1 Lagrange's Equations

From last time: we defined the path

$$S = \int_a^b f(x, y(x), y'(x))\mathrm{d}x$$

Goal: find $y(x)$ that minimizes S using EL

$$\text{EL:} \quad \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

where near the minimum $\delta S = 0$. From the EL, $y(x)$ is a stationary point of S(could also be a maximum!).

Lagrangian In Classical Mechanics, we use a specific form

$$\mathcal{L} = T - V$$

this has the units of energy and the action S has the units $[S] = [E \cdot T]$ similar to planck's constant $\hbar$.

3D Cartesian $x, y, z = q_1, q_2, q_3$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$U = U(x, y, z)$$

where the potential energy only depends on the position and T only depends on the velocity, so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

and the EL equation for the Lagrangian is

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

For the 3D case, we have 3 equations of motion: For x we have

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x}, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

and using the EL equation, we get

$$-\frac{\partial U}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x}$$

which is Newton's second law $F_x = ma_x$ where $\mathbf{F} = -\boldsymbol{\nabla} U$. We can now get the general form

$$\mathbf{F} = m\mathbf{a}$$

Polar Coordinates $q : (r, \phi)$ we know that

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

and

$$U = U(r, \phi), \qquad T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

first we find the parts EL equation for r

$$\frac{\partial\mathcal{L}}{\partial r} = mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$$
$$\frac{\partial\mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

and the EL equation is

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r})$$
$$m(\ddot{r} - r\dot{\phi}^2) = -\frac{\partial U}{\partial r}$$

which gives us N2L for r. For ϕ we have

$$\frac{\partial\mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi}$$
$$\frac{\partial\mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

and from the EL equation we get

$$-\frac{\partial U}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) = m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi})$$

dividing both sides by r

$$-\frac{1}{r}\frac{\partial U}{\partial \phi} = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = -(\boldsymbol{\nabla} U)_\phi$$

from both forms we know that the two parts of the EL represent the momentum and force:

$$\frac{\partial\mathcal{L}}{\partial \dot{q}_i} = p_i \quad \text{generalized momentum}$$
$$\frac{\partial\mathcal{L}}{\partial q_i} = F_i \quad \text{generalized force}$$

where $F_i = \frac{\mathrm{d}}{\mathrm{d}t}p_i$ is the generalized N2L.

Example: Mass m sliding down a frictionless *moving* ramp M. First we choose the coordinates x moving along withe the ramp and y down in the perpendicular direction. For the ramp M:

$$T_M = \frac{1}{2}M\dot{q}_2^2, \quad U_M = 0$$

and for the mass m: First we decompose the velocity of m into the x and y components

$$\mathbf{v}_m = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} = \hat{\mathbf{y}}(\dot{q}_1 \sin\alpha) + \hat{\mathbf{x}}(\dot{q}_1\cos\alpha + \dot{q}_2)$$

and the kinetic and potential energies are

$$T_m = \frac{1}{2}mv_m^2 = \frac{1}{2}m(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha + \dot{q}_2^2)$$
$$U_m = mgy = -mg(\dot{q}_1\sin\alpha)$$

using the Lagrangian $\mathcal{L} = T - U = T_M + T_m - U_M - U_m$ we get

$$\frac{\partial\mathcal{L}}{\partial q_2} = 0, \quad \frac{\partial\mathcal{L}}{\partial \dot{q}_2} = M\dot{q}_2 + m\dot{q}_2 + m\dot{q}_1\cos\alpha$$

and the EL equation gives us

$$(M+m)\ddot{q}_2 + m\ddot{q}_1 \cos\alpha = 0$$
$$a_2 = \ddot{q}_2 = -\frac{m\ddot{q}_1 \cos\alpha}{M+m}$$

and for q_1 we have

$$\frac{\partial \mathcal{L}}{\partial q_1} = mg\sin\alpha, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m(\dot{q}_1 + \dot{q}_2\cos\alpha)$$

and the EL equation gives us

$$mg\sin\alpha = m(\ddot{q}_1 + \ddot{q}_2\cos\alpha)$$

and since we have two equations and two unknowns, we can solve for $\ddot{q}_1$ and $\ddot{q}_2$.

$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{m+M}} = const$$
$$\ddot{q}_2 = -\frac{m\ddot{q}_1\cos\alpha}{M+m} = const$$

for $\alpha = 90°$, we get $\ddot{q}_1 = g$ and $\ddot{q}_2 = 0$ which is the same as a free falling. For and infinitely heavy ramp $M \to \infty$, we get $\ddot{q}_1 = g\sin\alpha$. For $M \to 0$ we get $\ddot{q}_1 = g/\sin\alpha$ which doesn't make sense because the force on the mass would be infinite. The normal force $N \to 0$ as $M \to 0$ and the mass would be in free fall.

Review Lagrangian: For a general integral

$$S \int f(x, y, y')\mathrm{d}x$$

find $y(x)$ minimizing S using the EL equation

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

For Classical Mechanics, we use the Lagrangian in the generalized coordinate system q_i we define the action S as

$$S = \int \mathcal{L}(q_i, \dot{q}_i, t)\mathrm{d}t \quad \text{find} \quad q(t)$$

and from the EL equation we get

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

for each degree of freedom. We define the Lagrangian in CM as the quantity $\mathcal{L} = T - U$

Examples, Examples, and more Examples: A pendulum but its spining on its axis. We first find the energies:

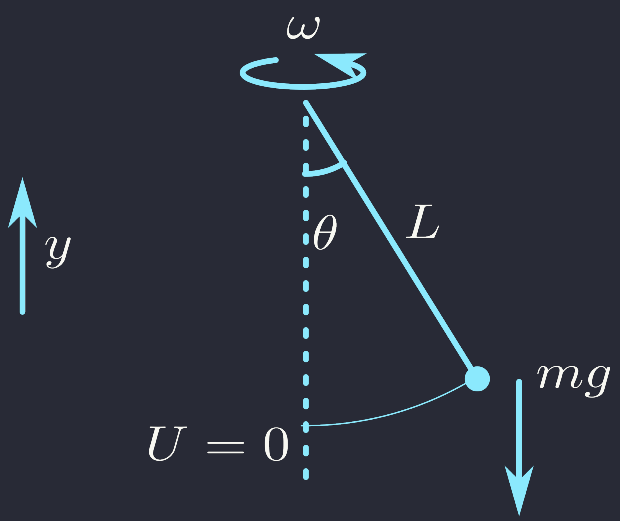

Figure 1.1: Pendulum

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m((\omega L\sin\theta)^2 + (L\dot{\theta})^2)$$
$$U = mgy = mgL(1 - \cos\theta)$$

from EL equation we get

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2}m\omega^2 L^2(2\sin\theta\cos\theta) - mgL\sin\theta = m\omega^2L^2\cos\theta\sin\theta - mgL\sin\theta$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\ddot{\theta}$$

so

$$mL^2\ddot{\theta} = m\omega^2 L^2 \cos\theta \sin\theta - mgL\sin\theta$$
$$\ddot{\theta} = \omega^2 \cos\theta\sin\theta - \frac{g}{L}\sin\theta$$

when $\omega = 0$ we get the simple pendulum $\ddot{\theta} = -\frac{g}{L}\sin\theta$. Identifying the the equilibrium points where $\ddot{\theta} = 0 \implies$

$$\sin\theta = 0 \implies \theta = 0, \pi$$

at $\theta = 0$ the pendulum is just hanging vertically down which we can physically deduce as a stable equilibrium point. To check this analytically we can assume a small deviation from the equilibrium point:

$$\theta = 0 + \epsilon$$
$$\cos(0+\epsilon) = 1 - \frac{\epsilon^2}{2} \approx 1$$
$$\sin(0+\epsilon) = \epsilon - \frac{\epsilon^3}{6} \approx \epsilon$$

and we get

$$\ddot{\theta} = (\omega^2 - \frac{g}{L})\theta$$
$$\ddot{\theta} = -\Omega^2\theta \implies \text{Stable}$$
$$\ddot{\theta} = \Omega^2\theta \implies \text{Unstable}$$

where

$$\omega^2 < \frac{g}{L} \implies \text{Stable}$$
$$\omega^2 > \frac{g}{L} \implies \text{Unstable}$$

when they are equal $\omega^2 = \frac{g}{L}$ we get a simple pendulum. Finding another equilibrium point at

$$\omega^2\cos\theta - \frac{g}{L} = 0$$
$$\cos\theta = \frac{g}{L\omega^2}, \qquad \theta = \pm\arccos\Big(\frac{g}{L\omega^2}\Big)$$

where there only exists a solution when

$$\omega^2 > \frac{g}{L}$$

since $\cos\theta \leq 1$. For this case, we can also look at the radial force in polar:

$$F_r = m\ddot{r} - mr\omega^2 \quad \text{or} \quad m\ddot{r} = F_r + mr\omega^2$$

where in the second equation we can see that the sum of the centrifugal force and F_r sums to zero so

$$\tan\theta = \frac{F_r}{mg} = \frac{mL\sin\theta\omega^2}{mg}$$
$$\implies \frac{L\omega^2}{g} = \frac{1}{\cos\theta}$$
$$\cos\theta = \frac{g}{L\omega^2}$$

from the force analysis we can see that the centrifugal force is balanced by the radial force. Substituing the equilibium position back into the EOM

$$\cos\theta_o = \frac{g}{L\omega^2} \to \theta = \theta_o + \epsilon$$

Using Taylor Expansion, $f(x) = f(x_o) + f'(x_o)(x - x_o)$, the sine and cosine terms are

$$\cos\theta = \cos(\theta_o + \epsilon) = \cos\theta_o - \sin\theta_o\epsilon$$
$$\sin\theta = \sin(\theta_o + \epsilon) = \sin\theta_o + \cos\theta_o\epsilon$$

so the EOM becomes

$$\ddot{\theta} = (-\omega^2 \sin\theta_o\epsilon)(\sin\theta_o + \cos\theta_o\epsilon)$$
$$\ddot{\epsilon} = -\omega^2 \sin^2(\theta_o)\epsilon$$

where we have Bifurcation at $\omega^2 = \frac{g}{L}$. We can see that the EOM for ϵ is similar to the harmonic oscillator so:

$$\epsilon = A\cos(\Omega t - \delta) \qquad \Omega = \omega\sin\theta_o$$

HW 5 Given $f(x, y, y')$. Independence of y means:

$$f(x, y') \implies \frac{\partial f}{\partial y'} = constant$$
$$\mathcal{L}(t, \dot{q}) \implies \frac{\partial \mathcal{L}}{\partial \dot{q}} = constant$$

so for the Lagrangian

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - U(x) \qquad \frac{\partial \mathcal{L}}{\partial q_i} = F_i \quad \text{generalized force}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = conserved, \qquad \frac{\partial U}{\partial x} = 0$$

if $\mathcal{L}$ doesn't depend on x, then p_x (momentum) is conserved. So for the generalized Lagrangian

$$\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$
$$\text{If} \quad \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{conserved}$$

So symmetry $\implies$ conservation (Noether's Theorem).

Conservation The two types:

- If $f(x, y')$ is independent of y, then

$$\frac{\partial f}{\partial y'} = \text{constant over } x$$

or if $\mathcal{L}$ is independent of q_i, then

$$\frac{\partial \mathcal{L}}{\partial q_i} = \text{constant over } t = p_i$$

- If $f(y, y')$ is independent of x, then

$$f - y' \frac{\partial f}{\partial y'} = \text{constant over } x$$

or if $\mathcal{L}$ is independent of t, then

$$\mathcal{L} - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant over } t$$

looking at this more closely:

$$\mathcal{L} = T - U = \frac{1}{2} m \dot{q}^2 - U(q)$$

where

$$\begin{aligned} \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} &= m\dot{q}^2; \\ \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} &= m\dot{q}^2 - \frac{1}{2} m \dot{q}^2 + U \\ &= \frac{1}{2} m \dot{q}^2 + U = T + U = E \end{aligned}$$

this is this Hamiltonian

$$\sum_i p_i \dot{q}_i - \mathcal{L} = \mathcal{H} = E$$

Noether's Theorem For a system independent of $t \leftrightarrow$ the system has time-translation symmetry $\implies$ conservation of energy

Dependence on t $U = U(q, t)$ e.g. Mass of sun is increasing over time, the potential energy is dependent on time, so the system is not conservative.

Pendulum thoughts: In our pendulum example, we chose $q = \theta$, but we could also choose $q_1 = x$ and $q_2 = y$. The truth lies in the fact that we intuitively chose $q_1 = r$ and $q_2 = \theta$. So in transforming from Cartesian coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = L$$

where we have a 'constraint' $r = L$...

Legal Terms: Formal Definition of Constraints In the beginning, we defined the first defined position with

$$\mathbf{r} = (x, y, z)$$

for the generalized coordinates we have

$$\mathbf{r} = \mathbf{r}(q_1, \ldots, q_n, t)$$

where we decided that in a 3D system $n = 3$. A constraint is an equation

$$f(q_1, \ldots, q_n) = 0$$

where this is a *holonomic* (whole) constraint and to find the number of generalized coordinates:

$$\begin{aligned} \#\text{ of generalized coordinates we need} &= \#\text{ of dimensions} - \#\text{ of constraints} \\ &= \#\text{ of degrees of freedom} \end{aligned}$$

this is only true for holonomic constraints. For *nonholonomic* constraints, it is more complicated e.g. A ball on a horizontal table: We can see that # of generalized coordinates = 2, but to describe the position of the ball i.e. a dot on the ball, we need 3 more coordinates (Euler angles). So the configuration of the ball is described by 5 coordinates $(x, y, \alpha, \beta, \gamma)$. In other words, the configuration is path dependent and we see a nonholonomic constraint.

Example: What are the constraints for the mass sliding down a moving mass? The holonomic constraints are the vertical position of the ramp $y_M = 0$, and from x_m, y_m, x_M we know the $x_{COM} =$ constant.

Fact! A constraint is enforced by a constraint force $\mathbf{F}_c \perp$ path(in the pendulum example, the normal force N). Finding this force where $f(q_i) = 0$ can be found by taking the gradient of the function $\boldsymbol{\nabla} f$. So

$$\mathbf{F}_c = \lambda \boldsymbol{\nabla} f$$

Lecture 15: 2/23/24

Review

- Convservation: Lagrangian is independent of time $\implies$ conservation of energy

Lagrange Multiplier Want to find $q_i(t)$ by minimizing $S = \int \mathcal{L}(q_i, \dot{q}_i, t)\mathrm{d}t$.

$\star$ Under holonomic constraints,

$$f(q_i) = 0$$

So we introduce a new unknown $\lambda(t)$ and the new minimizing integral becomes

$$I = \int (\mathcal{L} - \lambda f)\mathrm{d}t$$

The EL eqn for $\lambda(t)$: $\quad f = 0$

$$\frac{\partial(\mathcal{L} - \lambda f)}{\partial \lambda} = -f \qquad \frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{\lambda}} = 0$$

The EL eqn for $q_i(t)$:

$$F_i = \frac{\partial(\mathcal{L} - \lambda f)}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{q}_i}$$

or

$$F_i = \frac{\partial \mathcal{L}}{\partial q_i} - \lambda \frac{\partial f}{\partial q_i} \quad \text{where} \quad \frac{\partial \mathcal{L}}{\partial q_i} = p_i$$

So we are given $N+1$ unknowns and $N+1$ EL eqns with the addition of the lagrange multiplier.

Simple Pendulum (revisited) We have the Lagrangian $\mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U$ where

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$U = -mgy$$

so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

and using the constraint of the fixed length; $f(x, y) = x^2 + y^2 - L^2 = 0$ we get

$$\ell = \mathcal{L} - \lambda f = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy - \lambda(x^2 + y^2 - L^2)$$

and the EL eqns are

- x:

$$-2\lambda x = m\ddot{x}$$

- y:

$$mg - 2\lambda y = m\ddot{y}$$

- λ: Left as an exercise

We can see from force analysis of the pendulum:

$$m\ddot{x} = F_x = -2\lambda x \qquad m\ddot{y} = F_y = mg - 2\lambda y$$

so the lagrange multiplier quantities are equivalent to the tension

$$T_x = 2\lambda x \qquad T_y = 2\lambda y$$

where the negative sign indicates the correct direction of Tension.

Pendulum in Polar (r, ϕ)

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - mgr\cos\theta$$

where

$$f = r - L = 0$$

so we get the EL eqns

$$-\lambda + mg\cos\theta = m\ddot{r} \qquad \lambda = mg\cos\theta$$

Use cases of Lagrange Multipliers Although the previous example seems trivial, we consider its use in the example of a heavy chain hanging from two poles: The linear mass density is given by

$$M = \rho L$$

to find the shape, we need to minimize the potential energy

$$S = \int \mathrm{d}mgy$$

where $\mathrm{d}m = \rho \mathrm{d}s$ is the mass of a segment and under the constraint of chain length:

$$L = \int \mathrm{d}s = \int \mathrm{d}x\sqrt{1 + y'^2}$$

so

$$S = \int \rho gy\sqrt{1 + y'^2}\mathrm{d}x$$

and introducing λ we minimize

$$\int (\rho gy - \lambda)\sqrt{1 + y'^2}\mathrm{d}x = S - \lambda L$$

we can see that it is independent of x so

$$f = (\rho gy - \lambda)\sqrt{1 + y'^2}$$

and

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

so the EL eqn is:

$$\frac{\partial f}{\partial y'} = (\rho gy - \lambda)\frac{y'}{\sqrt{1 + y'^2}}$$

and therefore

$$f - y'\frac{\partial f}{\partial y'} = (\rho gy - \lambda)\left[\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}}\right] = \text{constant}$$

and quantity in brackets is

$$[\,] = \frac{1}{\sqrt{1 + y'^2}}$$

so

$$\left(\frac{\rho g y-\lambda}{\sqrt{1+y'^2}}\right)^2=C^2$$

$$1+y'^2=\frac{(\rho g y-\lambda)^2}{C^2}$$

for an easier solution we choose a change of variables

$$\tilde{y}=\frac{\rho g y-\lambda}{C} \implies \tilde{y}'=\frac{\rho g}{C}y'$$

and redifining the x

$$\tilde{y}'^2=1+y'^2 \begin{cases}\tilde{x}=\frac{\rho g}{C}x\\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}}=\frac{\mathrm{d}y}{\mathrm{d}x}\end{cases}$$

and we get

$$\tilde{y}=\pm\cosh(\tilde{x}-\tilde{x}_0)$$

we could have also used the Lagrange Multiplier for the Maximum Area Fixed Perimeter problem.