Homework 9

1. Inertia Tensor derivation using kronicker delta: The diagonal elements of the inertia tensor are given by

$$I_{xx} = \sum m_i(y_i^2 + z_i^2)$$
$$I_{yy} = \sum m_i(x_i^2 + z_i^2)$$
$$I_{zz} = \sum m_i(x_i^2 + y_i^2)$$

and the off-diagonal elements are given by

$$I_{xy} = -\sum m_i x_i y_i$$
$$I_{xz} = -\sum m_i x_i z_i$$
$$I_{yz} = -\sum m_i y_i z_i$$

For the diagonal element we can see that using the single equation

$$\begin{aligned}
I_{xx} &= \int \rho(r^2\delta_{xx} - r_x r_x)\,\mathrm{d}V \\
&= \int \rho(r^2 - x^2)\,\mathrm{d}V \\
&= \int \rho(x^2 + y^2 + z^2 - x^2)\,\mathrm{d}V \\
&= \int \rho(y^2 + z^2)\,\mathrm{d}V = \sum m_i(y_i^2 + z_i^2)
\end{aligned}$$

and similarly for the other diagonal elements. For the off-diagonal elements we can see that the kronecker delta will be zero and thus

$$\begin{aligned}
I_{xy} &= \int \rho(0 - xy)\,\mathrm{d}V \\
&= -\int \rho xy\,\mathrm{d}V = -\sum m_i x_i y_i
\end{aligned}$$

and similarly for the other off-diagonal elements.

2. (a) First we can see that

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})^2 &= (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) \\
&= (AB\sin\theta\hat{\mathbf{n}})(AB\sin\theta\hat{\mathbf{n}}) \\
&= A^2B^2(1-\cos^2\theta) \quad \text{using} \quad \mathbf{A}\cdot\mathbf{B} = AB\cos\theta \\
&= A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2
\end{aligned}$$

So the Kinetic Energy is

$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha \mathbf{v}_\alpha\cdot\mathbf{v}_\alpha \\
&= \frac{1}{2}\sum_\alpha m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha[(\omega r_\alpha)^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)^2]
\end{aligned}$$

(b) Angular momentum is given by

$$\begin{aligned}\mathbf{L} &= \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha \\ &= \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)\end{aligned}$$

using the BAC-CAB rule (or WRR-RRW rule in this case) we get

$$\begin{aligned}&= \sum_\alpha m_\alpha [\boldsymbol{\omega}(\mathbf{r}_\alpha \cdot \mathbf{r}_\alpha) - \mathbf{r}_\alpha(\mathbf{r}_\alpha \cdot \boldsymbol{\omega})] \\ &= \sum_\alpha m_\alpha [\boldsymbol{\omega} r_\alpha^2 - \mathbf{r}_\alpha(\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)]\end{aligned}$$

where dot products commute. (c) From the previous part we can see that

$$\begin{aligned}\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} &= \frac{1}{2}\boldsymbol{\omega} \cdot \sum_\alpha m_\alpha [\boldsymbol{\omega} r_\alpha^2 - \mathbf{r}_\alpha(\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)] \\ &= \frac{1}{2}\sum_\alpha m_\alpha [\boldsymbol{\omega} \cdot \boldsymbol{\omega} r_\alpha^2 - \boldsymbol{\omega} \cdot \mathbf{r}_\alpha(\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)] \\ &= \frac{1}{2}\sum_\alpha m_\alpha [\omega^2 r_\alpha^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_\alpha)^2] = T\end{aligned}$$

Since $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ we usually write the angular momentum as a column vector with three components

$$\mathbf{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$$

and taking the cross product of $\mathbf{L}$ with $\boldsymbol{\omega}$ we will have to transpose $\boldsymbol{\omega}$ to get the correct matrix multiplication:

$$\begin{aligned}\boldsymbol{\omega} \cdot \mathbf{L} &= \boldsymbol{\omega}^T \mathbf{L} \\ &= \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} \\ &= \omega_x L_x + \omega_y L_y + \omega_z L_z\end{aligned}$$

So the kinetic energy is also equivalent to

$$T = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{L} = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I}\boldsymbol{\omega}$$

3. (a) For a uniform hollow ice cream cone of radius R, height h, and mass M. The mass density is given by the mass per unit area

$$\begin{aligned} q &= \frac{M}{A} = \frac{M}{\pi R l} \qquad \frac{R}{l} = \sin\theta \\ &= \frac{M}{\pi R^2}\sin\theta \end{aligned}$$

where l is the the slant of the cone and using cylindrical coordinates

$$I_{zz} = q\int_A \mathrm{d}A\,(x^2+y^2) = q\int_A \mathrm{d}A\,\rho^2$$

The area element is a rectangular region with sides $\rho\,\mathrm{d}\phi$ and $\frac{1}{\sin\theta}\,\mathrm{d}\rho$ (the $\mathrm{d}\phi$ is projected onto the side of the cone) so the area element is

$$\mathrm{d}A = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\frac{1}{\sin\theta}$$

and the moment of inertia is

$$\begin{aligned} I_{zz} &= q\int_0^{2\pi}\mathrm{d}\phi\int_0^R \mathrm{d}\rho\,\frac{\rho^3}{\sin\theta} \\ &= \frac{M}{\pi R^2}\sin\theta(2\pi)\frac{R^4}{4\sin\theta} \\ &= \frac{1}{2}MR^2 \end{aligned}$$

$I_{xx} = I_{yy}$ since the cone is rotationally symmetric about the z-axis:

$$\begin{aligned} I_{xx} &= q\int_A \mathrm{d}A\,(y^2+z^2) \\ &= q\int_A \mathrm{d}A\,(\rho^2\sin^2\phi + z^2) \end{aligned}$$

Geometrically we have similar triangles where the ratio of the sides are

$$\frac{z}{\rho} = \frac{h}{R} \implies z = \frac{h}{R}\rho$$

so the integral becomes

$$\begin{aligned} I_{xx} &= q\int_0^{2\pi}\mathrm{d}\phi\left(\sin^2\phi + \frac{h^2}{R^2}\right)\int_0^R \mathrm{d}\rho\,\rho^3 \\ &= \frac{M}{\pi R^2}\sin\theta\left(\pi + 2\pi\frac{h^2}{R^2}\right)\frac{R^4}{4\sin\theta} \\ &= \frac{1}{4}MR^2\left(1 + 2\frac{h^2}{R^2}\right) \\ &= \frac{1}{4}M\big(R^2+2h^2\big) \end{aligned}$$

And the off-diagonal elements are all zero due to the rotational symmetry of the cone which gives the inertia tensor

$$\mathbf{I} = \begin{pmatrix} \frac{1}{4}M\big(R^2+2h^2\big) & 0 & 0 \\ 0 & \frac{1}{4}M\big(R^2+2h^2\big) & 0 \\ 0 & 0 & \frac{1}{2}MR^2 \end{pmatrix}$$

(b) An ellipsoid with volume $V = \frac{4}{3}\pi abc$ and mass M has a mass density of

$$q = \frac{M}{V} = \frac{3M}{4\pi abc}$$

Using a change of variables $x = ax', y = by', z = cz'$ the equation for the ellipsoid is

$$x'^2 + y'^2 + z'^2 = 1$$

and $\mathrm{d}x = a\,\mathrm{d}x'\,, \mathrm{d}y = b\,\mathrm{d}y'\,, \mathrm{d}z = c\,\mathrm{d}z'$ so the inertia tensor is

$$\begin{aligned} I_{zz} &= q \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,(x^2 + y^2) \\ &= qabc \int \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,(a^2x'^2 + b^2y'^2) \end{aligned}$$

and using spherical coordinates

$$\begin{aligned} x' &= r\sin\theta\cos\phi \\ y' &= r\sin\theta\sin\phi \\ z' &= r\cos\theta \end{aligned}$$

The integral becomes

$$\begin{aligned} I_{zz} &= qabc \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r\, r^2 \sin\theta (a^2r^2\sin^2\theta\cos^2\phi + b^2r^2\sin^2\theta\sin^2\phi) \\ &= q\int_0^{2\pi} \mathrm{d}\phi\,(a^2\cos^2\phi + b^2\sin^2\phi)\int_0^{\pi}\mathrm{d}\theta\sin^3\theta\int_0^1 r^4 \\ &= q(\pi a^2 + \pi b^2)\left(\frac{4}{3}\right)\frac{1}{5} \\ &= \frac{1}{5}M(a^2 + b^2) \end{aligned}$$

and the other diagonal elements will follow a similar pattern (due the rotational symmetry)

$$I_{yy} = \frac{1}{5}M(a^2 + c^2) \qquad I_{xx} = \frac{1}{5}M(b^2 + c^2)$$

and the off-diagonal elements are zero. The inertia tensor is then

$$\mathbf{I} = \begin{pmatrix} \frac{1}{5}M(b^2 + c^2) & 0 & 0 \\ 0 & \frac{1}{5}M(a^2 + c^2) & 0 \\ 0 & 0 & \frac{1}{5}M(a^2 + b^2) \end{pmatrix}$$

(c) The triangle is rotationally symmetrical $I_{xx} = I_{yy}$. Since the triangle lies on the xy-plane $z = 0$ so

$$I_{xx} = \sigma \int_A \mathrm{d}A\,(y^2 + z^2) = \sigma \int_A \mathrm{d}A\, y^2$$

the limits of integration are given by the lines $y = 0$ to $y = -x + 1$ and $x = 0 \to 1$ so

$$\begin{aligned} I_{xx} &= \sigma \int_0^1 \mathrm{d}x \int_0^{-x+1} \mathrm{d}y\, y^2 \\ &= \sigma \int_0^1 \mathrm{d}x\, \frac{1}{3}(-x + 1)^3 \\ &= \frac{1}{12}\sigma = 2 \end{aligned}$$

for I_{zz} we have

$$I_{zz} = \sigma \int_A \mathrm{d}A\,(x^2 + y^2) = I_{xx} + \sigma \int_A \mathrm{d}A\,x^2$$

and since $I_{yy} = \sigma \int_A \mathrm{d}A\,(x^2 + z^2) = \sigma \int_A \mathrm{d}_A x^2$

$$I_{zz} = I_{xx} + I_{yy} = 4$$

Sadly, only 4 diagonal elements (the ones containing z) are zero. So calculating the two off-diagonal elements $I_{xy} = I_{yx}$ we have

$$\begin{aligned}
I_{xy} &= -\sigma \int_A \mathrm{d}A\,xy \\
&= -\sigma \int_0^1 \mathrm{d}x \int_0^{-x+1} \mathrm{d}y\,xy \\
&= -\sigma \int_0^1 \mathrm{d}x\,x\frac{1}{2}(-x+1)^2 \\
&= -\sigma \int_0^1 \mathrm{d}x\,\frac{1}{2}(x^3 - 2x^2 + x) \\
&= -\frac{1}{24}\sigma = -1
\end{aligned}$$

So the inertia tensor is

$$\mathbf{I} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

To find the principal moments we find the eigenvalues for the matrix equation

$$\det(\mathbf{I} - \lambda\mathbf{1}) = 0$$
$$\begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0$$

Where we can see that the sum of the rows can give us the eigenvalues $\lambda = 1, 4$ and since the trace of the matrix is 8 the last eigenvalue is $\lambda = 3$. Plugging in the eigenvalues into the matrix equation does indeed give us the correct solutions. To find the first principal axis we plug in $\lambda = 1$ into the matrix equation

$$(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$$
$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

which gives us the equation $\omega_x = \omega_y = 1$ and $\omega_z = 0$ so the first principal axis is

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 3$ we have

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

which gives us the equation $\omega_x = -\omega_y$ and $\omega_z = 0$ so the second principal axis is

$$\hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

And for $\lambda = 4$ we have

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

Here, $\omega_x = \omega_y = 0$ is the only solution, so the third principal axis is

$$\hat{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where we just used 1 since it makes the normalization easier. So the principal moments are

$$\lambda_1 = 1 \quad \lambda_2 = 3 \quad \lambda_3 = 4$$

and the principal axes are

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \hat{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$