

Non-inertial Frames

In the *noninertial* frame, N1L and N2L do not hold true. In the simplest example of an accelerating frame, we have simple pendulum in a car moving in a straight line with velocity \mathbf{V} and acceleration $\dot{\mathbf{V}} = \mathbf{A}$. In the lab frame (inertial), \mathcal{S}_0 , we can relate the position of the moving frame \mathcal{S} by

$$\begin{aligned}\dot{\mathbf{r}}_0 &= \dot{\mathbf{r}} + \mathbf{V} \\ \ddot{\mathbf{r}}_0 &= \ddot{\mathbf{r}} + \mathbf{A} \\ \implies \ddot{\mathbf{r}}_0 &= \ddot{\mathbf{r}} - \mathbf{A}\end{aligned}$$

so N2L in \mathcal{S}_0 is always

$$\mathbf{F} = m\ddot{\mathbf{r}}_0 = m(\ddot{\mathbf{r}} - \mathbf{A})$$

so in the noninertial frame, \mathcal{S} , we have

$$m\ddot{\mathbf{r}} = F - m\mathbf{A}$$

The procedure of mechanics in noninertial frames is as follows:

1. write down equations in lab frame using \mathbf{r}_0
2. identify coordinates in noninertial frame

So in \mathcal{S} we have an effective gravity

$$\begin{aligned}m\vec{r} &= m\mathbf{g} - m\mathbf{A} \\ &= m\mathbf{g}_{\text{eff}}\end{aligned}$$

where the trigonometry tells us

$$\tan \phi_0 = \frac{A}{g}$$

so we have a new equation of motion

$$\ddot{\theta} = -\frac{g_{\text{eff}}}{L} \sin(\theta - \theta_0)$$

where

$$g_{\text{eff}} = \sqrt{g^2 - A^2}$$

with frequency

$$\omega = \sqrt{\frac{g_{\text{eff}}}{L}}$$

the force in the noninertial frame

$$F_{NI} = -m\mathbf{A}$$

is known as the inertial or *fictitious* force. From before we know that the gravitational force is actually

$$\begin{aligned}F_g &= -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \\ &= m\mathbf{g} \quad \mathbf{g} = -\frac{GM_E}{R_E^2}\hat{\mathbf{r}}\end{aligned}$$

where we assume a constant gravity near the surface of the Earth.

Einstein's Equivalence Principle Einstein thought that gravitational mass equals inertial mass. Thus Gravity is a fictitious force.

Tidal Forces Without considering the earth rotating on its axis or orbiting the sun, we can see that it is in a noninertial frame due to the gravitational attraction of the moon! So this lab frame (or God Frame as we must consider ourselves external to the motion of the earth) the N2L tells us

$$m\ddot{\mathbf{r}}_0 = m\mathbf{g} - \frac{GM_m m}{d^2} \hat{\mathbf{d}} + \mathbf{F}_{ext}$$

and in the noninertial frame we have

$$\mathbf{A} = -\frac{GM_m}{d_0^2} \hat{\mathbf{d}}_0$$

so N2L is

$$m\ddot{\mathbf{r}} = m\mathbf{g} - \frac{GM_m m}{d^2} \hat{\mathbf{d}} + \mathbf{F}_{ext} + \frac{GM_m m}{d_0^2} \hat{\mathbf{d}}_0$$

we call this difference of forces due to the moon the *tidal force*:

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{F}_{ext} + \mathbf{F}_{tid}$$

$$\mathbf{F}_{tid} = -GM_m m \left(\frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{d}}_0}{d_0^2} \right)$$

so on the far side of the earth (furthest from the moon) we have $d^2 < d_0^2$ so the tidal force points away from the moon. On the near side of the earth, the tidal force points towards the moon. we can also see that the tidal force is equivalent to the difference in gravitational force of the moon on the object minus the force of the moon at the center of mass of the earth. Earth as an extended object is subject to tidal forces, and applies to all extended objects such as black holes.

For example the tidal forces of a dinosaur near a black hole would stretch and compress the dinosaur.

Rotating Reference Frame What is rotation? u.r.t an axis i.e. angular velocity ω where the direction of the vector is the axis and magnitude is the speed of rotation ω . To determine the direction of rotation we use the Right Hand Rule.

Euler's Theorem The most general motion of a body about a fixed point is also a rotation about an axis that passes through the fixed point.

Common Notation:

- Ω is a fixed angular velocity
- ω is a specific (unknown) angular velocity.

From Taylor Figure 9.7, we can see that the distance from the point from the rotating axis ρ gives the relation $\rho = r \sin \theta$. And the magnitude of velocity is

$$\nu = \omega \rho = \omega r \sin \theta$$

and hence the velocity

$$\nu = \omega \times \mathbf{r} = \dot{\mathbf{r}}$$

So in general, the time derivative of a vector in a rotating frame is always in the form

$$\dot{\mathbf{u}} = \omega \times \mathbf{u}$$

Rotating Frame From last time, we have the two frames of reference:

- \mathcal{S}_0 the lab frame (God frame)
- \mathcal{S} the rotating frame

Where Ω is the angular velocity of \mathcal{S} in \mathcal{S}_0 .

For a rotation about the Earth we have

$$\Omega = \frac{2\pi}{24 \times 3600} \sim 7.3 \times 10^{-5} \text{ rad/s}$$

In \mathcal{S} we define coordinate basis vectors

$$\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$$

Which are orthogonal, and in \mathcal{S}_0 , $\hat{\mathbf{e}}_i$ rotates with ang. velocity Ω .

Goal: Write N2L in \mathcal{S} .

In \mathcal{S}_0 we have N2L

$$m\ddot{\mathbf{r}}_0 = \mathbf{F}$$

For a general vector

$$\mathbf{Q} = Q_1 \hat{\mathbf{e}}_1 + Q_2 \hat{\mathbf{e}}_2 + Q_3 \hat{\mathbf{e}}_3 = \sum_i Q_i \hat{\mathbf{e}}_i$$

In the rotating frame the change in unit vectors are constant

$$\begin{aligned} \dot{\hat{\mathbf{e}}}_i &= 0 \quad \text{in } \mathcal{S} \\ \implies \dot{\hat{\mathbf{e}}}_i &= \Omega \times \hat{\mathbf{e}}_i \quad \text{in } \mathcal{S}_0 \end{aligned}$$

So

$$\begin{aligned}\left(\frac{d\mathbf{Q}}{dt}\right)_S &= \sum_i \frac{dQ_i}{dt} \hat{\mathbf{e}}_i \\ \left(\frac{dQ}{dt}\right)_S &= \sum_i \frac{dQ_i}{dt} \hat{\mathbf{e}}_i + \sum Q_i \dot{\hat{\mathbf{e}}}_i\end{aligned}$$

And in the lab frame

$$\begin{aligned}\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} &= \sum_i \frac{dQ_i}{dt} \hat{\mathbf{e}}_i + \sum Q_i \boldsymbol{\Omega} \times \hat{\mathbf{e}}_i \\ &= \left(\frac{d\mathbf{Q}}{dt}\right)_S + \boldsymbol{\Omega} \times \mathbf{Q}\end{aligned}$$

So N2L in the lab frame is

$$\begin{aligned}m \left(\frac{d^2 \mathbf{r}}{dt^2}\right)_{S_0} &= \mathbf{F} \\ \left(\frac{d}{dt}\right)_{S_0} \left(\frac{d\mathbf{r}}{dt}\right)_{S_0} &= \left(\frac{d}{dt}\right)_{S_0} \left[\left(\frac{d\mathbf{r}}{dt}\right)_S + \boldsymbol{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_S \right] \\ &= \left(\frac{d}{dt}\right)_S \left[\left(\frac{d\mathbf{r}}{dt}\right)_S + \boldsymbol{\Omega} \times \mathbf{r} \right] + \boldsymbol{\Omega} \times \left[\left(\frac{d\mathbf{r}}{dt}\right)_S + \boldsymbol{\Omega} \times \mathbf{r} \right] \\ &= \left(\frac{d^2 \mathbf{r}}{dt^2}\right)_S + 2\boldsymbol{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_S + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})\end{aligned}$$

where we let

$$\dot{\mathbf{r}} = \left(\frac{d\mathbf{r}}{dt}\right)_S$$

which we can rewrite the N2L as

$$\mathbf{a}_{S_0} = \ddot{\mathbf{r}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

SO the N2L in the lab frame is simply

$$m\mathbf{a}_{S_0} = \mathbf{F}$$

and in the rotating frame we have

$$\begin{aligned}m(\ddot{\mathbf{r}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})) &= \mathbf{F} \\ m\ddot{\mathbf{r}} &= \mathbf{F} - 2m\boldsymbol{\Omega} \times \dot{\mathbf{r}} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\ &= \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} \\ &= \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}}\end{aligned}$$

where we have two extra terms: the Coriolis force and a Centrifugal force.

Coriolis Force From the Centrifugal force

$$\begin{aligned}\mathbf{F}_{\text{cf}} &= m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} \\ \mathbf{F}_{\text{cf}} &= m\mathbf{v} \times \boldsymbol{\Omega} \\ F_{\text{cf}} &= mv\Omega = m\rho\Omega^2 \\ &= \frac{m\rho^2\Omega^2}{\rho} = \frac{mv^2}{r}\end{aligned}$$

Example: St Louis The colatitude of St Louis is $\theta = 51.4^\circ$ so using

$$\Omega_E = 7.3 \times 10^{-5} \text{ rad/s} \quad R_E \approx 6400 \text{ km} = 6.4 \times 10^6 \text{ m}$$

we have the acceleration

$$\begin{aligned} a_{\text{cf}} &= (\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} = \rho \Omega^2 = r \sin \theta \Omega^2 \\ &\approx 2.66 \times 10^{-2} \text{ m/s}^2 \end{aligned}$$

and since we know that the acceleration due to Earth

$$a_g = \frac{GM_E}{r^2}$$

and using

$$\begin{aligned} G &= 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} \quad M_E = 5.9 \times 10^{24} \text{ kg} \\ r_{eq} &\approx 6378 \text{ km} \quad r_{pole} \approx 6356 \text{ km} \end{aligned}$$

the change in acceleration is

$$\begin{aligned} \Delta a_g &= GM_E \left(\frac{1}{r_{eq}^2} - \frac{1}{r_{pole}^2} \right) \\ &\approx 6.8 \times 10^{-2} \text{ m/s}^2 \end{aligned}$$

So we have a nonzero tangential component of acceleration due to gravity due to the rotation of the Earth (it isn't just radial). When we look at the acceleration of the Earth orbiting the sun we find

$$a_{\text{orbit}} \sim 6.0 \times 10^{-3} \text{ m/s}^2$$

which is only a third of the centrifugal force due to the Earth's axial rotation.

Review: N2L in a rotating frame:

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} \\ &= \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}} \end{aligned}$$

we have an additional Coriolis force and a Centrifugal force.

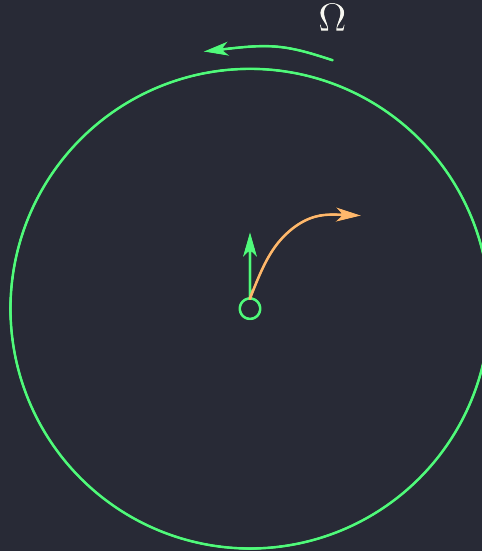


Figure 0.1: Turntable with frictionless puck. In the lab frame, the puck moves in a straight line, but in the noninertial frame—rotating with the turntable—the puck moves in a curved path.

Coriolis Force

$$\mathbf{F}_{\text{cor}} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega}$$

where

$$\mathbf{F}_{\text{cor}} \cdot \mathbf{v} = 0$$

when we look at the direction of the Coriolis force in the Earth's frame, at the North side the Coriolis force points clockwise, and at the South side the Coriolis force points counterclockwise—or the origin of cyclones.

Free Fall & Coriolis Force From Taylor Figure 9.15 we have the N2L in the rotating frame

$$\ddot{\mathbf{r}} = \mathbf{g} + 2\dot{\mathbf{r}} \times \boldsymbol{\Omega}$$

where we omit the centrifugal force. We can see the vectors are

$$\boldsymbol{\Omega} = (0, \Omega \sin \theta, \Omega \cos \theta)$$

$$\mathbf{r} = (x, y, z)$$

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$$

so

$$\dot{\mathbf{r}} \times \boldsymbol{\Omega} = (\dot{y}\Omega \cos \theta - \dot{z}\Omega \sin \theta, -\dot{x}\Omega \cos \theta, \dot{x}\Omega \sin \theta)$$

