1 Magnetostatics

In class demonstration Two parrell wires connected to high voltage source:

- Parallel current: The wires move towards each
- Antiparallel current: The wires move away from each other

This implies some sort of force acting on each wire which cannot not be explained by the electrostatics we have covered thus far...

1.1 Lorentz Force Law

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

where Q and $\mathbf{v}$ describe the moving test charge. Compared to the force of the electric field

$$\mathbf{F}_{\text{elec}} = Q\mathbf{E}$$

Thus the total force is give by the Lorentz Force Law

$$\boxed{\mathbf{F}_{\text{tot}} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

1.1.1 Magnetic Fields

Skip discussion so we can move straight to

1.1.2 Magnetic Forces

Cyclotron motion A $\mathbf{B}$ field is place into to page, and a test charge has velocity $\mathbf{v}$ moving up on the y-axis: So using the right-hand-rule (RHR) the force from the magnetic field points to the left with magnitude

$$F_{\text{mag}} = qvB = \text{acceleration} = \frac{mv^2}{R}$$

where we have a circular motion similar to large masses rotating under gravity i.e. the "cylotron" radius

$$R = \frac{mv}{qB}$$

Cycloid motion Add an $\mathbf{E} \perp \mathbf{B} \ldots \mathbf{B} = B\hat{\mathbf{x}} \quad \mathbf{E} = E\hat{\mathbf{z}}$

The velocity of the test charge is given by

$$\mathbf{v} = \left(0, \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \frac{\mathrm{d}z(t)}{\mathrm{d}t}\right) = (0, \dot{y}, \dot{z})$$

Where the charge is constrained in the yz plane. The math says

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\hat{\mathbf{y}} - B\dot{y}\hat{\mathbf{z}}$$

So the force is

$$\begin{aligned} \mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ &= q(E\hat{\mathbf{z}} + B\dot{z}\hat{\mathbf{y}}) = m\mathbf{a} = m(\ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}}) \end{aligned}$$

This gives us the equations of motion

$$qB\dot{z} = m\ddot{y}$$
$$qE - qB\dot{y} = m\ddot{z}$$

And using the "cyclotron frequency"

$$\omega \equiv \frac{qB}{m} \quad \sim \frac{\text{C.T}}{\text{kg}} \sim \frac{\text{rad}}{\text{s}}$$

We can rewrite the equations of motion as

$$\ddot{y} = \omega\dot{z}$$
$$\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

Which we can solve with the general solution

$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{E}{B}t + C_3$$
$$z(t) = C_2\cos(\omega t) - C_1\sin(\omega t) + C_4$$

with the intial conditions

$$\dot{y}(0) = \dot{z}(0) = 0$$
$$y(0) = z(0) = 0$$

The solution is

$$y(t) = \frac{E}{\omega B}(\omega t - \sin(\omega t))$$
$$z(t) = \frac{E}{\omega B}(1 - \cos(\omega t))$$

NOTE: $v = \omega R$ so the force $F = q(E + v \times B)$ we have a length scale

$$\frac{E}{B} \sim \omega R, \quad E \sim v \times B, \quad v \sim E/B$$

$$\boxed{R \equiv \frac{E}{\omega B}}$$

So the equations are simplified to

$$y(t) = R(\omega t - \sin(\omega t)) \quad z(t) = R(1 - \cos(\omega t))$$
$$(y - \omega tR)^2 = R^2\sin^2(\omega t) \quad (z - R)^2 = R^2\cos^2(\omega t)$$

thus

$$R^2(c^2 + s^2) = R^2 = (z - R)^2 + (y - \omega tR)^2$$

Since

$$\dot{y} = v = \omega R = \frac{E}{B}$$

and the equation is similar to the circle $x^2 + y^2 = R^2$ so the motion is a cycloid. So we can imagine the motion of the charge as rolling a circle along the y axis! [insert cycloid path]

NOTE:

$$\boxed{\textbf{magnetic fields do no work}}$$

As a charge q moves $\mathrm{d}l = \mathbf{v}\,\mathrm{d}t$ the change in word is

$$\mathrm{d}\omega_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot \mathrm{d}\mathbf{l} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,\mathrm{d}t = 0$$

But why did the wires move in the demonstration?

Currents

We usually define the current direction I using a positive charge (conventional current) because the the negative charge moving in the opposite direction of the positive charge is equivalent i.e.

$$q_+v_+ = (-q)(-v_+)$$

The current is charge/time

$$I \sim \mathrm{C/s} \sim \mathrm{A}$$

The the line charge has current

$$\mathbf{I} = \lambda\mathbf{v} \sim \mathrm{C/m\,m/s} = \mathrm{A/m}$$

In a neutral wire (or in general)

$$I = \lambda_+v_+ + \lambda_-v_-$$

The force on a current carrying wire:

$$\begin{aligned} F_{\text{mag}} &= \int (\mathbf{v}\times\mathbf{B})\,\mathrm{d}q = \int (\mathbf{v}\times\mathbf{B})\lambda\,\mathrm{d}l \\ &= \int (\mathbf{I}\times\mathbf{B})\,\mathrm{d}l = \int I(\mathrm{d}\mathbf{l}\times\mathbf{B}) \end{aligned}$$

where we assume that the current is constant *usually* so

$$\boxed{F_{\text{mag}} = I\int(\mathrm{d}\mathbf{l}\times\mathbf{B})}$$

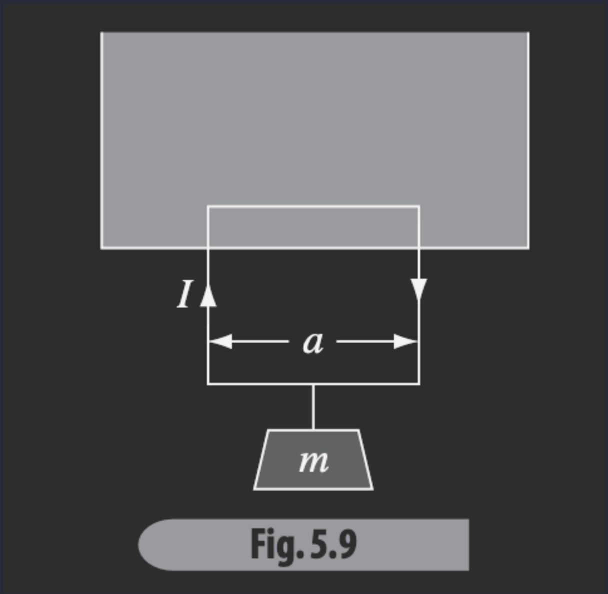

Figure 1.1: Magnetic field above point into page

Example: The current will move clockwise in the loop because on the top side in order to have the force point upwards to counteract the mass. The force on the sides of the loop will cancel out, so the magnitude is

$$|\mathbf{F}_{\text{mag}}| = IaB = mg$$

Now increasing I will increase the force which means the whole wire move upwards with velocity $\mathbf{u}$ and a change in height $\delta h = u\delta t$. This now looks like work

$$\delta W_{\text{mag}} = IaB\delta h$$

Who is doing the work to keep lifting the wire? The battery! This is analogous to normal force on applying force to a mass on an incline; the normal force does no work, but only redirects the force upwards. So magnetic force is like the normal force.

Surface Current Density When charge flows on a surface, this density

$$\mathbf{K} \equiv \frac{\mathrm{d}\mathbf{I}}{\mathrm{d}l_\perp}$$

where we can image $l_\perp$ as a thin ribbon (1.2) and K is the current per unit width thus

$$\mathbf{K} = \sigma \mathbf{v}$$

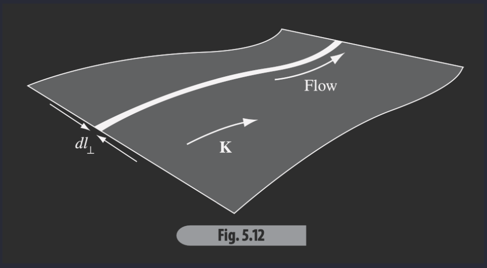

Figure 1.2: Surface current density

Magnetic force in 2D From the 1D $\mathbf{F}_{\text{mag}} = I \int \mathrm{d}\mathbf{l} \times \mathbf{B}$, so for 2D

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\sigma \,\mathrm{d}a = \int (\mathbf{K} \times \mathbf{B})\,\mathrm{d}a$$

For a charge dist. in a volume

$$\mathbf{J} \equiv \frac{\mathbf{I}}{a_\perp} \sim \text{current/unit area}$$

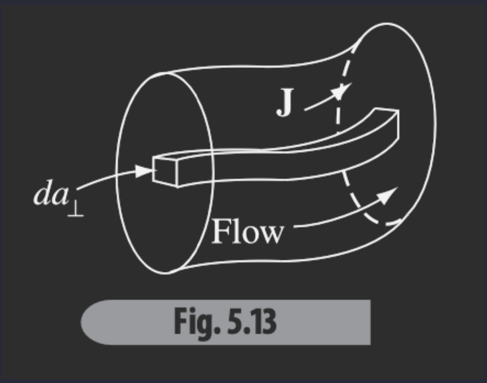

Figure 1.3: Volume current density

So we can find this (mobile) volume charge density with

$$\mathbf{J} = \rho \mathbf{v}$$

Figure 1.4: Volume current density

Think of this steady current as a column of water moving along this pipe. Rather than looking at each individual water molecule, we can look at the flow of the water as a whole.

In a $\mathbf{B}$ field there is a force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \rho \, \mathrm{d}\tau = \int (\mathbf{J} \times \mathbf{B}) \, \mathrm{d}\tau$$

With current I & uniform cross-sectional area (a wire), the volume charge density is

$$\mathbf{J} = \frac{I}{\pi a^2}$$

With $J \propto$ distance from the center of the wire thus

$$J = ks, \quad k \sim \text{constant}$$

The current is

$$\begin{aligned} I &= \int \mathrm{d}I = \int \mathbf{J} \cdot \mathrm{d}\mathbf{a}_\perp \\ &= \int (ks) s \, \mathrm{d}s \, \mathrm{d}(\phi) \\ &= 2\pi k \int_0^a s^2 \, \mathrm{d}s = 2\pi k \frac{a^3}{3} \end{aligned}$$

Total current crossing a surface

$$I = \int_S J \, \mathrm{d}\mathbf{a}_\perp = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{a}_\perp$$

If the surface is *closed*: if there is a source inside the surface, there is a net flow coming out of the surface and vice versa.

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{a} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}\tau = -\int_V \left(\frac{\mathrm{d}\rho}{\mathrm{d}t} \right) \mathrm{d}\tau$$

From the divergence theorem

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{a} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{J}) \, \mathrm{d}\tau$$

or also known as the "continuity equation"

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$$

1.2 Biot-Savart Law

- Stationare charge $\rightarrow$ const. $\mathbf{E} \rightarrow$ electrostatics (Coloumbs Law)
- Steady currents $\rightarrow$ const. $\mathbf{B} \rightarrow$ magnetostatics (Biot-Savart Law)

1.2.1 Steady Currents

ASSUME

$$\frac{\partial \rho}{\partial t} = 0 \implies \frac{\partial \mathbf{J}}{\partial t} = 0$$

For most technologies, computers, etc.

- freq $\rightarrow\sim$ 10-100 MHz; which is fast and reaches a limit due to the speed of light... (lets not worry about this for now)

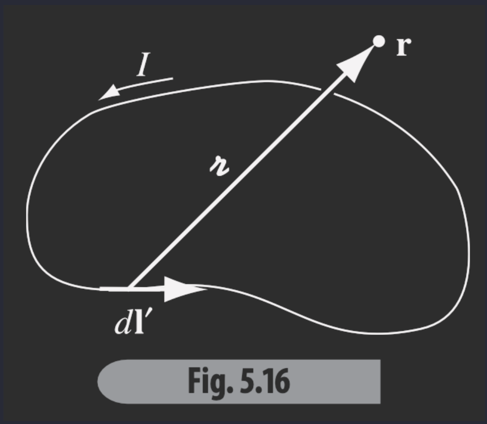

Figure 1.5: Biot-Savart Law along a wire

The $\mathbf{B}$ field of a steady current line currents is

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, \mathrm{d}l' \\ &= \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\end{aligned}$$

where we integrate along the current path and μ_0 is the permeability of free space

$$\mu_0 \approx 4\pi \times 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}$$

Units of magnetic field:

$$B \sim \frac{\mathrm{N}}{\mathrm{A.m}} \sim \frac{\mathrm{V.s}}{\mathrm{m}^2} \sim T \rightarrow \text{Tesla}$$

Where the Earth has a magnetic field of a few μ T compared to a NMR machine which has a field of a few T.

Example: Find $\mathbf{B}$ field a distance s from a long striaght current carrying wire I.

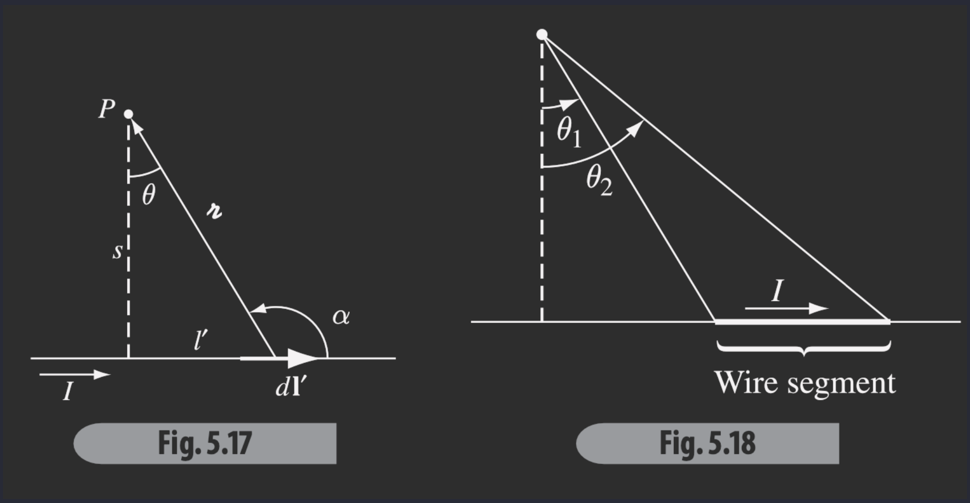

Figure 1.6: B-field a disance along a wire

From the Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

So looking at each part:

- $\mathrm{d}\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}$ at P tells us $\mathbf{B} \sim (-\hat{\mathbf{y}})$ (RHR) with magnitude

$$\mathrm{d}l' \sin\alpha = \mathrm{d}l' \cos\theta$$

- $l' = s\tan\theta$ so

$$\mathrm{d}l' = s\sec^2\theta\,\mathrm{d}\theta$$

- $s = \mathscr{r}\cos\theta$ so

$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

Thus

$$|\mathbf{B}| = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\cos\theta}{s}\, \mathrm{d}\theta$$
$$= \frac{\mu_0 I}{4\pi s} \int_{-\pi/2}^{\pi/2} \cos\theta\, \mathrm{d}\theta = \frac{\mu_0 I}{2\pi s}$$

We can imagine this as a circle around the wire capturing this B-field or wrapping your fingers around the wire where the thumb point in the direction of the current (RHR).

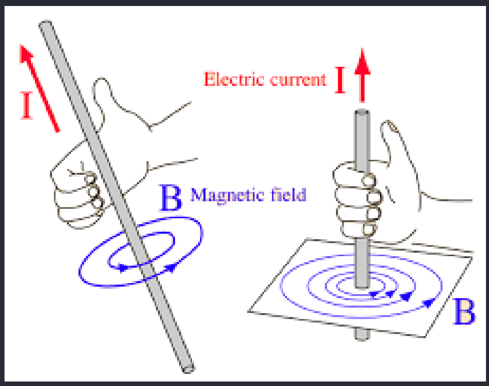

Figure 1.7: Right-hand rule

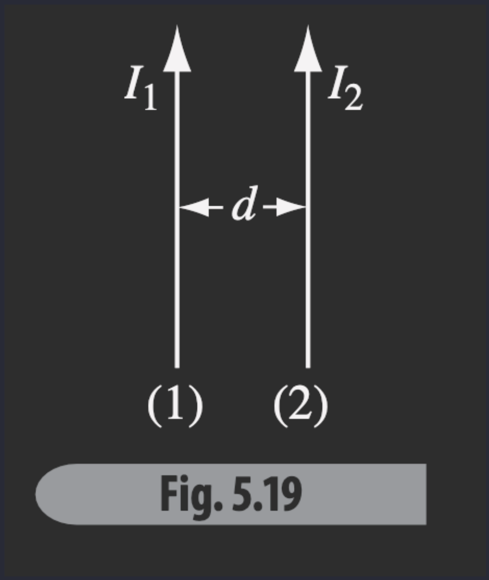

Figure 1.8: Demo

Back to demo We can now see the forces on the wires due to each other's magnetic fields.