Chapter 14: Plasmons, Polaritons, and Polarons

E&M Stuff

In E&M, we extensively study two fields: the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. We also have a vector

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$$

where $\mathbf{P}$ is the polarization vector, and $\mathbf{D}$ is the displacement vector. In a static field, we see that the divergence of the electric field

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

is equivalent to the ratio of the charge density ρ and the permittivity of free space ϵ_0. We also know that the curl

$$\nabla \times \mathbf{E} = 0$$

is zero as the electric field can be expressed as the gradient of a scalar(Hemholtz) potential. Looking at the displacement vector,

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f$$

where in CGS units we define

$$\mathbf{D} = \epsilon\mathbf{E}$$

where the dielectric function $\epsilon(\omega, \mathbf{K})$ has a dependence on frequency and wave vector which makes it a difficult problem to solve.

Plasmon The total charge density

$$\rho = \rho_{\text{ext}} + \rho_{\text{ind}}$$

is the sum of the external charge density and the induced charge density. In CGS units, the divergence of the two fields are

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho_{\text{ext}} \\
\nabla \cdot \mathbf{E} &= 4\pi(\rho_{\text{ext}} + \rho_{\text{ind}}) \\
&= 4\pi\rho
\end{aligned}$$

We define the following

$$D(\mathbf{K}) = \epsilon(\mathbf{K})\mathbf{E}(\mathbf{K})$$

so the divergence of the electric field and displacement vector are

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left[\sum \mathbf{E}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{r}}\right] = 4\pi\sum_K \rho(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{r}}$$

and

$$\nabla \cdot \mathbf{D} = \nabla \cdot \left[\sum_K \epsilon(\mathbf{K})\mathbf{E}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{r}}\right] = 4\pi\sum_K \rho_{\text{ext}}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{r}}$$

diving the two equations we find

$$\epsilon(\mathbf{K}) = \frac{\rho_{\text{ext}}(\mathbf{K})}{\rho(\mathbf{K})} = 1 - \frac{\rho_{\text{ind}}}{\rho(\mathbf{K})}$$

Free Electron In 1D, the EOM of an electron in an electric field is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -eE$$

where time dependence is harmonic i.e.

$$x = x_0e^{-i\omega t}$$
$$\implies -\omega^2 m x_0 = -eE; \qquad x_0 = \frac{eE}{m\omega^2}$$

The polarization, or dipole moment per unit volume of the electron, is

$$P = -nex_0 = \frac{ne^2E}{m\omega^2}$$

where n is the electron density. So the dielectric function is

$$\epsilon(\omega) = \frac{D}{E} = \frac{E + 4\pi P}{E} = 1 - \frac{4\pi ne^2}{m\omega^2}$$

We define the plasma frequency as

$$\omega_p^2 = \frac{4\pi ne^2}{m}$$

so

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

Example In the background the dielectric constant $\epsilon(\infty)$ then

$$\epsilon(\omega) = \epsilon(\infty)\left[1 - \frac{\bar{\omega}_p^2}{\omega^2}\right]$$

where

$$\bar{\omega}_p^2 = \frac{4\pi ne^2}{m\epsilon(\infty)}$$

Electromagnetic wave

From the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{B}$$

Aside: 3 Types of Differential Equations

- The wave equation

$$A\nabla^2 f = \frac{\partial^2 f}{\partial t^2}$$

- The diffusion equation

$$D\nabla^2 f = \frac{\partial f}{\partial t}$$

- The Poisson equation

$$\nabla^2 f = A$$

For EM waves, the wave equation is

$$\frac{\mathrm{d}^2 D}{\mathrm{d}t^2} = c^2 \nabla^2 \mathbf{E}$$

where we have a solution

$$E \propto e^{i\omega t} e^{i\mathbf{K}\cdot\mathbf{r}} \quad \text{and} \quad \mathbf{D} = \epsilon \mathbf{E}$$

so the wave equation tells us the dispersion relation

$$\omega^2 \epsilon(\omega, \mathbf{K}) = c^2 K^2$$

This tells us some intersting things

- ϵ is real, $\epsilon > 0$, and for real K and ω the wave propogates transversely with phase velocity

$$v_p = \frac{c}{\sqrt{\epsilon}}$$

- If ϵ is real and $\epsilon < 0$, then K is imaginary and the wave is damped.
- If ϵ is complex and ω is real, $\mathbf{K}$ is complex and is damped.

From the dispersion relation

$$\epsilon(\omega, \mathbf{K}) = 1 - \frac{\omega_p^2}{\omega^2}$$

if $\omega < \omega_p$ there is total reflection, and if $\omega > \omega_p$ the material is transparent.

Metal In a metal with positve charge density, we apply an electric field to slighlty displace the electrons and cause them to oscillate. The EOM is

$$nm\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -neE$$

This displaces the surface charge density $\sigma = \pm neu$ or a capacitor. Using a gaussian pillbox at the two surfaces of the capacitor, we know that

$$E \cdot S = \frac{\sigma \cdot s}{\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

and from Gauss's law

$$E = 4\pi nue$$

so the wave equation is

$$nm\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -neE = -4\pi ne^2 u$$
$$\implies \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega_p^2 u = 0$$

where the frequency is

$$\omega_p = \sqrt{\frac{4\pi ne^2}{m}}$$

We can approximately find that for 10^{23} electrons per cubic centimeter(Avogadro's number) we get a frequency of roughly 10^{16} Hz, and the energy is roughly

$$\hbar\omega_p \approx \frac{10^{-34} \cdot 10^{16}}{10^{-19}} = 1\,\mathrm{eV}$$

Experimentally we find that the plasmon energy is roughly $10\,\mathrm{eV}$ since the frequency is 10^{16} Hz.