4.1 Given the Polarizability of Hydrogen atom

$$\frac{\alpha}{4\pi\epsilon_0} = 0.667 \times 10^{-30}\,\mathrm{m}^3$$

of bohr radius $a = 0.5 \times 10^{-10}$ m, and the E-field between metal plates separated $\Delta D = 1 \times 10^{-3}$ m apart and connected to a 500 V battery is

$$E = \frac{V}{\Delta D} = \frac{500\,\mathrm{V}}{1 \times 10^{-3}\,\mathrm{m}} = 5 \times 10^5\,\mathrm{V/m}$$

The dipole moment is ($\epsilon_0 = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N.m}^2}$)

$$p = \alpha E = qd \implies d = \frac{\alpha E}{q} = \frac{(4\pi\epsilon_0)0.667 \times 10^{-30}\,\mathrm{m}^3 \times 5 \times 10^5\,\mathrm{V/m}}{1.6 \times 10^{-19}\,\mathrm{C}} = 2.32 \times 10^{-16}\,\mathrm{m}$$

So the separation distance d as a fraction of the atomic radius a is

$$\boxed{\frac{d}{a} = \frac{2.32 \times 10^{-16}\,\mathrm{m}}{0.5 \times 10^{-10}\,\mathrm{m}} = 4.6 \times 10^{-6}}$$

The estimate voltage we need to ionize the atom is

$$V = E\Delta D, \quad E = \frac{qd}{\alpha}$$

$$\implies V = \frac{qa}{\alpha}\Delta D = \frac{1.6 \times 10^{-19}\,\mathrm{C} \times 0.5 \times 10^{-10}\,\mathrm{m}}{(4\pi\epsilon_0)0.667 \times 10^{-30}\,\mathrm{m}^3} \times 1 \times 10^{-3}\,\mathrm{m} = \boxed{1.1 \times 10^8\,\mathrm{V}}$$

4.2 Given the charge density of a groundstate hydrogen atom

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

where q is the electron charge and a is the Bohr radius, find the atomic polarizability.

FIrst calculating the electric field of the electron cloud

$$E_e(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

where the enclosed charge Q is

$$\begin{aligned}
Q &= \int \rho(r')\mathrm{d}\tau, \quad \mathrm{d}\tau = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{q}{\pi a^3}(4\pi)\int_0^r r'^2 e^{-2r'/a}\mathrm{d}r', \quad u = -\frac{2r}{a}; \quad \mathrm{d}u = -\frac{2}{a}\mathrm{d}r \\
&= \frac{4q}{a^3}\left[-\frac{a^3}{8}\int u^2 e^u \mathrm{d}u\right] \\
&= -\frac{q}{2}\left[(u^2 - 2u + 2)e^u\right]\Big|_0^{-2r/a} \\
&= -\frac{q}{2}\left[\left(\frac{4r^2}{a^2} + \frac{4r}{a} + 2\right)e^{-2r/a} - 2\right] \\
&= q\left[1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right)e^{-2r/a}\right]
\end{aligned}$$

So

$$E_e(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right)e^{-2r/a}\right]$$

For $r \ll a$ we can Taylor expand $e^{-2r/a}$ since $-2r/a \approx 0$ to get

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\
\implies e^{-2r/a} &= 1 - \frac{2r}{a} + \frac{2r^2}{a^2} - \frac{4r^3}{3a^3} + \cdots
\end{aligned}$$

so

$$\begin{aligned}
1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right)e^{-2r/a} &= 1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1\right)\left(1 - \frac{2r}{a} + \frac{2r^2}{a^2} - \frac{4r^3}{3a^3}\right) \\
&= 1 - \frac{2r^2}{a^2} + \frac{4r^3}{a^3} - \frac{4r^4}{a^4} + \frac{8r^5}{3a^5} \\
&\quad - \frac{2r}{a} + \frac{4r^2}{a^2} - \frac{4r^3}{a^3} + \frac{8r^4}{3a^4} \\
&\quad - 1 + \frac{2r}{a} - \frac{2r^2}{a^2} + \frac{4r^3}{3a^3} \\
&= \frac{4r^3}{3a^3} + \ldots
\end{aligned}$$

where the higher order terms are negligble. In an external electric field E, the polarized atom will have a balanced internal field $E = E_e$, such that the field at a distance d from the center of the cloud is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left[\frac{4d^3}{3a^3}\right] = \frac{1}{3\pi\epsilon_0 a^3}(qd) = \frac{qd}{\alpha} \implies \boxed{\alpha = 3\pi\epsilon_0 a^3}$$

4.8 Given the energy of a permanent ideal dipole $\mathbf{p}$ in E-field $\mathbf{E}$ is

$$U = -\mathbf{p} \cdot \mathbf{E} \tag{4.6}$$

and the E-field of a perfect dipole

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \tag{3.104}$$

So to find the interaction energy of two ideal dipoles ($\mathbf{p}_1$ and $\mathbf{p}_2$) we use (4.6) and calculate the energy of dipole $\mathbf{p}_1$ in the E-field produced by the second dipole $\mathbf{E}_2$:

$$\begin{aligned}
U &= -\mathbf{p}_1 \cdot \mathbf{E}_2 \\
&= -\mathbf{p}_1 \cdot \left[\frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2] \right] \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [-\mathbf{p}_1 \cdot [3(\mathbf{p}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2]] \\
&= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})]
\end{aligned}$$

checkmark!

4.11 Given a short cylinder(radius a, length L) with "frozen-in" polarization $\mathbf{P}$ parallel to the axis. Since $\mathbf{P}$ is uniform, $\rho_b = 0$, but the surface bound charge density is

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

so the bound surface charge is $\sigma_b = P$ on one end and $\sigma_b = -P$ on the other end of the cylinder.

(i) E-field sketch for $L \gg a$:

(ii) E-field sketch for $L \ll a$:

(iii) E-field sketch for $L \approx a$:

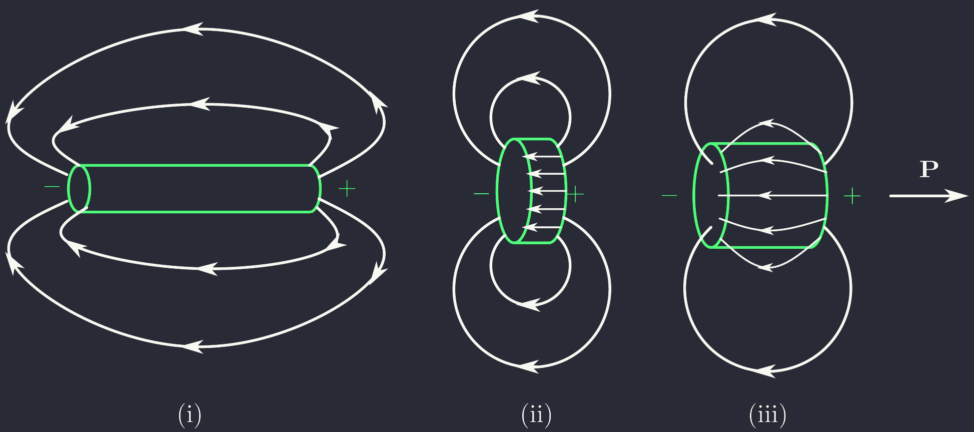

Figure 6.1: Sketch of E-field for a short cylinder

4.14 Given

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}, \quad \rho_b \equiv -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{4.11, 4.12}$$

Polarizing a neutral dielectric moves the charge a bit, so total bound charge is

$$\begin{aligned} Q &= \oint_S \sigma_b \, \mathrm{d}\mathbf{a} + \int_V \rho_b \, \mathrm{d}\tau \\ &= \oint_S \mathbf{P}\cdot \mathrm{d}a - \int_V \boldsymbol{\nabla}\cdot\mathbf{P}\, \mathrm{d}\tau = 0 \end{aligned}$$

from the divergence theorem $\int \boldsymbol{\nabla}\cdot\mathbf{P}\,\mathrm{d}\tau = \oint \mathbf{P}\cdot\mathrm{d}a$, so the total bound charge vanishes.

4.34 A dielectric cube of side a centered at the origin with a "frozen-in" polarization $\mathbf{P} = k\mathbf{r}$ where k is a constant and $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.

First the volume bound charge density is

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\boldsymbol{\nabla} \cdot (k\mathbf{r}) = -k\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z\right) = -3k$$

and the surface bound charge density is, e.g. on the face $\hat{\mathbf{n}} = \hat{\mathbf{x}}, \quad x = a/2$,

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \frac{ka}{2}$$

which is the same for all six faces. The total bound charge is

$$Q = \int_V \rho_b \, \mathrm{d}\tau + \oint_S \sigma_b \, \mathrm{d}a = -3ka^3 + \frac{ka}{2}(6a^2) = 0$$

where the volume integral is just the volume of the cube a^3 and the surface integral is the sum of the areas of the six faces $6a^2$.