# 1 Electric Fields in Matter

## 1.1 Polarization

### 1.1.1 Dielectrics

In dieletrics, "All charges are attached to specific atoms or molecules" (Griffith, pg.166)

### 1.1.2 Induced Dipoles

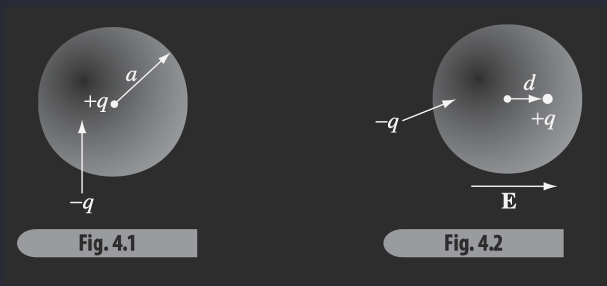



Figure 1.1: Left: Simple Nucleus $+q$ surrounded by spherical cloud $-q$ of radius $a$. Right: In an external electric field $E$ the nucleus shifts right $d$ and the cloud shifts left.

For a simple model of the atom (Fig. 1.1), the electric field at $d$ is

$$E_d = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

where the dipole moment is $p = qd$. We have equilibrium when

$$F_{\text{ext}} = qE_{\text{ext}} = q_- E_d$$

So using the dipole moment

$$|\mathbf{p}| = qd = 4\pi\epsilon_0 a^3 E_{\text{ext}} = \alpha E_{\text{ext}}$$

Here we have this "atomic polaritability"

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon v$$

where $v$ is the volume of the atom. Comment: this crude approximation is still accurate by a factor of 4.

In general we have a vector

$$\mathbf{p} = \hat{\alpha}\mathbf{E}$$

where $\hat{\alpha}$ is the polarizability tensor. For a linear dielectric relation between $E$ and $p$ we get

$$\hat{\alpha} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}$$

### 1.1.3 Alignment of Polar Molecules

The dipole will experience a torque in and $E$-field

$$\begin{aligned} \mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= \left(\frac{\mathbf{d}}{2} \times q\mathbf{E}\right) + \left(-\frac{\mathbf{d}}{2} \times q\mathbf{E}\right) \\ &= q\mathbf{d} \times \mathbf{E} \end{aligned}$$

thus

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

which implies that there is a force that acts to align $\mathbf{p} \parallel \mathbf{E}$.

**WHat if E is not uniform?**

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q\mathbf{E} + q\mathbf{E} = q\Delta E$$

assuming small $d$ in $E_x$, then

$$\Delta E_x = (\boldsymbol{\nabla}\mathbf{E}_x) \cdot \mathbf{d}$$

So the total change in the field is

$$\Delta \mathbf{E} = (\mathbf{d} \cdot \boldsymbol{\nabla})\mathbf{E}$$

thus

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$$

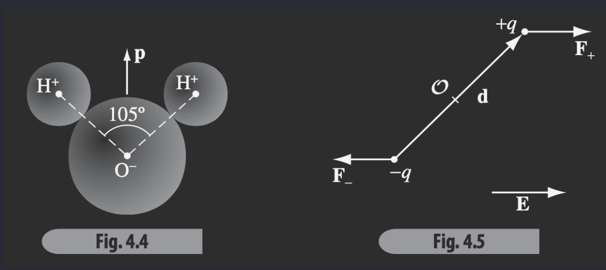



Figure 1.2: Dipole moment in oxygen molecule, and in an electric field.

**Example:** Problem 4.5 Using the method of images

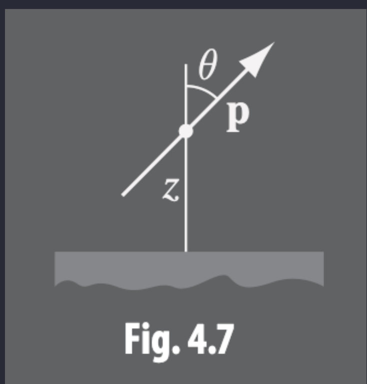



Figure 1.3: Infinitely grounded conductor with dipole at an angle $\theta$ from the normal plane and nailed in place.

Where look at $\mathbf{p}_i$ coordinate (pointing up) thus $2z$ away we have the dipole pointing perpendicular to the image dipole i.e.

$$\mathbf{E}_i = \frac{1}{4\pi\epsilon_0}\frac{p_i}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right)$$

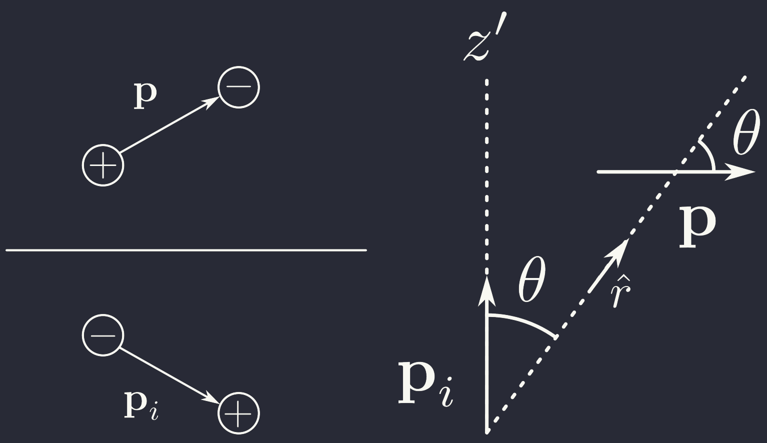



Figure 1.4: Left: Method of images using image dipole. Right: Coordinate using image dipole up as $z'$.

where

$$\mathbf{p} = p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\theta}$$

So the torque $\mathbf{N} = \mathbf{p}\times\mathbf{E}_i$ is

$$\begin{aligned}
\mathbf{N} &= \frac{1}{4\pi\epsilon_0}\frac{p^2}{(2z)^3}(\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) \times \left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{p^2}{(2z)^3}\left(\cos\theta\sin\theta\hat{\boldsymbol{\phi}} + 2\cos\theta\sin\theta(-\hat{\boldsymbol{\phi}})\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{p^2\cos\theta\sin\theta}{(2z)^3}(-\hat{\boldsymbol{\phi}}) \\
&= \frac{1}{4\pi\epsilon_0}\frac{p^2\sin(2\theta)}{8\pi\epsilon_0(8z^3)}(-\hat{\boldsymbol{\phi}})
\end{aligned}$$

So

$$\begin{cases} 0<\theta<\frac{\pi}{2} & N\sim -\hat{\boldsymbol{\phi}} \\ \frac{\pi}{2}<\theta<\pi & N\sim \hat{\boldsymbol{\phi}} \end{cases}$$

Which means the dipole can either align perdendicularly up or down depending on the angle $\theta$.

### 1.1.4 Polarization

$$\mathbf{P} \equiv \text{dipole moment per unit volume}$$

i.e. the little $\mathbf{p}$ is

$$\mathbf{p} = \mathbf{P}\,\mathrm{d}\tau$$

## 1.2 The Field of a Polarized Object

### 1.2.1 Bound Charges

For a single dipole

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

and using the dipole moment per unit volume def:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\mathrm{d}\tau'$$

recalling the math fact

$$\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

thus

$$V = \frac{1}{4\pi\epsilon_0}\int \mathbf{P}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\mathrm{d}\tau'$$

using another math fact

$$\boldsymbol{\nabla}\cdot(F\mathbf{A}) = F(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla}F)$$

So we can rewrtite the integral

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left[\int_V \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)\mathrm{d}\tau' - \int_V \frac{1}{\mathscr{r}}\boldsymbol{\nabla}\cdot\mathbf{P}\mathrm{d}\tau'\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{1}{\mathscr{r}}\mathbf{P}\cdot\mathrm{d}\mathbf{a}' - \int_V \frac{1}{\mathscr{r}}\boldsymbol{\nabla}\cdot\mathbf{P}\mathrm{d}\tau'\right] \end{aligned}$$

where we used the divergence theorem for the first term. For charge densities

$$\begin{cases} \text{surface charge} & \sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \\ \text{volume charge} & \rho_b \equiv -\boldsymbol{\nabla}\cdot\mathbf{P} \end{cases}$$

then

$$V = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\sigma_b}{\mathscr{r}}\mathrm{d}a' + \int_V \frac{\rho_b}{\mathscr{r}}\mathrm{d}\tau'\right]$$

### 1.2.2 Physical Interpretation of Bound Charges

So for a charge neutral sphere with an applied $E$-field, we can imagine this sphere as two oppositely charge spheres superimposed on each other but slighly shifted (Fig. 1.5). Thus we can imagine a collection of


Figure 1.5: Charge neutral sphere with applied $E$-field.

dipoles for each atom in a material with alternating charges. This is actually wrong (read Berry Phases in Electronic Structure Theory by David Vanderbilt).

**Example:** Find $\mathbf{E}$ of a uniformly polarized sphere of radius $R$.

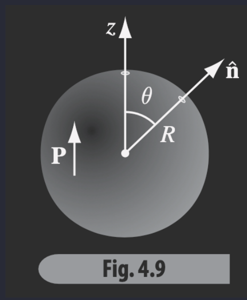



Figure 1.6: Uniformly polarized sphere of radius $R$.

We choose $\mathbf{P} \propto \hat{\mathbf{z}}$ as shown in Fig. 1.6. The bound volume and surface charges are

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = 0 \quad \text{but} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta$$

Thus using the result from before

$$V = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\sigma_b}{\mathscr{r}} \mathrm{d}a' + \int_V \frac{\rho_b}{\mathscr{r}} \mathrm{d}\tau'\right]$$

or

$$V(r\theta) = \begin{cases} \frac{P}{3\epsilon_0} r\cos\theta & r \leq R \\ \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & r > R \end{cases}$$

Recalling $z = r\cos\theta \quad \mathbf{E} = -\boldsymbol{\nabla} V$ we get

$$\mathbf{E}_{\text{in}} = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}$$

and for outside the potential is

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

with the total dipole moment

$$\mathbf{p} = \frac{4\pi}{3}R^3\mathbf{P}$$

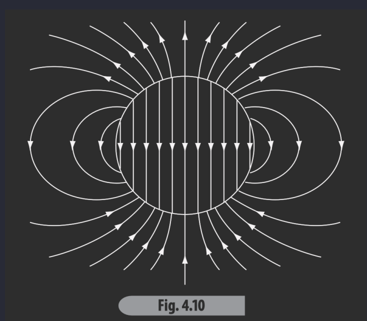



Figure 1.7: Polarized sphere in an external $E$-field.

So the polarized sphere is similar to the spherical conductor but with $E$-fields inside it (Fig. 1.7).

## 1.3 The Displacement Field

$$\rho_b = -\nabla \cdot \mathbf{P} \quad \text{and} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

The $\rho_{\text{free}} \to$ anything *not* due to Polarization

$$\rho = \rho_b + \rho_f$$

and from Gauss' Law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\implies \epsilon_0 \nabla \cdot \mathbf{E} = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

where $\mathbf{E}$ is the total electric field. Moving some terms around we get

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

where we now define the electric displacement field

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

which has the same laws as $\mathbf{E}$:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f} \qquad \boxed{\oint \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{\text{f, enc}}}$$

**Example:** Long straight wire, with unform $\lambda$ (charge per length), surrounded by rubber insulation out to radius $a$; Find $\mathbf{D}$

Using the Gaussian surface (cylinder) of length $l$ and radius $s$ so the enclosed charge is

$$\oint \mathbf{D} \cdot \mathrm{d}\mathbf{a} = 2\pi s l \mathbf{D} = \lambda l$$
$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$

$\implies$ holds for both $s \leq a$ and $s > a$.

For $s > a$

$$\mathbf{P}_{\text{out}} = 0 \implies \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

But for inside $s \leq a$ we can't determine $\mathbf{P}_{\text{in}}$ yet!

**Comments** While the two equations

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

are similar, the $E$-field has a Coulomb law

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$$

but there is no equivalent for $\mathbf{D}$

$$\cancel{\mathbf{D} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}}\hat{\mathscr{r}}$$

since there is this tensor relation $\mathbf{p} = \hat{\alpha}\mathbf{E}$. Furthermore, the curl is also different:

$$\nabla \times \mathbf{D} = \nabla \times (\epsilon_0 \mathbf{E} + \mathbf{P}) = \cancel{\epsilon_0 \nabla \times \mathbf{E}} + \nabla \times \mathbf{P}$$

since $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0$.

**What is D?** Units $\frac{\text{C}}{\text{m}^2}$ which is the same as $\sigma$ (surface charge density).

## 1.4 Linear Dielectrics

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

where $\chi_e$ is the electric susceptibility (dimensionless). For here, we will assume linear (isotropic & homogeneous) dielectrics.

- vacuum: $\chi_e = 0$
- air: 1.00054
- salt: $\sim 4.9$
- Si: $\sim 11$
- water: $\sim 80$ (water is already polarized!)
- $SrTiO_3$: $\sim 50,000$ at low temperatures

**E** is the *total* electric field.

Starting with

$$\begin{aligned}
\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\
&= \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} \\
&= \epsilon_0 (1 + \chi_e) \mathbf{E} \quad \epsilon = \epsilon_0 (1 + \chi_e) \\
&= \epsilon \mathbf{E}
\end{aligned}$$

where $\epsilon$ is the *permitivity* of the material and the relative permitivity or *dielectric constant* is

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

**Example:** Metal sphere with charge $Q$, radius $a$, surrounded by a linear dielectric, $\epsilon$, out to radius $b$. Find potential at the center (relative to $\infty$).

Inside the metal sphere $\mathbf{E} = \mathbf{P} = \mathbf{D} = 0$. Drawing the Gaussian surface between the sphere and the dielectric we get

$$\oint \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{\text{f, enc}}$$

for $r > a$

$$\mathbf{D} = \frac{Q}{4\pi r^2} \hat{\mathbf{r}}$$

Now finding $\mathbf{E}$ from $\mathbf{D} = \epsilon \mathbf{E}$ i.e. in the vacuum $\epsilon = \epsilon_0$ and in the dielectric $\epsilon = \epsilon$:

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon r^2} & a < r < b \\ \frac{Q}{4\pi\epsilon_0 r^2} & r > b \end{cases}$$

The potential at the origin is therefore

$$\begin{aligned}
V &= -\int_\infty^0 \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\int_\infty^b \frac{Q}{4\pi\epsilon_0 r^2} \mathrm{d}r - \int_b^a \frac{Q}{4\pi\epsilon r^2} \mathrm{d}r - \int_a^0 0 \mathrm{d}r \\
&= \frac{Q}{4\pi} \left( \frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right)
\end{aligned}$$

Now we can find $\mathbf{P}$ since $\mathbf{E}$ is fixed by $\mathbf{D}$:

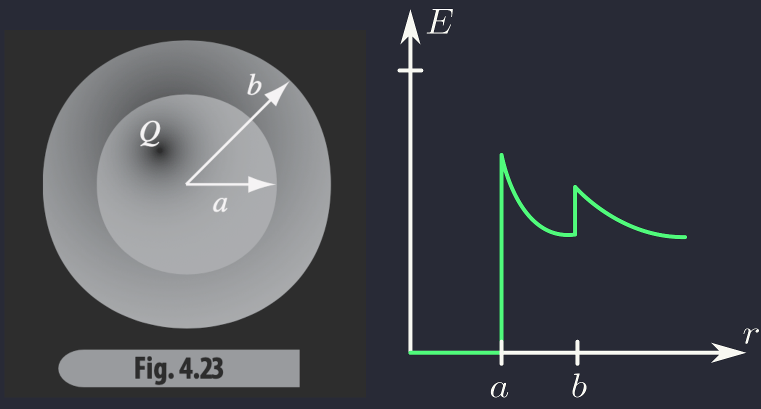
$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}} \quad \text{(in the dielectric)}$$

Thus we can get the volume bound charge

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = 0 \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 0 \quad \text{except at } r = 0$$

and

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2} & \text{outer} \\[2ex] \dfrac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2} & \text{inner} \end{cases}$$




(a) (b)

Figure 1.8: (a) Dielectric sphere surrounding metal sphere. (b) Electric field of resulting system.