Problem 1. Given

$$M\frac{\mathrm{d}^2u_s}{\mathrm{d}t^2} = C(u_{s+1} + u_{s-1} - 2u_s)$$

where the nearest neighbors is displaced by $\pm p$ which is small so using the Taylor expansion

$$u_{s+a} \approx u_s + a\frac{\partial u}{\partial x} + \frac{1}{2}a^2\frac{\partial^2 u}{\partial x^2}$$
$$u_{s-a} \approx u_s - a\frac{\partial u}{\partial x} + \frac{1}{2}a^2\frac{\partial^2 u}{\partial x^2}$$

so

$$C(u_{s+1} + u_{s-1} - 2u_s) \approx C\left(\cancel{u_s} + \cancel{a\frac{\partial u}{\partial x}} + \frac{1}{2}a^2\frac{\partial^2 u}{\partial x^2} + \cancel{u_s} - \cancel{a\frac{\partial u}{\partial x}} + \frac{1}{2}a^2\frac{\partial^2 u}{\partial x^2} - \cancel{2u_s}\right) = Ca^2\frac{\partial^2 u}{\partial x^2}$$

and

$$M\frac{\mathrm{d}^2u_s}{\mathrm{d}t^2} = Ca^2\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\mathrm{d}^2u_s}{\mathrm{d}t^2} = \frac{Ca^2}{M}\frac{\partial^2 u}{\partial x^2}$$

since the original differential equation can have solutions with time dependence $e^{i\omega t}$

$$\frac{\mathrm{d}^2u_s}{\mathrm{d}t^2} = -\omega^2 u_s$$

and the displacements are translationally symmetric so

$$u_s = ue^{iska} \qquad u_{s+1} = ue^{i(s+1)ka} = ue^{iska}e^{ika} \qquad u_{s-1} = ue^{i(s-1)ka} = ue^{iska}e^{-ika}$$

so

$$-M\omega^2 ue^{iska} = Cue^{iska}(e^{ika} + e^{-ika} - 2)$$
$$-M\omega^2 = C(e^{ika} + e^{-ika} - 2)$$

where we have the trigonometric identity

$$e^{ika} + e^{-ika} = 2\cosh(ika) = 2\cos(ka)$$

so

$$-M\omega^2 = C(2\cos(ka) - 2)$$
$$\omega^2 = \frac{2C}{M}(1 - \cos(ka))$$

and from the half angle identity

$$1 - \cos(ka) = 2\sin^2\left(\frac{ka}{2}\right)$$

so

$$\omega = \sqrt{\frac{4C}{M}}\sin\left(\frac{ka}{2}\right)$$

we can also find the group velocity

$$v = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \sqrt{\frac{4C}{M}}\cos\left(\frac{ka}{2}\right)\frac{a}{2} = \sqrt{\frac{Ca^2}{M}}\cos\left(\frac{ka}{2}\right)$$

and when $ka \ll 1$ or ≈ 0, cosine is 1 so the group velocity is

$$v = \sqrt{\frac{Ca^2}{M}} \quad \text{or} \quad v^2 = \frac{Ca^2}{M}$$

so we get the wave equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = v^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$$

Problem 2. There will be two equations for alternate force constants where the nearest neighbor for atom u_s are u'_s and u'_{s-1}, and for atom u'_s, the nearest neights are u_s and u_{s+1}. So the equations of motion are

$$M\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = C(u'_s - u_s) + C'(u'_{s-1} - u_s)$$
$$M\frac{\mathrm{d}^2u'}{\mathrm{d}t^2} = C'(u_{s+1} - u'_s) + C(u_s - u'_s)$$

where $C' = 10C$. The shifted parts are

$$u'_{s-1} = ue^{iska}e^{-ika} = ue^{iska}e^{-ika}$$
$$u_{s+1} = ue^{iska}e^{ika} = ue^{iska}e^{ika}$$

and from the previous problem e^{iska} will cancel out (but not u or u') so

$$-M\omega^2 u = C(u' - u) + C'(u'e^{-ika} - u)$$
$$-M\omega^2 u' = C'(ue^{ika} - u') + C(u - u')$$

rearranging the terms for the first EQ:

$$\begin{aligned} M\omega^2 u &= -C(u' - u) - C'(u'e^{-ika} - u) \\ &= (C + C')u + (-C - C'e^{-ika})u' \\ 0 &= (C + C' - M\omega^2)u + (-C - C'e^{-ika})u' \end{aligned}$$

and for the second EQ:

$$\begin{aligned} M\omega^2 u' &= -C'(ue^{ika} - u') - C(u - u') \\ &= (-C - C'e^{ika})u + (C + C')u' \\ 0 &= (-C - C'e^{ika})u + (C + C' - M\omega^2)u' \end{aligned}$$

so we have the matrix equation

$$\begin{pmatrix} C + C' - M\omega^2 & -C - C'e^{-ika} \\ -C - C'e^{ika} & C + C' - M\omega^2 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} = 0$$

we can solve for the eigenvalues of the matrix by finding the determinant:

For $K = 0$, the matrix is

$$\begin{pmatrix} C + C' - M\omega^2 & -C - C' \\ -C - C' & C + C' - M\omega^2 \end{pmatrix}$$

and we have a special case where $u = u'$ or

$$0 = (C + C' - M\omega^2)u + (-C - C')u = -M\omega^2 \implies \omega^2 = 0$$

and the other case where the determinant is zero

$$\begin{aligned} 0 &= (C + C' - M\omega^2)^2 - (-C - C')^2 \\ \text{using} \quad & (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \\ &= C^2 + C'^2 + M^2\omega^4 + 2CC' - 2C'M\omega^2 - 2CM\omega^2 \\ &- (C^2 + C'^2 + 2CC') \\ &= M^2\omega^4 - 2C'M\omega^2 - 2CM\omega^2 \\ &= M\omega^2 - 2(C' + C) \end{aligned}$$

and plugging in $C' = 10C$

$$0 = M\omega^2 - 2(11C) \implies \omega^2 = 22C/M$$

For $K = \pi/a$ and separation $a = \pi/2$

$$e^{ika} = e^{i\frac{\pi}{a}\frac{a}{2}} = e^{i\pi} = -1 \qquad e^{-ika} = -1$$

so the matrix is

$$\begin{pmatrix} C + C' - M\omega^2 & -C + C' \\ C - C' & C + C' - M\omega^2 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} = 0$$

so for the special case where $u = u'$:

$$\begin{aligned} 0 &= (C + C' - M\omega^2) + (-C + C') \\ &= 2C' - M\omega^2 \implies \omega^2 = 20C/M \end{aligned}$$

and the other case where the determinant is zero

$$\begin{aligned} 0 &= (11C - M\omega^2)^2 - (9C)^2 \\ &= 121C^2 - 22CM\omega^2 + M^2\omega^4 - 81C^2 \\ &= M^2\omega^4 - 22CM\omega^2 + 40C^2 \\ &= (M\omega^2)^2 - 22C(M\omega^2) + 40C^2 \end{aligned}$$

solving this quadratic equation where $a = 1, b = -22C, c = 40C^2$

$$\begin{aligned} M\omega^2 &= \frac{22C \pm \sqrt{(-22C)^2 - 4(40C^2)}}{2} = \frac{22C \pm \sqrt{484C^2 - 160C^2}}{2} \\ &= \frac{22C \pm \sqrt{324C^2}}{2} = \frac{22C \pm 18C}{2} = 20C, 2C \end{aligned}$$

so

$$\omega^2 = \frac{20C}{M}, \frac{2C}{M}$$

thus the dispersion relations are

$$\begin{aligned} K = 0: &\quad \omega = 0, \sqrt{\frac{22C}{M}} \\ K = \pi/a: &\quad \omega = \sqrt{\frac{20C}{M}}, \sqrt{\frac{2C}{M}} \end{aligned}$$

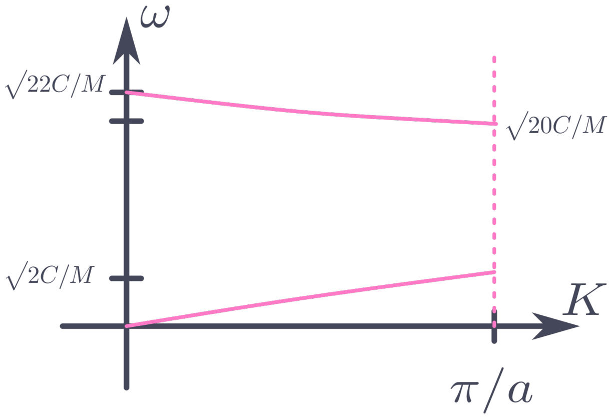

Figure 1: Dispersion relation for $K = 0, \pi/a$

Problem 3. Given

$$m\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{\tau}\right) = -eE$$

for time varying E and v

$$E = E_0 e^{-i\omega t} \qquad v = v_0 e^{-i\omega t}; \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -i\omega v$$

so solving for v

$$\begin{aligned} m\left(-i\omega v + \frac{v}{\tau}\right) &= -eE \\ v &= \frac{-eE}{m\left(-i\omega + \frac{1}{\tau}\right)}\left(\frac{\tau}{\tau}\right) \\ &= \frac{-eE}{m}\frac{\tau}{-i\omega\tau + 1} \end{aligned}$$

from Ohm's Law

$$\begin{aligned} j = \sigma E = nqv \quad &\text{or} \quad \sigma = \frac{-nev}{E} \\ &\text{where} \quad \sigma(0) = \frac{ne^2\tau}{m} \end{aligned}$$

where the charge is $q = -e$. Substituting v from the previous equation

$$\begin{aligned} \sigma &= \frac{-ne}{E}\frac{-eE}{m}\frac{\tau}{-i\omega\tau + 1} \\ &= \frac{ne^2\tau}{m}\frac{1}{-i\omega\tau + 1}\left(\frac{1 + i\omega\tau}{1 + i\omega\tau}\right) \\ \sigma(\omega) &= \sigma(0)\frac{1 + i\omega\tau}{1 + (\omega\tau)^2} \end{aligned}$$

where we multiply by the complex conjugate in the second step.