

2.22 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q . Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

The electric field outside the sphere is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

and from Problem 2.8, the electric field inside the sphere is

$$\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r \hat{\mathbf{r}}$$

For points outside the sphere ($r > R$),

$$V(r) = - \int_{\infty}^r \mathbf{E} \cdot d\mathbf{l} = - \frac{1}{4\pi\epsilon_0} \int_{\infty}^r \frac{q}{r'^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr' = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \Big|_{\infty}^r = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

For points inside the sphere ($r < R$),

$$\begin{aligned} V(r) &= - \int_{\infty}^R \mathbf{E} \cdot d\mathbf{l} - \int_R^r \mathbf{E} \cdot d\mathbf{l} \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{R} - \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \int_R^r r' dr' \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{R} - \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \left(\frac{r'^2}{2} \right) \Big|_R^r \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left(3 - \frac{r^2}{R^2} \right) \end{aligned}$$

The gradient of V for $r > R$:

$$-\nabla V = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} = \mathbf{E}_{out}$$

and for $r < R$:

$$-\nabla V = \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r \hat{\mathbf{r}} = \mathbf{E}_{in}$$

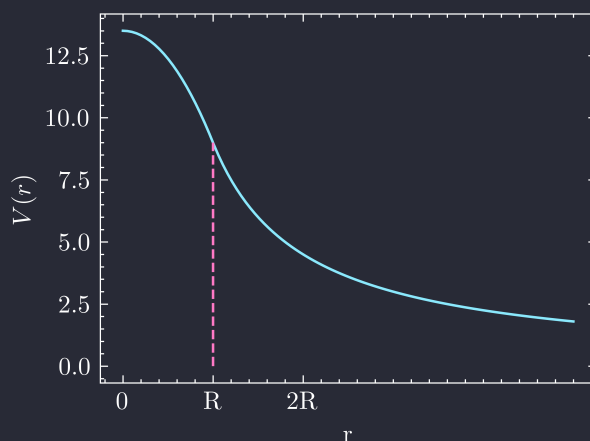


Figure 3.1: Plot of $V(r)$ as a function of r where $q = 1 \text{ nC}$ and $R = 1 \text{ m}$.

2.26 From Griffiths

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i} \quad (2.27)$$

and

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{r} d\ell' \quad \text{and} \quad V = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r} da' \quad (2.30)$$

- (a.1) Two point charges $+q$ a distance d apart: Find the potential a distance z above the center of the charges: Using Eq. (2.27), the potential is

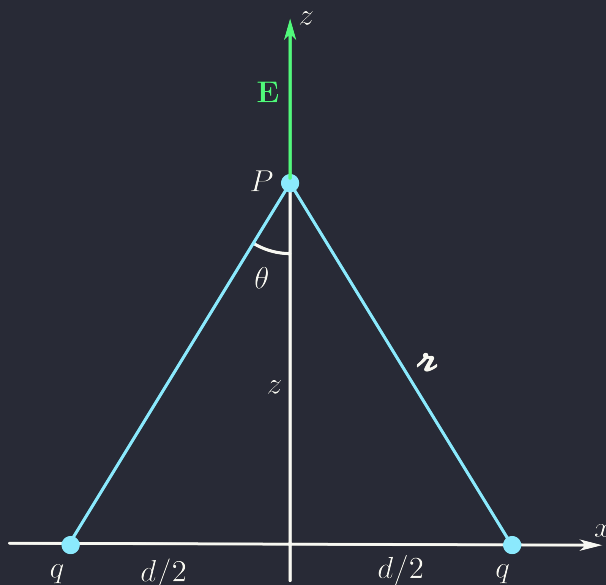


Figure 3.2: Two point charges $+q$ a distance d apart.

$$V_a = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{\sqrt{z^2 + \frac{d^2}{4}}} + \frac{q}{\sqrt{z^2 + \frac{d^2}{4}}} \right)$$

$$V_a = \frac{1}{4\pi\epsilon_0} \frac{2q}{\sqrt{z^2 + \frac{d^2}{4}}}$$

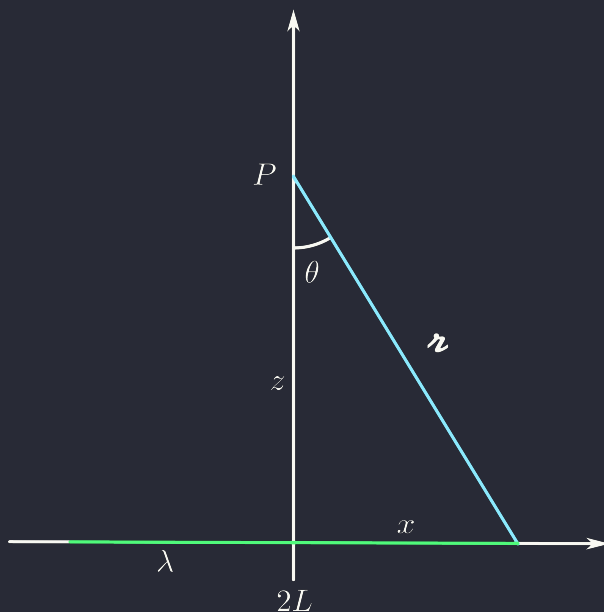
- (a.2) Computing the electric field $\mathbf{E} = -\nabla V$:

$$\begin{aligned} \mathbf{E}_a &= -\frac{\partial V_a}{\partial z} \hat{\mathbf{z}} \\ &= -\frac{1}{4\pi\epsilon_0} \frac{-1}{2} \frac{2q(2z)}{\left(z^2 + \frac{d^2}{4}\right)^{3/2}} \hat{\mathbf{z}} \end{aligned}$$

simplifying to

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{2qz}{\left(z^2 + \frac{d^2}{4}\right)^{3/2}} \hat{\mathbf{z}}$$

which is the same as Ex. 2.1

Figure 3.3: A line charge of density λ .

(b.1) Using Eq. (2.30), the potential is

$$V_b = \frac{1}{4\pi\epsilon_0} \lambda \int_{-L}^L \frac{1}{\sqrt{z^2 + x^2}} dx$$

To solve the integral, we can use the substitution from the trig identity

$$\begin{aligned} \cosh^2 u - \sinh^2 u &= 1 \\ \implies z^2 \cosh^2 u &= z^2 + z^2 \sinh^2 u \\ &= z^2 + x^2 \end{aligned}$$

where

$$\begin{aligned} x &= z \sinh u \implies u = \operatorname{arcsinh} \frac{x}{z} \\ dx &= z \cosh u du \end{aligned}$$

Thus the integral becomes

$$\begin{aligned} V_b &= \frac{1}{4\pi\epsilon_0} \lambda \int \frac{z \cosh u}{z \cosh u} du \\ &= \frac{1}{4\pi\epsilon_0} \lambda u \Big|_{-L}^L \\ &= \frac{1}{4\pi\epsilon_0} \lambda \left[\operatorname{arcsinh} \frac{L}{z} - \operatorname{arcsinh} \frac{-L}{z} \right] \end{aligned}$$

Using $\operatorname{arcsinh}(a) = \ln |a + \sqrt{a^2 + 1}|$:

$$\begin{aligned} \implies \operatorname{arcsinh}\left(\frac{L}{z}\right) &= \ln \left| \frac{L}{z} + \sqrt{\left(\frac{L}{z}\right)^2 + 1} \right| \\ &= \ln \left| \frac{1}{z} (L + \sqrt{L^2 + z^2}) \right| \end{aligned}$$

so the potential is

$$V_b = \frac{1}{4\pi\epsilon_0} \lambda \ln \left| \frac{L + \sqrt{L^2 + z^2}}{-L + \sqrt{L^2 + z^2}} \right|$$

(b.2) The electric field is

$$\begin{aligned} \mathbf{E}_b &= -\frac{\partial V_b}{\partial z} \hat{\mathbf{z}} \\ &= -\frac{1}{4\pi\epsilon_0} \lambda \left[\frac{1}{L + \sqrt{L^2 + z^2}} \left(\frac{1}{2} \frac{2z}{\sqrt{L^2 + z^2}} \right) - \frac{1}{-L + \sqrt{L^2 + z^2}} \left(\frac{1}{2} \frac{2z}{\sqrt{L^2 + z^2}} \right) \right] \hat{\mathbf{z}} \\ &= -\frac{1}{4\pi\epsilon_0} \lambda \frac{z}{\sqrt{L^2 + z^2}} \left[\frac{-L + \sqrt{L^2 + z^2}}{-L^2 + (L^2 + z^2)} - \frac{L + \sqrt{L^2 + z^2}}{-L^2 + (L^2 + z^2)} \right] \hat{\mathbf{z}} \\ &= -\frac{1}{4\pi\epsilon_0} \lambda \frac{-2Lz}{z^2 \sqrt{L^2 + z^2}} \hat{\mathbf{z}} \end{aligned}$$

simplifying to

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z \sqrt{L^2 + z^2}} \hat{\mathbf{z}}$$

which is the same as Ex. 2.2

(c.1) Using Eq. (2.30) and polar coordinates, the potential is

$$\begin{aligned} V_c &= \frac{1}{4\pi\epsilon_0} \sigma \int_0^{2\pi} \int_0^R \frac{1}{\sqrt{z^2 + r^2}} r \, dr \, d\theta \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma \int_0^R \frac{r}{\sqrt{z^2 + r^2}} \, dr \end{aligned}$$

substituting $u = z^2 + r^2$; $du = 2r \, dr$:

$$\begin{aligned} V_c &= \frac{1}{4\pi\epsilon_0} \pi\sigma \int \frac{1}{\sqrt{u}} \, du \\ &= \frac{1}{4\pi\epsilon_0} \pi\sigma 2\sqrt{z^2 + r^2} \Big|_0^R \end{aligned}$$

thus

$$V_c = \frac{\sigma}{2\epsilon_0} \left[\sqrt{z^2 + R^2} - z \right]$$

(c.2) The electric field is

$$\begin{aligned} \mathbf{E}_c &= -\frac{\partial V_c}{\partial z} \hat{\mathbf{z}} \\ &= -\frac{\sigma}{2\epsilon} \left[\frac{z}{\sqrt{z^2 + R^2}} - 1 \right] \hat{\mathbf{z}} \end{aligned}$$

thus

$$\mathbf{E}_c = \frac{\sigma}{2\epsilon_0} \left[1 - \frac{z}{\sqrt{z^2 + R^2}} \right] \hat{\mathbf{z}}$$

which is the same as Problem 2.6:

2.6 The electric field is only in the z -direction where $\cos \theta = z/\mathfrak{z}$:

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{\mathfrak{z}^2} \cos \theta \hat{\mathbf{z}} \, d\mathbf{a} \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \, d\mathbf{a}\end{aligned}$$

Using polar coordinates: since $d\mathbf{a} = r \, dr \, d\theta$

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \, r \, dr \, d\theta \\ &= \frac{1}{4\pi\epsilon_0} \sigma z (2\pi) \hat{\mathbf{z}} \int_0^R \frac{r}{(z^2 + r^2)^{3/2}} \, dr \\ &= \frac{\sigma}{2\epsilon_0} z \hat{\mathbf{z}} \left[-\frac{1}{\sqrt{z^2 + r^2}} \right]_0^R \\ &= \frac{\sigma}{2\epsilon_0} z \left[\frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}} \right] \hat{\mathbf{z}} \\ \mathbf{E} &= \frac{\sigma}{2\epsilon_0} \left[1 - \frac{1}{\sqrt{z^2 + R^2}} \right] \hat{\mathbf{z}}\end{aligned}$$

- (d) if the right-hand charge of Fig. 3.2 is replaced by a charge $-q$, the potential at P using Eq. (2.27) is

$$V_d = 0 \implies \mathbf{E}_d = 0$$

which contradicts the result from Prob 2.2. This is because point P does not give us any information about the electric field which points in the x -direction. In fact any reference point on the z -axis will give us the same result.

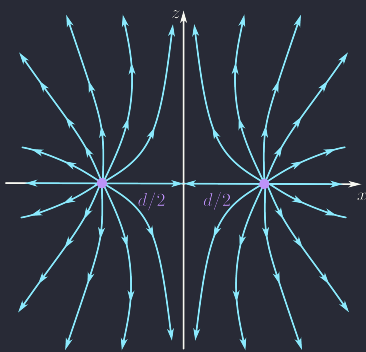


Figure 3.4: E-field for (a)

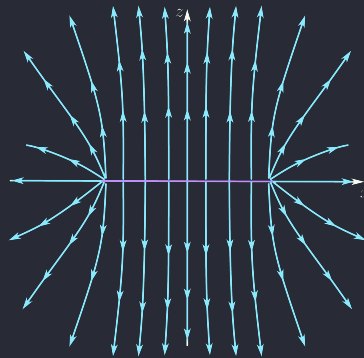


Figure 3.5: E-field for (b)

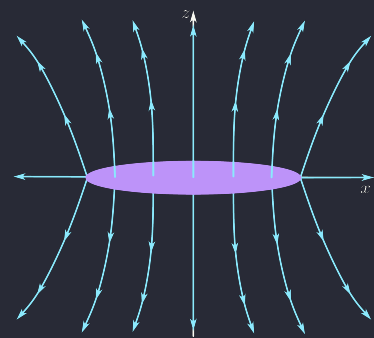
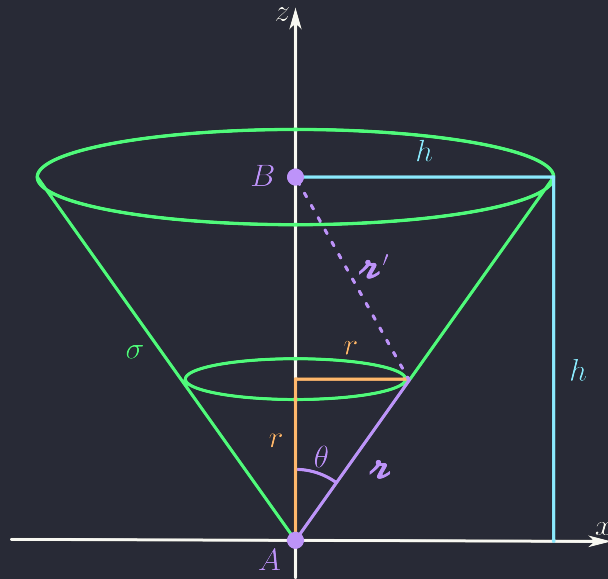


Figure 3.6: E-field for (c)



Figure 3.7: E-field for (c) E vs z

Figure 3.8: Empty ice cream cone with surface charge density σ .

2.27

- (i) Potential at A : Geometrically, we can see from the large right triangle that

$$\begin{aligned} z^2 &= h^2 + h^2 \\ \implies z &= h\sqrt{2}, \quad h = \frac{z}{\sqrt{2}} \end{aligned}$$

and from the smaller right triangle

$$z^2 = 2r^2 \implies r = \frac{z}{\sqrt{2}}$$

We can find the potential at A using Eq. (2.30) and integrate the rings of the cone along the slant length $0 \rightarrow h\sqrt{2}$ which gives us the area element $da = 2\pi r dz$:

$$\begin{aligned} V(A) &= \frac{1}{4\pi\epsilon_0} \int_0^{h\sqrt{2}} \frac{\sigma}{z} 2\pi r dz \\ &= \frac{\sigma}{2\epsilon_0\sqrt{2}} \int_0^{h\sqrt{2}} dz \\ &= \frac{\sigma}{2\epsilon_0\sqrt{2}} z \Big|_0^{h\sqrt{2}} \\ V(A) &= \frac{\sigma h}{2\epsilon_0} \end{aligned}$$

- (ii) Potential at B : Using the law of cosines,

$$z'^2 = h^2 + z^2 - 2hz \cos \theta$$

where

$$\begin{aligned} \cos \theta &= \frac{h}{z} \\ &= \frac{h}{h\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \implies z' &= \sqrt{h^2 + z^2 - hz\sqrt{2}} \end{aligned}$$

2.35 For a solid sphere radius R and charge q

(a) From Problem 2.22

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \left(3 - \frac{r^2}{R^2} \right)$$

and

$$W = \frac{1}{2} \int \rho V \, d\tau \quad (2.43)$$

So the energy is

$$\begin{aligned} W &= \frac{\rho}{2} \frac{1}{4\pi\epsilon_0} \frac{q}{2R} \int_0^{2\pi} \int_0^\pi \int_0^R \left(3 - \frac{r^2}{R^2} \right) r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= \frac{\rho}{2} \frac{1}{4\pi\epsilon_0} \frac{q}{2R} 4\pi \int_0^R \left(3r^2 - \frac{r^4}{R^2} \right) dr \\ &= \frac{\rho q}{4R\epsilon_0} \left[r^3 - \frac{r^5}{5R^2} \right]_0^R \\ &= \frac{\rho q}{4R\epsilon_0} \left[R^3 - \frac{R^3}{5} \right] \\ &= \frac{\rho q}{5\epsilon_0} R^2 \end{aligned}$$

where the charge over the volume of the sphere is $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, thus

$$W = \frac{q}{5\epsilon_0} R^2 \frac{q}{\frac{4}{3}\pi R^3}$$

$$W = \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R}$$

(b) Integrating over all space using

$$W = \frac{\epsilon_0}{2} \int E^2 \, d\tau \quad (2.45)$$

Where the electric field is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \quad \mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r \hat{\mathbf{r}}$$

so the energy is

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} 4\pi q^2 \left[\int_0^R \frac{r^2}{R^6} r^2 \, dr + \int_R^\infty \frac{1}{r^4} r^2 \, dr \right] \\ &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[\int_0^R \frac{r^4}{R^6} \, dr + \int_R^\infty \frac{1}{r^2} \, dr \right] \\ &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[\frac{r^5}{5R^6} \Big|_0^R - \frac{1}{R} \Big|_R^\infty \right] \\ &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[\frac{R^5}{5R^6} + \frac{1}{R} \right] \\ &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \frac{6}{5R} \\ W &= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R} \end{aligned}$$

checkmark.

(c) For a spherical volume of radius a and

$$W = \frac{\epsilon_0}{2} \left(\int_V E^2 d\tau + \oint_S V \mathbf{E} \cdot d\mathbf{a} \right) \quad (2.44)$$

we can assume the volume is outside the charged sphere so

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

From part (b), the first term is

$$\begin{aligned} \frac{\epsilon_0}{2} \int_V E^2 d\tau &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[\frac{1}{5R} - \frac{1}{a} + \frac{1}{R} \right] \\ &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[\frac{6}{5R} - \frac{1}{a} \right] \end{aligned}$$

the second term is at $r = a$

$$\begin{aligned} \frac{\epsilon_0}{2} \oint_V V \mathbf{E} \cdot d\mathbf{a} &= \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} \int \frac{q}{r} \frac{q}{r^2} r^2 \sin\theta d\theta d\phi \\ &= \frac{4\pi\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} \frac{1}{r} \Big|_{r=a} \\ &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2a} \end{aligned}$$

so the total energy is

$$\begin{aligned} W &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{2} \left[\frac{6}{5R} - \frac{1}{a} \right] + \frac{1}{4\pi\epsilon_0} \frac{q^2}{2a} \\ &= \frac{1}{4\pi\epsilon_0} \frac{3q^2}{5R} \end{aligned}$$

As $a \rightarrow \infty$ the $\oint V \mathbf{E} \cdot d\mathbf{a}$ term goes to zero.

2.40 Two cavities radii a and b in a conducting sphere of radius R with a point charge q_a and q_b respectively in each cavity.

- (a) Surface charge densities:

On the surface of cavity a the charge density is simply

$$\sigma_a = \frac{-q_a}{4\pi a^2}$$

and

$$\sigma_b = \frac{-q_b}{4\pi b^2}$$

respectively. For the surface of the conducting sphere, the charge density is positive and equal to the superposition of the two charges:

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

- (b) The field outside the conductor is equivalent to a point charge at the center of the sphere with the sum of the charges:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{\mathbf{r}}$$

- (c) The field in cavity a with respect to the center of the cavity is

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{a^2} \hat{\mathbf{a}}$$

and in cavity b is

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{b^2} \hat{\mathbf{b}}$$

- (d) The field due to the cavity charge is zero in the exterior of the cavity, so there is no Force on q_a or q_b .
- (e) If a charge q_c was brought near the conductor from outside, there would be a change in (a) σ_R and (b) \mathbf{E}_{out} .

2.48 Net force of of the southern hemisphere exerting on the northern hemisphere (solid sphere) with an inside Electric field (Problem 2.8)

$$E_{in} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \mathbf{r}$$

where the total force is

$$\mathbf{F} = Q\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q^2}{R^3} \mathbf{r}$$

Finding the net force exerted by the southern hemisphere: integrate $dF = \mathbf{F}/V$ over the southern hemisphere:

$$\begin{aligned} dF &= \frac{1}{\frac{4}{3}\pi R^3} \frac{1}{4\pi\epsilon_0} \frac{Q^2}{R^3} \mathbf{r} d\tau \\ &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \mathbf{r} d\tau \end{aligned}$$

The symmetry of the sphere implies that the Force is only in the z -direction i.e. $F_z = F \cos \theta$, so integrating over the southern hemisphere:

$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R F_z d\tau &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R r \cos \theta r^2 \sin \theta dr d\theta d\phi \\ &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} (2\pi) \left(\frac{r^4}{4} \right) \Big|_0^R \int_0^{\pi/2} \sin \theta \cos \theta d\theta d\phi \\ &= \frac{3Q^2}{32\pi\epsilon_0 R^2} \frac{\sin^2 x}{2} \Big|_0^{\pi/2} \\ &= \boxed{\frac{3Q^2}{64\pi\epsilon_0 R^2}} \end{aligned}$$