Homework 10

6.3 Force of attractive between two magnetic dipoles $\mathbf{m}_1, \mathbf{m}_2$ a distance r apart:

(a) Given

$$F = 2\pi IRB\cos\theta \tag{6.2}$$

and

$$m_2 = IA = \pi IR^2$$

The magnetic field due to m_1 is

$$\mathbf{B}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \tag{5.89}$$

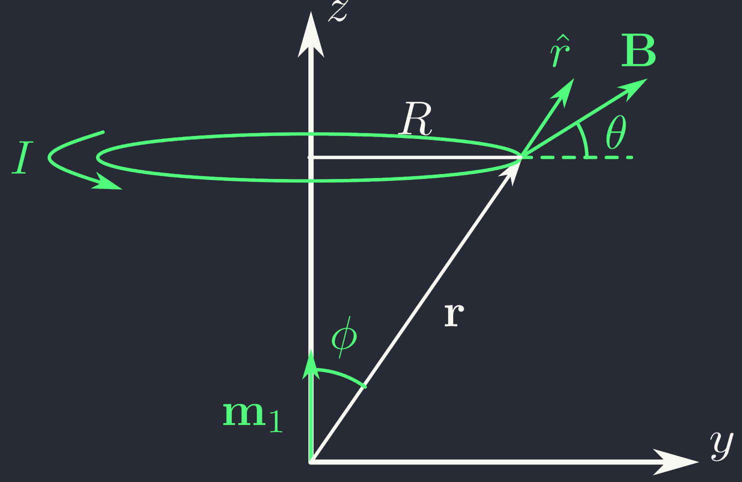


Figure 0.1: Magnetic field due to a magnetic dipole $\mathbf{m}_1$

$$\begin{aligned} B\cos\theta &= \mathbf{B}_1\cdot\hat{\mathbf{y}} \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}) - \mathbf{m}_1\cdot\hat{\mathbf{y}}] \end{aligned}$$

and from Fig. 0.1

$$\mathbf{m}_1\cdot\hat{\mathbf{r}} = m_1\cos\phi, \quad \hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\phi, \quad \hat{\mathbf{m}}_1\cdot\hat{\mathbf{y}} = 0$$
$$\text{and} \quad \sin\phi = \frac{R}{r}, \quad \cos\phi = \frac{z}{r} = \frac{\sqrt{r^2 - R^2}}{r}$$

we get

$$\begin{aligned} B\cos\theta &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3m_1\cos\phi\sin\phi - 0] \\ &= \frac{3\mu_0}{4\pi}\frac{m_1}{r^3}\sin\phi\cos\phi \\ &= \frac{3\mu_0}{4\pi}m_1R\frac{\sqrt{r^2 - R^2}}{r^5} \end{aligned}$$

So

$$F = 2\pi IRB\cos\theta$$
$$= 2(\pi IR^2)\frac{3\mu_0}{4\pi}m_1\frac{\sqrt{r^2-R^2}}{r^5} \qquad = \frac{3\mu_0}{2\pi}m_1m_2\frac{\sqrt{r^2-R^2}}{r^5}$$

where $r \gg R$ for the dipole so

$$\frac{\sqrt{r^2-R^2}}{r^5} \approx \frac{\sqrt{r^2}}{r^5} = \frac{1}{r^4}$$

Therefore

$$\boxed{F = \frac{3\mu_0}{2\pi}\frac{m_1m_2}{r^4}}$$

(b) Using

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) \tag{6.3}$$

and product rule (ii)

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}_2\cdot\mathbf{B})$$
$$= \mathbf{m}_2\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{m}_2) + (\mathbf{m}_2\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{m}_2$$

where the curl of $\mathbf{B}$ and $\mathbf{m}_2$ are zero

$$\implies \mathbf{F} = (\mathbf{m}_2\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{m}_2$$

and the second term is also zero for the dipole

$$(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})\mathbf{m}_2 = 0$$
$$\implies \mathbf{F} = (\mathbf{m}_2\cdot\boldsymbol{\nabla})\mathbf{B}$$

So using (5.89)

$$\mathbf{F} = \left[(0,0,m_2)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\right]\mathbf{B}$$
$$= m_2\frac{\partial}{\partial z}\left(\frac{\mu_0}{4\pi}\frac{1}{z^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{m}_1]\right)$$
$$= \frac{\mu_0}{4\pi}m_2\frac{\partial}{\partial z}\left(\frac{1}{z^3}[2m_1\hat{\mathbf{z}}]\right)$$
$$= \frac{\mu_0}{2\pi}m_1m_2\hat{\mathbf{z}}\left(-\frac{3}{z^4}\right)$$

or when $z \to r$

$$\boxed{F = -\frac{3\mu_0}{2\pi}\frac{m_1m_2}{r^4}\hat{\mathbf{z}}}$$

where the negative sign tells us the force is attractive.

6.5 A uniform current density $\mathbf{J} = J_0\hat{\mathbf{z}}$ fills a slab on the yz plane from $x = -a \to +a$ and a magnetic dipole $\mathbf{m} = m_0\hat{\mathbf{x}}$ placed at the origin:

(a) The force on the dipole using (6.3): Using a rectangular Amperian loop on the xy plane i.e. $I_{\text{enc}} = AJ_0 = lxJ_0$:

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}}$$

$$Bl = \mu_0 lxJ_0 \implies B = \mu_0 J_0 x$$

where the direction is in the y from the right hand rule for Bio-Savart's law $\mathbf{J} \times \hat{\boldsymbol{\mathscr{r}}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$;

$$\mathbf{B} = \mu_0 J_0 x\hat{\mathbf{y}}$$

So the force on the dipole is

$$\begin{aligned} \mathbf{F} &= \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}) \\ &= \boldsymbol{\nabla}(m_0\mu_0 J_0 x[\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}]) \end{aligned}$$

where $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$ so

$$\boxed{\mathbf{F} = 0}$$

(b) With $\mathbf{m} = m_0\hat{\mathbf{y}}$ the force on the dipole is

$$\begin{aligned} \mathbf{F} &= \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}) \\ &= \boldsymbol{\nabla}(m_0\mu_0 J_0 x[\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}]) \\ &= \hat{\mathbf{x}}\frac{\partial}{\partial x}(m_0\mu_0 J_0 x) \end{aligned}$$

which points in the x direction:

$$\boxed{\mathbf{F} = m_0\mu_0 J_0\hat{\mathbf{x}}}$$

(c) For a polarization $\mathbf{p}$ and E-field $\mathbf{E}$ we can use product rule (ii) again:

$$\begin{aligned} F &= \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E}) \\ &= \mathbf{p} \times (\boldsymbol{\nabla} \times \mathbf{E}) + \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{p}) + (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E} + (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{p} \end{aligned}$$

where $\boldsymbol{\nabla} \times \mathbf{p} = 0, \boldsymbol{\nabla} \times \mathbf{E} = 0, \quad \text{and} \quad \boldsymbol{\nabla}\mathbf{p} = 0$ so

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E} = \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E})$$

which is not the same as the magnetic case since Ampere's law states

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0\mathbf{J}$$

Using product rule (ii) once more,

$$\begin{aligned} F &= \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}) \\ &= \mathbf{m} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \cancel{\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{m})} + (\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{B} + \cancel{(\mathbf{B} \cdot \boldsymbol{\nabla})}\mathbf{m} \\ &= \mathbf{m} \times (\mu_0\mathbf{J}) + (\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{B} \end{aligned}$$

so for configuration (a),

$$(\mathbf{m}_a \cdot \boldsymbol{\nabla})\mathbf{B} = \left(m_0\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}\frac{\partial}{\partial x}\right)\mu_0 J_0 x\hat{\mathbf{y}} = m_0\mu_0 J_0\hat{\mathbf{y}}$$

and for configuration (b),

$$(\mathbf{m}_b \cdot \boldsymbol{\nabla})\mathbf{B} = \left(m_0\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\frac{\partial}{\partial y}\right)\mu_0 J_0 x\hat{\mathbf{y}} = 0$$

6.9 The bound current for a short cylinder of radius a and length L with a frozen in uniform magnetization parallel to its axis $\mathbf{M} = M\hat{\mathbf{z}}$ is

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}$$

This is the same as the bar electret but with opposite directions for the ($\mathbf{B}$ vs. $\mathbf{D}$) field produced

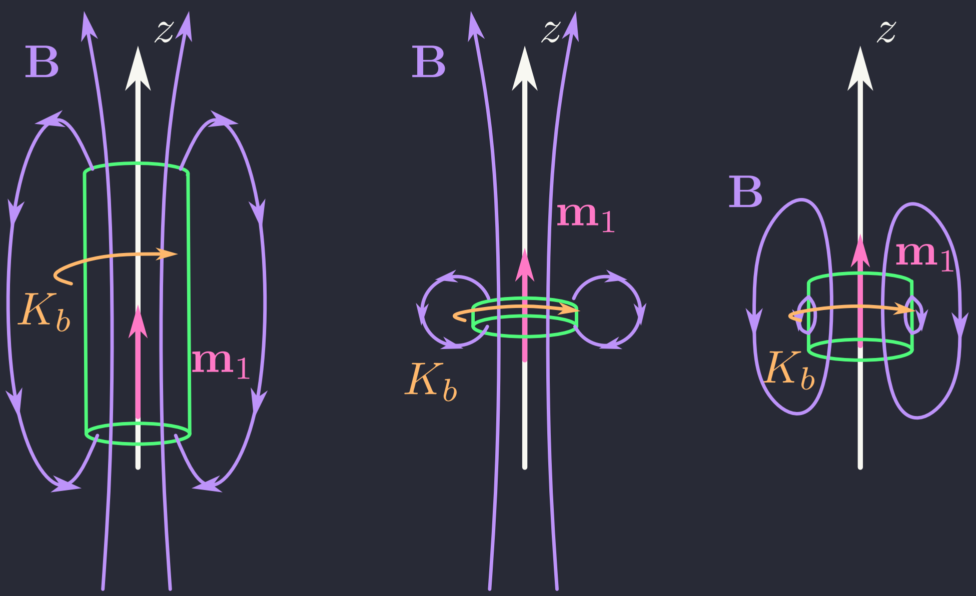
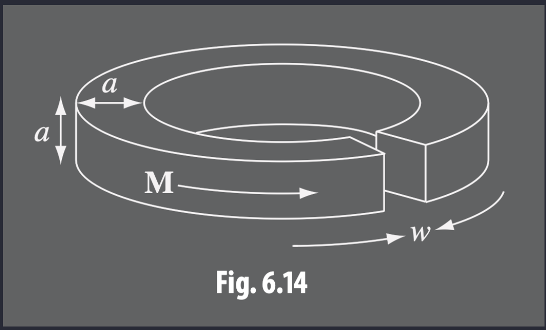


Figure 0.2: (left) $L \gg a$, (center) $L \ll a$, and (right) $L = a$



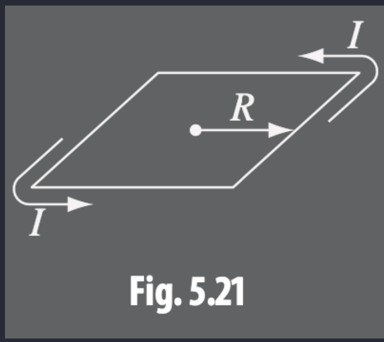


Figure 0.3: Superposition of torus and square loop with reversed current

6.10 For an iron rod of length L and square cross section of side length a with Magnetization $\mathbf{M}$ and bent around in a circle with gap w: Assuming $w \ll a \ll L$ find the magnetic field at the center of the gap: The magnetization of the torus is $\mathbf{M} = M\hat{\boldsymbol{\phi}}$ so the magnetic field of the torus is

$$\mathbf{B} = \mu_0 \mathbf{M}$$

The magnetic field at the center of the square loop is equivalent to four times the magnetic field of a straight wire (From Griffiths Example 5.5):

$$B = -4\frac{\mu_0 I}{4\pi R}\sin\theta\Big|_{\theta=-\pi/4}^{\pi/4} = \frac{\mu_0 I}{\pi R}\left(\sqrt{2}/2 + \sqrt{2}/2\right) = \frac{\sqrt{2}\mu_0 I}{\pi R}$$

where the square loop has a current $I = m/a$ where $M \equiv m/V$

$$\implies I = \frac{MV}{a} = Mw$$

and $R = a/2$ so

$$B = -\frac{\sqrt{2}\mu_0 M w}{\pi a/2} = -\frac{2\sqrt{2}\mu_0 M w}{\pi a}$$

The magnetic field is a superposition of the torus and square loop with reversed current

$$\boxed{\mathbf{B} = \mu_0 \mathbf{M} - \frac{2\sqrt{2}\mu_0 \mathbf{M} w}{\pi a}}$$

6.25 For two charged q magnetic dipoles $\mathbf{m}$ constrained to move on the z axis, they electrically repel and magnetically attract:

(a) The equilibrium separation distance: From Coloumbs law the electric force is

$$\mathbf{F}_e = \frac{1}{4\pi\epsilon_0}\frac{q^2}{z^2}\hat{\mathbf{z}}$$

and the magnetic force is (From Prob. 6.3)

$$\mathbf{F}_m = \nabla(\mathbf{m}\cdot\mathbf{B}) = -\frac{3\mu_0}{2\pi}\frac{m^2}{z^4}\hat{\mathbf{z}}$$

so at equilibrium

$$\mathbf{F}_e = -\mathbf{F}_m \implies \frac{1}{4\pi\epsilon_0}\frac{q^2}{z^2} = \frac{3\mu_0}{2\pi}\frac{m^2}{z^4} \implies z = \frac{m\sqrt{6\epsilon_0\mu_0}}{q}$$

where we can use the fact that $\epsilon_0\mu_0 = 1/c^2$ to rewrite the equilibrium separation distance as

$$\boxed{z = \sqrt{6}\frac{m}{qc}}$$

(b) For two electrons, we can use the magnetic moment of an electron, i.e, the "Bohr magneton" Wiki,

$$m = \frac{e\hbar}{2m_e}$$

withe with electron charge $q = e = 1.6\times 10^{-19}\,\text{C}$, mass $m_e = 9.11\times 10^{-31}\,\text{kg}$, reduced planck constant $\hbar = 1.05\times 10^{-34}\,\text{Js}$, and speed of light $c = 3\times 10^8\,\text{m/s}$:

$$\begin{aligned} z &= \sqrt{6}\frac{e\hbar}{2m_e qc} \\ &= \sqrt{6}\frac{\hbar}{2m_e c} \\ &= \sqrt{6}\frac{1.05\times 10^{-34}\,\text{Js}}{29.11\times 10^{-31}\,\text{kg}3\times 10^8\,\text{m/s}} \\ &= 4.72\times 10^{-13}\,\text{m} \end{aligned}$$

(c) Since the equilibrium separation distance is much smaller than the Bohr radius $a_0 = 5.29\times 10^{-11}\,\text{m}$, there is *no stable bound state of two electrons*