Homework 6

Due 2/28

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1. (a) Geometrically we find a constraint

$$\tan\alpha = \frac{r}{z} \quad \text{or} \quad z = r\cot\alpha; \qquad \dot{z} = \dot{r}\cot\alpha$$

where z is the vertical position of the bead. The position vector of the bead is a linear combination of this z position and polar position:

$$\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}; \quad \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

so the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{r}^2\cot^2\alpha)$$
$$\text{using} \quad 1 + \tan^2\alpha = \sec^2\alpha \implies \cot^2\alpha = \csc^2\alpha - 1$$
$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{r}^2(\csc^2\alpha - 1)) = \frac{1}{2}m(r^2\dot{\phi}^2 + \dot{r}^2\csc^2\alpha)$$

and the potential energy is

$$U = mgz = mgr\cot\alpha$$

so the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m(r^2\dot{\phi}^2 + \dot{r}^2\csc^2\alpha) - mgr\cot\alpha$$

(b) The EL eqn for ϕ is

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) \implies mr^2\dot{\phi} = \text{constant} = \ell$$

which states the conservation of angular momentum. The EL eqn for r is

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}}$$
$$mr\dot{\phi}^2 - mgr\cot\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{r}\csc^2\alpha\right)$$
$$= m\ddot{r}\csc^2\alpha$$

where the mass cancels out so we can simplify to

$$r\dot{\phi}^2 - gr\cot\alpha = \ddot{r}\csc^2\alpha$$
$$0 = \ddot{r} - r\dot{\phi}^2\sin^2\alpha + g\frac{\cos\alpha}{\sin\alpha}\sin^2\alpha$$
$$0 = \ddot{r} - r\dot{\phi}^2\sin^2\alpha + g\cos\alpha\sin\alpha$$

from the conservation of angular momentum

$$mr^2\dot{\phi} = \ell \implies \dot{\phi} = \frac{\ell}{mr^2}$$

so

$$0 = \ddot{r} - \frac{\ell^2}{m^2 r^3} \sin^2 \alpha + g \cos \alpha \sin \alpha$$

and solving for when $r = r_o \implies \ddot{r} = 0$:

$$\begin{aligned}
0 = 0 - \frac{\ell^2}{m^2 r_o^3} \sin^2 \alpha + g \cos \alpha \sin \alpha \\
\frac{\ell^2}{m^2 r_o^3} \sin^2 \alpha = g \cos \alpha \sin \alpha \\
r_o^3 = \frac{\ell^2}{m^2 g} \frac{\sin \alpha}{\cos \alpha} = \frac{\ell^2}{m^2 g} \tan \alpha \\
r_o = \left(\frac{\ell^2}{m^2 g} \tan \alpha \right)^{1/3}
\end{aligned}$$

we can analyze the stability of this solution by assuming a small deviation from the eq point:

$$r = r_o + \eta$$

and looking at how the second derivative behaves: rewriting the EL eqn,

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} \sin^2 \alpha - g \cos \alpha \sin \alpha$$

and we find a substitution to directly compare the two terms:

$$\frac{\ell^2}{m^2 r_o^3} \sin^2 \alpha = \frac{\ell^2}{m^2 \frac{\ell^2}{m^2 g} \tan \alpha} \sin^2 \alpha = \frac{g}{\tan \alpha} \sin^2 \alpha = g \cos \alpha \sin \alpha$$

so

$$\begin{aligned}
\ddot{r} &= \frac{\ell^2}{m^2 r^3} \sin^2 \alpha - \frac{\ell^2}{m^2 r_o^3} \sin^2 \alpha \\
&= \frac{\ell^2}{m^2} \sin^2 \alpha \left(\frac{1}{(r_o + \eta)^3} - \frac{1}{r_o^3} \right)
\end{aligned}$$

where

$$\frac{1}{(r_o + \eta)^3} < \frac{1}{r_o^3}$$

so when r slightly increases ($\eta > 0$), the bead tends back toward the eq point ($\ddot{r} < 0$) and when r slightly decreases ($\eta < 0$), the bead tends back toward the eq point i.e.

$$\frac{1}{(r_o - \eta)^3} > \frac{1}{r_o^3}$$

thus r_o is a stable equilibrium point.

2. From the