Physics 411: Mechanics

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Contents

1 Newtons Laws

The Four Horsemen of the Apocalypse (In Physics)

- Classical Mechanics
- Electromagnetism
- Statistical Mechanics
- Quantum Mechanics

Before 1900, there was no relativity or QM and the world was a simple place ...

Newton's 1st Law: The Law of Inertia
And object keeps going unless acted on by a force.
This only applies to and 'inertial frame'.

Newton's 2nd Law: $\mathbf{F} = m\mathbf{a}$
Sum notation: The position vector is

$$\mathbf{r} = (x, y, z) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$$

in the Cartesian coordinate system. The time derivative gives the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

and acceleration is the time derivative of velocity

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

Thus in vector notation, Newton's 2nd law is

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$$

where $\mathbf{r}(t)$ is and ordinary differential equation (ODE).
The basic idea of solving mechanics problems is writing down the ODEs and solving them.

What is mass? m is an 'inertial mass'.
In Newton's law of gravity

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

m is the 'gravitational mass' and $g \approx 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.
A larger mass has a larger inertia or 'resistance to being accelerated' (Taylor). Key fact: When acceleration is zero ($\mathbf{a} = 0$), the velocity is constant ($\mathbf{v} = \text{constant}$).

Momentum: $\mathbf{p} = m\mathbf{v}$
The third law of motion in terms of momentum is

$$\mathbf{F} = \dot{\mathbf{p}} = m\dot{\mathbf{v}}$$

Newton's Third Law: $\mathbf{F}_{12} = -\mathbf{F}_{21}$
In a two body system, the total force of the system is

$$\mathbf{F}_t = \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

From the second law,

$$\dot{\mathbf{p}}_1 = \mathbf{F}_{21} \qquad \dot{\mathbf{p}}_2 = \mathbf{F}_{12}$$

adding these two equations gives

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$$

thus the total momentum of the system is conserved.
For a system of N particles, the total momentum is

$$\mathrm{d}t \sum_i \mathbf{p}_i = \frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = \mathbf{F}_{ext}$$

sometimes $\mathbf{p}_{tot} = \mathbf{P}$ where the capital P denotes the total momentum of the system.

2 A pendulum

How to solve a problem:

1. Write down the eq
2. Solve it
3. Understand the solution

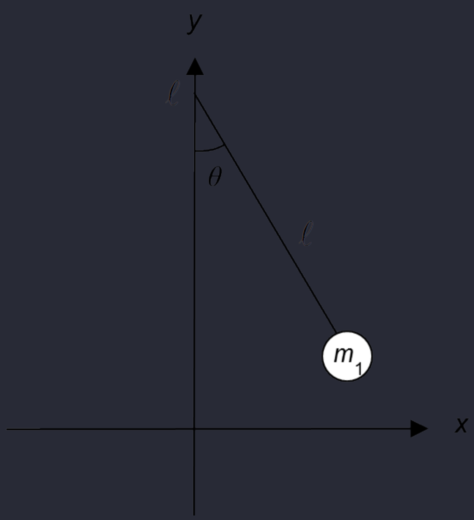

Figure 2.1: A pendulum with mass m and length l.

From Figure 2.1, we can write down Newton's 2nd law:

$$\begin{aligned}\mathbf{F} &= m\mathbf{a} = m\ddot{\mathbf{r}} \\ F_x &= -mg\sin\theta = m\ddot{x} \\ F_y &= -mg\cos\theta + T\cos\theta = m\ddot{x}\end{aligned}$$

Using a right triangle we can find the angle using $\tan\theta = x/y$. Furthermore, we can use the constrain that the length of the pendulum is constant thus $x^2 + y^2 = l^2$. But solving this system of equations is difficult. Instead we now use a new coordinate system.

Quick Hack Using the arc length $l = L\theta$ and choosing a coordinate in the direction of the pendulums path, we can write the force equation as

$$F_l = -mg\sin\theta = m\ddot{l} = mL\ddot{\theta}$$

Thus the equation of motion is

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

which is a second order ODE. This can only be solved with two conditions. We can use the initial conditions (at $t = 0$) of the position $\theta(t = 0) = \theta_0$ and velocity $\dot{\theta}(0) = 0$.

Polar Coordinates From Taylor:

$$x = r\cos\phi$$
$$y = r\sin\phi$$

For an arbitrary vector $\mathbf{v}$ it has the Cartesian vector components

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

Where the magnitude of the unit vectors are equivalent:

$$|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = 1$$

and the magnitude of the vector is

$$\begin{aligned} |\mathbf{v}| &= \sqrt{\mathbf{v}\cdot\mathbf{v}} \\ &= \sqrt{v_x^2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + 2v_xv_y\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}v_y^2\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}} \\ &= \sqrt{v_x^2 + v_y^2} \end{aligned}$$

The vector $\mathbf{v}$ can be written in polar coordinates as

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\phi}$$

where radial vector is

$$\mathbf{r} = r\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

taking the time derivative of $\mathbf{r}$ gives the velocity

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}}$$

but how do we find $\dot{\hat{r}}$? We can look at the change in the direction of the radial unit vector for a small change in time Δt. Thus,

$$\Delta\hat{\mathbf{r}} \approx r\Delta\phi\hat{\boldsymbol{\phi}}$$

dividing both sides by Δt gives

$$\frac{\Delta\hat{\mathbf{r}}}{\Delta t} \approx r\frac{\Delta\phi}{\Delta t}\hat{\boldsymbol{\phi}} = r\dot{\phi}\hat{\boldsymbol{\phi}}$$

Therefore, the vector in polar coordinates is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}}$$

where the polar components v_r and v_ϕ are related to the radial and angular velocity respectively. Taking the time derivative of $\dot{\mathbf{r}}$ gives the acceleration

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}} \\ &= \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\hat{r}} + \dot{v}_\phi\hat{\boldsymbol{\phi}} + v_\phi\dot{\hat{\phi}} \end{aligned}$$

3 Polar Coordinates

using the geometric relation $\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$, we can write the acceleration as

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}}\end{aligned}$$

where $r\dot{\phi}^2 = r\omega^2$ is the centripetal acceleration and $r\ddot{\phi} = r\dot{\omega}$ is the tangential acceleration.
From the Pendulum problem we know that the string is taut $r = L$ thus the radial velocity is zero $\dot{r} = 0$. Thus the force equation in the $\hat{\boldsymbol{\phi}}$ direction is

$$\begin{aligned}F_\phi &= mL\ddot{\phi} = -mg\sin\theta \\ \ddot{\phi} &= -\frac{g}{L}\sin\theta\end{aligned}$$

which is the same equation of motion.

Projectile in 2D The initial conditions of a general projectile is usually

$$\begin{aligned}F_x &= 0 = m\ddot{x} \\ F_y &= -mg = m\ddot{y}\end{aligned}$$

thus the equations of motion are

$$\begin{aligned}\ddot{x} &= 0 \\ \ddot{y} &= -g\end{aligned}$$

And solving these equations gives the position of the projectile

$$\begin{aligned}x(t) &= v_{ox}t \\ y(t) &= y_o v_{oy}t - \frac{1}{2}gt^2\end{aligned}$$

This can be expanded on with the addition of air resistance $\mathbf{f}$. This drag force is proportional to the velocity:

$$\mathbf{f} \propto -\hat{\mathbf{v}}$$

and there are two types of air resistance: linear

$$\mathbf{f}_l = -bv\hat{\mathbf{v}} = -b\mathbf{v}$$

and quadratic

$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

where we compare the terms with

$$\frac{f_l}{f_q} = \frac{cv}{b}$$

Linear $\mathbf{f}_l = -b\mathbf{v}$
From Newton's 2nd law

$$\begin{aligned} F_x &= -bv_x = m\ddot{x} = m\dot{v}_x \\ F_y &= -mg - bv_y = m\ddot{y} = m\dot{v}_y \end{aligned}$$

For the case of uncoupled differential equations (such as pure horizontal motion), we can solve the horizontal equation

$$\dot{v}_x = -\frac{b}{m}v_x$$

which has a general solution

$$v_x = Ae^{-kt}$$

where

$$k = \frac{b}{m}, \quad A = v_{ox}$$

to find the position we have to integrate $\dot{x} = v_x$:

$$\begin{aligned} x &= x_o + \int_0^t v_x(t')\,\mathrm{d}t' \\ &= x_o + \left[-\frac{v_{xo}}{k}e^{-kt}\right]\Bigg|_0^t \end{aligned}$$

where we have a limit of $t \to \infty$
For pure vertical motion, we solve the equation

$$\dot{v}_y = -g - \frac{b}{m}v_y \tag{3.1}$$

and with the initial condition $\dot{v}_y = 0$ we can solve for the velocity

$$v_y = -\frac{mg}{b} = v_{ter}$$

where v_{ter} is the terminal velocity. To get position, we use a trick by rewriting the equation as

$$m\dot{v}_y = -mg - bv_y = -mg - b(v_y - v_{ter})$$

and we can solve similar to the horizontal case using the general solution

$$v_y - v_{ter} = Ae^{-kt} = (v_{oy} - v_{ter})e^{-kt}bye$$

4 Air Resistance

Last time:

$$\mathbf{f}_l = -b\mathbf{v} \quad \dot{\mathbf{r}} = \mathbf{v}$$
$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

In the case of linear, x motino has a range, y velocity has a terminal velocity v_t.

Horizontal Quadratic Drag

$$\begin{aligned} F_y &= -mg - c|v_y|v_y \\ m\ddot{y} &= F_y \\ m\dot{v}_y &= -mg - c|v_y|v_y \end{aligned}$$

when $v_y = 0$ we have the terminal velocity

$$v_{ter} = \sqrt{\frac{mg}{c}} \quad \text{or} \quad c = \frac{mg}{v_{ter}^2}$$

thus the equation of motion is

$$\dot{v}_y = -g - \frac{c}{m}v_y^2 = -g(1 - \frac{v_y^2}{v_{ter}^2}) = \frac{\mathrm{d}v_y}{\mathrm{d}t}$$

using separation of variables

$$\frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\,\mathrm{d}t$$

integrating both sides

$$\int_{v_{oy}}^{v_y} \frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\int_0^t \mathrm{d}t$$

where we get the integral using the hyperbolic tangent

$$\begin{aligned} v_t \operatorname{arctanh}\frac{v_y}{v_t} &= -gt \\ v_y &= -v_t\tanh(gt) \end{aligned}$$

2D Motion For Quadratic

$$\begin{aligned} F_x &= -cvv_x = -c\sqrt{v_x^2 + v_y^2}v_x = m\dot{v}_x \\ F_y &= -mg - cvv_y = -mg - c\sqrt{v_x^2 + v_y^2}v_y = m\dot{v}_y \end{aligned}$$

where $v = \sqrt{v_x^2 + v_y^2}$. For linear, it is simply

$$\begin{aligned} F_x &= -bv_x = m\dot{v}_x \\ F_y &= -mg - bv_y = m\dot{v}_y \end{aligned}$$

5 Energy

Review: There are two requirements for conservation of angular momentum

1. Force is central
2. External torque is zero

Kinetici Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$. Taking the time derivative

$$\begin{aligned}\dot{T} &= \frac{1}{2}(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) \\ &= m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}\end{aligned}$$

and integrating over time t_1 to t_2

$$\int_{t_1}^{t_2} \dot{T}\,\mathrm{d}t = \Delta T = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = W(1 \to 2)$$

since $\mathbf{v}\cdot\mathrm{d}t = \mathrm{d}\mathbf{r}$ and $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ hints that this is a line integral.

Example:

$$\begin{aligned}\mathbf{F}(x,y) &= \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ \mathrm{d}vbr &= \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}}\end{aligned}$$

(a) $y = x$ from $a = (0,0)$ to $b = (1,1)$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + y\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 x\,\mathrm{d}x = 1\end{aligned}$$

(b) $y = x^2$ and $\mathrm{d}y = 2x\,\mathrm{d}x$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + x^2\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 2x^2\,\mathrm{d}x = 1\end{aligned}$$

thus the line integral is independent of the path.

Conservative force

1. Given $\mathbf{F}(\mathbf{r})$, there is no dependence on $\mathbf{v}$, t.
2. $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is independent of the path.

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}'$$

For gravity

$$U_g(\mathbf{r}) = -\int_a^b \mathbf{F}_g\cdot\mathrm{d}\mathbf{r} = -\int_a^b -mg\,\mathrm{d}y' = mg(y_a - y_b)$$

Work-Kinetic Energy Theorem:

$$W(1 \to 2) = W(1 \to \mathcal{O}) + W(\mathcal{O} \to 2) = U(1) - U(2) = \Delta T$$

From this we have

$$\Delta T = T_2 - T_1 = U_1 - U_2$$

rearranging terms give the mechanical energy

$$T_2 + U_2 = T_1 + U_1 = E$$

and for N conservative forces in a system

$$E = T + U_1 + U_2 + \cdots + U_N$$

Energy: Part 2

Conservative Force: Potential Energy The mechanical energy

$$E = T + U$$

is made up of the sum of the kinetic and potential energy. This is useful for

- obtaining equations of motion (EOM)

e.g. finding the EOM for a simple pendulum of mass m, length L and initial angle θ. The kinetic energy is

$$T = \frac{1}{2} m L^2 \dot{\theta}^2$$

where the magnitude of velocity is the tangential component $v = L\omega = L\dot{\theta}$. The potential energy is

$$U = -mgy = -mgL\cos\theta$$

and the conservation of energy tells us that the mechanical energy is constant

$$\begin{aligned} T + U &= \text{constant} = E \\ \frac{1}{2} m L^2 \dot{\theta}^2 - mgL\cos\theta &= E \end{aligned}$$

and in the intial condition we know that the velocity is zero $\dot{\theta} = 0$ and thus

$$-mgL\cos\theta_{max} = E$$

taking the time derivative of the energy equation gives

$$\begin{aligned} mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\dot{\theta} &= 0 \\ \ddot{\theta} + \frac{g}{L}\sin\theta &= 0 \\ \ddot{\theta} &= -\frac{g}{L}\sin\theta \end{aligned}$$

taking the time derivative of the energy is a useful trick for finding the EOM when we are trying to solve for $\dot{v}^2$.

Last time we found the potential energy for a position $\mathbf{r}$ in a conservative force field $\mathbf{F}(\mathbf{r})$ is

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}'$$

from the fundamental theorem of calculus (derivatives and intergrals are inverses) we want to find a function where the derivative equals the conservative force: First we take an infinitesimal change in the position $\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}$ and the change in potential energy is

$$\begin{aligned} U(\mathbf{r} + \mathrm{d}\mathbf{r}) &= -\int_{\mathbf{r}_0}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{\mathbf{r}}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= U(\mathbf{r}) - F(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \end{aligned}$$

where is know that the force is constant over a small distance. Moving the terms gives

$$\begin{aligned} U(\mathbf{r}+\mathrm{d}\mathbf{r}) - U(\mathbf{r}) &= -\mathbf{F}\cdot\mathrm{d}\mathbf{r} \\ &= -(F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z) \end{aligned}$$

where we use Cartesian Coordinates, and we know that the gradient of potential is

$$\begin{aligned} \boldsymbol{\nabla} U &= \hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z} \\ &= -(F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}) = -\mathbf{F} \end{aligned}$$

Example 3: 1D motion If we know what U is as a function of x, we can find the force! At points where $E = U$ we call these classical turning points. At a region of a relative minimum, a particle below the threshold of the turning point will oscillate between the two turning points. And at $E > U_max$ the particle is unbound and will escape the forces that attracted it.

Example 4:

$$\begin{aligned} E &= T + U(x) \text{ is constant} \\ T &= \frac{1}{2}m\dot{x}^2 = E - U(x) \\ \dot{x}^2 &= \frac{2}{m}(E - U(x)) \\ \dot{x} &= \pm\sqrt{\frac{2}{m}(E - U(x))} \end{aligned}$$

using seperation of variables

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\frac{2}{m}(E - U(x))} \\ \sqrt{\frac{m}{2}}\,\mathrm{d}t &= \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\ \int_{t_1}^{t_2}\sqrt{\frac{m}{2}}\,\mathrm{d}t &= \int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\ (t_2 - t_1) &= \sqrt{\frac{2}{m}}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E - U(x)}} \end{aligned}$$

where the sign of the velocity is positive within the oscillating bounds of the turning points and changes sign as the particle moves past the turning point.

Energy: Part 3

Last time: Conservative force as a negative gradient of potential:

$$\mathbf{F} = -\boldsymbol{\nabla} U$$

with classical turning points at $E = U$.

Conditions of a conservative force

- Only depends on position $\mathbf{r}$ (or just constant)
- Work done is path independent (this is sometimes hard to check) $\Leftrightarrow \boldsymbol{\nabla} \times \mathbf{F} = 0$

What is curl? In 3D Cartesian coordinates

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\hat{\mathbf{x}} + \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\hat{\mathbf{y}} + \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\hat{\mathbf{z}}() \right)\end{aligned}$$

Mathematically we know that if the force vector is a gradient of a scalar potential

$$\mathbf{F} = \boldsymbol{\nabla}\phi = -\boldsymbol{\nabla} U \quad \Leftrightarrow \quad \boldsymbol{\nabla} \times \mathbf{F} = 0$$

The curl of a gradient is always zero! Short 'proof':

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

and the curl

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial U}{\partial x} & -\frac{\partial U}{\partial y} & -\frac{\partial U}{\partial z} \end{vmatrix} = 0$$

Proving that the line integral is path independent is a little difficult, but in general we know that the two different paths a and b from points 1 to 2 we can write the work as

$$\int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_2 - \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_1 = \oint_{a-b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_A (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = 0$$

where we invoke Stokes' Theorem to find the integral of the curl over the surface A is zero.

Conservative Force: $\mathbf{F} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is conservative, but is

$$\mathbf{F}(\mathbf{r}) = F(\mathbf{r})\hat{\mathbf{r}}$$

a central force always conservative? We will need to check the curl of the force to find out. But first we define spherical coordinates (r, θ, ϕ)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

and the central force in spherical coordinates is

$$\mathbf{F} = F(\mathbf{r})\hat{\mathbf{r}} + 0\hat{\theta} + 0\hat{\phi}$$

the curl in spherical coordinates is

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} = &\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial\phi}\right]\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\hat{\theta} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]\hat{\phi}\end{aligned}$$

And since

$$\frac{\partial F}{\partial\theta} = \frac{\partial F}{\partial\phi} = 0$$

the curl is zero $\boldsymbol{\nabla} \times \mathbf{F} = 0$ and thus $\mathbf{F}$ is a conservative central force.

Gravity Conservative? The force due to gravity

$$\mathbf{F}_g = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}$$

is a central force as it only depends on $\mathbf{r}$. e.g. for a a two mass system:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Using Translational Invariance, we can shift the origin to the center of m_2

$$\mathbf{r}_2 = 0 \quad \mathbf{F}_{12} = -\frac{Gm_1m_2}{\mathbf{r}_1^3}\mathbf{r}_1$$

or

$$F_{12} = -\boldsymbol{\nabla}_1 U = -(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial z_1})$$

where the potential energy due to the interaction between 1 and 2 is

$$U_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

from newtons 3rd law, the force on 2 due to 1 is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

so

$$\begin{gathered}-\boldsymbol{\nabla}_1 U_{12} \to \mathbf{F}_{21} = \boldsymbol{\nabla}_1 U_{12} \\ \boldsymbol{\nabla}_1 U_{12}(\mathbf{r}_1 0 \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U_{12}(\mathbf{r}_2 0 \mathbf{r}_1) \\ u_{12}(\mathbf{x}) \quad \mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2 \\ \boldsymbol{\nabla}_1 U_{12}(\mathbf{x}) = \boldsymbol{\nabla}_x U_{12}(\mathbf{x}) = -\boldsymbol{\nabla}_2 U_{12}(\mathbf{x})\end{gathered}$$

so

$$\mathbf{F}_{12} = -\boldsymbol{\nabla}_1 U_{12} \qquad \mathbf{F}_{21} = -\boldsymbol{\nabla}_2 U_{12}$$

and for N particles

$$\mathbf{F}_i = -\boldsymbol{\nabla}_i U \qquad U = \sum_{i,j} U_{ij} + \sum_i U_i^{\text{ext}}$$

6 Oscillations

& Simple Harmonic Oscillators For the simple case of a mas on a spring, the spring force is $F_s = -k(x-x_o)$ where the force is conservative and the (elastic) potential energy is $U_s = \frac{1}{2}k(x-x_o)^2$.

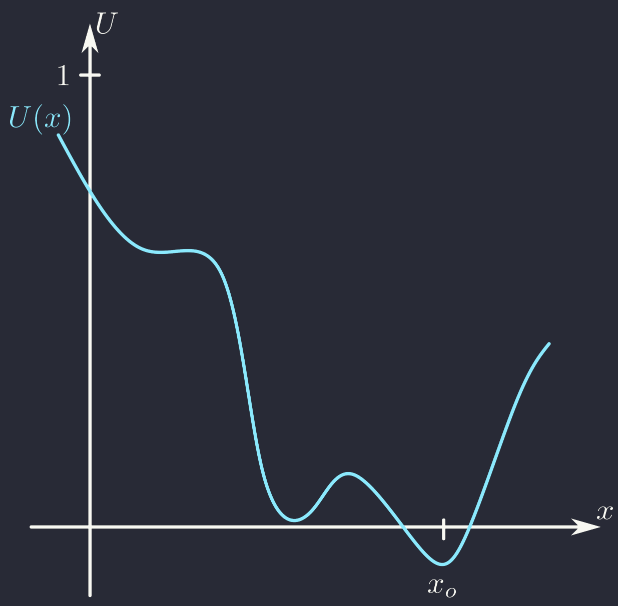

Figure 6.1: Arbitrary Potential Energy: $\mathbf{F} = -\boldsymbol{\nabla} U$

Aribtrary Potential energy For a equilibrium positon x_o we can take the taylor expansion of the potential energy

$$U(x) = U(x_o) + U'(x_o)\Delta x + \frac{1}{2}U''(x_o)\Delta x^2 + \dots$$

where $\Delta x = x - x_o$. Setting x_o to the reference point of U cancels the first term and the conservative nature tells us that the second term is also zero thus we are left with the third term where the spring constant is

$$k = U''(x_o)$$

To find the equations of motion, using N2L

$$m\ddot{x} = F = -k(x-x_o)$$
$$\ddot{x} = -\frac{k}{m}(x-x_o)$$

where we have a constant of angular frequency

$$\omega_o = \sqrt{\frac{k}{m}}$$

the solution could be a sinusoidal function

$$x(t) \approx \sin \omega_o t$$

but we are missing the initial value, so

$$x(t) \approx \sin \omega_o t + x_o$$

the general solution is linear combinations of the sine and cosine functions

$$x(t) = A\sin\omega_o t + B\cos\omega_o t + x_o$$
$$\dot{x}(t) = \omega_o A\cos\omega_o t - \omega_o B\sin\omega_o t$$

where we need 2 initial conditions to solve for A and B. e.g. $x(0)$ and $\dot{x}(0)$.

$$B = x(0) - x_o = \Delta x(0), \qquad A = \frac{\dot{x}(0)}{\omega_o}$$

Euler's Solution We can also use a general solution of the form

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where

$$\left|e^{i\theta}\right| = \cos^2\theta + \sin^2\theta = 1$$

taking the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{i\omega_o t} = i\omega_o e^{i\omega_o t}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{i\omega_o t} = -\omega_o^2 e^{i\omega_o t}$$

and the general solution is

$$x(t) = Ae^{i\omega_o t} + Be^{-i\omega_o t} + x_o$$

this does not mean that we have an imaginary solution, but rather we are using the geomtric nature of the solution.

Third Way We can also use a method where we introduce the phase

$$x(t) = A\cos(\omega_o t - \delta) + x_o$$
$$\dot{x}(t) = -A\omega_o\sin(\omega_o t - \delta)$$

for $t = 0$ we have

$$x(0) = A\cos(-\delta) + x_o = A\cos\delta + x_o$$
$$\dot{x}(0) = -A\omega_o\sin(-\delta) = A\omega_o\sin\delta$$

and the constants are found by squaring and adding the two equations

$$A^2 = (x(0) - x_o)^2 + \frac{\dot{x}(0)^2}{\omega_o^2} = \Delta x(0)^2 + \frac{\dot{x}(0)^2}{\omega_o^2}$$
$$\delta = \arctan\frac{\dot{x}(0)}{\omega_o(x(0) - x_o)} = \arctan\frac{\dot{x}(0)}{\omega_o\Delta x(0)}$$

Energy of the Oscillator The mechanical energy is $E = T + U$

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_o^2 A^2\sin^2(\omega_o t - \delta)$$
$$U = \frac{1}{2}k(x - x_o)^2 = \frac{1}{2}kA^2\cos^2(\omega_o t - \delta)$$

setting $x_o = 0$ we can work with a much simple case

$$U = \frac{1}{2}kx^2$$
$$T = \frac{1}{2}kx^2$$

using the third way where

$$x(t) = A\cos(\omega_o t - \delta)$$
$$\dot{x}(t) = -A\omega_o \sin(\omega_o t - \delta)$$

we have

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_o t - \delta)$$
$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_o^2\sin^2(\omega_o t - \delta) \qquad = \frac{1}{2}kA^2\sin^2(\omega_o t - \delta)$$

thus the total mechanical energy is

$$E = T + U = \frac{1}{2}kA^2$$

where this is the maximum potential energy of the system, or the potential energy at the maximum amplitude. This is also the classical turning point $E = U$. As time goes on, we can see that the energy oscillates betweeen being completely kinetic (T) and completely potential (U).

2D Oscillator We can have two cases of oscillation:

$$\mathbf{F} = -k(\mathbf{r} - \mathbf{r}_o) \quad \text{isotropic oscillator}$$

this is where each component share the same frequency, but different amplitudes and/or initial conditions

$$x(t) = A_x\cos(\omega_o t - \delta_x) + x_o$$
$$y(t) = A_y\cos(\omega_o t - \delta_y) + y_o$$

for the anisotropic oscillator

$$F_x = -k_x(x - x_o) \quad F_y = -k_y(y - y_o)$$

the frequency is decoupled thus

$$x(t) = A_x\cos(\omega_{ox} t - \delta_x) + x_o$$
$$y(t) = A_y\cos(\omega_{oy} t - \delta_y) + y_o$$

and if the ratio between the angular frequencies ω_{ox}/ω_{oy} are rational, the motion is periodic and the figure will be closed. But for irrational ratios, the motion is *quasiperiodic* and the figure is not closed (chaotic).

Oscillations: Damping

Damped Oscillator From last time the simple EOM for a spring is

$$m\ddot{x} = -k(x - x_o)$$

where the equilibrium position is x_o and the spring constant is k. When we add air resistance e.g. linear drag:

$$\mathbf{f} = -b\mathbf{v} \qquad m\ddot{x} = -kx - b\dot{x}$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

where we have two constants

$$\omega_o = \sqrt{\frac{k}{m}} \quad \beta = \frac{b}{2m}$$

where ω_o is the natural frequency and β is the damping coefficient. Rewriting in terms of the constants we get

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

where a general solution is

$$x = e^{rt}; \quad \dot{x} = re^{rt}; \quad \dot{x} = r^2 e^{rt}$$

plugging into the EOM gives

$$\begin{aligned} r^2 e^{rt} + 2\beta r e^{rt} + \omega_o^2 e^{rt} &= 0 \\ r^2 + 2\beta r + \omega_o^2 &= 0 \end{aligned}$$

which is the characteristic (or auxiliary) equation, and the solution is in the form of the quadratic formula

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$$

thus the position is a linear combination of the two solutions

$$x(t) = C_1 e^{(-\beta + \sqrt{\beta^2 - \omega_o^2})t} + C_2 e^{(-\beta - \sqrt{\beta^2 - \omega_o^2})t}$$

At $\beta = 0$ (no damping)

$$\sqrt{\beta^2 - \omega_o^2} = \sqrt{-\omega_o^2} = i\omega$$

thus the solution of a SHO

$$x(t) = C_1 \exp(i\omega t) + C_2 \exp(-i\omega t)$$

Weak Damping For the case $\beta < \omega_o$ (underdamping)

$$\sqrt{\beta^2 - \omega_o^2} = i\sqrt{\omega_o^2 - \beta^2}$$

thus the solution is

$$\begin{aligned} x(t) &= C_1 e^{(-\beta + i\sqrt{\omega_o^2 - \beta^2})t} + C_2 e^{(-\beta - i\sqrt{\omega_o^2 - \beta^2})t} \\ &= e^{-\beta t}(C_1 \cos\left(\sqrt{\omega_o^2 - \beta^2}t\right) + C_2 \sin\left(\sqrt{\omega_o^2 - \beta^2}t\right)) \end{aligned}$$

we can simplify this with a new frequency term $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ and therefore

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$

this is called underdamping because the amplitude oscillates and decays slowly.

Strong damping For the case $\beta > \omega_o$ we have the solution

$$x(t) = e^{-(\beta-\sqrt{\beta^2-\omega_o^2})t}\left(C_1 + C_2 e^{-2\sqrt{\beta^2-\omega_o^2}t}\right)$$

this called overdamping because the system cannot cannot complete a full oscillation, and decays exponentially to the equilibrium position. Thus we call the term

$$\text{decay parameter} = \beta - \sqrt{\beta^2 - \omega_o^2}$$

where the decaying tail is described by the decay parameter whereas the second term $-2\sqrt{\beta^2 - \omega_o^2}$ describes the fast initial damping of the system.

Large β For the case of $\beta \to \infty$ the decay parameter goes to zero:

$$\gamma = \beta - \sqrt{\beta^2 - \omega_o^2} \to 0$$

which is counter intuitive as the high damping coefficient results in a very slow exponential decay where it looks like a constant almost zero.

Critical Damping For the case $\beta = \omega_o$ we get

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t} \to x = e^{-\beta t}(C_1 + C_2 t)$$

where the extra factor of t comes from solving for a function $f(t)e^{-\beta t}$ to get the Constants. Pluggin this back into the initial EOM: $\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$

$$\dot{x} = e^{-\beta t} - te^{-\beta t} \qquad \ddot{x} = -2\beta e^{-\beta t} + te^{-\beta t}$$

so

$$-2\beta e^{-\beta t} + \beta^2 te^{-\beta t} + 2\beta e^{-\beta t} - 2\beta^2 te^{-\beta t} + \beta^2 te^{-\beta t} = 0$$

condition	γ
$\beta < \omega_o$	β
$\beta = \omega_o$	β
$\beta > \omega_o$	$\beta - \sqrt{\beta^2 - \omega_o^2}$

the critical damping will have the fastest decay of the system. The quickest way to stop an oscillating system is to apply a damping force at the natural frequency of the system.

NOTE: This all goes away when the magnitude of the damping force is not linear (e.g. quadratic drag). The linear EOM gives us something that can be easily analyzed, but for terms with higher powers (e.g. $\dot{x}^2$) the EOM becomes non-linear and the solutions are chaotic.

Driven Damped Oscillations

From last time: Note that the two parameters ω_o and β have the same units (rad/s) where we treat radians as a unitless quantity.

Time dependent force For the SHO we have a new EOM

$$m\ddot{x} + b\dot{x} + kx = F\cos(\omega t)$$

or in terms of the constants ω_o and β

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \frac{F(t)}{m} = f(t)$$

where $f(t)$ has the same units as acceleration/ force per unit mass. This is a inhomogeneous differential equation, but we can consider this as a combination of a homogeneous solution x_h and a particular solution x_p:

$$x_p(t) + x_h(t) = x(t)$$

denoting a differential operator

$$D = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_o^2$$

we know that

$$Dx_h(t) = 0 \qquad Dx_p(t) = f(t)$$

where from last time we know that the homogeneous solution is

$$x_h(t) = e^{-\beta t}(C_1\exp\left(t\sqrt{\beta^2 - \omega_o^2}\right) + C_2\exp\left(-t\sqrt{\beta^2 - \omega_o^2}\right))$$

and for the particular solution we can define the driving force as a sinusoidal function

$$f(t) = f_o\cos(\omega t) \quad \text{driving force}$$

where

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o\cos\omega t$$

or using Euler's formula we can define the EOM as the real part of the complex function

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o e^{i\omega t}$$

the particular solution is then

$$\begin{aligned} x &= Ce^{i\omega t} \\ \dot{x} = i\omega Ce^{i\omega t}, \qquad \ddot{x} &= -\omega^2 Ce^{i\omega t} \end{aligned}$$

subbing this back in to the EOM

$$-\omega^2 Ce^{i\omega t} + 2\beta i\omega Ce^{i\omega t} + \omega_o^2 Ce^{i\omega t} = f_o e^{i\omega t}$$

or

$$C = \frac{f_o}{\omega_o^2 - \omega^2 + 2i\beta\omega}$$

which is a complex number. Before taking the real part of the solution,

$$x(t) = \mathrm{Re}(Ce^{i\omega t})$$

we have to consider the complex part as well, so we rewrite C as

$$C = Ae^{-i\delta} = A\cos\delta - iA\sin\delta$$

so the real part of the solution is

$$x(t) = \mathrm{Re}(Ae^{i(\omega t-\delta)}) = A\cos(\omega t - \delta)$$

to find the amplitude A we have to multiply C by its complex conjugate

$$C^*C = |C|^2 = A^2$$

where we get the modulus squared. Thus we get

$$A^2 = \frac{f_o^2}{(\omega_o^2 - \omega^2)^2 + (2\beta\omega)^2}$$

to find δ we move some terms around

$$\begin{aligned}\frac{f_o}{\omega_o^2 - \omega^2 + 2i\beta\omega} &= Ae^{-i\delta}\\ f_o e^{i\delta} &= A(\omega_o^2 - \omega^2 + 2i\beta\omega)\\ f_o\cos\delta + if_o\sin\delta &= A(\omega_o^2 - \omega^2 + 2i\beta\omega)\end{aligned}$$

thus we have

$$\sin\delta = \frac{2A\beta\omega}{f_o} \qquad \cos\delta = \frac{A(\omega_o^2 - \omega^2)}{f_o}$$

dividing the two equations gives us the phase

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right)$$

The full solution is now

$$\begin{aligned} x(t) &= x_p(t) + x_h(t) \\ &= A\cos(\omega t - \delta) + C_1 e^{\left(-\beta+\sqrt{\beta^2-\omega_o^2}\right)t} + C_2 e^{\left(-\beta-\sqrt{\beta^2-\omega_o^2}\right)t} \end{aligned}$$

where ω is the driving frequency, and ω_o is the natural frequency. The last two exponential terms are known as the transient solution which decays very quick (exponentially) as shown in Figure 6.2 Finding

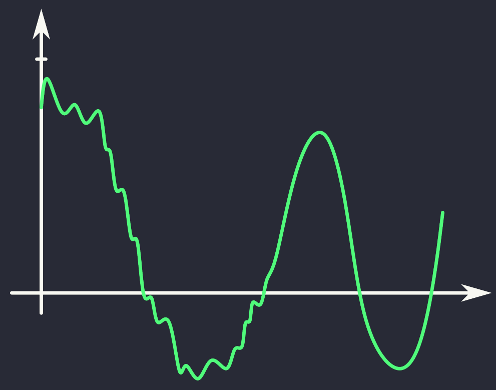

Figure 6.2: Driven Damped Oscillations

the maximum we look for where the derivative of A^2 is zero, or roughly

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left((\omega_o^2 - \omega^2)^2 + (2\beta\omega)^2\right) = 0$$

which gives us

$$\omega = \omega_2 = \sqrt{\omega_o^2 - 2\beta^2}$$

where at $B \ll \omega_o \to \omega \approx \omega_o$. Figure 6.3 shows that ω_2 is a resonant frequency where the amplitude is maximized.

$$A_{max} = \frac{f_o}{\sqrt{4\beta^2(\omega_o^2 - \omega^2)}} \approx \frac{f_o}{2\beta\omega} \quad \text{for} \quad \beta \ll \omega_o$$

What is δ? From the general solution, we can see that δ is a shift with respect to the driving force. This lag we can graph as a function of ω as shown in Figure 6.4

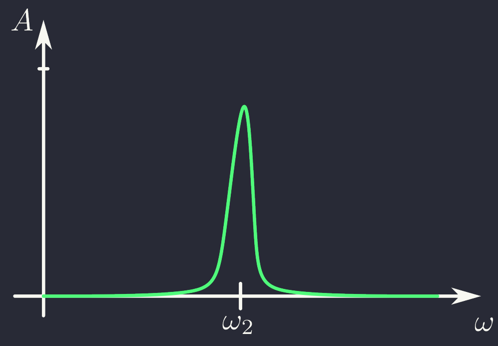

Figure 6.3: Resonance at ω_2

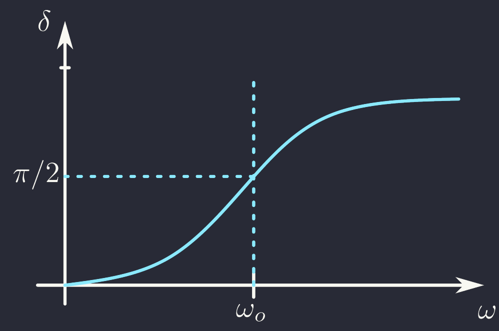

Figure 6.4: Phase shift δ

7 Calculus of Variations

Why do we care?

- What is the shortest distance between two points in a 2D plane?
- What is the shortest path between two points on a sphere?
- What is the fastest path for a ball to roll down a hill?
- For a car driving on a flat path $A \to B$, what shape of a pot hole will minimize the time it takes to get from $A \to B$?

For some path $a \to b$, we have a path defined as an integral

$$S = \int_a^b f(x, y, y')\mathrm{d}x$$

with a *Goal*: find $y(x)$ that minimizes S (path).

Path Length:

$$l = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{1 + y'^2}\mathrm{d}x$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. To minimize $y = f(x)$ it is equivalent to finding where

$$f'(x) = 0$$

where we note that this could be a maximium point, but it is usually a minimum in these cases. Another look at this function:

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

we can define a small change in the path $y(x)$ as

$$y(x) + \delta y(x)$$

where

$$\delta y(x_2) = 0 \quad \delta y(x_2) = 0$$

so the change in the path is

$$\delta S = \int_a^b \delta f \mathrm{d}x$$

and from the change of variables

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \qquad \delta y' = \frac{\mathrm{d}}{\mathrm{d}x}\delta y$$

thus we have

$$\delta S = \int_a^b \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x$$

this is the line integral of the change in the new path

$$\delta S = S_{new} - S_{old}$$

looking at the second term: using integration by parts

$$\int_a^b \left(\frac{\partial f}{\partial y}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x = \left[\frac{\partial f}{\partial y'}\delta y\right]\Bigg|_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y \mathrm{d}x$$

the first term is zero because $\delta y(a) = \delta y(b) = 0$. Thus we have

$$\delta S = \int_a^b \left[\frac{\partial f}{\partial y} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\right]\delta y \mathrm{d}x$$

Near a minimum, $\delta S = 0$ for any small δy. So the the terms in the brackets must be zero as well! This gives us the **Euler-Lagrange Equation**:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0$$

NOTE: δS is the variation of S(some number) under δy(a function).

Example: Shortest path between two points $a \to b$ in a 2D cartesian plane.

Goal: find $y(x)$ tha minimzes the path length $l = \int_a^b \sqrt{1+y'^2}\mathrm{d}x$ where $f(x,y,y') = \sqrt{1+y'^2}$.

$$\begin{aligned}\frac{\partial f}{\partial y} &= 0\\ \frac{\partial f}{\partial y'} &= \frac{2y'}{2\sqrt{1+y'^2}} = \frac{y'}{\sqrt{1+y'^2}}\end{aligned}$$

From the EL:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{\partial f}{\partial y} = 0$$

and

$$\begin{aligned}\frac{y'}{\sqrt{1+y'^2}} &= Const = C\\ y'^2 &= C(1+y'^2)\\ y'^2 &= \frac{C}{1-C}\\ y' &= \pm\sqrt{\frac{C}{1-C}} = \pm k\\ y &= \pm kx + b\end{aligned}$$

which is just a straight line as we expected.

Example: The Brachistochrone.

Goal: Find $y(x)$ that minimizes $t = \int_a^b \mathrm{d}t$ where

$$t = \frac{s}{v} \to \mathrm{d}t = \frac{\mathrm{d}s}{v}$$

and v is the velocity which can be found using the conservation of energy:

$$\frac{1}{2}mv^2 = mg(y_o - y) \to v = \sqrt{2g(y_o - y)}$$

thus we have

$$\mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o - y)}}\mathrm{d}x$$

where $f(x, y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o-y)}}$. Using EL:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}\sqrt{2g(y_o - y)}}$$
$$\frac{\partial f}{\partial y} = \frac{\sqrt{1+y^2}\sqrt{2g}}{(2g(y_o - y))^{3/2}}$$

From the initial conditions: $x = y = 0, \quad \dot{x} = \dot{y} = 0$ (assuming no friction). So

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \to \frac{\partial f}{\partial y} = \frac{\sqrt{1+y'^2}}{\sqrt{2gy^3}}$$

and if $y(x)$ satisfies EL, then we have a conserved quantity (prove in HW):

$$f - y'\frac{\partial f}{\partial y'} = const$$

we will soon find out that this is the energy of the system... we can solve for $y(x)$ using the conserved quantity:

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gy}}\frac{y'}{\sqrt{1+y'^2}}$$

so

$$f - y'\frac{\partial f}{\partial y'} = C$$
$$\frac{1}{\sqrt{2gy}}\left(\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right) = C$$
$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}}(1 + y'^2 - y'^2) = \sqrt{\frac{1}{4ga}}$$

using

$$\frac{1}{y(1+y'^2)} = \frac{1}{2a} \implies 1 + y'^2 = \frac{2a}{y} \implies y'^2 = \frac{2a}{y} - 1$$

this can be solved using separation of variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{2a}{y} - 1} \implies \mathrm{d}x = \mathrm{d}y\sqrt{\frac{y}{2a-y}}$$

and integration both sides:

$$\int_a^b \mathrm{d}y\sqrt{\frac{y}{2a-y}} = x - x_a = x$$

using a change of variables $y = a(1-\cos\theta)$; $\quad \mathrm{d}y = a\sin\theta\mathrm{d}\theta$ and a substitution $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{(1-\cos\theta)(1+\cos\theta)}$:

$$\int_a^b a\sin\theta\mathrm{d}\theta\sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}} = \int_a^b a(1-\cos\theta)\mathrm{d}\theta = a\theta - a\sin\theta$$

this is a parametric equation:

$$x = a(\theta - \sin\theta) = x(\theta)$$
$$y = a(1-\cos\theta) = y(\theta)$$

where $\theta = \omega t$. This is a curve traced by a point on a wheel AKA cycloid. When we choose a variable time we get

$$x(t) = a(\omega t - \sin\omega t)$$
$$y(t) = a(1-\cos\omega t)$$

and thus we get $\omega = \sqrt{\frac{g}{a}}$. To find a we use the coordiate of the lower second point to find the curve that goes through the two points.

Example: Find two functions $x(u)$, $y(u)$ where the path

$$S = \int_a^b f(x, x', y, y', u)\mathrm{d}u$$

is minimized/stationary. We will get two EL equations:

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial x'} = 0$$
$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial y'} = 0$$

e.g. for a distance between two points:

$$L = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{x'^2 + y'^2}\mathrm{d}u \quad \text{using} \quad \mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}u}\mathrm{d}u = y'\mathrm{d}u$$

and from the EL equations:

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial x'} = 0 = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{x'}{\sqrt{x'^2+y'^2}}\right)$$
$$\implies C_1 = \frac{x'}{\sqrt{x'^2+y'^2}} \qquad C_2 = \frac{y'}{\sqrt{x'^2+y'^2}}$$

this also tells us that

$$\frac{y'}{x'} = const = \frac{\mathrm{d}y}{\mathrm{d}x}$$

For N unknown functions in time t:

$$S = \int_a^b f(x_1, x_1', \ldots, x_N, x_N', u)\mathrm{d}u$$

where f has $2N+1$ variables.

Generalized Coordinates: $q_1, q_2, \ldots, q_N$ we would define the Lagrangian

$$\mathcal{L}(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t)$$

and minimize the action

$$S = \int \mathcal{L}\mathrm{d}t$$

and N EL equations gives the trajectory for the path of minimal action.

8 Lagrange's Equations

From last time: we defined the path

$$S = \int_a^b f(x, y(x), y'(x))\mathrm{d}x$$

Goal: find $y(x)$ that minimizes S using EL

$$\text{EL:} \quad \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

where near the minimum $\delta S = 0$. From the EL, $y(x)$ is a stationary point of S(could also be a maximum!).

Lagrangian In Classical Mechanics, we use a specific form

$$\mathcal{L} = T - V$$

this has the units of energy and the action S has the units $[S] = [E \cdot T]$ similar to planck's constant $\hbar$.

3D Cartesian $x, y, z = q_1, q_2, q_3$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$U = U(x, y, z)$$

where the potential energy only depends on the position and T only depends on the velocity, so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

and the EL equation for the Lagrangian is

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

For the 3D case, we have 3 equations of motion: For x we have

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x}, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

and using the EL equation, we get

$$-\frac{\partial U}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x}$$

which is Newton's second law $F_x = ma_x$ where $\mathbf{F} = -\boldsymbol{\nabla} U$. We can now get the general form

$$\mathbf{F} = m\mathbf{a}$$

Polar Coordinates $q : (r, \phi)$ we know that

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

and

$$U = U(r, \phi), \qquad T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

first we find the parts EL equation for r

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial r} &= mr\dot{\phi}^2 - \frac{\partial U}{\partial r} \\ \frac{\partial \mathcal{L}}{\partial \dot{r}} &= m\dot{r}\end{aligned}$$

and the EL equation is

$$\begin{aligned}mr\dot{\phi}^2 - \frac{\partial U}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) \\ m(\ddot{r} - r\dot{\phi}^2) &= -\frac{\partial U}{\partial r}\end{aligned}$$

which gives us N2L for r. For ϕ we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi} &= -\frac{\partial U}{\partial \phi} \\ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= mr^2\dot{\phi}\end{aligned}$$

and from the EL equation we get

$$-\frac{\partial U}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) = m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi})$$

dividing both sides by r

$$-\frac{1}{r}\frac{\partial U}{\partial \phi} = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = -(\boldsymbol{\nabla} U)_\phi$$

from both forms we know that the two parts of the EL represent the momentum and force:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} &= p_i \quad \text{generalized momentum} \\ \frac{\partial \mathcal{L}}{\partial q_i} &= F_i \quad \text{generalized force}\end{aligned}$$

where $F_i = \frac{\mathrm{d}}{\mathrm{d}t}p_i$ is the generalized N2L.

Example: Mass m sliding down a frictionless *moving* ramp M. First we choose the coordinates x moving along withe the ramp and y down in the perpendicular direction. For the ramp M:

$$T_M = \frac{1}{2}M\dot{q}_2^2, \quad U_M = 0$$

and for the mass m: First we decompose the velocity of m into the x and y components

$$\mathbf{v}_m = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} = \hat{\mathbf{y}}(\dot{q}_1 \sin\alpha) + \hat{\mathbf{x}}(\dot{q}_1\cos\alpha + \dot{q}_2)$$

and the kinetic and potential energies are

$$\begin{aligned}T_m = \frac{1}{2}mv_m^2 &= \frac{1}{2}m(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha + \dot{q}_2^2) \\ U_m = mgy &= -mg(\dot{q}_1\sin\alpha)\end{aligned}$$

using the Lagrangian $\mathcal{L} = T - U = T_M + T_m - U_M - U_m$ we get

$$\frac{\partial \mathcal{L}}{\partial q_2} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = M\dot{q}_2 + m\dot{q}_2 + m\dot{q}_1\cos\alpha$$

and the EL equation gives us

$$(M+m)\ddot{q}_2 + m\ddot{q}_1 \cos\alpha = 0$$
$$a_2 = \ddot{q}_2 = -\frac{m\ddot{q}_1 \cos\alpha}{M+m}$$

and for q_1 we have

$$\frac{\partial \mathcal{L}}{\partial q_1} = mg\sin\alpha, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m(\dot{q}_1 + \dot{q}_2 \cos\alpha)$$

and the EL equation gives us

$$mg\sin\alpha = m(\ddot{q}_1 + \ddot{q}_2\cos\alpha)$$

and since we have two equations and two unknowns, we can solve for $\ddot{q}_1$ and $\ddot{q}_2$.

$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{m+M}} = const$$
$$\ddot{q}_2 = -\frac{m\ddot{q}_1\cos\alpha}{M+m} = const$$

for $\alpha = 90°$, we get $\ddot{q}_1 = g$ and $\ddot{q}_2 = 0$ which is the same as a free falling. For and infinitely heavy ramp $M \to \infty$, we get $\ddot{q}_1 = g\sin\alpha$. For $M \to 0$ we get $\ddot{q}_1 = g/\sin\alpha$ which doesn't make sense because the force on the mass would be infinite. The normal force $N \to 0$ as $M \to 0$ and the mass would be in free fall.

Review Lagrangian: For a general integral

$$S\int f(x,y,y')\mathrm{d}x$$

find $y(x)$ minimizing S using the EL equation

$$\frac{\partial f}{\partial y}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

For Classical Mechanics, we use the Lagrangian in the generalized coordinate system q_i we define the action S as

$$S=\int\mathcal{L}(q_i,\dot{q}_i,t)\mathrm{d}t \quad \text{find} \quad q(t)$$

and from the EL equation we get

$$\frac{\partial\mathcal{L}}{\partial q_i}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$$

for each degree of freedom. We define the Lagrangian in CM as the quantity $\mathcal{L}=T-U$

Examples, Examples, and more Examples: A pendulum but its spining on its axis. We first find the energies:

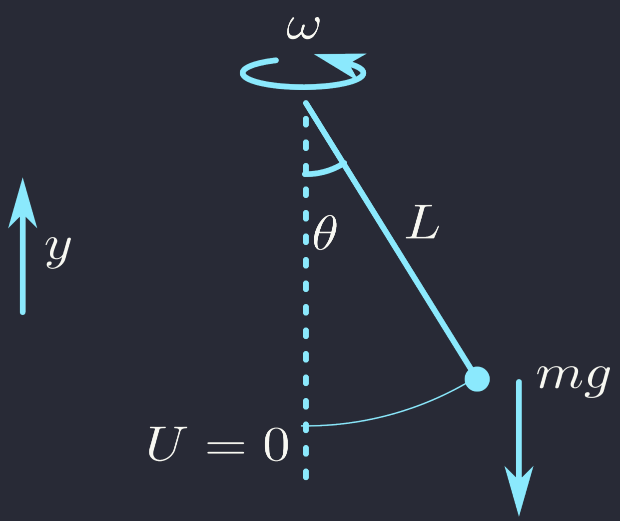

Figure 8.1: Pendulum

$$\begin{aligned}T&=\frac{1}{2}mv^2=\frac{1}{2}m((\omega L\sin\theta)^2+(L\dot{\theta})^2)\\U&=mgy=mgL(1-\cos\theta)\end{aligned}$$

from EL equation we get

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta}&=\frac{1}{2}m\omega^2L^2(2\sin\theta\cos\theta)-mgL\sin\theta=m\omega^2L^2\cos\theta\sin\theta-mgL\sin\theta\\\frac{\partial\mathcal{L}}{\partial\dot{\theta}}&=mL^2\dot{\theta}\\\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}&=mL^2\ddot{\theta}\end{aligned}$$

so

$$mL^2\ddot{\theta} = m\omega^2 L^2 \cos\theta \sin\theta - mgL\sin\theta$$
$$\ddot{\theta} = \omega^2 \cos\theta \sin\theta - \frac{g}{L}\sin\theta$$

when $\omega = 0$ we get the simple pendulum $\ddot{\theta} = -\frac{g}{L}\sin\theta$. Identifying the the equilibrium points where $\ddot{\theta} = 0 \implies$

$$\sin\theta = 0 \implies \theta = 0, \pi$$

at $\theta = 0$ the pendulum is just hanging vertically down which we can physically deduce as a stable equilibrium point. To check this analytically we can assume a small deviation from the equilibrium point:

$$\theta = 0 + \epsilon$$
$$\cos(0+\epsilon) = 1 - \frac{\epsilon^2}{2} \approx 1$$
$$\sin(0+\epsilon) = \epsilon - \frac{\epsilon^3}{6} \approx \epsilon$$

and we get

$$\ddot{\theta} = (\omega^2 - \frac{g}{L})\theta$$
$$\ddot{\theta} = -\Omega^2\theta \implies \text{Stable}$$
$$\ddot{\theta} = \Omega^2\theta \implies \text{Unstable}$$

where

$$\omega^2 < \frac{g}{L} \implies \text{Stable}$$
$$\omega^2 > \frac{g}{L} \implies \text{Unstable}$$

when they are equal $\omega^2 = \frac{g}{L}$ we get a simple pendulum. Finding another equilibrium point at

$$\omega^2\cos\theta - \frac{g}{L} = 0$$
$$\cos\theta = \frac{g}{L\omega^2}, \qquad \theta = \pm\arccos\Big(\frac{g}{L\omega^2}\Big)$$

where there only exists a solution when

$$\omega^2 > \frac{g}{L}$$

since $\cos\theta \leq 1$. For this case, we can also look at the radial force in polar:

$$F_r = m\ddot{r} - mr\omega^2 \quad \text{or} \quad m\ddot{r} = F_r + mr\omega^2$$

where in the second equation we can see that the sum of the centrifugal force and F_r sums to zero so

$$\tan\theta = \frac{F_r}{mg} = \frac{mL\sin\theta\omega^2}{mg}$$
$$\implies \frac{L\omega^2}{g} = \frac{1}{\cos\theta}$$
$$\cos\theta = \frac{g}{L\omega^2}$$

from the force analysis we can see that the centrifugal force is balanced by the radial force. Substituing the equilibium position back into the EOM

$$\cos\theta_o = \frac{g}{L\omega^2} \to \theta = \theta_o + \epsilon$$

Using Taylor Expansion, $f(x) = f(x_o) + f'(x_o)(x - x_o)$, the sine and cosine terms are

$$\begin{aligned} \cos\theta = \cos(\theta_o + \epsilon) = \cos\theta_o - \sin\theta_o\epsilon \\ \sin\theta = \sin(\theta_o + \epsilon) = \sin\theta_o + \cos\theta_o\epsilon \end{aligned}$$

so the EOM becomes

$$\begin{aligned} \ddot{\theta} = (-\omega^2 \sin\theta_o\epsilon)(\sin\theta_o + \cos\theta_o\epsilon) \\ \ddot{\epsilon} = -\omega^2 \sin^2(\theta_o)\epsilon \end{aligned}$$

where we have Bifurcation at $\omega^2 = \frac{g}{L}$. We can see that the EOM for ϵ is similar to the harmonic oscillator so:

$$\epsilon = A\cos(\Omega t - \delta) \qquad \Omega = \omega\sin\theta_o$$

HW 5 Given $f(x, y, y')$. Independence of y means:

$$\begin{aligned} f(x, y') \implies \frac{\partial f}{\partial y'} = constant \\ \mathcal{L}(t, \dot{q}) \implies \frac{\partial \mathcal{L}}{\partial \dot{q}} = constant \end{aligned}$$

so for the Lagrangian

$$\begin{aligned} \mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - U(x) \qquad \frac{\partial \mathcal{L}}{\partial q_i} = F_i \quad \text{generalized force} \\ \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = conserved, \quad \frac{\partial U}{\partial x} = 0 \end{aligned}$$

if $\mathcal{L}$ doesn't depend on x, then p_x (momentum) is conserved. So for the generalized Lagrangian

$$\begin{aligned} \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \\ \text{If} \quad \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{conserved} \end{aligned}$$

So symmetry $\implies$ conservation (Noether's Theorem).

Conservation The two types:

- If $f(x, y')$ is independent of y, then

$$\frac{\partial f}{\partial y'} = \text{constant over } x$$

or if $\mathcal{L}$ is independent of q_i, then

$$\frac{\partial \mathcal{L}}{\partial q_i} = \text{constant over } t = p_i$$

- If $f(y, y')$ is independent of x, then

$$f - y' \frac{\partial f}{\partial y'} = \text{constant over } x$$

or if $\mathcal{L}$ is independent of t, then

$$\mathcal{L} - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant over } t$$

looking at this more closely:

$$\mathcal{L} = T - U = \frac{1}{2} m\dot{q}^2 - U(q)$$

where

$$\begin{aligned} \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} &= m\dot{q}^2; \\ \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} &= m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + U \\ &= \frac{1}{2}m\dot{q}^2 + U = T + U = E \end{aligned}$$

this is this Hamiltonian

$$\sum_i p_i \dot{q}_i - \mathcal{L} = \mathcal{H} = E$$

Noether's Theorem For a system independent of $t \leftrightarrow$ the system has time-translation symmetry $\implies$ conservation of energy

Dependence on t $U = U(q, t)$ e.g. Mass of sun is increasing over time, the potential energy is dependent on time, so the system is not conservative.

Pendulum thoughts: In our pendulum example, we chose $q = \theta$, but we could also choose $q_1 = x$ and $q_2 = y$. The truth lies in the fact that we intuitively chose $q_1 = r$ and $q_2 = \theta$. So in transforming from Cartesian coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = L$$

where we have a 'constraint' $r = L\ldots$

Legal Terms: Formal Definition of Constraints In the beginning, we defined the first defined position with

$$\mathbf{r} = (x, y, z)$$

for the generalized coordinates we have

$$\mathbf{r} = \mathbf{r}(q_1, \ldots, q_n, t)$$

where we decided that in a 3D system $n = 3$. A constraint is an equation

$$f(q_1, \ldots, q_n) = 0$$

where this is a *holonomic* (whole) constraint and to find the number of generalized coordinates:

$$\begin{aligned} \#\text{ of generalized coordinates we need} &= \#\text{ of dimensions} - \#\text{ of constraints} \\ &= \#\text{ of degrees of freedom} \end{aligned}$$

this is only true for holonomic constraints. For *nonholonomic* constraints, it is more complicated e.g. A ball on a horizontal table: We can see that # of generalized coordinates = 2, but to describe the position of the ball i.e. a dot on the ball, we need 3 more coordinates (Euler angles). So the configuration of the ball is described by 5 coordinates $(x, y, \alpha, \beta, \gamma)$. In other words, the configuration is path dependent and we see a nonholonomic constraint.

Example: What are the constraints for the mass sliding down a moving mass? The holonomic constraints are the vertical position of the ramp $y_M = 0$, and from x_m, y_m, x_M we know the $x_{COM} =$ constant.

Fact! A constraint is enforced by a constraint force $\mathbf{F}_c \perp$ path(in the pendulum example, the normal force N). Finding this force where $f(q_i) = 0$ can be found by taking the gradient of the function $\boldsymbol{\nabla} f$. So

$$\mathbf{F}_c = \lambda \boldsymbol{\nabla} f$$

Review

- Convservation: Lagrangian is independent of time $\implies$ conservation of energy

Lagrange Multiplier Want to find $q_i(t)$ by minimizing $S = \int \mathcal{L}(q_i, \dot{q}_i, t)\mathrm{d}t$.

$\star$ Under holonomic constraints,

$$f(q_i) = 0$$

So we introduce a new unknown $\lambda(t)$ and the new minimizing integral becomes

$$I = \int (\mathcal{L} - \lambda f)\mathrm{d}t$$

The EL eqn for $\lambda(t)$: $\quad f = 0$

$$\frac{\partial(\mathcal{L} - \lambda f)}{\partial \lambda} = -f \qquad \frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{\lambda}} = 0$$

The EL eqn for $q_i(t)$:

$$F_i = \frac{\partial(\mathcal{L} - \lambda f)}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{q}_i}$$

or

$$F_i = \frac{\partial \mathcal{L}}{\partial q_i} - \lambda\frac{\partial f}{\partial q_i} \quad \text{where} \quad \frac{\partial \mathcal{L}}{\partial q_i} = p_i$$

So we are given $N+1$ unknowns and $N+1$ EL eqns with the addition of the lagrange multiplier.

Simple Pendulum (revisited) We have the Lagrangian $\mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U$ where

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$U = -mgy$$

so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

and using the constraint of the fixed length; $f(x, y) = x^2 + y^2 - L^2 = 0$ we get

$$\ell = \mathcal{L} - \lambda f = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy - \lambda(x^2 + y^2 - L^2)$$

and the EL eqns are

- x:
$$-2\lambda x = m\ddot{x}$$
- y:
$$mg - 2\lambda y = m\ddot{y}$$
- λ: Left as an exercise

We can see from force analysis of the pendulum:

$$m\ddot{x} = F_x = -2\lambda x \qquad m\ddot{y} = F_y = mg - 2\lambda y$$

so the lagrange multiplier quantities are equivalent to the tension

$$T_x = 2\lambda x \qquad T_y = 2\lambda y$$

where the negative sign indicates the correct direction of Tension.

Pendulum in Polar (r, ϕ)

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - mgr\cos\theta$$

where

$$f = r - L = 0$$

so we get the EL eqns

$$-\lambda + mg\cos\theta = m\ddot{r} \qquad \lambda = mg\cos\theta$$

Use cases of Lagrange Multipliers Although the previous example seems trivial, we consider its use in the example of a heavy chain hanging from two poles: The linear mass density is given by

$$M = \rho L$$

to find the shape, we need to minimize the potential energy

$$S = \int \mathrm{d}mgy$$

where $\mathrm{d}m = \rho \mathrm{d}s$ is the mass of a segment and under the constraint of chain length:

$$L = \int \mathrm{d}s = \int \mathrm{d}x\sqrt{1 + y'^2}$$

so

$$S = \int \rho gy\sqrt{1 + y'^2}\mathrm{d}x$$

and introducing λ we minimize

$$\int (\rho gy - \lambda)\sqrt{1 + y'^2}\mathrm{d}x = S - \lambda L$$

we can see that it is independent of x so

$$f = (\rho gy - \lambda)\sqrt{1 + y'^2}$$

and

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

so the EL eqn is:

$$\frac{\partial f}{\partial y'} = (\rho gy - \lambda)\frac{y'}{\sqrt{1 + y'^2}}$$

and therefore

$$f - y'\frac{\partial f}{\partial y'} = (\rho gy - \lambda)\left[\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}}\right] = \text{constant}$$

and quantity in brackets is

$$[\,] = \frac{1}{\sqrt{1 + y'^2}}$$

so

$$
\begin{aligned}
\left(\frac{\rho g y-\lambda}{\sqrt{1+y'^2}}\right)^2 &= C^2 \\
1+y'^2 &= \frac{(\rho g y-\lambda)^2}{C^2}
\end{aligned}
$$

for an easier solution we choose a change of variables

$$\tilde{y}=\frac{\rho g y-\lambda}{C} \implies \tilde{y}'=\frac{\rho g}{C}y'$$

and redifining the x

$$\tilde{y}'^2=1+y'^2\begin{cases}\tilde{x}=\frac{\rho g}{C}x\\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}}=\frac{\mathrm{d}y}{\mathrm{d}x}\end{cases}$$

and we get

$$\tilde{y}=\pm\cosh(\tilde{x}-\tilde{x}_0)$$

we could have also used the Lagrange Multiplier for the Maximum Area Fixed Perimeter problem.

Review Constraint – holonomic $f(q,\dots,q_n)=0\to$ Lagrange Multiplier

Example Simple pendulum spinning on its vertical axis. We have the Lagrangian

$$\begin{aligned}T&=\frac{1}{2}m(L^2\dot{\theta}^2+L^2\sin^2\theta\omega^2)\\U&=mgy=mgL(1-\cos\theta)\\\mathcal{L}&=T-U\\&=\frac{1}{2}mL^2\dot{\theta}^2+\frac{1}{2}mL^2\sin^2\theta\omega^2+mgL(1-\cos\theta)\end{aligned}$$

but we can see the derivatives are also conserved:

$$\begin{aligned}T'&=\frac{1}{2}mL^2\dot{\theta}^2,\quad U'=-\frac{1}{2}mL^2\omega^2\sin^2\theta+mg(1-\cos\theta)\\\mathcal{L}&=T'-U'\end{aligned}$$

where U' is called the effective potential. E' is conserved, so

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}&=ml^2\dot{\theta}\quad\text{angular momentum}\\\dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}-\mathcal{L}&=E'\\&=\frac{1}{2}mL^2\dot{\theta}^2-\frac{1}{2}mL^2\omega^2\sin^2\theta+mgL(1-\cos\theta)=T'+U'\\T+U&=\frac{1}{2}mL^2\dot{\theta}^2+\frac{1}{2}mL^2\omega^2\sin^2\theta+mgL(1-\cos\theta)\end{aligned}$$

we can see that the conserved quantity is different the mechanical energy $E=T+U$. We should be careful with finding what is the conserved quantity in noninertial frames (mechanical energy is not always conserved). In order to study the the function, we should look at the E' term.

Equilibrium points:

$$\begin{aligned}\frac{\mathrm{d}U'}{\mathrm{d}\theta}&=0\\(g-L\omega^2\cos\theta)\sin\theta&=0\end{aligned}$$

- if $\omega^2<\frac{g}{L}$ then only 1 equilibrium point
- if $\omega^2>\frac{g}{L}$ then 2 equilibrium points

Figure 8.2 shows the effective potential $U'(\theta)$ for the two cases. Sidenote: spinning the green curve around the E' axis gives rise to the Mexican Hat potential in particle physics(Higgs Boson!).

Lagrangian for a charged particle $\mathbf{E},\mathbf{B}$: we have a Lorentz force

$$m\mathbf{a}=F=q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

where we can define a vector potential $\mathbf{A}$ such that

$$\mathbf{E}=-\boldsymbol{\nabla}\phi-\frac{\partial\mathbf{A}}{\partial t},\quad\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$$

Goal: to find $\mathcal{L}$ that gives

$$\frac{\partial\mathcal{L}}{\partial q}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)$$

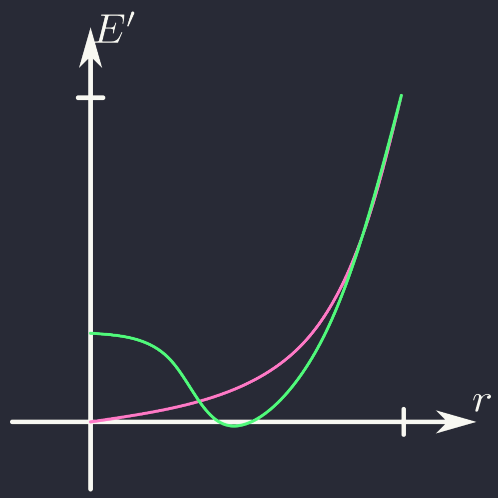

Figure 8.2: Red shows 1 eq point, green shows 2 eq points

so

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^2 - q(\phi - \mathbf{v}\cdot\mathbf{A})$$

the generalized coordinate is $q(x, y, z)$ and we just look at x:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= -q\frac{\mathrm{d}\phi}{\mathrm{d}x} + q\mathbf{v}\cdot\frac{\partial \mathbf{A}}{\partial x} \\
&= -q\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) \\
\frac{\partial \mathcal{L}}{\partial \dot{x}} &= mv_x + qA_x
\end{aligned}$$

so from the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\dot{v}_x + q(\cancel{v_x\frac{\partial A_x}{\partial x}} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t})$$

so

$$\begin{aligned}
m\dot{v}_x = ma_x = &-q\left(\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) && \equiv qE_x \\
&+ qv_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) && \equiv B_z \\
&- qv_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) && \equiv B_y \\
= &\; qE_x + qv_yB_z - qv_zB_y
\end{aligned}$$

which is the Lorentz force. We can think of the momentum as

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \qquad \mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

and from QM we know

$$\mathbf{p} \to -i\hbar\boldsymbol{\nabla} - q\mathbf{A}$$

And reviewing from Chapter 2, the linear drag force

$$\mathbf{f} = -b\mathbf{v}$$

is obviously nonconservative. So can get the EL eqn quickly by adding this force to the generalized force

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) &= \frac{\partial \mathcal{L}}{\partial q_i} + f \\ p_i &= F + f \end{aligned}$$

this comes from Rayliegh dissipation function.

$$R(\dot{q}) = \frac{1}{2}b\dot{q}^2 \qquad f = -\frac{\partial R}{\partial \dot{q}}$$

so the EL eqn is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial R}{\partial \dot{q}}$$

where we have functions

$$T(\dot{q}), U(q), R(\dot{q})$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial R}{\partial \dot{q}} + \frac{\partial U}{\partial q} = 0$$

Lecture 17: 2/28/24

Midterm Review

- Newton's Laws
  - 1. Inertial: Keep on going and it won't stop coming, so much to do so much to see.
  - 2. $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}$
  - 3. $\mathbf{F}_{12} = -\mathbf{F}_{21}$
- Polar Coordinates (r, ϕ)

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

and unit vectors are orthogonal

$$\begin{cases} \hat{r} = \cos\phi\hat{x} + \sin\phi\hat{y} \\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \end{cases} \quad \rightarrow \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$$

such that the unit vector time derivatives are

$$\dot{\hat{r}} = \dot{\phi}\hat{\boldsymbol{\phi}}$$
$$\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$$

so the velocity and acceleration is actually

$$\begin{aligned} \mathbf{v} = \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \\ &= v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} \\ \mathbf{a} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}} \end{aligned}$$

where the forces are

$$F_r = ma_r \qquad F_\phi = ma_\phi$$

- Momentum $\mathbf{p} = m\mathbf{v}$ and in relation to force $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. For a collection of particles, the total external force is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{p}_i = \mathbf{F}_{ext}$$

- Angular momentum

$$\ell = \mathbf{r} \times \mathbf{p}$$

- Center of mass

$$\mathbf{R} = \frac{1}{M}\sum m_i\mathbf{r}_i \qquad M = \sum m_i$$
$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,\mathrm{d}m$$

- Energy: Kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$$

and for the two coordinate systems:

$$v^2 = v_x^2 + v_y^2 = v_r^2 + v_\phi^2$$

- Work-KE Theorem:

$$T_2 - T_1 = \int_1^2 \mathbf{F}\cdot \mathrm{d}\mathbf{r} = W(1\to 2)$$

and if $\mathbf{F}$ is conservative—only depends on position: $\mathbf{F}(\mathbf{r})$ & $\boldsymbol{\nabla}\times\mathbf{F} = 0$ thus $\mathbf{F} = -\boldsymbol{\nabla}U$—then

$$\begin{aligned} W(1\to 2) &= -\Delta U = U_1 - U_2 \\ E &= T_1 + U_1 = T_2 + U_2 \end{aligned}$$

and more closely finding the critical points of U i.e.

$$\frac{\partial U}{\partial x} = 0 \quad \text{or} \quad \boldsymbol{\nabla}U = 0$$

we also have classical turning points when $E = U$. (Not too important) For the case

$$\begin{aligned} \frac{1}{2}m\dot{x}^2 &= E - U(x) \\ \implies \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\frac{2}{m}}\sqrt{E - U(x)} \end{aligned}$$

- Oscillators

$$\ddot{x} = -\frac{k}{m}x = -\omega_o^2 x \qquad \omega_o = \sqrt{\frac{k}{m}}$$

the solution is written in several forms:

$$\begin{aligned} x(t) &= B_1\cos(\omega_o t) + B_2\sin(\omega_o t) \\ &= \mathrm{Re}[C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t}] \\ &= A\cos(\omega_o t - \delta) \end{aligned}$$

where we solve for the constants using the initial conditions $x(0) = x_o, \quad \dot{x}(0) = v_o$

- Damped Oscillators:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

where we have a homogeneous solution for the three cases:

  - $\beta = \omega_o$: Critical damping

$$x_h(t) = e^{-\beta t}(C_1 + C_2 t)$$

  - $\beta > \omega_o$: Overdamping

$$x_h(t) = e^{-(\beta - \sqrt{\beta^2-\omega_o^2})t}\left(C_1 + C_2 e^{-2\sqrt{\beta^2-\omega_o^2}t}\right)$$

  - $\beta < \omega_o$: Underdamping (weak damping)

$$x_h(t) = e^{-\beta t}A\cos(\omega t - \delta) \qquad \omega = \sqrt{\omega_o^2 - \beta^2}$$

- Driven Damped Oscillators:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_0\cos(\omega t)$$

where f_o has units of acceleration and β has units of frequency. The solution is always

$$x(t) = A\cos(\omega t - \delta) + x_h(t)$$

where $x_h(t)$ is the transient solution and the constants are

$$A^2 = \frac{f_0^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right)$$

where we have a resonance frequency around $\omega = \omega_o$.

- Calculus of Variations: minimizing the action

$$S = \int f(x, y, y')\mathrm{d}x$$

to find $y(x)$ from the EL eqn

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

- Lagrangian (Application of CoV)

$$\mathcal{L} = T - U$$

where the EL eqns are

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

- Conservation: Two special cases
  - If $\mathcal{L}$ is independent of $q_i \Leftrightarrow \frac{\partial \mathcal{L}}{\partial q_i} = 0$

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{is conserved}$$

  - If $\mathcal{L}$ is independent of t

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L} = \text{constant}$$

Lecture 18: **3/4/24**

9 Central Force Problems

Two-Body Considering a two-body system of masses m_1, m_2 we know that under the influence of gravitational potential

$$U = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = -\frac{Gm_1m_2}{\boldsymbol{\mathscr{r}}}$$

so the force on each mass is

$$\begin{aligned}\mathbf{F}_{12} &= -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_1 U \\ \mathbf{F}_{21} &= +\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U\end{aligned}$$

computing the Lagrangian:

$$\begin{aligned}T &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 \\ U &= -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\end{aligned}$$

in 3D we have 6 degrees of freedom, so we have 6 generalized coordinates:

$$\mathbf{r}_1 = (x_1, y_1, z_2) \quad \mathbf{r}_2 = (x_2, y_2, z_2)$$

and from the separation vector

$$\boldsymbol{\mathscr{r}} = \mathbf{r}_1 - \mathbf{r}_2$$

the center of mass is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \qquad M = m_1 + m_2$$

we can rewrite the position vectors in terms of the COM and separation vector:

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\boldsymbol{\mathscr{r}} \\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\boldsymbol{\mathscr{r}}\end{aligned}$$

and thus the derivatives are

$$\begin{aligned}\dot{\mathbf{r}}_1 &= \dot{\mathbf{R}} + \frac{m_2}{M}\dot{\boldsymbol{\mathscr{r}}} \\ \dot{\mathbf{r}}_2 &= \dot{\mathbf{R}} - \frac{m_1}{M}\dot{\boldsymbol{\mathscr{r}}}\end{aligned}$$

so the Lagrangian is rewritten as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\boldsymbol{\mathscr{r}}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\boldsymbol{\mathscr{r}}}\right)^2 - U \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1m_2}{M}\dot{\boldsymbol{\mathscr{r}}}^2 - U\end{aligned}$$

where we have the reduced mass

$$\mu = \frac{m_1m_2}{M} = \frac{m_1m_2}{m_1 + m_2}$$

here we can see that $\mathcal{L}$ does not depend on $\mathbf{R}$

$$\implies \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{R}}_i} = \text{const} \implies M\dot{\mathbf{R}} = \text{const} \quad \text{or} \quad M\ddot{\mathbf{R}} = 0$$

this is the igorable coordinate, so Transforming into the COM frame

$$\begin{aligned} \mathbf{r}_1' &= \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\boldsymbol{\mathscr{r}} \\ \mathbf{r}_2' &= \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\boldsymbol{\mathscr{r}} \end{aligned}$$

and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\mu\dot{\boldsymbol{\mathscr{r}}}^2 - U(\boldsymbol{\mathscr{r}})$$

which is basically a single particle leaving us with 3 coordinates(Degrees of freedom).

Angular momentum in the COM frame is

$$\begin{aligned} L &= \sum_i \boldsymbol{\mathscr{r}}_i' \times \mathbf{p}_i' \\ &= \boldsymbol{\mathscr{r}}' \times m_i\dot{\boldsymbol{\mathscr{r}}}' \\ &= m_1\boldsymbol{\mathscr{r}}_1' \times \dot{\boldsymbol{\mathscr{r}}}_1' + m_2\boldsymbol{\mathscr{r}}_2' \times \dot{\boldsymbol{\mathscr{r}}}_2' \\ &= \frac{m_1 m_2^2}{M}\boldsymbol{\mathscr{r}} \times \dot{\boldsymbol{\mathscr{r}}} + \frac{m_1^2 m_2}{M}\boldsymbol{\mathscr{r}} \times \dot{\boldsymbol{\mathscr{r}}} \\ &= \frac{m_1 m_2}{M}\boldsymbol{\mathscr{r}} \times \dot{\boldsymbol{\mathscr{r}}} = \mu\boldsymbol{\mathscr{r}} \times \dot{\boldsymbol{\mathscr{r}}} \end{aligned}$$

which is the same as the angular momentum of a single particle with reduced mass μ.

- If $m_2 \gg m_1$ then $\mathbf{R} \approx \mathbf{r}_2$ and $\mu \approx m_2$.
- If $m_1 \gg m_2$ then $\mathbf{R} \approx \mathbf{r}_1$ and $\mu \approx m_1$.
- If $m_1 = m_2$ then $\mathbf{R}$ is directly in the middle of the two particles and $\mu = \frac{m_1}{2} = \frac{m_2}{2}$.

We can see that for two vectors, any linear combination will result in a vector on a plane, so we can turn this into a 2D problem. Using polar coordinates we can write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\mu(\dot{\boldsymbol{\mathscr{r}}}^2 + \boldsymbol{\mathscr{r}}^2\dot{\phi}^2) - U(r)$$

where we can see that it does not depend on ϕ, so we have the conserved quantity

$$\implies \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{const} = \mu\boldsymbol{\mathscr{r}}\dot{\phi} = \ell$$

and the EL equation is only needed for r:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} &= \mu\boldsymbol{\mathscr{r}}\dot{\phi}^2 - \frac{\partial U}{\partial \boldsymbol{\mathscr{r}}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\mathscr{r}}}}\right) &= \mu\ddot{\boldsymbol{\mathscr{r}}} \\ \implies \mu\ddot{\boldsymbol{\mathscr{r}}} &= \mu\boldsymbol{\mathscr{r}}\dot{\phi}^2 - \frac{\partial U}{\partial \boldsymbol{\mathscr{r}}} \qquad U = -\frac{Gm_1m_2}{\boldsymbol{\mathscr{r}}} \\ &= \frac{l^2}{\mu\boldsymbol{\mathscr{r}}^3} - \frac{\partial U}{\partial \boldsymbol{\mathscr{r}}} \\ &= \frac{l^2}{\mu\boldsymbol{\mathscr{r}}^3} - \frac{Gm_1m_2}{\boldsymbol{\mathscr{r}}^2} \end{aligned}$$

Solving the 1D Problem First we note the first term is equivalent to the minus gradient of the centrifugal potential:

$$\begin{aligned} m\ddot{\mathscr{r}} &= -\frac{\partial U_{cf}}{\partial \mathscr{r}} - \frac{\partial U}{\partial \mathscr{r}} \\ &= -\frac{\partial}{\partial \mathscr{r}}\left(\frac{\ell^2}{2\mu \mathscr{r}^2}\right) - \frac{\partial U}{\partial \mathscr{r}} \\ &= -\frac{\partial}{\partial \mathscr{r}}(U_{cf} + U) \end{aligned}$$

where $U_{cf} = \frac{\ell^2}{2\mu \mathscr{r}^2}$ is the centrifugal potential. We can see that the effective potential is

$$U_{eff} = \frac{\ell^2}{2\mu \mathscr{r}^2} - \frac{Gm_1 m_2}{\mathscr{r}}$$

From Last Time For a 2-Body problem where $M = m_1 + m_2$ and the COM

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)$$
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mu = \frac{m_1 m_2}{M}$$

we found the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\\ &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)\end{aligned}$$

where $\mathcal{L}$ is independent of ϕ, so we have the conserved quantity

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2\dot{\phi} = \ell \implies \dot{\phi} = \frac{l}{mr^2}$$

so the EL equation for r is

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

where the centrifugal force is

$$F_{cf} = \frac{\ell^2}{\mu r^3}$$

and the effective potential is

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = U_{cf} + U$$

From the graph of this effective potential, there is a centrifugal barrier for finite ℓ for $\ell = \mathbf{r}\times\mathbf{p}$ and for $r \to 0$ the potential is dominated by the centrifugal term.

Conservation of Energy If this problem is independent of time we know that

$$\begin{aligned}E &= \sum_i \dot{q}_i p_i - \mathcal{L}\\ &= \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L}\\ &= \mu\dot{r}^2 + \frac{\ell^2}{\mu r^2} - \frac{1}{2}\mu\dot{r}^2 - \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)\\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = T + U\end{aligned}$$

we can find the equilibrium point at

$$\begin{aligned}\frac{\partial U_{eff}}{\partial r} &= 0\\ &= -\frac{\ell^2}{\mu r^3} + \frac{\gamma}{r^2} \qquad \gamma = Gm_1m_2\\ \implies r_o &= \frac{\ell^2}{\gamma\mu}\end{aligned}$$

this radius is related to a perfectly circular *orbit.* and at

$$r = r_o, \qquad \dot{\phi} = \frac{\ell\mu^2\gamma^2}{\mu\ell^4} = \frac{\mu\gamma^2}{\ell^3}$$

so

$$\phi(t) = \int_0^t \dot{\phi}(t')\,\mathrm{d}t'$$

For $E < 0$ we have a bound (bounded) orbit, and for $E > 0$ we have an unbounded orbit. For $E = 0$ we also have an unbounded orbit.

What does the orbit look like? Find $r(\phi)$ using a differential equation (For a circular orbit we know $r = r_o$). First we introduce a variable transformation

$$q = \frac{1}{r}, \qquad r = \frac{1}{q}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{q}\right) = -\frac{1}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\phi}, \qquad q' = \frac{\mathrm{d}q}{\mathrm{d}\phi}$$

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\cdot\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\ell}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi} = -\frac{\ell}{\mu r^2}\frac{1}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\phi} = -\frac{\ell}{\mu}\frac{\mathrm{d}q}{\mathrm{d}\phi}$$

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}\dot{r}}{\mathrm{d}\phi} = -\dot{\phi}\frac{\ell}{\mu}q'' = -\frac{\ell^2 q^2}{\mu^2}q''$$

and the central force is

$$\begin{aligned} \mu\ddot{r} &= \frac{\ell^2}{\mu r^3} + F \\ -\mu\frac{\ell^2 q^2}{\mu^2}q'' &= \frac{\ell^2 q^3}{\mu r^3} + F \\ q'' &= -q - \frac{\mu}{q^2\ell^2}F \end{aligned}$$

and since the force is

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{\gamma}{r^2} = -\gamma q^2$$

so the differential equation is just

$$q'' = -q + \frac{\gamma\mu}{\ell^2}$$

and the RHS vanishes when

$$q = \frac{\gamma\mu}{\ell^2} \quad \text{or} \quad r_o = \frac{\ell^2}{\gamma\mu}$$

we can redefine the constant

$$\omega = q - \frac{\gamma\mu}{\ell^2} \implies \omega'' = q'' = -\omega$$

so

$$\omega(\phi) = A\cos(\phi - \delta)$$

and choosing initial conditions so that $\delta = 0$

$$\omega(\phi) = A\cos(\phi) \implies q(\phi) = A\cos(\phi) + \frac{\gamma\mu}{\ell^2} = \frac{1}{r(\phi)}$$

and thus

$$r(\phi) = \frac{\ell^2/\gamma\mu}{1 + \epsilon\cos(\phi)} = \frac{C}{1 + \epsilon\cos(\phi)} \qquad \epsilon = \frac{A}{C}$$

we can check and see that r has the unit of length and the denominator is unitless, so C has the unit of length. We can see that ϵ only depends on the initial conditions, and at

$$\epsilon = 0 \implies r(\phi) = C = r_o$$

so

$$r(\phi) = \frac{r_o}{1 + \epsilon\cos(\phi)}$$

and ϵ is the eccentricity of the orbit.

- If $\epsilon = 0$ then $r = r_o$ and we have a circular orbit.
- If $\epsilon > 1$ then the denominator can $\to 0$ and we have $r \to \infty$ or hyperbolic orbit.
- If $0 < \epsilon < 1$ then we have a bounded orbit or ellipse.
- IF $\epsilon = 1$ then we have a parabolic orbit.

Lecture 20: 3/8/24

Missed Lecture: