Physics 411: Mechanics

Homework: Junseo Shin

Contents

Homework 1

Due 1/24 9pm

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1. Given: the 2D Cartesian relation to polar coordinates

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \quad \hat{\phi} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \tag{1}$$

We can write v as a linear combination of $\hat{\mathbf{r}}$ and $\hat{\phi}$

$$\begin{aligned}
\mathbf{v} &= v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} \\
&= v_r(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi) + v_\phi(-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi) \\
&= (v_r\cos\phi - v_\phi\sin\phi)\hat{\mathbf{x}} + (v_r\sin\phi + v_\phi\cos\phi)\hat{\mathbf{y}}
\end{aligned}$$

and since we know the vector in Cartesian coordinates is

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

we can equate the components to get

$$\begin{aligned}
v_x &= v_r\cos\phi - v_\phi\sin\phi \\
v_y &= v_r\sin\phi + v_\phi\cos\phi
\end{aligned}$$

multiplying the first equation by $\cos\phi$ and the second by $\sin\phi$ and adding them together

$$\begin{aligned}
v_x\cos\phi &= v_r\cos^2\phi - v_\phi\sin\phi\cos\phi \\
v_y\sin\phi &= v_r\sin^2\phi + v_\phi\sin\phi\cos\phi \\
v_x\cos\phi + v_y\sin\phi &= v_r(\cos^2\phi + \sin^2\phi)
\end{aligned}$$

or simply

$$v_r = v_x\cos\phi + v_y\sin\phi$$

Likewise,

$$\begin{aligned}
v_y\cos\phi &= v_r\sin\phi\cos\phi + v_\phi\sin^2\phi \\
v_x\sin\phi &= v_r\sin\phi\cos\phi - v_\phi\cos^2\phi
\end{aligned}$$

and subtracting the second equation from the first

$$v_y\cos\phi - v_x\sin\phi = v_\phi(\sin^2\phi + \cos^2\phi)$$

Therefore we get the components of $\mathbf{v}$ in polar coordinates

$$\boxed{v_r = v_x\cos\phi + v_y\sin\phi}$$

$$\boxed{v_\phi = -v_x\sin\phi + v_y\cos\phi}$$

Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are *constant*, the time derivatives of (1) are

$$\begin{aligned}
\dot{\hat{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\phi) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\phi) \\
&= (-\dot{\phi}\sin\phi)\hat{\mathbf{x}} + (\dot{\phi}\cos\phi)\hat{\mathbf{y}} = \dot{\phi}\hat{\phi}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\hat{\phi}} &= \frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi} = -\hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\phi) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\phi) \\
&= (-\dot{\phi}\cos\phi)\hat{\mathbf{x}} + (-\dot{\phi}\sin\phi)\hat{\mathbf{y}} = -\dot{\phi}\hat{\mathbf{r}}
\end{aligned}$$

2. From Taylor Problem 1.45: [*Hint:* Consider the derivative of $\mathbf{v}^2$].
Since the magnitude of $\mathbf{v}(t)$ is also $\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$, the derivative of $\mathbf{v}^2$ is tells us if the magnitude is constant.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\cdot\mathbf{v}(t)) \\ &= 2\dot{\mathbf{v}}(t)\cdot\mathbf{v}(t)\end{aligned}$$

The magnitude of $\mathbf{v}(t)$ is constant if $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 = 0$. Since the dot product is zero, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$.

3. Using the product rule for the dot product $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}\cdot\mathbf{y}) = \dot{\mathbf{x}}\cdot\mathbf{y} + \mathbf{x}\cdot\dot{\mathbf{y}}$

$$\begin{aligned}\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] &= \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\frac{d\mathbf{r}}{dt} \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\mathbf{a}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\mathbf{v} \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r})\end{aligned}$$

The cross product of a vector with itself is zero ($\mathbf{v}\times\mathbf{v} = 0$), and the dot product of orthogonal vectors are zero (acceleration and position are perpendicular based on the result from Problem 2) QED

4. Given

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \tag{3}$$

(a) Moving everything to one side:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

multiplying by $\dot{\theta}$ and grouping terms:

$$\begin{aligned}\dot{\theta}\ddot{\theta} + \frac{g}{l}\dot{\theta}\sin\theta &= 0 \\ \dot{\theta}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}\right) + \frac{g}{l}\sin\theta\frac{\mathrm{d}}{\mathrm{d}t}(\theta) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\theta}^2\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g}{l}\cos\theta\right) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta\right) &= 0\end{aligned}$$

so the integral constant is

$$\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = C$$

Initially the pendulum starts at rest, $\dot{\theta}(t=0) = 0$ and $\theta(0) = \theta_o$ thus

$$\boxed{C = -\frac{g}{l}\cos\theta_o}$$

(b) Rewriting $X = C$

$$\begin{aligned}\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta &= -\frac{g}{l}\cos\theta_o \\ \dot{\theta} &= \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_o)}\end{aligned}$$

using separation of variables

$$\mathrm{d}\theta = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_o)}\,\mathrm{d}t$$

Integrating gives the analytic solution

$$\boxed{\theta(t) = \sqrt{\frac{2g}{l}}\int_0^T \sqrt{\cos\theta - \cos\theta_o}\,\mathrm{d}t}$$

where T is the period of the pendulum.
(c) The period of the pendulum is the time it takes to complete one cycle. Since

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

using separation of variables

$$\mathrm{d}t = \frac{1}{\dot{\theta}}\,\mathrm{d}\theta$$

Integrating both sides gives the period

$$\int \mathrm{d}t = \int \frac{1}{\dot{\theta}}\,\mathrm{d}\theta$$

$$\boxed{T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_o} \frac{1}{\sqrt{\cos\theta - \cos\theta_o}}\,\mathrm{d}\theta}$$

where the constant '4' comes from the fact that the total cycle is 4 times the period it takes to go from path from θ_o to 0.

Homework 2

Due 1/31

Check for the code. asdfasdf stuff here
why is this not wokring here

Homework 3

Due 2/7 9pm

1. The Center of Mass of the system is

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,\mathrm{d}m$$

and the mass element is the mass denisty times the volume element

$$\mathrm{d}m = \rho\,\mathrm{d}V = \frac{M}{2\pi R^2}R^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta = \frac{M}{2\pi}\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$$

where the there is no $\mathrm{d}r$ term because the radius is constant. Since the center of mass is symmetric about the x and y axes, $X_{cm} = Y_{cm} = 0$. The z component of the center of mass is

$$\begin{aligned} Z_{cm} &= \frac{1}{M}\int z\,\mathrm{d}m \\ &= \frac{1}{M}\frac{M}{2\pi}\iint z\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \end{aligned}$$

where $z = R\cos\phi$ so

$$\begin{aligned} Z_{cm} &= \frac{R}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\cos\phi\sin\phi\,\mathrm{d}\phi \\ &\text{using}\quad u = \sin\phi \implies \mathrm{d}u = \cos\phi\,\mathrm{d}\phi \\ &= \frac{R}{2\pi}[2\pi]\int_0^1 u\,\mathrm{d}u \\ Z_{cm} &= \frac{R}{2} \end{aligned}$$

So the COM is at $\boxed{\left(0, 0, \frac{R}{2}\right)}$

2. (a) From N3L the force of the jettisoned fuel on the rocket is equal and opposite to the thrust on the rocket from the jettisoned fuel:

$$\begin{aligned} F_{\text{fuel}} &= -F_{\text{thrust}} \\ \dot{m}v_{ex} &= -\dot{m}v_{ex} \end{aligned}$$

So using N2L, the sum of the forces on the rocket is the thrust and air resistance:

$$F = m\dot{v} = F_{\text{thrust}} - f = -\dot{m}v_{ex} - bv$$

(b) Using $\dot{m} = -k$

$$\begin{aligned} m\dot{v} &= kv_{ex} - bv \\ \frac{m}{b}\dot{v} &= \frac{kv_{ex}}{b} - v \end{aligned}$$

defining the constant $a = \frac{kv_{ex}}{b}$ and using separation of variables

$$\frac{1}{a-v}\,\mathrm{d}v = \frac{b}{m}\,\mathrm{d}t$$

we can write an expression for m as a function of time through using separtion of variables again:

$$\begin{aligned} \frac{\mathrm{d}m}{\mathrm{d}t} &= -k \\ \int_{m_o}^{m} \mathrm{d}m' &= -k\int_0^t \mathrm{d}t' \\ m - m_o &= -kt \\ m &= m_o - kt \end{aligned}$$

where m_o is the initial mass of the rocket. Substituting this back into main expression and integrating both sides:

$$\begin{aligned} \int_0^v \frac{1}{a-v'}\,\mathrm{d}v' &= b\int_0^t \frac{1}{m_o - kt'}\,\mathrm{d}t' \\ -\ln(a-v')\Big|_0^v &= -\frac{b}{k}\ln(m_o - kt')\Big|_0^t \\ -\ln(a-v) + \ln(a) &= \frac{b}{k}[-\ln(m_o - kt) + \ln(m_o)] \\ \ln\left(\frac{a}{a-v}\right) &= \frac{b}{k}\ln\left(\frac{m_o}{m_o - kt}\right) \end{aligned}$$

substituting back in $m = m_o - kt$ and exponentiating both sides:

$$\begin{aligned} \frac{a}{a-v} &= \left(\frac{m_o}{m}\right)^{\frac{b}{k}} \\ a\left(\frac{m_o}{m}\right)^{-\frac{b}{k}} &= a - v \\ v &= a - a\left(\frac{m_o}{m}\right)^{-\frac{b}{k}} \\ v &= a\left[1 - \left(\frac{m}{m_o}\right)^{\frac{b}{k}}\right] \end{aligned}$$

subbing back in $a = \frac{kv_{ex}}{b}$ we get the final expression

$$v(m) = \frac{kv_{ex}}{b}\left[1 - \left(\frac{m}{m_o}\right)^{\frac{b}{k}}\right]$$

3. (a) The angular momentum vector is

$$\begin{aligned}\boldsymbol{\ell} &= \mathbf{r} \times \mathbf{p} \\ &= \mathbf{r} \times m\dot{\mathbf{r}} \\ &= m\mathbf{r} \times \dot{\mathbf{r}}\end{aligned}$$

from HW 1, we know that

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

so

$$\begin{aligned}\boldsymbol{\ell} &= m(r\hat{\mathbf{r}}) \times (\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) \\ &= m[r\dot{r}(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) + r(r\dot{\phi})(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}})] \\ &= m[0 + r^2\dot{\phi}\hat{\mathbf{z}}] = mr^2\omega\hat{\mathbf{z}}\end{aligned}$$

where $\omega = \dot{\phi}$ and the magnitude of the angular momentum is

$$\ell = |\boldsymbol{\ell}| = mr^2\omega$$

(b) The area swept by an infinitesimal change in the planets position is equivalent to the area of a triangle as shown in Figure 2, so

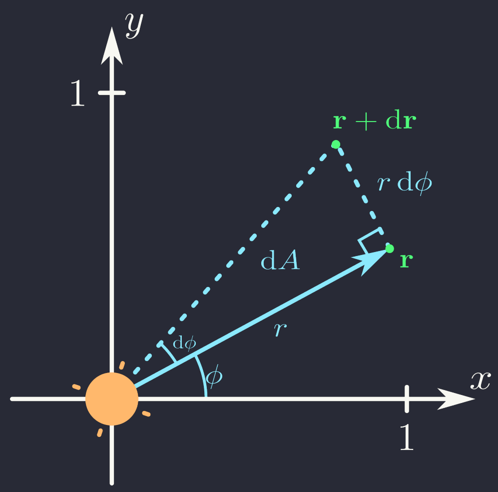

Figure 3.1: Area swept by planet

$$\mathrm{d}A = \frac{1}{2}r(r\,\mathrm{d}\phi) = \frac{1}{2}r^2\,\mathrm{d}\phi$$

dividing both sides by $\mathrm{d}t$ gives us

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{2}r^2\omega$$

and from part (a) we know that $\ell = mr^2\omega$ or $\omega = \frac{\ell}{mr^2}$ so

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\ell}{mr^2} = \frac{\ell}{2m}$$

therefore, the rate in change of the area swept by the planet is a constant that is proportional to ℓ.

4. $\mathbf{F} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$ from $P = (1,0)$ to $Q = (0,1)$
(a) For a straight line the path is given by

$$y = 1 - x \qquad \mathrm{d}y = -\,\mathrm{d}x$$

so the work done is

$$\begin{aligned} W_a &= \int_P^Q F_x\,\mathrm{d}x + F_y\,\mathrm{d}y = \int_P^Q -y\,\mathrm{d}x + x\,\mathrm{d}y \\ &= \int_{x=1}^{0} -(1-x)\,\mathrm{d}x + x(-\,\mathrm{d}x) \\ &= \int_1^0 -1\,\mathrm{d}x = 1 \end{aligned}$$

(b) For a circular path of radius 1, the path in polar coordinates is

$$\begin{aligned} y &= \sin\phi \to \mathrm{d}y = \cos\phi\,\mathrm{d}\phi \\ x &= \cos\phi \to \mathrm{d}x = -\sin\phi\,\mathrm{d}\phi \end{aligned}$$

from the equation of a circle $x^2 + y^2 = 1$. The limits of integration are $\phi = 0 \to \pi/2$, and the work is

$$\begin{aligned} W_b &= \int_{\phi=0}^{\pi/2} -\sin\phi(-\sin\phi)\,\mathrm{d}\phi + \cos\phi\cos\phi\,\mathrm{d}\phi \\ &= \int_{\phi=0}^{\pi/2} 1\,\mathrm{d}\phi = \frac{\pi}{2} \end{aligned}$$

(c) Splitting this into two paths: For path 1, $y = 0$; $\mathrm{d}y = 0$; and $x = 1 \to 0$ so

$$W_1 = \int_{x=1}^{0} 0\,\mathrm{d}x + x(0) = 0$$

For path 2, $x = 0$; $\mathrm{d}x = 0$; $y = 0 \to 1$ so

$$W_1 = \int_{y=0}^{1} -y(0) + 0\,\mathrm{d}y = \int_{y=0}^{1} 0\,\mathrm{d}y = 0$$

And the work done is $W_c = W_1 + W_2 = 0$
(d) The force is not conservative because the work done is path dependent! We can also double check by taking the curl:

$$\boldsymbol{\nabla} \times \mathbf{F} = \det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & x \end{vmatrix} = \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right)\hat{\mathbf{z}} = 2\hat{\mathbf{z}}$$

which is not zero, so the force is not conservative.

5. (a) Using the time derivatives of the polar unit vectors

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \dot{\phi}\hat{\boldsymbol{\phi}} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$$

Acceleration in polar coordinates is

$$\begin{aligned}
\mathbf{a} = \dot{\mathbf{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}\right)\\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + (\dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}(-\dot{\phi}\hat{\mathbf{r}}))\\
&= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}
\end{aligned}$$

so the radial and angular components of the force are

$$\begin{aligned}
F_r &= ma_r = m(\ddot{r} - r\dot{\phi}^2)\\
F_\phi &= ma_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})
\end{aligned}$$

and since the spring force is conservative with magnitude $F_s = -k(r-a)\hat{\mathbf{r}}$ the equations of motion are

$$\begin{aligned}
m(\ddot{r} - r\dot{\phi}^2) &= -k(r-a)\\
m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) &= 0
\end{aligned}$$

or

$$\begin{aligned}
m\ddot{r} - mr\dot{\phi}^2 + k(r-a) &= 0\\
r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0
\end{aligned}$$

(b) The initial angular momentum of the system is

$$\ell_o = mv_o a$$

and after some time the angular momentum is (from Problem 3)

$$\ell = mr^2\dot{\phi}$$

and using the conservation of angular momentum

$$\begin{aligned}
\ell_o &= \ell\\
mv_o a &= mr^2\dot{\phi}\\
\dot{\phi} &= \frac{v_o a}{r^2}
\end{aligned}$$

(c) First the initial mechanical energy of the system is purely kinetic given by the initial velocity:

$$E_o = T_o = \frac{1}{2}mv_o^2$$

the total mechanical energy of the system after some time will be the sum of the kinetic and potential energies:

$$\begin{aligned}
U &= -\int_0^r \mathbf{F}\cdot\mathrm{d}\mathbf{r}' = \int_0^r k(r-a)\,\mathrm{d}r' = \frac{1}{2}k(r-a)^2\\
T &= \frac{1}{2}mv^2 = \frac{1}{2}m(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}m(\dot{\mathbf{r}}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}})\cdot(\dot{\mathbf{r}}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)
\end{aligned}$$

And from the conservation of energy

$$\begin{aligned}
E_o &= E = T + U\\
\frac{1}{2}mv_o^2 &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}k(r-a)^2\\
v_o^2 &= \dot{r}^2 + r^2\dot{\phi}^2 + \frac{k}{m}(r-a)^2
\end{aligned}$$

substituting the result from part (b) and solving for $\dot{r}$:

$$\begin{aligned}
v_o^2 &= \dot{r}^2 + r^2\left(\frac{v_o a}{r^2}\right)^2 + \frac{k}{m}(r-a)^2 \\
v_o^2 &= \dot{r}^2 + \frac{v_o^2 a^2}{r^2} + \frac{k}{m}(r-a)^2 \\
\dot{r}^2 &= v_o^2 - \frac{v_o^2 a^2}{r^2} - \frac{k}{m}(r-a)^2 \\
\dot{r} &= \sqrt{v_o^2\left(1 - \frac{a^2}{r^2}\right) - \frac{k}{m}(r-a)^2}
\end{aligned}$$

Homework 5

Due 2/21

1. (a) If f is independent of y, then

$$\frac{\partial f}{\partial y} = 0$$

and using the Euler-Lagrange (EQ) equation, we have

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

so

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

and for the derivative to be zero,

$$\frac{\partial f}{\partial y'} = \text{constant}$$

(b) Since f is independent of x,

$$\frac{\partial f}{\partial x} = 0$$

Using the Euler-Lagrange equation, we have

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

Using chain rule to differentiate f with respect to x for a general function $f(x, y, y')$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x, y, y') &= \frac{\partial f}{\partial x}\left(\frac{\mathrm{d}}{\mathrm{d}x} x\right) + \frac{\partial f}{\partial y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{\partial f}{\partial y'}\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
\frac{\mathrm{d}}{\mathrm{d}x} f(y, y') &= 0 + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''
\end{aligned}$$

and substituting what we got from the EL equation,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(y, y') = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] y' + \frac{\partial f}{\partial y'} y'' = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] y' + \frac{\partial f}{\partial y'}\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x} f = \frac{\mathrm{d}}{\mathrm{d}x}\left(y' \frac{\partial f}{\partial y'}\right)$$

from chain rule. Moving everything to one side:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f - \frac{\mathrm{d}}{\mathrm{d}x}\left(y' \frac{\partial f}{\partial y'}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(f - y' \frac{\partial f}{\partial y'}\right) &= 0
\end{aligned}$$

which is only true if

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}$$

2. (a) In 2D, we know the length of a short segment is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

and in 3D

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$$

transforming to spherical coordinates:

$$\begin{aligned}
&x = r\cos\phi\sin\theta \quad \mathrm{d}x = \mathrm{d}r\cos\phi\sin\theta - r\sin\phi\sin\theta\mathrm{d}\phi + r\cos\phi\cos\theta\mathrm{d}\theta \\
&y = r\sin\phi\sin\theta \quad \mathrm{d}y = \mathrm{d}r\sin\phi\sin\theta + r\cos\phi\sin\theta\mathrm{d}\phi + r\sin\phi\cos\theta\mathrm{d}\theta \\
&z = r\cos\theta \quad \mathrm{d}z = \mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta
\end{aligned}$$

for the sphere of radius $r \to R$ and $\mathrm{d}r = 0$ since radius is constant, so

$$\begin{aligned}
\mathrm{d}x &= -R\sin\phi\sin\theta\mathrm{d}\phi + R\cos\phi\cos\theta\mathrm{d}\theta \\
\mathrm{d}y &= R\cos\phi\sin\theta\mathrm{d}\phi + R\sin\phi\cos\theta\mathrm{d}\theta \\
\mathrm{d}z &= -R\sin\theta\mathrm{d}\theta
\end{aligned}$$

and squaring each term:

$$\begin{aligned}
\mathrm{d}x^2 &= R^2\sin^2\phi\sin^2\theta\mathrm{d}\phi^2 + R^2\cos^2\phi\cos^2\theta\mathrm{d}\theta^2 - 2R^2\sin\phi\sin\theta\cos\phi\cos\theta\mathrm{d}\phi\mathrm{d}\theta \\
\mathrm{d}y^2 &= R^2\cos^2\phi\sin^2\theta\mathrm{d}\phi^2 + R^2\sin^2\phi\cos^2\theta\mathrm{d}\theta^2 + 2R^2\sin\phi\sin\theta\cos\phi\cos\theta\mathrm{d}\phi\mathrm{d}\theta \\
\mathrm{d}z^2 &= R^2\sin^2\theta\mathrm{d}\theta^2
\end{aligned}$$

when we add all three equations we can see that the last term in $\mathrm{d}x^2$ and $\mathrm{d}y^2$ cancel out, and grouping the like terms we get

$$\begin{aligned}
R^2\sin^2\phi\sin^2\theta\mathrm{d}\phi^2 + R^2\cos^2\phi\sin^2\theta\mathrm{d}\phi^2 &= R^2\sin^2\theta\mathrm{d}\phi^2(\sin^2\phi + \cos^2\phi) \\
&= R^2\sin^2\theta\mathrm{d}\phi^2
\end{aligned}$$

and

$$\begin{aligned}
&R^2\cos^2\phi\cos^2\theta\mathrm{d}\theta^2 + R^2\sin^2\phi\cos^2\theta\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\theta^2 \\
&\qquad = R^2\cos^2\theta\mathrm{d}\theta^2(\cos^2\phi + \sin^2\phi) + R^2\sin^2\theta\mathrm{d}\theta^2 \\
&\qquad = R^2\mathrm{d}\theta^2(\cos^2\theta + \sin^2\theta) \\
&\qquad = R^2\mathrm{d}\theta^2
\end{aligned}$$

so the length of a short segment in spherical coordinates is

$$\begin{aligned}
\mathrm{d}s &= \sqrt{R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2} \\
&= \sqrt{R^2\mathrm{d}\theta^2\left(\frac{\mathrm{d}\theta^2}{\mathrm{d}\theta^2} + \sin^2\theta\frac{\mathrm{d}\phi^2}{\mathrm{d}\theta^2}\right)} \\
\text{using}\quad \frac{\mathrm{d}\phi}{\mathrm{d}\theta} &= \phi'(\theta) \\
&= R\sqrt{1 + \sin^2\theta\phi'(\theta)^2}\mathrm{d}\theta
\end{aligned}$$

and the total path length L is found by integrating $\mathrm{d}s$ from θ_a to θ_b:

$$L = R\int_{\theta_a}^{\theta_b}\sqrt{1 + \sin^2\theta\phi'(\theta)^2}\mathrm{d}\theta$$

(b) We find that the integral function is independent of ϕ or

$$f = f(\theta, \phi') = \sqrt{1 + \sin^2\theta\phi'(\theta)^2}$$

so from Problem 1a, we know that

$$\frac{\partial f}{\partial \phi'} = \text{constant} = C$$
$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta\phi'(\theta)}{\sqrt{1+\sin^2\theta\phi'(\theta)^2}}$$

setting one of the points to be the north pole, $\theta_a = 0$, so the constant is

$$\frac{\sin^2(0)\phi'(0)}{\sqrt{1+\sin^2(0)\phi'(0)^2}} = 0 = C$$

solving for $\phi'(\theta)$

$$\frac{\sin^2\theta\phi'(\theta)}{\sqrt{1+\sin^2\theta\phi'(\theta)^2}} = 0$$
$$\phi'(\theta) = 0$$

using separation of variables,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = 0$$
$$\int \mathrm{d}\phi = \int 0\mathrm{d}\theta$$
$$\phi(\theta) = C_2$$

Since ϕ is a constant, this is equivalent to to a slice of the sphere through the north pole, which has a cross section of a circle with radius R. The path follows the circumference of the circle, from the north pole to the point at θ_b.

3. (a) Finding the 2D length element in polar coordinates:

$$x = r\cos\phi \quad \mathrm{d}x = \mathrm{d}r\cos\phi - r\sin\phi\mathrm{d}\phi$$
$$y = r\sin\phi \quad \mathrm{d}y = \mathrm{d}r\sin\phi + r\cos\phi\mathrm{d}\phi$$

so the length element is

$$\begin{aligned}
\mathrm{d}x^2 + \mathrm{d}y^2 &= (\mathrm{d}r\cos\phi - r\sin\phi\mathrm{d}\phi)^2 + (\mathrm{d}r\sin\phi + r\cos\phi\mathrm{d}\phi)^2 \\
&= \mathrm{d}r^2\cos^2\phi + r^2\sin^2\phi\mathrm{d}\phi^2 - 2r\sin\phi\mathrm{d}r\mathrm{d}\phi \\
&+ \mathrm{d}r^2\sin^2\phi + r^2\cos^2\phi\mathrm{d}\phi^2 + 2r\cos\phi\mathrm{d}r\mathrm{d}\phi \\
&= \mathrm{d}r^2(\cos^2\phi + \sin^2\phi) + r^2\mathrm{d}\phi^2(\sin^2\phi + \cos^2\phi) \\
&= \mathrm{d}r^2 + r^2\mathrm{d}\phi^2
\end{aligned}$$

so the length element in polar coordinates is

$$\mathrm{d}l = \sqrt{\mathrm{d}r^2 + r^2\mathrm{d}\phi^2}$$

and solving for $\mathrm{d}\phi$,

$$\begin{aligned}
\mathrm{d}l^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \\
\mathrm{d}\phi^2 &= \frac{\mathrm{d}l^2 - \mathrm{d}r^2}{r^2} \\
\mathrm{d}\phi &= \frac{1}{r}\sqrt{\mathrm{d}l^2 - \mathrm{d}r^2} \\
\mathrm{d}\phi &= \frac{1}{r}\sqrt{1 - \left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)^2}\mathrm{d}l
\end{aligned}$$

plugging into the area integral:

$$\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2}r^2\mathrm{d}\phi \\
&= \int_0^L \frac{1}{2}r^2\left[\frac{1}{r}\sqrt{1 - \left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)^2}\mathrm{d}l\right] \\
&= \frac{1}{2}\int_0^L r\sqrt{1 - \left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)^2}\mathrm{d}l
\end{aligned}$$

so the function in the integrand is

$$f = f(r, r') = r\sqrt{1 - r'^2}$$

(b) From Problem 1b we know that the conserved quantity we know that f is independent of l, so

$$f - r'\frac{\partial f}{\partial r'} = \text{constant} = K$$

using the partial derivative

$$\frac{\partial f}{\partial r'} = \frac{r}{2\sqrt{1 - r'^2}}(-2r') = -\frac{rr'}{\sqrt{1 - r'^2}}$$

So the conserved quantity is

$$\begin{aligned}
K &= r\sqrt{1 - r'^2} + \frac{rr'^2}{\sqrt{1 - r'^2}} \\
&= \frac{r(1 - r'^2) + rr'^2}{\sqrt{1 - r'^2}} \\
K &= \frac{r}{\sqrt{1 - r'^2}}
\end{aligned}$$

rearranging for r';

$$
\begin{aligned}
K\sqrt{1-r'^2} &= r \\
K^2(1-r'^2) &= r^2 \\
K^2 - K^2r'^2 &= r^2 \\
K^2r'^2 &= r^2 - K^2 \\
r'^2 &= \frac{r^2}{K^2} - 1 \\
r' &= \sqrt{\frac{r^2}{K^2} - 1} = \frac{\mathrm{d}r}{\mathrm{d}l}
\end{aligned}
$$

using separation of variables:

$$
\mathrm{d}l = \frac{\mathrm{d}r}{\sqrt{r^2/K^2 - 1}}
$$

using the substitution $u = r/K$ and $\mathrm{d}u = \mathrm{d}r/K$:

$$
\begin{aligned}
\mathrm{d}l &= \frac{K\mathrm{d}u}{\sqrt{u^2-1}} \\
\int \mathrm{d}l &= K\int \frac{\mathrm{d}u}{\sqrt{u^2-1}} \\
l &= K\,\mathrm{arccosh}(r/K) + C \\
\mathrm{arccosh}(r/K) &= \frac{l-C}{K} \\
r &= K\cosh\left(\frac{l-C}{K}\right)
\end{aligned}
$$

Since the constraint l is constant as the total length of the curve (K & C are also constants), $r =$ constant is a solution to the equation or the radius of the curve is constant. This is only true for circles which have a constant radial distance from the origin, so circles leads to the maximum area integral.

4. (a) The kinetic energy of the particle is

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$$

so the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - U$$

where $U = U(r,\theta,\phi)$ and the Euler-Lagrange equations are

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right)\\
\frac{\partial\mathcal{L}}{\partial\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)\\
\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right)
\end{aligned}$$

For the first equation:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial r} &= mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - \frac{\partial}{\partial r}U(r)\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = m\ddot{r}
\end{aligned}$$

so

$$\begin{aligned}
m\ddot{r} &= mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - \frac{\partial U}{\partial r}\\
-\frac{\partial U}{\partial r} &= m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)
\end{aligned}$$

For the second equation:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= mr^2\dot{\phi}^2\sin\theta\cos\theta - \frac{\partial U}{\partial\theta}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}
\end{aligned}$$

so

$$\begin{aligned}
mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= mr^2\dot{\phi}^2\sin\theta\cos\theta - \frac{\partial U}{\partial\theta}\\
-\frac{\partial U}{\partial\theta} &= m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\dot{\phi}^2\sin\theta\cos\theta)\\
-\frac{1}{r}\frac{\partial U}{\partial\theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)
\end{aligned}$$

and for the third equation:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= -\frac{\partial U}{\partial\phi}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\sin^2\theta\right)\\
&= 2mr\dot{r}\dot{\phi}\sin^2\theta + mr^2\ddot{\phi}\sin^2\theta + 2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta
\end{aligned}$$

so

$$\begin{aligned}
mr^2\ddot{\phi}\sin^2\theta &= -2mr\dot{r}\dot{\phi}\sin^2\theta - 2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta - \frac{\partial U}{\partial\phi}\\
-\frac{\partial U}{\partial\phi} &= m(r^2\ddot{\phi}\sin^2\theta + 2r\dot{r}\dot{\phi}\sin^2\theta + 2r^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta)\\
-\frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi} &= m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)
\end{aligned}$$

where we have the 3 equations of motion:

$$
\begin{aligned}
-\frac{\partial U}{\partial r} &= m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta) \\
-\frac{1}{r}\frac{\partial U}{\partial \theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta) \\
-\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} &= m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)
\end{aligned}
$$

where from N2L in spherical coordinates, the components of acceleration are

$$
\begin{aligned}
a_r &= \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta \\
a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta \\
a_\phi &= r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta
\end{aligned}
$$

and the conservative force is

$$
\mathbf{F} = -\boldsymbol{\nabla} U = -\frac{\partial U}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}
$$

To compare with N2L we start with the unit vectors in spherical coordinates:

$$
\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
\hat{\theta} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\
\hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}
\end{aligned}
$$

and from the velocity equation we know that the derivative of the radial unit vector is

$$
\dot{\hat{r}} = \dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}
$$

for the colatitude unit vector in the θ direction

$$
\begin{aligned}
\dot{\hat{\theta}} &= (-\dot{\theta}\sin\theta\cos\phi - \dot{\phi}\cos\theta\sin\phi)\hat{\mathbf{x}} + (-\dot{\theta}\sin\theta\sin\phi + \dot{\phi}\cos\theta\cos\phi)\hat{\mathbf{y}} - \dot{\theta}\cos\theta\hat{\mathbf{z}} \\
&= \dot{\phi}\cos\theta\hat{\boldsymbol{\phi}} - \dot{\theta}\hat{\mathbf{r}}
\end{aligned}
$$

(linear combination of unit vectors) and for the azimuthal unit vector in the ϕ direction

$$
\begin{aligned}
\dot{\hat{\phi}} &= -\dot{\phi}\sin\phi\hat{\mathbf{x}} - \dot{\phi}\cos\phi\hat{\mathbf{y}} \\
&= -\dot{\phi}\sin\theta\hat{\mathbf{r}} - \dot{\phi}\cos\theta\hat{\boldsymbol{\theta}}
\end{aligned}
$$

with this in hand taking the time derivative of velocity:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right)
$$

the first term is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{\mathbf{r}}) &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}
\end{aligned}
$$

the second term is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\theta}\hat{\theta}\right) &= (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + r\dot{\theta}(\dot{\phi}\cos\theta\hat{\boldsymbol{\phi}} - \dot{\theta}\hat{\mathbf{r}}) \\
&= (-r\dot{\theta}^2)\hat{\mathbf{r}} + (\dot{r}\ddot{\theta})\hat{\boldsymbol{\theta}} + (r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}}
\end{aligned}
$$

and the third term is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}\right) &= (\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}} + r\dot{\phi}\sin\theta(-\dot{\phi}\sin\theta\hat{\mathbf{r}} - \dot{\phi}\cos\theta\hat{\boldsymbol{\theta}}) \\ &= (-r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + (-r\dot{\phi}^2\sin\theta\cos\theta)\hat{\boldsymbol{\theta}} + (\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}}\end{aligned}$$

so combing all the terms:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\boldsymbol{\theta}} \\ &\quad + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}}\end{aligned}$$

5. (a) In the frame where the cart is at rest at $x' = x - v_o t$ this is simply the Brachistochrone where the time of travel is

$$T = \int_A^B \frac{\mathrm{d}s}{v}$$

and the short segment length is

$$\mathrm{d}s = \sqrt{\mathrm{d}x'^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x'}\right)^2}\,\mathrm{d}x' \quad \text{or} \quad \sqrt{1+y'^2}\mathrm{d}x$$

and from the conservation of energy

$$\frac{1}{2}mv^2 = mgy \quad \text{or} \quad v = \sqrt{2gy}$$

so

$$T = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\mathrm{d}x'$$

where the integral function is independent of x;

$$f = f(y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

so using the second form of the EL equation from Problem 1b, we have

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

using using the partial derivative

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gy}}\frac{y'}{\sqrt{1+y'^2}}$$

so the conserved quantity is

$$\begin{aligned}
C &= \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - \frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} \\
&= \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}}\left[\frac{\cancel{\sqrt{2gy}}\sqrt{1+y'^2}\sqrt{1+y'^2}}{\cancel{\sqrt{2gy}}} - y'^2\right] \\
&= \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} \\
\text{or} \quad 2C^2 g &= \frac{1}{y(1+y'^2)}
\end{aligned}$$

setting a new constant

$$2C^2 g = \frac{1}{2a} \quad \text{where} \quad C = \sqrt{\frac{1}{4ga}}$$

we can now solve for y':

$$\begin{aligned}
\frac{1}{2a} &= \frac{1}{y(1+y'^2)} \\
1 + y'^2 &= \frac{2a}{y} \\
y'^2 &= \frac{2a}{y} - 1 \\
y' &= \sqrt{\frac{2a}{y} - 1} \quad \text{or} \quad \sqrt{\frac{2a-y}{y}}
\end{aligned}$$

using separation of variables:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x'} &= \sqrt{\frac{2a-y}{y}} \\
\int \mathrm{d}y \sqrt{\frac{y}{2a-y}} &= \int \mathrm{d}x'
\end{aligned}
$$

and using the substitution $y = a(1-\cos\theta)$; $\quad \mathrm{d}y = \mathrm{d}\theta a \sin\theta$ and

$$
\begin{aligned}
\sin^2\theta &= 1 - \cos^2\theta \\
\sin\theta &= \sqrt{1-\cos^2\theta} = \sqrt{1-\cos\theta}\sqrt{1+\cos\theta}
\end{aligned}
$$

so

$$
\begin{aligned}
x' &= \int \mathrm{d}\theta a \sin\theta \sqrt{\frac{a(1-\cos\theta)}{2a - a(1-\cos\theta)}} \\
&= \int \mathrm{d}\theta a \sqrt{1-\cos\theta}\cancel{\sqrt{1+\cos\theta}}\frac{\sqrt{1-\cos\theta}}{\cancel{\sqrt{1+\cos\theta}}} \\
&= a \int \mathrm{d}\theta(1-\cos\theta) = a(\theta - \sin\theta)
\end{aligned}
$$

since the $\theta = \omega t$ we have the parametric equation for the path in the reference frame

$$
\begin{aligned}
x'(t) &= a(\omega t - \sin(\omega t)) \\
y(t) &= a(1 - \cos(\omega t))
\end{aligned}
$$

since $x = x' + v_o t$ the x position in the original frame is

$$
x(t) = a(\omega t - \sin(\omega t)) + v_o t
$$

(b) Using the initial conditions

$$
x = y = 0,\ \dot{x} = v_o,\ \dot{y} = 0
$$

we first solve for ω:

$$
\begin{aligned}
\dot{x} &= a\omega(1-\cos(\omega t)) + v_o \\
\dot{y} &= a\omega \sin(\omega t)
\end{aligned}
$$

we know at $t = 0$ that $\dot{x} = v_o$ and $\dot{y} = 0$. Since $\ddot{y} = g$ from the gravitational force we can solve for ω:

$$
\begin{aligned}
\ddot{y}(t) &= a\omega^2 \cos(\omega t) \\
\ddot{y}(0) &= a\omega^2 = g \implies \omega = \sqrt{\frac{g}{a}}
\end{aligned}
$$

At the boundary point B we know that the cycloid completes one cycle so $\omega t = 2\pi$:

$$
\begin{aligned}
x'(t_B) &= a(2\pi - \sin(2\pi)) = L \\
L = 2\pi a &\implies a = \frac{L}{2\pi} \implies \omega = \sqrt{\frac{2\pi g}{L}}
\end{aligned}
$$

so

$$
\begin{aligned}
x(t) &= \frac{L}{2\pi}\left[\sqrt{\frac{2\pi g}{L}}t - \sin\left(\sqrt{\frac{2\pi g}{L}}t\right)\right] + v_o t \\
y(t) &= \frac{L}{2\pi}\left[1 - \cos\left(\sqrt{\frac{2\pi g}{L}}t\right)\right]
\end{aligned}
$$

(c) Sketching the path

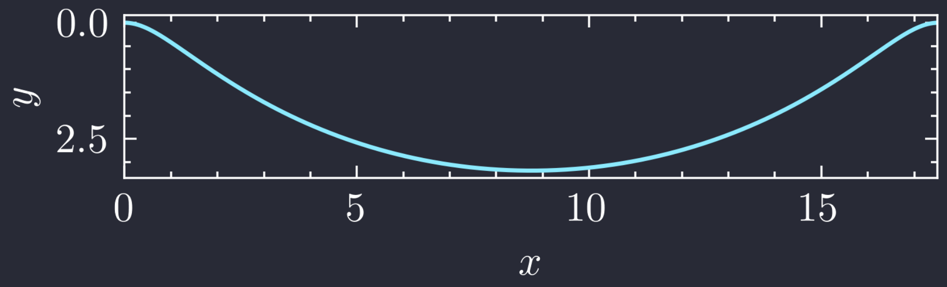

Figure 5.2: Numerically computed track shape