistic System

$$\begin{aligned}\mathrm{d}\Gamma &\propto |\mathcal{M}|^2\mathrm{d}\Pi \\ \mathrm{d}\sigma &\propto |\mathcal{M}|^2\mathrm{d}\Pi\end{aligned}$$

where $\mathrm{d}\Pi$ is the phase space. For the two body decay

$$\begin{aligned}1 &\to 2+3 \\ m_1 &> m_2+m_3\end{aligned}$$

Wigner-Eckart Theorem For spherically symmetric systems we can split the amplitude into two parts: the symmetric and dynamic parts.

$$\langle f|\,0\,|\propto\rangle\,\text{symmetric}\times\text{dynamic}$$

Quiz Review

- The decay formula gives us

$$N(t) = N_0 e^{-t/\tau} = 10^6 e^{-10} \approx 45$$

- The probability of 1 particle still being there after 10 average lifetimes is directly equal to

$$e^{-t/\tau} = e^{-10} \approx 4.5 \times 10^{-5}$$

- Dirac Delta Function

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

or

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

We can also think of a rectangle with area 1 at $x = 0$ and we keep shortening the width and increasing the height to keep the area 1. As the width gets infinitesimally small, the height gets infinitely large.

- From the heaviside step function

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\theta(x)\mathrm{d}x = \theta(x)\Big|_{-\infty}^{\infty} &= 1 \\ &= \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x \\ \implies \delta(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\theta(x) \end{aligned}$$

Fermi Golden Rule (again) We know that the phase space is dependent of the kinematics i.e. it only depends on the number of paritcles involved. The amplitude $\mathcal{M}$ is dependent on the dynamics or the type of interaction.

Decay $1 \to 2 + 3 + \cdots + n$

$$\begin{aligned} \Gamma = \frac{S}{2m_1\hbar} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - \cdots - p_n) \\ \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2c^2)\theta(p_j^0) \cdot \frac{\mathrm{d}^4 p_j}{(2\pi)^4} \end{aligned}$$

Is the decay rate where S is the symmetry factor

$$S = \frac{1}{\prod_i k_i!}$$

e.g. $a \to b + b + c + c + c$

$$S = \frac{1}{2!3!} = \frac{1}{12}$$

and we also have the phase space part which is in a 4-dimensional component i.e.

$$\begin{aligned} \delta^3(\mathbf{r}) &= \delta(x)\delta(y)\delta(z) \\ \delta^4(p) &= \delta(p^0)\delta^3(\mathbf{p}) \end{aligned}$$

Phase space parts

1. In the first part
$$\delta^4(p_1 - p_2 - p_3 - \cdots - p_n)$$
we have a non-zero value *only* when
$$\begin{aligned} &p_1 - p_2 - p_3 - \cdots - p_n = 0 \\ \implies &\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3 + \cdots + \mathbf{p}_n \end{aligned}$$
or the Energy-momentum conservation.

2. In the second part
$$\delta(p_j^2 - m_j^2c^2)$$
we have a non-zero value *only* when
$$\begin{aligned} &p_j^2 - m_j^2c^2 = 0 \\ \implies &p_j^2 = m_j^2c^2 \qquad \forall j = 2, 3, \ldots, n \end{aligned}$$
which is true for all real particles (on-shell condition). If this is not true i.e. $p_j^2 \neq m_j^2c^2$ we have a virtual particle.

3. In the third part
$$\theta(p_j^0)$$
is non-zero *only* when $p_j^0 > 0$ or $E_j > 0$ (positivity of energy). So from the energy momentum relation
$$\begin{aligned} &E_j^2 = \mathbf{p}_j^2c^2 + m_j^2c^4 \\ \implies &E_j = \pm\sqrt{\mathbf{p}_j^2c^2 + m_j^2c^4} > 0 \end{aligned}$$

Evaluating the integral From the delta function
$$\int \mathrm{d}x\, \delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)]$$
so
$$\begin{aligned} \delta(p_j^2 - \mathbf{p}_j^2 - m_j^2c^2) &= \delta(p_j^0 - a^2) \quad a = \sqrt{\mathbf{p}_j^2 + m_j^2c^2} \\ &= \frac{1}{2a}[\delta(p_j^0 - a) + \delta(p_j^0 + a)] \\ &= \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right)\right] \end{aligned}$$
the second term does not contribute so we are left with
$$\int \mathrm{d}p_j^0\, \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\delta\left(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right) = \frac{\mathrm{d}^3\mathbf{p}_j}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}$$
so we have removed one of the integrals. Now we are left with the integral
$$\begin{aligned} \Gamma = \frac{S}{2m_1\hbar}\int &|\mathcal{M}|^2(2\pi)^4\delta(m_1c - p_2^0 - p_3^0 - \cdots - p_n^0) \\ &\delta^3(\mathbf{0} - \mathbf{p}_2 - \mathbf{p}_3 - \cdots - \mathbf{p}_n) \\ &\times \prod_{j=2}^{n}\frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}} \end{aligned}$$
and from the energy-momentum relation
$$\frac{E_j}{c} = \sqrt{\mathbf{p}_j^2 + m_j^2c^2}$$

Example Two-body decay $1 \to 2 + 3$

Sidenote: we cannot have $1 \to 2$ as it would violate the conservation of 4-momentum Since the delta function is even, $\delta(\mathbf{x}) = \delta(-\mathbf{x})$, so

$$\begin{aligned}\Gamma = \frac{S}{2m_1\hbar}\int &|\mathcal{M}|^2(2\pi)^4\delta(m_1c - E_2/c - E_3/c)\delta^3(\mathbf{p}_2 + \mathbf{p}_3) \\ &\times \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_2^2 + m_2^2c^2}}\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_3^2 + m_3^2c^2}}\end{aligned}$$

We have nonzero values when $\mathbf{p}_2 = -\mathbf{p}_3$ and $E_2 = E_3 = \frac{m_1 c}{2}$. We can use the delta function to remove the integral over $\mathbf{p}_3$ and we are left with

$$= \frac{S}{2m_1\hbar}\int |\mathcal{M}|^2(2\pi)^4\delta(m_1c - (E_2 + E_3)/c)\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^6}\frac{1}{2\sqrt{\mathbf{p}_2^2 + m_2^2c^2}}\frac{1}{2\sqrt{\mathbf{p}_2^2 + m_3^2c^2}}$$

and now we can remove one more integral using

$$\mathrm{d}^3\mathbf{p}_2 = |\mathbf{p}_2|^2\,\mathrm{d}p_2\,\mathrm{d}\Omega \qquad \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and we also know that

$$E_2 = c\sqrt{|\mathbf{p}_2|^2 + m_2^2c^2} \qquad E_3 = c\sqrt{|\mathbf{p}_2|^2 + m_3^2c^2}$$

so

$$\Gamma = \frac{S}{2m_1\hbar}\frac{1}{4(2\pi)^2}\int |\mathcal{M}|^2\delta(m_1c - (E_2 + E_3)/c)\frac{|\mathbf{p}_2|^2\,\mathrm{d}|\mathbf{p}_2|\,\mathrm{d}\Omega}{\sqrt{|\mathbf{p}_2|^2 + m_2^2c^2}\sqrt{|\mathbf{p}_2|^2 + m_3^2c^2}}$$

we know the momentums are

$$p_1 = (m_1c, \mathbf{0}) \qquad p_2 = (E_2/c, \mathbf{p}_2) \qquad p_3 = (E_3/c, -\mathbf{p}_2)$$

we can construct a scalar out of two vectors using the dot product which is always dependent on $|\mathbf{p}_2|^2$ (there is no angular dependence) so

$$|\mathcal{M}|^2(\mathbf{p}_2) = f(|\mathbf{p}_2|^2)$$

so we are left with one integral and one delta function

$$\Gamma = \frac{S}{2m_1\hbar}\frac{1}{4\pi^2 4}(4\pi)\int_0^\infty |\mathcal{M}|^2\delta(m_1c - (E_2 + E_3)/c)|\mathbf{p}_2|^2\frac{\mathrm{d}|\mathbf{p}_2|}{\sqrt{|\mathbf{p}_2|^2 + m_2^2c^2}\sqrt{|\mathbf{p}_2|^2 + m_3^2c^2}}$$

using a change of variables we can use

$$\begin{aligned}u &= \sqrt{|\mathbf{p}_2|^2 + m_2^2c^2} + \sqrt{|\mathbf{p}_2|^2 + m_3^2c^2} \\ \mathrm{d}u &= \frac{2|\mathbf{p}_2|\,\mathrm{d}|\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_2|^2 + m_2^2c^2}} + \frac{2|\mathbf{p}_2|\,\mathrm{d}|\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_2|^2 + m_3^2c^2}}\end{aligned}$$

and thus we get

$$\begin{aligned}\Gamma &= \frac{S}{8m_1\pi\hbar}\int_{(m_2+m_3)c}^\infty |\mathcal{M}|^2\delta(m_1c - u)\,\mathrm{d}u\,\frac{|\mathbf{p}_2|^2}{u} \\ &= \frac{S|\mathbf{p}|}{8\pi\hbar m_1^2c}|\mathcal{M}|^2\end{aligned}$$

Quiz review

- A simple delta function integral tells us

$$\int_{a-e}^{a+e} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

- If the the non zero term is out of bounds of the integral, then the integral is zero!
- From the theta function (step function) we know that

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

  and thus

$$\theta(2x-4) = \begin{cases} 1 & x > 2 \\ 0 & x < 2 \end{cases}$$

  so we can split the integral from $-1 \to 2$ and $2 \to 5$ and we get

$$\int_{-1}^{2} 0e^{-3x}\,\mathrm{d}x = 0$$
$$\begin{aligned} \int_{2}^{5} \theta(2x-4)e^{-3x}\,\mathrm{d}x &= \int_{2}^{5} e^{-3x}\,\mathrm{d}x \\ &= -\frac{1}{3}e^{-3x}\Big|_{2}^{5} \end{aligned}$$

- For integration over a sphere we can just find if the magnitude of distance is less than the radius of the sphere 1.5:

$$|(2,2,2)-(3,2,1,)| = \sqrt{2} \approx 1.4 < 1.5$$

  so we find the function

$$\begin{aligned} \oint \mathrm{d}V\,\mathbf{r}\cdot(\mathbf{a}-\mathbf{r})\delta^3(\mathbf{r}-\mathbf{b}) &= \int \mathrm{d}V\,f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{b}) \\ &= f(\mathbf{b}) \end{aligned}$$

  which is

$$\begin{aligned} f(\mathbf{b}) &= \mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) \\ &= (3,2,1)\cdot[(1,2,3)-(3,2,1)] \\ &= -4 \end{aligned}$$

- The decay rate using dimensional analysis from last time

$$\Gamma = \frac{1}{[\mathrm{J\,s\,kg^2\,m/s}]}\cdot\mathrm{kg\,m/s}\cdot\mathcal{M}$$

  and since $\mathrm{J} = \mathrm{kg\,m^2/s^2}$ we can see that the amplitude has units of $\mathrm{kg\,m/s}$ or momentum. Thus the number of particles involved is the only thing that is dependent on the number of particles involved.

Scattering

(2 $\to$ n Scattering)

$$1 + 2 \to 3 + 4 + \cdots + n$$

the cross section is given by

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \cdots - p_n)$$
$$\times \prod_{j=3}^{n} \frac{\mathrm{d}^4\mathbf{p}_j}{(2\pi)^4} (2\pi)\delta(p_j^2 - m_j^2c^2)\theta(p_j^0)$$

From momentum conservation we have

$$p^2 = (p^0)^2 - \mathbf{p}^2$$

so the delta function can be rewritten as

$$\delta(p_j^2 - m_j^2c^2) = \delta((p_j^0)^2 - \mathbf{p}_j^2 - m_j^2c^2)$$

and using the same trick as last time we can split

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)]$$

or in the general form

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

so defining =

$$x = p_j^0 \qquad a = \sqrt{\mathbf{p}_j^2 + m_j^2c^2}$$

we can rewrite the delta function as

$$\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right)\right]$$

and we can remove the second term becase the theta function removes negative energies! So we are left with

$$\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\delta(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2})$$

Now we we are left with an integral

$$\int \frac{\mathrm{d}p_j^0}{\cancel{(2\pi)}}\cancel{(2\pi)}\delta(p_j^2 - m_j^2c^2)\theta(p_j^0)f(p_j^0) = f(\sqrt{\mathbf{p}_j^2 + m_j^2c^2})$$

which removes the zeroth component of the 4-momentum in the original integral which leaves us with

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \cdots - p_n)$$
$$\prod_{j=3}^{n} \frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}$$

with

$$p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2c^2} = \frac{E_j}{c}$$

2 - 2 Scattering $1 + 2 \to 3 + 4$
In the center of mass frame the total 3-momentum is zero (HW In the lab frame with one particle at rest initially i.e. $p_2 = (m_2 c, \mathbf{0})$). We have two momenta of the *beam* of particles (LHC)

$$p_1 = (E_1/c, \mathbf{p}_1) \qquad p_2 = (m_2 c, \mathbf{p}_2)$$

where

$$p_1 + p_2 = \mathbf{0} \implies \mathbf{p}_1 = -\mathbf{p}_2$$

which means

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = \frac{|\mathbf{p}_1|^2}{c}\sqrt{S} \qquad S = (E_1 + E_2)^2$$

where S is the Mandelstam variable. So the cross section is

$$\sigma = \frac{S\hbar^2}{4\frac{|\mathbf{p}_1|^2}{c}\sqrt{S}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$
$$\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{p}_4}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_3^2 + m_3^2c^2}}\frac{1}{2\sqrt{\mathbf{p}_4^2 + m_4^2c^2}}$$

and we can remove the delta function by using the energy-momentum relation

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta\left(\frac{E_1 + E_2}{c} - \frac{E_3 + E_4}{c}\right)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

but since the total momentum is zero i.e.

$$\mathbf{p}_1 + \mathbf{p}_2 = 0$$

we can replace the $\mathrm{d}^3\mathbf{p}_4$ with the $\mathrm{d}^3\mathbf{p}_3 + \mathbf{p}_4$ and we are left with

$$\sigma = \frac{S\hbar^2}{4\frac{|\mathbf{p}_1|^2}{c}\sqrt{S}} \int |\mathcal{M}|^2 (2\pi)^4 \delta\left(\frac{E_1 + E_2}{c} - \frac{E_3 + E_4}{c}\right)$$
$$\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3}\frac{1}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_3^2 + m_3^2c^2}}\frac{1}{2\sqrt{\mathbf{p}_3^2 + m_4^2c^2}}$$

and since

$$\mathrm{d}^3\mathbf{p}_3 = |\mathbf{p}_3|^2\,\mathrm{d}|\mathbf{p}_3|\,\mathrm{d}\Omega \qquad \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

We know that

$$\begin{aligned} & \mathbf{p}_4 = -\mathbf{p}_3 \\ \implies E_4 = \sqrt{\mathbf{p}_4^2 c^2 + m_4^2 c^4} = & \sqrt{\mathbf{p}_3^2 c^2 + m_4^2 c^4} \end{aligned}$$

so we can represent

$$\begin{aligned} |\mathcal{M}|^2(p_1, p_2, p_3, p_4) &= |\mathcal{M}|^2(p_3, p_4) \\ &= |\mathcal{M}|^2(\mathbf{p}_3, \theta, \phi) \end{aligned}$$

which can't be written as a function of $|\mathbf{p}_3|$ so we must use the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{4|\mathbf{p}_1^0|\sqrt{S}}\frac{1}{(2\pi)^4}\frac{1}{4}\int |\mathcal{M}|^2 \delta\left(\frac{E_1 + E_2}{c} - \frac{E_3 + E_4}{c}\right)$$
$$|\mathbf{p}_3|^2\,\mathrm{d}|\mathbf{p}_3|\,\frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2c^2}\sqrt{\mathbf{p}_3^2 + m_4^2c^2}}$$

using the change of variables we can use

$$
\begin{aligned}
u &= \frac{E_3 + E_4}{c} \\
&= \sqrt{\mathbf{p}_3^2 + m_3^2 c^2} + \sqrt{\mathbf{p}_3^2 + m_4^2 c^2} \\
\mathrm{d}u &= \frac{2|\mathbf{p}_3|\,\mathrm{d}|\mathbf{p}_3|}{2\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} + \frac{2|\mathbf{p}_3|\,\mathrm{d}|\mathbf{p}_3|}{2\sqrt{\mathbf{p}_3^2 + m_4^2 c^2}} \\
&= |\mathbf{p}_3|\,\mathrm{d}|\mathbf{p}_3|\,\frac{u}{\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}\sqrt{\mathbf{p}_3^2 + m_4^2 c^2}}
\end{aligned}
$$

which is the last part of the integral So

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{S\hbar^2}{4|\mathbf{p}_1^0|\sqrt{S}}\frac{1}{(2\pi)^4}\frac{1}{4}\int |\mathcal{M}|^2 \delta\left(\frac{E_1 + E_2}{c} - \frac{E_3 + E_4}{c}\right)\mathrm{d}u\,\frac{1}{u}|\mathbf{p}_3| \\
&= \frac{S\hbar^2 c}{64\pi^2|\mathbf{p}_1|(E_1 + E_2)}\frac{|\mathcal{M}|^2|\mathbf{p}_3|}{\frac{E_1+E_2}{c}} \\
&= \left(\frac{hc}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2}\frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}
\end{aligned}
$$

We find that this cross section is proportional to many things:

$$
\sigma \propto \frac{1}{S}, \qquad \sigma \propto \frac{|p_f|}{|p_i|}
$$

But why use the collider like this?

- In the past we used $\sqrt{S} = 91\,\mathrm{GeV}$ (LEP)
- $\sqrt{S} = 1.96\,\mathrm{TeV}$ (Tevatron)
- $\sqrt{S} = 13.6\,\mathrm{TeV}$ (LHC)
- $\sqrt{S} = 100\,\mathrm{TeV}$ (FCC/SPPC)

But we can only find the cross section to grow with S if $|\mathcal{M}|^2$ is independent of S.‘’