Homework 11

6.20 To demagnetize a permanent magnet, we can heat it above its Curie temperature T_C

6.21

(a) Show that the energy of the magnetic dipole in a magnetic field is

$$U = -\mathbf{m} \cdot \mathbf{B} \tag{6.34}$$

Moving the dipole from infinity to a point on the origin:
From the force on a magnetic dipole in a magnetic field $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$, the work done is given by

$$\begin{aligned} U &= -\int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{l} \\ &= -\int_{\infty}^{\mathbf{r}} \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}) \cdot \mathrm{d}\mathbf{l} \\ \text{using} \quad & \text{Fundamental theorem of gradients} \\ &= -[\mathbf{m} \cdot \mathbf{B}(\mathbf{r}) - \mathbf{m} \cdot \mathbf{B}(\infty)] \end{aligned}$$

Since the magnetic field goes to zero at infinity $\mathbf{B}(\infty) = 0$,

$$\boxed{U = -\mathbf{m} \cdot \mathbf{B}}$$

(b) For two magnetic dipoles separated by $\mathbf{r}$, find the interaction energy:
Given the coordinate-free form of the magnetic field of a dipole

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

The interaction energy of dipole $\mathbf{m}_1$ in the magnetic field of dipole 2 $\mathbf{B}_2$ is

$$\begin{aligned} U &= -\mathbf{m}_1 \cdot \mathbf{B}_2 \\ &= -\mathbf{m}_1 \cdot \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_2] \\ &= \frac{\mu_0}{4\pi} \frac{1}{r^3} [\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})] \end{aligned}$$

(c) In the dipoles are left free to rotate, From the figure above,

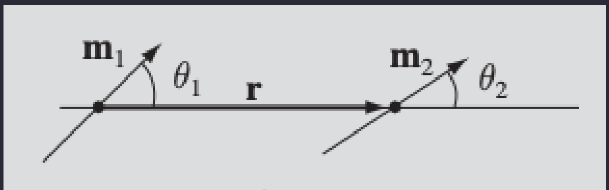


Fig. 6.30

$$\begin{aligned} \mathbf{m}_1 \cdot \mathbf{m}_2 &= m_1 m_2 \cos(\theta_1 - \theta_2) \\ \mathbf{m}_1 \cdot \hat{\mathbf{r}} &= m_1 \cos\theta_1 \\ \mathbf{m}_2 \cdot \hat{\mathbf{r}} &= m_2 \cos\theta_2 \end{aligned}$$

And using the cos difference formula

$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$

The interaction energy is

$$\begin{aligned}
U &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[m_1m_2\cos(\theta_1-\theta_2) - 3m_1m_2\cos\theta_1\cos\theta_2] \\
&= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) - 3\cos\theta_1\cos\theta_2] \\
&= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2[\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2]
\end{aligned}$$

The stable configuration happens when

$$\frac{\partial U}{\partial\theta_1} = \frac{\partial U}{\partial\theta_2} = 0$$

So

$$\begin{aligned}
\frac{\partial U}{\partial\theta_1} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2[\cos\theta_1\sin\theta_2 + 2\sin\theta_1\cos\theta_2] = 0 \\
&\implies 2\sin\theta_1\cos\theta_2 = -\cos\theta_1\sin\theta_2
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial U}{\partial\theta_2} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2[\sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2] = 0 \\
&\implies 2\cos\theta_1\sin\theta_2 = -\sin\theta_1\cos\theta_2
\end{aligned}$$

Which implies that both

$$\sin\theta_1\cos\theta_2 = \cos\theta_1\sin\theta_2 = 0$$

or two sets of solutions

$$\sin\theta_1 = \sin\theta_2 = 0 \quad \text{or} \quad \cos\theta_1 = \cos\theta_2 = 0$$

For the first set, we can have either $\theta_1 = \theta_2 = 0$ (aligned horizontal dipoles $\rightarrow\rightarrow$) or $\theta_1 = 0$ and $\theta_2 = \pi$ (anti aligned horizontal dipoles $\rightarrow\leftarrow$), but for the anti aligned case, the magnetic field $\mathbf{B}_2$ will not be parallel to $\mathbf{m}_1$ so it is not a stable configuration.

For the second set, we can have either $\theta_1 = \theta_2 = \frac{\pi}{2}$ (aligned vertical dipoles $\uparrow\uparrow$) or $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = -\frac{\pi}{2}$ (anti aligned vertical dipoles $\uparrow\downarrow$), but for the aligned case, the magnetic field $\mathbf{B}_2$ will not be parallel to $\mathbf{m}_1$ so it is not a stable configuration.

Therefore, looking at the two stable configurations:

$$\begin{aligned}
U_{\rightarrow\rightarrow} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2[\sin 0\sin 0 - 2\cos 0\cos 0] \\
&= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2[-2] \\
U_{\uparrow\downarrow} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2\left[\sin\frac{\pi}{2}\sin\frac{-\pi}{2} - 2\cos\frac{\pi}{2}\cos\frac{-\pi}{2}\right] \\
&= \frac{\mu_0}{4\pi}\frac{1}{r^3}m_1m_2[-1]
\end{aligned}$$

So the lower energy is the more stable configuration

$$U_{\rightarrow\rightarrow} < U_{\uparrow\downarrow}$$

thus the dipoles line up horizontally parallel $\rightarrow\rightarrow$.

(d) If a bunch of compasses were lined up along a straight line, they would align in straight line similarly to part (c): $\rightarrow\rightarrow\rightarrow\dots$

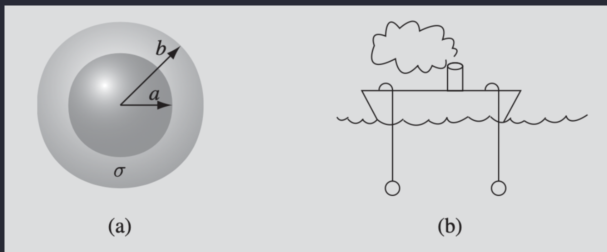


Fig. 7.4

7.1

(a) Between the two shells we have an electric field

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$$

So we have a potential difference

$$\begin{aligned} V &= -\int_b^a \mathbf{E} \cdot \mathrm{d}\mathbf{l} \\ &= -\frac{1}{4\pi\epsilon_0}Q\int_b^a \frac{1}{r^2}\,\mathrm{d}r \\ &= -\frac{1}{4\pi\epsilon_0}Q\left(-\frac{1}{a} + \frac{1}{b}\right) \\ &= \frac{1}{4\pi\epsilon_0}Q\left(\frac{1}{a} - \frac{1}{b}\right) \\ \implies Q &= \frac{4\pi\epsilon_0 V}{1/a - 1/b} \end{aligned}$$

The current is therefore

$$\begin{aligned} I &= \int \mathbf{J} \cdot \mathrm{d}\mathbf{a} = \sigma \int \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{\sigma}{\epsilon_0}Q \\ &= \frac{\sigma}{\epsilon_0}\frac{4\pi\epsilon_0 V}{1/a - 1/b} \end{aligned}$$

or

$$\boxed{I = \frac{4\pi\sigma V}{1/a - 1/b}}$$

(b) The resistance between the shells is

$$\boxed{R = \frac{V}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

(c) If $b \gg a$, $\frac{1}{b} \to 0$ so the resistance is simply

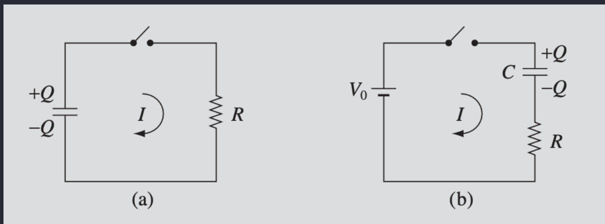
$$R = \frac{1}{4\pi\sigma a}$$

For two shells immersed really deep in the sea with potential difference V, the we can add the resistance in series i.e.

$$R_{\text{sea}} = 2 \times \frac{1}{4\pi\sigma a} = \frac{1}{2\pi\sigma a}$$

Thus the current flowing between the two metal spheres is

$$\boxed{I = \frac{V}{R_{\text{sea}}} = 2\pi\sigma a V}$$



7.2

(a) Inititally, $V_0 = \frac{Q_0}{C}$ As the capacitor discharges the current decreases,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -I$$

and the sum of the voltages across the capacitor and the resistor is

$$\begin{aligned}\sum V &= V_C + V_R = 0\\ &= \frac{Q}{C} + IR\\ \implies \frac{\mathrm{d}Q}{\mathrm{d}t} &= -I = -\frac{Q}{RC}\end{aligned}$$

The solution to this differential equation $\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ$ is

$$\begin{aligned}Q(t) &= Q_0 e^{-kt}\\ \implies I(t) &= -\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ_0 e^{-kt}\\ &= -\frac{Q_0}{RC} e^{-t/RC}\end{aligned}$$

and from the initial condition $Q_0 = CV_0$,

$$\boxed{I(t) = -\frac{V_0}{R} e^{-t/RC}}$$

(b) The original energy stored in the capacitor is (2.55)

$$\boxed{W = \frac{1}{2} C V_0^2}$$

So integrating the power equation (7.7)

$$\begin{aligned}\int_0^\infty P &= \int_0^\infty I^2 R \,\mathrm{d}t\\ &= \frac{V_0^2}{R} \int_0^\infty e^{-2t/RC}\,\mathrm{d}t\\ &= \frac{V_0^2}{R}\left[-\frac{RC}{2} e^{-2t/RC}\right]\Bigg|_0^\infty\\ &= \frac{V_0^2}{R}\left[0 + \frac{RC}{2}\right]\\ &= \frac{1}{2} C V_0^2\end{aligned}$$

(c) Initially connecting the battery of V_0:

$$V_0 = \sum V = V_C + V_R = \frac{Q}{C} + IR$$

where the current will start to increase, $\frac{\mathrm{d}Q}{\mathrm{d}t} = I$, so

$$\begin{aligned} V_0 &= \frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t} \\ CV_0 &= Q + RC\frac{\mathrm{d}Q}{\mathrm{d}t} \\ \implies \frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{CV_0 - Q}{RC} \end{aligned}$$

And rewriting to get $-1/RC$ on to the right side:

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= -\frac{Q - CV_0}{RC} \\ \frac{\mathrm{d}Q}{Q - CV_0} &= -\frac{\mathrm{d}t}{RC} \end{aligned}$$

Integrating both sides

$$\begin{aligned} \ln(Q - CV_0) &= -\frac{t}{RC} + k \\ Q - CV_0 &= e^{-t/RC+k} \\ Q &= CV_0 + e^{-t/RC+k} \\ \implies Q(t) &= CV_0 + ke^{-t/RC} \end{aligned}$$

And from the initial condition $Q(0) = 0$,

$$\begin{aligned} Q(0) &= CV_0 + k = 0 \\ \implies k &= -CV_0 \end{aligned}$$

So the charge on the capacitor is

$$\boxed{Q(t) = CV_0(1 - e^{-t/RC})}$$

and the current is

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} = CV_0\left(\frac{1}{RC}e^{t/RC}\right) = \boxed{\frac{V_0}{R}e^{-t/RC}}$$

(d) Integrating power to get the energ from the battery:

$$\begin{aligned} \int P &= \int_0^\infty IV_0 \,\mathrm{d}t \\ &= \frac{V_0^2}{R}\int_0^\infty e^{-t/RC}\,\mathrm{d}t \\ &= \frac{V_0^2}{R}\left[-RCe^{-t/RC}\right]\Big|_0^\infty \\ &= \frac{V_0^2}{R}[0 + RC] = \boxed{CV_0^2} \end{aligned}$$

The heat delivered to the resistor is still $\boxed{\frac{1}{2}CV_0^2}$, so the final energy stored in the capacitor is $\boxed{\frac{1}{2}CV_0^2}$ i.e. 50% of the work done by the battery shows up as energy in the capacitor.

7.7 Magnetic field points into the page

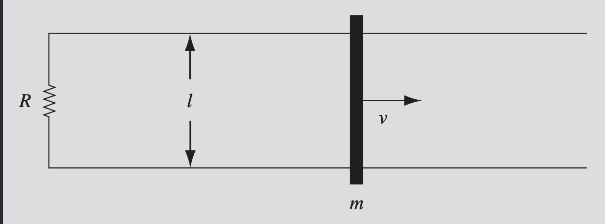


Fig. 7.17

(a) If the bar moves to the right with speed v the EMF is

$$\mathcal{E} = B\ell v$$

And since $\mathcal{E} = IR$, the current is

$$\boxed{I = \frac{B\ell v}{R}}$$

where RHR points upwards (or counterclockwise) so the current points downwards in the resistor.

(b) The magnetic force on the bar is

$$\boxed{F = I\ell B = \frac{B^2\ell^2 v}{R}}$$

and from RHR, the force points left

(c) Starting at speed v_0 at $t = 0$, to find the speed at later time t we can just use Newton's second law

$$\begin{aligned} F = m\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{B^2\ell^2 v}{R} \\ \frac{\mathrm{d}v}{\mathrm{d}t} = \left(\frac{B^2\ell^2}{mR}\right)v \quad &\text{or} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -kv \\ \implies v(t) = v_0 e^{-kt} &= \boxed{v_0 e^{-\frac{B^2\ell^2}{mR}t}} \end{aligned}$$

(d) Given the initial kinetic energy $\frac{1}{2}mv_0^2$, and we can check this by integratin the power to get the work done by the magnetic field

$$\begin{aligned} \int_0^\infty P\,\mathrm{d}t &= \int_0^\infty I^2 R\,\mathrm{d}t \\ &= \frac{B^2\ell^2}{R}v_0^2 \int_0^\infty e^{-2kt}\,\mathrm{d}t \quad k = \frac{B^2\ell^2}{mR} \\ &= \frac{B^2\ell^2}{R}v_0^2 \left[-\frac{1}{2k}e^{-2kt}\right]\Bigg|_0^\infty \\ &= \frac{B^2\ell^2}{R}v_0^2 \left[0 + \frac{1}{2k}\right] \\ &= \frac{1}{2}mv_0^2 \end{aligned}$$

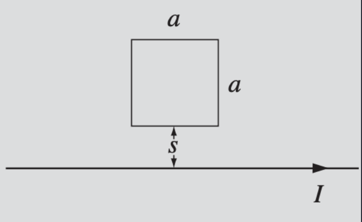


Fig. 7.18

7.8

(a) The flux of **B** through the loop: From HW 5b, the magnetic field of a long straight wire is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

So the flux is simply the integral

$$\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

where the surfance element is $\mathrm{d}\mathbf{a} = a\,\mathrm{d}s$ from $s \to s + a$:

$$\begin{aligned}\Phi &= \frac{\mu_0 I a}{2\pi}\int_s^{s+a}\frac{1}{s}\,\mathrm{d}s \\ &= \boxed{\frac{\mu_0 I a}{2\pi}\ln\left(\frac{s+a}{s}\right)}\end{aligned}$$

(b) If the loop is moved away from the wire at velocity v, the EMF is

$$\begin{aligned}\mathcal{E} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu_0 I a}{2\pi}[\ln(s+a) - \ln(s)] \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = v \\ &= -\frac{\mu_0 I a}{2\pi}\left[\frac{v}{s+a} - \frac{v}{s}\right] \\ &= \frac{\mu_0 I a v}{2\pi}\left[\frac{1}{s} - \frac{1}{s+a}\right] \\ &= \frac{\mu_0 I a v}{2\pi}\left[\frac{s+a}{s(s+a)} - \frac{s}{s(s+a)}\right] \\ &= \boxed{\frac{\mu_0 I a^2 v}{2\pi s(s+a)}}\end{aligned}$$

From RHR ($\mathbf{v}\times\mathbf{B}$)—$\mathbf{v}$ points up and **B** point out of the page—points to the right, but the magitude of force is less on the top side of the loop so the current flows counterclockwise.

(c) No EMF is generated if the loop is moved parallel to the wire!