

Physics 463: Statistical Mechanics and Thermodynamics

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1 Intro to statistical methods

- Goal: Study systems consist of many particles (magnitude of moles) that interact with each other.
- Stat Mech bridges the gap between the Macroscopic and Microscopic Description of a system.

Macroscopic Description $> \mu\text{m}$:

- Temperature, Pressure, Volume Entropy, etc.

Microscopic Description \AA :

Worksheet

- (1) Avogadro's number: $N_A = 6.022 \times 10^{23}$ e.g. in 12g of carbon-12, there are N_A atoms!
 1 mole of air : 22.4 L at 273 K, 1 atm.
 e.g. Say a room is 5m x 5m x 8m = 200 m³ = 200×10^3 L, how many moles of air are in the room?
 $\sim 10000 N_A$

- (2) $k_B = 1.38 \times 10^{-23}$ J/K
 Physical Meaning: $k_B T$ will roughly gives us the energy in one atom

- (3) On a number line with 1 and $+\infty$, where is N_A ?
 In mathematics, we would place N_A closer to 1, but in physics we would place it closer to $+\infty$ because this number is huge in the context of physics.

- (4) In the physics convention, we use θ as the polar angle and ϕ as the azimuthal angle. So a volume element in a sphere is

$$r^2 \sin \theta dr d\theta d\phi$$

Thus the volume of a sphere is

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_0^R r^2 \sin \theta dr d\theta d\phi = \frac{4}{3} \pi R^3$$

- (5) The ideal gas law comes in two forms:

$$PV = N k_B T$$

$$PV = nRT$$

where $n = \frac{N}{N_A}$, and N is the number of particles in the system.

- (6) The container with gas confined to half a container at $t = 0$ releases the gas to fill the whole container at $t > 0$. What is the change in entropy?

The equation for the change in entropy is

$$dS = \frac{dQ}{T}$$

but this doesn't tell us much...

Basic Statistical Concepts : “statistical ensemble”

Example: Fair coin toss (50/50) N times. The expected value of heads is $N/2$. Repeating this many times gives a Gaussian distribution centered at $N/2$.

Random walk in 1D Starting at $x = 0$, we have a probability p to move one unit to the right and probability $(1 - p) = q$ to move left.

For a ‘trajectory’

- n_L : # of steps left
- n_R : # of steps right
- $N = n_L + n_R$
- Displacement: $x = n_R - n_L$

Each step is independent: “no memory”, “Markovian/Markov process”

The probability of a specific trajectory is

$$p \cdot p \cdots p \cdot q \cdots q = p^{n_R} q^{n_L}$$

How many ways this (n_R, n_L) can be arranged?

$$\binom{N}{n_R} = \frac{N!}{n_R! n_L!}$$

So the probability of taking n_R steps to the right is

$$W_N(n_R) = \frac{N!}{n_R! n_L!} p^{n_R} q^{n_L}$$

Finishing the Random Walk

$$W_N(n_R) = \frac{N!}{n_R!n_L!} p^{n_R} q^{n_L}$$

is indeed the “Binomial distribution”.

The mean displacement (or expected value) is

$$\bar{m} = \bar{n}_R - \bar{n}_L = pN - qN = N(p - q)$$

How do we define variance/dispersion?

$$\begin{aligned} \overline{(\Delta n_R)^2} &= \overline{(n_R - \bar{n}_R)^2} = \overline{n_R^2 - 2n_R\bar{n}_R + \bar{n}_R^2} \\ &= \overline{n_R^2} - 2\bar{n}_R^2 + \bar{n}_R^2 \\ &= \overline{n_R^2} - \bar{n}_R^2 = Npq \end{aligned}$$

So the deviation or width is roughly $\sim \sqrt{Npq}$

For large N , the distribution can be approximated a continuous:

$$\left. \frac{dW(n_R)}{dn_R} \right|_{\bar{n}_R} = 0$$

or equivalently

$$\left. \frac{d \ln W(n_R)}{dn_R} \right|_{\bar{n}_R} = 0$$

And using

$$n_R \equiv \bar{n}_R + \xi$$

where ξ is the deviation from the mean.

So now we can Taylor expand $\ln W$:

$$\ln W(n_R) = \ln W(\bar{n}_R) + \cancel{\left. \frac{d \ln W(n_R)}{dn_R} \right|_{\bar{n}_R} (n_R - \bar{n}_R)} + \frac{1}{2} B_2 \xi^2 + \dots$$

where

$$W(n_R) \equiv W_{max} e^{-\frac{1}{2} B_2 \xi^2}, \quad B_2 = \frac{1}{Npq}$$

This yields the Gaussian distribution approximation.

$$P(m) = W(n_R) = (2\pi Npq)^{-1/2} e^{-\frac{[m - N(p-q)]^2}{8Npq}}$$

Worksheet

1. If a coin is flipped 400 times, what's the probability of getting 215 heads?

$$N = 215 + 185 = 400, \quad p = 0.5, \quad q = 0.5, \quad m = 215 - 185 = 30$$

Plugging in the numbers gives $P(30) = 1.295\%$

2 Statistical description of systems of particles

2.1 Statistical formulation

Essential ingredients:

1. state of the system:
 - single spin-1/2 particle. \uparrow, \downarrow
 - a bunch of spin-1/2 particles. $\uparrow\uparrow\downarrow \dots$
 - a simple 1D Harmonic Oscillator: $E = (n + 1/2)\hbar\omega$, with states $|n\rangle$
 - a bunch of 1D HO: $|n_1, n_2, \dots, n_N\rangle$
2. Statistical ensemble: Instead of a simple experiments, we consider an ensemble of many exps.
3. Basic postulate about a priori probabilities (relative prob of finding the system in any of its accessible states)
4. Calculate probabilities

Example: 3 spin-1/2

State	Spin	Energy	$\Omega(E)$	$y_k = \uparrow, \downarrow$ $\Omega(-\mu H, \uparrow)$
$\uparrow\uparrow\uparrow$	3/2	$-3\mu H$	1	
$\uparrow\uparrow\downarrow$	1/2	$-\mu H$	3	
$\uparrow\downarrow\uparrow$				
$\downarrow\uparrow\uparrow$				
$\uparrow\downarrow\downarrow$	-1/2	μH	3	
$\downarrow\uparrow\downarrow$				
$\downarrow\downarrow\uparrow$				
$\downarrow\downarrow\downarrow$	-3/2	$3\mu H$	1	

Table 1: Energy levels of 3 spin-1/2 particles

System: *isolated*: energy cannot change *equilibrium*: prob of finding the system in any one accessible state is constant in time

A fundamental postulate:

An isolated system in equilibrium is equally likely to be in any of its accessible states

In calculating probabilities, e.g., isolated system with energy in range $[E, E + \delta E]$

$\Omega(E)$: total number of states of the system in this range

$\Omega(E, y_k)$: in this energy range and some other property y_k where the probability of having this property is

$$P(y_k) = \frac{\Omega(E, y_k)}{\Omega(E)}$$

Density of states (DOS)

$$\Omega(E) = w(E)dE, \quad w(E) \sim E$$

where $w(E)$ is the density of states.

2.2 Interactions between macroscopic systems

In general: specify some macroscopic measurable parameters x_1, x_2, \dots, x_n

- Microstate: A particular quantum state: γ of the system with energy E_γ

$$E_\gamma = E_\gamma(x_1, x_2, \dots, x_n)$$

- Macrostate (Macroscopic state): Specify external parameters and any other conditions, and includes all the possible microstates—e.g., from Table above the macrostate of $-\mu H$ has 3 microstates. “Microstate” is one particular specific state consistent with the macrostate.

Consider two macro systems A, A' ; they can interact with each other to exchange energy.

Q: what are the different ways to exchange E ? HEAT, WORK. e.g. If A, A' are in a box separated by a wall, then the wall moving due to pressure exchanges energy as work. If the wall cannot move, then there is no work exchanged. [insert image of two boxes with a wall]

Two Cases:

- *thermal interaction*: If all the external parameters are fixed

$$\Delta E = Q, \quad \Delta E' = Q'$$

where Q, Q' are the heat absorbed by each macrosystem, and the energy of the whole system is unchanged, i.e.,

$$\Delta E + \Delta E' = 0 \implies Q + Q' = 0, \quad Q = -Q'$$

- *mechanical interaction* (thermal isolation): no heat exchange “adiabatic”. I do work, negative work is done!

Example: Beaker of water, A , and a wheel attached to a pulley with a weight, A' (2.7 Example 2). The work done by the pulley decreases the energy of system A' by ws (weight times distance).

In general energy can be exchanged both as Heat and Work.

$$Q \equiv \Delta E - \mathcal{W}$$

where \mathcal{W} is the work done to the system. And

$$W = \mathcal{W}$$

is the work done by the system, i.e.,

$$Q \equiv \Delta \bar{E} + W$$

Case of small amounts interaction: Infinitesimal changes

$$dQ = d\bar{E} + dW$$

where the bar through the differential indicates the process as path dependent.

Worksheet

- (1) For the infinitesimal quantity

$$dG = \alpha dx + \beta \frac{x}{y} dy$$

it is path dependent:

General interaction process: energy is exchanged both as heat and work

$$Q = \Delta E + W$$

where Q is the heat added to the system (positive ΔE adds energy) and W is the work done by the system

Very very small work/heat: infinitesimal

$$dQ = d\bar{E} + dW$$

where d is an exact differential (path independent) and $d\bar{}$ is an inexact differential (path dependent).

Math: multivariable differential

A differential form is exact if its equal to the general differential dF for some function $F(x, y)$

e.g. $A(x, y)dx + B(x, y)dy = dF(x, y)$

From last times worksheet:

$$\frac{a}{x}dx + \frac{b}{y}dy = d(a \ln x + b \ln y)$$

How to check if its exact? Assume F exists:

$$dF(x, y) : \text{ is exact } \iff \left(\frac{\partial A}{\partial y} \right)_x = \left(\frac{\partial B}{\partial x} \right)_y$$

where \iff means iff or if and only if. e.g. from the worksheet:

$$dG = a dx + b \frac{x}{y} dy, \quad A = a, \quad B = b \frac{x}{y}$$

so

$$\frac{\partial A}{\partial y} = 0, \quad \frac{\partial B}{\partial x} = \frac{b}{y}$$

thus it is inexact.

Quasi-static process: A system interacts with other systems in a process that is so slow that A remains arbitrarily close to equilibrium at all stages!

e.g. a piston pushing very slowly in a cylinder; when the system is not in equilibrium, then the ideal gas law $pV = nRT$ does not hold.

“relaxational time τ ”: time system requires to reach equilibrium if it experiences a sudden change.

Recall we denote the external parameters of an isolated system

$$x_1, x_2, \dots, x_n$$

and the energy of a microstate r

$$E_r = E_r(x_1, x_2, \dots, x_n)$$

When we start to change the external parameter, energy of state r will change:

$$x_\alpha \rightarrow x_\alpha + dx_\alpha$$

and the change in energy is

$$dE_r = \sum_{\alpha=1}^n \frac{\partial E_r}{\partial x_\alpha} dx_\alpha$$

Now in isolated case $dQ = 0$ so

$$\begin{aligned} dE_r + dW_r &= 0 \\ \implies dW_r &= -dE_r = - \sum_{\alpha=1}^n \left(\frac{\partial E_r}{\partial x_\alpha} \right) dx_\alpha \end{aligned}$$

where

$$X_{\alpha,r} = -\frac{\partial E_r}{\partial x_\alpha}$$

is the “generalized force”— e.g. if x is a distance, then X is a force; if x is a volume, then X is a pressure.

NOTE all discussion above are for : state r

Consider an ensemble: in a quasi static process, $X_{\alpha,r}$ has definite value, so

$$dW = \sum_{\alpha} \bar{X}_{\alpha,r} dx_{\alpha}$$

where $\bar{X}_{\alpha,r}$ is mean of the generalized force.

Example: Cylindrical chamber in state r (height s , circular area A , pressure P_r) with a piston pushing in ds

Force on the piston: $P_r A$

Volume: $V = AS$

Thus work done is

$$\begin{aligned} dW &= F ds = (P_r A) ds \\ &= P_r dV \end{aligned}$$

and

$$dE_r = -dW_r = -P_r dV, \quad P_r = -\frac{\partial E_r}{\partial V}$$

Worksheet

1. The mean pressure p of thermally insulated gas varies with volume V by

$$pV^\gamma = K$$

where K and γ are constants. Find work from p_i, V_i to p_f, V_f .

$$\begin{aligned} \int dW &= \int_{V_i}^{V_f} p dV \\ &= \int_{V_i}^{V_f} \frac{K}{V^\gamma} dV \\ W &= \left. \frac{KV^{1-\gamma}}{1-\gamma} \right|_{V_i}^{V_f} \end{aligned}$$

And since $p_i V_i^\gamma = p_f V_f^\gamma = K$, then

$$\begin{aligned} W &= \frac{K}{1-\gamma} (V_f^{1-\gamma} - V_i^{1-\gamma}) \\ &= \frac{1}{1-\gamma} (p_f V_f - p_i V_i) \end{aligned}$$

3 Statistical thermodynamics

Irreversibility and attainment of equilibrium

3.1 Equilibrium conditions and constraints

Equilibrium condition: The system is equally likely to be found in any accessible states.

“accessible states”: some specific conditions/constraints of system, these limit the number of states the system can be possibly found.

Furthermore, how does the change of constraints change the number of accessible states?

Examples:

- Box divided (partition) into two equal parts: left half is filled with gas, and the right half is empty. After removing the partition (constraint), the gas spreads, but the probability of the gas being in the left half is much smaller, $\frac{1}{2^N}$. Rather, we would expect an equal number of particles on each side for $N \rightarrow N_A$.
- Box with insulating wall constrained to move: If the barrier freely moves, we would expect the Pressures to equalize $P = P'$
- Box with noninsulating wall (can't move): We would expect temperature to be equal $T = T'$

After the states reach equilibrium, if we added the constraint back in, the system would not go back to the original state! (irreversible process)

But what is temperature???

- Kinetic energy? Heat transfer?
- Perhaps macroscopically: flow of heat from one system to another by touch (thermal contact)

3.2 Distribution of energy between systems via heat

Consider two systems A and A' :

- A : Energy E , Number of states $\Omega(E)$
- A' : Energy E' , Number of states $\Omega(E')$

where $\Omega(E)$ is the # of states in A with energy range $(E, E + \delta E)$

The total combined system $A^{(0)}$, with number of states $\Omega^{(0)}$, has a constant total energy,

$$E^{(0)} = E + E' = \text{constant}$$

where we define: $\Omega^{(0)}(E)$: # of states accessible to $A^{(0)}$ when the subsystem A has energy $(E, E + \delta E)$.

When $A^{(0)}$ is in equilibrium the probability is proportional to the number of accessible states:

$$P(E) \propto \Omega^{(0)}(E), \quad \text{or} \quad P(E) = \frac{\Omega(E)\Omega'(E^{(0)} - E)}{\sum_E \Omega^{(0)}(E)} = C\Omega^{(0)}(E)$$

where C is a constant.

Multiplicity

$$\Omega^{(0)}(E) = \Omega(E)\Omega'(E^{(0)} - E)$$

Now, the probability $P(E)$ with E is

$$P(E) = C\Omega(E)\Omega'(E^{(0)} - E)$$

Graphically, we would expect E vs. $\Omega(E)$ to increase (as E increases, $\Omega(E)$ increases), and same with E' vs. $\Omega'(E')$. But E vs. $\Omega'(E^{(0)} - E)$ would decrease. In addition, the probability $P(E)$ as a function of E would have a sharp peak near the equilibrium \tilde{E} .

Finding maximum Take the derivative (of the log because multiplication becomes addition):

$$\frac{\partial \ln P(E)}{\partial E} = 0, \quad \ln P(E) = \ln C + \ln \Omega(E) + \ln \Omega'(E^{(0)} - E)$$

hence

$$\frac{\partial \ln P(E)}{\partial E} = \frac{\partial \ln(\Omega(E))}{\partial E} - \frac{\partial \ln(\Omega'(E'))}{\partial E'} = 0$$

Thermodynamic beta (Wikipedia) Define β :

$$\beta = \frac{\partial \ln \Omega(E)}{\partial E}$$

where at equilibrium, $\beta(\tilde{E}) = \beta'(\tilde{E}')$

Then we introduce a *dimensionless* parameter T such that

$$\boxed{kT = \frac{1}{\beta}}$$

where k (k_B everywhere else) is the Boltzmann constant. Therefore, temperature characterizes the variation of density of state with energy.

Entropy From temperature and defining entropy $S(E)$:

$$\begin{aligned} \frac{1}{T} = k\beta &= \frac{\partial k \ln \Omega(E)}{\partial E}, \quad S(E) = k \ln \Omega(E) \\ &= \frac{\partial S}{\partial E} \end{aligned}$$

Worksheet

1. # of energy levels $\Phi_1(\epsilon) \leq \frac{\epsilon}{\Delta\epsilon} = C\epsilon$
2. Average energy per molecule is $\epsilon = E/f$ (f molecules)

$$\Phi(E) = (\Phi_1(\epsilon))^f$$

3.

$$\begin{aligned} \Omega(E) &= \Phi(E + \delta E) - \Phi(E) \\ &= \frac{\partial \Phi(E)}{\partial E} \delta E, = \Phi_1^{f-1} \frac{\partial \Phi_1}{\partial \epsilon} \delta E \end{aligned}$$

4. If f is very large

$$\begin{aligned} \ln \Omega &= (f-1) \ln \Omega(\epsilon) + \dots \\ &\approx f \ln \Omega(\epsilon) \end{aligned}$$

5. So

$$\Omega \propto \phi_1(\epsilon)^f \propto E^f$$

Review of last time: What is temperature?

For two systems A and A' that present heat exchange:

- $E^{(0)} = E + E'$: total energy is constant
- At thermal equilibrium, the temperature of the systems are the same.
- $P(E) \propto \Omega^{(0)}(E)$: probability of finding system A to have energy E is proportional to the number of accessible states in the total system A^0 .
- $\Omega^{(0)}(E) = \Omega(E)\Omega'(E^{(0)} - E)$: multiplicity
- To find the maximum, take the derivative to zero:

$$\begin{aligned} \frac{\partial \ln P(E)}{\partial E} &= 0 \\ \Rightarrow \frac{\partial \ln \Omega(E)}{\partial E} &= \frac{\partial \ln \Omega'(E^{(0)} - E)}{\partial E'} \end{aligned}$$

where

$$\beta = \frac{\partial \ln \Omega(E)}{\partial E} \quad \text{and} \quad kT = \frac{1}{\beta}$$

and at equilibrium, $\beta(\tilde{E}) = \beta'(\tilde{E}')$ Finally, we define entropy $S(E)$ as

$$S(E) = k \ln \Omega(E)$$

where $S + S' = \text{maximum at equilibrium}$

[insert figure graph of $P(E), \Omega(E), \Omega'(E')$]

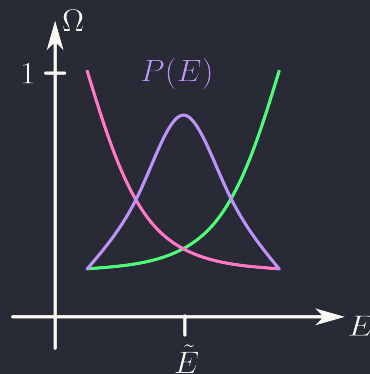


Figure 3.1: Graph of $P(E), \Omega(E)$ (green), $\Omega'(E')$

What is kT ?

We replace $\Omega \sim E^f$ so

$$\begin{aligned} \frac{1}{kT} = \beta &= \frac{\partial \ln \Omega}{\partial E} \approx f \frac{\partial \ln E}{\partial E} = \frac{f}{E} \\ \Rightarrow kT &\approx \frac{\bar{E}}{f} \end{aligned}$$

so kT roughly represents energy per atom. Or

kT is a measure of the mean energy above the ground state per atom

- In the hydrogen atom, if $kT \geq \Delta$ (energy difference between the ground state and the first excited state), then the atom can be excited.

- Room temperature (300 K) is roughly $kT \approx 1/40 \text{ eV} = 25 \text{ meV}$
- A superconducting qubit of energy (usually in GHz) $25 \mu\text{eV}$ uses temperature in the magnitude of 15 mK: Backwards calculation: Going from 300K to 25 meV means we need to go to 0.3 K, but 0.03 K for the precision, to be in the range of the qubit.

3.3 Approach to equilibrium

Given A, A' we have average initial energy \bar{E}_i, \bar{E}'_i : At equilibrium

$$\begin{aligned}\bar{E}_f &= \tilde{E} \\ \bar{E}'_f &= \tilde{E}' = E^{(0)} - \tilde{E}\end{aligned}$$

The heat exchange is

$$\begin{aligned}Q &= \bar{E}_f - \bar{E}_i \\ Q' &= \bar{E}'_f - \bar{E}'_i\end{aligned}$$

where the total heat is constant, $Q + Q' = 0$.

- The system that absorbs heat is “colder”
- The system that releases (gives off) heat is “hotter”

3.4 Temperature

Properties of T :

- If the two systems have same T , they will remain in equilibrium when brought together.
- Zeroth Law of Thermodynamics: If two systems are in thermal equilibrium with a 3rd system, then they must be in equilibrium with each other.

This allows us to use a test system as a “thermometer”: a small system that has a macroscopic parameter that varies when brought into contact with another system.

3.5 Heat Reservoir

A heat reservoir A' with temp T' transfers heat to a smaller system A with temp T :

- $T \rightarrow T'$
- T' does not change!

Using the parameter $\beta' = \beta'(E')$ (where it is inversely related to temp),

$$\left| \frac{\partial \beta'}{\partial E'} Q' \right| \ll \beta'$$

which pretty much tells us that $\Delta \beta' \ll \beta'$, so the temperature of the reservoir does not change.

The change of the density of states of the reservoir is

$$\ln \Omega'(E' + Q') - \ln \Omega'(E')$$

where the Taylor expansion gives us

$$= \frac{\partial \ln \Omega'}{\partial E'} Q' = \beta' Q'$$

This is proportional to the change of entropy:

$$\begin{aligned}\Delta S' &= k(\ln \Omega'(E' + Q) - \ln \Omega'(E')) \\ \beta' Q' k &= \frac{Q'}{T'}\end{aligned}$$

Simply,

$$\Delta S' = \frac{Q'}{T'} \quad (\text{for a heat reservoir})$$

If one assumes an infinitesimal amount of heat dQ

$$dS' = \frac{dQ'}{T'}$$

which is the 2nd law of thermodynamics.

The first law of thermodynamics is (as we discovered)

$$dE + dW = dQ$$

3.6 Dependence on Density of States (DoS) on external parameters

$\Omega(E, x)$ where x is an external parameter. The change of energy is

$$\frac{\partial E}{\partial x} dx \quad \text{where} \quad X = -\frac{\partial E}{\partial x} \quad \text{“generalized force”}$$

Review

$$\frac{\partial \ln \Omega(E)}{\partial E} = \beta = \frac{1}{kT}$$

$$ds = \frac{dQ}{T}, \quad S = k \ln \Omega$$

3.7 Density of states on external parameters

When a parameter $x \rightarrow x + dx$ the energy of each microstate r changes by

$$\frac{\partial E_r}{\partial x} dx$$

where

$$X = -\frac{\partial E_r}{\partial x}$$

is the generalized force.

- $\Omega(E, x)$: external parameter x for the # of states in $[E, E + \delta E]$.
- $\Omega_Y(E, x)$: # of states in $[E, E + \delta E]$ with $Y = \frac{\partial E_r}{\partial x}$ in $[Y, Y + \delta Y]$

The total number of states is now

$$\Omega(E, x) = \sum_Y \Omega_Y(E, x)$$

Consider energy E when $x \rightarrow x + dx$:

some states r originally with energy $< E$ Can now acquire energy $> E$:

- $\sigma(E)$: # of state of originally
- $\sigma_Y(E)$: # of states ... with $Y = \frac{\partial E_r}{\partial x}$ in $[Y, Y + \delta Y]$

For a given E, Y ; the work done is the generalized force times the change in the parameter $Y dx$ which is like the width of the rectangle below the energy E .

Above the energy $[E, E + \delta E]$ we can treat the total area as the total # of states $\Omega_Y(E, x)$, and dividing by the width δE kind of gives us a linear density of states per unit energy. Finally we can find σ_Y by getting the area

$$\sigma_Y(E) = \frac{\Omega_Y(E, x)}{\delta E} Y dx$$

and thus

$$\begin{aligned} \sigma(E, x) &= \sum_Y \sigma_Y(E, x) = \sum_Y \frac{\Omega_Y(E, x)}{\delta E} Y dx \\ &= \frac{\Omega(E, x)}{\delta E} \bar{Y} dx \\ \bar{Y} &= \frac{1}{\Omega(E, x)} \sum_Y Y \Omega_Y(E, x) \end{aligned}$$

The change of the DoS is

$$\frac{\partial \Omega(E, x)}{\partial x} dx = \sigma(E) - \sigma(E + \delta E) = -\frac{\partial \sigma(E)}{\partial E} \delta E$$

on the RHS

$$\frac{\partial \sigma(E, x)}{\partial E} = \frac{dx}{\delta E} \frac{\partial \Omega \bar{Y}}{\partial E}$$

and thus

$$\frac{\partial \Omega(E, x)}{\partial x} = -\frac{\partial \Omega \bar{Y}}{\partial E} = -\frac{\partial \Omega}{\partial E} \bar{Y} - \Omega \frac{\partial \bar{Y}}{\partial E}$$

dividing both sides by Ω

$$\frac{1}{\Omega} \frac{\partial \Omega}{\partial x} = -\frac{1}{\Omega} \frac{\partial \Omega}{\partial E} \bar{Y} - \frac{\partial \bar{Y}}{\partial E}$$

where

$$\frac{\partial \ln \Omega}{\partial x} = \frac{1}{\Omega} \frac{\partial \Omega}{\partial x}, \quad \frac{\partial \ln \Omega}{\partial E} = \frac{1}{\Omega} \frac{\partial \Omega}{\partial E}$$

and another trick: the second term is $\sim \bar{Y}/E$ and $\Omega \sim E^f$ so we can drop the second term

$$\frac{\partial \ln \Omega}{\partial x} = -\frac{\partial \ln \Omega}{\partial E} \bar{Y} = \beta \bar{X}$$

Recap: For a system with states $\Omega(E, x)$ we are give

$$\boxed{\frac{\partial \ln \Omega}{\partial E} = \beta}, \quad \boxed{\frac{\partial \ln \Omega}{\partial x} = \beta \bar{X}}$$

The second equation tells us that doing work will obviously increase Ω .

The rubber band example: Stretching the rubber band (work done) adds energy or heat to the system.

3.8 Equilibrium of interacting systems

$\Omega^{(0)}(E, x)$ will be maximum at \tilde{E}, \tilde{x} .

Consider an infinitesimal quasi-static process where A and A' are brought from equilibrium to another state $\tilde{E} + d\tilde{E}, \tilde{x} + d\tilde{x}$.

The differential of the log Dos is (using math)

$$\begin{aligned} d \ln \Omega(E, x) &= \frac{\partial \ln \Omega}{\partial E} d\tilde{E} + \frac{\partial \ln \Omega}{\partial x} d\tilde{x} \\ &= \beta(d\tilde{E} + X d\tilde{x}) \quad X d\tilde{x} = dW \\ &= \beta dQ \\ k d \ln \Omega(E, x) &= \frac{dQ}{T} \end{aligned}$$

where $S = k \ln \Omega$ or equivalently

$$dS = \frac{dQ}{T}$$

Case 1 : Thermally isolated and quasi-static process

$$dQ = 0 \implies \Delta S = 0$$

thus a **reversible process**

Worksheet A gas with N atoms confined to half a box is released to fill the whole container:
Given

- Ideal gas: $PV = NkT$
 - Average energy: $\bar{E}(T) = \frac{3}{2}NkT$
- (i) Change in entropy $S = k \ln \Omega$: Originally we have only 1 macrostate or Ω_0 microstates, and now the total number of arrangements is $\Omega_0 2^N$ so

$$\Delta S = k \ln \Omega_0 2^N - k \ln \Omega_0 = Nk \ln 2$$

where we have a volume independence $\Omega \sim V^N X(E)$.

- (ii) Defining a quasi-static process where energy doesn't change from initial to final state: We can now use

$$dS = \frac{dQ}{T}$$

and for a quasi-static process

$$\begin{aligned} dQ &= d\tilde{E} + dW \\ dQ &= dW = PdV \end{aligned}$$

So

$$\Delta S = \int ds = \int_{V_0}^{2V_0} \frac{PdV}{T} = \int_{V_0}^{2V_0} \frac{Nk}{V} dV = Nk \ln 2$$

3.9 Fundamental Results

3.9.1 Thermodynamic laws

0th law If two systems are in equilibrium with a 3rd system, they're in equilibrium with each other. (allows us to compare temp of systems via thermometer)

1st law Energy is conserved: $dQ = d\bar{E} + dW$

2nd law An equilibrium macrostate can be characterized by S (entropy), which has properties that

- a. In any thermally isolated process goes from one macrostate to another, entropy tends to increase

$$\Delta S \geq 0$$

- b. If the system is not isolated and undergoes a quasi-static process (absorbs heat),

$$dS = \frac{dQ}{T}$$

3rd law The entropy S of a system has the limiting properties that

$$T \rightarrow 0, \quad S \rightarrow S_0 \quad (\text{Absolute entropy})$$

Remarks All four laws are *macroscopic*. In the context of three parameters \bar{E}, S, T .

3.9.2 Statistical calculation of thermodynamic quantities

Statistical Relations (Microscopic nature)

- $S = k \ln \Omega$
- $\beta = \frac{\partial \ln \Omega}{\partial E}, \quad X_\alpha = \frac{1}{\beta} \frac{\partial \ln \Omega}{\partial x_\alpha}$

For the free expansion of an idea gas:

$$\Omega \propto V^N X(E)$$

where the log of the DoS is

$$\ln \Omega = N \ln V + \ln X(E) + C$$

So from the mean pressure

$$P = \frac{1}{\beta} \frac{\partial \ln \Omega}{\partial V} = \frac{1}{\beta} \frac{\partial \ln V^N}{\partial V} = \frac{1}{\beta} \frac{N}{V} = \frac{NkT}{V}$$

$$PV = NkT$$

From the beta relation

$$\beta = \frac{\ln X(E)}{E}$$

thus it is only a function of energy and not volume V :

$$\beta(\bar{E}) = E \implies \bar{E}(T) = \bar{E}$$

Extensive vs Intensive parameters For macroscopic parameters y_1, y_2 we have two cases:

- Case 1: Extensive if $y_1 + y_2 = y$
- Case 2: Intensive if $y_1 = y_2 = y$

Extensive	Intensive
Volume	T
Energy	Pressure
entropy	specific heat
mass	
heat capacity	

3.10 Heat capacity and specific heat

Suppose we add dQ to the system while other parameters are fixed, the system temperature raised by dT

$$\text{“Heat capacity”} \quad \left(\frac{dQ}{dT} \right)_y = C_y$$

In general The heat capacity is defined by temperature and the parameter

$$C_y = C_y(T, y)$$

We can also relate it to the entropy change $ds = \frac{dQ}{T}$:

$$C_y = \left(\frac{dQ}{dT} \right)_y = T \left(\frac{ds}{dT} \right)_y$$

Specific heat: Intensive parameter

$$C_y = \frac{C_V}{V} \quad \text{per mole}$$

$$C_y = \frac{C_P}{\text{mass}} \quad \text{per gram}$$

or the “specific heat/heat capacity per mole/gram”

Defining some units:

- Calorie: Heat required to raise temperature of 1 g of water @ 1 atm from 14.5 \rightarrow 15.5 $^{\circ}\text{C}$.
- Joule: Applied work

$$1 \text{ cal} = 4.1840 \text{ J}$$

Worksheet

1. Increase in entropy of a cup of water as it is heated from room temp to boiling:

$$\begin{aligned} \Delta S &= \int dS = \int \frac{dQ}{T} = \int_{T_1}^{T_2} \frac{C_P}{T} dT \\ &= C_P \ln \frac{T_2}{T_1} = C_P \ln \frac{373}{298} \end{aligned}$$

where $q = mC_P\Delta T$

4 Macroscopic Parameters and their measurement

4.1 Work & internal energy

From the first law of thermodynamics we always talk about

$$Q = \Delta \bar{E} + W$$

Given a system, work is *easy* measure i.e. we integrate

$$W = - \int p dV$$

Measure of internal energy

- Thermal isolation case: $Q = 0$

$$\Delta \bar{E} = \bar{E}_b - \bar{E}_a = -W_{ab} = \int_a^b dW$$

e.g. a thermally isolated piston goes from state a to b .

4.2 Heat

The heat absorbed by a system going from macrostate a to b is simply

$$Q_{ab} = (\bar{E}_b - \bar{E}_a) + W_{ab}$$

Example A superconducting circuit A is connected to the circuit B with a resistor.

Adding 20 μW of heat to the system: we actually are doing work on a resistor.

Method of Mixers (Comparison Method)

Bring system A into contact with system B that has a known relation between its internal energy and some parameters (T).

$$Q_A = \Delta \bar{E}_B = -Q_B$$

e.g. system A is submerged in water B and we can measure the change in internal energy of water quite easily.

4.3 Entropy

We define entropy S

$$dS = \frac{dQ}{T}$$

and **Absolute entropy** from the 3rd law

$$T \rightarrow 0, \quad S \rightarrow S_0$$

Example: Tin

Two structures of a solid:

1. White tin—a metal \rightarrow stable $> 298 \text{ K}$
2. Grey tin—semiconductor \rightarrow stable $< 298 \text{ K}$

Thus it requires some amount of heat Q to transform from grey to white tin.

- Case 1: a mole of white tin from $T = 0 \rightarrow T_0$ with specific heat $C^{(w)}(T)$

$$S^{(w)}(T_0) = S^{(w)}(T = 0) + \int_0^{T_0} \frac{C^{(w)}(T)}{T} dT$$

- Case 2: Grey tin from 0 K $\rightarrow T_0$ and then it transforms to white tin quasi-statically. It absorbs heat Q and the entropy change is

$$S^{(w)}(T_0) = S^{(g)}(T = 0) + \int_0^{T_0} \frac{C^{(g)}(T)}{T} dT + \frac{Q}{T_0}$$

where

$$S^{(g)}(T = 0) = S^{(w)}(T = 0) = S_0$$



Figure 4.1: Mole of Tin (DALL-E 3)

5 Simple Applications of macroscopic thermodynamics

5.1 General relationship of thermodynamics

Fundamental thermodynamic relation for a *quasi-static process*:

$$dS = \frac{\delta Q}{T}$$

where

$$\delta Q = dE + \delta W = dE + p dV$$

The only external parameter of change is V

$$\implies dE = T dS - p dV$$

This specifies certain relationship between T, S, p, V i.e. S & V are independent variables

$$E = E(S, V)$$

So we have a pure mathematical relationship

$$dE = \left(\frac{\partial E}{\partial S} \right)_V dS + \left(\frac{\partial E}{\partial V} \right)_S dV$$

where

$$\begin{cases} T = \left(\frac{\partial E}{\partial S} \right)_V \\ -p = \left(\frac{\partial E}{\partial V} \right)_S \end{cases}$$

which we already know! Because dE is an exact differential

$$\left(\frac{\partial T}{\partial V} \right)_S = - \left(\frac{\partial p}{\partial S} \right)_V$$

this is known as the first Maxwell relation ([wiki](#)).

How about S, P ?

From our favorite starting point

$$dE = T dS - p dV$$

we need to change $dV \rightarrow dp$ so from chain rule

$$d(pV) = p dV + V dp \implies p dV = d(pV) - V dp$$

so

$$dE = T dS - d(pV) + V dp$$

or

$$d(E + pV) = T dS + V dp$$

lets call this new parameter $H = E + pV$ the **enthalpy** i.e.

$$H = H(S, p)$$

So

$$\begin{cases} T = \left(\frac{\partial H}{\partial S} \right)_p \\ V = \left(\frac{\partial H}{\partial p} \right)_S \end{cases}$$

where dH is an exact differential

$$\left(\frac{\partial T}{\partial p} \right)_S = \left(\frac{\partial V}{\partial S} \right)_p$$

or the second Maxwell relation!

Worksheet We can derive the Helmholtz free energy $F = F(T, V)$ by starting with

$$d(TS) = TdS + SdT \implies TdS = d(TS) - SdT$$

so

$$\begin{aligned} dE &= d(TS) - SdT - pdV \\ d(E - TS) &= -SdT - pdV \end{aligned}$$

1. Thus the Helmholtz free energy $F \equiv E - TS$ so

$$\begin{aligned} dF &= dE - (TdS + SdT) \\ &= TdS - pdV - TdS - SdT \\ &= -SdT - pdV \end{aligned}$$

2. So $F = F(T, V)$ then we know that

$$\begin{cases} -S = \left(\frac{\partial F}{\partial T}\right)_V \\ -p = \left(\frac{\partial F}{\partial V}\right)_T \end{cases}$$

and dF is an exact differential

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

Finally for independent parameters T, p :

$$dE = TdS - pdV$$

we need to change $dV \rightarrow dp$ so from chain rule

$$d(pV) = pdV + Vdp \implies pdV = d(pV) - Vdp$$

so also using $TdS = d(TS) - SdT$

$$\begin{aligned} dE &= (d(TS) - SdT) - (d(pV) - Vdp) \\ d(E - TS + pV) &= -SdT + Vdp \end{aligned}$$

where $G = E - TS + pV$ is the Gibbs free energy $G = G(T, p)$

$$\begin{cases} -S = \left(\frac{\partial G}{\partial T}\right)_p \\ V = \left(\frac{\partial G}{\partial p}\right)_T \end{cases}$$

and dG is an exact differential

$$\left(\frac{\partial V}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T$$

Summary of Maxwell relations

$$\begin{aligned} \left(\frac{\partial T}{\partial V}\right)_S &= -\left(\frac{\partial p}{\partial S}\right)_V \\ \left(\frac{\partial T}{\partial p}\right)_S &= \left(\frac{\partial V}{\partial S}\right)_p \\ \left(\frac{\partial S}{\partial V}\right)_T &= \left(\frac{\partial p}{\partial T}\right)_V \\ \left(\frac{\partial V}{\partial T}\right)_p &= -\left(\frac{\partial S}{\partial p}\right)_T \end{aligned}$$

or in box form Where the components are horizontal (TS) and vert (pV) give us the relations.
e.g. something

$$\begin{array}{c|c} E & F \\ \hline H & G \end{array}$$

Review of Maxwell relations the DoS and external parameters of the system

$$(T, S) \quad \text{and} \quad (p, V)$$

are not independent, but related through

$$dE = Tds - pdV$$

From this we can get the Maxwell relations

$$\begin{aligned} \left(\frac{\partial T}{\partial V}\right)_S &= -\left(\frac{\partial p}{\partial S}\right)_V \\ \left(\frac{\partial T}{\partial p}\right)_S &= \left(\frac{\partial V}{\partial S}\right)_p \\ \left(\frac{\partial S}{\partial V}\right)_T &= \left(\frac{\partial p}{\partial T}\right)_V \\ \left(\frac{\partial V}{\partial T}\right)_p &= -\left(\frac{\partial S}{\partial p}\right)_T \end{aligned}$$

which can be derived from the Thermodynamic functions

$$\begin{aligned} E &= E(S, V) \\ H &= H(S, p) = E + pV \\ F &= F(T, V) = E - TS \\ G &= G(T, p) = E - TS + pV \end{aligned}$$

5.2 Specific Heats

- Molar specific heat at constant volume $dV = 0$

$$C_V = \frac{1}{n} \left(\frac{dQ}{dT} \right)_V \quad dE = dQ = nC_V dT$$

- Molar specific heat for constant pressure $dp = 0$

$$C_p = \frac{1}{n} \left(\frac{dQ}{dT} \right)_p = C_V + \frac{1}{n} p \left(\frac{dV}{dT} \right)_p$$

When comparing the two specific heats we can infer that $C_p > C_V$ because the heat dQ has to both increase the internal energy and do mechanical work to expand the volume:

$$dQ = dE + pdV = nC_V dT + pdV$$

- For an ideal gas

$$\begin{aligned} pV &= nRT \\ pdV &= nRdT \\ \Rightarrow \left(\frac{dV}{dT} \right) &= \frac{nR}{p} \end{aligned}$$

thus

$$C_p = C_V + R$$

where we define

$$\gamma = \frac{C_p}{C_v} = 1 + \frac{R}{C_V}$$

For the idea gas molecule the energy is given by

$$E(T) = \frac{3}{2}nRT$$

where there is 3 degrees of freedom thus

$$C_V = \frac{1}{n} \frac{dE}{dT} = \frac{3}{2}R$$

$$C_p = C_V + R = \frac{5}{2}R$$

For a diatomic molecule there are 2 extra degrees of freedom for rotation so

$$\gamma = \frac{C_p}{C_V} = \frac{5}{3}$$

5.3 Adiabatic expansion or compression

Some definitions:

- “Isothermal”: T is constant $\implies pV = \text{Constant}$.
- “Adiabatic”: $dQ = 0$

$$\implies 0 = dE + pdV$$

$$= nC_V dT + pdV$$

So from the ideal gas law $pV = nRT$ we can get

$$VdP + pdV = nRdT$$

and substituting dT into the adiabatic expression

$$dQ = 0 = V \frac{C_V}{R} dP + \frac{C_V P}{R} dV + pdV$$

which can be rewritten as

$$0 = (C_V + R)pdV + C_V VdP = C_p pdV + C_v Vdp$$

or dividing by $C_V PV$ we get

$$\gamma \frac{dV}{V} + \frac{dP}{P} = 0$$

Integration then gives

$$\gamma \ln V + \ln P = \text{Constant} \implies \ln(PV^\gamma) = \text{Constant}$$

or

$$PV^\gamma = \text{Constant}$$

Worksheet

- Pumping a bike tire, a liter of air at 1 atm is compressed *adiabatically* to 7 atm. (Air is mostly diatomic gas)

- For diatomic gas

$$E(T) = \frac{5}{2}nRT$$

So the specific heats are

$$C_V = \frac{1}{n} \frac{dE}{dT} = \frac{5}{2}RC_p = C_V + R = \frac{7}{2}R$$

which gives us

$$\gamma = \frac{C_p}{C_V} = \frac{7}{5}$$

- The final volume after compression is

$$p_i V_i^\gamma = p_f V_f^\gamma$$

or

$$(1 \text{ atm})(1 \text{ L})^{7/5} = (7 \text{ atm})V_f^{7/5} \implies V_f = 0.25 \text{ L}$$

- Work done compressing air: using $p_i V_i^\gamma = 1 \implies p = \frac{1}{V^\gamma}$

$$\begin{aligned} W &= \int_{V_i}^{V_f} p dV = \int_{V_i}^{V_f} \frac{1}{V^\gamma} dV \\ &= \frac{1}{1-\gamma} (V_f^{1-\gamma} - V_i^{1-\gamma}) \end{aligned}$$

- If initial temp is 300 K, the final temp is

$$\begin{aligned} P_i V_i &= nRT_i \quad P_f V_f = nRT_f \\ \implies \frac{P_f V_f}{P_i V_i} &= \frac{T_f}{T_i} \end{aligned}$$

- If the compression is isothermal (pumping very slowly) how does the answers change?

General case:

$$\begin{aligned} C_V &= \left(\frac{dQ}{dT} \right)_V = T \left(\frac{\partial S}{\partial T} \right)_V \\ C_P &= \left(\frac{dQ}{dT} \right)_P = T \left(\frac{\partial S}{\partial T} \right)_P \end{aligned}$$

5.4 Entropy

Consider $S = S(T, P)$

$$\begin{aligned} dQ &= T dS = T \left[\left(\frac{\partial S}{\partial T} \right)_P dT + \left(\frac{\partial S}{\partial P} \right)_T dP \right] \\ &= C_P dT + T \left(\frac{\partial S}{\partial P} \right)_T dP \end{aligned}$$

5.5 Specific heats again

General relation between C_v and C_p for non-ideal gas ($C_p - C_v = R$),

$$C_V = \left(\frac{dQ}{dT} \right)_V = T \left(\frac{\partial S}{\partial T} \right)_V$$

$$C_P = \left(\frac{dQ}{dT} \right)_P = T \left(\frac{\partial S}{\partial T} \right)_P$$

where

$$\begin{aligned} dQ = TdS &= T \left[\left(\frac{\partial S}{\partial T} \right)_P dT + \left(\frac{\partial S}{\partial P} \right)_T dP \right] \\ &= C_P dT + T \left(\frac{\partial S}{\partial P} \right)_T dP \end{aligned}$$

The pressure $P(T, V)$ with temp and volume dependence has a differential

$$dP = \left(\frac{\partial P}{\partial T} \right)_V dT + \left(\frac{\partial P}{\partial V} \right)_T dV$$

and since C_V acknowledges fixed volume $dV = 0$ we get

$$C_V = \left(\frac{dQ}{dT} \right)_V = C_P + T \left(\frac{\partial S}{\partial P} \right)_T \left(\frac{\partial P}{\partial T} \right)_V$$

where we can replace the entropy term with the Maxwell relation

$$\left(\frac{\partial S}{\partial P} \right)_T = - \left(\frac{\partial V}{\partial T} \right)_P$$

We define the “volume coefficient of expansion”

$$\alpha \equiv \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P$$

so

$$\left(\frac{\partial S}{\partial P} \right)_T = -\alpha V$$

The second term is not well defined since fixing volume while increasing pressure is hard to do (e.g. filling a water bottle with more and more water). Now using the volume dependence $V(P, T)$ i.e.

$$dV = \left(\frac{\partial V}{\partial T} \right)_P dT + \left(\frac{\partial V}{\partial P} \right)_T dP = 0$$

and moving thing around we get

$$\left(\frac{dP}{dT} \right)_V = - \frac{\left(\frac{\partial V}{\partial T} \right)_P}{\left(\frac{\partial V}{\partial P} \right)_T}$$

where we define another term

$$\kappa = - \frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T$$

AKA the “isothermal compressibility” thus

$$C_P - C_V = VT \frac{\alpha^2}{\kappa}$$

One check we can do is use the Ideal gas law to calculate α and κ then see if the above equation gives us the relation $C_P - C_V = R$

5.6 Entropy and Internal energy

$S(T, V)$ doing the same thing

$$\begin{aligned} dS &= \left(\frac{\partial S}{\partial T} \right)_V dT + \left(\frac{\partial S}{\partial V} \right)_T dV \\ &= \frac{C_V}{T} dT + \left(\frac{\partial P}{\partial T} \right)_N dV \end{aligned}$$

etc.

5.7 Free expansion of a gas

For the general case

$$\begin{aligned} dE &= 0 \\ E &= E(T, V) \end{aligned}$$

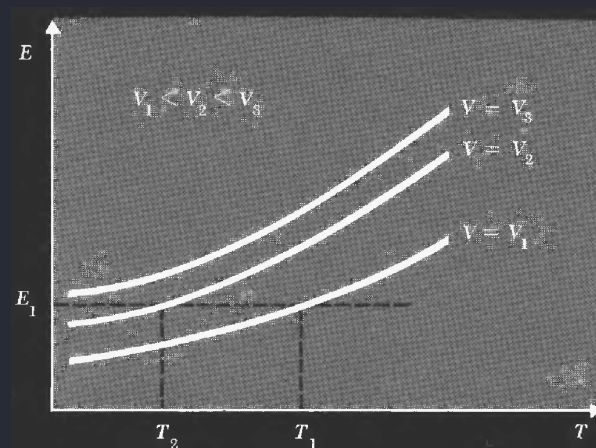


Figure 5.1: Free expansion of a gas

Van de Waals Gas

$$\begin{aligned} \left(P + \frac{a}{v^2} \right) (v - b) &= RT \quad E = E(T, V) \\ v &= \frac{V}{n} \end{aligned}$$

5.8 Heat Engine

What is a heat engine? Heat \rightarrow Work.

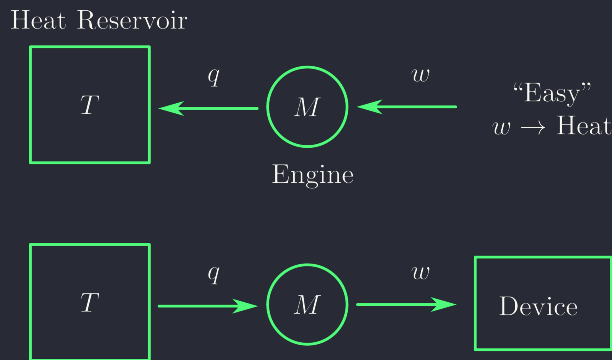


Figure 5.2: (top) reverse “easy” process converting work to heat. (bottom) Heat Engine converting heat from a reservoir into work.

- For a heat engine: whatever mechanisms needed to retune to the same original condition; go through a cycle; otherwise, engine cannot continuously operate.
- Ideally perfect engine: $q = w$ or 100% efficiency

A perfect engine *violates* the 2nd law of thermodynamics $\Delta S \geq 0$.

- entropy change for engine and external device $\Delta S = 0$
- entropy change for heat reservoir $\Delta S = -q/T$ thus

$$\Delta S_{\text{tot}} = -\frac{q}{T} < 0$$

Building a heat engine: Using two heat reservoirs

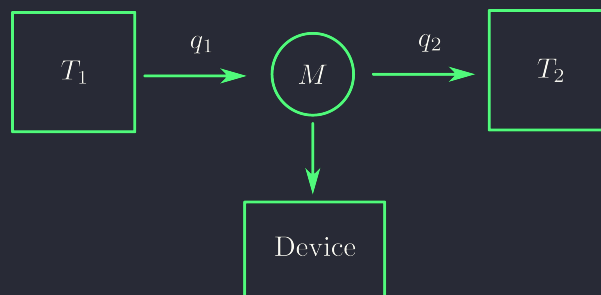


Figure 5.3: Heat engine using two heat reservoirs

Where we make $q_1 > q_2$ so that heat flows from $T_1 \rightarrow T_2$ so that

$$W = q_1 - q_2 \implies q_2 = q_1 - W$$

so the total entropy change is

$$\begin{aligned} \Delta S_{\text{tot}} &= -\frac{q_1}{T_1} - \frac{q_2}{T_2} \geq 0 \\ &= -\frac{q_1}{T_1} + \frac{q_1 - W}{T_2} \\ \implies \frac{W}{T_2} &\leq q_1 \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \end{aligned}$$

thus the efficiency of the engine

$$\begin{aligned}\eta = \frac{W}{q_1} &\leq T_2 \left(\frac{1}{T_2} - \frac{1}{T_1} \right) \\ &= 1 - \frac{T_2}{T_1}\end{aligned}$$

Carnot Engine

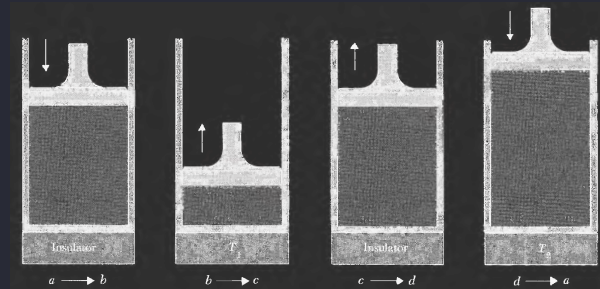


Figure 5.4: 4 stages of Carnot Engine

- $a \rightarrow b$: adiabatic (no heat exchange) bringing colder $T_2 \rightarrow T_1$
- $b \rightarrow c$: isothermal (constant temp) adding heat q_1 from hot reservoir
- $c \rightarrow d$: adiabatic (no heat exchange) cooling down $T_1 \rightarrow T_2$ as work is done through expansion
- $d \rightarrow a$: isothermal (constant temp) releasing heat q_2 to cold reservoir

Worksheet Carnot Engine using an ideal gas:

The work done is

$$W = \int_a^b P dV + \int_b^c P dV + \int_c^d P dV + \int_d^a P dV$$

- $b \rightarrow c$: Using the ideal gas law $PV = nRT$ we get

$$W = \int P dV = \int_{V_b}^{V_c} \frac{nRT_1}{V} dV = nRT_1 \ln \frac{V_c}{V_b}$$

and since $\Delta E = 0$ we know that

$$q_1 = W = nRT_1 \ln \frac{V_c}{V_b}$$

- $d \rightarrow a$: Same as above

$$q_2 = nRT_2 \ln \frac{V_a}{V_d}$$

so

$$W = q_1 - q_2$$

where the work done should be the total heat of the system as we go from $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$, i.e.

$$\Delta E = 0 \implies W = \Delta Q$$

The efficiency of the Carnot engine is by definition

$$\eta = \frac{W}{q_1} = \frac{q_1 - q_2}{q_1} = 1 - \frac{q_2}{q_1} = 1 - \frac{T_2 \ln(V_d/V_a)}{T_1 \ln(V_c/V_b)}$$

Argument:

$$\frac{V_d}{V_a} = \frac{V_c}{V_b}$$

From the adiabatic process

$$PV^\gamma = \text{Constant}$$

so

$$\begin{aligned} P_a V_a^\gamma &= P_b V_b^\gamma \\ P_d V_d^\gamma &= P_c V_c^\gamma \\ \implies \frac{P_a V_a^\gamma}{P_d V_d^\gamma} &= \frac{P_b V_b^\gamma}{P_c V_c^\gamma} \end{aligned}$$

And from the Isothermal process

$$PV = \text{Constant} \implies \frac{P_a V_a}{P_b V_b} = \frac{P_d V_d}{P_c V_c}$$

From the former equation we can rewrite it as

$$\frac{P_a V_a V_a^{\gamma-1}}{P_d V_d V_d^{\gamma-1}} = \frac{P_b V_b V_b^{\gamma-1}}{P_c V_c V_c^{\gamma-1}}$$

which cancels out some terms

$$\frac{V_a^{\gamma-1}}{V_d^{\gamma-1}} = \frac{V_b^{\gamma-1}}{V_c^{\gamma-1}} \implies \frac{V_d}{V_a} = \frac{V_c}{V_b}$$

5.9 Refrigerators

[insert figure of 2 fridges]

From the Thermo Laws

- 1st law:

$$W + q_2 + q_1$$

- 2nd law:

$$\Delta S = \frac{q_1}{T_1} + \frac{-q_2}{T_2} \geq 0$$

or

$$\frac{q_1}{q_2} \geq \frac{T_1}{T_2}$$

We define the “coefficient of performance”

$$CoP \equiv \frac{q_2}{W} = \frac{q_2}{q_1 - q_2} = \frac{1}{\frac{q_1}{q_2} - 1} \leq \frac{1}{\frac{T_1}{T_2} - 1} = \frac{T_2}{T_1 - T_2}$$

Case:

- Room temp: $T_1 = 300K$
- Freezer: $T_2 = 255K$

so the CoP is roughly

$$\frac{255}{45} \approx 5.67$$

since less work is required to move heat rather than converting into heat, the number is greater than one $CoP > 1$.

	$-TS \rightarrow$	
$+pV \downarrow$	E	F
	H	G

Recap: Classical thermodynamics only focuses on macroscopic parameters and their measures.
Key Points:

- $dE = dQ + dW$ from 1st law.
- $dE = TdS - PdV$, and from 2nd law $dS = dQ/T$
- Relationship between (T, S) and (P, V) where F is the Hemholtz, G is Gibbs, and H is enthalpy.

6 Stat Mech Results and Methods

Our Return to th stat mech part... with systems A and heat reservoir A' where

$$A \ll A'$$

What is the prob of finding system A in a ny particular microstate r with energy E_r ?

$$E_r + E' = E^{(0)}, \implies E' = E^{(0)} - E_r$$

And from the DoS the number of states in A' is

$$\Omega'(E^{(0)} - E_r)$$

or the Multiplicity of A' given E_r . The prob P_r has a proportionality

$$P_r \propto C' \Omega'(E^{(0)} - E_r)$$

Since $A \ll A'$ and $E_r \ll E^{(0)}$ we can take the log and Taylor expand

$$\ln \Omega'(E^{(0)} - E_r) = \ln \Omega'(E^{(0)}) - \left. \frac{\partial \ln \Omega'}{\partial E'} \right|_{E^{(0)}} E_r$$

where the derivative is the thermodynamic beta

$$\left. \frac{\partial \ln \Omega'}{\partial E'} \right|_{E^{(0)}} = \frac{1}{kT} = \beta$$

which is independent of E_r . So taking the exponential again...

$$\Omega'(E^{(0)} - E_r) = \Omega'(E^{(0)}) e^{-\beta E_r} = C e^{-\beta E_r}$$

where the $\Omega'(E^{(0)})$ is a constant, i.e.

$$P_r = C e^{-\beta E_r}$$

This must be normalized by

$$\sum P_r = 1$$

or

$$C = \frac{1}{\sum e^{-\beta E_r}}$$

where the “Partition Function” is

$$Z \equiv \sum_r e^{-\beta E_r}$$

coined by Planck (1920) as “Zustandsumme” or “Sum over all states”.

The probability is

$$P_r = \frac{e^{-\beta E_r}}{Z}$$

Where we have a “Boltzmann funtion” $e^{-\beta E_r}$ and P_r is the cannonical distribution.

1.

$$\begin{aligned}
W(n) &= \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \cdots \sum_{m=1}^2 w_i w_j w_k \cdots w_m \\
&= \sum_{i=1}^2 w_i \sum_{j=1}^2 w_j \sum_{k=1}^2 w_k \cdots \sum_{m=1}^2 w_m \\
&= (w_1 + w_2)(w_1 + w_2)(w_1 + w_2) \cdots (w_1 + w_2) \\
&= (w_1 + w_2)^N
\end{aligned}$$

since there are N factors from $i \rightarrow m$. Using binomial theorem:

$$\begin{aligned}
(w_1 + w_2)^N &= \sum_{n=0}^N \binom{N}{n} w_1^n w_2^{N-n} \\
&= \sum_{n=0}^N \frac{N!}{n!(N-n)!} w_1^n w_2^{N-n}
\end{aligned}$$

where the term involving w_1^n is simply

$$W(n) = \frac{N!}{n!(N-n)!} w_1^n w_2^{N-n}$$

2. N_0 molecules in container of volume V_0 . N molecules in subvolume V .

(a) Probability that any given molecule is in the subvolume V :

The probability of *one single* molecule being in the subvolume is $p = \frac{V}{V_0}$, and $q = 1 - p$ for the opposite case. Then for any given molecule, we use the binomial theorem:

$$P(n) = \frac{N_0!}{N!(N_0 - N)!} \left(\frac{V}{V_0}\right)^N \left(1 - \frac{V}{V_0}\right)^{N_0 - N}$$

(b) the mean # of molecules in V for a binomial distribution is simply [From Information Theory, Mackay eq 1.7]

$$\bar{N} = N_0 p = N_0 \frac{V}{V_0}$$

(c) The relative dispersion:

$$\begin{aligned} \frac{\overline{(N - \bar{N})^2}}{\bar{N}^2} &= \frac{\overline{N^2 - 2N\bar{N} + \bar{N}^2}}{\bar{N}^2} \\ &= \frac{\overline{N^2} - 2\bar{N}^2 + \bar{N}^2}{\bar{N}^2} \\ &= \frac{\overline{N^2} - \bar{N}^2}{\bar{N}^2} \end{aligned}$$

where the top term is the variance/dispersion $\overline{N^2} - \bar{N}^2 = N_0 p q$, so

$$\begin{aligned} &= \frac{N_0 p q}{\bar{N}^2} \\ &= \frac{N_0 \frac{V}{V_0} \left(1 - \frac{V}{V_0}\right)}{(N_0 \frac{V}{V_0}) \bar{N}} \\ \text{relative dispersion} &= \frac{1 - \frac{V}{V_0}}{\bar{N}} \end{aligned}$$

(d) When $V \ll V_0$, $\frac{V}{V_0} \approx 0$, so

$$\text{relative dispersion} \approx \frac{1}{\bar{N}} \rightarrow \infty$$

(e) When $V \rightarrow V_0$:

$$\begin{aligned} \overline{(N - \bar{N})^2} &= N_0 p q \approx N_0 \frac{V_0}{V_0} \left(1 - \frac{V_0}{V_0}\right) \\ &= 0 \end{aligned}$$

which agrees with part (c) since

$$\text{relative dispersion} \rightarrow \frac{1 - \frac{V_0}{V_0}}{\bar{N}} = 0$$

3. N antennas with em radiation of wavelength λ and velocity c . Antennas are on the x -axis separated λ apart. Observer on x -axis measures intensity I from one antenna.

(a) Total intensity of all antennas:

All of the antennas are in phase, so the amplitudes add up i.e.

$$E_T = NE$$

and since intensity is proportional to the square of the amplitude $I \propto E^2$, the total intensity is

$$I_T = N^2 I$$

(b) For completely random phases (but same freq), the total amplitude as a vector is

$$E_T = \sum_{i=1}^N \mathbf{E}_i$$

so the mean square amplitude is [from Reif eq (1.9.9)]

$$\begin{aligned} \overline{E_T^2} &= \overline{\sum_{i=1}^N \mathbf{E}_i \cdot \sum_{j=1}^N \mathbf{E}_j} \\ &= \sum_{i=1}^N \overline{E_i^2} + \sum_{i \neq j} \sum \overline{\mathbf{E}_i \cdot \mathbf{E}_j} \\ &= NE^2 \end{aligned}$$

where the (second) cross terms add up to zero since the phases are random— there are just as many positive and negative values. So the mean intensity is

$$\bar{I}_T \propto \overline{E_T^2} = NI$$

4. N particles of spin $1/2$. Magnetic moment μ which points parallel or antiparallel in an applied field H . Energy E in the field is then $E = -(n_1 - n_2)\mu H$ where n_1 is parallel and n_2 is antiparallel.

(a) In the energy range $[E, E + \delta E]$ the total # of states $\Omega(E)$ in the range:

A single particle can have spin $\pm\mu H$, so in the range of δE there are $\delta E/2\mu H$ different states. So the total number of states for a large number N is

$$\Omega(E) = \binom{N}{n_1} \frac{\delta E}{2\mu H} = \frac{N!}{n_1!n_2!} \frac{\delta E}{2\mu H}$$

And using $n_1 + n_2 = N$ or $n_2 = N - n_1$ and $n_1 = N - n_2$ we can get

$$\begin{aligned} E &= -(n_1 - n_2)\mu H \\ \frac{E}{\mu H} &= -(n_1 - (N - n_1)) = -2n_1 + N \\ \Rightarrow n_1 &= \frac{1}{2} \left(N - \frac{E}{\mu H} \right), \quad n_2 = \frac{1}{2} \left(N + \frac{E}{\mu H} \right) \end{aligned}$$

So the total number of states is

$$\Omega(E) = \frac{N!}{\left[\frac{1}{2} \left(N - \frac{E}{\mu H} \right) \right]! \left[\frac{1}{2} \left(N + \frac{E}{\mu H} \right) \right]!} \frac{\delta E}{2\mu H}$$

(b) Using Stirling's approximation ($\ln N! \approx N \ln N - N$):

$$\ln \Omega(E) \approx N \ln N - N - [n_1 \ln n_1 - n_1] - [n_2 \ln n_2 - n_2] + \ln \frac{\delta E}{2\mu H}$$

simplifying some terms:

$$\begin{aligned} -[n_1 \ln n_1 - n_1] - [n_2 \ln n_2 - n_2] &= -n_1 \ln n_1 - n_2 \ln n_1 + n_1 + n_2 \\ \text{where } n_1 + n_2 &= \frac{1}{2} \left(N - \frac{E}{\mu H} \right) + \frac{1}{2} \left(N + \frac{E}{\mu H} \right) = N \end{aligned}$$

so we can cancel out a term:

$$\ln \Omega(E) = N \ln N - n_1 \ln n_1 - n_2 \ln n_2 + \ln \frac{\delta E}{2\mu H}$$

(c) A Gaussian approximation to part (a): From (a)

$$\begin{aligned} \Omega(E) &= \frac{N!}{n_1!n_2!} \frac{\delta E}{2\mu H} \\ &= \frac{N!}{n_1!(N - n_1)!} \frac{\delta E}{2\mu H} \\ &= W(n_1) \frac{\delta E}{2\mu H}, \quad W(n_1) = \frac{N!}{n_1!(N - n_1)!} \end{aligned}$$

Using $n_1 \equiv \bar{n}_1 + \xi$ the Taylor expansion gives [From lecture notes...]

$$\begin{aligned} \ln W(n_1) &\approx \ln W(\bar{n}_1) + \frac{1}{2} B_2 \xi^2 \\ \Rightarrow W(n_1) &= W(\bar{n}_1) e^{-\frac{1}{2} B_2 \xi^2} \end{aligned}$$

where

$$\begin{aligned} B_2 &= \frac{1}{N p q} \quad \text{using } p = \frac{1}{2}, q = \frac{1}{2} \\ B_2 &= \frac{4}{N} \end{aligned}$$

and using $\bar{n}_1 = N/2$

$$\begin{aligned}\xi &= n_1 - \bar{n}_1 = \frac{1}{2} \left(N - \frac{E}{\mu H} \right) - \frac{N}{2} \\ \Rightarrow \xi^2 &= \left(\frac{E}{2\mu H} \right)^2\end{aligned}$$

To find $W(\bar{n}_1)$ we must satisfy the normalization condition: the integral of $W(n_1)$ over all n_1 must equal the total number of possible spins 2^N (like N coin flips) i.e.

$$\begin{aligned}\int_{-\infty}^{\infty} W(n_1) \, dn_1 &= 2^N \\ \int_{-\infty}^{\infty} W(\bar{n}_1) e^{-\frac{1}{2} B_2 \xi^2} \, dn_1 &= 2^N\end{aligned}$$

and since [From Randy Harris Modern Physics Front Page]

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ax^2} &= \sqrt{\frac{\pi}{a}} \\ \Rightarrow W(\bar{n}_1) &= \frac{2^N}{\sqrt{2\pi/B_2}} = \frac{2^N}{\sqrt{\pi N/2}}\end{aligned}$$

So the Gaussian approximation is

$$\begin{aligned}W(n_1) &= \frac{2^N}{\sqrt{\pi N/2}} e^{-\frac{1}{2} B_2 \xi^2} \\ &= \frac{2^N}{\sqrt{\frac{\pi N}{2}}} e^{-\frac{2}{N} \left(\frac{E}{2\mu H} \right)^2}\end{aligned}$$

Finally, we get the total number of states from $\Omega(E) = W(n_1) \frac{\delta E}{2\mu H}$:

$$\boxed{\Omega(E) = \frac{2^N}{\sqrt{\frac{\pi N}{2}}} e^{-\frac{2}{N} \left(\frac{E}{2\mu H} \right)^2} \frac{\delta E}{2\mu H}}$$

5. $A dx + B dy \equiv dF$

- (a) Show that
- $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$
- :

Since dF is an exact differential

$$\frac{\partial F}{\partial x} = A, \quad \frac{\partial F}{\partial y} = B$$

so

$$\frac{\partial A}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y} \right) = \frac{\partial B}{\partial x}$$

- (b) Show that
- $\int dF$
- on any closed path in
- xy
- plane is zero:

For an exact differential

$$\int_a^b dF = F(b) - F(a)$$

so for a closed path $a \rightarrow b$ then back $b \rightarrow a$:

$$\int_a^b dF + \int_b^a dF = F(b) - F(a) + F(a) - F(b) = 0$$

6. From $A \rightarrow B$ the mean pressure is

$$\bar{p} = \alpha V^{-5/3}$$

- (a) Work done when system expanded to final volume, heat added to maintain pressure ($V = 1 \rightarrow 8$), $\bar{p} = 32$. Heat extracted to reduce pressure to $10^6 \text{ dynes cm}^{-2}$:

First finding α at macrostate B :

$$\alpha = \bar{p} V^{5/3} = 1 * 8^{5/3} = 32$$

so the work done is

$$\begin{aligned} W_a &= \int dW = \int_{V_i}^{V_f} \bar{p} dV \\ &= 32V \Big|_1^8 \\ &= 224 \times 10^9 \text{ dynes cm} \end{aligned}$$

From wikipedia, $1 \text{ dynes} = 10^{-5} \text{ N}$, so the units of work is

$$10^6 \text{ dynes cm}^{-2} \times 10^3 \text{ cm}^3 = 10^9 \text{ dynes cm} * \frac{10^{-5} \text{ N}}{1 \text{ dynes}} * \frac{1 \text{ m}}{10^2 \text{ cm}} = 100 \text{ J}$$

so

$$W_a = 22\,400 \text{ J}$$

To find the net heat absorbed we use the first law of thermodynamics:

$$\Delta E = Q - W \implies Q = \Delta E + W$$

where from macro state A to B

$$\begin{aligned} \Delta E &= \int dE = - \int \bar{p} dV \\ &= -32 \int_1^8 V^{-5/3} dV \\ &= 32 \frac{3}{2} V^{-2/3} \Big|_1^8 \\ &= 48(8^{-2/3} - 1) \\ &= -36 \times 10^9 \text{ dynes cm} = -3600 \text{ J} \end{aligned}$$

Finally the net heat absorbed is

$$Q = \Delta E + W = -3600 \text{ J} + 22\,400 \text{ J} = 18\,800 \text{ J}$$

- (b) Volume increase and heat added to cause linear decrease in pressure:

New pressure equation is in the form $p = mV + b$, where the slope $m = \frac{-31}{7}$ and the intercept is at

$$32 = \frac{-31}{7} + b \implies b = \frac{255}{7}$$

thus

$$p = \frac{-31}{7} V + \frac{255}{7}$$

The work done is

$$\begin{aligned}W_b &= \int_1^8 p \, dV = \int_1^8 \left(\frac{-31}{7}V + \frac{255}{7} \right) dV \\&= -\frac{31}{14}V^2 + \frac{255}{7}V \Big|_1^8 \\&= 11\,550 \text{ J}\end{aligned}$$

and using the energy change found in part (a)

$$Q = \Delta E + W = -3600 \text{ J} + 11\,550 \text{ J} = 7950 \text{ J}$$

(c) Part (a) but in reverse:

First the pressure is reduced to $1 \times 10^6 \text{ dynes/cm}^2$, then expanding the volume from $V = 1 \rightarrow 8$ amounts to work

$$\begin{aligned}W_c &= \int_1^8 p \, dV = \int_1^8 1 \, dV \\&= 700 \text{ J}\end{aligned}$$

and the net heat absorbed is

$$Q = \Delta E + W = -3600 \text{ J} + 700 \text{ J} = -2900 \text{ J}$$

7. 3D particle in a box with energy level

$$E = \frac{\hbar^2}{2m} \pi^2 \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right)$$

(a) Force by particle on wall perpendicular to x axis:

$$dW = -dE = -\frac{\partial E}{\partial L_x} dL_x = F_x dL_x$$

where F_x is the generalized force on the wall perpendicular to the x axis. This holds true as long as $dQ = 0$.

(b) Pressure on the wall, and the mean pressure:

The unit area for the pressure on the wall perpendicular to the x axis is $a = L_y L_z$, so the pressure is

$$p_x = \frac{F_x}{a} = -\frac{1}{L_y L_z} \frac{\partial E}{\partial L_x}$$

where

$$\frac{\partial E}{\partial L_x} = \frac{\hbar^2}{2m} \pi^2 n_x^2 \frac{\partial}{\partial L_x} \left(\frac{1}{L_x^2} \right) = -\frac{\hbar^2}{m} \pi^2 \frac{n_x^2}{L_x^3}$$

so

$$p_x = \frac{\hbar^2}{mV} \pi^2 \frac{n_x^2}{L_x^2}$$

where $V = L_x L_y L_z$ is the volume. The mean pressure is then

$$\bar{p} = \frac{\hbar^2}{mV} \pi^2 \frac{\overline{n_x^2}}{L_x^2}$$

Since $\overline{n_x^2} = \overline{n_y^2} = \overline{n_z^2}$ and $L_x = L_y = L_z$ by *symmetry* we can rewrite the mean energy as

$$\begin{aligned} \bar{E} &= \frac{\hbar^2}{2m} \pi^2 \left(\frac{\overline{n_x^2}}{L_x^2} + \frac{\overline{n_y^2}}{L_y^2} + \frac{\overline{n_z^2}}{L_z^2} \right) = \frac{\hbar^2}{2m} \pi^2 \left(\frac{3\overline{n_x^2}}{L_x^2} \right) \\ \implies \frac{2}{3} \bar{E} &= \frac{\hbar^2}{m} \pi^2 \frac{\overline{n_x^2}}{L_x^2} \end{aligned}$$

Thus we can substitute \bar{E} into the mean pressure equation:

$$\bar{p} = \frac{2}{3} \frac{\bar{E}}{V}$$

1.

(a) Given

$$\Omega(E, E + \delta E) = \frac{N!}{(N/2 - E/(2\mu H))!(N/2 + E/(2\mu H))!} \frac{\delta E}{2\mu H}$$

and using $\beta = \frac{\partial \ln \Omega}{\partial E}$ & stirling approx $\ln N! = N \ln N - N$, so taking the log of the DoS:

$$\begin{aligned} \ln \Omega(E) &= \ln N! - \ln(N/2 - E/(2\mu H))! - \ln(N/2 + E/(2\mu H))! + \ln \frac{\delta E}{2\mu H} \\ &= \ln N! - (N/2 - E/(2\mu H)) \ln(N/2 - E/(2\mu H)) + (N/2 - \cancel{E/(2\mu H)}) \\ &\quad - (N/2 + E/(2\mu H)) \ln(N/2 + E/(2\mu H)) + (N/2 + \cancel{E/(2\mu H)}) + \ln \frac{\delta E}{2\mu H} \\ &= \ln N! + N - (N/2 - E/(2\mu H)) \ln(N/2 - E/(2\mu H)) \\ &\quad - (N/2 + E/(2\mu H)) \ln(N/2 + E/(2\mu H)) + \ln \frac{\delta E}{2\mu H} \end{aligned}$$

Finally taking the partial derivative

$$\begin{aligned} \frac{\partial \ln \Omega}{\partial E} &= \frac{1}{2\mu H} \ln(N/2 - E/(2\mu H)) - \frac{\cancel{N/2 - E/(2\mu H)}}{\cancel{N/2 - E/(2\mu H)}} \left(-\frac{1}{2\mu H} \right) \\ &\quad - \frac{1}{2\mu H} \ln(N/2 + E/(2\mu H)) - \frac{1}{2\mu H} \\ \beta &= \frac{1}{2\mu H} [\ln(N/2 - E/(2\mu H)) - \ln(N/2 + E/(2\mu H))] \\ &= \frac{1}{2\mu H} \ln \left(\frac{N/2 - E/(2\mu H)}{N/2 + E/(2\mu H)} \right) \\ &= \frac{1}{2\mu H} \ln \left(\frac{N/2 \frac{1 - E/(N\mu H)}{1 + E/(N\mu H)}}{N/2 \frac{1 + E/(N\mu H)}{1 - E/(N\mu H)}} \right) \\ &= \frac{1}{2\mu H} \ln \left(\frac{1 - E/(N\mu H)}{1 + E/(N\mu H)} \right) \end{aligned}$$

or using the inverse hyperbolic tangent

$$\operatorname{arctanh}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

So we set $x = -E/(N\mu H)$

$$\beta = \frac{1}{\mu H} \operatorname{arctanh}(-E/(N\mu H))$$

Thus using $\beta = 1/kT$ and the odd function $\tanh(-x) = -\tanh(x)$

$$\begin{aligned} \frac{\mu H}{kT} &= \operatorname{arctanh}\left(-\frac{E}{N\mu H}\right) \\ \implies \tanh\left(\frac{\mu H}{kT}\right) &= -\frac{E}{N\mu H} \end{aligned}$$

and finally

$$\boxed{E = -N\mu H \tanh\left(\frac{\mu H}{kT}\right)}$$

- (b) For $T = -T_0 < 0$ the sign of the hyperbolic tangent is negative

$$E = -N\mu H \tanh\left(-\frac{\mu H}{kT_0}\right) = N\mu H \tanh\left(\frac{\mu H}{kT_0}\right)$$

$$\Rightarrow \boxed{E > 0}$$

- (c) The total magnetic moment is proportional to the difference of parallel and antiparallel spins

$$M = \mu(n_1 - n_2) = \mu(n_1 - (N - n_1)) = \mu(2n_1 - N)$$

and from the previous HW 1 we know that

$$n_1 = \frac{1}{2}\left(N - \frac{E}{\mu H}\right)$$

thus

$$M = \mu\left[\left(N - \frac{E}{\mu H}\right) - N\right] = -\frac{E}{H}$$

and therefore

$$\boxed{M(H, T) = N\mu \tanh\left(\frac{\mu H}{kT}\right)}$$

2.

(a) Once again the numnber of ways is

$$\Omega(N, n_+) = \frac{N!}{n_+!(N - n_+)!}$$

And the total length is

$$\ell = (n_+ - n_-)d = (2n_+ - N)d \implies n_+ = \frac{1}{2}(N + \ell/d)$$

so

$$\Omega(\ell) = \frac{N!}{\left[\frac{1}{2}(N + \ell/d)\right]! \left[\frac{1}{2}(N - \ell/d)\right]!}$$

(b) And using

$$x = \frac{\ell}{Nd} \implies \ell = xNd$$

we can equate the two expressions above:

$$xNd = (2n_+ - N)d \implies xN = 2n_+ - N \implies n_+ = \frac{N}{2}(1 + x)$$

So

$$\Omega = \frac{N!}{\left[\frac{N}{2}(1 + x)\right]! \left[N - \frac{N}{2}(1 + x)\right]!} = \frac{N!}{\left[\frac{N}{2}(1 + x)\right]! \left[\frac{N}{2}(1 - x)\right]!}$$

and using stirling's approximation

$$\begin{aligned} \ln \Omega &= N \ln N - \cancel{N} - \frac{N}{2}(1 + x) \ln \left(\frac{N}{2}(1 + x) \right) + \cancel{\frac{N}{2}(1 + x)} \\ &\quad - \frac{N}{2}(1 - x) \ln \left(\frac{N}{2}(1 - x) \right) + \cancel{\frac{N}{2}(1 - x)} \\ &= N \ln N - \frac{N}{2}(1 + x) \ln \left(\frac{N}{2}(1 + x) \right) - \frac{N}{2}(1 - x) \ln \left(\frac{N}{2}(1 - x) \right) \end{aligned}$$

and from

$$\begin{aligned} -\left(\frac{\partial S}{\partial \ell}\right)_E &= -k \left(\frac{\partial \ln \Omega}{\partial \ell}\right)_E \\ \implies S &= k \ln \Omega \end{aligned}$$

or

$$S = k \left[N \ln N - \frac{N}{2}(1 + x) \ln \left(\frac{N}{2}(1 + x) \right) - \frac{N}{2}(1 - x) \ln \left(\frac{N}{2}(1 - x) \right) \right]$$

4. A glass bulb with air at room temp and 1 atm is placed in to a chamber with helium at 1 atm. Since the glass bulb is only permeable to helium, so over time, the pressure of helium outside the bulb will have to equal the partial pressure of helium inside the bulb. Thus the final pressure inside the bulb after equilibrium is

$$P_0 + P_{\text{Helium out}} = 2 \text{ atm}$$

5. $m_c = 750$ g copper calorimeter can contains $m_w = 200$ g of water in equilibrium at $T_i = 293$ K. $m_{ice} = 30$ g of ice at $T_{ice} = 273$ K is placed in the calorimeter and enclosed in a heat-insulating shield.

(a) Given

$$c_w = 4.18 \text{ J/gK} \quad c_c = 0.418 \text{ J/gK}$$

After the ice melts and reaches equilibrium the final temperature of the water and calorimeter must be equal

$$\begin{aligned} \Delta Q &= Q_{ice} + Q_{\text{melted ice}} + Q_{\text{water}} + Q_{\text{copper}} = 0 \\ &= m_{ice}L_f + m_{ice}c_w(T_f - T_{ice}) + m_w c_w(T_f - T_i) + m_c c_c(T_f - T_i) \\ &= 30 \text{ g} \times 333 \text{ J/g} + 30 \text{ g} \times 4.18 \text{ J/gK}(T_f - 273 \text{ K}) \\ &\quad + 200 \text{ g} \times 4.18 \text{ J/gK}(T_f - 293 \text{ K}) + 750 \text{ g} \times 0.418 \text{ J/gK}(T_f - 293 \text{ K}) \\ &= 9990 + 125.4(T_f - 273) + 836(T_f - 293) + 313.5(T_f - 293) \\ \Rightarrow T_f &= 283 \text{ K} \end{aligned}$$

(b) The total entropy for ice melting at $T = 273$ K is

$$\Delta S_{ice} = \frac{Q_{ice}}{T_{ice}} = \frac{m_{ice}L_f}{T_{ice}}$$

and for three other processes

$$\Delta S_a = \int_{T_0}^{T_f} \frac{m_a c_a}{T} dT = m c \ln \frac{T_f}{T_i}$$

so

$$\begin{aligned} \Delta S &= \frac{m_{ice}L_f}{273} + m_{ice}c_w \ln \frac{283}{273} + m_w c_w \ln \frac{283}{293} + m_c c_c \ln \frac{283}{293} \\ \Delta S &= 1.19 \text{ J/K} \end{aligned}$$

(c) The work required to bring the water back to $T_i = 293$ K i.e. $\Delta T = T_f - T_i = 293 - 283 = 10$ K

$$\begin{aligned} W &= \Delta Q \\ &= (m_w + m_i)c_w(T_f - T_i) + m_c c_c(T_f - T_i) \\ &= 230(4.18)10 + 750(0.418)10 \\ W &= 12\,749 \text{ J} \end{aligned}$$