4.15 Thick spherical shell of inner radius a, outer radius b, with polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}}$$

(a) E-field in all regions using bound charges: The volume bound charge is

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$$

and the surface bound charges are ($\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ at $r = a$)

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} -\dfrac{k}{a} & r = a \\ \dfrac{k}{b} & r = b \end{cases}$$

Using Gauss's Law

$$\oint \mathbf{E}\cdot \mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

(i) $r < a$: $Q_{\text{enc}} = 0$, so $\mathbf{E} = 0$

(ii) $a < r < b$: The enclosed charge is the inner surface charge plus the volume charge:

$$\begin{aligned} Q_{\text{enc}} &= \oint_S \sigma_b \,\mathrm{d}\mathbf{a} + \int_V \rho_b \,\mathrm{d}\tau \\ &= \int -\frac{k}{a}a^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi + \int -\frac{k}{r^2}r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -4\pi k a - 4\pi k(r-a) = -4\pi k r \end{aligned}$$

So using Gauss's Law

$$\begin{aligned} \oint \mathbf{E}\cdot\mathrm{d}\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ |\mathbf{E}|\oint \mathrm{d}a &= -\frac{4\pi k r}{\epsilon_0} \\ |\mathbf{E}|4\pi r^2 &= -\frac{4\pi k r}{\epsilon_0} \\ |\mathbf{E}| &= -\frac{k}{\epsilon_0 r} \end{aligned}$$

or

$$\mathbf{E} = -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}}$$

(iii) $r > b$: The total enclosed charge of a dielectric is zero (from last HW 4.14), so $\mathbf{E} = 0$

(b) Using

$$\oint D\cdot \mathrm{d}\mathbf{a} = Q_{\text{free}} \tag{4.23}$$

and

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \tag{4.21}$$

the total free enclosed charge is zero, so

$$\mathbf{D} = 0$$

Thus

$$\epsilon_0 \mathbf{E} + \mathbf{P} = 0$$

or

$$\mathbf{E} = -\frac{1}{\epsilon_0}\mathbf{P} = \begin{cases} 0 & r < a \quad \text{and} \quad r > b \\ -\dfrac{k}{\epsilon_0 r}\hat{\mathbf{r}} & a < r < b \end{cases}$$

4.17 Bar electret from Prob. 4.11 has $\rho_b = 0$: From divergence theorem

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{D})\, \mathrm{d}\tau = \oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{\text{free}} = 0 \implies \boldsymbol{\nabla} \cdot \mathbf{D} = 0 \tag{4.22}$$

So, the field lines for $\mathbf{D}$ are closed loops as shown in Fig. 7.1.

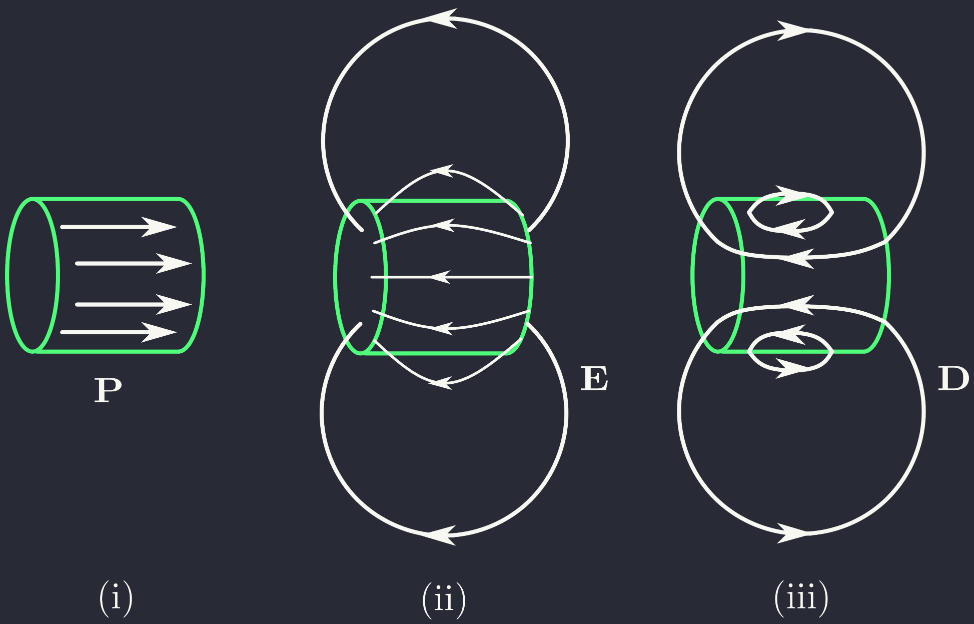

Figure 7.1: Field lines for (i) $\mathbf{P}$ (ii) $\mathbf{E}$ (iii) $\mathbf{D}$

4.19 For a parallel plate capacitor, the E-field in the air between the plates is two times the E-field of a single plate

(a)

$$\mathbf{E}_{\text{air}} = 2\mathbf{E}_{\text{plate}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

where $\sigma = \pm Q/A$ is the surface charge density on the plates. The displacement field is

$$\oint D \cdot \mathrm{d}\mathbf{a} = Q_{\text{free}}$$
$$D(A) = \sigma A$$
$$\implies D = \sigma$$

In the linear dielectric medium, $\mathbf{D}$ is proportional to $\mathbf{E}$ by

$$\mathbf{D} = \epsilon\mathbf{E} \tag{4.32}$$

So

$$\implies E = \frac{D}{\epsilon} = \frac{\sigma}{\epsilon}$$

and the potential difference between the plates is the sum of the potential differences in the air and the dielectric

$$\begin{aligned} V &= Ed \\ &= E_{\text{air}}d_{\text{air}} + E_{\text{die}}d_{\text{die}} \\ &= \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} \quad \text{using} \quad \sigma = \frac{Q}{A} \\ &= \frac{Qd}{2A\epsilon_0}\left[1 + \frac{\epsilon_0}{\epsilon}\right] \end{aligned}$$

Finally the capacitance of the half filled capacitor (a) is

$$C_a = \frac{Q}{V} = \frac{2A\epsilon_0}{d}\frac{1}{1+\frac{\epsilon_0}{\epsilon}}$$

Compared to the capacitance of a fully filled capacitor

$$C = \frac{A\epsilon_0}{d} \tag{2.54}$$

So the increase in capacitance is

$$\frac{C_a}{C} = \frac{2}{1+\frac{\epsilon_0}{\epsilon}}$$

Using the relative permittivity $\epsilon_r = \epsilon/\epsilon_0$, the capacitance of the half filled capacitor (a) increases by a factor of

$$\boxed{\frac{C_a}{C} = \frac{2\epsilon_r}{1+\epsilon_r}}$$

(i) In air: the E-field increases by the same factor so for a given potential $V = Ed$,

$$\boxed{\mathbf{E}_{\text{air}} = \frac{2\epsilon_r}{1+\epsilon_r}\frac{V}{d}\hat{\mathbf{n}}}$$

And since $\boxed{\mathbf{P}_{\text{air}} = 0}$ in air, the displacement field $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ is

$$\boxed{\mathbf{D}_{\text{air}} = \frac{2\epsilon_r}{1+\epsilon_r}\epsilon_0\frac{V}{d}\hat{\mathbf{n}}}$$

(ii) In the dielectric: The E-field in the dielectric is proportional to the E-field in the air (vacuum) by

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}} \tag{4.35}$$

so

$$\boxed{\mathbf{E}_{\text{die}} = \frac{2}{1+\epsilon_r}\frac{V}{d}\hat{\mathbf{n}}}$$

and from (4.35)

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \epsilon = \epsilon_r\epsilon_0$$

$$\implies \boxed{\mathbf{D}_{\text{die}} = \frac{2\epsilon_r}{1+\epsilon_r}\epsilon_0\frac{V}{d}\hat{\mathbf{n}} = \mathbf{D}_a}$$

Finally, we can find the polarization directly from either (4.21) or

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \epsilon_0(\epsilon_r - 1)\mathbf{E} \tag{4.30}$$

so we might as well use (4.30) and get the polarization in the dielectric as

$$\boxed{\mathbf{P}_{\text{die}} = \frac{2}{1+\epsilon_r}\epsilon_0(\epsilon_r - 1)\frac{V}{d}\hat{\mathbf{n}}}$$

(iii) The free bound surface charge is simply $\sigma_f = D$ so for the top plate

$$\boxed{\sigma_{f_{\text{top}}} = \frac{2\epsilon_r}{1+\epsilon_r}\epsilon_0\frac{V}{d}}$$

and $\sigma_{f\text{bot}} = -\sigma_f$ for the bottom plate. And using $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ points in the negative direction for the top plate, so

$$\boxed{\sigma_{b_{\text{top}}} = -\frac{2}{1+\epsilon_r}\epsilon_0(\epsilon_r - 1)\frac{V}{d} \quad \sigma_{b_{\text{bot}}} = -\sigma_{b\text{top}}}$$

(b) Again, from Gauss's law

$$E_{\text{air}} = \frac{\sigma_{f_{\text{air}}}}{\epsilon_0} \quad \text{and} \quad V = E_{\text{air}}d$$

$$\implies \boxed{\sigma_{f_{\text{air}}} = \frac{\epsilon_0 V}{d}}$$

for the top plate ($-\sigma_{f_{\text{air}}}$ in the bottom plate) but in the dielectric, using $\mathbf{D}$

$$\oint \mathbf{D}\cdot d\mathbf{a} = Q_{\text{free}}$$

$$\implies D = \sigma_{f_{\text{die}}}$$

and from $D = \epsilon E_{\text{die}}$

$$E_{\text{die}} = \frac{\sigma_{f_{\text{die}}}}{\epsilon} = \frac{\sigma_f}{\epsilon_r\epsilon_0}$$

Then using the potential difference $V = E_{\text{die}}d$

$$\implies \boxed{\sigma_{f_{\text{die}}} = \epsilon_r\epsilon_0\frac{V}{d}}$$

for the top plate (negative for bottom). From this, we can find the capacitance $C = Q/V$ where

$$\begin{aligned} Q &= \sigma_{f_{\text{air}}} \frac{A}{2} + \sigma_{f_{\text{die}}} \frac{A}{2} \\ &= \frac{\epsilon_0 V A}{2d} + \frac{\epsilon_r \epsilon_0 V A}{2d} \end{aligned}$$

so

$$\implies C = \frac{A\epsilon_0}{2d}(1 + \epsilon_r)$$

From (2.54), the capacitance increases by a factor of

$$\boxed{\frac{C_b}{C} = \frac{1 + \epsilon_r}{2}}$$

(i) The E-field in the air is just the E-field between two plates

$$\boxed{\mathbf{E}_{\text{air}} = \frac{V}{d}\hat{\mathbf{n}}}$$

and $\boxed{\mathbf{P}_{\text{air}} = 0}$ for air thus the displacement field (4.21) $\mathbf{D} = \epsilon_0 \mathbf{E} + 0$, or

$$\boxed{\mathbf{D}_{\text{air}} = \epsilon_0 \frac{V}{d}\hat{\mathbf{n}}}$$

(ii) The E-field in the dielectric using (4.32):

$$\mathbf{E}_{\text{die}} = \frac{1}{\epsilon}\mathbf{D}$$

where $D = \sigma_{f_{\text{die}}}$ so

$$\boxed{\mathbf{D} = \epsilon_r \epsilon_0 \frac{V}{d}\hat{\mathbf{n}}}$$

Then we can use $\epsilon = \epsilon_r \epsilon_0$ so

$$\boxed{\mathbf{E}_{\text{die}} = \frac{V}{d}\hat{\mathbf{n}} = \mathbf{E}_{\text{air}}}$$

Now the polarization is given by (4.30)

$$\boxed{\mathbf{P}_{\text{die}} = \epsilon_0(\epsilon_r - 1)\frac{V}{d}\hat{\mathbf{n}}}$$

Finally, the bound charge is $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ so

$$\boxed{\sigma_{b_{\text{top}}} = -\epsilon_0(\epsilon_r - 1)\frac{V}{d} \quad \sigma_{b_{\text{bot}}} = -\sigma_{b_{\text{top}}}}$$

4.37 Between two linear dielectric, show that the E-field lines bend by

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\epsilon_2}{\epsilon_1} \tag{4.68}$$

Since $\sigma_f = 0$ so from the boundary conditions

$$D_1^{\perp} - D_2^{\perp} = \sigma_b = 0 \implies D_1^{\perp} = D_2^{\perp}$$

or from (4.32)

$$D_{y_1} = D_{y_2} \implies \epsilon_1 E_{y_1} = \epsilon_2 E_{y_2}$$

Also from the boundary conditions, the parallel components of $\mathbf{E}$ are continuous so

$$E_{x_1} = E_{x_2} \implies \tan\theta_1 = \frac{E_{y_1}}{E_{x_1}}, \quad \text{and} \quad \tan\theta_2 = \frac{E_{y_2}}{E_{x_2}}$$

Thus

$$\begin{aligned}
\frac{\tan\theta_2}{\tan\theta_1} &= \frac{E_{y_2}/E_{x_2}}{E_{y_1}/E_{x_1}} \\
&= \frac{E_{y_2}}{E_{y_1}} \quad \text{using} \quad E_{y_1} = \frac{\epsilon_2}{\epsilon_1} E_{y_2} \\
\frac{\tan\theta_2}{\tan\theta_1} &= \frac{\epsilon_2}{\epsilon_1}
\end{aligned}$$

From Snell's Law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where a convex lens has $n_2 > n_1$ which makes the angle of refraction smaller than the angle of incidence $\theta_2 < \theta_1$ thus the light ray bends towards the normal or "focuses" the light. For the dielectric interface, its convex lens

$$\epsilon_2 > \epsilon_1 \implies \tan\theta_2 > \tan\theta_1$$

So $\theta_2 > \theta_1$, thus the E-field lines bend away from the normal which will "defocus" the E-field.