

4.2 Given the charge density of a groundstate hydrogen atom

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

where q is the electron charge and a is the Bohr radius, find the atomic polarizability.

First calculating the electric field of the electron cloud

$$E_e(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

where the enclosed charge Q is

$$\begin{aligned} Q &= \int \rho(r') d\tau, \quad d\tau = r'^2 \sin\theta dr' d\theta d\phi \\ &= \frac{q}{\pi a^3} (4\pi) \int_0^r r'^2 e^{-2r'/a} dr', \quad u = -\frac{2r}{a}; \quad du = -\frac{2}{a} dr \\ &= \frac{4q}{a^3} \left[-\frac{a^3}{8} \int u^2 e^u du \right] \\ &= -\frac{q}{2} \left[(u^2 - 2u + 2) e^u \right] \Big|_0^{-2r/a} \\ &= -\frac{q}{2} \left[\left(\frac{4r^2}{a^2} + \frac{4r}{a} + 2 \right) e^{-2r/a} - 2 \right] \\ &= q \left[1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1 \right) e^{-2r/a} \right] \end{aligned}$$

So

$$E_e(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \left[1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1 \right) e^{-2r/a} \right]$$

For $r \ll a$ we can Taylor expand $e^{-2r/a}$ since $-2r/a \approx 0$ to get

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \\ \implies e^{-2r/a} &= 1 - \frac{2r}{a} + \frac{2r^2}{a^2} - \frac{4r^3}{3a^3} + \dots \end{aligned}$$

so

$$\begin{aligned} 1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1 \right) e^{-2r/a} &= 1 - \left(\frac{2r^2}{a^2} + \frac{2r}{a} + 1 \right) \left(1 - \frac{2r}{a} + \frac{2r^2}{a^2} - \frac{4r^3}{3a^3} \right) \\ &= 1 - \frac{2r^2}{a^2} + \frac{4r^3}{a^3} - \frac{4r^4}{a^4} + \frac{8r^5}{3a^5} \\ &\quad - \frac{2r}{a} + \frac{4r^2}{a^2} - \frac{4r^3}{a^3} + \frac{8r^4}{3a^4} \\ &= 1 + \frac{2r}{a} - \frac{2r^2}{a^2} + \frac{4r^3}{3a^3} \\ &= \frac{4r^3}{3a^3} + \dots \end{aligned}$$

where the higher order terms are negligible. In an external electric field E , the polarized atom will have a balanced internal field $E = E_e$, such that the field at a distance d from the center of the cloud is

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \left[\frac{4d^3}{3a^3} \right] = \frac{1}{3\pi\epsilon_0 a^3} (qd) = \frac{qd}{\alpha} \implies \boxed{\alpha = 3\pi\epsilon_0 a^3}$$

4.8 Given the energy of a permanent ideal dipole \mathbf{p} in E-field \mathbf{E} is

$$U = -\mathbf{p} \cdot \mathbf{E} \quad (4.6)$$

and the E-field of a perfect dipole

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \quad (3.104)$$

So to find the interaction energy of two ideal dipoles (\mathbf{p}_1 and \mathbf{p}_2) we use (4.6) and calculate the energy of dipole \mathbf{p}_1 in the E-field produced by the second dipole \mathbf{E}_2 :

$$\begin{aligned} U &= -\mathbf{p}_1 \cdot \mathbf{E}_2 \\ &= -\mathbf{p}_1 \cdot \left[\frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2] \right] \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [-\mathbf{p}_1 \cdot [3(\mathbf{p}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_2]] \\ &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})] \end{aligned}$$

checkmark!

4.11 Given a short cylinder(radius a , length L) with “frozen-in” polarization \mathbf{P} parallel to the axis. Since \mathbf{P} is uniform, $\rho_b = 0$, but the surface bound charge density is

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

so the bound surface charge is $\sigma_b = P$ on one end and $\sigma_b = -P$ on the other end of the cylinder.

- (i) E-field sketch for $L \gg a$:
- (ii) E-field sketch for $L \ll a$:
- (iii) E-field sketch for $L \approx a$:

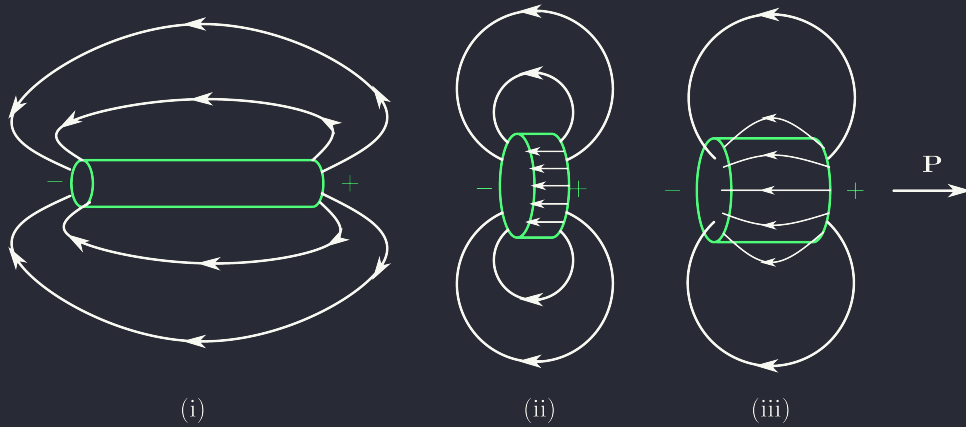


Figure 6.1: Sketch of E-field for a short cylinder

4.34 A dielectric cube of side a centered at the origin with a “frozen-in” polarization $\mathbf{P} = k\mathbf{r}$ where k is a constant and $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.

First the volume bound charge density is

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot (k\mathbf{r}) = -k \left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z \right) = -3k$$

and the surface bound charge density is, e.g. on the face $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, $x = a/2$,

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \frac{ka}{2}$$

which is the same for all six faces. The total bound charge is

$$Q = \int_V \rho_b d\tau + \oint_S \sigma_b da = -3ka^3 + \frac{ka}{2}(6a^2) = 0$$

where the volume integral is just the volume of the cube a^3 and the surface integral is the sum of the areas of the six faces $6a^2$.