Title

Contents

1 Intro to Particle Physics

Four Fundamental Forces

- Strong (gluon)
- Weak (W, Z)
- Electromagnetic (photon)
- Gravity (graviton?)

The 'Standard Model' describe the first three forces and unifies the Strong and Weak Forces known as the 'Electroweak' force. So, the Standard Model does not include gravity.

The Standard Model (SM)

- Basic building blocks: spin 1/2 particles (fermions)
- Interaction between then are mediated by force carriers: spin 1 particles (vector bosons)
- How particles get mass? $\rightarrow$ Higgs Boson (spin 0)

The Range of Forces:

- Strong: 10^{-15} m
- Weak: $10^{-18} \quad 10^{-16}$ m
- EM: $1/r^2$
- Gravity: $1/r^2$

The ranges of forces are related by

$$R \quad \frac{e^{-r/a}}{r^2}$$

where $a \approx 10^{-15}$ m for the Strong and Weak forces.

The Rise of Quantum Field Theory (QFT) Relativity + Quantum Mechanics $\rightarrow$ QFT

	Macroscopic	Micro
SLOW	CM	Quantum Mechanics
FAST	Special Relativity	QFT

QFT Discoveries

- Existence of anti-particles
- Spin-statistics theorem
- CPT Theorem (Charge conjugation, Parity, Time reversal)

Units!

- Mass: (kg) $\rightarrow$ (eV) from $E = mc^2$

$$m_e = 0.5 \times 10^6 \,\text{eV/c}^2 \quad E_n = \frac{-13.6\,\text{eV}}{n^2}$$
$$m_p = 1\,\text{GeV/c}^2 \quad 1\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}$$

- Momentum: $\frac{eV}{C} \rightarrow p = \frac{E}{c}$
- Energy: eV

Matter Fermions are divided into two groups:

- Leptons (electrons, muon, tau, neutrinos): Doesn't have the strong force
- Quarks (up, down, charm, strange, top, bottom): Feels the strong force

e.g. the proton is made of 2 up quarks and 1 down quark (uud) and the Neurtron is (udd).

Quarks make up composite subparticles (Hadrons) are held together by the strong force.

- Mesons: 1 quark + 1 anti-quark ($q\bar{q}$) e.g. pion, kaon...
- Baryons: 3 quarks (qqq) e.g proton, neutron

Quark charges:

- $Q = +2/3$ (up, charm, top)
- $Q = -1/3$ (down, strange, bottom)

Leptons are fundamental particles

- Charged electrically (-1)
  - electron (0.5 MeV)
  - muon (105 MeV)
  - tau (1.8 GeV)
- Neutral (neutrinos)
  - electron neutrino ν_e
  - mueon neutrino ν_μ
  - tau neutrino ν_τ

Crossing Symmetry

$$\begin{aligned} A + B &\to C + D \quad \text{Scattering} \\ A &\to B + C + D \quad \text{Decay} \\ A + \bar{C} &\to \bar{B} + D \end{aligned}$$

e.g. Neutron Decay

$$n \to p + e^- + \bar{\nu}_e$$

Sum rules to think about:

- Baryon Number Conservation
- Lepton Number Conservation
- Electric Charge Conservation

another example:

$$n + e^+ \to p + \bar{\nu}_e$$
$$p + e^- \to n + \nu_e$$

Particle Conservation Laws

2 Relativistic Kinematics

Quiz 2 Review

1. The Baryon, Lepton, and Electric Charge are conserved in the Standard Model.
2. The Baryon and Lepton number ensure the stability of the proton.
3. In Neutron Decay $n \to p + e^- + \bar{\nu}_e$, the weak force is responsible for the decay.

	Strong	EM	Weak	Gravity
Strength	1	10^{-2}	10^{-7}	10^{-40}
Time scale	10^{-23} sec	10^{-16}	10^{-10}	$>$ yr

The decay rate is proportional to the coupling strength of the force $\Gamma \propto \alpha^2$. For the time scale is τ it is inversely proportional:

$$\tau \propto \frac{1}{\Gamma}$$

4. The strong force is responsible for holding the nucleus together.
5.

Experimental Discoveries To discover and observe particles, there are typically three ways:

1. Scattering (cross section)
2. Decay (decay rate or lifetime)
3. Bound states (binding energy/mass)

Relativistic Kinematics 4-vectors

$$x^\mu = (ct, x, y, z) \quad \text{space-time}$$
$$p^\mu = (E/c, p_x, p_y, p_z) \quad \text{momentum}$$

where x^μ and p^μ are the space-time (position) four-vector and energy-momentum four-vector.

NOTE: Totak four-momentum is conserved in all interactions.
Starting with the lorentz invariant

$$p^\mu p_\mu = p^2$$

using the Einstein-summation convention

$$p^\mu p_\mu = \sum_{\mu=0}^{3} p^\mu p_\mu = p^2$$

and the metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we can write the lower momentum vector as

$$p_\mu = p^\nu g_{\mu\nu}$$

thus

$$\begin{aligned} p^\mu p_\mu &= p^\mu p^\nu g_{\mu\nu} \\ &= \left(\frac{E}{c}\right)^2 + \mathbf{p}\cdot\mathbf{p}(-1) \\ &= \left(\frac{E}{c}\right)^2 - |\mathbf{p}|^2 \\ &= m^2c^2 \end{aligned}$$

Using

$$E = \sqrt{|\mathbf{p}|^2 + m^2c^4} \tag{2.1}$$

Lorentz Transformation At rest $\mathbf{p} = 0$ and $E = mc^2$.
In the Galilean transformation in the x direction:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

where we assume absolute time, but in the Lorentz transformation:

$$\begin{aligned} x' &= \gamma(\beta ct + x) \quad \beta = \frac{v}{c} \\ ct' &= \gamma(t - \beta x) \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \\ y' &= y \\ z' &= z \end{aligned}$$

In matrix form:

$$\Lambda = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

and thus $p^\mu p_\mu$ is invariant under Lorentz transformation.

Massless particle: From the energy momentum relation

$$E^2 = |\mathbf{p}|^2 c^2 + m^2c^4$$

The massless particle has energy $E = |\mathbf{p}|c$. But we have to include the frequency (Planck) relation from quantum mechanics as well:

$$E = h\nu = \hbar\omega$$

And in the SM photons and neutrinos are massless thus

$$p^2 = p^\mu p_\mu = m^2c^2 = 0$$

Collisions Non-relativistic vs. Relativistic
Non-relativistic:

- Elastic (KE conserved)
- Inelastic (KE not conserved)

Relativistic:

- Elastic (KE conserved) e.g. particle splitting into two
- Inelastic (KE not conserved) or Rest energy and mass e.g. colliding two particles to form a new particle
  - KE increases (Explosive)
  - KE decreases (Sticky)

In the extreme case:

$$\begin{aligned} A + B \to C &\quad \text{inverse decay} \\ A \to B + C &\quad \text{decay} \end{aligned}$$

Example $\pi^+ \to \mu^+ + \nu_\mu$ (decay)
The Rest energies are $m_{\pi^+} = 135\,\mathrm{MeV/c^2}$, $m_{\mu^+} = 105\,\mathrm{MeV/c^2}$, and $m_{\nu_\mu} = 0$. But this energy is lost through the kinetic energy of the muon and muon-neutrino.
The momentum before is just the momentum of the pion

$$p_i = p_\pi = 0$$

since it is startionary. Afterward the momentum is split between the muon and neutrino

$$p_f = p_\mu + p_{\nu_\mu}$$

where energy and momentum is conserved:

$$\begin{aligned} \mathbf{p}_\mu &= -\mathbf{p}_\nu \\ m_\pi c^2 &= E_\mu + E_{\nu_\mu} \end{aligned}$$

4-momentum conservation

$$\begin{aligned} p_{before} &= p_{after} \\ p_\pi &= p_\mu + p_{\nu_\mu} \end{aligned}$$

since the massless particle has no momentum from the energy momentum relation

$$\begin{aligned} p_\nu &= p_\pi - p_\mu \\ p_\nu^2 &= (p_\pi - p_\mu)^2 \\ &= p_\pi^2 - 2p_\pi p_\mu + p_\mu^2 \\ 0 &= m_\pi^2 c^2 + m_\mu^2 c^2 - 2\frac{m_\pi c^2}{c}\frac{E_\mu}{c} \\ 2E_\mu m_\pi &= (m_\pi^2 + m_\mu^2)c^2 \\ E_\mu &= \frac{m_\pi^2 + m_\mu^2 - m_\nu^2}{2m_\pi}c^2 \end{aligned}$$

Another way is finding

$$p_\pi = p_\mu + p_\nu$$

rewritten as

$$p_\mu = p_\pi - p_\nu$$

squaring both sides gives

$$p_\mu^2 = p_\pi^2 - 2p_\pi p_\nu + p_\nu^2$$

and since $p_\nu^2 = 0$ we have

$$p_\mu^2 = p_\pi^2 - 2p_\pi p_\nu$$

which implies

$$m_\mu^2 c^2 = m_\pi^2 c^2 - 2m_\pi E_\nu$$

the Planck relation tells us

$$E_\nu = |\mathbf{p}_\nu| c = |\mathbf{p}_\mu| c$$

thus

$$2m_\pi |\mathbf{p}_\mu| c = (m_\pi^2 - m_\mu^2)c^2$$

and

$$|\mathbf{p}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c$$

Scattering experiments

- Head-on collision: (LHC)
- Fixed target collision: Beam of protons hitting a target (e.g. Carbon) (SLAC)

From momentum conservation, the head-on collision is more energy efficient as it loses the minimum amount of energy. The created particle is at rest, thus the energy is the rest energy. But the Fixed target collision has a higher energy loss since the particle loses energy since the created particle has kinetic energy.
e.g. The Anti-proton Discovery is due to the Bevatron colliding two protons to create an anti-proton

$$p + p \to p + p + p + \bar{p}$$

HW HINT: $E_{cm} < E_{fixed}$

3 Symmetries

Quiz review:

3. The Energy of the large mass is

$$Mc^2 = E_1 + E_2 = 2\gamma mc^2$$

where the energy of the smaller masses are

$$E = \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}$$

where $|\mathbf{p}| = \gamma mv$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Thus the mass $M > 2m$.

4. Using the same thought from 3. we know that the rest mass of M is greater.

Lorentz Invariant

$$p^2 = m^2 c^2$$

From Wikipedia: this is the lightlike vector. For the timelike $p^2 > 0$ and spacelike $p^2 < 0$.

Symmetries

Equilateral triangles are symmetric under 3 axes where we can flip the triangle and it is still the same. For the square, we have 4 axes, and so and so forth. All of these objects are studied in Group Theory.

Group Theory Group is a set of objects satisfying certain properies under an operation.

Properties

1. Closure: For $a, b \in G, \quad a \cdot b \in G$
2. Identity: For any $a \in G, \quad a \cdot I = I \cdot a = a$
3. Inverse: For each $a \in G, \quad a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. Associativity: For $a, b, c \in G, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (optional) Commutativity: For $a, b \in G, \quad a \cdot b = b \cdot a$ AKA Abelian Group. Not all groups are commutatiive and thus are called non-Abelian groups.

Two Types of Groups

1. Finite: Finite number of elements. e.g. $Z_2 = \{1, -1\} = \{I, r\}$ where $r^2 = I$
2. Infinite: Descrete or continuous. e.g. set of integers under addition (discrete), set of real numbers under multiplication (continuous), $U(1)$ (continuous)

Examples For an isoscale triangle $Z_2 = \{1, -1\}$ and for an equilateral triangle $Z_3 = \{0, 1, 2\}$ or the operation mod 3. Which is isomorphic to

$$\equiv \left\{1, \omega, \omega^2\right\}, \quad \omega = e^{2\pi i/3}$$

For the square

$$Z_4 = \{0, 1, 2, 3, \} \equiv \{1, i, -1, -i\} \quad \text{or} \quad \left\{1, \omega, \omega^2, \omega^3\right\}$$

Thus for n elements.

$$Z_n = \left\{e^{i2\pi j/n}\right\}, \quad j = 0, 1, \ldots, n-1$$

where all of these groups are Abelian.

For $n \to \infty$ We get a circle as it has an infinite number of symmetries.
In addition $j \to \infty$

$$\frac{2\pi j}{n} = \theta$$

we get

$$U = e^{i\theta} = \cos\theta + i\sin\theta$$

where $\theta \in [0, 2\pi]$, and we have the $U(1)$ group.

$$U^\dagger U = I \qquad U^\dagger = (U^*)^T$$

where the dagger is the transpose of the complex conjugate (conjugate transpose).

Standard Model

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_y$$

$U(N)$ set of unitary $N \times N$ matrices (non-Abelian in general except for $N > 1$). Taking the determinant of the matrix

$$\det\left(U^\dagger U\right) = \det I = 1$$

and

$$\det\left(U^\dagger\right)\det(U) = 1 \qquad \det\left(U^{*T}\right) = \det(U^*) = (\det U)$$

and

$$|\det U|^2 = 1$$
$$\det U = e^{i\alpha} \quad \alpha \in [0, 2\pi]$$

Choosing the phase angle $\alpha = 0$ we get

$$\det U = 1 \qquad SU(N)CU(N)$$

$\otimes$ is a direct product: Two groups F and G. For $f \in F$ and $g \in G$ we have

$$(f, g) \in F \otimes G$$

The $U(1)$ group is related to the photon γ, the $SU(2)$ group is related to the weak force $W^\pm, Z^0$, and the $SU(3)$ group is related to the strong force g (gluon).

SU(2) A set of 2×2 matrices with a determinant of 1.
Given the theorem

$$U = e^{iH}$$

for the hermitian matrix H where

$$U^\dagger U = 1 \to e^{-iH^\dagger} e^{iH} = 1$$

thus

$$H^\dagger = H$$

we take the determinant of U:

$$\det U = \det\left(e^{iH}\right) = e^{i \operatorname{Tr} H} = 1 = e^0$$

thus $\operatorname{Tr} H = 0$. This means that the Hermitian H is traceless.

Pauli Matrices

traceless matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus we can write the Hermitian matrix as

$$H = \frac{1}{2} \sum_i \theta_i \sigma_i = \frac{1}{2} \theta \cdot \sigma$$

where we have the group element of $SU(2)$

$$U = e^{i\theta \cdot \sigma/2}$$

From QM

$$\mathbf{S} = \frac{\hbar}{2} \sigma$$

$$\begin{aligned}
[S_y, S_z] &= iS_x \\
[S_z, S_x] &= iS_y \\
[S_x, S_y] &= iS_z [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k
\end{aligned}$$

where ϵ_{ijk} is the Levi-Civita symbol.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ interchange any two indices } (3,2,1) \\ 0 & \text{otherwise any index is repeated} \end{cases}$$

thus

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

The Lie Algebra for $SU(2)$ is $SO(3)$ where both groups are isomorphic.

$$[L_i, L_j] = i\epsilon_{ijk} L_k \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

the generators of $SU(2)$ is $\sigma/2$. For $SU(3)$

$$U = e^{i\theta \cdot \lambda/2}$$

where we have the Gell-Mann matrices λ. In general for $SU(N)$

Addition of Angular Momenta

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and

$$[J^2, J_i] = 0 \qquad J^2 = J_x^2 + J_y^2 + J_z^2 T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_o} \frac{1}{\sqrt{\cos\theta - \cos\theta_o}} \, \mathrm{d}\theta$$

where J^2 is the Casimir operator. Since we have simultaneous eigenstates of J^2 and J_z we can write

$$|j, j_z\rangle$$

Symmetries Part 2: Spin & Isospin

Quiz 3 Review SU(2) is the group of 2x2 unitary matrices with determinant 1. Using the basisc vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ we can write the group element as

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

or the linear combination of the basis vectors. Thus the transformation is

$$\begin{pmatrix}a'\\b'\end{pmatrix} = U(\theta)\begin{pmatrix}a\\b\end{pmatrix} = e^{i\theta\cdot\sigma/2}\begin{pmatrix}a\\b\end{pmatrix}$$

THe Lie Algebra for SU(2) is

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

and

$$[J^2, J_i] = 0$$

for simultaneous eigenstates of $|j,m\rangle$.

$$J_z|j,m\rangle = m\hbar|j,m\rangle \qquad J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$$

from the ladder operators

$$J_\pm = J_x \pm iJ_y$$

where these are not Hermitian (does not commute). Thus

$$\begin{aligned} J^2 &= J_x^2 + J_y^2 + J_z^2 \\ &= J_+J_- + J_+J_-J_z^2 \end{aligned}$$

furthermore

$$J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m)}\,|j,m\pm 1\rangle$$

where going up the ladder $m \to m+1$ and going down the ladder $m \to m-1$. For fixed j there is a maximum and minimum m value

$$m_{max} = j \qquad m_{min} = -j$$

so for example

$$J_+|j,j\rangle = 0 \qquad J_-|j,-j\rangle = 0$$

Spin

$$j \equiv s = 1/2, \qquad m \equiv m_s = \pm 1/2$$

The basis states are

$$\begin{aligned} (1/2, 1/2) &= \begin{pmatrix}1\\0\end{pmatrix} = |\uparrow\rangle \qquad m_s = 1/2 \\ (1/2, -1/2) &= \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle\, m_s = -1/2 \end{aligned}$$

For the addition of spin

$$\frac{1}{2}\otimes\frac{1}{2} = ? \qquad \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \qquad S_{tot} = (S_1+S_2), ..., (S_1-S_2) = 1, 0 \qquad m_{s,tot} = 1, 0, -1, 0$$

General Addition of Angular Momentum

$$|1,1\rangle = |\uparrow\uparrow\rangle \qquad |1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle \qquad |1,-1\rangle = |\downarrow\downarrow\rangle$$

finding the linear combination through basis transformation by using the resolution of the identity

$$\begin{aligned} |j,m\rangle &\to |j_1,m_1\rangle \otimes |j_2,m_2\rangle \\ &= \sum_{m_1,m_2} |j_1,m_1,j_2,m_2\rangle \langle j_1,m_1,j_2,m_2|j,m\rangle \end{aligned}$$

where the bra-ket is the Clebsch-Gordan coefficient. thus

$$= \sum_{m_1,m_2} c^{j,j_1,j_2}_{m,m1,m2} |j_1,m_1,j_2,m_2\rangle$$

where $m = m_1 + m_2$ and $c^{j,j_1,j_2}_{m,m1,m2}$ is the Clebsch-Gordan coefficient.

Example For the $S = 1$ state $m = 1$

$$\begin{aligned} |1,1,\rangle &= |1/2,1/2\rangle \otimes |1/2,1/2\rangle \\ &= |1/2,1/2,1/2,1/2\rangle \\ &= |\uparrow\uparrow\rangle \end{aligned}$$

For $m = 0$ we have a linear combination of the basis states

$$\begin{aligned} J_- |1,1\rangle &= \hbar\sqrt{2}|1,0\rangle \\ \text{or} \quad |1,0\rangle &= \frac{1}{\hbar\sqrt{2}} J_- |1,1\rangle \end{aligned}$$

the sum of the basis states is

$$\begin{aligned} J_-(|1/2,1/2\rangle \otimes |1/2,1/2\rangle) &= \hbar\sqrt{(1/2+1/2)(1/2-1/2+1)}\,|1/2,-1/2\rangle \otimes |1/2,1/2\rangle \\ &+ |1/2,1/2\rangle \otimes |1/2,-1/2\rangle \\ &= \hbar(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \end{aligned}$$

or

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

for $m = -1$ we have

$$J_- |1,0\rangle = \hbar\sqrt{2}|1,-1\rangle$$

where

$$|1,-1\rangle = |1/2,-1/2\rangle \otimes |1/2,-1/2\rangle \qquad = |\downarrow\downarrow\rangle$$

Now for $S = 0$, $m = 0$ we have

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

since it is the way to make it orthogonal to $|1,0\rangle$. Therefore

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

Thus the there are 3 triplet states $m_s = 1,0,-1$ and 1 singlet state $m_s = 0$.

Isospin

$$m_p = 938.3 \text{ MeV/c}^2 \qquad m_n = 939.6 \text{ MeV/c}^2$$

why are they so close? Heisenberg postulated an isospin state of a nucleon N as

$$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha |p\rangle + \beta |n\rangle$$

with

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the isospin state of the proton and neutron are

$$|p\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \qquad |n\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

1. Strong interactions preserve isospin symmetry
2. EM & Weak interactions do not preserve isospin symmetry

Examples

Pions: π^+, π^0, π^- where the approximate symmetry is a triplet state

$$\begin{aligned} \pi^+ &= |1,1\rangle \qquad I = 1, \quad I_3 = 1 \\ \pi^- &= |1,0\rangle \qquad I = 1, \quad I_3 = 0 \\ \pi^0 &= |1,-1\rangle \qquad I = 1, \quad I_3 = -1 \end{aligned}$$

Δ-baryons:

$$\begin{aligned} \Delta^{++} &= |3/2, 3/2\rangle \qquad I = 3/2, \quad I_3 = 3/2 \\ \Delta^{+} &= |3/2, 1/2\rangle \qquad I = 3/2, \quad I_3 = 1/2 \\ \Delta^{0} &= |3/2, -1/2\rangle \qquad I = 3/2, \quad I_3 = -1/2 \\ \Delta^{-} &= |3/2, -3/2\rangle \qquad I = 3/2, \quad I_3 = -3/2 \end{aligned}$$

where Δ^{--} is an antiparticle of Δ^{++}. We write from the highest to lowest from the empirical Gellman-Nishijima formula

$$Q = I_3 + \frac{1}{2}(B + S)$$

where Q is the charge, I_3 is the third component of isospin, B is the baryon number, and S is the strangeness.

Pions

Since a Pion is a *meson* and not a baryon, it has a baryon number of 0. Thus with no strangeness

$$S = 0 \qquad B = 0$$

Nucleons

$$S = 0 \qquad B = 1$$

$$Q = \begin{cases} 1/2 + 1/2(1+0) = 1 & \text{proton} \\ -1/2 + 1/2(1+0) = 0 & \text{neutron} \end{cases}$$

For all elementary particles there is a general formula

$$Q = I_3 + \frac{Y}{2}$$

where Y is the hyper charge $U(1)_Y$.

Power of Symmetry: Applications

1. Deuteron (neutron of deuterium): Two-Nucleon system

$$I = 1 \quad \text{or} \quad 0 \qquad I_3 = 1, 0, -1 \quad \text{or} \quad 0 \text{ (singlet)}$$

$$\begin{aligned} |1,1\rangle &= |p,p\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle + |n,p\rangle) \\ |1,-1\rangle &= |n,n\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle - |n,p\rangle) \end{aligned}$$

experimentally, we only see the singlet state because we see only one deuteron state. Thus we can only see a isospin state of $I = 0$.

Two-nucleon potential $\propto \mathbf{I}_1 \cdot \mathbf{I}_2$ where we hae the total isospin

$$\mathbf{I}^2 = (\mathbf{I}_1 + \mathbf{I}_2)^2 = \mathbf{I}_1^2 + \mathbf{I}_2^2 + 2\mathbf{I}_1 \cdot \mathbf{I}_2$$

where the s^2 term is

$$s^2 = 1/2(1/2+1)\hbar^2 = \frac{3}{4}\hbar^2$$

Thus

$$\mathbf{I}_1^2 + \mathbf{I}_2^2 = \frac{3}{2}$$

and

$$\begin{aligned} \mathbf{I}_1 \cdot \mathbf{I}_2 &= \frac{1}{2}\left(\mathbf{I}^2 - 3/2\right)^{3/2} \\ &= \begin{cases} 1/2(1(1+1) - 3/2) & = 1/4 \quad \text{triplet} \\ 1/2(0(0+1) - 3/2) & = -3/4 \quad \text{singlet} \end{cases} \end{aligned}$$

Symmetries Part 3: Scattering

Quiz 5 Review For j

$$\frac{1}{2}\otimes\frac{1}{2}=1\oplus 0$$

For $2j+1$

$$2\otimes 2=3\oplus 1$$

Isospins of particles

1. pion: 1
2. deuteron: 0
3. Δ-baryons: 3/2
4. nucleons: 1/2

The strong ineteraction preserves I and I_3, and the weak interactions do not preserve I and I_3 (e.g. in beta decay the iso spin of the neutron (-1/2) go to an iso spin of the proton (1/2)). In E&M the isospin preserves only I and not I_3 (e.g. π_o decay to two photons $\gamma\gamma$: $I_3=0$ for the π_o and $I_3=0$ for the two photons).

Applications of Isospin: Nucleon-nucleon Scattering

$$p+p\to D+\pi^+$$
$$p+n\to D+\pi^0$$
$$n+n\to D+\pi^-$$

The relative probabilities of these processes: we get this from the amplitude A where the probability $|A|^2$ is proportional to the cross section $\sigma=\pi r^2$ (the cross section of a sphere, but this is not a solid sphere and rather a 'fuzzy' sphere). With the fact that 'strong interactions preserve isospin' we have the the ratio of the cross sections

$$\sigma_a:\sigma_b:\sigma_c$$

For all three processes the RHS the isospin is

$$I_{tot}=0\otimes 1=1$$

on the left hand side

$$I_{tot}=\frac{1}{2}\otimes\frac{1}{2}=0\quad\text{or}\quad 1$$

(a) The ratio of getting an isospin of 1 on the left hand side for the first process

$$|pp\rangle=|11\rangle$$

(c) for the third proccess

$$|nn\rangle=|1,-1\rangle$$

(b) The second is the linear combination of $|10\rangle$ and $|00\rangle$

$$|pn\rangle=\frac{1}{\sqrt{2}}(|10\rangle+|00\rangle)$$

the $|00\rangle$ does not contribute to the isospin of 1. Thus the ratio of the probability is

$$A_a:A_b:A_c=1:\frac{1}{\sqrt{2}}:1$$

and the ratio of the cross sections is

$$\sigma_a:\sigma_b:\sigma_c=1:\frac{1}{2}:1$$

Example 3 Pion-nucleon Scattering

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \quad I = 1, \qquad \begin{pmatrix} p \\ n \end{pmatrix} \quad I = 1/2$$

So the total isospin j is

$$I_{tot} = 1/2 \otimes 1 = 3/2 \oplus 1/2$$

and for $2j + 1$

$$3 \otimes 2 = 4 \oplus 2$$

The elastic processes are (from $a \to f$)

$$\begin{gathered} \pi^+ + p \to \pi^+ + p \\ \pi^0 + p \to \pi^0 + p \\ \pi^- + p \to \pi^- + p \\ \pi^+ + n \to \pi^+ + n \\ \pi^0 + n \to \pi^0 + n \\ \pi^- + n \to \pi^- + n \end{gathered}$$

and the charge-exchange processes are (from $g \to j$)

$$\begin{gathered} \pi^+ + n \to \pi^0 + p \\ \pi^0 + p \to \pi^+ + n \\ \pi^- + p \to \pi^0 + n \\ \pi^0 + n \to \pi^- + p \end{gathered}$$

The states of the 3/2 isospin are

$$|3/2, 3/2\rangle, |3/2, 1/2\rangle, |3/2, -1/2\rangle, |3/2, -3/2\rangle$$

and the states of the 1/2 isospin are

$$|1/2, 1/2\rangle, |1/2, -1/2\rangle$$

so we have the following states

$$\begin{aligned} \left|\pi^+ p\right\rangle &= |1, 1\rangle \otimes |1/2, 1/2\rangle = |3/2, 3/2\rangle \\ \left|\pi^- n\right\rangle &= |1, -1\rangle \otimes |1/2, -1/2\rangle = |3/2, -3/2\rangle \end{aligned}$$

for the obvious highest and lowest isospin states. Carrying on...

$$\left|\pi^+ n\right\rangle = |1, 1\rangle \otimes |1/2, -1/2\rangle$$

this is the linear combination of $|3/2, 1/2\rangle$ and $|1/2, 1/2\rangle$, and so on. To find the proportional cross sections we know that

$$\langle i|f\rangle \propto A \qquad |\langle i|f\rangle|^2 \propto \sigma$$

We can use the Clebsch-Gordan coefficients to find the linear combination of the states. For example

$$\left|\pi^+ n\right\rangle = |3/2, 1/2\rangle + |1/2, 1/2\rangle$$

where the Clebsch-Gordan coefficient is

$$\langle 3/2, 1/2, 1/2, 1/2 | 3/2, 1/2 \rangle = \sqrt{\frac{2}{3}}$$

e.g. for the $\pi^+ p$ state

$$\left|\pi^+ p\right\rangle = |3/2, 3/2\rangle$$
$$\langle 3/2, 3/2, 1/2, 1/2 | 3/2, 3/2 \rangle = 1$$

using the lowering operator

$$J_- |j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle$$

so

$$J_- |3/2, 3/2\rangle = \hbar\sqrt{3}\, |3/2, 1/2\rangle$$

applying the lower operator to $J_{1-} + J_{2-}$ we get

$$\begin{aligned} J_-(|11\rangle \otimes |1/2, 1/2\rangle) &= \hbar\sqrt{2}\, |10\rangle \otimes |1/2, 1/2\rangle + \hbar\sqrt{1}\, |11\rangle \otimes |1/2, -1/2\rangle \\ &= \hbar\sqrt{2}\, |10\rangle \otimes |1/2, 1/2\rangle + \hbar\, |11\rangle \otimes |1/2, -1/2\rangle \end{aligned}$$

we then get

$$\begin{aligned} |3/2, 1/2\rangle &= \sqrt{2/3}\, |11\rangle \otimes |1/2, 1/2\rangle + \sqrt{1/3}\, |10\rangle \otimes |1/2, 1/2\rangle \\ &= \sqrt{2/3}\, \left|\pi^+ p\right\rangle + \sqrt{1/3}\, \left|\pi^+ n\right\rangle \end{aligned}$$

and the orthogonal state is

$$|1/2, 1/2\rangle = \sqrt{2/3}\, \left|\pi^+ p\right\rangle - \sqrt{1/3}\, \left|\pi^+ n\right\rangle$$

and so on for the other states. At the end we will find that the ratio of the total cross sections (adding up the matching elastic and exchange processes) is 3.
The amplitude has a factor

$$\left\langle \pi^+ p \middle| \pi^+ p \right\rangle = \langle 3/2, 3/2 | 3/2, 3/2 \rangle = M_3$$

where for example

$$\begin{aligned} (\sqrt{2}3\, \langle 3/2, 1/2| - 1/\sqrt{3}\, \langle 1/2, 1/2|)(\sqrt{2}3\, |3/2, 1/2\rangle - 1/\sqrt{3}\, |1/2, 1/2\rangle) = \\ = 2/3\, \langle 3/2, 1/2 | 3/2, 1/2 \rangle - 1/3\, \langle 1/2, 1/2 | 1/2, 1/2 \rangle \\ = 2/3 M_3 - 1/3 M_1 \end{aligned}$$

for $M_3 >> M_1$ the ratio is 4/9, and for $M_3 << M_1$ the ratio is 1/3.

$SU(3)$

$$\begin{pmatrix} p \\ n \end{pmatrix} \quad SU(2)\text{doublet}$$

where the spins are

$$p : uud \quad Q_u = 2/3$$
$$n : udd \quad Q_d = -1/3$$

For the two spins

$$\begin{pmatrix} u \\ d \end{pmatrix}$$

the isospins are

$$I = 1/2, \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

for the up and down quarks respectively. In reality we have six quarks

- Light quarks: u, d, s
- Heavy quarks: c, b, t

For the light quaks we have a $SU(3)$ symmetry

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

the masses are all different:

$$m_u \approx 2\,\mathrm{MeV/c^2} \qquad m_d \approx 4\,\mathrm{MeV/c^2} \qquad m_s \approx 95\,\mathrm{MeV/c^2}$$

so we have to add a flavor symmetry to the $SU(2)$ isospin symmetry:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} \oplus s$$

or the $SU(3)$ symmetry

$$SU(3)_f \to SU(2)_I \otimes U(1)_y$$

From $SU(2)$ algebra:

$$[J_i, J_j] = i\epsilon_{ijk} I_k \qquad J_i = \sigma_i/2$$

For the three pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now for $SU(3)$: We know that the generators

$$U = e^{i\theta\cdot\lambda/2}$$

For $SU(N)$ we have $N^2 - 1$ generators. For $SU(3)$ we have 8 generators. The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Symmetries Part 4: Young's Tableaux & Eightfold Way

Quiz 6 Review

(1) With respect to the QCD scale (≈ 200 MeV) the masses of the quarks are divided into light and heavy quarks.

(2) (2) The effective mass mass is much larger than the mass of the light quark.

(3) $SU(2)$ is a subgroup of $SU(3)$.

For the $SU(3)$: a 3x3 unitary matrix with determinant 1. There are $n^2 - 1 = 8$ generators where

$$U = e^{iH}$$

where H is the Hermitian matrix:

$$\begin{aligned} U^\dagger U &= 1 \qquad H^\dagger = H \\ \det U &= 1 \qquad \operatorname{tr} H = 0 \\ \det M = e^{\operatorname{tr} \ln M} \end{aligned}$$

or Hermitian matrices are traceless.

Gell-Mann Matrices Starting with the Pauli matrices but in 3x3 form

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and moving the sectors of the matrices we also get

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \cancel{\lambda_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

and

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

but λ_9 is not linearly independent since it can be written as a linear combination of $\lambda_3 + \lambda_8$.

Commutation Relation For $SU(2)$ we know

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and for $SU(3)$ we have

$$[J_i, J_j] = i f_{ijk} J_k$$

where f_{ijk} are the structure constants.

Subgroup We know that $SU(2) \leq SU(3)$ (where $\leq$ means 'is a subgroup of'). So $\{\lambda_1, \lambda_2, \lambda_3\}$ forms an $SU(2)$ sub-algebra. Also

$$\begin{gathered} \{\lambda_4, \lambda_5, a\lambda_3 + b\lambda_8\} \\ \{\lambda_6, \lambda_7, a\lambda_3 + b\lambda_8\} \end{gathered}$$

are also $SU(2)$ sub-algebras. NOTE that

$$SU(3) \neq SU(2) \otimes SU(2) \otimes SU(2)$$

Isospin and Strangeness

$$\lambda_3/2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I = 1/2$$

and the isospins are

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

and the strangeness is

$$S : I = 0$$

For λ_8 we define a hypercharge y such that

$$\lambda_8/2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}} \frac{2}{\sqrt{3}}$$

and

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad y = 1/3$$

and for the strangeness $S : y = -2/3$. This is because the Gell-Mann-Nishijima formula

$$Q = I_3 + \frac{y}{2}$$
$$2/3 = 1/2 + 1/2(1/3) \qquad -1/3 = 0 - 1/2(2/3)$$

For the triplet

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

and the anti-triplet

$$\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

Mesons $(q, \bar{q})$ in $SU(3)$ is

$$3 \otimes \bar{3} = 8 \oplus 1$$

where the 8 is the octet and the 1 is the singlet. We can do this using the Young Tableaux. For $SU(3)$ we have a 3 fundamental and $\bar{3}$ anti-fundamental.

N	$N+1$
$N-1$	N
$N-2$	

Using Hook Law:

$$\dim = \frac{\Pi_i N_i}{\Pi_i h_i}$$

$$\frac{N(N+1)(N-1)N(N-2)}{1 \cdot 3 \cdot 4 \cdot 1 \cdot 2}$$

so

$$3 \otimes \bar{3} = \boxed{3} \otimes \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} =$$

or

$$\boxed{a} \otimes \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} = \frac{3 \cdot 4 \cdot 2}{1 \cdot 3 \cdot 1} \oplus \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}$$

Goin from $SU(3)$ to $SU(2)$ we have

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} + s$$

or

$$3 \to 2_1 + 1_{-2}$$

where the hypercharges are subscripts. So the octet is

$$\begin{aligned} 3 \otimes \bar{3} &= (2_1 + 1_{-2}) \otimes (2_{-1} + 1_2) \\ &= (2_1 \otimes 2_{-1}) \oplus (2_1 \otimes 1_2) \oplus (1_{-2} \otimes 2_{-1}) \oplus (1_{-2} \otimes 1_2) \\ &= (3_0 \oplus 1_0) \oplus 2_3 \oplus 2_{-3} \oplus 1_0 \\ &= 8 \oplus 1 \end{aligned}$$

This is called the eightfold way.

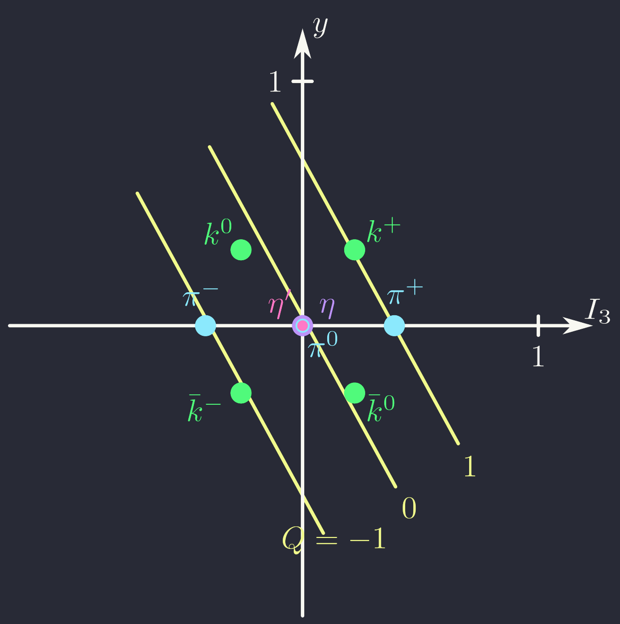

Figure 3.1: The eightfold way

Eightfold way Where η is a $SU(2)$ singlet but a $SU(3)$ octet. and η' is a $SU(3)$ singlet.
If $SU(3)_f$ was a good symmetry, expet all these 8 mesons to have similar mass. All of these obey up to a factor of 2.

Baryons (222) or $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$. The baryons are antisymmetric as each quark is a fermion. Using the Young Tablauex

$$3 \otimes 3 = \boxed{a} \otimes \boxed{b} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array} = \frac{3 \cdot 4}{1 \cdot 3} \oplus \frac{3 \cdot 2}{1 \cdot 2} = 6$$

so

$$6 \otimes 3 = \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \otimes \boxed{c} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \frac{3 \cdot 4 \cdot 5}{3 \cdot 2 \cdot 1} \oplus \frac{3 \cdot 4 \cdot 2}{1 \cdot 3 \cdot 1} = 10 \oplus 8$$

This is a 10-plet as shown in the figure.

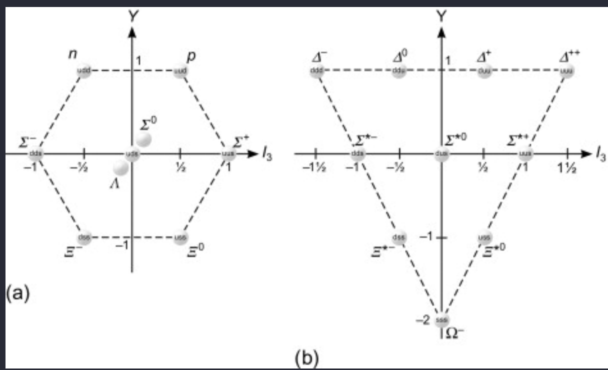

Figure 3.2: The 10-plet of baryons

Symmetries Part 5: Parity

Parity (Discrete Symmetry) For a simple reflection on a z axis the point $A = (x, y, z)$ goes to $A' = (x, -y, z)$. But for parity operation we go to $P(A) = (-x, -y, -z)$ or in the general form

$$P(\mathbf{a}) = -\mathbf{a}$$

also known as inversion. Taking the parity again we get

$$P^2(\mathbf{a}) = P(-\mathbf{a}) = \mathbf{a}$$

Thus it has a discrete Z_2 symmetry.

Pseudo-vector (axial vector) for a pseudo vector $\mathbf{c}$

$$P(\mathbf{c}) = \mathbf{c}$$

where cross products (of two vectors) are pseudo-vectors. For example

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

and

$$P(\mathbf{c}) = P(\mathbf{a} \times \mathbf{b}) = (-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{c}$$

e.g. of pseudo-vectors:

- Torque: $\tau = \mathbf{r} \times \mathbf{F}$
- Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- Magnetic field: $\mathbf{B} = \mathbf{E} \times \mathbf{v}$

But for the lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

the cross product of a vector and a pseudo-vector is a vector, so the lorentz force is a vector. Also from the general definition

$$P(\mathbf{F}) = \frac{q}{c}P(\mathbf{v}) \times P(\mathbf{B}) = \frac{q}{c}(-\mathbf{v}) \times (-\mathbf{B}) = -\mathbf{F}$$

The weak interaction violates parity...

Scalar For a scalar $s = \mathbf{a} \cdot \mathbf{b}$ is invariant under parity:

$$P(s) = P(\mathbf{a} \cdot \mathbf{b}) = (-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = s$$

for a pseudo-scalar p (a dot product of a vector and pseudo-vector):

$$P(p) = P(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})) = P(\mathbf{a})P(\mathbf{b} \times \mathbf{c}) = -\mathbf{a}(\mathbf{b} \times \mathbf{c}) = -p$$

So the partities of the four types of quantities:

- Scalar: $P(s) = s$
- Pseudo-scalar: $P(p) = -p$
- Vector: $P(\mathbf{v}) = -\mathbf{v}$
- Pseudo-vector: $P(\mathbf{c}) = \mathbf{c}$

Intrinsic Parity The parity of a fermion is

$$P(\text{fermion}) = -P(\text{anti-fermion})$$

for bosons

$$P(\text{boson}) = P(\text{anti-boson})$$

For composite particles i.e. mesons $q\bar{q}$ and baryons qqq:

$$P(\text{meson}) = -1 \quad \text{or} \quad (+1)(-1) = -1$$

Since mesons are two pairs (particle, antiparticle) the parity is always negative. For baryons we can only have a positive parity:

$$P(\text{baryon}) = (+1)^3 = +1$$

For spherical harmonics $Y_l^m(\theta, \phi)$ under parity of each term:

$$\mathbf{r} \to -\mathbf{r} \qquad \theta \to -\theta \qquad \phi \to \pi + \phi$$

so

$$P(Y_l^m(\theta, \phi)) = (-1)^l Y_l^m(\theta, \phi)$$

and for excited states

$$P = (-1)^l \times P(\text{ground state})$$

where l is the orbital angular momentum.

Parity Violation THe $\theta - \tau$ puzzle: given two particles

$$\begin{aligned} \theta^+ &\to \pi^+ + \pi^0 & P = +1 \\ \tau^+ &\to \pi^+ + \pi^0 + \pi^0 & P = -1 \end{aligned}$$

the same two particles are found to be the same particle K^+ having the same mass and lifetime, but this violates the parity. To solve this puzzle came from the Colmbia University known as Wu's experiment (wikipedia):

$$^{60}_{27}\text{Co} \to ^{60}_{28}\text{Ni} + e^- + \bar{\nu}_e$$

The Cobalt has a spin state of $J = 5$, the Nickel has spin $J = 4$ and the spin states of the electron-antielectron pair is $J = 1$. The electron is always emitted in the direction opposite of the Cobalt spin, and when the magnetic field was inverted, the electrons were emitted in the opposite direction of nuclear spin. This breaks parity, because in the mirror world, the spin of the electron would be in the same direction as the nuclear spin.

Helicity From spin $\mathbf{s}$ and momentum $\mathbf{p}$ we can define the helicity

$$\begin{aligned} \lambda &= \frac{\mathbf{s} \cdot \mathbf{p}}{|\mathbf{s}||\mathbf{p}|} \\ &= \begin{cases} +1 & \text{right-handed} \\ -1 & \text{left-handed} \end{cases} \end{aligned}$$

But this depends on the reference frame. e.g. for a case where $\mathbf{p}$ is faster and in the same direction of $\mathbf{s}$ the helicity is $\lambda = +1$, but in the reference frame faster that $\mathbf{p}$ the helicity is $\lambda = -1$.

Massless Particles For massless particles the helicity is the same in all reference frames because the speed is always c and thus the helicity is well-defined. For an electron, we can get a frame where the momentum is different for changes in reference frames.

Back to Wu's experiment The spin of the electron and neutrino are in the same direction as the spin of the Cobalt and Ni.

Note: In the SM

- neutrinos are always left-handed; $\lambda = -1$
- anti-neutrinos are always right-handed; $\lambda = +1$

Thus the electron momentum is in the opposite direction of spin as shown in Figure 3.3

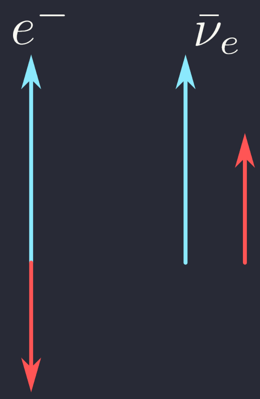

Figure 3.3: Wu's experiment

Another Example Pions and muons

$$\pi^+ \to \mu^+ + \nu_\mu \quad \text{or} \quad \to e^+ + \nu_e$$

From the comparison of masses

$$m_\pi = 140\,\text{MeV} \qquad m_\mu = 105\,\text{MeV} \qquad m_e = 0.511\,\text{MeV}$$

we would think the small mass reaction would be more likely due to the higher velocity, but this is not the case. For the pion the spin is 0, so the combination of spin will be in opposite directions. From Figure 3.4 we can see that the anti-leption(anti particle) must be left-handed and thus the lepton must also be left-handed. This is a parity violation. Thus the less favored reaction $u^+ + \nu_\mu$ is seen 99.7% of

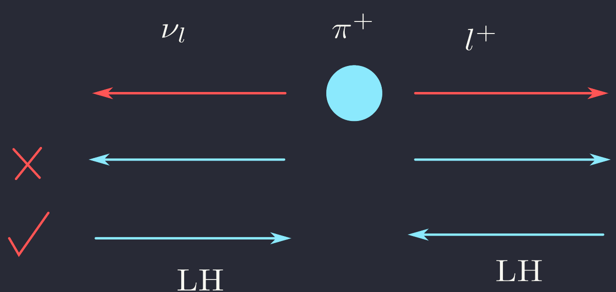

Figure 3.4: Pion decay

the time.

Anti-charged lepton has to be left-handed in this process of appoximate

$$\Gamma \propto m_\ell^\beta$$

where Γ is the decay rate.

Muon decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

This 3 body decay for a polarized muon (choosing the handedness of the muon) we have the following possiblities:

1. LH
2. RH

For maximum energy to the electron, the electron goes in one direction while the neutrinos go in the opposite direction as shown in Figure 3.5. From this the RH case is less favored than the LH case because of the helicity of the neutrinos.

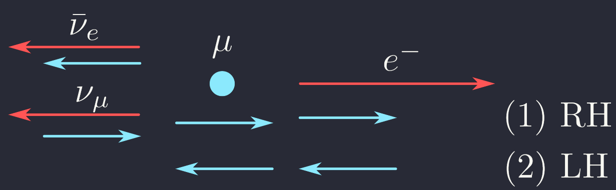

Figure 3.5: Muon decay

Symmetries Part 6: G-Parity and CP Violation

Quiz Review :

1. The neutrino is an eigenstate of partity (neutrino has handedness)

$$P\left|\nu\right\rangle_L = \pm\left|\nu\right\rangle_L x \qquad A\left|\psi\right\rangle = \lambda\left|\psi\right\rangle \qquad P^2 = 1$$

but $\left|\nu\right\rangle_R$ does not exists!

2. Charge Conjugation (C)

$$C\left|m\right\rangle = \left|\bar{m}\right\rangle$$

taking the charge conjugation twice

$$C^2\left|m\right\rangle = C(C\left|m\right\rangle) = C\left|\bar{m}\right\rangle = \left|m\right\rangle$$
$$\implies C^2 = 1 \implies C = \pm 1 \quad Z_z\text{-symmetry}$$

So if $\left|m\right\rangle$ is an eigenstate of C then

$$C\left|m\right\rangle = \pm\left|m\right\rangle = \left|\bar{m}\right\rangle \implies \left|m\right\rangle = \pm\left|\bar{m}\right\rangle$$

e.g. For charged particles:

$$C\left|\pi^+\right\rangle = \left|\pi^-\right\rangle$$

and for some neutral particles the charge conjugation is the same as the particle. One exception is the neutron

$$C\left|n\right\rangle = \left|\bar{n}\right\rangle$$

and for the violation of Charge conjugation: the neutrino

$$C\left|\nu\right\rangle_L = \left|\nu_L\right\rangle \times$$

here C must be violated (in weak interactions) but preserved in strong & EM interactions.

3. G-parity: defined as

$$G = CR$$

where R is a rotation. From the 8-fold way, we can get from π^+ to π^- by reflecting it about the hypercharge axis or rotation by 2π in the I_2 axis and then taking its charge conjugation:

$$G = Ce^{i\pi I_2}$$

so

$$G\left|\pi^+\right\rangle = Ce^{i\pi I_2}\left|\pi^+\right\rangle = C\left|\pi^-\right\rangle = \left|\pi^+\right\rangle$$

finding the G-parity of K^+: from the 8-fold way(Figure 3.1) we know that rotating K^+ actually gives us K^0 so

$$G\left|K^+\right\rangle = Ce^{i\pi I_2}\left|K^+\right\rangle = C\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle$$

Neutral Pion Decay $\pi^0 \to \gamma + \gamma$ is a EM interaction (particle antiparticle pair: $(u\bar{u} - d\bar{d})/\sqrt{2}$).

$$\pi^0 \to \gamma + \gamma$$

The partity on the LHS is $P = -1$ and on the RHS $P = (-1)^2 = +1$. For the photon of spin:

$$S = 1, \qquad S_z = +1, \cancel{0}, -1$$

there is no longitudinal polarization $S_z = 0$, and the transverse polatization of the EM wave $S_z = \pm 1$ is the helicity. So the helicity of the photons must be the same; either $\lambda = |{+}{+}\rangle$ or $\lambda = |{-}{-}\rangle$. This is not an eigenstate of partity. The two photons must have aligned polarizations. We also need to find p in

$$\lambda = \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{S}||\mathbf{p}|}$$

where we know

$$p\,|+\rangle = |-\rangle \quad p\,|-\rangle = |+\rangle \quad p\,|{+}{+}\rangle = |{-}{-}\rangle$$

we can write a linear combination of the two states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|{+}{+}\rangle - |{-}{-}\rangle)$$

for

$$P\,|\psi_1\rangle = |\psi_1\rangle \quad \text{(even parity)}$$
$$P\,|\psi_2\rangle = -\,|\psi_2\rangle \quad \text{(odd parity)}$$

This is similar to quantum optics where polarization is used:

$$|+\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle)$$

we know the G parity is

$$G = (-1)^I$$

4. Pion decay to muon and neutrino: For the mesons

$$C = (-1)^{l+s}$$

where we have two types: Pseudoscalars ($s = 0$) e.g π, K with $C = (-1)^l$ and vector mesons ($s = 1$) e.g. ρ, K^* with $C = (-1)^{l+1}$. For ρ we know that $I(\rho) = 1$ from the 8-fold way, as well as $I(\eta) = 0$ so $\rho \to 3\pi$ and $\eta \to 2\pi$ are allowed.

CP

$$P\,|\nu_L\rangle = |\nu_R\rangle \times$$
$$CP\,|\nu_L\rangle = C\,|\nu_L\rangle = |\bar{\nu}_R\rangle$$

where the P is violated due to the handedness of the neutrino, but the CP is conserved. applying CP on to the charged pion decay:

$$\pi^+ \to \mu^+ + \nu_\mu$$

CP Violation

For an oscillation of a neutral kaon $K^0(d\bar{s})$ and $\bar{K}^0(\bar{d}s)$. Because K's are pseudoscalars:

$$P\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad P\left|\bar{K}^0\right\rangle = \left|\bar{K}^0\right\rangle$$

and the charge conjugation is

$$C\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle \qquad C\left|\bar{K}^0\right\rangle = \left|K^0\right\rangle$$

and the CP is

$$CP\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad CP\left|\bar{K}^0\right\rangle = -\left|K^0\right\rangle$$

which are not eigenstates of CP, so taking a linear combination of the two states

$$|K_1\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) \qquad |K_2\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right)$$

the CP of the two states are

$$CP\,|K_1\rangle = |K_1\rangle \qquad CP\,|K_2\rangle = -\,|K_2\rangle$$

which are eigenstates of CP. Karons decay to pions(lightest meson): to either 2 or 3 pions. From the phase space, the 2 pion decay is more likely than the 3 pion decay because faster particles are more likely to decay: A kaon with 490 MeV to $2\times140 = 280$ MeV is more likely than $3\times140 = 420$ MeV. The 2 pion decay has CP

$$CP\left|\pi^+\pi^-\right\rangle = (-1)^2\left|\pi^+\pi^-\right\rangle = \left|\pi^+\pi^-\right\rangle$$
$$CP\left|\pi^+\pi^-\pi^0\right\rangle = (-1)^3\left|\pi^+\pi^-\pi^0\right\rangle = -\left|\pi^+\pi^-\pi^0\right\rangle$$

so the 2 pion decay is CP even and the 3 pion decay is CP odd. If CP were conserved,

$$|K_1\rangle \to |2\pi\rangle \qquad |K_2\rangle \to |3\pi\rangle$$

so we call this fast decay K_S meaning short $(9\times10^{-11}sec)$ and the slow decay K_L meaning long $(5\times10^{-8}sec)$ or

$$|K_1\rangle \equiv \left|K_S^0\right\rangle \qquad |K_2\rangle \equiv \left|K_L^0\right\rangle$$

from a nobel prize winning experiment, a detector far away would be expected to see mostly K_L but the experiment showed that we saw some amount to the 2 pion decay:

$$\left|K_L^0\right\rangle = \frac{1}{\sqrt{1+\epsilon^2}}\left(\left|K_S^0\right\rangle + \epsilon\left|K_L^0\right\rangle\right)$$

where $\epsilon = 2.3\times10^{-3}$ characterizes CP violation.

Symmetries Part 7: CP Violation

Quiz Reviw:

- matter-antimatter asymmetry: the ϵ parameter measures the CP violation in the neutral kaon (hadronic decay mode). For the semi leptonic decay mode:

$$K_L^0 \to \pi^+ + e^- + \bar{\nu}_e$$

The CP symmetry shows

$$\to \pi^- + e^+ + \nu_e$$

And the rates for the decay modes differ by ϵ. This relates to the matter-antimatter symmetry. (barygenesis) $\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} = 6 \times 10^{-10}$ where n is the number of baryons/antibaryons/photons. In the SM $\eta_{SM} \approx 10^{-20}$, so the SM does not explain the matter-antimatter asymmetry fully. Finding the change in the rate of the process:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} \propto \Gamma(X \to Y + B) - \Gamma(\bar{X} \to \bar{Y} + \bar{B})$$

where the first decay rate Γ violates B(baryon number) for the charge conjugation

$$C(X \to Y + B) = \bar{X} \to \bar{Y} + \bar{B}$$

if C is conserved then

$$\Gamma(X \to Y + B) = \Gamma(\bar{X} \to \bar{Y} + \bar{B})$$

For CP violation: Quarks are chiral e.g. $X \to q_L q_L, q_R q_R$ with (chirality:L and R). We consider X as a linear combination:

$$X \to qq = q_L q_L + q_R q_R$$

The CP operation is

$$CP(X \to q_L q_L) = \bar{X} \to \bar{q}_R \bar{q}_R$$
$$CP(X \to q_R q_R) = \bar{X} \to \bar{q}_L \bar{q}_L$$

so we can write this as a linear combination ($r = \Gamma$)

$$r(X \to qq) = r(X \to q_L q_L) + r(X \to q_R q_R)$$
$$r(\bar{X} \to \bar{q}\bar{q}) = r(\bar{X} \to \bar{q}_R \bar{q}_R) + r(\bar{X} \to \bar{q}_L \bar{q}_L)$$

these are known as the Sakharov conditions:

1. Baryon number violation
2. C and CP violation
3. Departure from thermal equilibrium (forward process $\neq$ backward process)

thus $r(X \to Y + B) \neq r(Y + B \to X)$

- Rotation angles needed to describe mixing n generations of quarks: Cahibbo angles...in 2D we have 1 angle, in 3D we have 3 angles, and in 4D we have 6 angles. This is related to the number of orthogonal places in the space so $n(n-1)/2$ angles are needed.

nxn **unitary matrix:** How many free parameters are there? n^2 complex parts and n^2 real parts thus $2n^2$ parts. But the unitary matrix has n^2 constraints (conditions) e.g. $UU^\dagger = 1$ so we are left with n^2 parameters. To find the number of physical parameters we add a phase which does not change the physical observables so we can have $2n-1$ phases e.g.

$$\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} e^{i\alpha} u \\ e^{i\beta} d \end{pmatrix} e^{i\gamma}$$

here we can always add an overall phase hence the -1. so the free parameters are

$$n^2 - (2n-1) = (n-1)^2$$

and out these free parameters we subtract the number of rotation angles to get the number of physical phase angles.

$$(n-1)^2 - \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2}$$

nxn unitary matrix is described by $n(n-1)/2$ rotation angles and $(n-1)(n-2)/2$ phase angles. e.g. for $n = 2$: 1 rotation and 0 phase, for $n = 3$: 3 rotation and 1 phase. The 1 phase angle in the 3D case is the CP violation angle. This is known as the CKM matrix.

For SU(3) The euler rotation is:

$$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$c_{12} \equiv \cos\theta_{12} \qquad s_{12} \equiv \sin\theta_{12}$$

If the V_{CKM} is close to the identity, the rotation angles would be small, so the two bases are roughly the same. Experimentally we find that the rotation angles

$$\theta_{12} \approx 13^\circ \qquad \theta_{23} \approx 2.4^\circ \qquad \theta_{13} \approx 0.2^\circ \qquad \delta \approx 70^\circ$$

which gives us the CKM matrix

$$V_{CKM} = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

or from the Feynman diagram

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

theses are called the Cabbibo-suppressed processes. There also is a Wolfenstein parameterization

$$V_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

for the mixing of the quarks of neutrinos: ν_e, ν_μ, ν_τ we have the PMNS matrix

$$\begin{pmatrix} \nu_e' \\ \nu_\mu' \\ \nu_\tau' \end{pmatrix} = V_{PMNS} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

parametrized by $(\theta_{12}, \theta_{23}, \theta_{13}, \delta)$ again. This is due to the angles of the rotations being much larger than the CKM matrix:

$$\theta_{12} \approx 34^\circ \qquad \theta_{23} \approx 45^\circ \qquad \theta_{13} \approx 8^\circ \qquad \delta \approx ?$$

this is known as the flavor puzzle.

Time reversal (T) $t \to^{T} -t$
In principle we can have a reversable process $n \leftrightarrow p + e^- + \bar{\nu}_e$ such as in stars, but we can't do this in lab.

T violation Dipole Moment:

Symmetries Part 8: Time Reversal

Time Reversal Symmetry

$$n \leftrightarrow p + e^- + \bar{\nu}_e$$

the reverse process is hard to observe, but in principle it is possible in dipole moments.

Dipole Moment Two opposite charges $\pm q$ separated by a distance d, we have electric field lines going from the positive to the negative charge which have a dipole moment

$$\mathbf{p} = q\mathbf{r}$$

for a uniform charge distribution we usually have a displacement vector

$$\mathbf{d} = \int \mathrm{d}^3 r \rho \mathbf{r}$$

Magnetic dipole Moment

$$\mu = \int \mathrm{d}^3 r \mathbf{r} \times \mathbf{j}_q$$

where $\mathbf{j}_q$ is the current density. A charged particle with a spin has a electric dipole moment

$$\mathbf{d} = d\frac{\mathbf{J}}{|\mathbf{J}|} \qquad \mathbf{J} = \mathbf{L} + \mathbf{S}$$

and a magnetic dipole moment

$$\mu = u\frac{\mathbf{J}}{|\mathbf{J}|}$$

from Wigner-Eckart theorem. The Hamiltonian is

$$H = -(\mu \cdot \mathbf{B} + \mathbf{d} \cdot \mathbf{E}) = -\frac{1}{|\mathbf{J}|}(\mu \mathbf{J} \cdot \mathbf{B} + d\mathbf{J} \cdot \mathbf{E})$$

This is T-even because from the Schrodinger Eq $H\psi = E\psi$ and invariant under T. Using Using Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ t \to -t: \qquad \mathbf{E} &\to \mathbf{E} \quad \text{T-even} \end{aligned}$$

and for second equation

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ t \to -t: \quad &\text{LHS doesn't change} \\ \text{RHS} \quad \mathbf{B} &\to -\mathbf{B} \quad \text{T-odd} \end{aligned}$$

To find if the angular momentum is T-even or T-odd we can use the definition of angular momentum:

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} \\ &= \mathbf{r} \times m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\ t \to -t: \mathbf{v} &\to -\mathbf{v} \\ \mathbf{L} &\to -\mathbf{L} \quad \text{T-odd} \end{aligned}$$

for spin angular momentum we can visualize that from the right hand rule, the spin is T-odd i.e. $\mathbf{S} \to -\mathbf{S}$ (T-odd). Adding the orbital and spin angular momentum we get the total angular momentum $\mathbf{J}$ as T-odd. Since $\mathbf{J}$ and $\mathbf{B}$ are T-odd, we can deduce that μ is T-even (odd times odd is even, and even times odd is even). By the same logic $\mathbf{d}$ is T-odd.

ACME Experiment The limit of the electron dipole moment is

$$d_e < 8 \times 10^{-30} e.cm$$

and for the neutron dipole moment

$$d_n < 10^{-27} e.cm$$

Axions(Not in Textbook) A hypothetical particle: The Langrangian for fundamental particle is

$$\mathcal{L} = \theta G^{\mu\nu} G_{\mu\nu}$$

the electromagnetic field tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

where the magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

and

$$A^0 = \rho = F^{\mu\nu} F_{\mu\nu}$$

the Gluon field tensor

$$G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

and θ is T-odd. Experimentally we find that $\theta < 10^{-10}$ (Strong CP problem). ADMX experiment is currently ongoing.

CPT Theorem All Observables must be CPT invariant in a Lorentz-invariant theory. Sometimes C, P, or T are violated, but CPT is always conserved. As a result, all anti-particles have the same mass as particles (Tested by LHC alpha experiment).

4 Bound States

Two Types:

- Binding Energy < Rest mass energy: Nonrelativistic bound state e.g. Hydrogen atom (-13.6 eV < 1GeV rest mass of proton).
- Binding Energy > Rest mass energy: Relativistic bound state e.g. light meson.

Hydrogen Atom: The potential energy is given by

$$V(r) = -\frac{e^2}{r}$$

or the coulomb potential. The Hamiltonian is given by the Schrödinger equation

$$H\psi = -\frac{\hbar}{2m}\nabla^2\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

where $V(r)$ is the central potential with spherical symmetry SO(3). But there also is an enhanced symmetry.

Noether's Theorem: Symmetry $\leftrightarrow$ Conservation Law. e.g.

- SO(3) $\leftrightarrow$ Conservation of Angular momentum.
- SO(1,3) $\leftrightarrow$ linear momentum (Poincare symmetry)
- T-reversal $\leftrightarrow$ energy
- U(1)$_{em}$ $\leftrightarrow$ electric charge

so from the central potential, we know that angular momentum $\mathbf{L}$ is conserved. But for $1/r$ there is a SO(4) symmetry from the LRL (Laplace-Runge-Lenz) vector

$$\mathcal{L} = \frac{1}{m}\mathbf{L}\times\mathbf{p} + \frac{\kappa\mathbf{r}}{r}$$

where

$$V(r) = -\frac{\kappa}{r}$$

the energy eigenvalues of the hydrogen atom are given by

$$E_n = -\frac{13.6\,\text{eV}}{n^2} = -\frac{me^4}{2\hbar^2 n^2} = -\frac{1}{2}\frac{\alpha m_e c^2}{n^2}$$

where $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine structure constant.

Degeneracy n^2 e.g. For SO(3), $(2l+1)$ degeneracy

$$\sum_{l=0}^{n-1}(2l+1) = 2\sum_{l=0}^{n-1} l + n = 2\frac{(n-1)(n)}{2} + n = n^2$$

n	l	m	degeneracy
1	0	0	1
2	0	0	1
	1	-1,0,1	3
3	0	0	1
	1	-1,0,1	3
	2	-2,-1,0,1,2	5

Positronium (e^+e^- bound state) has the same energy levels as the hydrogen atom the energy eigen value is given by first looking at the reduced mass

$$\mu = \frac{1}{\frac{1}{m_e} + \frac{1}{m_e}} = \frac{m_1 m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_1 \ll m_2$$

but here $m_1 = m_2 = m_e$ so $\mu = \frac{m_e}{2}$. The energy eigenvalues are given by

$$E_n = \frac{1}{2} - \frac{13.6\,\text{eV}}{n^2} = -\frac{6.8\,\text{eV}}{n^2}$$

We can do this for Muonium (μ^+e^- bound state) and Pionic Hydrogen(π^+e^- bound state).

Fine Structure

1. Relativistic Correction

$$T = E - m_e c^2$$

2. Spin-Orbit Coupling
3. Lambd Shift (QED)
4. Hyperfine Splitting aka zeeman effect

Quiz Review

- For the Positronium:

$$C : (-1)^{l+s} = (-1)^n$$

where $l + s = n$ (the selection rule for Positronium decay). *for n photons, $C = (-1)^n$. FOr the ground states $l = 0$ so the spin is

$$S : \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

where we have a triplet state $S = 1$ and a singlet state $S = 0$. For this singlet:

$$S = 0 \implies (-1)^0 = 1 = (-1)^2$$

or two photons can be emitted (para-positronium). For the triplet state:

$$S = 1 \implies (-1)^1 = -1 = (-1)^3$$

or three photons can be emitted (ortho-positronium). The mass of each photon for two photons is roughtly a half of the mass of the positronium $E_\gamma = 511\,\text{keV}$. For three photons $E_\gamma < 511$ keV.

- Binding Energy vs. Rest Mass Energy: Quarkonium ($q\bar{q}$): *uds* light quarks, *cbt* heavy quarks.

  - Heavy Quarkonium: $c\bar{c}$: Charmonium (J/ψ), $b\bar{b}$: Bottomonium (Υ), $t\bar{t}$: Toponium *does not exist* (very heavy so it decays really fast $\sim 10^{-25}$s vs $\tau_{\text{bound state}} \sim 10^{-23}$ sec).

For Charmoniun, the reduced mass is

$$\mu = \frac{m_c m_c}{m_c + m_c} \approx \frac{m_c}{2}$$

and the energy of the Hydrogen atom is

$$E_n = -\frac{me^4}{2\hbar^2 n^2} = -\frac{1}{2}\frac{\alpha mc^2}{n^2}$$

and for the Charmonium:

$$E_n = -\frac{4}{9}\frac{1}{2}\frac{\alpha m_c c^2}{n^2} \quad \text{incorrect}$$

where we have to adjust for the charge of the quark $e \to \frac{2}{3}e$ and the potential: For electron coulomb potential we know that

$$V = -\frac{e^2}{r} = -\frac{e^2}{\hbar r}\frac{\hbar c}{r} = -\frac{\alpha \hbar c}{r}$$

but for quarks there is a different potential from the strong interaction (gluon)

$$V(r) = -\frac{\alpha_s \hbar c}{r} - \frac{4}{9}\frac{\alpha \hbar c}{r} \qquad \alpha_s = \frac{g_s^2}{\hbar c} \gg \alpha$$

which is much larger than the coulomb potential (suppressed second term), but there is a transition to a linear potential as the distance get very large.

$$V(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} + F_o r \quad \text{QCD Potential}$$

there also is a color factor $\frac{4}{3}$ based on the three colors of the quarks. So the energy is given by

$$E_n = -\frac{4}{3}\frac{1}{2}\frac{\alpha_s m_c c^2}{n^2}$$

- Decay of Charmonium:

$$J/\psi \to \pi^+\pi^-\pi^0 \quad \text{or} \quad D^+D^-$$

For the ground state, $m_{J/\psi} = 3.1$ GeV. And the total rest mass of D^+D^- is kinematically forbidden $m_{D^+} + m_{D^-} = 3.7$ GeV. We have a decay to 3 pions due to the G-parity conservation $(-1)^I C$ or $(-1)^n$.

OZI rule (Okubo, Zweig, Iizuka) Cutting a hard gluon line in the Feynman diagram separates the quarks and the decay is suppressed. For soft gluon lines, cutting a line does not separate the quarks and the decay is not suppressed.

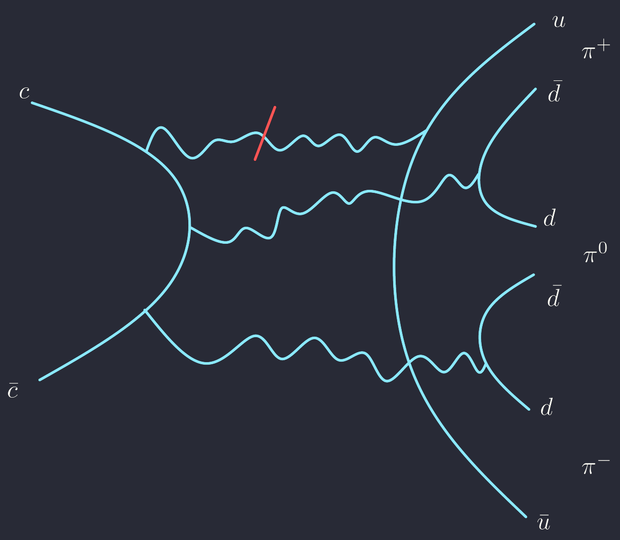

Figure 4.1: OZI Rule

- Light Mesons: $q\bar{q}$ where $q = u, d, s$. There are nine spin-0 (pseudo scalar) mesons and nine spin-1 (vector) mesons. (insert figure 5.11 from Griffiths). From the lie algebra of the spin-0 nonet

$$3 \otimes \bar{3} = 8 \oplus 1$$

where the 1 is the η' meson. and we break down the 8 into

$$8 \to 2 \oplus 3 \oplus 2 \oplus 1$$

where they refer to the top row, middle row pions, bottom row and η meson. For the vector mesons. For the isospin doublet:

$$u = \left|\frac{1}{2}\frac{1}{2}\right\rangle \qquad d = \left|\frac{1}{2} - \frac{1}{2}\right\rangle$$

and for the antiquarks:

$$\bar{u} = \left|\frac{1}{2} - \frac{1}{2}\right\rangle \qquad \bar{d} = -\left|\frac{1}{2}\frac{1}{2}\right\rangle$$

and the pions are given by

$$\begin{aligned}
\pi^+ &= \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle = -u\bar{d} \\
\pi^- &= \left|\frac{1}{2} - \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} - \frac{1}{2}\right\rangle = d\bar{u} \\
\pi^0 &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2} - \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle + \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} - \frac{1}{2}\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})
\end{aligned}$$

for the corner mesons:

$$K^0 = d\bar{s} \qquad \bar{K}^0 = \bar{d}s \qquad K^+ = u\bar{s} \qquad K^- = \bar{u}s$$

and the η mesons are

$$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$
$$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

For spin-1 mesons, in terms of the flavor

$$\rho^+, \rho^0, \rho^- = \pi^+, \pi^0, \pi^-$$

and the same for the K^* mesons. The difference is in the center

$$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \qquad \phi = s\bar{s}$$

Quiz Review: For the kinetic energy of a particle

$$\begin{aligned} T &= E - mc^2 \\ &= \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4} - mc^2 \\ &= mc^2 \left(1 + \frac{\mathbf{p}^2}{m^2 c^2}\right)^{1/2} - mc^2 \end{aligned}$$

and using the binomial expansion

$$(1+x)^n \approx 1 + nx + \frac{1}{2}\frac{n(n-1)}{2}x^2 \ldots \quad \text{for} \quad x \ll 1$$

so

$$\begin{aligned} T &= mc^2 \left(\cancel{1} + \frac{\mathbf{p}^2}{2m^2c^2} - \frac{\mathbf{p}^4}{8m^4c^4} + \ldots \right) - \cancel{mc^2} \\ &= \frac{\mathbf{p}^2}{2m} - \frac{1}{8}\frac{\mathbf{p}^4}{m^3c^2} + \ldots \end{aligned}$$

From last time: Light mesons (u, d, s) with $q\bar{q}$ bound states and spin

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

where we for the spin-0 pseudoscalar mesons: π, K, η we have 9 states. And for the spin-1 vector mesons: ρ, K^*, ω, ϕ we have 9 states. The flavor states of K mesons:

$$K^+ : (u\bar{s}) \quad K^- : (\bar{u}s) \quad K^0 : (d\bar{s}) \quad \bar{K}^0 : (\bar{d}s)$$

and for vector K mesons

$$K^{*+} : (u\bar{s}) \quad \text{etc.}$$

so the Isospin states

$$\begin{pmatrix} u \\ d \end{pmatrix} \otimes \begin{pmatrix} \bar{d} \\ \bar{u} \end{pmatrix} = 1 \oplus 0$$

so

- spin-0: $1 \sim \pi^+, \pi^-, \pi^0$, $0 \sim \eta$
- spin-1: $1 \sim \rho^+, \rho^0, \rho^-$, $0 \sim \omega$

and in $SU(3)$:

$$3 \otimes \bar{3} = 8 \oplus 1$$

where η' is the spin-0 singlet (1). We can break down the spin algebra to isospin

$$\begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \otimes \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix} \otimes \begin{pmatrix} \bar{d} \\ \bar{u} \end{pmatrix} = 1 \oplus 0$$

where

$$\begin{aligned} |1,1\rangle &= |\uparrow\uparrow\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1,-1\rangle &= |\downarrow\downarrow\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{aligned}$$

and the charge conjugation operator

$$
\begin{aligned}
Cu \to \bar{u} \qquad Cd \to \bar{d} \\
C = i\sigma^2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
C\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}
\end{aligned}
$$

so

$$
\begin{aligned}
|1,1\rangle &= -u\bar{d} = \left|\pi^+\right\rangle, \left|\rho^+\right\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) = \left|\pi^0\right\rangle, \left|\rho^0\right\rangle \\
|1,-1\rangle &= \bar{u}d = \left|\pi^-\right\rangle, \left|\rho^-\right\rangle \\
|0,0\rangle &= \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) = |\cancel{\eta}\rangle, |\omega\rangle
\end{aligned}
$$

where the η is not correct. We know that ψ is orthgonal to ω so

$$
|\psi\rangle = |s\bar{s}\rangle
$$

we know that η' is an $SU(3)$ singlet so

$$
|\eta'\rangle = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})
$$

and since η is orthogonal to η' and all other states:

$$
|\eta\rangle = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})
$$

this is a special case for the pseudoscalar light mesons. Even though the spin-0 and spin-1 particles are made of the same quarks, their masses are very different! So the true meson mass is

$$
M = m_1 + m_2 + A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}
$$

this third term is the spin-spin interaction from the Hydrogen Atom (n^2 degeneracry) hyperfine splitting. The spin-spin interaction breaks the degeneracy of the n^2 states. Finding

$$
\begin{aligned}
\mathbf{S}_T &= \mathbf{S}_1 + \mathbf{S}_2 \\
\mathbf{S}^2 &= (\mathbf{S}_1 + \mathbf{S}_2)^2 \\
&= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2
\end{aligned}
$$

Where from the operator

$$
\begin{aligned}
[J^2, J_z] = 0 \qquad |j,m\rangle \\
J_z\,|j,m\rangle + \hbar m\,|j,m\rangle \\
J^2\,|j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle
\end{aligned}
$$

so the eigenvalues of $\mathbf{S}$ are $\frac{1}{2}(\frac{1}{2}+1)\hbar^2$:

$$
\begin{aligned}
\mathbf{S}^2 &= \frac{3}{2}\hbar^2 + \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \\
&= \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2
\end{aligned}
$$

so for the scalar case $s = 0$, $\mathbf{S}^2 = 0$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = -\frac{3}{4}\hbar^2$$

and for the vector case $s = 1$, $\mathbf{S}^2 = 2\hbar^2$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(2 - \frac{3}{2}\hbar^2\right)$$

So

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \begin{cases} -\frac{3}{4}\hbar^2 & \text{spin-0} \\ \frac{1}{4}\hbar^2 & \text{spin-1} \end{cases}$$

e.g. we have meson masses

- $(u, d) : m_\rho = 775\,\mathrm{MeV/c^2}$
- $(u, d) : m_\pi = 140\,\mathrm{MeV/c^2}$

And we have an effective mass (MIT bag model):

$$m_{eff} \geq \Lambda_{QCD} \sim 200\,\mathrm{MeV/c^2}$$

Quarks are always in bound states, and do not feel the strong interaction. And the bare mass is different from the effective mass. Some bare masses:

$$m_{u,d} \sim 300\,\mathrm{MeV/c^2}$$
$$m_s \sim 400\,\mathrm{MeV/c^2}$$

Baryons... (light): qqq bound states. We can treat the 3 body system as a 2 body system(CM of 2 quarks plus the third quark). For the ground state, $l = l' = 0$ for simplicity.

Spin configurations

$$\begin{aligned} \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= (1 \oplus 0) \otimes \frac{1}{2} \\ &= (1 \otimes \frac{1}{2}) \oplus (0 \otimes \frac{1}{2}) \\ &= (\frac{3}{2} \oplus \frac{1}{2}) \oplus (\frac{1}{2}) \\ &= 4 \oplus 2 \oplus 2 \end{aligned}$$

(the 3/2 spin has 4 states: $3/2, 1/2, -1/2, -3/2$) where

$$j_1 \otimes j_2 = (j_1 + j_2), \ldots, |j_1 - j_2|$$

we also have 8 spin states for the 3 quarks:

$$\begin{aligned} &\uparrow\uparrow\uparrow \\ \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, &\downarrow\uparrow\uparrow \\ \uparrow\downarrow\downarrow, \downarrow\uparrow\downarrow, &\downarrow\downarrow\uparrow \\ &\downarrow\downarrow\downarrow \end{aligned}$$

so

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = |\uparrow\uparrow\uparrow\rangle$$
$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)$$
$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)$$
$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = |\downarrow\downarrow\downarrow\rangle$$

which are all symmetric states! But Baryons are fermions so the wavefunction must be antisymmetric! And from 3 spins, we cant make an antisymmetric state, but we can make a partically antisymmetric state

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\downarrow$$

where the subscript 12 denotes that the first two spins are antisymmetric and the third spin is free. So we can also get

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{23} =\uparrow \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{23} =\downarrow \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

but we cant get an antisymmetric state for the first and third spins:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{13} = \frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{13} = \frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)$$

this is just a linear combination of the other states:

$$|\ \rangle_{13} = |\ \rangle_{12} + |\ \rangle_{23}$$

Pauli Exclusion Principle: For probability of wavefunction to be the same

$$\psi(1,2) \to \psi(2,1)$$
$$|\psi(1,2)|^2 = |\psi(2,1)|^2$$
$$\implies \psi(1,2) = e^{i\phi}\psi(2,1)$$
$$\psi(1,2) \to \psi(2,1)e^{i\phi} \to \psi(1,2)e^{2i\phi}$$
$$\implies e^{2i\phi} = 1 \implies e^{i\phi} = \pm 1$$

so

$$\psi(1,2) = \begin{cases} +\psi(2,1) & \text{even} \\ -\psi(2,1) & \text{odd} \end{cases}$$

And distinguishable particles

$$\psi_\alpha(1)\psi_\beta(2)$$

and for indistinguishable particles

$$\psi(1,2) = \frac{1}{\sqrt{2}}[\psi_\alpha(1)\psi_\beta(2) \pm \psi_\alpha(2)\psi_\beta(1)]$$

and from the *Spin-Statistics Theorem*:

- Bosons $\rightarrow$ Even wavefunctions(+)
- Fermions $\rightarrow$ Odd wavefunctions(-)

where if $\alpha = \beta$ we then know $\psi(1,2) = 0$ for fermions.

Quiz Review:

- For the baryon wavefunction its a little more complicated:
  - Fermion → Pauli Exclusion Principle
  - Three body system
  - Color Quantum Number

  *The baryon wavefunction must be antisymmetric under the inerchange of any two constituent quarks. So the baryon wavefunction is a combination of 4 parts:

$$\psi(baryon) = \psi(space)\psi(spin)\psi(flavor)\psi(color)$$

  So for total antisymmetry

  - For the space part, we can assume a ground state where $\ell = 0$ thus sphericaly symmetric.
  - The spin part can be symmetric (3/2) or partially antisymmetric (1/2).
  - For the flavor part we have $n^3 = 27$ states, or from group theory we have a SU(3):

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

    or from Young Tableux for mesons

$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline & \\ \hline\end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$$

$$= \frac{3*4*2}{1*3*1} \oplus \frac{3*2*1}{1*2*3} = 8 \oplus 1$$

    but for the Bosons

$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array}$$

$$= \frac{3*4}{1*2} \oplus \frac{3*2}{1*2} = 6 \oplus \bar{3}$$

    where the $\bar{3}$ comes from the antisymmetric part of now expanding again

$$6 \otimes 3 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline\end{array} = 10 \oplus 8$$

    thus

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

    How do we arrange this 10 states? From the mesons we know that the octet splits like

$$8 \to 2 + 3 + 2 + 1$$

    and we are told that the decuplet splits to

$$10 \to 4 + 3 + 2 + 1$$

    The figure has isospin I_3 on the x-axis and spin S on the y-axis. We know that the quark charges are

$$u = \frac{2}{3} \qquad d = -\frac{1}{3} \qquad s = -\frac{1}{3}$$

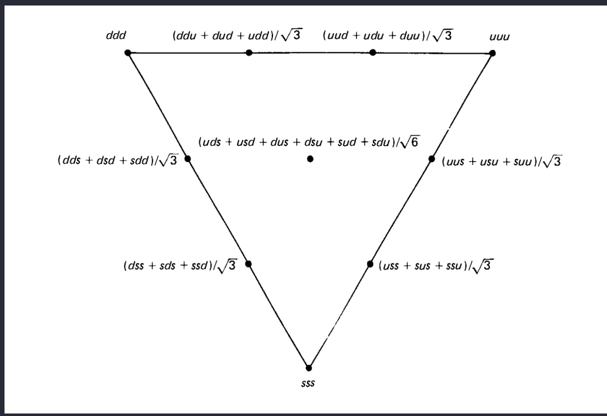

Figure 4.2: Decuplet

So the top right is (uuu) and etc. for the other corners. But these states are symmetric and the spin is also symmetric so we need something else that needs to be antisymmetric. Thus we have colors in $SU(3)_c$:

$$q = \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

this also by $SU(3)$ algebra has 27 states of color:

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

but....

Every Naturally Occuring Particle is a Color Singlet - Griffiths

AKA the *Color Confinement* principle. So the singlet is always antisymmetric:

$$\psi = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)$$

So since everything except color is must be symmetric for the total to be antisymmetric i.e.

$$\psi(spin)\psi(flavor) = \text{symmetric}$$

So how do we get from Δ^{++} to Δ^{+}? We use the I_- operator: From the lowering operator

$$J_- \left|j, m\right\rangle = \hbar\sqrt{(j+m)(j-m+1)}\left|j, m-1\right\rangle$$

so in isospin-space

$$u = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$$

thus

$$I\_u = I_- \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \hbar\sqrt{1}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = d$$

and we also get

$$I\_d = 0 \quad \text{lowest } I_3 \text{ state}$$
$$I\_s = 0 \quad \text{singlet}$$

SO

$$I_-\Delta^{++} = I_-(uuu) = (uud + udu + duu)$$
$$I_-\left|\frac{3}{2},\frac{3}{2}\right\rangle = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle$$
$$= \sqrt{3}\Delta^+$$
$$\implies \Delta^+ = \frac{1}{\sqrt{3}}(uud + udu + duu)$$

applying the lowering operator again we get

$$\Delta^0 = \frac{1}{\sqrt{3}}(udd + dud + ddu)$$

and again

$$\Delta^- = ddd$$

and for the Σ (strangeness $S = -1$) states we have the following:

$$\Sigma^{*+} = \frac{1}{\sqrt{3}}(uus + usu + suu)$$
$$\Sigma^{*0} = \frac{1}{\sqrt{3}}(uds + dus + dsu + sud + sdu + uds)$$
$$\Sigma^{*-} = \frac{1}{\sqrt{3}}(dds + dsd + sdd)$$

for $S = -2$ we have the 2 Ξ states:

$$\Xi^{*0} = \frac{1}{\sqrt{3}}(uss + sus + ssu)$$
$$\Xi^{*-} = \frac{1}{\sqrt{3}}(dss + sds + ssd)$$

and a singlet

$$\Omega^- = sss$$

so the flavor is symmetric for the decuplet states

Octet States In the first octet we have 8 states

- n, p
- $\Sigma^+, \Sigma^0, \Sigma^-$
- Ξ^0, Ξ^-
- Λ^0

now we need to do the same partial antisymmetry thing we did for spins onto the flavor states:

$$\psi(flavor) = |\,\rangle_{12} \quad \text{or} \quad |\,\rangle_{23} \quad \text{or} \quad |\,\rangle_{13}$$

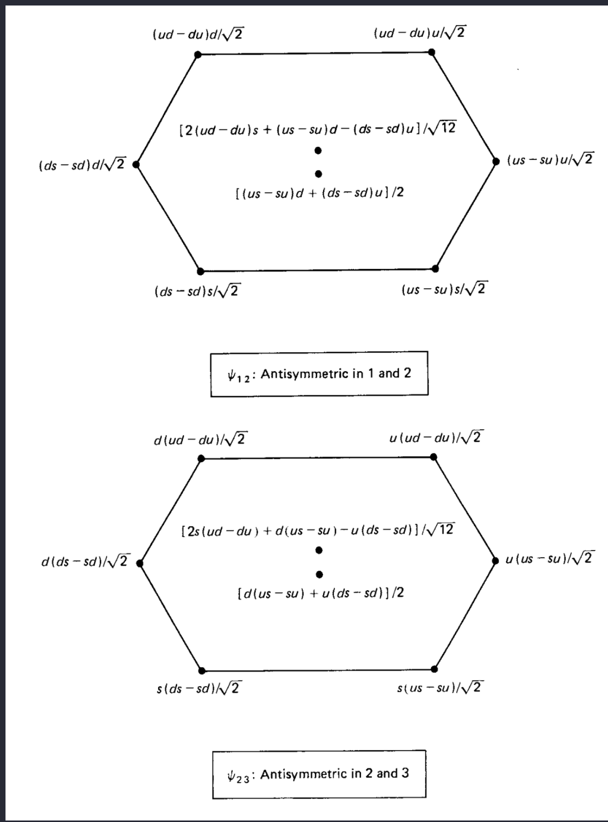

Figure 4.3: 1st and 2nd Octets

since the proton is a quark made of uud we know that

$$
\begin{aligned}
|\ \rangle_{12} &= \frac{1}{\sqrt{2}}(ud - du)u \\
\left|p(s_z = \frac{1}{2})\right\rangle &= \frac{1}{\sqrt{2}}(ud - du)u \otimes \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow \\
&= \frac{1}{2}[u(\uparrow)d(\downarrow)u(\uparrow) \\
&\quad - u(\downarrow)d(\uparrow)u(\uparrow) \\
&\quad - d(\uparrow)u(\downarrow)u(\uparrow) \\
&\quad + d(\downarrow)u(\uparrow)u(\uparrow)]
\end{aligned}
$$

and we can do this for the flavor and spin wavefunctions:

$$
\begin{aligned}
\psi(flavor)\psi(spin) = \frac{\sqrt{2}}{3}[&\psi_{12}(f)\psi_{12}(s) \\
+ &\psi_{23}(f)\psi_{23}(s) \\
+ &\psi_{13}(f)\psi_{13}(s)]
\end{aligned}
$$

where the will have to find this constant in HW. Thus the proton is

$$\begin{aligned}|p(\text{spin up})\rangle &= \frac{1}{2}\sqrt{2}3[(ud-du)u(\uparrow\downarrow-\downarrow\uparrow)\uparrow \\ &\quad + u(ud-da)\uparrow(\uparrow\downarrow-\downarrow\uparrow) \\ &\quad + (uud-duu)(\uparrow\uparrow\downarrow-\downarrow\uparrow\uparrow)] \\ &= \frac{1}{3\sqrt{2}}[uud(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow) \\ &\quad + udu(2\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow-\uparrow\uparrow\downarrow) \\ &\quad + duu(-\uparrow\downarrow\uparrow+2\downarrow\uparrow\uparrow-\uparrow\uparrow\downarrow)]\end{aligned}$$

so for n we have

$$|n\rangle_{12} = \frac{1}{\sqrt{2}}(ud-du)d$$

and for sigma

$$\left|\Sigma^+\right\rangle_{12} = \frac{1}{\sqrt{2}}(us-su)u$$
$$\left|\Xi^0\right\rangle_{12} = \frac{1}{\sqrt{2}}(us-su)s$$

Quiz Review (From last time) We found for the baryon wavefunction that the space part is symmetric ($\ell = 0$), the spin & flavor parts are symmetric, and the color part is antisymmetric. For the spin up proton:

$$\psi = \frac{\sqrt{2}}{3}[\psi_{12}(spin)\psi_{12}(flavor) + \psi_{23}(spin)\psi_{23}(flavor) + \psi_{13}(spin)\psi_{13}(flavor)]$$

where

$$\langle\psi_{12}|\psi_{23}\rangle \neq 0 \qquad \langle\psi_{12}|\psi_{13}\rangle \neq 0$$

and

$$\begin{aligned} \psi_{12}(spin) &= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow \\ \psi_{12}(flavor) &= \frac{1}{\sqrt{2}}(ud - du)u \end{aligned}$$

so the wavefunction of the two parts is

$$\psi = \frac{\sqrt{2}}{3}[2u(\uparrow)u(\uparrow)d(\downarrow) - u(\uparrow)u(\downarrow)d(\uparrow) - u(\downarrow)u(\uparrow)d(\uparrow) + \text{permuations}]$$

Magnetic Moment The experimental test, the magnetic moment

$$\boldsymbol{\mu} = \frac{q}{mc}\mathbf{S}$$

where the Hamiltonian is

$$H = -\boldsymbol{\mu}\cdot\mathbf{B}$$

and the magnetic moment of a baryon is a vector sum of the three quarks

$$\boldsymbol{\mu}(\text{baryon}) = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3$$

so the z-component of the magnetic moment is

$$\mu_z(\text{baryon}) = \sum_i \mu_{i_z} = \sum_i \frac{q_i}{mc}S_{i_z}$$

where

$$\mu_{i_z} = \frac{q_i}{mc}\frac{\hbar}{2}$$

so for each quark

$$\mu_u = \frac{2}{3}\frac{\hbar}{2m_uc} \qquad \mu_d = -\frac{1}{3}\frac{\hbar}{2m_dc} \qquad \mu_s = -\frac{1}{3}\frac{\hbar}{2m_sc}$$

so the expectation value of the magnetic moment operator μ_z is

$$\begin{aligned} \mu_{\text{baryon},\uparrow} &= \langle B(\uparrow)|\,\mu_z\,|B(\uparrow)\rangle \\ &= \sum_i \langle B(\uparrow)|\,\mu_{i_z}\,|B(\uparrow)\rangle \\ &= \frac{2}{\hbar}\sum_i \langle B(\uparrow)|\,\mu_i S_{i_z}\,|B(\uparrow)\rangle \end{aligned}$$

So calculating the expectation value for the proton

$$\sum_i \mu_i S_{i_z}\,|u(\uparrow)u(\uparrow)d(\downarrow)\rangle = \left(\mu_u\frac{\hbar}{2} + \mu_u\frac{\hbar}{2} + \mu_d\frac{-\hbar}{2}\right)|u(\uparrow)u(\uparrow)d(\downarrow)\rangle$$

the first term is

$$\begin{aligned}\mu_1 &= \sum_i \langle p(\uparrow)| \mu_i S_{i_z} |p(\uparrow)\rangle \\ &= \left(\frac{2}{3\sqrt{2}}\right)^2 \frac{\hbar}{2}(2\mu_u - \mu_d)\frac{2}{\hbar} \langle u(\uparrow)u(\uparrow)d(\downarrow)|u(\uparrow)u(\uparrow)d(\downarrow)\rangle \\ &= \frac{2}{9}(2\mu_u - \mu_d)\end{aligned}$$

the second term $-u(\uparrow)u(\downarrow)d(\uparrow)$ is

$$\frac{\hbar}{2}(\mu_u - \mu_u + \mu_d) = \frac{\hbar}{2}\mu_d$$

so

$$\left(\frac{1}{3\sqrt{2}}\right)^2 \mu_d = \frac{1}{18}\mu_d$$

and same for the third term. We get the magnetic moment as

$$\begin{aligned}\mu_{p(\uparrow)} &= 3\left[\frac{2}{9}(2\mu_u - \mu_d) + \frac{1}{18}\mu_d + \frac{1}{18}\mu_d\right] \\ &= \frac{3}{18}(4(2\mu_u - \mu_d) + 2\mu_d) \\ &= \frac{1}{6}(8\mu_u - 2\mu_d) \\ &= \frac{1}{3}(4\mu_u - \mu_d)\end{aligned}$$

and from the experimental values:

$$m_p = 2.79 m_u = 2.79 m_d$$

we get

$$\mu_p = 2.79\frac{e\hbar}{2m_p c} = 2.79\mu_B$$

where $\mu_B = \frac{e\hbar}{2m_p c}$ is the Bohr magneton. And for the neutron we know

$$p : uud \qquad n : udd$$

so all we have to do is replace the u with d and thus:

$$\begin{aligned}\mu_n &= \frac{1}{3}(4\mu_d - \mu_u) \\ &= \frac{1}{3}\left(4(-1/3)\frac{\hbar}{2m_d c} - 2/3\frac{\hbar}{2m_u c}\right) \\ &= -\frac{2}{3}(2.79)\frac{e\hbar}{2m_p c} = -1.86\mu_B\end{aligned}$$

experimentally, we got the proton and neutron magnetic moments to be

$$\mu_p = 2.793\mu_B \qquad \mu_n = -1.913\mu_B$$

we can also take the ratio of the magnetic moments

$$\frac{\mu_n}{\mu_p} = -\frac{2}{3}$$

which is independent of the quark masses! This is a (robust) prediction of the quark model

If there was no color Then the wavefunction would be

$$\psi = \psi_{space}\psi_{spin}\psi_{flavor}$$

so

$$\psi(spin)\psi(flavor) = \text{anti-symmetric}$$

and we would get the ratio of the magnetic moments to be

$$\frac{\mu_n}{\mu_p} = -2$$

Baryon Masses For the meson masses we had the mass as the sum of the masses and the spin-spin interaction:

$$M = m_1 + m_2 + A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}$$

But for the baryons we have 3 quarks so we have

$$M = m_1 + m_2 + m_3 + A'\left(\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2} + \frac{\mathbf{S}_1 \cdot \mathbf{S}_3}{m_1 m_3} + \frac{\mathbf{S}_2 \cdot \mathbf{S}_3}{m_2 m_3}\right)$$

For the decuplet we have the inverted triangle from Figure 4.2 and the octet from Figure 4.3. Taking $m_u = m_d$

- For Baryons with no S,

  $$M = 3\mu_u + \frac{A'}{m_\mu^2}(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3)$$

  so the total spin is

  $$\begin{aligned} \mathbf{S}^2 &= (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3)^2 \\ &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 + 2(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3) \\ \implies \mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3 &= \frac{1}{2}(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2 - \mathbf{S}_3^2) = S' \end{aligned}$$

  where the eigen values are

  $$\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$$

  i.e.

  $$\begin{aligned} s = 3/2 : \mathbf{S}^2 &= \frac{3}{2}(\frac{3}{2}+1)\hbar^2 = \frac{15}{4}\hbar^2 \\ s = 1/2 : \mathbf{S}^2 &= \frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2 \end{aligned}$$

  so

  $$\begin{aligned} &= \frac{1}{2}\left(\mathbf{S}^2 - 3\frac{3}{4}\hbar^2\right) \\ &= \frac{1}{2}\begin{cases} \frac{3}{2}\hbar^2 & S = \frac{3}{2} \quad \text{(decuplet)} \\ -\frac{3}{2}\hbar^2 & S = \frac{1}{2} \quad \text{(octet)} \end{cases} \end{aligned}$$

  so the masses are

  $$M_N = 3m_u - \frac{3}{4}\frac{\hbar^2}{m_u^2}A'$$

  $$M_\Delta = 3m_u + \frac{3}{4}\frac{\hbar^2}{m_u^2}A' M_\Omega = 3m_s + \frac{3}{4}\frac{\hbar^2}{m_s^2}A'$$

For the decuplet case all spin are parallel so for $s = 3/2$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \mathbf{S}_1 \cdot \mathbf{S}_3 = \mathbf{S}_2 \cdot \mathbf{S}_3 = \frac{1}{4}\hbar^2$$

and

$$\begin{aligned} (\mathbf{S}_1 + \mathbf{S}_2)^2 &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \\ \mathbf{S}_1 \cdot \mathbf{S}_2 &= \frac{1}{2}(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2) = \frac{3}{4}\hbar^2 & &= \frac{1}{2}\left(1(1+1)\hbar^2 - 2\frac{3}{4}\hbar^2\right) = \frac{1}{4}\hbar^2 \end{aligned}$$

so the maasses are calculated by:

$$M = m_1 + m_2 + m_3 + \frac{A'}{4}\hbar^2\left(\frac{1}{m_1 m_3} + \frac{1}{m_2 m_3} + \frac{1}{m_2 m_3}\right)$$

For the octet case

$$\begin{aligned} M_\Sigma &: (ud) \quad I = 1 \\ M_\Lambda &: (ud) \quad I = 0 \end{aligned}$$

so for example the Σ^+ case:

$$M_{\Sigma^+} = 2m_u + m_s + A'\left(\frac{\mathbf{S}_u \cdot \mathbf{S}_u}{m_u^2} + \frac{\mathbf{S}_u \cdot \mathbf{S}_s}{m_u m_s} + \frac{\mathbf{S}_u \cdot \mathbf{S}_s}{m_u m_s}\right)$$

and since we have already calculated

$$\mathbf{S}_u \cdot \mathbf{S}_u = \frac{1}{4}\hbar^2$$

and we know the spin for the octet is

$$\mathbf{S}_u \cdot \mathbf{S}_u + \mathbf{S}_{u1} \cdot \mathbf{S}_s + \mathbf{S}_{u2} \cdot \mathbf{S}_s = -\frac{3}{4}\hbar^2 \mathbf{S}_{u1} \cdot \mathbf{S}_s + \mathbf{S}_{u2} \cdot \mathbf{S}_s = -\hbar^2$$

and from anti symmetry we know that

$$\mathbf{S}_{u1} = \mathbf{S}_{u2} = -\frac{3}{2}\hbar^2$$

and we can do the same for the other octet states.

Midterm: Multiple choice 25pt, bound states(general structure why $1/n^2$) heavy masses, wavefunctions, relativistic kinematics.

Homework 1

Due 1/24

1. Gravity vs. E&M
Given the force of gravitational attraction

$$\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$$

and the force of electrostatic repulsion

$$\mathbf{F}_e = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\hat{\mathbf{r}}$$

The ratio of the two forces between two electrons is

$$\frac{F_g}{F_e} = \frac{4\pi\epsilon_0 G m_e^2}{q_e^2}$$

Using the values for the constants

$$G = 6.67\times10^{-11}\,\mathrm{N\,m^2/kg^2}, \quad \epsilon_0 = 8.85\times10^{-12}\,\mathrm{C^2/N\,m^2}$$
$$m_e = 9.11\times10^{-31}\,\mathrm{kg}, \quad q_e = 1.60\times10^{-19}\,\mathrm{C}$$

we find

$$\boxed{\frac{F_g}{F_e} = 2.40\times10^{-43}}$$

this tells us that the denominator (the electrostatic force) is *much* larger than the numerator (the gravitational force), which convinces us that gravitaional forces are negligible for elementary particles.

2. Mesons and Baryons

(a) For mesons, you can have n possible quarks and n possible antiquarks, thus there are $\boxed{n^2}$ combinations.

For baryons order does not matter we have to make sure not to double count instances such as (uud) and (udu): This is essentially a combination problem with the solution

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \boxed{\frac{n(n-1)(n-2)}{6}}$$

(b) Given the 6 flavors of quarks, we would expect $6^2 = 36$ mesons and $\binom{6}{3} = 20$ baryons.

(c) We haven't found all of them because of energy required to observe the heavier particles. In the Particle Data Group website, the heaviest baryon is in the order of 6000 MeV and the LHC has a beam energy of 6.5 TeV and the energy consumption is about 1.3 TWh per year compared to the global energy consumption of 20 000 TWh per year (source). In addition to the enourmous energy required to produce these particles, they are also very unstable and decay very quickly thus detecting them require us to measure at very small time scales.

3. Global Conservation Laws

(a) $n \to \bar{p} + e^+ + \nu_e$

not valid: violates Baryon number conservation

(b) $\nu_e + n \to p + e^-$
valid

(c) $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$
valid

(d) $\mu^- \to e^- + \gamma$
not valid because it violates electron and muon lepton number conservation

(e) $e^+ + e^- \to \gamma$
valid

4. Nuclear β-decay

(a)

$$^A_Z X \to ^A_{Z+1} Y + e^-$$

From the conservation of momentum

$$p^\mu_X = p^\mu_Y + p^\mu_e \quad \text{or} \quad p^\mu_Y = p^\mu_X - p^\mu_e$$

squaring both sides

$$p_Y^2 = p_X^2 + p_e^2 - 2p_X \cdot p_e$$

and since

$$p_X^2 = m_X^2 c^2, \quad p_Y^2 = m_Y^2 c^2, \quad p_e^2 = m_e^2 c^2$$

and

$$p_X p_e = \frac{E_X}{c}\frac{E_e}{c} - \mathbf{p}_X \cdot \mathbf{p}_e$$

but we know that particle X has momentum $\mathbf{p}_X = 0$ and rest mass $E_X = m_X c^2$ so

$$m_Y^2 c^2 = m_X^2 c^2 + m_e^2 c^2 - 2m_X E_e$$

solving for the Energy of the outgoing particle is

$$\boxed{E_e = \frac{m_X^2 + m_e^2 - m_Y^2}{2m_X} c^2}$$

to find the momentum of the outgoing electron we use energy-momentum relation

$$E_e^2 = |\mathbf{p}_e|^2 c^2 + m_e^2 c^4$$

or

$$|\mathbf{p}_e|^2 = \frac{E_e^2}{c^2} - m_e^2 c^2$$

using the energy of the outgoing electron we found earlier:

$$\begin{aligned}
|\mathbf{p}_e|^2 &= c^2 \left(\frac{m_X^2 + m_e^2 - m_Y^2}{2m_X}\right)^2 - m_e^2 c^2 \\
&= \frac{c^2}{4m_X^2}(m_X^2 + m_e^2 - m_Y^2)^2 - \left(\frac{c^2}{4m_X^2}\right) 4m_X^2 m_e^2 \\
&= \frac{c^2}{4m_X^2}(m_X^4 + m_e^4 + m_Y^4 + 2m_X^2 m_e^2 - 2m_X^2 m_Y^2 - 2m_e^2 m_Y^2 - 4m_X^2 m_e^2) \\
&= \frac{c^2}{4m_X^2}(m_X^4 + m_e^4 + m_Y^4 - 2m_X^2 m_e^2 - 2m_X^2 m_Y^2 - 2m_e^2 m_Y^2)
\end{aligned}$$

and therefore the momentum of the outgoing electron is

$$\boxed{|\mathbf{p}_e| = \frac{c}{2m_X}\sqrt{m_X^4 + m_e^4 + m_Y^4 - 2m_X^2 m_e^2 - 2m_X^2 m_Y^2 - 2m_e^2 m_Y^2}}$$

(b) For the decay including an anti-neutrino

$$^A_Z X \rightarrow ^A_{Z+1} Y + e^- + \bar{\nu}_e$$

For the massless neutrino, the energy is

$$E_\nu = |\mathbf{p}_\nu| c$$

or from planck's relation

$$E_\nu = h\nu = \frac{hc}{\lambda}$$

so the energy of the neutrino is

$$\boxed{E_\nu = \frac{6.63 \times 10^{-34}\,\mathrm{J\,s} \cdot 3 \times 10^8\,\mathrm{m\,s^{-1}}}{10^{-15}\,\mathrm{m}} = 1.99 \times 10^{-10}\,\mathrm{J} = 1240\,\mathrm{MeV}}$$

and the momentum of the neutrino is

$$\boxed{|\mathbf{p}_\nu| = \frac{E_\nu}{c} = \frac{1.99 \times 10^{-10}\,\mathrm{J}}{3 \times 10^8\,\mathrm{m\,s^{-1}}} = 6.63 \times 10^{-19}\,\mathrm{kg\,m\,s^{-1}} \quad \text{or} \quad 1240\,\frac{\mathrm{MeV}}{\mathrm{c}}}$$

This is much larger compared to the typical neutrino energy (keV). This means that the neutrino could not have come from inside the nucleus.
Using the Heisenberg uncertainty principle

$$\Delta p \Delta x \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta p \geq \frac{h}{4\pi \Delta x}$$

and the typical size of a nucleus is $\Delta x \approx 1 \times 10^{-15}\,\mathrm{m}$ so

$$\Delta p \geq \frac{6.63 \times 10^{-34}\,\mathrm{J\,s}}{4\pi \cdot 10^{-15}\,\mathrm{m}} = 0.53\,\frac{\mathrm{J}}{\mathrm{m/s}}$$

or in more convenient units $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$ and $\mathrm{c} = 3 \times 10^8\,\mathrm{m/s}$:

$$\Delta p \geq 0.53\,\frac{\mathrm{J}}{\mathrm{m/s}}\,\frac{1\,\mathrm{eV}}{1.6 \times 10^{-19}\,\mathrm{J}}\,\frac{3 \times 10^8\,\mathrm{m/s}}{\mathrm{c}}$$

$$\boxed{\Delta p \geq 99\,\frac{\mathrm{MeV}}{\mathrm{c}}}$$

and the energy of the neutrino (a massless particle) is

$$\boxed{E_\nu = |\mathbf{p}_\nu| c \geq 99\,\mathrm{MeV}}$$

Compared to the typical neutrino energy of 1 keV, this is much larger and thus the neutrino could not have come from inside the nucleus.

Homework 2

Due 1/31

1. Muon Decay
The minimum energy of the electron would be equivalent to the rest mass

$$E_{\text{min}} = m_e c^2 = 9.11 \times 10^{-31}\,\text{kg} \cdot (3 \times 10^8\,\text{m/s})^2 = 8.2 \times 10^{-14}\,\text{J} * \frac{1\text{eV}}{1.6 \times 10^{-19}\,\text{J}} = \boxed{0.512\,\text{MeV}}$$

The maximum energy of the electron would be when the electron moves in one direction and both neutrinos move together in the opposing direction. Thus we can treat this like a two body decay with the energy of the electron:

$$E_e = \frac{m_\mu^2 + m_e^2 - M^2}{2m_u} c^2$$

where M is the sum of the masses of the neutrinos $M = 0$ (neutrinos are massless lol). Using the mass of the muon and electron:

$$m_e = 9.11 \times 10^{-31}\,\text{kg} \cdot \frac{(3 \times 10^8\,\text{m/s})^2}{1.6 \times 10^{-19}\,\text{J/eV}} = 0.51\,\text{MeV}$$
$$m_\mu = 1.88 \times 10^{-28}\,\text{kg} = 106\,\text{MeV}$$

The maximum energy of the electron is then

$$E_{\text{max}} = \frac{m_\mu^2 + m_e^2}{2m_\mu} c^2 = \boxed{53\,\text{MeV}}$$

2. (a) For the head-on collision, we know that the initial energy is the sum of the rest masses of the protons and the minimum energy required to produce the antiproton. Thus

$$\begin{aligned} E_i &= E_f \\ 2m_p c^2 + E_{\text{min}} &= 4m_p c^2 \\ E_{\text{min}} &= 2m_p c^2 = \boxed{1.88\,\text{GeV}} \end{aligned}$$

(b) For the fixed target, we first look at the total momentum four vector before the collision: The zeroth component of the total momentum is

$$p^0 = \frac{E}{c} + \frac{E_{rest}}{c} = \frac{E}{c} + m_p c$$

thus the four-vector before the collision is

$$p^\mu = \left(\frac{E}{c} + m_p c, |\mathbf{p}|\right)$$

to find the four-vector after the collision, we can use the center of momentum frame (CM) where the two protons are viewed as going towards each other at the same speed like part (a), so the total three-vector momentum is zero. Thus the zeroth component of the four-vector is jus the sum of the rest masses of the protons and the antiproton

$$p^{\mu'} = (4mc, 0)$$

we can then exploit the invariant dot product of the four-vectors to find the minimum energy

$$\begin{aligned} p^\mu p_\mu &= p^{\mu'} p'_\mu \\ \left(\frac{E}{c} + m_p c\right)^2 - |\mathbf{p}|^2 &= (4mc)^2 \end{aligned}$$

where the can use the energy momentum relation

$$E^2 = |\mathbf{p}|^2c^2 + m^2c^4$$
$$|\mathbf{p}|^2 = \frac{1}{c^2}(E^2 - m^2c^4)$$

to solve for the minimum energy

$$\left(\frac{E}{c} + m_pc\right)^2 - \frac{1}{c^2}(E^2 - m_p^2c^4) = (4mc)^2$$
$$\frac{E^2}{c^2} + 2m_pE + m_p^2c^2 - \frac{E^2}{c^2} + m_p^2c^2 = 16m^2c^2$$
$$2m_pE = 14m_pc^2$$
$$E = 7m_pc^2 = \boxed{6.6\,\mathrm{GeV}}$$

Thus the head-on collision requires less energy to produce the antiproton.

3. (a) Given

$$s \equiv \frac{(p_A + p_B)^2}{c^2}, \quad t \equiv \frac{(p_A - p_C)^2}{c^2}, \quad u \equiv \frac{(p_A - p_D)^2}{c^2}$$

the sums of the Mandelstroms variables are

$$s + t + u = \frac{1}{c^2}\left[(p_A + p_B)^2 + (p_A - p_C)^2 + (p_A - p_D)^2\right]$$

where we expand the squares

$$[\,] = p_A^2 + p_B^2 + 2p_A \cdot p_B + p_A^2 + p_C^2 - 2p_A \cdot p_C + p_A^2 + p_D^2 - 2p_A \cdot p_D$$
$$= 3p_A^2 + p_B^2 + p_C^2 + p_D^2 - 2p_A \cdot p_B - 2p_A \cdot p_C - 2p_A \cdot p_D$$

spliting the $3p_A^2$ into $p_A^2 + 2p_A^2$ we can factor out the dot products

$$[\,] = p_A^2 + p_B^2 + p_C^2 + p_D^2 - 2p_A \cdot (p_A + p_B - p_C - p_D)$$

and from the conservation of momentum

$$p_A + p_B = p_C + p_D$$

so we are left with

$$[\,] = p_A^2 + p_B^2 + p_C^2 + p_D^2$$
$$= c^2(m_A^2 + m_B^2 + m_C^2 + m_D^2)$$

thus plugging [ ] back into the sum of the Mandelstroms variables gives

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

(b) In the CM frame, the total momentum is zero, so the four-vector of the total momentum is and the total energy is

$$E_T = E_A + E_B$$

we know that the momentum of the two particles are zero:

$$\mathbf{p}_A + \mathbf{p}_B = 0$$

so from the first Mandelstroms variable

$$s = \frac{(p_A + p_B)^2}{c^2} = \frac{1}{c^2}\left[\left(\frac{E_A}{c} + \frac{E_b}{c}\right)^2 + (\mathbf{p}_A + \mathbf{p}_B)^2\right]$$
$$s = \frac{E_T^2}{c^4}$$

so

$$E_T = c^2\sqrt{s}$$

(c) Since $\mathbf{p}_A + \mathbf{p}_B = 0$ and $E_A = E_B = E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$, the Mandelstroms variable s is

$$\begin{aligned}\frac{1}{c^2}(p+p)^2 &= \frac{1}{c^2}\left[\frac{(E_A + E_B)^2}{c^2} - (\mathbf{p}_A + \mathbf{p}_B)^2\right] \\ &= \frac{1}{c^2}\left[\frac{4E^2}{c^2}\right] \\ &= \frac{1}{c^2}\left[\frac{\mathbf{p}^2c^2 + m^2c^4}{c^2}\right] \\ s &= \frac{\mathbf{p}^2c^2 + m^2c^4}{c^2}\end{aligned}$$

for t we assume the angle between $\mathbf{p}_A$ and $\mathbf{p}_C$ is θ so

$$\begin{aligned}t &= \frac{1}{c^2}(p_A - p_C)^2 \\ &= \frac{1}{c^2}\left[\left(\frac{E_A}{c} - \frac{E_C}{c}\right)^2 - (\mathbf{p}_A - \mathbf{p}_C)^2\right]\end{aligned}$$

and we know that $E_A = E_C$ so the first term is zero and the second term is

$$(\mathbf{p}_A - \mathbf{p}_C)^2 = \mathbf{p}_A^2 + \mathbf{p}_C^2 - 2\mathbf{p}_A \cdot \mathbf{p}_C$$

where $\mathbf{p}_A^2 = \mathbf{p}_C^2 = \mathbf{p}^2$ and $\mathbf{p}_A \cdot \mathbf{p}_C = \mathbf{p}^2 \cos\theta$ so

$$(\mathbf{p}_A - \mathbf{p}_C)^2 = 2\mathbf{p}^2 - \mathbf{p}^2\cos\theta = 2\mathbf{p}^2(1 - \cos\theta)$$

thus

$$t = \frac{1}{c^2}[0 - 2\mathbf{p}^2(1 - \cos\theta)] = \frac{-2\mathbf{p}^2(1 - \cos\theta)}{c^2}$$

for u everything is the same but $\mathbf{p}_A \cdot \mathbf{p}_D = -\mathbf{p}^2\cos\theta$ so

$$(\mathbf{p}_A - \mathbf{p}_D)^2 = 2\mathbf{p}^2 + \mathbf{p}^2\cos\theta = 2\mathbf{p}^2(1 + \cos\theta)$$

and thus

$$u = \frac{-2\mathbf{p}^2(1 + \cos\theta)}{c^2}$$

4. First we know that the energy and momentum of a photon γ are

$$E_\gamma = h\nu = pc \quad p = \frac{h\nu}{c}$$

from the planck relation and energy-mass relation. Using conservation of energy we know that the before the collision it is the energy of the photon plus the rest mass of the electron:

$$E_i = E_f$$
$$E_\gamma + m_ec^2 = E'_\gamma + E_e$$

and using the energy momentum relation for the electron

$$h\nu + m_e c^2 = h\nu' + \sqrt{p_e^2 c^2 + (m_e c^2)^2} \tag{1}$$

and from the conservation of momentum we know that

$$\mathbf{p} = \mathbf{p}' + \mathbf{p}_e \quad \text{or} \quad \mathbf{p}_e = \mathbf{p} - \mathbf{p}'$$

where the momentum of the electron is initially zero, squaring both sides

$$\mathbf{p}_e^2 = (\mathbf{p} - \mathbf{p}')^2$$
$$p_e^2 = p^2 + p'^2 - 2\mathbf{p} \cdot \mathbf{p}'$$

where we know that the dot product of the two momenta is

$$\mathbf{p} \cdot \mathbf{p}' = pp' \cos\theta$$

so

$$p_e^2 = p^2 + p^2 - 2pp' \cos\theta \tag{2}$$

now we relate the two equations by first solving (1) for $p_e^2 c^2$

$$p_e^2 c^2 = (h\nu^2 - h\nu'^2 + m_e c^2)^2 - (m_e c^2)^2$$

and then multiplying (2) by c^2

$$p_e^2 c^2 = (pc)^2 + (p'c)^2 - 2pp'c^2 \cos\theta$$

and substituting the momentum of the photon from the energy relation $p = \frac{h\nu}{c}$

$$p_e^2 c^2 = (h\nu)^2 + (h\nu')^2 - 2h^2\nu\nu' \cos\theta$$

thus we can set the two equations equal to each other

$$(h\nu^2 - h\nu'^2 + m_e c^2)^2 - (m_e c^2)^2 = (h\nu)^2 + (h\nu')^2 - 2h^2\nu\nu' \cos\theta$$
$$(h\nu)^2 + (h\nu')^2 + (m_e c^2)^2 + 2(h^2\nu\nu' - h\nu' m_e c^2 + h\nu m_e c^2) - (m_e c^2)^2 = (h\nu)^2 + (h\nu')^2 - 2h^2\nu\nu' \cos\theta$$

where the first three terms cancel out

$$2(h^2\nu\nu' - h\nu' m_e c^2 + h\nu m_e c^2) = -2h^2\nu\nu' \cos\theta$$

dividing both sides by $2h$ and rearranging terms gives

$$m_e c^2(\nu - \nu') = h\nu\nu'(1 - \cos\theta)$$

dividing both sides again but by $m_e c\nu\nu'$ gives

$$\left(\frac{c}{\nu'} - \frac{c}{\nu}\right) = \frac{h}{m_e c}(1 - \cos\theta)$$

and since the wavelength is $\lambda = \frac{c}{\nu}$ we can solve for the outgoing wavelength

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

$$\boxed{\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\theta)}$$

Homework 3

Due 2/7

1

(a) An electron has spin $s = \frac{1}{2}$ so

$$j = \frac{3}{2}, \frac{1}{2}$$

for the 6 possible states of $|j, j_z\rangle$, the $1 \otimes \frac{1}{2}$ C-B coefficients are

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = 1\left|1, \frac{1}{2}\right\rangle = |m_l, m_s\rangle$$
$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|0, \frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|0, -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|-1, \frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle = 1\left|-1, -\frac{1}{2}\right\rangle$$

and

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1, -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|0, \frac{1}{2}\right\rangle$$
$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|0, -\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|-1, \frac{1}{2}\right\rangle$$

(b) We can see for the $\left|j = \frac{3}{2}, j_z = \frac{1}{2}\right\rangle$ state, the probability of measuring a spin $s_z = \frac{1}{2}$ is proportional to the coefficient squared

$$P = \frac{2}{3}$$

2

(a) For the following processes

- The elastic processes are (from $a \to f$)

$$\pi^+ + p \to \pi^+ + p \quad (a)$$
$$\pi^0 + p \to \pi^0 + p \quad (b)$$
$$\pi^- + p \to \pi^- + p \quad (c)$$
$$\pi^+ + n \to \pi^+ + n \quad (d)$$
$$\pi^0 + n \to \pi^0 + n \quad (e)$$
$$\pi^- + n \to \pi^- + n \quad (f)$$

- The inelastic processes are (from $g \to j$)

$$\pi^+ + n \to \pi^0 + p \quad (g)$$
$$\pi^0 + p \to \pi^+ + n \quad (h)$$
$$\pi^- + p \to \pi^0 + n \quad (i)$$
$$\pi^0 + n \to \pi^- + p \quad (j)$$

The states are linear combinations of the states from Problem 1

$$(a) \to \left|1, \frac{1}{2}\right\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle$$
$$(b) \to \left|0, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}, \frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle$$
$$(c) \to \left|-1, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$
$$(d) \to \left|1, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{3}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle$$
$$(e) \to \left|0, -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$
$$(f) \to \left|-1, -\frac{1}{2}\right\rangle = \left|-1, -\frac{1}{2}\right\rangle = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$$

Looking at the coefficients and the Isospin states, we can see the amplitudes as

$$\begin{aligned} M_a = M_f &= M_3 \\ M_b = M_e &= \frac{2}{3}M_3 + \frac{1}{3}M_1 \\ M_c = M_d &= \frac{1}{3}M_3 + \frac{2}{3}M_1 \\ M_g = M_h = M_i = M_j &= \frac{\sqrt{2}}{3}(M_3 - M_1) \end{aligned}$$

and the cross sections are proportional to the square of the amplitudes (coefficient square) or $\sigma \propto |M|^2$, but...

$$\begin{aligned} \sigma_a : \sigma_c = |M_3|^2 &: \left|\frac{1}{3}M_3 + \frac{2}{3}M_1\right|^2 \\ 9|M_3|^2 &: |M_3 + 2M_1|^2 \end{aligned}$$

so the total ratios are

$$\begin{aligned} \sigma_a : \sigma_b : \sigma_c : \sigma_d : \sigma_e : \sigma_f : \sigma_g : \sigma_h : \sigma_i : \sigma_j = \\ 9|M_3|^2 : |2M_3 + M_1|^2 : |M_3 + 2M_1|^2 : |M_3 + 2M_1|^2 : |2M_3 + M_1|^2 \\ : 9|M_3|^2 : 2|M_3 - M_1|^2 : 2|M_3 - M_1|^2 : 2|M_3 - M_1|^2 : 2|M_3 - M_1|^2 \end{aligned}$$

and for $M_3 \gg M_1$ the ratios are

$$\boxed{9 : 4 : 1 : 1 : 4 : 9 : 2 : 2 : 2 : 2}$$

(b) For $M_3 \ll M_1$ the ratios are

$$\boxed{0 : 1 : 4 : 4 : 1 : 0 : 2 : 2 : 2 : 2}$$

where (a) and (f) are very, very small cross sections in comparison.

3

(a) for protons $I = \frac{1}{2}$ and neutrons, $I = -\frac{1}{2}$ so the isospin of the α particle is $I = 0$.
(b) On the LHS the isospin of the deuteron is $I = 0$, and on the RHS the isospin of the α particle is $I = 0$ so the isospin of the pion is $I = 1$. Since the isospin is not conserved $0 \not\to 1$ the reaction is not allowed.
(c) The 4-proton state has isospin $I = 2$ and this is *does not exist* since the isospin $I = 1$ of the ^{4}Li does not exist. The 4-neutron state with isospin $I = -2$ *does not exist* as well due since the ^{4}H isotope of $I = -1$ does not exist. There can only be one possible 4-nucleon state: ^{4}He with isospin $I = 0$.

Homework 4

Due 2/14

1. (a) Imagining the electron as solid sphere, the moment of inertia is $I = \frac{2}{5}m_e r^2$. The speed at a point on its 'equator' is given by the tangential velocity $v = \omega r \implies \omega = \frac{v}{r}$. And if this electron is spinning with angular momentum $\ell = \hbar/2$, the speed of this point is

$$\ell = I\omega$$
$$\frac{\hbar}{2} = \frac{2}{5}m_e r^2 \frac{v}{r}$$
$$v = \frac{5\hbar}{4m_e r}$$

(b) If we have probed down to 10^{-18} and still haven't found any structure, then we would think that the radius of this electron solid sphere is $r < 10^{-18}$. Then the speed of the point on the equator is roughly

$$v > \frac{5 \times 10^{-34}}{4 \times 9 \times 10^{-31} \times 10^{-18}} \approx 10^{14}\,\text{m/s}$$

which is much faster than the speed of light $c = 3 \times 10^8$ m/s. So an electron is probably not spinning.

2. Given that the neutron, proton, and electron are all spin $S = 1/2$ particles, the total spin of the beta decay

$$n \to p + e^-$$

on the left side is $S_L = 1/2$ and on the right side we have $S_R = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. So the angular momentum is not conserved in this process. For the correct conservation of angular momentum, we need to include the neutrino in the decay

$$n \to p + e^- + \bar{\nu}_e$$

if we were to suppose the electron antineutrino had spin $S = 1/2$, then the total spin states could be 1/2 or 3/2 and the angular momentum would be conserved. We could also suppose it has spin $S = 3/2$ and the possible total spin states could be $1/2, 3/2$ or, $5/2$ which also conserves angular momentum. This means that any half integer spin would conserve angular momentum in the beta decay.

3. If the J_i's are Hermitian, then

$$J_i^\dagger = J_i, \quad J_j^\dagger = J_i, \quad J_k^\dagger = J_k$$

and the commutator is defined as

$$[J_i, J_j] = J_i J_j - J_j J_i$$

taking the Hermitian conjugate of the commutator

$$\begin{aligned}
[J_i, J_j]^\dagger &= (J_i J_j - J_j J_i)^\dagger \\
&= (J_i J_j)^\dagger - (J_j J_i)^\dagger \\
&= J_j^\dagger J_i^\dagger - J_i^\dagger J_j^\dagger \\
&= J_j J_i - J_i J_j \\
&= -(J_i J_j - J_j J_i) = -[J_i, J_j]
\end{aligned}$$

where on the 3rd step we know that the Hermitian adjoint (conjugate *transpose*) of a product of matrices is the product of the Hermitian adjoints in reverse order i.e. $(AB)^\dagger = B^\dagger A^\dagger$ because transposes do this. taking the Hermitian adjoint of the right hand side:

$$(i f_{ijk} J_k)^\dagger = -i f_{ijk}^\dagger J_k$$

so

$$-[J_i, J_j] = -if^{\dagger}_{ijk}J_k \rightarrow [J_i, J_j] = if^{\dagger}_{ijk}J_k$$

and in order for the commutator relation to be true, the structure constants must be real i.e. $f_{ijk} = f^{\dagger}_{ijk}$ (real numbers are Hermitian).

4. f as a sum:

$$f(x, y, z) = f_+(x, y, z) + f_-(x, y, z)$$

Parities are:

$$P(f_+) = +f_+, \quad P(f_-) = -f_- \quad P(f(x, y, z)) = f(-x, -y, -z)$$

where the parity of f is just the inversion(reflection + 180 degree rotation). Thus the parity of the RHS:

$$\begin{aligned} P(f(x, y, z)) &= P(f_+(x, y, z)) + P(f_-(x, y, z)) \\ f(-x, -y, z) &= (+f_+(x, y, z)) + (-f_-(x, y, z)) \end{aligned}$$

solving for f_+ and substituting $f_- = f - f_+$:

$$\begin{aligned} f_+(x, y, z) &= f(-x, -y, z) + f_-(x, y, z) \\ &= f(-x, -y, z) + f(x, y, z) - f_+(x, y, z) \\ 2f_+(x, y, z) &= f(-x, -y, z) + f(x, y, z) \\ f_+(x, y, z) &= \frac{1}{2}(f(x, y, z) + f(-x, -y, z)) \end{aligned}$$

and similarly by substituting the result back into $f_- = f - f_+$:

$$f_-(x, y, z) = \frac{1}{2}(f(x, y, z) - f(-x, -y, z))$$

we can see that

$$f_+(x, y, z) + f_-(x, y, z) = f(x, y, z)$$

where f_+ and f_- is an eigenfunction of the parity operator with eigenvalues $+1$ and -1 respectively.

5. (a) Given that for EM & Strong interactions, the parity must be conserved. For the decay

$$\eta \rightarrow 2\pi$$

the parity of η is $P(\eta) = -1$ and the parity of 2π is $P_{tot} = (P(\pi))^2 = (-1)^2 = 1$. So the parity is not conserved in this decay and thus forbidden for EM & Strong interactions.
(b) For the decay

$$\eta \rightarrow 3\pi$$

we can see that the parity is conserved: $P(\eta) = P_{tot} = (P(\pi))^3 = -1$. Since a G-parity violation forbids decay under Strong interactions, the G-parity of the two sides are:

$$G(\eta) = (-1)^0 C = 1(+1) = +1, \quad G(3\pi) = (-1)^3 = -1$$

so G-parity conservation is violated and the decay is forbidden under Strong interactions, but allowed for EM interactions due to Parity conservation.

1. (a) From the Gell-Mann-Nishijima formula

$$Q = I_3 + \frac{1}{2}(A+S)$$

For the uds quarks, the isospin is $I_3 = \frac{1}{2}, -\frac{1}{2}, 0$, the baryon number is $A = \frac{1}{3}$ and the strangeness is $S = 0, 0, -1$ respectively. The charges are then

$$\begin{aligned} Q_u &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} + 0\right) = \frac{2}{3} \\ Q_d &= -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} + 0\right) = -\frac{1}{3} \\ Q_s &= 0 + \frac{1}{2}\left(\frac{1}{3} - 1\right) = -\frac{1}{3} \end{aligned}$$

(b) The antiparticle will have the opposite charge $Q_{\bar{q}} = -Q_q$, baryon number $A_{\bar{q}} = -\frac{1}{3}$ and strangeness $S_{\bar{q}} = 0, 0, 1$, so the isospin states are

$$\begin{aligned} Q_{\bar{u}} &= -\frac{2}{3} = I_3 + \frac{1}{2}\left(-\frac{1}{3} + 0\right) \implies I_3 = -\frac{1}{2} \\ Q_{\bar{d}} &= \frac{1}{3} = I_3 + \frac{1}{2}\left(-\frac{1}{3} + 0\right) \implies I_3 = \frac{1}{2} \\ Q_{\bar{s}} &= \frac{1}{3} = I_3 + \frac{1}{2}\left(-\frac{1}{3} + 1\right) \implies I_3 = 0 \end{aligned}$$

so the isospin assignments $|I, I_3\rangle$ are

$$\bar{u} = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \quad \bar{d} = \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \quad \bar{s} = |0, 0\rangle$$

2. (a) For a the charged kaon

$$K^- \Leftrightarrow K^+$$

the charge is not conserved, so they cannot interconvert, so only the neutral mesons can mix. (b) We don't observe baryon-antibaryon interconversion because it violates baryon number conservation. (c) There is no mixing of neutral strange vector mesons because the K^{*0} and $\bar{K}^{*0}$ have different strangeness $S = +1, -1$, so they cannot mix due to strangeness conservation.

3. From the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

and the time reversal operator $|\psi(t)\rangle = T|\psi(-t)\rangle$, so the equation is

$$i\hbar\frac{\partial}{\partial(t)}T|\psi(-t)\rangle = HT|\psi(-t)\rangle$$

and the time derivative of the time-reversed state using chain rule

$$i\hbar\frac{\partial}{\partial(t)}|\psi(-t)\rangle = -i\hbar\frac{\partial}{\partial t}|\psi(-t)\rangle$$

for the right side, since T and H commute

$$\begin{aligned} -i\hbar\frac{\partial}{\partial t}|\psi(-t)\rangle &= TH|\psi(-t)\rangle \\ -i\hbar\frac{\partial}{\partial t}T|\psi(t)\rangle &= THT|\psi(t)\rangle = TTH|\psi(t)\rangle \end{aligned}$$

and since $T^2 = 1$

$$-i\hbar\frac{\partial}{\partial t}T\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

or

$$Tc = c^* T$$

4. Given the Hamiltonian

$$H = -\frac{1}{|\mathbf{J}|}(\mu\mathbf{J}\cdot\mathbf{B} + d\mathbf{J}\cdot\mathbf{E})$$

(a) From Maxwells equations

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\rho \\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\end{aligned}$$

under time reversal $t \to -t$ the electric field is T-even $E \to E$ and the magnetic field is T-odd $B \to -B$. From angular momentum

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = \mathbf{r}\times m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

so under time reversal $t \to -t$ angular momentum is T-odd $\mathbf{L} \to -\mathbf{L}$, and since spin angular momentum is T-odd by the right hand rule, the total angular momentum is T-odd $\mathbf{J} \to -\mathbf{J}$.

For the parity, the magnetic field and angular momentum are even under parity since they are pseudovectors, and the electric field is odd under parity as a vector.

The charge conjugation of the electric field is $E \to -E$ and the magnetic field is $B \to -B$ since the antiparticle will have the opposite charge. The total angular momentum is invariant under charge conjugation $\mathbf{J} \to \mathbf{J}$ since the antiparticle will have the same spin.

$$\begin{aligned}C &: \mathbf{E} \to -\mathbf{E}, \quad \mathbf{B} \to -\mathbf{B}, \quad \mathbf{J} \to \mathbf{J} \\ P &: \mathbf{E} \to -\mathbf{E}, \quad \mathbf{B} \to \mathbf{B}, \quad \mathbf{J} \to \mathbf{J} \\ T &: \mathbf{E} \to \mathbf{E}, \quad \mathbf{B} \to -\mathbf{B}, \quad \mathbf{J} \to -\mathbf{J}\end{aligned}$$

(b) Since the Hamiltonian is invariant under time reversal so μ is T-even (odd times odd is even and even times even is even) and d is T-odd. The Hamiltonian is also invariant under parity so μ is P-even and d is P-odd. For charge conjugation, the Hamiltonian is invariant so μ is C-odd and d is C-odd.

$$\begin{aligned}C &: \mu \to \mu, \quad d \to -d \\ P &: \mu \to \mu, \quad d \to -d \\ T &: \mu \to -\mu, \quad d \to -d\end{aligned}$$

1. From the Lagrangian

$$\mathcal{L} = \frac{1}{m}(\mathbf{L} \times \mathbf{p}) + \frac{\kappa \mathbf{r}}{r} = \frac{1}{2m}(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) + \frac{\kappa \mathbf{r}}{r}$$

taking the derivative of a cross product is given by the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L} \times \mathbf{p}) = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{L} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

so

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \frac{1}{m}\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{L} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa \mathbf{r}}{r}\right)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa \mathbf{r}}{r}\right) = \kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right)$$

and since the angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L} \times \mathbf{p}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) \times \mathbf{p} + (\mathbf{r} \times \mathbf{p}) \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

and since

$$\begin{aligned} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} = -\boldsymbol{\nabla} V = -\frac{\kappa}{r^2} &= -\frac{\kappa \mathbf{r}}{r^3} \\ \frac{\mathrm{d}L}{\mathrm{d}t} &= 0 \end{aligned}$$

we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L} \times \mathbf{p}) &= -m\frac{\kappa}{r^3}(r \times \dot{\mathbf{r}}) \times \mathbf{r} \\ &= -m\frac{\kappa}{r^3}[\dot{\mathbf{r}}r^2 - \mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}})] \end{aligned}$$

where

$$\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \cdot \mathbf{r}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = r\dot{r}$$

so

$$\begin{aligned} \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \frac{1}{m}\left[-m\frac{\kappa}{r^3}\left(\dot{\mathbf{r}}r^2 - \mathbf{r}(r\dot{r})\right)\right] + \kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right) \\ &= -\kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right) + \kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right) \\ &= 0 \end{aligned}$$

2. (a) Given

$$M_{\text{meson}} = m_1 + m_2 + A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}$$

and

$$A = \left(\frac{2m_u}{\hbar}\right)^2 159\,\text{MeV/c}^2, \quad m_u = m_d = 308\,\text{MeV/c}^2, \quad m_s = 483\,\text{MeV/c}^2$$

Finding $S_1 \cdot S_2$:

$$\begin{aligned}\mathbf{S}_T &= \mathbf{S}_1 + \mathbf{S}_2\\ \mathbf{S}^2 &= (\mathbf{S}_1 + \mathbf{S}_2)^2\\ &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2\end{aligned}$$

Where from the operator

$$\begin{aligned}[J^2, J_z] = 0 \qquad |j, m\rangle\\ J_z\,|j,m\rangle + \hbar m\,|j,m\rangle\\ J^2\,|j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle\end{aligned}$$

so the eigenvalues of $\mathbf{S}$ are $\frac{1}{2}(\frac{1}{2}+1)\hbar^2$:

$$\begin{aligned}\mathbf{S}^2 &= \frac{3}{2}\hbar^2 + \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2\\ &= \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2\end{aligned}$$

so for the scalar case $s = 0$, $\mathbf{S}^2 = 0$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = -\frac{3}{4}\hbar^2$$

and for the vector case $s = 1$, $\mathbf{S}^2 = 2\hbar^2$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(2 - \frac{3}{2}\hbar^2\right) = \frac{1}{4}\hbar^2$$

So

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \begin{cases} -\frac{3}{4}\hbar^2 & \text{spin-0}\\ \frac{1}{4}\hbar^2 & \text{spin-1}\end{cases}$$

So for pseudoscalar cases $\mathbf{S}_1 \cdot \mathbf{S}_2 = -\frac{3}{4}\hbar^2$:

- π (ud)

$$\begin{aligned}M_\pi &= 2m_u + A\frac{-3}{4m_u m_d} = 2(308) + 4(308)^2 159\frac{-3}{4(308)(308)}\\ &= 139\,\text{MeV/c}^2\end{aligned}$$

- $K+$ (us)

$$\begin{aligned}M_{K+} &= (308) + 483 - (308)^2 159\frac{3}{308(483)}\\ &= 487\,\text{MeV/c}^2\end{aligned}$$

- K^0 (ds)

$$\begin{aligned} M_{K^0} &= (308) + 483 - (308)^2 159 \frac{3}{308(483)} \\ &= 487\,\mathrm{MeV/c^2} \end{aligned}$$

- η The masses of constituent parts:
  - $u\bar{u}$ and $d\bar{d}$:

$$M_{u\bar{u}} = M_{d\bar{d}} = 139\,\mathrm{MeV/c^2}$$

  - $s\bar{s}$:

$$\begin{aligned} M_{s\bar{s}} &= 2(483) - (308)^2 159 \frac{3}{483^2} \\ &= 772\,\mathrm{MeV/c^2} \end{aligned}$$

  so

$$\begin{aligned} M_\eta &= \frac{1}{6}(139) + \frac{1}{6}(139) + \frac{4}{6}(772) \\ &= 561\,\mathrm{MeV/c^2} \end{aligned}$$

- η'

$$\begin{aligned} M_{\eta'} &= \frac{1}{3}(139) + \frac{1}{3}(139) + \frac{1}{3}(772) \\ &= 350\,\mathrm{MeV/c^2} \end{aligned}$$

And for vector cases $\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{4}\hbar^2$:

- ρ (ud):

$$\begin{aligned} M_\rho &= 2(308) + 4(308)^2 159 \frac{1}{4(308)(308)} \\ &= 775\,\mathrm{MeV/c^2} \end{aligned}$$

- K^{*+} (us):

$$\begin{aligned} M_{K^{*+}} &= (308) + 483 + (308)^2 159 \frac{1}{308(483)} \\ &= 892\,\mathrm{MeV/c^2} \end{aligned}$$

- K^{*0} (ds):

$$M_{K^{*0}} = 892\,\mathrm{MeV/c^2}$$

- $\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$:

$$\begin{aligned} M_\omega &= \frac{1}{2}(775) + \frac{1}{2}(775) \\ &= 775\,\mathrm{MeV/c^2} \end{aligned}$$

- $\phi = s\bar{s}$:

$$\begin{aligned} M_\phi &= 2(483) + 308^2 159 \frac{1}{483^2} \\ &= 1031\,\mathrm{MeV/c^2} \end{aligned}$$

(b) With $m_c = 1250\,\mathrm{MeV/C^2}$ and the same stuff from part (a) the pseudoscalars are:

- $\eta_c(c\bar{c})$:

$$\begin{aligned} M_{\eta_c} &= 2(1250) - (308)^2 159 \frac{3}{1250^2} \\ &= 2471\,\mathrm{MeV/c^2} \end{aligned}$$

- $D^0(c\bar{u})$:

$$\begin{aligned} M_{D^0} &= 1250 + 308 - (308)^2 159 \frac{3}{308(1250)} \\ &= 1440\,\mathrm{MeV/c^2} \end{aligned}$$

- $D_s^+(c\bar{s})$:

$$\begin{aligned} M_{D_s^+} &= 1250 + 483 - (308)^2 159 \frac{3}{483(1250)} \\ &= 1658\,\mathrm{MeV/c^2} \end{aligned}$$

and the vector mesons are:

- $J/\psi(c\bar{c})$:

$$\begin{aligned} M_{J/\psi} &= 2(1250) + (308)^2 159 \frac{1}{1250^2} \\ &= 2510\,\mathrm{MeV/c^2} \end{aligned}$$

- $D^{*0}(c\bar{u})$:

$$\begin{aligned} M_{D^{*0}} &= 1250 + 308 + (308)^2 159 \frac{1}{308(1250)} \\ &= 1597\,\mathrm{MeV/c^2} \end{aligned}$$

- $D_s^{*+}(c\bar{s})$:

$$\begin{aligned} M_{D_s^{*+}} &= 1250 + 483 + (308)^2 159 \frac{1}{483(1250)} \\ &= 1758\,\mathrm{MeV/c^2} \end{aligned}$$

(c) Now the beauty mesons with $m_b = 4.5\,\mathrm{GeV/c^2} = 4500\,\mathrm{MeV/c^2}$: Pseudoscalars

- $\eta_b(b\bar{b})$:

$$\begin{aligned} M_{\eta_b} &= 2(4500) - (308)^2 159 \frac{3}{4500^2} \\ &= 8998\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^+(u\bar{b})$:

$$\begin{aligned} M_{B^+} &= 308 + 4500 - (308)^2 159 \frac{3}{308(4500)} \\ &= 4775\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^0(d\bar{d})$:

$$M_{B^0} = 4775\,\mathrm{MeV/c^2}$$

- $B_c^+(c\bar{b})$:

$$\begin{aligned} M_{B_c^+} &= 1250 + 4500 - (308)^2 159 \frac{3}{1250(4500)} \\ &= 5742\,\mathrm{MeV/c^2} \end{aligned}$$

and vector mesons are:

- $\Upsilon(b\bar{b})$:

$$\begin{aligned} M_{\Upsilon} &= 2(4500) + (308)^2 159 \frac{1}{4500^2} \\ &= 9001\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^{*+}(u\bar{b})$:

$$\begin{aligned} M_{B^{*+}} &= 308 + 4500 + (308)^2 159 \frac{1}{308(4500)} \\ &= 4819\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^{*0}(d\bar{b})$:

$$M_{B^{*0}} = 4819\,\mathrm{MeV/c^2}$$

- $B_c^{*+}(c\bar{b})$:

$$\begin{aligned} M_{B_c^{*+}} &= 1250 + 4500 + (308)^2 159 \frac{1}{1250(4500)} \\ &= 5752\,\mathrm{MeV/c^2} \end{aligned}$$

(d) Comparing all these masses compared to the PDB

Meson	Calculated Mass	PDB Mass
π	$139\,\mathrm{MeV/c^2}$	$139.570\,61\,\mathrm{MeV/c^2}$
K^+	$487\,\mathrm{MeV/c^2}$	$493.677\,\mathrm{MeV/c^2}$
K^0	$487\,\mathrm{MeV/c^2}$	$497.614\,\mathrm{MeV/c^2}$
η	$561\,\mathrm{MeV/c^2}$	$547.862\,\mathrm{MeV/c^2}$
η'	$350\,\mathrm{MeV/c^2}$	$957.78\,\mathrm{MeV/c^2}$
ρ	$775\,\mathrm{MeV/c^2}$	$775.26\,\mathrm{MeV/c^2}$
K^{*+}	$892\,\mathrm{MeV/c^2}$	$891.66\,\mathrm{MeV/c^2}$
K^{*0}	$892\,\mathrm{MeV/c^2}$	$895.55\,\mathrm{MeV/c^2}$
ω	$775\,\mathrm{MeV/c^2}$	$782.65\,\mathrm{MeV/c^2}$
ϕ	$1031\,\mathrm{MeV/c^2}$	$1019\,\mathrm{MeV/c^2}$
η_c	$2471\,\mathrm{MeV/c^2}$	$2980\,\mathrm{MeV/c^2}$
D^0	$1440\,\mathrm{MeV/c^2}$	$1864\,\mathrm{MeV/c^2}$
D_s^+	$1658\,\mathrm{MeV/c^2}$	$1968\,\mathrm{MeV/c^2}$
J/ψ	$2510\,\mathrm{MeV/c^2}$	$3096\,\mathrm{MeV/c^2}$
D^{*0}	$1597\,\mathrm{MeV/c^2}$	$2006\,\mathrm{MeV/c^2}$
D_s^{*+}	$1758\,\mathrm{MeV/c^2}$	$2112\,\mathrm{MeV/c^2}$
η_b	$8998\,\mathrm{MeV/c^2}$	$9398\,\mathrm{MeV/c^2}$
B^+	$4775\,\mathrm{MeV/c^2}$	$5279\,\mathrm{MeV/c^2}$
B^0	$4775\,\mathrm{MeV/c^2}$	$5279\,\mathrm{MeV/c^2}$
B_c^+	$5742\,\mathrm{MeV/c^2}$	$6274\,\mathrm{MeV/c^2}$
Υ	$9001\,\mathrm{MeV/c^2}$	$9460\,\mathrm{MeV/c^2}$
B^{*+}	$4819\,\mathrm{MeV/c^2}$	$5325\,\mathrm{MeV/c^2}$
B^{*0}	$4819\,\mathrm{MeV/c^2}$	$5324\,\mathrm{MeV/c^2}$
B_c^{*+}	$5752\,\mathrm{MeV/c^2}$	-

The light mesons are all pretty good estimates except for η'... For heavier mesons the estimates are not as good, but they are within the ballpark of the actual masses. Why is the η' so far off?