1 Electric Fields in Matter

1.1 Polarization

1.1.1 Dielectrics

In dieletrics, "All charges are attached to specific atoms or molecules" (Griffith, pg.166)

1.1.2 Induced Dipoles

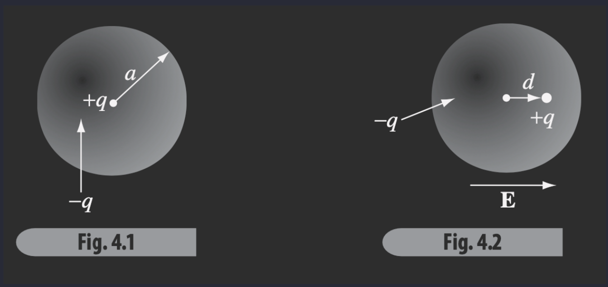


Figure 1.1: Left: Simple Nucleus $+q$ surrounded by spherical cloud $-q$ of radius a. Right: In an external electric field E the nucleus shifts right d and the cloud shifts left.

For a simple model of the atom (Fig. 1.1), the electric field at d is

$$E_d = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

where the dipole moment is $p = qd$. We have equilibrium when

$$F_{\text{ext}} = qE_{\text{ext}} = q_- E_d$$

So using the dipole moment

$$|\mathbf{p}| = qd = 4\pi\epsilon_0 a^3 E_{\text{ext}} = \alpha E_{\text{ext}}$$

Here we have this "atomic polaritability"

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon v$$

where v is the volume of the atom. Comment: this crude approximation is still accurate by a factor of 4.

In general we have a vector

$$\mathbf{p} = \hat{\alpha}\mathbf{E}$$

where $\hat{\alpha}$ is the polarizability tensor. For a linear dielectric relation between E and p we get

$$\hat{\alpha} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}$$

1.1.3 Alignment of Polar Molecules

The dipole will experience a torque in and E-field

$$\begin{aligned} \mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= \left(\frac{\mathbf{d}}{2} \times q\mathbf{E}\right) + \left(-\frac{\mathbf{d}}{2} \times q\mathbf{E}\right) \\ &= q\mathbf{d} \times \mathbf{E} \end{aligned}$$

thus

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

which implies that there is a force that acts to align $\mathbf{p} \parallel \mathbf{E}$.

WHat if E is not uniform?

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q\mathbf{E} + q\mathbf{E} = q\Delta E$$

assuming small d in E_x, then

$$\Delta E_x = (\boldsymbol{\nabla}\mathbf{E}_x) \cdot \mathbf{d}$$

So the total change in the field is

$$\Delta\mathbf{E} = (\mathbf{d} \cdot \boldsymbol{\nabla})\mathbf{E}$$

thus

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$$

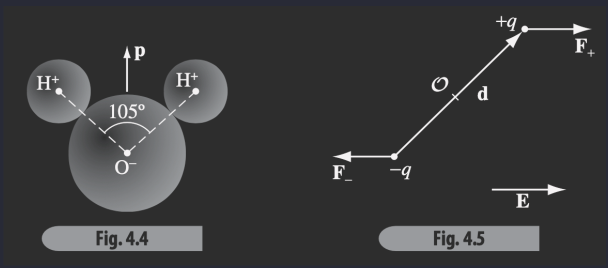


Figure 1.2: Dipole moment in oxygen molecule, and in an electric field.

Example: Problem 4.5 Using the method of images

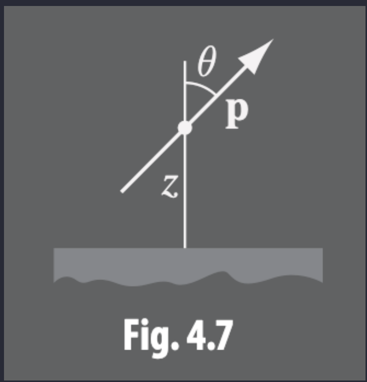


Figure 1.3: Infinitely grounded conductor with dipole at an angle θ from the normal plane and nailed in place.

Where look at $\mathbf{p}_i$ coordinate (pointing up) thus $2z$ away we have the dipole pointing perpendicular to the image dipole i.e.

$$\mathbf{E}_i = \frac{1}{4\pi\epsilon_0}\frac{p_i}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right)$$

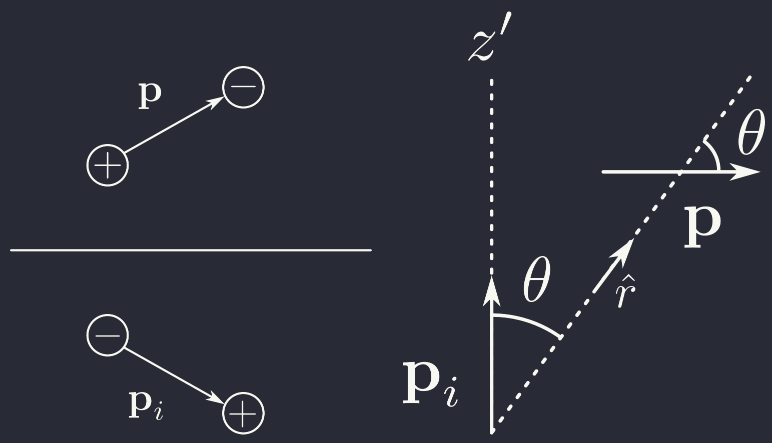


Figure 1.4: Left: Method of images using image dipole. Right: Coordinate using image dipole up as z'.

where

$$\mathbf{p} = p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\theta}$$

So the torque $\mathbf{N} = \mathbf{p} \times \mathbf{E}_i$ is

$$\begin{aligned}
\mathbf{N} &= \frac{1}{4\pi\epsilon_0}\frac{p^2}{(2z)^3}(\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) \times \left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{p^2}{(2z)^3}\left(\cos\theta\sin\theta\hat{\boldsymbol{\phi}} + 2\cos\theta\sin\theta(-\hat{\boldsymbol{\phi}})\right) \\
&= \frac{1}{4\pi\epsilon_0}\frac{p^2\cos\theta\sin\theta}{(2z)^3}(-\hat{\boldsymbol{\phi}}) \\
&= \frac{1}{4\pi\epsilon_0}\frac{p^2\sin(2\theta)}{8\pi\epsilon_0(8z^3)}(-\hat{\boldsymbol{\phi}})
\end{aligned}$$

So

$$\begin{cases} 0 < \theta < \frac{\pi}{2} & N \sim -\hat{\boldsymbol{\phi}} \\ \frac{\pi}{2} < \theta < \pi & N \sim \hat{\boldsymbol{\phi}} \end{cases}$$

Which means the dipole can either align perdendicularly up or down depending on the angle θ.

1.1.4 Polarization

$$\mathbf{P} \equiv \text{dipole moment per unit volume}$$

i.e. the little $\mathbf{p}$ is

$$\mathbf{p} = \mathbf{P}\mathrm{d}\tau$$

1.2 The Field of a Polarized Object

1.2.1 Bound Charges

For a single dipole

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

and using the dipole moment per unit volume def:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\mathrm{d}\tau'$$

recalling the math fact

$$\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

thus

$$V = \frac{1}{4\pi\epsilon_0}\int \mathbf{P}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\mathrm{d}\tau'$$

using another math fact

$$\boldsymbol{\nabla}\cdot(F\mathbf{A}) = F(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla}F)$$

So we can rewrtite the integral

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left[\int_V \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)\mathrm{d}\tau' - \int_V \frac{1}{\mathscr{r}}\boldsymbol{\nabla}\cdot\mathbf{P}\mathrm{d}\tau'\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{1}{\mathscr{r}}\mathbf{P}\cdot\mathrm{d}\mathbf{a}' - \int_V \frac{1}{\mathscr{r}}\boldsymbol{\nabla}\cdot\mathbf{P}\mathrm{d}\tau'\right] \end{aligned}$$

where we used the divergence theorem for the first term. For charge densities

$$\begin{cases} \text{surface charge} & \sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \\ \text{volume charge} & \rho_b \equiv -\boldsymbol{\nabla}\cdot\mathbf{P} \end{cases}$$

then

$$V = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\sigma_b}{\mathscr{r}}\mathrm{d}a' + \int_V \frac{\rho_b}{\mathscr{r}}\mathrm{d}\tau'\right]$$

1.2.2 Physical Interpretation of Bound Charges

So for a charge neutral sphere with an applied E-field, we can imagine this sphere as two oppositely charge spheres superimposed on each other but slighly shifted (Fig. 1.5). Thus we can imagine a collection of

Figure 1.5: Charge neutral sphere with applied E-field.

dipoles for each atom in a material with alternating charges. This is actually wrong (read Berry Phases in Electronic Structure Theory by David Vanderbilt).

Example: Find $\mathbf{E}$ of a uniformly polarized sphere of radius R.

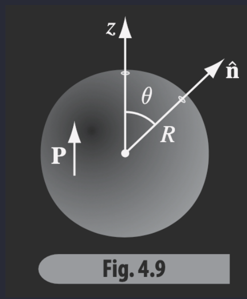


Figure 1.6: Uniformly polarized sphere of radius R.

We choose $\mathbf{P} \propto \hat{\mathbf{z}}$ as shown in Fig. 1.6. The bound volume and surface charges are

$$\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = 0 \quad \text{but} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta$$

Thus using the result from before

$$V = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\sigma_b}{\mathscr{r}} \mathrm{d}a' + \int_V \frac{\rho_b}{\mathscr{r}} \mathrm{d}\tau'\right]$$

or

$$V(r\theta) = \begin{cases} \frac{P}{3\epsilon_0} r\cos\theta & r \leq R \\ \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta & r > R \end{cases}$$

Recalling $z = r\cos\theta \quad \mathbf{E} = -\boldsymbol{\nabla} V$ we get

$$\mathbf{E}_{\text{in}} = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}$$

and for outside the potential is

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

with the total dipole moment

$$\mathbf{p} = \frac{4\pi}{3}R^3\mathbf{P}$$

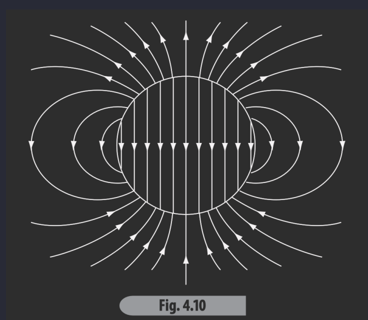


Figure 1.7: Polarized sphere in an external E-field.

So the polarized sphere is similar to the spherical conductor but with E-fields inside it (Fig. 1.7).

1.3 The Displacement Field

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} \quad \text{and} \quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$$

The $\rho_{\text{free}} \to$ anything *not* due to Polarization

$$\rho = \rho_b + \rho_f$$

and from Gauss' Law

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\implies \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} = \rho_b + \rho_f = -\boldsymbol{\nabla}\cdot\mathbf{P} + \rho_f$$

where $\mathbf{E}$ is the total electric field. Moving some terms around we get

$$\boldsymbol{\nabla}\cdot(\epsilon_0\mathbf{E} + \mathbf{P}) = \rho_f$$

where we now define the electric displacement field

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P}$$

which has the same laws as $\mathbf{E}$:

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_f} \qquad \boxed{\oint \mathbf{D}\cdot d\mathbf{a} = Q_{\text{f, enc}}}$$

Example: Long straight wire, with unform λ (charge per length), surrounded by rubber insulation out to radius a; Find $\mathbf{D}$

Using the Gaussian surface (cylinder) of length l and radius s so the enclosed charge is

$$\oint \mathbf{D}\cdot d\mathbf{a} = 2\pi s l\mathbf{D} = \lambda l$$

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$

$\implies$ holds for both $s \le a$ and $s > a$.

For $s > a$

$$\mathbf{P}_{\text{out}} = 0 \implies \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

But for inside $s \le a$ we can't determine $\mathbf{P}_{\text{in}}$ yet!

Comments While the two equations

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_f \quad \text{and} \quad \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$$

are similar, the E-field has a Coulomb law

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

but there is no equivalent for $\mathbf{D}$

$$\cancel{\mathbf{D} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}}$$

since there is this tensor relation $\mathbf{p} = \hat{\alpha}\mathbf{E}$. Furthermore, the curl is also different:

$$\boldsymbol{\nabla}\times\mathbf{D} = \boldsymbol{\nabla}\times(\epsilon_0\mathbf{E} + \mathbf{P}) = \cancel{\epsilon_0\boldsymbol{\nabla}\times\mathbf{E}} + \boldsymbol{\nabla}\times\mathbf{P}$$

since $\boldsymbol{\nabla}\times\mathbf{E} = \boldsymbol{\nabla}\times(-\boldsymbol{\nabla}V) = 0$.

What is D? Units $\frac{\mathrm{C}}{\mathrm{m}^2}$ which is the same as σ (surface charge density).

1.4 Linear Dielectrics

1.4.1 Susceptibility, Permittivity, and Dielectric constant

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

where χ_e is the electric susceptibility (dimensionless). For here, we will assume linear (isotropic & homogeneous) dielectrics.

- vacuum: $\chi_e = 0$
- air: 1.00054
- salt: ~ 4.9
- Si: ~ 11
- water: ~ 80 (water is already polarized!)
- $\mathrm{SrTiO_3}$: $\sim 50,000$ at low temperatures

E is the *total* electric field.
Starting with

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ &= \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} \\ &= \epsilon_0 (1 + \chi_e) \mathbf{E} \quad \epsilon = \epsilon_0 (1 + \chi_e) \\ &= \epsilon \mathbf{E}\end{aligned}$$

where ϵ is the *permitivity* of the material and the relative permitivity or *dielectric constant* is

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

Example: Metal sphere with charge Q, radius a, surrounded by a linear dielectric, ϵ, out to radius b. Find potential at the center (relative to ∞).

Inside the metal sphere $\mathbf{E} = \mathbf{P} = \mathbf{D} = 0$. Drawing the Gaussian surface between the sphere and the dielectric we get

$$\oint \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{\text{f, enc}}$$

for $r > a$

$$\mathbf{D} = \frac{Q}{4\pi r^2} \hat{\mathbf{r}}$$

Now finding $\mathbf{E}$ from $\mathbf{D} = \epsilon \mathbf{E}$ i.e. in the vacuum $\epsilon = \epsilon_0$ and in the dielectric $\epsilon = \epsilon$:

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi \epsilon r^2} & a < r < b \\ \frac{Q}{4\pi \epsilon_0 r^2} & r > b \end{cases}$$

The potential at the origin is therefore

$$\begin{aligned} V &= -\int_\infty^0 \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\int_\infty^b \frac{Q}{4\pi\epsilon_0 r^2} \mathrm{d}r - \int_b^a \frac{Q}{4\pi \epsilon r^2} \mathrm{d}r - \int_a^0 0 \mathrm{d}r \\ &= \frac{Q}{4\pi} \left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right) \end{aligned}$$

Now we can find $\mathbf{P}$ since $\mathbf{E}$ is fixed by $\mathbf{D}$:

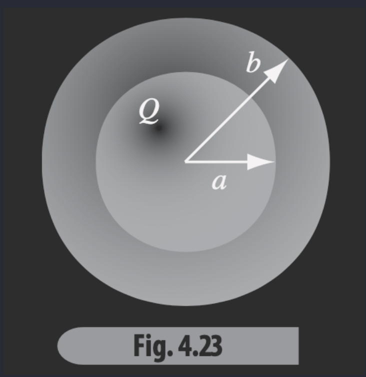
$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}} \quad \text{(in the dielectric)}$$

Thus we can get the volume bound charge

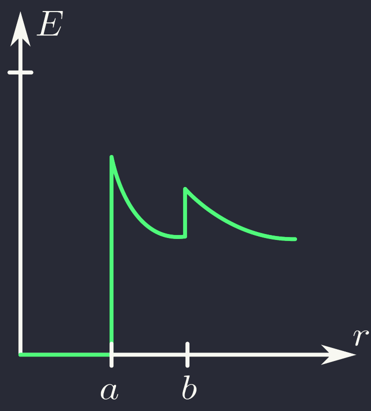
$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = 0 \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 0 \quad \text{except at } r = 0$$

and

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2} & \text{outer} \\[2ex] \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2} & \text{inner} \end{cases}$$



(a)



(b)

Figure 1.8: (a) Dielectric sphere surrounding metal sphere. (b) Electric field of resulting system.

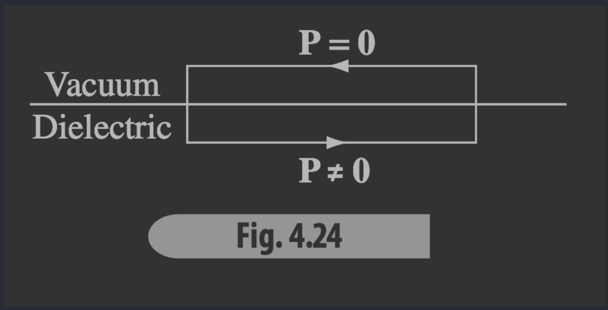


Figure 1.9: Interface between polarized dielectric and vacuum.

Since $\mathbf{P}$ is zero on the vacuum side but not on the dielectric side

$$\oint \mathbf{P} \cdot \mathrm{d}\boldsymbol{\ell} \neq 0$$

In addition, the displacement field $\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$ implies

$$\oint \mathbf{D} \cdot \mathrm{d}\mathbf{a} \neq 0 \neq \boldsymbol{\nabla} \times \mathbf{D}$$

Furthermore, the proportionality factor $\epsilon_0\chi_e$ is different on both sides. For the homogeneous linear dielectric

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \boldsymbol{\nabla} \times \mathbf{D} = 0$$

where

$$\begin{cases} \chi_f \mathbf{E}_{\text{vac}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \\ \chi_c \mathbf{D} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \end{cases} \implies \mathbf{E} = \epsilon\mathbf{E}$$

or

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D}$$

Example: Parallel plate capacitor with insulating material of dielectric constant ϵ_r

$$\mathbf{E} \to \frac{\mathbf{E}}{\epsilon_r}$$

and the potential differnce between the plates is

$$V = -\int \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -Ed \to \frac{Ed}{\epsilon_r}$$

And the capacitance is

$$C = \frac{Q}{V} \to \frac{\epsilon_r Q}{V} = \epsilon_r C$$

Boundary Value Problems with Linear Dielectrics

From the given properties of the displacement vector

$$\begin{aligned} \mathbf{D} &= \epsilon_0\mathbf{E} + \mathbf{P} \\ &= \epsilon_0(1 + \chi_e)\mathbf{E} \end{aligned}$$

where $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$ so

$$\mathbf{D} = \epsilon\mathbf{E}$$

We can get the bound charge densities

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\boldsymbol{\nabla}\cdot\left(\frac{\epsilon_0\chi_e}{\epsilon}\mathbf{D}\right), \quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$$

which implies

$$\boxed{\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f}$$

Since the net charge resides in the surface of the dielectric, i.e. $\rho = 0$ so from Laplace's equation

$$\epsilon_{\text{above}}\mathbf{E}^{\perp}_{\text{above}} - \epsilon_{\text{below}}\mathbf{E}^{\perp}_{\text{below}} = \sigma_f$$

or in terms of voltage (from $\mathbf{E} = -\boldsymbol{\nabla}V$)

$$\epsilon_{\text{above}}\frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}}\frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f$$

And since the potential is still continuous

$$V_{\text{above}} = V_{\text{below}}$$

Example: Sphere of homogeneous linear dielectric material in a uniform $\mathbf{E}_0$. Find $\mathbf{E}$ in sphere.

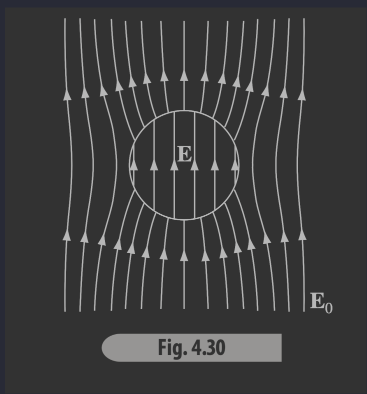


Figure 1.10: Homogeneous linear dielectric sphere in uniform E-field $\mathbf{E}_0$.

Using B.C. to solve Laplace's equation

(i) $V_{\text{in}} = V_{\text{out}}$ at $r = R$

(ii) $\epsilon\frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0\frac{\partial V_{\text{out}}}{\partial r}$ at $r = R$ where $\epsilon = \epsilon_0\epsilon_r$

(iii) $V_{\text{out}} = -E_0 r\cos\theta$ at $r \gg R$

From the Legendre polynomials

$$V_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

and outside the sphere

$$V_{\text{out}} = \underbrace{-E_0 r\cos\theta}_{\text{external field}} + \underbrace{\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)}_{\text{due to sphere}}$$

and at $r = R \to V_{\text{in}} = V_{\text{out}}$ so

$$\sum_l A_l R^l P_l(\cos\theta) = -E_0 R\cos\theta + \sum_l \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

Remembering that $P_0 = 1$ and $P_1 = \cos\theta$ we get from (i)

$$\begin{cases} A_1 R = -\epsilon_0 R + \frac{B_1}{R^2} & l = 1 \\ A_l R_l = \frac{B_l}{R^{l+1}} & l \neq 1 \end{cases}$$

and from (ii)

$$\epsilon_r \sum_l A_l R^{l+1} P_l(\cos\theta) = -\epsilon_0 \cos\theta - \sum_l \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta)$$

so the orthogonality of the Legendre polynomials gives

$$\begin{cases} \epsilon_r A_1 = -\epsilon_0 - \frac{2B_1}{R^3} & l = 1 \\ \epsilon_r l A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} & l \neq 1 \end{cases}$$

After staring at it for some time and looking through the math

$$\begin{cases} A_l = B_l = 0 & l \neq 1 \\ A_1 = -\frac{3}{\epsilon_r + 2} E_0 \quad \text{and} \quad B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0 & l = 1 \end{cases}$$

Thus the potentials are

$$V_{\text{in}}(r,\theta) = -\frac{3\epsilon_0}{\epsilon_r + 2} r\cos\theta = -\frac{3\epsilon_0}{\epsilon_r + 2} z$$

so the electric field is

$$\mathbf{E}_{\text{in}} = -\boldsymbol{\nabla} V = \frac{3}{\epsilon_r + 2}\mathbf{E}_0$$

The field inside the sphere will go to zero if the dielectric constant is infinite.

Another example: Everything below plane $z = 0$ is linear dielectric with susceptibility χ_e. Calculating the force on a charge q a distance d above the origin.

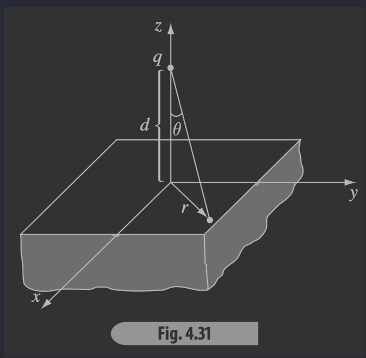


Figure 1.11: Everything below $z = 0$ is a dielectric

Earlier we found a relationship in the volume of a dielectric sphere being $\rho_b \propto \rho_f = 0$, so we only need to worry about the bound surface charge density

$$\begin{aligned} \sigma_b &= \mathbf{P}\cdot\hat{\mathbf{n}} = P_z, \quad \mathbf{P} = \epsilon_0\chi_e\mathbf{E} \\ &= P_z = \epsilon_0\chi_e E_z \end{aligned}$$

where E_z is the z-component of the TOTAL field. The total field due to q and σ_b:

- z-comp of the charge q

$$E_q = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2+d^2}\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}}$$

- z-comp of the bound surface charge

$$E_{\sigma_b} = -\frac{\sigma_b}{2\epsilon_0}$$

So

$$\sigma_b = \epsilon_0\chi_e\left[-\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0}\right]$$

and solving for σ_b we get

$$\sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}}$$

Which is exactly the same as the surface charge of an induced surface charge on an infinite conducting plane...

The total bound charge is therefore

$$q_b = -\left(\frac{\chi_e}{\chi_e+2}\right)q$$

1.4.2 Energy in Dielectrics

The work it takes to charge a capacitor is

$$W = \frac{1}{2}CV^2$$

where the a capacitor filled with a linear dielectric has a capacitance

$$C = \epsilon_r C_0$$

Thus the energy stored in this system is

$$W_{\text{die}} = \frac{\epsilon_0}{\int} \epsilon_r E^2 \, \mathrm{d}\tau$$

and using $\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E}$ we get

$$\boxed{W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E} \, \mathrm{d}\tau}$$

which is integrated over all space.

Example Sphere of radius R filled with dielectric ϵ_r and free charge ρ_f has energy (from Gauss's Law)

$$\oint D \cdot \mathrm{d}\mathbf{a} = Q_{\text{f, enc}}$$

$$\implies \mathbf{D}(\mathbf{r}) = \begin{cases} \dfrac{\rho_f}{3}\mathbf{r} & r < R \\ \dfrac{\rho_f}{3}\dfrac{R^3}{r^2}\mathbf{r} & r > R \end{cases}$$

Thus the correlated E-field

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & r < R \\ \dfrac{\rho_f}{3\epsilon_0\epsilon_r}\dfrac{R^3}{r^2}\mathbf{r} & r > R \end{cases}$$

The purely electrostatic energy is $W_{\text{es}} = \frac{\epsilon_0}{2}\int E^2 \, \mathrm{d}\tau$ or

$$\begin{aligned} W_{\text{es}} &= \frac{\epsilon_0}{2}\left[\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)^2 \int_0^R r^2(4\pi r^2)\,\mathrm{d}r + \left(\frac{\rho_f}{3\epsilon_0}\right)^2 R^6 \int_R^\infty \frac{1}{r^4}(4\pi r^2)\,\mathrm{d}r\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r^2} + 1\right) \end{aligned}$$

but the total energy is

$$\begin{aligned} W_{\text{tot}} &= \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E}\,\mathrm{d}\tau \\ &= \frac{1}{2}\left[\frac{\rho_f^2}{9\epsilon_0\epsilon_r}\int_0^R r^2(4\pi r^2)\,\mathrm{d}r + \left(\frac{\rho_f}{3}R^3\right)\left(\frac{\rho_f}{3\epsilon_0}R^3\right)\int_R^\infty \frac{1}{r^4}(4\pi r^2)\,\mathrm{d}r\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r} + 1\right) \end{aligned}$$

Thus

$$W_{\text{es}} < W_{\text{tot}}$$

So starting with the unpolarized dielectric sphere and using the "method of assembly" i.e. adding charge $\mathrm{d}q$ to each layer of the sphere, we have the field in three regions

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & r < r' \\ \frac{\rho_f}{3\epsilon_0\epsilon_r}\frac{R^3}{r^2}\mathbf{r} & r' < r < R \\ \frac{\rho_f}{3\epsilon_0}\frac{R^3}{r^2}\mathbf{r} & r > R \end{cases}$$

And bringing down $\mathrm{d}q$ from $\infty \to r'$ the infinitesimal work is

$$\begin{aligned} \mathrm{d}W &= -\,\mathrm{d}q\left[\int_\infty^R \mathbf{E}\cdot\mathrm{d}\mathbf{l} + \int_R^{r'} \mathbf{E}\cdot\mathrm{d}\mathbf{l}\right] \\ &= -\,\mathrm{d}q\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_\infty^R \frac{1}{r^2}\,\mathrm{d}r + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_R^{r'} \frac{1}{r^2}\,\mathrm{d}r\right] \\ &= \frac{\rho_f r'^3}{3\epsilon_0}\left(\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right)\mathrm{d}q \end{aligned}$$

which increases the radius by

$$\mathrm{d}q = \rho\,\mathrm{d}V = \rho_f(4\pi'^2)\,\mathrm{d}r'$$

thus

$$\mathrm{d}W = \frac{\rho_f}{3\epsilon_0}\left[\frac{r'^3}{R}\left(1 - \frac{1}{\epsilon_r}\right) + \frac{r'^2}{\epsilon_r}\right]\mathrm{d}q$$

So the total work is done by integrating over $0 \to R$

$$\begin{aligned} W &= \int \mathrm{d}W = \int (\,)\,\mathrm{d}q \to \int (\,)\,\mathrm{d}r' \\ &= \frac{4\pi\rho_f^2}{3\epsilon_0}\left[\frac{1}{R}\left(1 - \frac{1}{\epsilon_r}\right)\int_0^R r'^5\,\mathrm{d}r' + \frac{1}{\epsilon_r}\int_0^R r'^4\,\mathrm{d}r'\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^3 R^5\left[\frac{1}{5\epsilon_r} + 1\right] = W_{\text{tot}} \end{aligned}$$

So the energy of deformation is

$$W_{\text{def}} = W_{\text{tot}} - W_{\text{es}} = \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2 R^5(\epsilon_r - 1)$$

Another way. . . We can pretend that this deformation energy is analogous to a spring (separated by distance d and spring constant k) connecting two charges where one charge $+q$ is nailed down so

$$qE = kd$$

here

$$\mathbf{E} = \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} \implies \quad \text{field of free charge, inside dielectric}$$

thus the dipole $p = qd$ and polarization $P = p/\,\mathrm{d}\tau$. The spring constant and infinitesimal change in work is

$$\begin{aligned} k &= \frac{\rho_f}{3\epsilon_0\epsilon_r d^2}Pr\,\mathrm{d}\tau \\ \mathrm{d}W &= \frac{1}{2}kd^2 = \frac{\rho_f}{6\epsilon_0\epsilon_r}Pr\,\mathrm{d}\tau \end{aligned}$$

This is will give us the energy stored in the dipole "spring" i.e.

$$W_{\text{sp}} = \frac{\rho_f}{6\epsilon_0\epsilon_r}\int Pr\,\mathrm{d}\tau$$

and the polarization is

$$\mathbf{P} = \epsilon_0(1+\chi_e)\mathbf{E} = \epsilon_0\chi_e\frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} = \frac{(\epsilon_r-1)\rho_f}{3\epsilon_r}\mathbf{r}$$

So

$$\begin{aligned}W_{\text{sp}} &= \frac{\rho_f}{6\epsilon_0\epsilon_r}\int\frac{(\epsilon_r-1)\rho_f}{3\epsilon_r}(4\pi)\int_0^R r^4\,\mathrm{d}r\\ &= \frac{2\pi}{45\epsilon_0\epsilon_r^2}\rho_f^2R^5(\epsilon_r-1)\end{aligned}$$

which is the exact same thing! This is because our approximation for the polarization $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$ is linear i.e. we were approximating for a spring the whole time!