Homework 3

Due 2/7 9pm

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1. The Center of Mass of the system is

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,\mathrm{d}m$$

and the mass element is the mass denisty times the volume element

$$\mathrm{d}m = \rho\,\mathrm{d}V = \frac{M}{2\pi R^2}R^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta = \frac{M}{2\pi}\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$$

where the there is no $\mathrm{d}r$ term because the radius is constant. Since the center of mass is symmetric about the x and y axes, $X_{cm} = Y_{cm} = 0$. The z component of the center of mass is

$$\begin{aligned} Z_{cm} &= \frac{1}{M}\int z\,\mathrm{d}m \\ &= \frac{1}{M}\frac{M}{2\pi}\iint z\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \end{aligned}$$

where $z = R\cos\phi$ so

$$\begin{aligned} Z_{cm} &= \frac{R}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi/2}\cos\phi\sin\phi\,\mathrm{d}\phi \\ &\text{using}\quad u = \sin\phi \implies \mathrm{d}u = \cos\phi\,\mathrm{d}\phi \\ &= \frac{R}{2\pi}[2\pi]\int_0^1 u\,\mathrm{d}u \\ Z_{cm} &= \frac{R}{2} \end{aligned}$$

So the COM is at $\boxed{\left(0, 0, \frac{R}{2}\right)}$

2. (a) From N3L the force of the jettisoned fuel on the rocket is equal and opposite to the thrust on the rocket from the jettisoned fuel:

$$
\begin{aligned}
F_{\text{fuel}} &= -F_{\text{thrust}} \\
\dot{m}v_{ex} &= -\dot{m}v_{ex}
\end{aligned}
$$

So using N2L, the sum of the forces on the rocket is the thrust and air resistance:

$$F = m\dot{v} = F_{\text{thrust}} - f = -\dot{m}v_{ex} - bv$$

(b) Using $\dot{m} = -k$

$$
\begin{aligned}
m\dot{v} &= kv_{ex} - bv \\
\frac{m}{b}\dot{v} &= \frac{kv_{ex}}{b} - v
\end{aligned}
$$

defining the constant $a = \frac{kv_{ex}}{b}$ and using separation of variables

$$\frac{1}{a-v}\,\mathrm{d}v = \frac{b}{m}\,\mathrm{d}t$$

we can write an expression for m as a function of time through using separtion of variables again:

$$
\begin{aligned}
\frac{\mathrm{d}m}{\mathrm{d}t} &= -k \\
\int_{m_o}^{m} \mathrm{d}m' &= -k\int_0^t \mathrm{d}t' \\
m - m_o &= -kt \\
m &= m_o - kt
\end{aligned}
$$

where m_o is the initial mass of the rocket. Substituting this back into main expression and integrating both sides:

$$
\begin{aligned}
\int_0^v \frac{1}{a-v'}\,\mathrm{d}v' &= b\int_0^t \frac{1}{m_o - kt'}\,\mathrm{d}t' \\
-\ln(a-v')\Big|_0^v &= -\frac{b}{k}\ln(m_o - kt')\Big|_0^t \\
-\ln(a-v) + \ln(a) &= \frac{b}{k}[-\ln(m_o - kt) + \ln(m_o)] \\
\ln\left(\frac{a}{a-v}\right) &= \frac{b}{k}\ln\left(\frac{m_o}{m_o - kt}\right)
\end{aligned}
$$

substituting back in $m = m_o - kt$ and exponentiating both sides:

$$
\begin{aligned}
\frac{a}{a-v} &= \left(\frac{m_o}{m}\right)^{\frac{b}{k}} \\
a\left(\frac{m_o}{m}\right)^{-\frac{b}{k}} &= a - v \\
v &= a - a\left(\frac{m_o}{m}\right)^{-\frac{b}{k}} \\
v &= a\left[1 - \left(\frac{m}{m_o}\right)^{\frac{b}{k}}\right]
\end{aligned}
$$

subbing back in $a = \frac{kv_{ex}}{b}$ we get the final expression

$$v(m) = \frac{kv_{ex}}{b}\left[1 - \left(\frac{m}{m_o}\right)^{\frac{b}{k}}\right]$$

3. (a) The angular momentum vector is

$$\begin{aligned}\boldsymbol{\ell} &= \mathbf{r} \times \mathbf{p} \\ &= \mathbf{r} \times m\dot{\mathbf{r}} \\ &= m\mathbf{r} \times \dot{\mathbf{r}}\end{aligned}$$

from HW 1, we know that

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

so

$$\begin{aligned}\boldsymbol{\ell} &= m(r\hat{\mathbf{r}}) \times (\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) \\ &= m[r\dot{r}(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) + r(r\dot{\phi})(\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}})] \\ &= m[0 + r^2\dot{\phi}\hat{\mathbf{z}}] = mr^2\omega\hat{\mathbf{z}}\end{aligned}$$

where $\omega = \dot{\phi}$ and the magnitude of the angular momentum is

$$\ell = |\boldsymbol{\ell}| = mr^2\omega$$

(b) The area swept by an infinitesimal change in the planets position is equivalent to the area of a triangle as shown in Figure 2, so

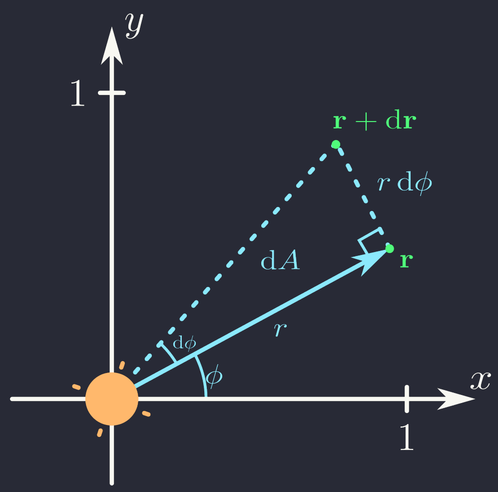

Figure 3.1: Area swept by planet

$$\mathrm{d}A = \frac{1}{2}r(r\,\mathrm{d}\phi) = \frac{1}{2}r^2\,\mathrm{d}\phi$$

dividing both sides by $\mathrm{d}t$ gives us

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{2}r^2\omega$$

and from part (a) we know that $\ell = mr^2\omega$ or $\omega = \frac{\ell}{mr^2}$ so

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\ell}{mr^2} = \frac{\ell}{2m}$$

therefore, the rate in change of the area swept by the planet is a constant that is proportional to ℓ.

4. $\mathbf{F} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$ from $P = (1, 0)$ to $Q = (0, 1)$
(a) For a straight line the path is given by

$$y = 1 - x \qquad \mathrm{d}y = -\,\mathrm{d}x$$

so the work done is

$$\begin{aligned}
W_a &= \int_P^Q F_x\,\mathrm{d}x + F_y\,\mathrm{d}y = \int_P^Q -y\,\mathrm{d}x + x\,\mathrm{d}y \\
&= \int_{x=1}^{0} -(1-x)\,\mathrm{d}x + x(-\,\mathrm{d}x) \\
&= \int_1^0 -1\,\mathrm{d}x = 1
\end{aligned}$$

(b) For a circular path of radius 1, the path in polar coordinates is

$$\begin{aligned}
y = \sin\phi &\to \mathrm{d}y = \cos\phi\,\mathrm{d}\phi \\
x = \cos\phi &\to \mathrm{d}x = -\sin\phi\,\mathrm{d}\phi
\end{aligned}$$

from the equation of a circle $x^2 + y^2 = 1$. The limits of integration are $\phi = 0 \to \pi/2$, and the work is

$$\begin{aligned}
W_b &= \int_{\phi=0}^{\pi/2} -\sin\phi(-\sin\phi)\,\mathrm{d}\phi + \cos\phi\cos\phi\,\mathrm{d}\phi \\
&= \int_{\phi=0}^{\pi/2} 1\,\mathrm{d}\phi = \frac{\pi}{2}
\end{aligned}$$

(c) Splitting this into two paths: For path 1, $y = 0$; $\mathrm{d}y = 0$; and $x = 1 \to 0$ so

$$W_1 = \int_{x=1}^{0} 0\,\mathrm{d}x + x(0) = 0$$

For path 2, $x = 0$; $\mathrm{d}x = 0$; $y = 0 \to 1$ so

$$W_1 = \int_{y=0}^{1} -y(0) + 0\,\mathrm{d}y = \int_{y=0}^{1} 0\,\mathrm{d}y = 0$$

And the work done is $W_c = W_1 + W_2 = 0$
(d) The force is not conservative because the work done is path dependent! We can also double check by taking the curl:

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & x \end{vmatrix} = \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right)\hat{\mathbf{z}} = 2\hat{\mathbf{z}}$$

which is not zero, so the force is not conservative.

5. (a) Using the time derivatives of the polar unit vectors

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \dot{\phi}\hat{\boldsymbol{\phi}} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$$

Acceleration in polar coordinates is

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}\right)\\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + (\dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}(-\dot{\phi}\hat{\mathbf{r}}))\\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}\end{aligned}$$

so the radial and angular components of the force are

$$\begin{aligned}F_r &= ma_r = m(\ddot{r} - r\dot{\phi}^2)\\ F_\phi &= ma_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\end{aligned}$$

and since the spring force is conservative with magnitude $F_s = -k(r-a)\hat{\mathbf{r}}$ the equations of motion are

$$\begin{aligned}m(\ddot{r} - r\dot{\phi}^2) &= -k(r-a)\\ m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) &= 0\end{aligned}$$

or

$$\begin{aligned}m\ddot{r} - mr\dot{\phi}^2 + k(r-a) &= 0\\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0\end{aligned}$$

(b) The initial angular momentum of the system is

$$\ell_o = mv_o a$$

and after some time the angular momentum is (from Problem 3)

$$\ell = mr^2\dot{\phi}$$

and using the conservation of angular momentum

$$\begin{aligned}\ell_o &= \ell\\ mv_o a &= mr^2\dot{\phi}\\ \dot{\phi} &= \frac{v_o a}{r^2}\end{aligned}$$

(c) First the initial mechanical energy of the system is purely kinetic given by the initial velocity:

$$E_o = T_o = \frac{1}{2}mv_o^2$$

the total mechanical energy of the system after some time will be the sum of the kinetic and potential energies:

$$\begin{aligned}U &= -\int_0^r \mathbf{F}\cdot\mathrm{d}\mathbf{r}' = \int_0^r k(r-a)\,\mathrm{d}r' = \frac{1}{2}k(r-a)^2\\ T &= \frac{1}{2}mv^2 = \frac{1}{2}m(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}m(\dot{\mathbf{r}}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}})\cdot(\dot{\mathbf{r}}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)\end{aligned}$$

And from the conservation of energy

$$\begin{aligned}E_o &= E = T + U\\ \frac{1}{2}mv_o^2 &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}k(r-a)^2\\ v_o^2 &= \dot{r}^2 + r^2\dot{\phi}^2 + \frac{k}{m}(r-a)^2\end{aligned}$$

substituting the result from part (b) and solving for $\dot{r}$:

$$
\begin{aligned}
v_o^2 &= \dot{r}^2 + r^2\left(\frac{v_o a}{r^2}\right)^2 + \frac{k}{m}(r-a)^2 \\
v_o^2 &= \dot{r}^2 + \frac{v_o^2 a^2}{r^2} + \frac{k}{m}(r-a)^2 \\
\dot{r}^2 &= v_o^2 - \frac{v_o^2 a^2}{r^2} - \frac{k}{m}(r-a)^2 \\
\dot{r} &= \sqrt{v_o^2\left(1 - \frac{a^2}{r^2}\right) - \frac{k}{m}(r-a)^2}
\end{aligned}
$$