

Physics 411: Mechanics

Homework: Junseo Shin

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Homework 1

Due 1/24 9pm

1. Given: the 2D Cartesian relation to polar coordinates

$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi, \quad \hat{\phi} = -\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi \quad (1)$$

We can write \mathbf{v} as a linear combination of $\hat{\mathbf{r}}$ and $\hat{\phi}$

$$\begin{aligned} \mathbf{v} &= v_r \hat{\mathbf{r}} + v_\phi \hat{\phi} \\ &= v_r (\hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi) + v_\phi (-\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi) \\ &= (v_r \cos \phi - v_\phi \sin \phi) \hat{\mathbf{x}} + (v_r \sin \phi + v_\phi \cos \phi) \hat{\mathbf{y}} \end{aligned}$$

and since we know the vector in Cartesian coordinates is

$$\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$$

we can equate the components to get

$$\begin{aligned} v_x &= v_r \cos \phi - v_\phi \sin \phi \\ v_y &= v_r \sin \phi + v_\phi \cos \phi \end{aligned}$$

multiplying the first equation by $\cos \phi$ and the second by $\sin \phi$ and adding them together

$$\begin{aligned} v_x \cos \phi &= v_r \cos^2 \phi - v_\phi \sin \phi \cos \phi \\ v_y \sin \phi &= v_r \sin^2 \phi + v_\phi \sin \phi \cos \phi \\ v_x \cos \phi + v_y \sin \phi &= v_r (\cos^2 \phi + \sin^2 \phi) \end{aligned}$$

or simply

$$v_r = v_x \cos \phi + v_y \sin \phi$$

Likewise,

$$\begin{aligned} v_y \cos \phi &= v_r \sin \phi \cos \phi + v_\phi \sin^2 \phi \\ v_x \sin \phi &= v_r \sin \phi \cos \phi - v_\phi \cos^2 \phi \end{aligned}$$

and subtracting the second equation from the first

$$v_y \cos \phi - v_x \sin \phi = v_\phi (\sin^2 \phi + \cos^2 \phi)$$

Therefore we get the components of \mathbf{v} in polar coordinates

$$\begin{aligned} v_r &= v_x \cos \phi + v_y \sin \phi \\ v_\phi &= -v_x \sin \phi + v_y \cos \phi \end{aligned}$$

Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are *constant*, the time derivatives of (1) are

$$\begin{aligned} \dot{\hat{\mathbf{r}}} &= \frac{d}{dt} \hat{\mathbf{r}} = \hat{\mathbf{x}} \frac{d}{dt} (\cos \phi) + \hat{\mathbf{y}} \frac{d}{dt} (\sin \phi) \\ &= (-\dot{\phi} \sin \phi) \hat{\mathbf{x}} + (\dot{\phi} \cos \phi) \hat{\mathbf{y}} = \dot{\phi} \hat{\phi} \end{aligned}$$

and

$$\begin{aligned} \dot{\hat{\phi}} &= \frac{d}{dt} \hat{\phi} = -\hat{\mathbf{x}} \frac{d}{dt} (\sin \phi) + \hat{\mathbf{y}} \frac{d}{dt} (\cos \phi) \\ &= (-\dot{\phi} \cos \phi) \hat{\mathbf{x}} + (-\dot{\phi} \sin \phi) \hat{\mathbf{y}} = -\dot{\phi} \hat{\mathbf{r}} \end{aligned}$$

Homework 2

Due 1/31

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Homework 3

Due 2/7 9pm

1. The Center of Mass of the system is

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r} \, dm$$

and the mass element is the mass density times the volume element

$$dm = \rho \, dV = \frac{M}{2\pi R^2} R^2 \sin \phi \, d\phi \, d\theta = \frac{M}{2\pi} \sin \phi \, d\phi \, d\theta$$

where there is no dr term because the radius is constant. Since the center of mass is symmetric about the x and y axes, $X_{cm} = Y_{cm} = 0$. The z component of the center of mass is

$$\begin{aligned} Z_{cm} &= \frac{1}{M} \int z \, dm \\ &= \frac{1}{M} \frac{M}{2\pi} \iint z \sin \phi \, d\phi \, d\theta \end{aligned}$$

where $z = R \cos \phi$ so

$$\begin{aligned} Z_{cm} &= \frac{R}{2\pi} \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \\ &\quad \text{using } u = \sin \phi \implies du = \cos \phi \, d\phi \\ &= \frac{R}{2\pi} [2\pi] \int_0^1 u \, du \\ Z_{cm} &= \frac{R}{2} \end{aligned}$$

So the COM is at $\boxed{\left(0, 0, \frac{R}{2}\right)}$

2. (a) From N3L the force of the jettisoned fuel on the rocket is equal and opposite to the thrust on the rocket from the jettisoned fuel:

$$\begin{aligned} F_{\text{fuel}} &= -F_{\text{thrust}} \\ \dot{m}v_{ex} &= -\dot{m}v_{ex} \end{aligned}$$

So using N2L, the sum of the forces on the rocket is the thrust and air resistance:

$$F = m\dot{v} = F_{\text{thrust}} - f = -\dot{m}v_{ex} - bv$$

(b) Using $\dot{m} = -k$

$$\begin{aligned} m\dot{v} &= kv_{ex} - bv \\ \frac{m}{b}\dot{v} &= \frac{kv_{ex}}{b} - v \end{aligned}$$

defining the constant $a = \frac{kv_{ex}}{b}$ and using separation of variables

$$\frac{1}{a-v} dv = \frac{b}{m} dt$$

we can write an expression for m as a function of time through using separation of variables again:

$$\begin{aligned} \frac{dm}{dt} &= -k \\ \int_{m_o}^m dm' &= -k \int_0^t dt' \\ m - m_o &= -kt \\ m &= m_o - kt \end{aligned}$$

where m_o is the initial mass of the rocket. Substituting this back into main expression and integrating both sides:

$$\begin{aligned} \int_0^v \frac{1}{a-v'} dv' &= b \int_0^t \frac{1}{m_o - kt'} dt' \\ -\ln(a-v') \Big|_0^v &= -\frac{b}{k} \ln(m_o - kt') \Big|_0^t \\ -\ln(a-v) + \ln(a) &= \frac{b}{k} [-\ln(m_o - kt) + \ln(m_o)] \\ \ln\left(\frac{a}{a-v}\right) &= \frac{b}{k} \ln\left(\frac{m_o}{m_o - kt}\right) \end{aligned}$$

substituting back in $m = m_o - kt$ and exponentiating both sides:

$$\begin{aligned} \frac{a}{a-v} &= \left(\frac{m_o}{m}\right)^{\frac{b}{k}} \\ a\left(\frac{m_o}{m}\right)^{-\frac{b}{k}} &= a-v \\ v &= a - a\left(\frac{m_o}{m}\right)^{-\frac{b}{k}} \\ v &= a \left[1 - \left(\frac{m}{m_o}\right)^{\frac{b}{k}} \right] \end{aligned}$$

subbing back in $a = \frac{kv_{ex}}{b}$ we get the final expression

$$v(m) = \frac{kv_{ex}}{b} \left[1 - \left(\frac{m}{m_o}\right)^{\frac{b}{k}} \right]$$

3. (a) The angular momentum vector is

$$\begin{aligned}\ell &= \mathbf{r} \times \mathbf{p} \\ &= \mathbf{r} \times m\dot{\mathbf{r}} \\ &= m\mathbf{r} \times \dot{\mathbf{r}}\end{aligned}$$

from HW 1, we know that

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}$$

so

$$\begin{aligned}\ell &= m(r\hat{\mathbf{r}}) \times (\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}) \\ &= m[r\dot{r}(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) + r(r\dot{\phi})(\hat{\mathbf{r}} \times \hat{\phi})] \\ &= m[0 + r^2\dot{\phi}\hat{\mathbf{z}}] = mr^2\omega\hat{\mathbf{z}}\end{aligned}$$

where $\omega = \dot{\phi}$ and the magnitude of the angular momentum is

$$\ell = |\ell| = mr^2\omega$$

(b) The area swept by an infinitesimal change in the planets position is equivalent to the area of a triangle as shown in Figure 2, so

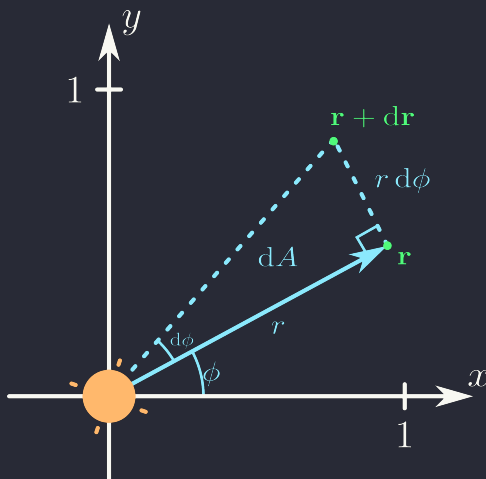


Figure 3.1: Area swept by planet

$$dA = \frac{1}{2}r(r d\phi) = \frac{1}{2}r^2 d\phi$$

dividing both sides by dt gives us

$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{d\phi}{dt} = \frac{1}{2}r^2\omega$$

and from part (a) we know that $\ell = mr^2\omega$ or $\omega = \frac{\ell}{mr^2}$ so

$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{\ell}{mr^2} = \frac{\ell}{2m}$$

therefore, the rate in change of the area swept by the planet is a constant that is proportional to ℓ .

4. $\mathbf{F} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$ from $P = (1, 0)$ to $Q = (0, 1)$

(a) For a straight line the path is given by

$$y = 1 - x \quad dy = -dx$$

so the work done is

$$\begin{aligned} W_a &= \int_P^Q F_x dx + F_y dy = \int_P^Q -y dx + x dy \\ &= \int_{x=1}^0 -(1-x) dx + x(-dx) \\ &= \int_1^0 -1 dx = 1 \end{aligned}$$

(b) For a circular path of radius 1, the path in polar coordinates is

$$\begin{aligned} y &= \sin \phi \rightarrow dy = \cos \phi d\phi \\ x &= \cos \phi \rightarrow dx = -\sin \phi d\phi \end{aligned}$$

from the equation of a circle $x^2 + y^2 = 1$. The limits of integration are $\phi = 0 \rightarrow \pi/2$, and the work is

$$\begin{aligned} W_b &= \int_{\phi=0}^{\pi/2} -\sin \phi (-\sin \phi) d\phi + \cos \phi \cos \phi d\phi \\ &= \int_{\phi=0}^{\pi/2} 1 d\phi = \frac{\pi}{2} \end{aligned}$$

(c) Splitting this into two paths: For path 1, $y = 0$; $dy = 0$; and $x = 1 \rightarrow 0$ so

$$W_1 = \int_{x=1}^0 0 dx + x(0) = 0$$

For path 2, $x = 0$; $dx = 0$; $y = 0 \rightarrow 1$ so

$$W_2 = \int_{y=0}^1 -y(0) + 0 dy = \int_{y=0}^1 0 dy = 0$$

And the work done is $W_c = W_1 + W_2 = 0$

(d) The force is not conservative because the work done is path dependent! We can also double check by taking the curl:

$$\nabla \times \mathbf{F} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & x \end{vmatrix} = \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial y} (-y) \right) \hat{\mathbf{z}} = 2\hat{\mathbf{z}}$$

which is not zero, so the force is not conservative.

5. (a) Using the time derivatives of the polar unit vectors

$$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\phi}\hat{\phi} \quad \frac{d}{dt}\hat{\phi} = -\dot{\phi}\hat{\mathbf{r}}$$

Acceleration in polar coordinates is

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} &= \frac{d}{dt}\dot{\mathbf{r}} = \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\phi} + (\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}(-\dot{\phi}\hat{\mathbf{r}})) \\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}\end{aligned}$$

so the radial and angular components of the force are

$$\begin{aligned}F_r &= ma_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi &= ma_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\end{aligned}$$

and since the spring force is conservative with magnitude $F_s = -k(r - a)\hat{\mathbf{r}}$ the equations of motion are

$$\begin{aligned}m(\ddot{r} - r\dot{\phi}^2) &= -k(r - a) \\ m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) &= 0\end{aligned}$$

or

$$\begin{aligned}m\ddot{r} - mr\dot{\phi}^2 + k(r - a) &= 0 \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0\end{aligned}$$

(b) The initial angular momentum of the system is

$$\ell_o = mv_o a$$

and after some time the angular momentum is (from Problem 3)

$$\ell = mr^2\dot{\phi}$$

and using the conservation of angular momentum

$$\begin{aligned}\ell_o &= \ell \\ mv_o a &= mr^2\dot{\phi} \\ \dot{\phi} &= \frac{v_o a}{r^2}\end{aligned}$$

(c) First the initial mechanical energy of the system is purely kinetic given by the initial velocity:

$$E_o = T_o = \frac{1}{2}mv_o^2$$

the total mechanical energy of the system after some time will be the sum of the kinetic and potential energies:

$$\begin{aligned}U &= -\int_0^r \mathbf{F} \cdot d\mathbf{r}' = \int_0^r k(r - a) dr' = \frac{1}{2}k(r - a)^2 \\ T &= \frac{1}{2}mv^2 = \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}m(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}) \cdot (\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)\end{aligned}$$

And from the conservation of energy

$$\begin{aligned}E_o &= E = T + U \\ \frac{1}{2}mv_o^2 &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}k(r - a)^2 \\ v_o^2 &= \dot{r}^2 + r^2\dot{\phi}^2 + \frac{k}{m}(r - a)^2\end{aligned}$$

Homework 5

Due 2/21

1. (a) If f is independent of y , then

$$\frac{\partial f}{\partial y} = 0$$

and using the Euler-Lagrange (EQ) equation, we have

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial y} = \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)$$

so

$$0 = \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)$$

and for the derivative to be zero,

$$\frac{\partial f}{\partial y'} = \text{constant}$$

- (b) Since f is independent of x ,

$$\frac{\partial f}{\partial x} = 0$$

Using the Euler-Lagrange equation, we have

$$\frac{\partial f}{\partial y} = \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)$$

Using chain rule to differentiate f with respect to x for a general function $f(x, y, y')$,

$$\begin{aligned} \frac{d}{dx} f(x, y, y') &= \frac{\partial f}{\partial x} \left(\frac{d}{dx} x \right) + \frac{\partial f}{\partial y} \left(\frac{dy}{dx} \right) + \frac{\partial f}{\partial y'} \left(\frac{dy'}{dx} \right) \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\ \frac{d}{dx} f(y, y') &= 0 + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \end{aligned}$$

and substituting what we got from the EL equation,

$$\frac{d}{dx} f(y, y') = \left[\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] y' + \frac{\partial f}{\partial y'} y'' = \left[\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] y' + \frac{\partial f}{\partial y'} \left(\frac{dy'}{dx} \right)$$

which is equivalent to

$$\frac{d}{dx} f = \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right)$$

from chain rule. Moving everything to one side:

$$\begin{aligned} \frac{d}{dx} f - \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) &= 0 \\ \frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) &= 0 \end{aligned}$$

which is only true if

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}$$

2. (a) In 2D, we know the length of a short segment is

$$ds = \sqrt{dx^2 + dy^2}$$

and in 3D

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

transforming to spherical coordinates:

$$\begin{aligned} x &= r \cos \phi \sin \theta & dx &= dr \cos \phi \sin \theta - r \sin \phi \sin \theta d\phi + r \cos \phi \cos \theta d\theta \\ y &= r \sin \phi \sin \theta & dy &= dr \sin \phi \sin \theta + r \cos \phi \sin \theta d\phi + r \sin \phi \cos \theta d\theta \\ z &= r \cos \theta & dz &= dr \cos \theta - r \sin \theta d\theta \end{aligned}$$

for the sphere of radius $r \rightarrow R$ and $dr = 0$ since radius is constant, so

$$\begin{aligned} dx &= -R \sin \phi \sin \theta d\phi + R \cos \phi \cos \theta d\theta \\ dy &= R \cos \phi \sin \theta d\phi + R \sin \phi \cos \theta d\theta \\ dz &= -R \sin \theta d\theta \end{aligned}$$

and squaring each term:

$$\begin{aligned} dx^2 &= R^2 \sin^2 \phi \sin^2 \theta d\phi^2 + R^2 \cos^2 \phi \cos^2 \theta d\theta^2 - 2R^2 \sin \phi \sin \theta \cos \phi \cos \theta d\phi d\theta \\ dy^2 &= R^2 \cos^2 \phi \sin^2 \theta d\phi^2 + R^2 \sin^2 \phi \cos^2 \theta d\theta^2 + 2R^2 \sin \phi \sin \theta \cos \phi \cos \theta d\phi d\theta \\ dz^2 &= R^2 \sin^2 \theta d\theta^2 \end{aligned}$$

when we add all three equations we can see that the last term in dx^2 and dy^2 cancel out, and grouping the like terms we get

$$\begin{aligned} R^2 \sin^2 \phi \sin^2 \theta d\phi^2 + R^2 \cos^2 \phi \sin^2 \theta d\phi^2 &= R^2 \sin^2 \theta d\phi^2 (\sin^2 \phi + \cos^2 \phi) \\ &= R^2 \sin^2 \theta d\phi^2 \end{aligned}$$

and

$$\begin{aligned} R^2 \cos^2 \phi \cos^2 \theta d\theta^2 + R^2 \sin^2 \phi \cos^2 \theta d\theta^2 + R^2 \sin^2 \theta d\theta^2 \\ &= R^2 \cos^2 \theta d\theta^2 (\cos^2 \phi + \sin^2 \phi) + R^2 \sin^2 \theta d\theta^2 \\ &= R^2 d\theta^2 (\cos^2 \theta + \sin^2 \theta) \\ &= R^2 d\theta^2 \end{aligned}$$

so the length of a short segment in spherical coordinates is

$$\begin{aligned} ds &= \sqrt{R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2} \\ &= \sqrt{R^2 d\theta^2 \left(\frac{d\theta^2}{d\theta^2} + \sin^2 \theta \frac{d\phi^2}{d\theta^2} \right)} \\ \text{using } \frac{d\phi}{d\theta} &= \phi'(\theta) \\ &= R \sqrt{1 + \sin^2 \theta \phi'(\theta)^2} d\theta \end{aligned}$$

and the total path length L is found by integrating ds from θ_a to θ_b :

$$L = R \int_{\theta_a}^{\theta_b} \sqrt{1 + \sin^2 \theta \phi'(\theta)^2} d\theta$$

(b) We find that the integral function is independent of ϕ or

$$f = f(\theta, \phi') = \sqrt{1 + \sin^2 \theta \phi'(\theta)^2}$$

so from Problem 1a, we know that

$$\frac{\partial f}{\partial \phi'} = \text{constant} = C$$

$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2 \theta \phi'(\theta)}{\sqrt{1 + \sin^2 \theta \phi'(\theta)^2}}$$

setting one of the points to be the north pole, $\theta_a = 0$, so the constant is

$$\frac{\sin^2(0)\phi'(0)}{\sqrt{1 + \sin^2(0)\phi'(0)^2}} = 0 = C$$

solving for $\phi'(\theta)$

$$\frac{\sin^2 \theta \phi'(\theta)}{\sqrt{1 + \sin^2 \theta \phi'(\theta)^2}} = 0$$

$$\phi'(\theta) = 0$$

using separation of variables,

$$\frac{d\phi}{d\theta} = 0$$

$$\int d\phi = \int 0 d\theta$$

$$\phi(\theta) = C_2$$

Since ϕ is a constant, this is equivalent to to a slice of the sphere through the north pole, which has a cross section of a circle with radius R . The path follows the circumference of the circle, from the north pole to the point at θ_b .

rearranging for r' ;

$$\begin{aligned}
K\sqrt{1-r'^2} &= r \\
K^2(1-r'^2) &= r^2 \\
K^2 - K^2r'^2 &= r^2 \\
K^2r'^2 &= r^2 - K^2 \\
r'^2 &= \frac{r^2}{K^2} - 1 \\
r' &= \sqrt{\frac{r^2}{K^2} - 1} = \frac{dr}{dl}
\end{aligned}$$

using separation of variables:

$$dl = \frac{dr}{\sqrt{r^2/K^2 - 1}}$$

using the substitution $u = r/K$ and $du = dr/K$:

$$\begin{aligned}
dl &= \frac{K du}{\sqrt{u^2 - 1}} \\
\int dl &= K \int \frac{du}{\sqrt{u^2 - 1}} \\
l &= K \operatorname{arccosh}(r/K) + C \\
\operatorname{arccosh}(r/K) &= \frac{l - C}{K} \\
r &= K \cosh\left(\frac{l - C}{K}\right)
\end{aligned}$$

Since the constraint l is constant as the total length of the curve (K & C are also constants), $r = \text{constant}$ is a solution to the equation or the radius of the curve is constant. This is only true for circles which have a constant radial distance from the origin, so circles leads to the maximum area integral.

where we have the 3 equations of motion:

$$\begin{aligned} -\frac{\partial U}{\partial r} &= m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta) \\ -\frac{1}{r} \frac{\partial U}{\partial \theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta) \\ -\frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} &= m(r\ddot{\phi} \sin \theta + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\phi}\dot{\theta} \cos \theta) \end{aligned}$$

where from N2L in spherical coordinates, the components of acceleration are

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta \\ a_\phi &= r\ddot{\phi} \sin \theta + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\phi}\dot{\theta} \cos \theta \end{aligned}$$

and the conservative force is

$$\mathbf{F} = -\nabla U = -\frac{\partial U}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\boldsymbol{\theta}} - \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \hat{\boldsymbol{\phi}}$$

To compare with N2L we start with the unit vectors in spherical coordinates:

$$\begin{aligned} \hat{\mathbf{r}} &= \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \end{aligned}$$

and from the velocity equation we know that the derivative of the radial unit vector is

$$\dot{\hat{\mathbf{r}}} = \dot{\theta} \hat{\boldsymbol{\theta}} + \dot{\phi} \sin \theta \hat{\boldsymbol{\phi}}$$

for the colatitude unit vector in the θ direction

$$\begin{aligned} \dot{\hat{\boldsymbol{\theta}}} &= (-\dot{\theta} \sin \theta \cos \phi - \dot{\phi} \cos \theta \sin \phi) \hat{\mathbf{x}} + (-\dot{\theta} \sin \theta \sin \phi + \dot{\phi} \cos \theta \cos \phi) \hat{\mathbf{y}} - \dot{\theta} \cos \theta \hat{\mathbf{z}} \\ &= \dot{\phi} \cos \theta \hat{\boldsymbol{\phi}} - \dot{\theta} \hat{\mathbf{r}} \end{aligned}$$

(linear combination of unit vectors) and for the azimuthal unit vector in the ϕ direction

$$\begin{aligned} \dot{\hat{\boldsymbol{\phi}}} &= -\dot{\phi} \sin \phi \hat{\mathbf{x}} - \dot{\phi} \cos \phi \hat{\mathbf{y}} \\ &= -\dot{\phi} \sin \theta \hat{\mathbf{r}} - \dot{\phi} \cos \theta \hat{\boldsymbol{\theta}} \end{aligned}$$

with this in hand taking the time derivative of velocity:

$$\frac{d}{dt} \mathbf{v} = \frac{d}{dt} (\dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}} + r \dot{\phi} \sin \theta \hat{\boldsymbol{\phi}})$$

the first term is

$$\begin{aligned} \frac{d}{dt} (\dot{r} \hat{\mathbf{r}}) &= \ddot{r} \hat{\mathbf{r}} + \dot{r} \dot{\hat{\mathbf{r}}} \\ &= \ddot{r} \hat{\mathbf{r}} + \dot{r} \dot{\theta} \hat{\boldsymbol{\theta}} + \dot{r} \dot{\phi} \sin \theta \hat{\boldsymbol{\phi}} \end{aligned}$$

the second term is

$$\begin{aligned} \frac{d}{dt} (r \dot{\theta} \hat{\boldsymbol{\theta}}) &= (\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\boldsymbol{\theta}} + r \dot{\theta} (\dot{\phi} \cos \theta \hat{\boldsymbol{\phi}} - \dot{\theta} \hat{\mathbf{r}}) \\ &= (-r \dot{\theta}^2) \hat{\mathbf{r}} + (\dot{r} \ddot{\theta}) \hat{\boldsymbol{\theta}} + (r \dot{\phi} \dot{\theta} \cos \theta) \hat{\boldsymbol{\phi}} \end{aligned}$$

and the third term is

$$\begin{aligned}\frac{d}{dt}(r\dot{\phi}\sin\theta\hat{\phi}) &= (\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\phi} + r\dot{\phi}\sin\theta(-\dot{\phi}\sin\theta\hat{\mathbf{r}} - \dot{\phi}\cos\theta\hat{\theta}) \\ &= (-r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + (-r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} + (\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\phi}\end{aligned}$$

so combining all the terms:

$$\begin{aligned}\frac{d}{dt}\mathbf{v} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} \\ &\quad + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\hat{\phi}\end{aligned}$$

5. (a) In the frame where the cart is at rest at $x' = x - v_o t$ this is simply the Brachistochrone where the time of travel is

$$T = \int_A^B \frac{ds}{v}$$

and the short segment length is

$$ds = \sqrt{dx'^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx'}\right)^2} dx' \quad \text{or} \quad \sqrt{1 + y'^2} dx$$

and from the conservation of energy

$$\frac{1}{2}mv^2 = mgy \quad \text{or} \quad v = \sqrt{2gy}$$

so

$$T = \int_A^B \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} dx'$$

where the integral function is independent of x ;

$$f = f(y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}$$

so using the second form of the EL equation from Problem 1b, we have

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} = C$$

using using the partial derivative

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gy}} \frac{y'}{\sqrt{1 + y'^2}}$$

so the conserved quantity is

$$\begin{aligned} C &= \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} - \frac{y'}{\sqrt{2gy}\sqrt{1 + y'^2}} \\ &= \frac{1}{\sqrt{2gy}\sqrt{1 + y'^2}} \left[\frac{\cancel{\sqrt{2gy}}\sqrt{1 + y'^2}\sqrt{1 + y'^2}}{\cancel{\sqrt{2gy}}} - y'^2 \right] \\ &= \frac{1}{\sqrt{2gy}\sqrt{1 + y'^2}} \\ \text{or } 2C^2g &= \frac{1}{y(1 + y'^2)} \end{aligned}$$

setting a new constant

$$2C^2g = \frac{1}{2a} \quad \text{where} \quad C = \sqrt{\frac{1}{4ga}}$$

we can now solve for y' :

$$\begin{aligned} \frac{1}{2a} &= \frac{1}{y(1 + y'^2)} \\ 1 + y'^2 &= \frac{2a}{y} \\ y'^2 &= \frac{2a}{y} - 1 \\ y' &= \sqrt{\frac{2a}{y} - 1} \quad \text{or} \quad \sqrt{\frac{2a - y}{y}} \end{aligned}$$

using separation of variables:

$$\frac{dy}{dx'} = \sqrt{\frac{2a-y}{y}}$$

$$\int dy \sqrt{\frac{y}{2a-y}} = \int dx'$$

and using the substitution $y = a(1 - \cos \theta)$; $dy = d\theta a \sin \theta$ and

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \cos \theta} \sqrt{1 + \cos \theta}$$

so

$$x' = \int d\theta a \sin \theta \sqrt{\frac{a(1 - \cos \theta)}{2a - a(1 - \cos \theta)}}$$

$$= \int d\theta a \sqrt{1 - \cos \theta} \sqrt{1 + \cos \theta} \frac{\sqrt{1 - \cos \theta}}{\sqrt{1 + \cos \theta}}$$

$$= a \int d\theta (1 - \cos \theta) = a(\theta - \sin \theta)$$

since the $\theta = \omega t$ we have the parametric equation for the path in the reference frame

$$x'(t) = a(\omega t - \sin(\omega t))$$

$$y(t) = a(1 - \cos(\omega t))$$

since $x = x' + v_o t$ the x position in the original frame is

$$x(t) = a(\omega t - \sin(\omega t)) + v_o t$$

(b) Using the initial conditions

$$x = y = 0, \quad \dot{x} = v_o, \quad \dot{y} = 0$$

we first solve for ω :

$$\dot{x} = a\omega(1 - \cos(\omega t)) + v_o$$

$$\dot{y} = a\omega \sin(\omega t)$$

we know at $t = 0$ that $\dot{x} = v_o$ and $\dot{y} = 0$. Since $\ddot{y} = g$ from the gravitational force we can solve for ω :

$$\ddot{y}(t) = a\omega^2 \cos(\omega t)$$

$$\ddot{y}(0) = a\omega^2 = g \implies \omega = \sqrt{\frac{g}{a}}$$

At the boundary point B we know that the cycloid completes one cycle so $\omega t = 2\pi$:

$$x'(t_B) = a(2\pi - \sin(2\pi)) = L$$

$$L = 2\pi a \implies a = \frac{L}{2\pi} \implies \omega = \sqrt{\frac{2\pi g}{L}}$$

so

$$x(t) = \frac{L}{2\pi} \left[\sqrt{\frac{2\pi g}{L}} t - \sin \left(\sqrt{\frac{2\pi g}{L}} t \right) \right] + v_o t$$

$$y(t) = \frac{L}{2\pi} \left[1 - \cos \left(\sqrt{\frac{2\pi g}{L}} t \right) \right]$$

(c) Sketching the path

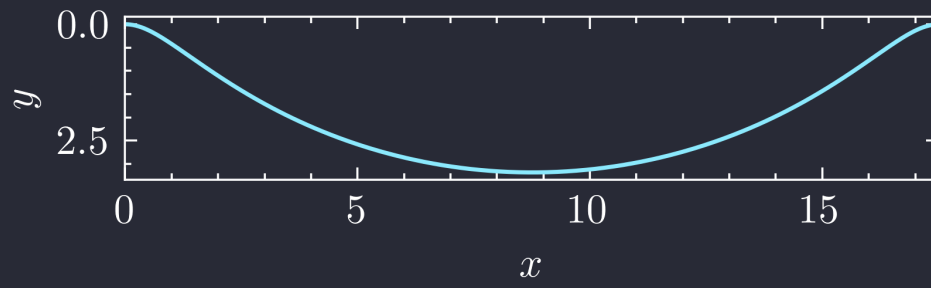


Figure 5.2: Numerically computed track shape