1 Lagrange's Equations

From last time: we defined the path

$$S = \int_a^b f(x, y(x), y'(x))\mathrm{d}x$$

Goal: find $y(x)$ that minimizes S using EL

$$\text{EL:} \quad \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

where near the minimum $\delta S = 0$. From the EL, $y(x)$ is a stationary point of S(could also be a maximum!).

Lagrangian In Classical Mechanics, we use a specific form

$$\mathcal{L} = T - V$$

this has the units of energy and the action S has the units $[S] = [E \cdot T]$ similar to planck's constant $\hbar$.

3D Cartesian $x, y, z = q_1, q_2, q_3$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$U = U(x, y, z)$$

where the potential energy only depends on the position and T only depends on the velocity, so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

and the EL equation for the Lagrangian is

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

For the 3D case, we have 3 equations of motion: For x we have

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x}, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

and using the EL equation, we get

$$-\frac{\partial U}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x}$$

which is Newton's second law $F_x = ma_x$ where $\mathbf{F} = -\boldsymbol{\nabla} U$. We can now get the general form

$$\mathbf{F} = m\mathbf{a}$$

Polar Coordinates $q : (r, \phi)$ we know that

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

and

$$U = U(r, \phi), \qquad T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

first we find the parts EL equation for r

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

and the EL equation is

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r})$$
$$m(\ddot{r} - r\dot{\phi}^2) = -\frac{\partial U}{\partial r}$$

which gives us N2L for r. For ϕ we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

and from the EL equation we get

$$-\frac{\partial U}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) = m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi})$$

dividing both sides by r

$$-\frac{1}{r}\frac{\partial U}{\partial \phi} = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = -(\boldsymbol{\nabla} U)_\phi$$

from both forms we know that the two parts of the EL represent the momentum and force:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \quad \text{generalized momentum}$$
$$\frac{\partial \mathcal{L}}{\partial q_i} = F_i \quad \text{generalized force}$$

where $F_i = \frac{\mathrm{d}}{\mathrm{d}t}p_i$ is the generalized N2L.

Example: Mass m sliding down a frictionless *moving* ramp M. First we choose the coordinates x moving along withe the ramp and y down in the perpendicular direction. For the ramp M:

$$T_M = \frac{1}{2}M\dot{q}_2^2, \quad U_M = 0$$

and for the mass m: First we decompose the velocity of m into the x and y components

$$\mathbf{v}_m = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} = \hat{\mathbf{y}}(\dot{q}_1 \sin\alpha) + \hat{\mathbf{x}}(\dot{q}_1\cos\alpha + \dot{q}_2)$$

and the kinetic and potential energies are

$$T_m = \frac{1}{2}mv_m^2 = \frac{1}{2}m(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha + \dot{q}_2^2)$$
$$U_m = mgy = -mg(\dot{q}_1\sin\alpha)$$

using the Lagrangian $\mathcal{L} = T - U = T_M + T_m - U_M - U_m$ we get

$$\frac{\partial \mathcal{L}}{\partial q_2} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = M\dot{q}_2 + m\dot{q}_2 + m\dot{q}_1\cos\alpha$$

and the EL equation gives us

$$(M+m)\ddot{q}_2 + m\ddot{q}_1 \cos\alpha = 0$$
$$a_2 = \ddot{q}_2 = -\frac{m\ddot{q}_1 \cos\alpha}{M+m}$$

and for q_1 we have

$$\frac{\partial \mathcal{L}}{\partial q_1} = mg\sin\alpha, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m(\dot{q}_1 + \dot{q}_2 \cos\alpha)$$

and the EL equation gives us

$$mg\sin\alpha = m(\ddot{q}_1 + \ddot{q}_2 \cos\alpha)$$

and since we have two equations and two unknowns, we can solve for $\ddot{q}_1$ and $\ddot{q}_2$.

$$\ddot{q}_1 = \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{m+M}} = const$$
$$\ddot{q}_2 = -\frac{m\ddot{q}_1 \cos\alpha}{M+m} = const$$

for $\alpha = 90°$, we get $\ddot{q}_1 = g$ and $\ddot{q}_2 = 0$ which is the same as a free falling. For and infinitely heavy ramp $M \to \infty$, we get $\ddot{q}_1 = g\sin\alpha$. For $M \to 0$ we get $\ddot{q}_1 = g/\sin\alpha$ which doesn't make sense because the force on the mass would be infinite. The normal force $N \to 0$ as $M \to 0$ and the mass would be in free fall.

Review Lagrangian: For a general integral

$$S\int f(x,y,y')\mathrm{d}x$$

find $y(x)$ minimizing S using the EL equation

$$\frac{\partial f}{\partial y}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

For Classical Mechanics, we use the Lagrangian in the generalized coordinate system q_i we define the action S as

$$S=\int \mathcal{L}(q_i,\dot{q}_i,t)\mathrm{d}t \quad \text{find} \quad q(t)$$

and from the EL equation we get

$$\frac{\partial \mathcal{L}}{\partial q_i}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

for each degree of freedom. We define the Lagrangian in CM as the quantity $\mathcal{L}=T-U$

Examples, Examples, and more Examples: A pendulum but its spining on its axis. We first find the energies:

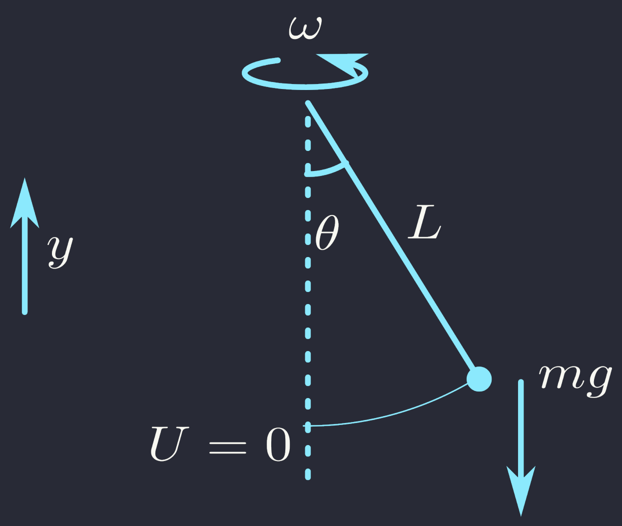

Figure 1.1: Pendulum

$$T=\frac{1}{2}mv^2=\frac{1}{2}m((\omega L\sin\theta)^2+(L\dot{\theta})^2)$$
$$U=mgy=mgL(1-\cos\theta)$$

from EL equation we get

$$\frac{\partial \mathcal{L}}{\partial \theta}=\frac{1}{2}m\omega^2L^2(2\sin\theta\cos\theta)-mgL\sin\theta=m\omega^2L^2\cos\theta\sin\theta-mgL\sin\theta$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}}=mL^2\dot{\theta}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}=mL^2\ddot{\theta}$$

so

$$mL^2\ddot{\theta} = m\omega^2 L^2 \cos\theta \sin\theta - mgL\sin\theta$$
$$\ddot{\theta} = \omega^2 \cos\theta \sin\theta - \frac{g}{L}\sin\theta$$

when $\omega = 0$ we get the simple pendulum $\ddot{\theta} = -\frac{g}{L}\sin\theta$. Identifying the the equilibrium points where $\ddot{\theta} = 0 \implies$

$$\sin\theta = 0 \implies \theta = 0, \pi$$

at $\theta = 0$ the pendulum is just hanging vertically down which we can physically deduce as a stable equilibrium point. To check this analytically we can assume a small deviation from the equilibrium point:

$$\theta = 0 + \epsilon$$
$$\cos(0+\epsilon) = 1 - \frac{\epsilon^2}{2} \approx 1$$
$$\sin(0+\epsilon) = \epsilon - \frac{\epsilon^3}{6} \approx \epsilon$$

and we get

$$\ddot{\theta} = (\omega^2 - \frac{g}{L})\theta$$
$$\ddot{\theta} = -\Omega^2\theta \implies \text{Stable}$$
$$\ddot{\theta} = \Omega^2\theta \implies \text{Unstable}$$

where

$$\omega^2 < \frac{g}{L} \implies \text{Stable}$$
$$\omega^2 > \frac{g}{L} \implies \text{Unstable}$$

when they are equal $\omega^2 = \frac{g}{L}$ we get a simple pendulum. Finding another equilibrium point at

$$\omega^2 \cos\theta - \frac{g}{L} = 0$$
$$\cos\theta = \frac{g}{L\omega^2}, \qquad \theta = \pm\arccos\left(\frac{g}{L\omega^2}\right)$$

where there only exists a solution when

$$\omega^2 > \frac{g}{L}$$

since $\cos\theta \leq 1$. For this case, we can also look at the radial force in polar:

$$F_r = m\ddot{r} - mr\omega^2 \quad \text{or} \quad m\ddot{r} = F_r + mr\omega^2$$

where in the second equation we can see that the sum of the centrifugal force and F_r sums to zero so

$$\tan\theta = \frac{F_r}{mg} = \frac{mL\sin\theta\omega^2}{mg}$$
$$\implies \frac{L\omega^2}{g} = \frac{1}{\cos\theta}$$
$$\cos\theta = \frac{g}{L\omega^2}$$

from the force analysis we can see that the centrifugal force is balanced by the radial force. Substituing the equilibium position back into the EOM

$$\cos\theta_o = \frac{g}{L\omega^2} \rightarrow \theta = \theta_o + \epsilon$$

Using Taylor Expansion, $f(x) = f(x_o) + f'(x_o)(x - x_o)$, the sine and cosine terms are

$$\cos\theta = \cos(\theta_o + \epsilon) = \cos\theta_o - \sin\theta_o\epsilon$$
$$\sin\theta = \sin(\theta_o + \epsilon) = \sin\theta_o + \cos\theta_o\epsilon$$

so the EOM becomes

$$\ddot{\theta} = (-\omega^2 \sin\theta_o\epsilon)(\sin\theta_o + \cos\theta_o\epsilon)$$
$$\ddot{\epsilon} = -\omega^2 \sin^2(\theta_o)\epsilon$$

where we have Bifurcation at $\omega^2 = \frac{g}{L}$. We can see that the EOM for ϵ is similar to the harmonic oscillator so:

$$\epsilon = A\cos(\Omega t - \delta) \qquad \Omega = \omega\sin\theta_o$$

HW 5 Given $f(x, y, y')$. Independence of y means:

$$f(x, y') \implies \frac{\partial f}{\partial y'} = constant$$
$$\mathcal{L}(t, \dot{q}) \implies \frac{\partial \mathcal{L}}{\partial \dot{q}} = constant$$

so for the Lagrangian

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - U(x) \qquad \frac{\partial \mathcal{L}}{\partial q_i} = F_i \quad \text{generalized force}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = conserved, \qquad \frac{\partial U}{\partial x} = 0$$

if $\mathcal{L}$ doesn't depend on x, then p_x (momentum) is conserved. So for the generalized Lagrangian

$$\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$
$$\text{If} \quad \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{conserved}$$

So symmetry $\implies$ conservation (Noether's Theorem).

Conservation The two types:

- If $f(x, y')$ is independent of y, then

$$\frac{\partial f}{\partial y'} = \text{constant over } x$$

or if $\mathcal{L}$ is independent of q_i, then

$$\frac{\partial \mathcal{L}}{\partial q_i} = \text{constant over } t = p_i$$

- If $f(y, y')$ is independent of x, then

$$f - y' \frac{\partial f}{\partial y'} = \text{constant over } x$$

or if $\mathcal{L}$ is independent of t, then

$$\mathcal{L} - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant over } t$$

looking at this more closely:

$$\mathcal{L} = T - U = \frac{1}{2} m\dot{q}^2 - U(q)$$

where

$$\begin{aligned}\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} &= m\dot{q}^2; \\ \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} &= m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + U \\ &= \frac{1}{2}m\dot{q}^2 + U = T + U = E\end{aligned}$$

this is this Hamiltonian

$$\sum_i p_i \dot{q}_i - \mathcal{L} = \mathcal{H} = E$$

Noether's Theorem For a system independent of $t \leftrightarrow$ the system has time-translation symmetry $\implies$ conservation of energy

Dependence on t $U = U(q, t)$ e.g. Mass of sun is increasing over time, the potential energy is dependent on time, so the system is not conservative.

Pendulum thoughts: In our pendulum example, we chose $q = \theta$, but we could also choose $q_1 = x$ and $q_2 = y$. The truth lies in the fact that we intuitively chose $q_1 = r$ and $q_2 = \theta$. So in transforming from Cartesian coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = L$$

where we have a 'constraint' $r = L\ldots$

Legal Terms: Formal Definition of Constraints In the beginning, we defined the first defined position with

$$\mathbf{r} = (x, y, z)$$

for the generalized coordinates we have

$$\mathbf{r} = \mathbf{r}(q_1, \ldots, q_n, t)$$

where we decided that in a 3D system $n = 3$. A constraint is an equation

$$f(q_1, \ldots, q_n) = 0$$

where this is a *holonomic* (whole) constraint and to find the number of generalized coordinates:

$$\begin{aligned} \text{\# of generalized coordinates we need} &= \text{\# of dimensions} - \text{\# of constraints} \\ &= \text{\# of degrees of freedom} \end{aligned}$$

this is only true for holonomic constraints. For *nonholonomic* constraints, it is more complicated e.g. A ball on a horizontal table: We can see that # of generalized coordinates = 2, but to describe the position of the ball i.e. a dot on the ball, we need 3 more coordinates (Euler angles). So the configuration of the ball is described by 5 coordinates $(x, y, \alpha, \beta, \gamma)$. In other words, the configuration is path dependent and we see a nonholonomic constraint.

Example: What are the constraints for the mass sliding down a moving mass? The holonomic constraints are the vertical position of the ramp $y_M = 0$, and from x_m, y_m, x_M we know the $x_{COM} =$ constant.

Fact! A constraint is enforced by a constraint force $\mathbf{F}_c \perp$ path(in the pendulum example, the normal force N). Finding this force where $f(q_i) = 0$ can be found by taking the gradient of the function $\boldsymbol{\nabla} f$. So

$$\mathbf{F}_c = \lambda \boldsymbol{\nabla} f$$

Lecture 15: 2/23/24

Review

- Convservation: Lagrangian is independent of time $\implies$ conservation of energy

Lagrange Multiplier Want to find $q_i(t)$ by minimizing $S = \int \mathcal{L}(q_i, \dot{q}_i, t)\mathrm{d}t$.

$\star$ Under holonomic constraints,

$$f(q_i) = 0$$

So we introduce a new unknown $\lambda(t)$ and the new minimizing integral becomes

$$I = \int (\mathcal{L} - \lambda f)\mathrm{d}t$$

The EL eqn for $\lambda(t)$: $f = 0$

$$\frac{\partial(\mathcal{L} - \lambda f)}{\partial \lambda} = -f \qquad \frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{\lambda}} = 0$$

The EL eqn for $q_i(t)$:

$$F_i = \frac{\partial(\mathcal{L} - \lambda f)}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{q}_i}$$

or

$$F_i = \frac{\partial \mathcal{L}}{\partial q_i} - \lambda \frac{\partial f}{\partial q_i} \quad \text{where} \quad \frac{\partial \mathcal{L}}{\partial q_i} = p_i$$

So we are given $N+1$ unknowns and $N+1$ EL eqns with the addition of the lagrange multiplier.

Simple Pendulum (revisited) We have the Lagrangian $\mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U$ where

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$U = -mgy$$

so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

and using the constraint of the fixed length; $f(x, y) = x^2 + y^2 - L^2 = 0$ we get

$$\ell = \mathcal{L} - \lambda f = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy - \lambda(x^2 + y^2 - L^2)$$

and the EL eqns are

- x:

$$-2\lambda x = m\ddot{x}$$

- y:

$$mg - 2\lambda y = m\ddot{y}$$

- λ: Left as an exercise

We can see from force analysis of the pendulum:

$$m\ddot{x} = F_x = -2\lambda x \qquad m\ddot{y} = F_y = mg - 2\lambda y$$

so the lagrange multiplier quantities are equivalent to the tension

$$T_x = 2\lambda x \qquad T_y = 2\lambda y$$

where the negative sign indicates the correct direction of Tension.

Pendulum in Polar (r, ϕ)

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - mgr\cos\theta$$

where

$$f = r - L = 0$$

so we get the EL eqns

$$-\lambda + mg\cos\theta = m\ddot{r} \qquad \lambda = mg\cos\theta$$

Use cases of Lagrange Multipliers Although the previous example seems trivial, we consider its use in the example of a heavy chain hanging from two poles: The linear mass density is given by

$$M = \rho L$$

to find the shape, we need to minimize the potential energy

$$S = \int \mathrm{d}mgy$$

where $\mathrm{d}m = \rho \mathrm{d}s$ is the mass of a segment and under the constraint of chain length:

$$L = \int \mathrm{d}s = \int \mathrm{d}x\sqrt{1 + y'^2}$$

so

$$S = \int \rho gy\sqrt{1 + y'^2}\mathrm{d}x$$

and introducing λ we minimize

$$\int (\rho gy - \lambda)\sqrt{1 + y'^2}\mathrm{d}x = S - \lambda L$$

we can see that it is independent of x so

$$f = (\rho gy - \lambda)\sqrt{1 + y'^2}$$

and

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

so the EL eqn is:

$$\frac{\partial f}{\partial y'} = (\rho gy - \lambda)\frac{y'}{\sqrt{1 + y'^2}}$$

and therefore

$$f - y'\frac{\partial f}{\partial y'} = (\rho gy - \lambda)\left[\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}}\right] = \text{constant}$$

and quantity in brackets is

$$[\,] = \frac{1}{\sqrt{1 + y'^2}}$$

so

$$\left(\frac{\rho g y-\lambda}{\sqrt{1+y'^2}}\right)^2=C^2$$
$$1+y'^2=\frac{(\rho g y-\lambda)^2}{C^2}$$

for an easier solution we choose a change of variables

$$\tilde{y}=\frac{\rho g y-\lambda}{C} \implies \tilde{y}'=\frac{\rho g}{C}y'$$

and redifining the x

$$\tilde{y}'^2=1+y'^2 \begin{cases}\tilde{x}=\frac{\rho g}{C}x \\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}}=\frac{\mathrm{d}y}{\mathrm{d}x}\end{cases}$$

and we get

$$\tilde{y}=\pm\cosh(\tilde{x}-\tilde{x}_0)$$

we could have also used the Lagrange Multiplier for the Maximum Area Fixed Perimeter problem.

Review Constraint – holonomic $f(q,\dots,q_n)=0 \to$ Lagrange Multiplier

Example Simple pendulum spinning on its vertical axis. We have the Lagrangian

$$\begin{aligned}
T &= \frac{1}{2}m(L^2\dot{\theta}^2 + L^2\sin^2\theta\omega^2) \\
U &= mgy = mgL(1-\cos\theta) \\
\mathcal{L} &= T - U \\
&= \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mL^2\sin^2\theta\omega^2 + mgL(1-\cos\theta)
\end{aligned}$$

but we can see the derivatives are also conserved:

$$\begin{aligned}
T' &= \frac{1}{2}mL^2\dot{\theta}^2, \quad U' = -\frac{1}{2}mL^2\omega^2\sin^2\theta + mg(1-\cos\theta) \\
\mathcal{L} &= T' - U'
\end{aligned}$$

where U' is called the effective potential. E' is conserved, so

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= ml^2\dot{\theta} \quad \text{angular momentum} \\
\dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \mathcal{L} &= E' \\
&= \frac{1}{2}mL^2\dot{\theta}^2 - \frac{1}{2}mL^2\omega^2\sin^2\theta + mgL(1-\cos\theta) = T' + U' \\
T + U &= \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mL^2\omega^2\sin^2\theta + mgL(1-\cos\theta)
\end{aligned}$$

we can see that the conserved quantity is different the mechanical energy $E = T + U$. We should be careful with finding what is the conserved quantity in noninertial frames (mechanical energy is not always conserved). In order to study the the function, we should look at the E' term.

Equilibrium points:

$$\begin{aligned}
\frac{\mathrm{d}U'}{\mathrm{d}\theta} &= 0 \\
(g - L\omega^2\cos\theta)\sin\theta &= 0
\end{aligned}$$

- if $\omega^2 < \frac{g}{L}$ then only 1 equilibrium point
- if $\omega^2 > \frac{g}{L}$ then 2 equilibrium points

Figure 1.2 shows the effective potential $U'(\theta)$ for the two cases. Sidenote: spinning the green curve around the E' axis gives rise to the Mexican Hat potential in particle physics(Higgs Boson!).

Lagrangian for a charged particle $\mathbf{E}, \mathbf{B}$: we have a Lorentz force

$$m\mathbf{a} = F = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

where we can define a vector potential $\mathbf{A}$ such that

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A}$$

Goal: to find $\mathcal{L}$ that gives

$$\frac{\partial\mathcal{L}}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)$$

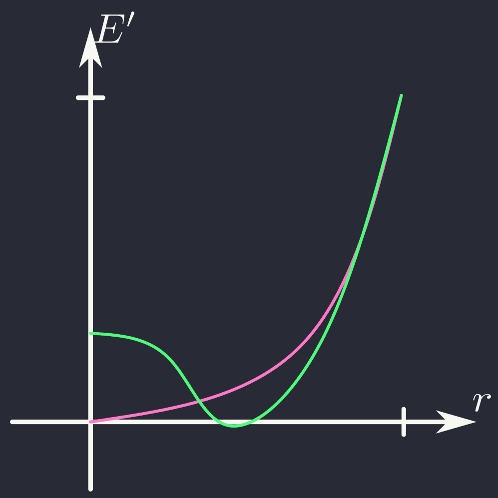

Figure 1.2: Red shows 1 eq point, green shows 2 eq points

so

$$\mathcal{L} = \frac{1}{2} m\mathbf{v}^2 - q(\phi - \mathbf{v}\cdot\mathbf{A})$$

the generalized coordinate is $q(x, y, z)$ and we just look at x:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= -q\frac{\mathrm{d}\phi}{\mathrm{d}x} + q\mathbf{v}\cdot\frac{\partial \mathbf{A}}{\partial x} \\
&= -q\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) \\
\frac{\partial \mathcal{L}}{\partial \dot{x}} &= mv_x + qA_x
\end{aligned}$$

so from the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\dot{v}_x + q(\cancel{v_x\frac{\partial A_x}{\partial x}} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t})$$

so

$$\begin{aligned}
m\dot{v}_x = ma_x = -q&\left(\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) && \equiv qE_x \\
&+ qv_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) && \equiv B_z \\
&- qv_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) && \equiv B_y \\
= qE_x &+ qv_yB_z - qv_zB_y
\end{aligned}$$

which is the Lorentz force. We can think of the momentum as

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \qquad \mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

and from QM we know

$$\mathbf{p} \to -i\hbar\boldsymbol{\nabla} - q\mathbf{A}$$

And reviewing from Chapter 2, the linear drag force

$$\mathbf{f} = -b\mathbf{v}$$

is obviously nonconservative. So can get the EL eqn quickly by adding this force to the generalized force

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} + f$$
$$p_i = F + f$$

this comes from Rayliegh dissipation function.

$$R(\dot{q}) = \frac{1}{2}b\dot{q}^2 \qquad f = -\frac{\partial R}{\partial \dot{q}}$$

so the EL eqn is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial R}{\partial \dot{q}}$$

where we have functions

$$T(\dot{q}), U(q), R(\dot{q})$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial R}{\partial \dot{q}} + \frac{\partial U}{\partial q} = 0$$

Midterm Review

- Newton's Laws
  - 1. Inertial: Keep on going and it won't stop coming, so much to do so much to see.
  - 2. $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}$
  - 3. $\mathbf{F}_{12} = -\mathbf{F}_{21}$
- Polar Coordinates (r, ϕ)

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

  and unit vectors are orthogonal

$$\begin{cases} \hat{r} = \cos\phi\hat{x} + \sin\phi\hat{y} \\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \end{cases} \quad \rightarrow \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$$

  such that the unit vector time derivatives are

$$\dot{\hat{r}} = \dot{\phi}\hat{\boldsymbol{\phi}}$$
$$\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$$

  so the velocity and acceleration is actually

$$\begin{aligned} \mathbf{v} = \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \\ &= v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} \\ \mathbf{a} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}} \end{aligned}$$

  where the forces are

$$F_r = ma_r \qquad F_\phi = ma_\phi$$

- Momentum $\mathbf{p} = m\mathbf{v}$ and in relation to force $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. For a collection of particles, the total external force is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{p}_i = \mathbf{F}_{ext}$$

- Angular momentum

$$\ell = \mathbf{r} \times \mathbf{p}$$

- Center of mass

$$\mathbf{R} = \frac{1}{M}\sum m_i\mathbf{r}_i \qquad M = \sum m_i$$
$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\mathrm{d}m$$

- Energy: Kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$$

  and for the two coordinate systems:

$$v^2 = v_x^2 + v_y^2 = v_r^2 + v_\phi^2$$

- Work-KE Theorem:

$$T_2 - T_1 = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = W(1 \to 2)$$

and if $\mathbf{F}$ is conservative—only depends on position: $\mathbf{F}(\mathbf{r})$ & $\boldsymbol{\nabla} \times \mathbf{F} = 0$ thus $\mathbf{F} = -\boldsymbol{\nabla} U$—then

$$W(1 \to 2) = -\Delta U = U_1 - U_2$$
$$E = T_1 + U_1 = T_2 + U_2$$

and more closely finding the critical points of U i.e.

$$\frac{\partial U}{\partial x} = 0 \quad \text{or} \quad \boldsymbol{\nabla} U = 0$$

we also have classical turning points when $E = U$. (Not too important) For the case

$$\frac{1}{2} m \dot{x}^2 = E - U(x)$$
$$\implies \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

- Oscillators

$$\ddot{x} = -\frac{k}{m} x = -\omega_o^2 x \qquad \omega_o = \sqrt{\frac{k}{m}}$$

the solution is written in several forms:

$$x(t) = B_1 \cos(\omega_o t) + B_2 \sin(\omega_o t)$$
$$= \mathrm{Re}[C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t}]$$
$$= A \cos(\omega_o t - \delta)$$

where we solve for the constants using the initial conditions $x(0) = x_o, \quad \dot{x}(0) = v_o$

- Damped Oscillators:

$$\ddot{x} + 2\beta \dot{x} + \omega_o^2 x = 0$$

where we have a homogeneous solution for the three cases:

  - $\beta = \omega_o$: Critical damping

$$x_h(t) = e^{-\beta t}(C_1 + C_2 t)$$

  - $\beta > \omega_o$: Overdamping

$$x_h(t) = e^{-(\beta - \sqrt{\beta^2 - \omega_o^2})t} \left(C_1 + C_2 e^{-2\sqrt{\beta^2 - \omega_o^2}t} \right)$$

  - $\beta < \omega_o$: Underdamping (weak damping)

$$x_h(t) = e^{-\beta t} A \cos(\omega t - \delta) \qquad \omega = \sqrt{\omega_o^2 - \beta^2}$$

- Driven Damped Oscillators:

$$\ddot{x} + 2\beta \dot{x} + \omega_o^2 x = f_0 \cos(\omega t)$$

where f_o has units of acceleration and β has units of frequency. The solution is always

$$x(t) = A \cos(\omega t - \delta) + x_h(t)$$

where $x_h(t)$ is the transient solution and the constants are

$$A^2 = \frac{f_0^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2 \omega^2} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2} \right)$$

where we have a resonance frequency around $\omega = \omega_o$.

- Calculus of Variations: minimizing the action

$$S = \int f(x, y, y')\mathrm{d}x$$

to find $y(x)$ from the EL eqn

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

- Lagrangian (Application of CoV)

$$\mathcal{L} = T - U$$

where the EL eqns are

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

- Conservation: Two special cases
  - If $\mathcal{L}$ is independent of $q_i \Leftrightarrow \frac{\partial \mathcal{L}}{\partial q_i} = 0$

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{is conserved}$$

  - If $\mathcal{L}$ is independent of t

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L} = \text{constant}$$