Homework 7

Due 3/20

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1. (a) The center of mass is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \qquad M = m_1 + m_2$$

we can rewrite the position vectors in terms of the COM and separation vector:

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}$$
$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

Transforming into the CM frame since $\mathbf{R}$ is an ignorable coordinate ($\mathcal{L}$ doesn't depend on $\mathbf{R}$)

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\mathbf{r}$$
$$\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\mathbf{r}$$

so The kinetic energy of the two particles are

$$T_1 = \frac{1}{2}m_1\left(\frac{m_2}{M}\dot{\mathbf{r}}\right)^2 \qquad T_2 = \frac{1}{2}m_2\left(-\frac{m_1}{M}\dot{\mathbf{r}}\right)^2$$

and the potential energy is

$$U = \frac{1}{2}kr^2$$

The Lagrangian in polar coordinates is

$$\begin{aligned}
\mathcal{L} &= T_1 + T_2 - U \\
&= \frac{1}{2}\frac{m_1 m_2^2}{M^2}\dot{\mathbf{r}}^2 + \frac{1}{2}\frac{m_1^2 m_2}{M^2}\dot{\mathbf{r}}^2 - \frac{1}{2}kr^2 \\
&= \frac{1}{2}\frac{m_1 m_2(m_1+m_2)}{M(m_1+m_2)}\dot{\mathbf{r}}^2 - \frac{1}{2}kr^2 \\
&= \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{1}{2}kr^2 \qquad \mu = \frac{m_1 m_2}{m_1+m_2} \\
\mathcal{L} &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{1}{2}kr^2
\end{aligned}$$

(b) From the EL equation for we find a conserved quantity

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r\dot{\phi} = \ell$$

and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) &= \mu\ddot{r} \\
\implies \mu\ddot{r} &= \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} \\
&= \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r} \qquad U = \frac{1}{2}kr^2
\end{aligned}$$

rewriting the first term as the negative gradient of a potential i.e.

$$\mu\ddot{r} = -\frac{\partial}{\partial r}(U_{cf} + U)$$

$$\frac{\ell^2}{\mu r^3} = -\frac{\partial U_{cf}}{\partial r}$$
$$U_{cf} = \frac{\ell^2}{2\mu r^2}$$

so we have an effective potential

$$U_{\text{eff}} = U + U_{cf}$$
$$= \frac{1}{2}kr^2 + \frac{\ell^2}{2\mu r^2}$$

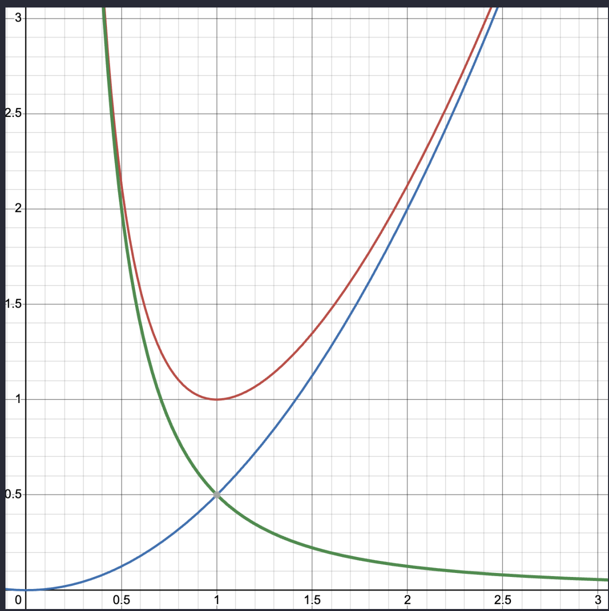

Figure 7.1: Effective potential U_{eff} for the reduced mass in Red. U_{cf} is in Green and U is in Blue.

To find the equilibrium point, we find the minimum of U_{eff} or when the derivative is zero:

$$U'_{\text{eff}}(r_0) = kr_0 - \frac{\ell^2}{\mu r_0^3} = 0$$
$$kr_0 = \frac{\ell^2}{\mu r_0^3}$$
$$r_0^4 = \frac{\ell^2}{\mu k}$$
$$r_0 = \left(\frac{\ell^2}{\mu k}\right)^{1/4}$$

We can see from the sketch that r_0 is a stable equilibrium point.
(c) Taylor expanding the effective potential about r_0:

$$U_{\text{eff}}(r) \approx U_{\text{eff}}(r_0) + (r - r_0)U'_{\text{eff}}(r_0) + \frac{1}{2}(r - r_0)^2 U''_{\text{eff}}(r_0)$$

When we set the reference point at r_0, the first term is zero, the second term is zero from part (b), so the third term is

$$
\begin{aligned}
U''_{\text{eff}}(r_0) &= k + \frac{3\ell^2}{\mu r_0^4} \\
&= k + \frac{3\ell^2}{\mu}\frac{1}{\left(\frac{\ell^2}{\mu k}\right)} \\
&= k + 3k \\
&= 4k \\
\implies U_{\text{eff}}(r) &\approx \frac{1}{2}(4k)\Delta r^2 \qquad \Delta r = r - r_0
\end{aligned}
$$

which would give the equation of motion

$$\mu \ddot{\Delta r} = -4k\Delta r$$

with a frequency of

$$\omega = \sqrt{\frac{4k}{\mu}}$$

The periodicity of the motion implies closed orbit?

2. (a) Given $U = kr^n$, the negative gradient of the potential is the force:

$$F = -\boldsymbol{\nabla} U = -\frac{\partial U}{\partial r} = -knr^{n-1} = -\frac{n}{r}U$$

where the magnitude of force is also equivalent to the centripetal force $F = -mv^2/r$ (the sign specifies the inward direction). Therefore,

$$\begin{aligned} -\frac{mv^2}{r} &= -\frac{n}{r}U \\ mv^2 &= nU \end{aligned}$$

Subbing into the KE equation

$$T = \frac{1}{2}mv^2 = \frac{1}{2}nU$$

(b) Given $G = \mathbf{r} \cdot \mathbf{p}$, taking the time derivative of G:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}G &= \dot{\mathbf{r}} \cdot \mathbf{p} + \mathbf{r} \cdot \dot{\mathbf{p}} && (7.1) \\ &= mv^2 + \mathbf{F} \cdot \mathbf{r} && (7.2) \\ &= 2T + \mathbf{F} \cdot \mathbf{r} && (7.3) \end{aligned}$$

and integrating from $t = 0 \to t$ ():

$$\begin{aligned} \int_0^t 2T + \mathbf{F} \cdot \mathbf{r} \, \mathrm{d}t &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} G \, \mathrm{d}t \\ &= G(t) - G(0) \end{aligned}$$

and diving by t:

$$\begin{aligned} \frac{G(t) - G(0)}{t} &= 2\frac{1}{t}\int_0^t T \, \mathrm{d}t + \frac{1}{t}\int_0^t \mathbf{F} \cdot \mathbf{r} \, \mathrm{d}t \\ &= 2\langle T \rangle + \langle \mathbf{F} \cdot \mathbf{r} \rangle \end{aligned}$$

(c) Letting $U = kr^n$, we can find the force:

$$\mathbf{F} = -\boldsymbol{\nabla} U = -\frac{\partial U}{\partial r} = -knr^{n-1}\hat{\mathbf{r}} = -\frac{n}{r}U\hat{\mathbf{r}}$$

so the work done is

$$\mathbf{F} \cdot \mathbf{r} = -nU$$

So in the equation from (b) for $t \to \infty$

$$\frac{G(t) - G(0)}{t} = 0$$

the left side goes to zero and

$$0 = 2\langle T \rangle + \langle -nU \rangle$$

and since n is some constant

$$\begin{aligned} 0 &= 2\langle T \rangle - n\langle U \rangle \\ \implies \langle T \rangle &= \frac{n}{2}\langle U \rangle \end{aligned}$$

3. (a)