Physics 421: Intro to Electrodynamics

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Contents

1 Vector Analysis

1.1 What is a Vector?

In type we use boldface $\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}$, where we we can do some simple operations as such:

- Adding and Subtraction: $\mathbf{A} \pm \mathbf{B} = \mathbf{C}$ or aligning the head to the tail
- Multiplication:
  - Scalar: $\mathbf{A} \to 2\mathbf{A}$
  - Dot Product: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = AB\cos\theta$
  - Cross Product: $\mathbf{A} \times \mathbf{B} = AB\sin\theta$, and $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$

Components of a Vector In 3D space, we often use the familiar Cartesian coordinates, e.g.

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

and we can add components by adding the components:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

and likewise for subtraction. For the dot product:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

or more shortly

$$= \sum_{i,j} A_iB_j\delta_{ij}$$

where δ_{ij} is the Kronecker delta. The cross product is a bit trickier...

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{\mathbf{x}} - (A_xB_z - A_zB_x)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}$$

This can also be written in short form using the Levi-Civita symbol (look it up)

Scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

This is the also the volume of the parallelepiped formed by the three vectors. NOTE that

$$\cancel{(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}}$$

since you can't cross a scalar with a vector.

Vector triple product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Using the BAC-CAB rule.

Some important vectors We define a position vector

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = r\hat{\mathbf{r}}$$

where the unit vector is

$$\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}$$

and an infinitesimal displacement vector

$$\mathrm{d}l = \mathrm{d}x\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}} + \mathrm{d}z\hat{\mathbf{z}}$$

In EM we define a source point $\mathbf{r}'$ (e.g. a charge) and a field point $\mathbf{r}$ that give us the seperation vector

$$\boldsymbol{\mathscr{r}} = \mathbf{r} = \mathbf{r}'$$

with magnitude

$$|\boldsymbol{\mathscr{r}}| = |\mathbf{r} - \mathbf{r}'|$$

and unit vector

$$\hat{\mathbf{r}} = \frac{\mathbf{r} - \mathbf{r}'}{\mathbf{r} - \mathbf{r}'}$$

1.2 Differential Calculus

And ordinary derivative $\frac{\mathrm{d}F}{\mathrm{d}x}$ is a change in $F(x)$ in $\mathrm{d}x$

$$\mathrm{d}F = \left(\frac{\partial F}{\partial x}\right)\mathrm{d}x$$

. . . geometrically, it's the slope

Gradient for functions of 2 or more variables, generalize for $h(x, y)$

$$\mathrm{d}h = \left(\frac{\partial h}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial h}{\partial y}\right)\mathrm{d}y$$

it's a scalar so $\mathrm{d}h = (\boldsymbol{\nabla} h) \cdot (\mathrm{d}l)$ where

$$\boldsymbol{\nabla} h = \frac{\partial h}{\partial x}\hat{\mathbf{x}} + \frac{\partial h}{\partial y}\hat{\mathbf{y}}$$

In 3D

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

If $\boldsymbol{\nabla} u = 0$, we are at an extremum (max, min, or shoulder/saddle point) Rewriting:

$$\boldsymbol{\nabla} T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)T(x, y, z)$$

wehere we can assume the ∇ as an "operator" acting on T:

1. Scalars like T: $\boldsymbol{\nabla} T$, "grad T", generalized slope
2. Dot product on $\mathbf{V}$: $\boldsymbol{\nabla} \cdot \mathbf{V}$, "divergence" or "div"
3. Cross product : $\boldsymbol{\nabla} \times \mathbf{V}$, "curl" or "rotatation"

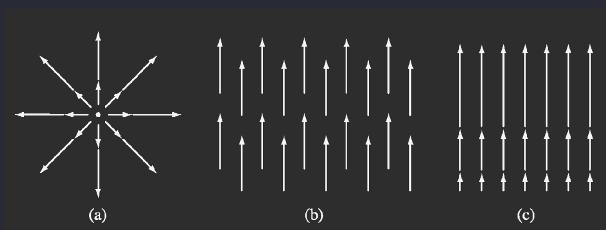

Figure 1.1: Divergence of field lines

Divergence From the Figure, we can see that (a) & (c) diverges, and (b) does not.

Geometrical Interpretation: Sources of positive divergence is a source or "faucet", and negative divergence is a sink or "drain".

Curl

$$\boldsymbol{\nabla} \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

E.g. for $\mathbf{V} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, $\boldsymbol{\nabla} \times \mathbf{V} = 2\hat{\mathbf{z}}$.

Combining Multiple Operations Two ways to get scalar from two functions:

$$fg \quad \text{or} \quad \mathbf{A} \cdot \mathbf{B}$$

Two ways to get vector from two functions:

$$f\mathbf{A} \quad \text{or} \quad \mathbf{A} \times \mathbf{B}$$

And we have 3 'derivatives': div, grad, and curl.

Product rule:

$$\partial_x(fg) = f\partial_x g + g\partial_x f$$

i $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f$

ii $\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + B \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}$

. . . (see Griffiths for more)

Second Derivatives Combining $\boldsymbol{\nabla}, \boldsymbol{\nabla}\cdot, \boldsymbol{\nabla}\times$

$\boldsymbol{\nabla} T$ is a vector

i

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} T) &= (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}} + \partial_y + \hat{\mathbf{z}}\partial_z)\cdot(\hat{\mathbf{x}}\partial_x T + \hat{\mathbf{y}}\partial_y T + \hat{\mathbf{z}}\partial_z T)\\ &= \partial_x^2 T + \partial_y^2 T + \partial_z^2 T\\ &= \nabla^2 T\end{aligned}$$

ii $\boldsymbol{\nabla}\times(\boldsymbol{\nabla} T) = 0$

iii $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \ldots$ ignored

iv $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = 0$

v $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

1.3 Integral Calculus:

line, surface and volume integrals

"Fundamental theorem for gradients" Start with a scalar $T(x,y,z)$: from $a\to b$, in small steps $\mathrm{d}T = \boldsymbol{\nabla}\cdot T\,\mathrm{d}\boldsymbol{\ell}$

Total change in T:

$$\boxed{\int_a^b \mathrm{d}T = \int_a^b \boldsymbol{\nabla} T\cdot\mathrm{d}\boldsymbol{\ell} = T(b) - T(a)}$$

This line integral is path independent but $\int_a^b \mathbf{F}\cdot\mathrm{d}\ell$ is *not*!

Divergence Theorem, "Gauss' Theorem", or "Green's Theorem"

$$\boxed{\int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\mathrm{d}\tau = \oint_S v\cdot\mathrm{d}\mathbf{a}}$$

where V is the volume enclosed by the surface S. The divergence of a field in a volume is equivalent to the flux of the field at the boundary or surface.

Geometrical Interpretation: The "source"(or faucet) should present a flux (or flow) out through the surface.

Fundamental Theorem of Curls: Stokes' Theorem

$$\boxed{\oint_S (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{a} = \oint_P \mathbf{v}\cdot\mathrm{d}\boldsymbol{\ell}}$$

We have a 2D surfaces S bounded by a closed 1D perimeter P.

In 3D, imagine a soap bubble in a wire frame. We can change the surface of the bubble by moving the wire frame and making almost making a bubble, but the perimeter that bounds the bubble is still the wireframe.

Example:

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + 4yz^2\hat{\mathbf{z}}$$

On a surface S square on the $y - z$ plane:

$$\oint \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} =$$

RHS: We can split this into 4 integrals going counterclockwise around the square: First $x = 0, z = 0, y : 0 \to 1$: $dx = dz = 0$ $\int_0^1 3y^2 \mathrm{d}y = 1$

Second $\int_0^1 4z^2 \mathrm{d}z = 4/3$

Third: -1

Fourth: 0

Summing them all gives: $\oint \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} = 4/3$

LHS: The curl gives: $4z^2 - 2x, -(0-0), 2z$ so

$$\oint (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint$$

...

1.4 Dirac Delta Function

Considering a vector function

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}$$

[insert picture of field lines]

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = 0$$

The div theorem doesn't obviously tell what volume and surface to choose, but we can choose a sphere of radius R and its corresponding surface:

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \frac{1}{R^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = 4\pi$$

where the polar angle is $\theta : 0 \to \pi$ and the azimuthal angle is $\phi : 0 \to 2\pi$.

We think that the divergence is zero (thus the theorem does not hold), but we make a tiny mistake:

$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ everywhere except at the origin $r \to 0$

and we stumbled on the Dirac delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

where

$$\int f(x)\delta(x)\mathrm{d}x = f(0)$$

Shifting the delta function:

$$\delta(x-a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

so

$$\int f(x)\delta(x-a)\mathrm{d}x = f(a)$$

Multiplying by a constant:

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Generalizing to 3D:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

Examples:

$$\int_V (\boldsymbol{\nabla} \cdot (\mathbf{v}))\mathrm{d}\tau = \int 4\pi\delta^3(\mathbf{r}) = 4\pi$$

and

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\boldsymbol{\mathscr{r}}^2} \right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

2 Electrostatics

2.1 The Electric Field

given charge q: find force on Q , where $\mathbf{F}$ depends on $\boldsymbol{\mathscr{r}}, \mathbf{v}_i, \mathbf{a}_i$

2.1.1 Coulombs law

Coulomb's Law empirically,

$$\mathbf{F}_Q = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

where $k = \frac{1}{4\pi\epsilon_0}$ and the permittivity of free space is $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{Nm}^2$

The force is attractive if $sgn(qQ) = -1$ and repulsive if $= +1$.

Principal of superposition:

$$\begin{aligned}\mathbf{F}_T &= \mathbf{F}_{Q1} + \mathbf{F}_{Q2} + \dots \\ &= \frac{1}{4\pi\epsilon_0}Q\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \dots\right) \\ &= Q\mathbf{E}_T\end{aligned}$$

where $\mathbf{E}_T$ is the total electric field due to all of the source (point) charges.

$\mathbf{E}$ doesn't depend on Q

- $\mathbf{E} \sim F/Q$

Example: $\mathbf{E}$ field midway above two charges q: The electric fields are zero in the x and y direction:

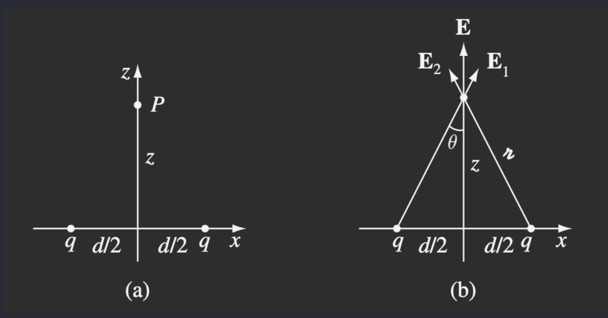

Figure 2.1: Griffiths Example 2.1

$$E_x = E_y = 0$$

But we can sum the fields in the z direction:

$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta$$

where

$$\mathscr{r} = \left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2} \qquad \cos\theta = \frac{z}{\mathscr{r}}$$

so

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}$$

Far away: $z \gg d$, so $d \to 0$ thus

$$E_z \approx \frac{1}{4\pi\epsilon_0}\frac{2qz}{z^3} = \frac{1}{4\pi\epsilon_0}\frac{2}{z^2}$$

Continuous Charge Distributions

- line: charge per unit length λ; $\quad \mathrm{d}q = \lambda \mathrm{d}\ell$
- surface: charge per unit area σ; $\quad \mathrm{d}q = \sigma \mathrm{d}a$
- volume: charge per unit volume ρ; $\quad \mathrm{d}q = \rho \mathrm{d}\tau$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\mathrm{d}q$$

e.g. for a volume charge:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\mathrm{d}\tau'$$

where $'$ denotes the source charge in (no $'$ is a field point)

Example: Find $\mathbf{E}$ at z above a straight line segment of length $2L$ with uniform line charge λ. If we

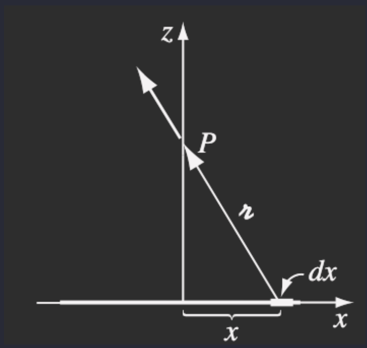

Figure 2.2: Griffiths Example 2.2

treat $\mathrm{d}q$ as a point particle, then we can use Ex 2.1 likewise but integrate over the line segment.

First we catalog what we know:

- Field point P is at $\mathbf{r} = z\hat{\mathbf{z}}$
- Sources at $\mathbf{r}' = x\hat{\mathbf{x}}$; $\quad \mathrm{d}\ell' = \mathrm{d}x$
- $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = z\hat{\mathbf{z}} - x\hat{\mathbf{x}}$
- $\mathscr{r} = \sqrt{x^2 + z^2}$
- $\hat{\mathscr{r}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2+z^2}}$

The electric field is then (line charge)

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{+L}\frac{\lambda}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\mathrm{d}x = \frac{1}{4\pi\epsilon_0}\lambda\int_{-L}^{+L}\frac{z\hat{\mathbf{z}}-x\hat{\mathbf{x}}}{(z^2+x^2)^{3/2}}\mathrm{d}x \\
&= \frac{1}{4\pi\epsilon_0}\lambda\left[z\hat{\mathbf{z}}\int_{-L}^{L}\frac{\mathrm{d}x}{(z^2+x^2)^{3/2}} - \hat{\mathbf{x}}\int_{-L}^{L}\frac{x\mathrm{d}x}{(z^2+x^2)^{3/2}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\lambda\left[z\hat{\mathbf{z}}\frac{x}{z^2\sqrt{z^2+x^2}}\bigg|_{-L}^{L} - \cancel{\hat{\mathbf{x}}\frac{1}{\sqrt{z^2+x^2}}\bigg|_{-L}^{L}}\right]
\end{aligned}
$$

we can easily see that the x component is zero through the geometrical symmetry of the line centered at the origin (like Ex 2.1). Simplifying gives us

$$
\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{\mathbf{z}}
$$

Checks and balances:

- $\hat{\mathbf{z}}$ is expected!
- $$z \gg L \quad \sqrt{z^2+L^2}\approx z \quad E(P, z\gg L) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2}$$
  where we can treat this as a point charge $q = 2\lambda L$ when we are far away.

2.2 Divergence and curl of E: Gauss' Law

'flux' of field lines

$$
\Phi = \oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{a}
$$

What is Φ for point charge at origin surrounded by a spherical surface?

$$
\begin{aligned}
\Phi &= \int \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \int\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{q_{enc}}{\epsilon_0}
\end{aligned}
$$

A bunch of charges surrounded by a surface: $\mathbf{E}_T = \sum\mathbf{E}_i$

$$
\Phi = \oint\mathbf{E}_T\cdot\mathrm{d}\mathbf{a} = \sum_i\oint\mathbf{E}_i\cdot\mathrm{d}\mathbf{a} = \sum_i\frac{q_i}{\epsilon_0}
$$

Thus we have an integral form of Gauss's law:

$$
\boxed{\oint\mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}
$$

where $Q = \sum q_i$.

From the theorem of divergence:

$$
\oint_S\mathbf{v}\cdot\mathrm{d}\mathbf{a} = \int_V(\boldsymbol{\nabla}\cdot\mathbf{v})\mathrm{d}\tau \quad \text{and} \quad Q = \int_V\rho\mathrm{d}\tau
$$

so

$$
\int_V(\boldsymbol{\nabla}\cdot\mathbf{E})\mathrm{d}\tau = \int_V\rho\mathrm{d}\tau \to \text{good for all volume}
$$

therefore we have the differential form of Gauss' Law:

$$
\boxed{\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}}
$$

Three ways Gauss's law makes life nice: Gaussian surfaces

- spherical: gaussian sphere
- cylindrical: gaussian cylinder
- planar: gaussian pillbox

2.2.1 Applications of Gauss's Law

$$\int \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{q_{enc}}{\epsilon_0} \to \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

1. (Simple spherical) What is $\mathbf{E}$ outside a uniformly charged solid sphere of radius R and total charge Q? The spherical Gaussian surface implies a symmetry where we should *only have a radial component* $\mathbf{E} = E_r$.

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q}{\epsilon_0}$$
$$E \oint \mathrm{d}\mathbf{a} = E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$
$$\implies \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

where the integral is equivalent to the surface area of the sphere. This is also $\implies$ a field of a *point*.

2. (Simple cylindrical) A long cylinder (radius a) of charge density $\rho = ks$ ($\propto$ distance from axis) where k is a constant and s is the radial distance from the axis. What is $\mathbf{E}$ inside the cylinder? The cylindrical Gaussian surface has radius s and length ℓ:

$$\oint \mathbf{E}.\mathrm{d}\mathbf{a} = \frac{Q_enc}{\epsilon_0}; \quad Q_{enc} = \int \rho \mathrm{d}\tau = \int (ks')\mathrm{d}s'\mathrm{d}\phi\mathrm{d}z = \frac{2}{3}\pi k\ell s^3$$

When using the divergence theorem, note that only the curved part of the cylinder contributes to the flux. Thus,

$$\int \mathbf{E}\mathrm{d}\mathbf{a} \to E\int \mathrm{d}a = E(2\pi s\ell)$$
$$\implies \mathbf{E} = \frac{1}{3\epsilon_0}ks^2\hat{\mathbf{s}}$$

If we were to find the field outside the cylinder we would find that the enclosed charge is constant Q_{enc} thus the field is proportional to $1/s$.

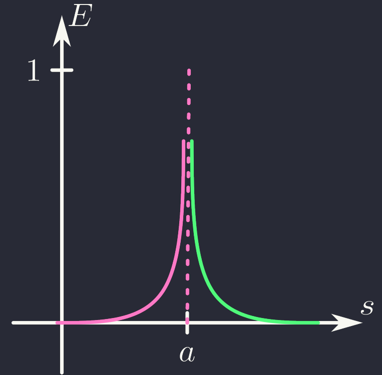

Figure 2.3: Electric field as a function of s

3. (Simple infinite plane) with uniform surface charge σ. Symmetry implies that $\mathbf{E}$ is perpendicular to the plane. The Gaussian pillbox (either box or cylinder) will have a field of

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

2.2.2 The curl of E

$$\nabla \times \mathbf{E} = 0, \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

calculating

$$\int_a^b \mathbf{E}\cdot \mathrm{d}\ell, \quad \mathrm{d}\ell = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\theta\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\phi\hat{\boldsymbol{\phi}}$$

So the integral is

$$\frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a} - \frac{q}{b}\right)$$

This means:

- path independent!
- if $a = b$ then $\oint \mathbf{E}\cdot \mathrm{d}\ell = 0$ (ℓ is a vector but I don't know how to bold it)

We can now use Stokes' theorem: $\oint \mathbf{v}\cdot\mathrm{d}\ell = \int_S (\nabla\times\mathbf{v})\cdot\mathrm{d}\mathbf{a}$ or

$$\oint \mathbf{E}\cdot\mathrm{d}\ell = \int_S (\nabla\times\mathbf{E})\cdot\mathrm{d}\mathbf{a} = 0 \implies \nabla\times\mathbf{E} = 0$$

2.3 Electric potential

Any function f with zero curl is the gradient of a scalar function: $\nabla\times(\nabla f) = 0$ (curl of gradient is always 0!)

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot\mathrm{d}\ell$$

implies all paths give some value.

$V \sim$ "electric potential"

$$\begin{aligned} V(\mathbf{b}) - V(\mathbf{a}) &= -\left(\int_{\mathcal{O}}^{b}\mathbf{E}\cdot\mathrm{d}\ell\right) - \left(-\int_{\mathcal{O}}^{a}\mathbf{E}\cdot\mathrm{d}\ell\right) \\ &= -\int_{\mathcal{O}}^{b} - \int_{\dashv} O\mathbf{E}\cdot\mathrm{d}\ell \\ &= -\int_a^b \mathbf{E}\cdot\mathrm{d}\ell \end{aligned}$$

And from the fundamental theorem for gradients: $T(\mathbf{b}) - T(\mathbf{a}) = \int_a^b (\nabla T)\cdot\mathrm{d}\ell$

$$\implies \mathbf{E} = -\nabla V$$

i "potential" is a terrible name

ii $\mathbf{E} = (E_x, E_y, E_z)$ vs V with only *one* value at every point in space! Otherwise we would have to deal with

$$(\nabla\times\mathbf{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

iii

$$V'(\mathbf{r}) = -\int_{O'}^{\mathbf{r}}\mathbf{E}\cdot\mathrm{d}\ell = -\int_{O'}^{O}\mathbf{E}\cdot\mathrm{d}\ell - \int_{O}^{\mathbf{r}}\mathbf{E}\cdot\mathrm{d}\ell = C + V(\mathbf{r})$$

$\implies \mathbf{E} = -\nabla V$

Electric Potential cont.

2.3.1 Poisson's and Laplace's equations

The divergence and curl of $\mathbf{E}$ in terms of the potential V:

- Divergence: $\boldsymbol{\nabla} \cdot \mathbf{E} = \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = -\nabla^2 V$, or **Poisson's equation**:

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

  and in regions of no charge $\rho = 0$ we have **Laplace's equation**:

$$\boxed{\nabla^2 V = 0}$$

- Curl: It's always zero, which doesn't give us any info about V...

2.3.2 Potential of localized charge distributions

For a point charge q we can easily find the potential using the electric field $\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$:

$$\begin{aligned} V(r) &= -\int_{O}^{r} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{q}{r'^2}\mathrm{d}r' \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \end{aligned}$$

where in general, the potential of a point charge is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

And using the principle of superposition, the potential of a collection of charges can be a sum

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\mathscr{r}_i}$$

or for a continuous charge distribution, and integral:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\mathrm{d}q$$

e.g. for a volume charge distribution:

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\mathrm{d}\tau'}$$

2.3.3 Boundary conditions

From the Gaussian pillbox the E-field is given by Gauss's law:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

and the only components that contribute to the field are the top and bottom:

$$\mathbf{E}^{\perp}_{\text{above}} - \mathbf{E}^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

But we can also use the path integral of the E-field (which should be zero):

$$\oint \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = 0$$

so the parallel components should be equal:

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0$$
$$\implies \boxed{E^{\parallel}_{\text{above}} = E^{\parallel}_{\text{below}}}$$

This also implies that the potential "V is continous across boundaries":

$$V_{\text{above}} = V_{\text{below}}$$

which also implies the gradient of the potential is

$$\boldsymbol{\nabla} V_{\text{above}} - \boldsymbol{\nabla} V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

2.4 Work and energy in electrostatics

$[V] = \text{J/C} = \text{energy/charge}$, so the work is equivalent to

$$W = QV(\mathbf{r})$$

2.4.1 Energy of point charge distribution

Collection of 3 charges First we assume we start with $W_1 = 0$, or we are bringing the two other charges towards q_1; Work done to bring in $q_2 = V_1 q_2 = W_2$;

$$\begin{aligned} W_3 &= V_1 q_3 + V_2 q_3 \\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}} \\ &= \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) \end{aligned}$$

where we can use the superposition principle to find the total work:

$$W = W_1 + W_2 + W_3$$

In general

$$\begin{aligned} W &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{r_{ij}} \\ &= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}\sum_{j\neq i}\frac{q_i q_j}{r_{ij}} \\ &= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1} q_i \sum_{j\neq i}\frac{q_j}{r_{ij}} \\ &= \frac{1}{2}\sum_{i=1} q_i \left[\sum_{j\neq i}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right] \end{aligned}$$

where the term in the brackets is the potential energy due to all other charges, $[\,] = V(\mathbf{r}_i)$, thus

$$W = \frac{1}{2}\sum_{i=1} q_i V(\mathbf{r}_i)$$

2.4.2 Energy of continuous charge distribution

$$W = \frac{1}{2}\int \rho V \mathrm{d}\tau$$

but what if we don't know the potential? First using Gauss's law:

$$\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E}$$

so the work done is (using integration by parts):

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int (\boldsymbol{\nabla} \cdot \mathbf{E}) V \mathrm{d}\tau \\ &= \frac{\epsilon_0}{2}[-\int \mathbf{E}(\boldsymbol{\nabla} V)\mathrm{d}\tau + \oint V\mathbf{E} \cdot \mathrm{d}\mathbf{A}] \\ &= \frac{\epsilon_0}{2}[\int E^2 \mathrm{d}\tau + \oint V\mathbf{E} \cdot \mathrm{d}\mathbf{A}] \end{aligned}$$

but we know that

- $E \propto \frac{1}{r^2} \quad \mathrm{d}\tau \propto r^3$
- $V \propto \frac{1}{r} \quad \mathrm{d}A \propto r^2$

So the second surface integral term goes to 0 much faster, thus

$$W = \frac{\epsilon_0}{2}\int E^2 \mathrm{d}\tau$$

Energy Review:

$$W = \frac{1}{2}\sum q_i V(\mathbf{r}_i)$$
$$W = \frac{\epsilon_0}{2}\int E^2 \mathrm{d}\tau$$

Where did we go wrong in going from the point charge distribution to the continuous distribution?

2.4.3 Comments on Electrostatic energy

Energy of a point charge?:

$$W = \frac{\epsilon_0}{2}\int \frac{q^2}{(r^2)^2} r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \infty$$

Uh oh, as $r \to 0$ the energy goes to infinity! And this problem persists everywhere. (Solution is renormalization. . .)

Where is the energy stored?

- In the first case: we have a charge in a potential (from other charges); somehow the relative position of the charge implies its "electric potential"

We don't really know where it is, but we can book keep this energy i.e. ρ, V, E

Superposition (again) If we try to superimpose two electric fields,

$$W = \frac{\epsilon_0}{2}\int |\mathbf{E}_1 + \mathbf{E}_2|^2 \mathrm{d}\tau$$

the terms in the integral are

$$E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2$$

so

$$W \neq W_1 + W_2$$

2.5 Conductors in electrostatics

2.5.1 Basic Properties

What is a conductor? Material where charge can move; in the perfect case, the charges move freely and the resistance goes to zero $R \to 0$. Thus no *resistance* to the motion of charge.

i $\mathbf{E} = 0$ inside a conductor

if $E + q \to \mathbf{F} = q\mathbf{E}$: a charge in a field $\to$ charges move to canncel $\mathbf{E}_{\text{in}}$

ii $\rho = 0$ inside a conductor; so from divergence

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon}, \quad \text{if} \quad \mathbf{E} = 0,\ \rho = 0$$

iii any non-zero ρ resides on surface (only in 3D); in 2D & 1D, its not exactly at the boundary, but quickly peaks and decays before the boundary.

iv conductors are equipotentials

v $\mathbf{E}$ is perpendicular to a surface outside the conductor

The silly experiment: An aluminum beverage can (of ingenious design) is a conductor, i.e., it has free electrons. The plastic rod is given negative charge (via cloth/wool) and when it is brought near the can, the can moves towards the rod.

How does the can move, even though it has no charge? The rod induces a charge by pushing the negative charges away from the can. Then the positive charges close to the rod are attracted hence the attractive movement (force).

2.5.2 Induced charge distributions

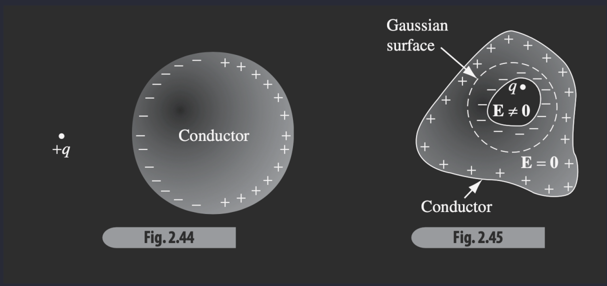

Figure 2.4: From Griffiths

Case 1: A point charge in a cavity surrounded by a conductor

- Flux $\Phi = \int \mathbf{E} \mathrm{d}\mathbf{A} \neq 0$ for a Gaussian surface in the cavity
- $\Phi = \int \mathbf{E} \mathrm{d}\mathbf{A} = 0$ for a surface where that encloses no charge (positive point charge cancels negative charges surrounding it in the conductor)
- $\int \mathbf{E} \mathrm{d}\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{q}{\epsilon_0}$ for a Gaussian surface outside the conductor

Case 2: Spherical conductor, with a weird cavity containing a point charge $+q$

What is $\mathbf{E}(\mathbf{P})$ at $\mathbf{P}$ for $P > r$ (outside the conductor)? The electric field will uniformly distribute itself in the conductor, so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

What about a point charge $+q$ outside the conductor, what is the E-field in the cavity? It may put minus charges close, and plus charges away from further away on the surface of the conductor...but in the cavity, you can't transmit information in the cavity since the E-field is zero inside a conductor. Screening out an electric field AKA the Faraday cage.

2.5.3 Capacitors

Imagine 2 conductors (of weird shapes) with charges $+Q$ and $-Q$ has a potential of $+V$ and $-V$ respectively. The potential difference (Theorem of gradients) is

$$\Delta V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell}$$

where we know that there is an electric field $\mathbf{E} \propto Q$ or more precisely from Coloumb's law:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\mathscr{r}^2} \hat{\mathscr{r}} \mathrm{d}\tau$$

thus

$$\Delta V \propto Q \implies C \equiv \frac{Q}{\Delta V}$$

where C is the "capacitance".

Example: Parallel plate capacitor Two very large plates with area A, charge $+Q$ & $-Q$, a separation d such that $d \ll \sqrt{A}$, and the surface charge density is $\sigma_\pm = \pm\frac{Q}{A}$

We can easily find the electric field of a plate $\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$, and in the presence of two oppositely charged plates, there is only an electric field between the plates with double the magnitude:

$$E = \frac{\sigma}{\epsilon_0}$$

The potential difference is then

$$\begin{aligned} \Delta V &= \int \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} \\ &= \frac{\sigma}{\epsilon_0} d = \frac{d}{\epsilon_0}\frac{Q}{A} \end{aligned}$$

so the capacitance of the parallel plates is

$$C_\parallel = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d} = []$$

3 Potentials

3.1 Laplace's Equation

3.1.1 Intro

In principle, electrostatis is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\mathbf{r}')\mathrm{d}\tau', \quad \mathscr{r} = \mathbf{r} - \mathbf{r}'$$

And simplifying with potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\mathrm{d}\tau'$$

So we often use Poisson's equation e.g.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Even better, Laplace's equation

$$\boxed{\nabla^2 V = 0}$$

or in Cartesian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

3.1.2 Start in 1D

$$\frac{\partial^2 V}{\partial x^2} = 0 \implies V = mx + b$$

where we have two undetermined constants m & b. We can determine these constants by *boundary conditions.*

- e.g. $V(1) = 4 \quad V(5) = 0$; we get a line $V = -\frac{4}{5}x + 4$

Two features:

1. $V(x)$ is average of $V(x+a)$ and $V(x-a)$

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$$

2. **NO** local minima or maxima (no curvature!)

3.1.3 On to 2D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{no general solution}$$

and no requirement on the # of constants. But we can note common properties e.g. soap film on a wireframe assumes the same shape. The solutions are called *harmonic functions*:

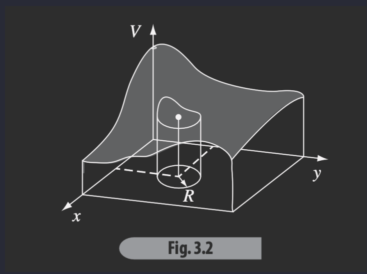

Figure 3.1

1. value $V(x, y)$ is average of nearby values; more precisely, for a circle of radius R (Fig. 3.1)
$$V(x, y) = \frac{1}{2\pi R}\oint V \,\mathrm{d}\ell$$
where $2\pi R$ is the circumference of the circle.
2. **NO** local minima or maxima

3.1.4 In 3D

$$\nabla^2 V = 0$$

Holds same properties as 2D:

1. Average over spherical surface of radius R centered at $\mathbf{r}$:
$$V(\mathbf{r}) = \frac{1}{4\pi R^2}\oint_S V \,\mathrm{d}a$$
where $4\pi R^2$ is the surface area of the sphere.

Example: Point charge outside sphere; The potential at da

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

and from law of cosines

$$\mathscr{r}^2 = z^2 + R^2 - 2rR\cos\theta$$

so the average potential is

$$\begin{aligned}
V_{\text{avg}} &= \frac{q}{4\pi R^2}\frac{1}{4\pi\epsilon_0}\int \frac{R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi}{\sqrt{z^2 + R^2 - 2zR\cos\theta}} \\
&= \frac{q}{2zR}\frac{1}{4\pi\epsilon_0}\left[z^2 + R^2 - 2zR\cos\theta\right]^{1/2}\Big|_0^{\pi} \\
&= \frac{q}{2zR}\frac{1}{4\pi\epsilon_0}[(z+R) - (z-R)] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{z}
\end{aligned}$$

which is just the potential of a point charge q in the center of the sphere.

Question: Is it possible to stably trap a charged particle using electrostatic forces alone?

Answer Earnshaw's theorem: A charged particle cannot be held in stable equilibrium by electrostatic forces alone.

3.1.5 Boundary Conditions & Uniqueness Theorem

Laplace's eq requires boundary conditions (b.c.c)

1st uniqueness theorem: "Solutions to L's eq in volume V is uniquely determined if potential is specied in surface S bounding V."

How is the solution unique?

Have solution in V_1 s.l.

$$\nabla^2 V_1 = 0 \quad \text{also} \quad \nabla^2 V_2 = 0$$

Then

$$V_3 \equiv \Delta V = V_1 - V_2$$

which means

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0 - 0 = 0$$

thus

$$\begin{aligned} \implies \nabla^2 V_1 &= \nabla^2 V_2 \\ V_1 &= V_2 \end{aligned}$$

We should emphasize that V_1 defined on the boundary S is also the same b.c.s as V_2 while V_3 on the boundary equals zero. That is V_3 is zero everywhere in space.

3.1.6 2nd uniqueness theorem (conductors):

"In a volume V surrounded by conductors, and containing a specifed charge density ρ, then the $\mathbf{E}$ field is uniquely determined if the total charge on each conductor is given."

"This proof was not easy" - Griffiths

Example: Connecting two pairs of opposite charges with a conductor; what is the final charge config and E-field?

Answer: $\mathbf{E} = 0$ everywhere. The total charge of each conductor is zero.

3.1.7 Boundary conditions pt. II

(Griffiths 2.3.5 pg 85) Given a sheet of charge $\sigma = Q/A$ and using the Gaussian pillbox method

$$\begin{aligned} \oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{q_{\text{enc}}}{\epsilon_0} \\ E_a A - E_b A &= \sigma \frac{A}{\epsilon_0} \\ E_a - E_b &= \frac{\sigma}{\epsilon_0} \end{aligned}$$

For the same surface we know that

$$\boldsymbol{\nabla} \times \mathbf{E} = \oint \mathbf{E} \cdot \mathrm{d}\ell = 0$$

So going around the loop we have

$$\begin{aligned} E_a \ell - E_b \ell &= 0 \\ E_a &= E_b \end{aligned}$$

3.2 Method of Images

$\nabla^2 V = -\rho/\epsilon_0$ *is* electrostatics. AND when we have $\nabla^2 V = 0$, Uniqueness theorems tell us there is a solution, but doesn't tell us how to find it...thus we have a set of "easily" solvable problems.

3.2.1 Classic image problem:

"Ground" is infinite conducting plane $V = 0$ at $z = 0$. For a charge q at $z = d$ what is the potential in $z > 0$? We know that the point charge will induce a charge on the plane which will effect the electic potential in the region $z > 0$.

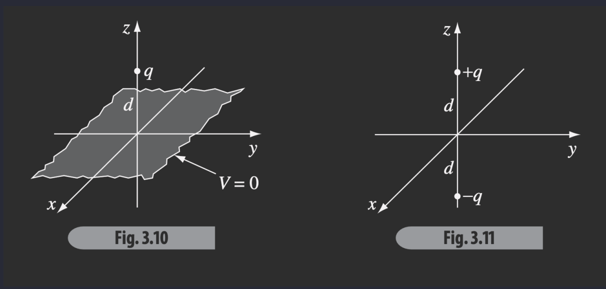

Figure 3.2

GOAL: solve $\nabla^2 V = -\rho/\epsilon_0$ for $z \geq 0$ with $+q$ at $(0, 0, d)$ subject to boundary conditions (b.c.s)

1. $V(z = 0) = 0$
2. $V \to 0$ for far away

The next step is to replace the conducting plane with an equivalent charge $-q$ at $z = -d$.

NOTE: We only care about $z \geq 0$ and ignore $z < 0$ region; furthermore, the potential below the plane should be zero as the boundary of the plane sort of "wraps" around the charge

Thus the solution is a superposition of charges

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

we can see that in the replacement problem, the region below the grounded plane is nonzero which is different from the real problem which is why we ignore it!

3.2.2 Induced surface charge

What is σ? Recall on the sheet σ we have a field

$$\mathbf{E}_{\text{ab}} - \mathbf{E}_{\text{be}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

where the $\hat{\mathbf{n}}$ tells us that we are dealing with the perpendicular planes; thus the same eq

$$\boldsymbol{\nabla} V_{\text{ab}} - \boldsymbol{\nabla} V_{\text{be}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \implies \mathbf{E} = -\boldsymbol{\nabla} V$$

We can infer that

$$\frac{\partial V_a}{\partial n} - \cancel{\frac{\partial V_b}{\partial n}} = -\frac{\sigma}{\epsilon_0}$$

since anything below the plane is zero from the previous example. We know have

$$\left.\frac{\partial V_a}{\partial z}\right|_{z\to 0} = \left.\frac{1}{4\pi\epsilon_0}\left[-\frac{q(z-d)}{(x^2 + y^2 + (z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2 + y^2 + (z+d)^2)^{3/2}}\right]\right|_{z\to 0}$$

So

$$\sigma = -\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}}$$

We can check that the total induced charge in the plane is infact

$$Q = \int_0^{2\pi}\int_0^{\infty} \sigma r \mathrm{d}r \mathrm{d}\phi = -q$$

where $r^2 = x^2 + y^2$.

This charge comes from the reservoir of charge given by ground.

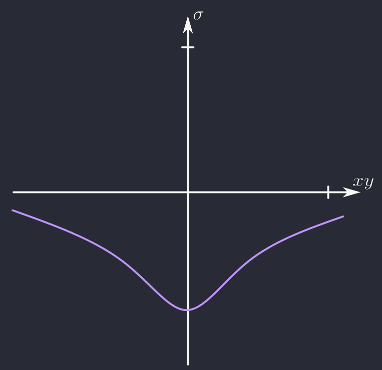

Figure 3.3: Just a cool graph of xy vs σ

3.2.3 Force and energy

Calcuating the force of attraction because of the negative induced charge:

$$F = qE = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}(-\hat{\mathbf{z}})$$

Naively, we calculate the energy/work done as

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d} \quad (= q\Delta V)$$

but we only have a single charge and the total work is half of this value

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

This is because the ideal conductor requires no work to build up a charge distribution σ.

Thus we have two ODEs

3.3 Seperation of Variables

3.3.1 In Cartesian

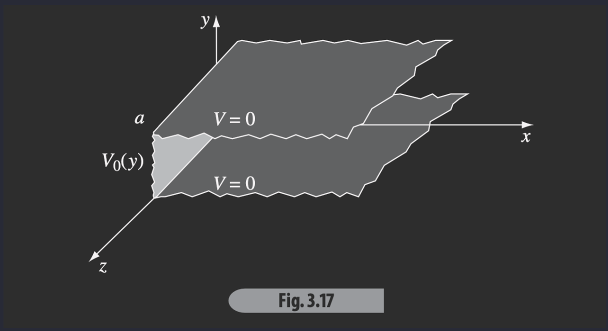

Figure 3.4: 3 planes where top and bottom are grounded and the middle is at $V_0(y)$

Find potential V in $x > 0$, $0 < y < a$, $-\infty < z < \infty$:
This is a 2D problem $V \to V(x, y)$
No change in $V \to \nabla^2 V = 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$.
From b.c.s

(i) $V(y = 0) = 0$

(ii) $V(y = a) = 0$

(iii) $V(x = 0) = V_0(y)$

(iv) $V(x \to \infty) = 0$

PROPOSE: $V(x, y) = X(x)Y(y)$

$$\nabla^2 V = Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$
$$\frac{\nabla^2 V}{V} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

thus we have two independent functions

$$f(x) + g(y) = 0$$

which is *only* possible if both f and g are constants

$$C_1 + C_2 = 0 \implies C_1 = -C_2 \quad \text{or} \quad C_1 = C_2 = 0$$

We now claim

$$\text{sgn}(C_1) = 1, \quad \text{sgn}(C_2) = -1, \quad \text{and} \quad |C_1| = |C_2| = k^2$$

$$\frac{\partial^2 X}{\partial x^2} = k^2 X$$
$$\frac{\partial^2 Y}{\partial y^2} = -k^2 Y$$

with general solutions

$$X(x) = Ae^{kx} + Be^{-kx}, \quad Y(y) = C\sin(ky) + D\cos(ky)$$

so the original potential are the product of the two solutions

$$\begin{aligned} V(x,y) &= \left(Ae^{kx} + Be^{-kx}\right)(C\cos(ky) + D\sin(ky)) \\ &= e^{-kx}(C\cos(ky) + D\sin(ky)) \end{aligned}$$

where condition (iv) requires $A = 0$ and condition (i) requires $D = 0$. From condition (ii) we require $\sin(ka) = 0$ thus

$$k = \frac{n\pi}{a}, \quad n = 1, 2, 3, \dots$$

Finally from condition (iii) we start with the rewritten potential

$$V(x,y) = Ce^{-kx}\sin(ky) \qquad k = \frac{n\pi}{a}$$

We now have an infinite number of solutions

$$V_1, V_2, V_3, \dots \quad \text{if} \quad V = \alpha_1 V_1 + \alpha_2 V_2 + \dots$$

where the general solution is

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a)$$

Now from (iii) we want $V(0,y) \implies V_0(y)$, so we multiply both sides by $\sin(n\pi y/a)$ and integrate over y

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = \int_0^a V_0(y)\sin(n'\pi y/a)dy$$

where the left integral is evaluated as

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases}$$

This gives us the coefficient

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy$$

We let $V_0(y) = V_0 \neq 0$ so

$$C_n = \frac{2}{a}V_0\int_0^a \sin(n\pi y/a)dy = \frac{2V_0}{n\pi}(1 - \cos(n\pi)) = \begin{cases} 0 & n \text{ even} \\ \frac{4V_0}{n\pi} & n \text{ odd} \end{cases}$$

so

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{\text{odd } n}\frac{1}{n}e^{-n\pi x/a}\sin\left(\frac{n\pi y}{a}\right)}$$

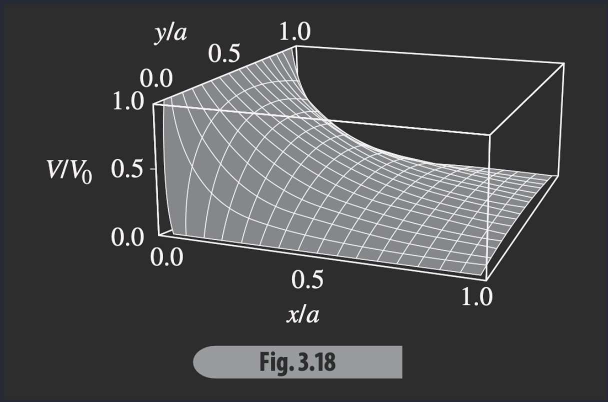

Figure 3.5: Graph of $V(x, y)$

Furthermore, the infinite series can be simplified to

$$V(x, y) = \frac{2V_0}{\pi} \arctan\left(\frac{\sin\left(\frac{\pi y}{a}\right)}{\sinh\left(\frac{\pi x}{a}\right)}\right)$$

We can see this graphed in 3D space in Fig. 3.5.

3.3.2 In Spherical

The laplacian now becomes

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

For the subject matter of undergraduate E&M, we will assume azimuthal symmetry i.e.

$$V = V(r, \theta)$$

so that V is independent of ϕ:

$$\nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

So we look for solutions that are a product two independent functions

$$V(r, \theta) = R(r)\Theta(\theta)$$

thus dividing the laplacian by V we get

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)}_{C_1} + \underbrace{\frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right)}_{C_2} = 0$$

where C_1 and C_2 are constants. We now choose

$$C_1 = \ell(\ell+1), \quad C_2 = -\ell(\ell+1)$$

so that the radial & polar equation becomes

$$R(r) = Ar^\ell + \frac{B}{r^{\ell+1}}$$
$$\Theta(\theta) = P_\ell(\cos\theta)$$

where P_ℓ is a Legendre polynomial defined by the Rodrigues formula (there are multiple ways to define this polynomial)

$$\begin{aligned} P_\ell(x) &= \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (x^2-1)^\ell \\ P_0 &= 1 \\ P_1 &= x \\ P_2 &= \frac{1}{2}(3x^2-1)\ldots \end{aligned}$$

The most general solution is a linear combination

$$V_{\text{gen}}(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Example Uncharged conducting sphere sitting in a background E-field $\mathbf{E} = E_0\hat{\mathbf{z}}$, so far away, the field points directly up. The induced charge on the sphere has positive charges at the top and negative charges at the bottom. The surface of the sphere is an equipotential of

$$V_{\text{sph}} = 0$$

and

$$V \to -E_0 r\cos\theta$$

as we go to ∞ i.e. $r \gg R$. So

$$\begin{aligned} V &= -\int \mathbf{E}\cdot \mathrm{d}\ell \\ &= E_0 z + C = 0 \end{aligned}$$

The general solution is given by

$$V = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Using B.C (i) when $r = R$

$$\begin{aligned} &\implies 0 = A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} \\ &\implies B_\ell = -A_\ell R^{2\ell+1} \end{aligned}$$

so

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell\left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right)\right]P-\ell(\cos\theta)$$

For $r \gg R$ we can ignore $\frac{1}{r^{\ell+1}}$ and we have

$$\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r\cos\theta$$

$\implies$ only $\ell = 1$ contributes to the sum on the left thus

$$A_1 r\cos\theta = -Er\cos\theta \implies A_1 = -E_0$$

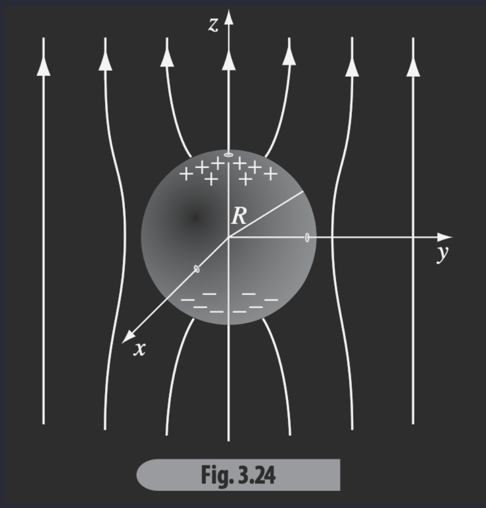

Figure 3.6: Graph of $V(r, \theta)$

Therefore the potential

$$\boxed{V(r, \theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta}$$

We can see from Fig. 3.6 that the conductor acts as an attractive lens for the field. Furthermore, the charge distribution on the sphere is

$$\begin{aligned} \sigma &= -\epsilon_0 \frac{\partial V}{\partial n} = \epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} \\ &= 3\epsilon_0 E_0 \cos \theta \end{aligned}$$

which is shown in Fig. 3.7.

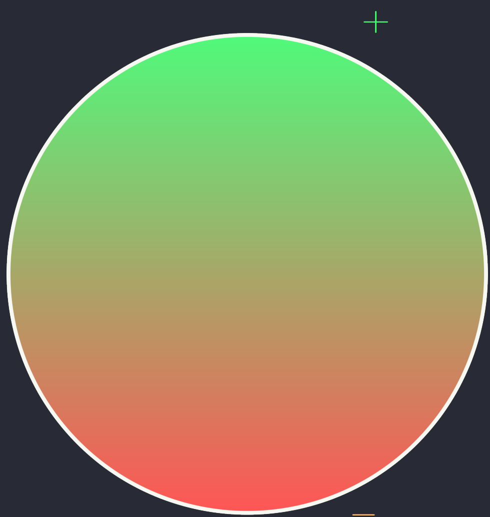

Figure 3.7: Charge distribution $\sigma(\theta)$

3.4 Multipole Expansion

3.4.1 To approximate potential at great distances

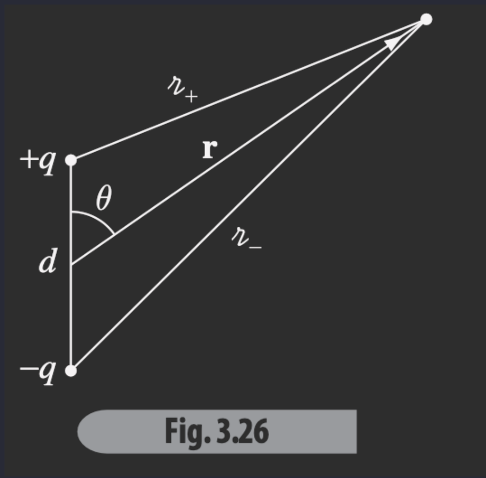

Figure 3.8: Dipole

For a dipole of opposite charge (Fig. 3.8) we are trying to find the potential $V(\mathbf{r})$ for $r \gg d$. Using superposition

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

And using Law of Cosines

$$\begin{aligned}\mathscr{r}_\pm^2 &= r^2 + (d/2)^2 \mp rd\cos\theta \\ &= r^2\left(1 \mp \frac{d}{r}\cos\theta + \cancel{\frac{d^2}{4r^2}}\right)\end{aligned}$$

So we solve for $\frac{1}{\mathscr{r}_\pm}$ and use binomial expansion:

$$\frac{1}{\mathscr{r}_\pm} = \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

Thus

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

We can abstract this to general multipoles such as the monoponole, quadropole, octopole, hexadecapole, etc. (Fig. 3.9).

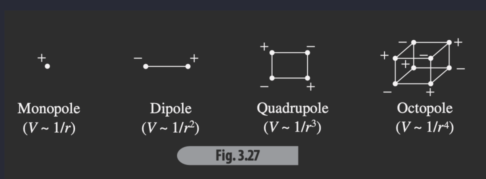

Figure 3.9: Multipole Expansion

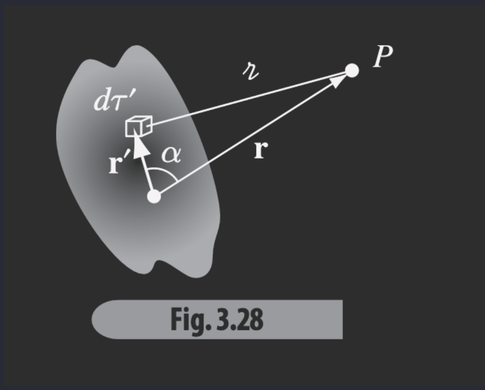

Figure 3.10: General local charge distribution

Expanding the potential to any local distribution of charge Thus

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\mathrm{d}\tau'$$

using LoC again

$$\begin{aligned}\mathscr{r}^2 &= r^2 + r'^2 - 2rr'\cos\alpha \\ &= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha\right]\end{aligned}$$

where we let $\mathscr{r} = r\sqrt{1+\epsilon}$ so

$$\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$$

Far away $\epsilon \ll 1$ so we can binomial expand

$$\begin{aligned}\frac{1}{\mathscr{r}} &= \frac{1}{r}(1+\epsilon)^{-1/2} \\ &= \frac{1}{r}\left(1 - \frac{\epsilon}{2} + 3/8\epsilon^2 - \dots\right) \\ &= \frac{1}{r}\left(1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3\right) \\ &= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \dots\right]\end{aligned}$$

all these terms are related to the Legendre polynomial $P_\ell(\cos\alpha)$ thus we can condense this to

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

We can substitute this back into the potential

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\mathrm{d}\tau'}$$

where each term of the sum is a multipole expansion of V i.e. the monopole $(1/r)$, dipole $(1/r^2)$, etc.

3.4.2 Monopole & Dipole

Obviously the $1/r$ dominates for the monopole i.e.

$$V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

The dipole is the $n = 1$ term

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\rho(\mathbf{r}')\mathrm{d}\tau'$$

where $r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}'$ so

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\mathrm{d}\tau'$$

And we define the integral as the dipole moment

$$\boxed{\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')\mathrm{d}\tau'}$$

So the dipole potential is simplified to

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}}$$

For *discrete* charges:

$$\mathbf{p} = \sum_{i=1}^{n} q_i\mathbf{r}'_i$$

where for a dipole

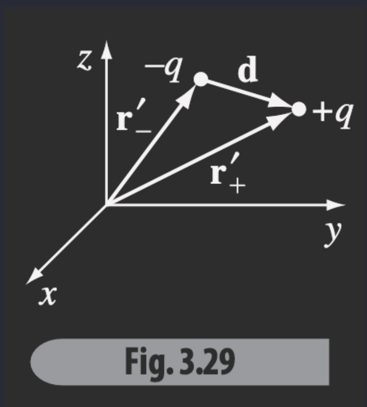

Figure 3.11: Dipole

$$\begin{aligned}\mathbf{p} &= q\mathbf{r}'_+ - q\mathbf{r}'_- \\ &= q(\mathbf{r}'_+ - \mathbf{r}'_-) \\ &= q\mathbf{d}\end{aligned}$$

For a physical or a pure dipole where $\mathbf{d} \to 0$ while $q \to \infty$

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

And looking at the E-field $\mathbf{E} = -\boldsymbol{\nabla} V$; the components are

$$\begin{aligned} E_r &= -\frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{r^3} \\ E_\theta &= -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\epsilon_0}\frac{p\sin\theta}{r^3} \\ E_\phi &= 0 \end{aligned}$$

So

$$\mathbf{E}_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$$

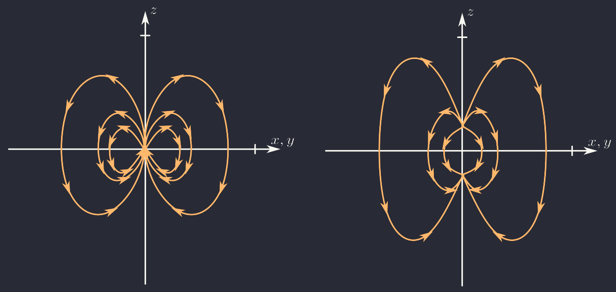

Figure 3.12: Pure dipole (left) and physical dipole (right)

4 Electric Fields in Matter

4.1 Polarization

4.1.1 Dielectrics

In dieletrics, "All charges are attached to specific atoms or molecules" (Griffith, pg.166)

4.1.2 Induced Dipoles

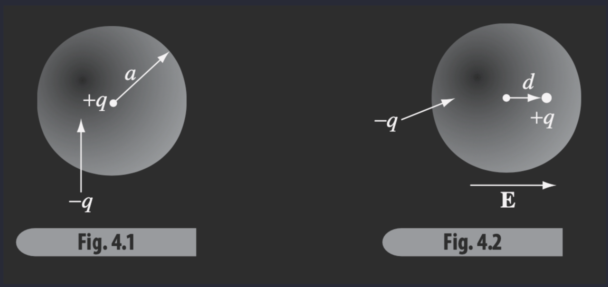

Figure 4.1: Left: Simple Nucleus $+q$ surrounded by spherical cloud $-q$ of radius a. Right: In an external electric field E the nucleus shifts right d and the cloud shifts left.

For a simple model of the atom (Fig. 4.1), the electric field at d is

$$E_d = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3}$$

where the dipole moment is $p = qd$. We have equilibrium when

$$F_{\text{ext}} = qE_{\text{ext}} = q_- E_d$$

So using the dipole moment

$$|\mathbf{p}| = qd = 4\pi\epsilon_0 a^3 E_{\text{ext}} = \alpha E_{\text{ext}}$$

Here we have this "atomic polaritability"

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon v$$

where v is the volume of the atom. Comment: this crude approximation is still accurate by a factor of 4.

In general we have a vector

$$\mathbf{p} = \hat{\alpha}\mathbf{E}$$

where $\hat{\alpha}$ is the polarizability tensor. For a linear dielectric relation between E and p we get

$$\hat{\alpha} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}$$

4.1.3 Alignment of Polar Molecules

The dipole will experience a torque in and E-field

$$\begin{aligned} \mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= \left(\frac{\mathbf{d}}{2} \times q\mathbf{E}\right) + \left(-\frac{\mathbf{d}}{2} \times q\mathbf{E}\right) \\ &= q\mathbf{d} \times \mathbf{E} \end{aligned}$$

thus

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

which implies that there is a force that acts to align $\mathbf{p} \parallel \mathbf{E}$.

WHat if E is not uniform?

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q\mathbf{E} + q\mathbf{E} = q\Delta E$$

assuming small d in E_x, then

$$\Delta E_x = (\boldsymbol{\nabla}\mathbf{E}_x) \cdot \mathbf{d}$$

So the total change in the field is

$$\Delta\mathbf{E} = (\mathbf{d} \cdot \boldsymbol{\nabla})\mathbf{E}$$

thus

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$$

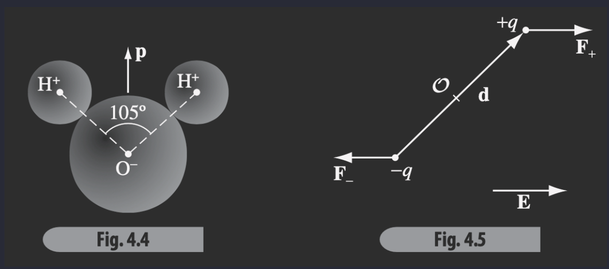

Figure 4.2: Dipole moment in oxygen molecule, and in an electric field.

Example: Problem 4.5 Using the method of images

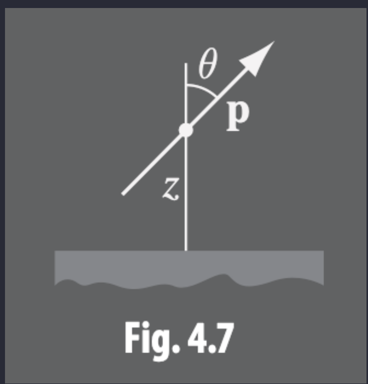

Figure 4.3: Infinitely grounded conductor with dipole at an angle θ from the normal plane and nailed in place.

Where look at $\mathbf{p}_i$ coordinate (pointing up) thus $2z$ away we have the dipole pointing perpendicular to the image dipole i.e.

$$\mathbf{E}_i = \frac{1}{4\pi\epsilon_0}\frac{p_i}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right)$$

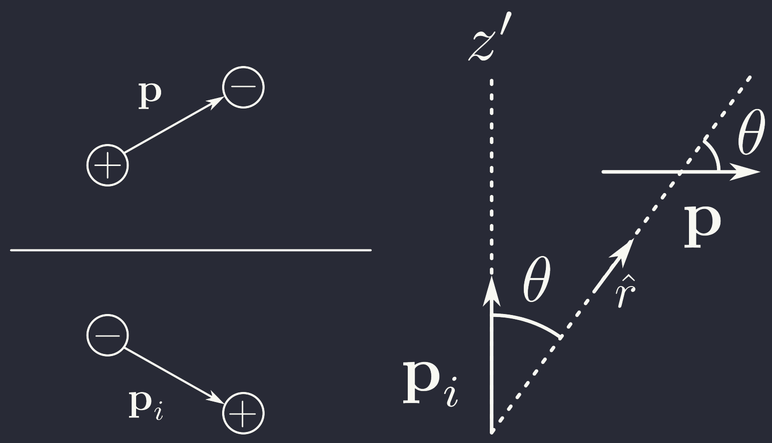

Figure 4.4: Left: Method of images using image dipole. Right: Coordinate using image dipole up as z'.

where

$$\mathbf{p} = p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\theta}$$

So the torque $\mathbf{N} = \mathbf{p} \times \mathbf{E}_i$ is

$$\begin{aligned}\mathbf{N} &= \frac{1}{4\pi\epsilon_0}\frac{p^2}{(2z)^3}(\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) \times \left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{p^2}{(2z)^3}\left(\cos\theta\sin\theta\hat{\boldsymbol{\phi}} + 2\cos\theta\sin\theta(-\hat{\boldsymbol{\phi}})\right) \\ &= \frac{1}{4\pi\epsilon_0}\frac{p^2\cos\theta\sin\theta}{(2z)^3}(-\hat{\boldsymbol{\phi}}) \\ &= \frac{1}{4\pi\epsilon_0}\frac{p^2\sin(2\theta)}{8\pi\epsilon_0(8z^3)}(-\hat{\boldsymbol{\phi}})\end{aligned}$$

So

$$\begin{cases} 0 < \theta < \frac{\pi}{2} & N \sim -\hat{\boldsymbol{\phi}} \\ \frac{\pi}{2} < \theta < \pi & N \sim \hat{\boldsymbol{\phi}} \end{cases}$$

Which means the dipole can either align perdendicularly up or down depending on the angle θ.

4.1.4 Polarization

$$\mathbf{P} \equiv \text{dipole moment per unit volume}$$

i.e. the little $\mathbf{p}$ is

$$\mathbf{p} = \mathbf{P}\mathrm{d}\tau$$

4.2 The Field of a Polarized Object

4.2.1 Bound Charges

For a single dipole

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

and using the dipole moment per unit volume def:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\mathrm{d}\tau'$$

recalling the math fact

$$\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

thus

$$V = \frac{1}{4\pi\epsilon_0}\int \mathbf{P}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\mathrm{d}\tau'$$

using another math fact

$$\boldsymbol{\nabla}\cdot(F\mathbf{A}) = F(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla}F)$$

So we can rewrtite the integral

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left[\int_V \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)\mathrm{d}\tau' - \int_V \frac{1}{\mathscr{r}}\boldsymbol{\nabla}\cdot\mathbf{P}\mathrm{d}\tau'\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{1}{\mathscr{r}}\mathbf{P}\cdot\mathrm{d}\mathbf{a}' - \int_V \frac{1}{\mathscr{r}}\boldsymbol{\nabla}\cdot\mathbf{P}\mathrm{d}\tau'\right] \end{aligned}$$

where we used the divergence theorem for the first term. For charge densities

$$\begin{cases} \text{surface charge} & \sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \\ \text{volume charge} & \rho_b \equiv -\boldsymbol{\nabla}\cdot\mathbf{P} \end{cases}$$

then

$$V = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{\sigma_b}{\mathscr{r}}\mathrm{d}a' + \int_V \frac{\rho_b}{\mathscr{r}}\mathrm{d}\tau'\right]$$

4.2.2 Physical Interpretation of Bound Charges

So for a charge neutral sphere with an applied E-field, we can imagine this sphere as two oppositely charge spheres superimposed on each other but slighly shifted (Fig. 4.5). Thus we can imagine a collection of

Figure 4.5: Charge neutral sphere with applied E-field.

dipoles for each atom in a material with alternating charges. This is actually wrong (read Berry Phases in Electronic Structure Theory by David Vanderbilt).

1.5 Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}\\ &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x\end{vmatrix}\end{aligned}$$

where the x component is

$$A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

From the "BAC-CAB" rule,

$$\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)$$

where x component simplifes to

$$B_x(\cancel{A_xC_x} + A_yC_y + A_zC_z) - C_x(\cancel{A_xB_x} + A_yB_y + A_zB_z) = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

So,

$$[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_x = [\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})]_x$$

and similary for the y and z components. Therefore, the "BAC–CAB" rule holds true.

1.11 (a) Finding gradient of $f(x,y,z) = x^2 + y^3 + z^4$:

$$\begin{aligned}\boldsymbol{\nabla} f &= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\ &= 2x\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + 4z^3\hat{\mathbf{z}}\end{aligned}$$

(b) Gradient of $f(x,y,z) = x^2y^3z^4$:

$$\boldsymbol{\nabla} f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$$

(c) Gradient of $f(x,y,z) = e^x \sin(y)\ln(z)$:

$$\boldsymbol{\nabla} f = e^x\hat{\mathbf{x}} + e^x\cos(y)\ln(z)\hat{\mathbf{y}} + \frac{e^x\sin(y)}{z}\hat{\mathbf{z}}$$

1.13 Given the seperation vector

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(a) Show that $\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$:

$$\mathscr{r}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

and each component of the gradient is

$$\begin{aligned} \frac{\partial}{\partial x}\left(\mathscr{r}^2\right) &= 2(x - x') \\ \frac{\partial}{\partial y}\left(\mathscr{r}^2\right) &= 2(y - y') \\ \frac{\partial}{\partial z}\left(\mathscr{r}^2\right) &= 2(z - z') \end{aligned}$$

so

$$\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$$

(b) Show $\boldsymbol{\nabla}(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$:

$$\frac{1}{\mathscr{r}} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2} = [(\ )]^{-1/2}$$

looking at the x component of the gradient (using chain rule),

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right) &= -\frac{1}{2}[(\ )]^{-3/2} \cdot 2(x - x') \\ &= -\frac{(x - x')}{\mathscr{r}^3} \end{aligned}$$

and similarly for the y and z components:

$$\begin{aligned} \boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{z}} \\ &= -\frac{1}{\mathscr{r}^3}[(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}] = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} \end{aligned}$$

Finally, substituting the unit vector $\hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}/\mathscr{r}$ gives us

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(c) The general formula for $\boldsymbol{\nabla}(\mathscr{r}^n)$:

$$\boldsymbol{\nabla}(\mathscr{r}^n) = n\,\mathscr{r}^{n-1} \cdot \boldsymbol{\nabla}(\mathscr{r})$$

where

$$\begin{aligned} \boldsymbol{\nabla}(\mathscr{r}) &= \frac{\partial}{\partial x}(\mathscr{r})\hat{\mathbf{x}} + \frac{\partial}{\partial y}(\mathscr{r})\hat{\mathbf{y}} + \frac{\partial}{\partial z}(\mathscr{r})\hat{\mathbf{z}} \\ &= \frac{1}{2}[(\ )]^{-1/2} \cdot 2(x - x')\hat{\mathbf{x}} + \dots \quad \text{[similar to part (b)]} \\ &= \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \hat{\boldsymbol{\mathscr{r}}} \end{aligned}$$

So the general formula is

$$\boldsymbol{\nabla}(\mathscr{r}^n) = n\,\mathscr{r}^{n-1}\,\hat{\boldsymbol{\mathscr{r}}}$$

1.15 (a) Calculating divergence of $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{v}_a &= \frac{\partial v_{ax}}{\partial x} + \frac{\partial v_{ay}}{\partial y} + \frac{\partial v_{az}}{\partial z} \\ &= 2x + 0 - 2x = 0\end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\boldsymbol{\nabla} \cdot \mathbf{v}_b = y + 2z + 3x$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\boldsymbol{\nabla} \cdot \mathbf{v}_c = 0 + 2x + 2y = 2(x + y)$$

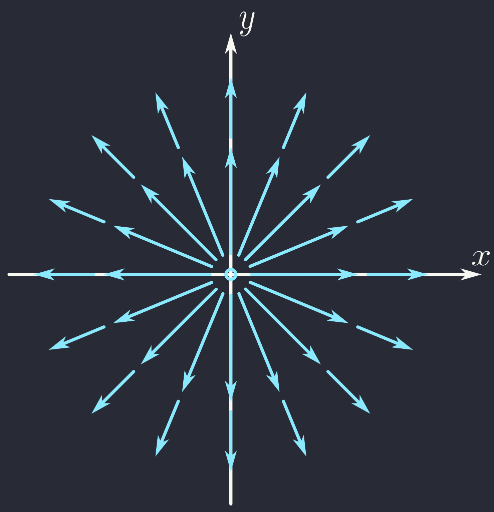

Figure 1.16: The vector field $\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$

1.16 Given

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector. The vector functions is

$$v = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

where the components are

$$v_x = \frac{x}{r^3}, \quad v_y = \frac{y}{r^3}, \quad \text{and} \quad v_z = \frac{z}{r^3}$$

Looking at the x component of the divergence,

$$\begin{aligned}
[\boldsymbol{\nabla} \cdot \mathbf{v}]_x &= \frac{\partial v_x}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) \quad \text{using chain rule...} \\
&= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{1}{r^3} - \frac{3x^2}{r^5}
\end{aligned}$$

therefore, the divergence of $\mathbf{v}$ is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v} &= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \\
&= \frac{3}{r^3} - 3\frac{x^2 + y^2 + z^2}{r^5} \\
&= \frac{3}{r^3} - 3\frac{r^2}{r^5} = 0
\end{aligned}$$

The divergence is zero everywhere except at the origin where $r = 0$ because division by r^3 tells us that the divergence is infinite at the origin.

1.18 Curl of vector functions from Problem 1.15: (a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_a &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0) \\
&= -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}
\end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_b &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x) \\
&= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}
\end{aligned}$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_c &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\
&= \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) \\
&= 0
\end{aligned}$$

1.32 Given $T = x^2 + 4xy + 2yz^3$,

$$\frac{\partial T}{\partial x} = 2x + 4y, \quad \frac{\partial T}{\partial y} = 4x + 2z^3, \quad \text{and} \quad \frac{\partial T}{\partial z} = 6yz^2$$

therefore

$$\boldsymbol{\nabla} T = \hat{\mathbf{x}}(2x + 4y) + \hat{\mathbf{y}}(4x + 2z^3) + \hat{\mathbf{z}}(6yz^2)$$

Checking the fundamental theorem for gradients using the points $a = (0, 0, 0) \to b = (1, 1, 1)$:

$$\int_a^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = T(b) - T(a) = 1^2 + 4(1)(1) + 2(1)(1)^3 - 0 = 7$$

For the three paths:

(a) $a \to (1, 0, 0) \to (1, 1, 0) \to b$;

(i) $a \to (1, 0, 0)$:

$$x : 0 \to 1; \quad y = z = \mathrm{d}y = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 2x\,\mathrm{d}x$$

and

$$\int_a^c \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2x\,\mathrm{d}x = 1$$

(ii) $(1, 0, 0) \to (1, 1, 0)$:

$$y : 0 \to 1; \quad x = 1, \quad z = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 4\,\mathrm{d}y$$

and

$$\int_c^d \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 4\,\mathrm{d}y = 4$$

(iii) $(1, 1, 0) \to b$:

$$z : 0 \to 1; \quad x = y = 1, \quad \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 6z^2\,\mathrm{d}z$$

and

$$\int_d^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 6z^2\,\mathrm{d}z = 2$$

therefore

$$\int_a^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 1 + 4 + 2 = 7$$

(b) $a \to (0, 0, 1) \to (0, 1, 1) \to b$;

(i) $a \to (0, 0, 1)$:

$$z : 0 \to 1; \quad x = y = \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 0$$

and

$$\int_a^c \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 0 = 0$$

(ii) $(0,0,1) \to (0,1,1)$:

$$z = 1, \quad x = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 2\,\mathrm{d}y$$

and

$$\int_c^d \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2\,\mathrm{d}y = 2$$

(iii) $(0,1,1) \to b$:

$$z : 0 \to 1; \quad y = z = 1, \quad \mathrm{d}y = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = (2x+4)\,\mathrm{d}x$$

and

$$\int_d^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 (2x+4)\,\mathrm{d}x = 5$$

therefore

$$\int_a^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 0 + 2 + 5 = 7$$

(c) the parabolic path $z = x^2$; $y = x$:

$$\mathrm{d}x = \mathrm{d}y\,, \quad \text{and} \quad \mathrm{d}z = 2x\,\mathrm{d}x\,; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}x\,\hat{\mathbf{y}} + 2x\,\mathrm{d}x\,\hat{\mathbf{z}}$$

and

$$\begin{aligned} \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} &= (2x+4y)\,\mathrm{d}x + (4x+2z^3)\,\mathrm{d}x + (6yz^2)2x\,\mathrm{d}x \\ &= 6x\,\mathrm{d}x + (4x+2x^6)\,\mathrm{d}x + (12x^6)\,\mathrm{d}x \\ &= 10x\,\mathrm{d}x + 14x^6\,\mathrm{d}x \end{aligned}$$

therefore

$$\begin{aligned} \int_a^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} &= \int_0^1 (10x + 14x^6)\,\mathrm{d}x \\ &= 5x^2 + 2x^7 \Big|_0^1 = 7 \end{aligned}$$

1.33 Testing the divergence theorem: For the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

the divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{v} = y + 2z + 3x$$

so the volume integral is

$$\begin{aligned}\int_V \boldsymbol{\nabla} \cdot \mathbf{v}\, \mathrm{d}\tau &= \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ &= \int_0^2 \int_0^2 (2y + 4z + 6)\, \mathrm{d}y\, \mathrm{d}z \\ &= \int_0^2 (4 + 8z + 12)\, \mathrm{d}z \\ &= 8 + 16 + 24 \\ \int_V \boldsymbol{\nabla} \cdot \mathbf{v}\, \mathrm{d}\tau &= 48\end{aligned}$$

The surface integral is evaluated over the six faces of the cube:

(i) $x = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}y\, \mathrm{d}z\, \hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 2y\, \mathrm{d}y\, \mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 2y\, \mathrm{d}y\, \mathrm{d}z = 8$$

(ii) $x = 0$, $\mathrm{d}\mathbf{A} = -\, \mathrm{d}y\, \mathrm{d}z\, \hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 0\, \mathrm{d}y\, \mathrm{d}z = 0$$

(iii) $y = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x\, \mathrm{d}z\, \hat{\mathbf{y}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 4z\, \mathrm{d}x\, \mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 4z\, \mathrm{d}x\, \mathrm{d}z = 16$$

(iv) $y = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

(v) $z = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x\, \mathrm{d}y\, \hat{\mathbf{z}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 6x\, \mathrm{d}x\, \mathrm{d}y$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 6x\, \mathrm{d}x\, \mathrm{d}y = 24$$

(vi) $z = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

So the total flux is

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 8 + 0 + 16 + 0 + 24 + 0 = 48$$

therefore, the divergence theorem is verified.

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{v})\, \mathrm{d}\tau = \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A}$$

1.34 Testing Stokes' theorem for the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

using the triangular shaded area bounded by the vertices $O = (0,0,0)$, $A = (0,2,0)$, and $B = (0,0,2)$:

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{v} &= (0 - 2y)\hat{\mathbf{x}} - (3z - 0)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}} \quad \text{and} \quad \mathrm{d}\mathbf{A} = \mathrm{d}z\,\mathrm{d}y\,\hat{\mathbf{x}} \\ &= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}\end{aligned}$$

$x = 0$ on this surface, and the limits of integration are $y : 0 \to 2$ and $z = 0 \to z = 2 - y$:

$$(\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = -2y\,\mathrm{d}z\,\mathrm{d}y$$

Thus, the flux of the curl through the surface is

$$\begin{aligned}\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} &= \int_0^2 \int_0^{2-y} -2y\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_0^2 -2y(2-y)\,\mathrm{d}y \\ &= -2y^2 + \frac{2}{3}y^3 \Big|_0^2 = -8/3\end{aligned}$$

The line integral is evaluated over the three sides of the triangle:

(i) On the path OA:
$x = z = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 2yz\,\mathrm{d}y = 0$;

$$\int_{OA} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

(ii) On the path AB:
$x = 0$, $y = 2 - z$; $\mathrm{d}y = -\,\mathrm{d}z$; $\mathrm{d}\mathbf{l} = -\,\mathrm{d}z\,(\hat{\mathbf{y}} - \hat{\mathbf{z}})$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = -2yz\,\mathrm{d}z = -2(2-z)z\,\mathrm{d}z = (2z^2 - 4z)\,\mathrm{d}z$;

$$\int_{AB} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^2 (2z^2 - 4z)\,\mathrm{d}z = -8/3$$

(iii) On the path BO:
$x = y = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$;

$$\int_{BO} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

So, the circulation of $\mathbf{v}$ around the triangle is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0 + -8/3 + 0 = -8/3$$

thus,

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

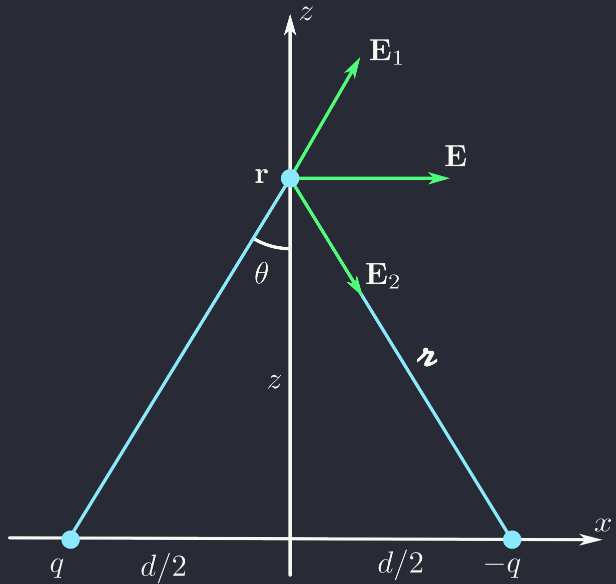

Figure 2.7: An electric field at a distance z from the midpoint between equal and opposite charges ($\pm q$) separated by a distance d. The charge at $x = d/2$ is $-q$.

2.2 The vertical componets of the electric field cancel out and the horizontal components add up:

$$E_x = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\sin\theta$$

where $E_x = E\cos\theta$, $\mathscr{r} = \sqrt{z^2 + (d/2)^2}$, and $\sin\theta = d/(2\mathscr{r})$, so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{[z^2 + (d/2)^2]^{3/2}}\hat{\mathbf{x}}$$

2.5 The horizontal components of the electric field cancel out, and the vertical components conspire:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda}{\mathscr{r}^2} \cos\theta \hat{\mathbf{z}} \, \mathrm{d}l$$

where geometrically $\mathscr{r} = \sqrt{z^2 + r^2}$ and $\cos\theta = z/\mathscr{r}$. So,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \, \mathrm{d}l$$

and the line integral is over the circumference of the circle, so $\mathrm{d}\mathbf{l} = r \, \mathrm{d}\theta$ and the limits of integration are $[0, 2\pi]$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{\lambda z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \int_0^{2\pi} r \, \mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda z (2\pi r)}{(z^2 + r^2)^{3/2}}
\end{aligned}$$

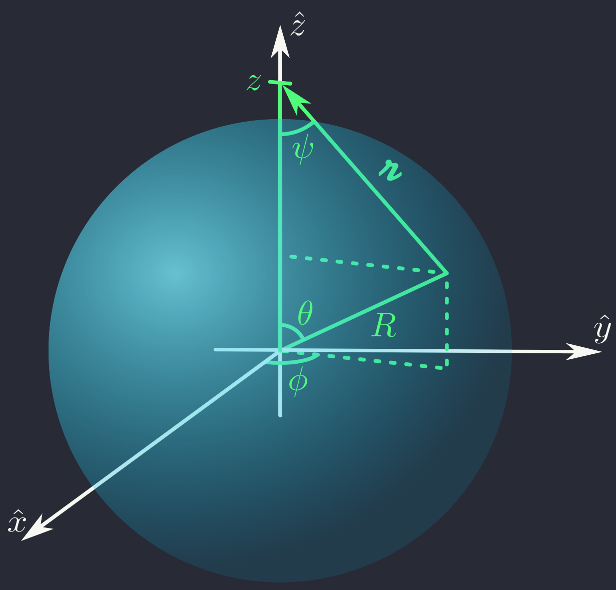

Figure 2.8: An electric field a distance z from the center of a spherical surface of radius R that carries a charge density σ.

2.7 Once again, the electric field is in the z-direction:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}^2}\cos\psi\,\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a} \tag{2.1}$$

From the law of cosines, $\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$; Geometrically, $\cos\psi = \dfrac{z - R\cos\theta}{\mathscr{r}}$; the surface area element is $\mathrm{d}\mathbf{a} = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\theta$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{\sigma R^2(z - R\cos\theta)}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}(2\pi\sigma R^2)\int_0^{\pi}\frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\,\mathrm{d}\theta\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}(2\pi\sigma R^2)f(\theta)\hat{\mathbf{z}}
\end{aligned}$$

using the substitution $u = \cos\theta$: $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$, and the limits of integration are $[\cos 0, \cos\pi]$. So,

$$f(\theta) = \int_{-1}^{1}\frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}}\,\mathrm{d}u = f(u)$$

substituting again with $v = \sqrt{z^2 + R^2 - 2zRu}$; $\mathrm{d}v = -\dfrac{zR}{v}\,\mathrm{d}u$; and $u = \dfrac{1}{2zR}(z^2 + R^2 - v^2)$:

$$\begin{aligned}
f(v) &= -\frac{1}{zR}\int\frac{z - \frac{1}{2z}(z^2 + R^2 - v^2)}{v^3}v\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{2z^2 - (z^2 + R^2 - v^2)}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{v^2 + z^2 - R^2}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\left(1 + \frac{z^2 - R^2}{v^2}\right)\mathrm{d}v \\
&= -\frac{1}{2z^2R}\left(v - \frac{z^2 - R^2}{v}\right)
\end{aligned}$$

back substituting $v = \sqrt{z^2 + R^2 - 2zRu}$,

$$\begin{aligned}
f(u) &= -\frac{1}{2z^2R}\left(\frac{z^2 + R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}} - \frac{z^2 - R^2}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= -\frac{1}{2z^2R}\left(\frac{2R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{zu - R}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} - \frac{-z - R}{\sqrt{z^2 + R^2 + 2zR}}\right)
\end{aligned}$$

Taking the positive square root: $\sqrt{z^2 + R^2 - 2zR} = (R - z)$ if $R > z$, but $(z - R)$ if $R < z$. So, for the case $z < R$ (inside the sphere) the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} - \frac{-z - R}{R + z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + \frac{z + R}{R + z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + 1\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + \frac{R - z}{R - z}\right)\hat{\mathbf{z}} \\
&= 0
\end{aligned}$$

For the case $z > R$ (outside the sphere) the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{z - R} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{4\pi\sigma R^2}{z^2}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}
\end{aligned}$$

This makes sense: From outside the sphere, the point charge q is the charge-per-area σ times the surface area of the sphere $4\pi R^2$, or simply $q = 4\pi R^2\sigma$.

2.8 Finding the field inside and outside a solid sphere of radius R with a uniform volume charge density ρ is similar to Prob. 2.7. Outside the solid sphere the total charge q contributes to the electric field as if it were a point charge:

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Inside the solid sphere, only the volume of the solid sphere less than r contributes to the electric field. The volume of the total sphere is $V = \frac{4}{3}\pi R^3$, and the volume of the sphere less than r is $V' = \frac{4}{3}\pi r^3$. So, electric field inside the solid sphere is

$$\begin{aligned}\mathbf{E}_{in} &= \frac{V'}{V}\mathbf{E}_{out} \\ &= \frac{r^3}{R^3}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}\end{aligned}$$

or

$$\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\mathbf{r}$$

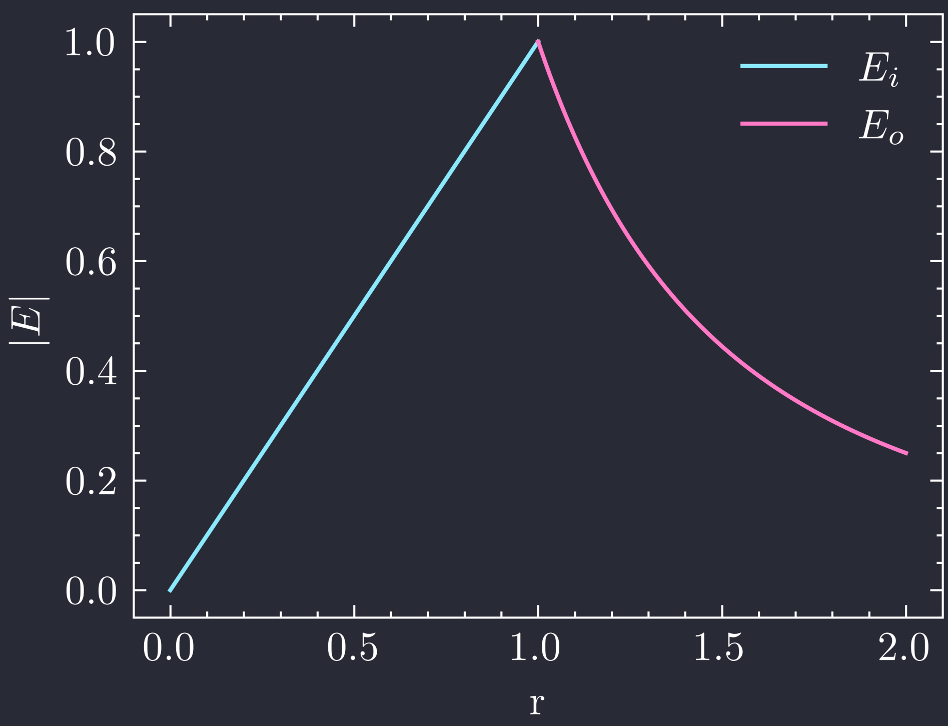

Figure 2.9: Magnitude of Electric field $|E|$ as a function of r inside and outside a solid. Where $q = 9\text{nC}$ and $R = 1\text{m}$.

2.12 For a spherical shell of radius R with a uniform surface charge density σ, the enclosed charge in side the sphere is $Q_{enc} = 0$, thus the electric field inside the sphere is

$$\mathbf{E}_i = 0$$

and using the sphericla symmetry of a Gaussian surface, the electric field outside the sphere is

$$\begin{aligned}
\oint \mathbf{E}_o \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\
|\mathbf{E}_0| \int \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o}(4\pi\sigma R^2) \\
\mathbf{E}_o(4\pi r^2) &= \frac{1}{\epsilon_o}(4\pi\sigma R^2)\hat{\mathbf{r}} \\
\mathbf{E}_o &= \frac{\sigma R^2}{\epsilon_o r^2}\hat{\mathbf{r}}
\end{aligned}$$

2.16 A thick spherical shell with charge density

$$\rho = \frac{k}{r^2} \quad (a \le r \le b)$$

The electric field in the three regions:
(i) $r < a$

$$Q_{enc} = 0; \mathbf{E} = 0$$

(ii) $a \le r \le b$

$$Q_{enc} = \int_0^{2\pi}\int_0^{\pi}\int_a^r \rho(r^2\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi\int_a^r \frac{k}{r^2}(r^2)\,\mathrm{d}r = 4\pi k(r-a)$$

And from Gauss's law,

$$\oint \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o}Q_{enc}$$
$$|\mathbf{E}|\int \mathrm{d}a = \frac{1}{\epsilon_o}4\pi k(r-a)$$
$$E(4\pi r^2) = \frac{1}{\epsilon_o}4\pi k(r-a)$$

or

$$\mathbf{E} = \frac{k(r-a)}{\epsilon_o r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{4\pi k(r-a)}{r^3}\mathbf{r}$$

(iii) $r > b$

$$Q_{enc} = \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho(r^2\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi k(b-a)$$

And from Gauss's law,

$$\oint \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o}Q_{enc}$$
$$|\mathbf{E}|\int \mathrm{d}a = \frac{1}{\epsilon_o}4\pi k(b-a)$$
$$E(4\pi r^2) = \frac{1}{\epsilon_o}4\pi k(b-a)$$

or

$$\mathbf{E} = \frac{k(b-a)}{\epsilon_o r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{4\pi k(b-a)}{r^3}\mathbf{r}$$

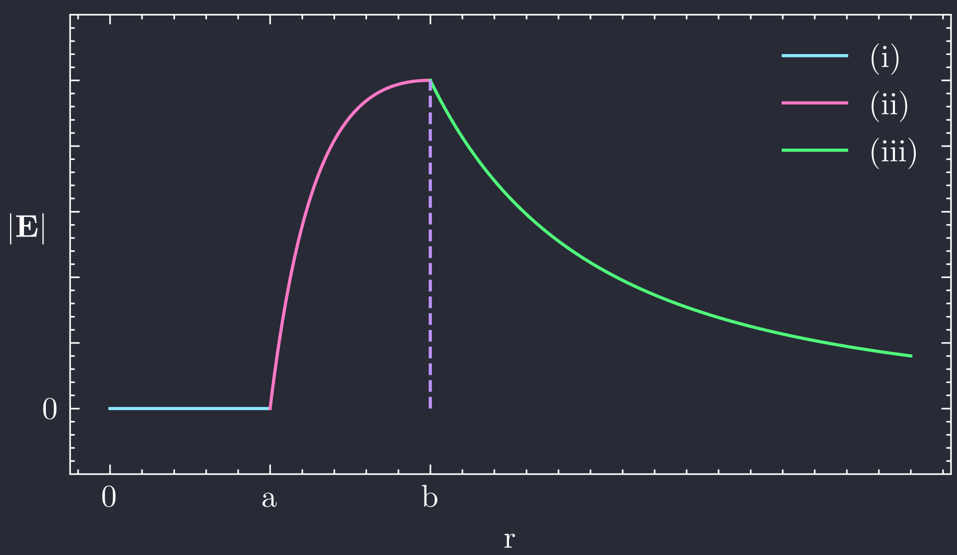

Figure 2.10: Plot of $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

2.18 Finding the electric field, as a function of y, where $y = 0$ is the center of an infinite plane slab, of thickness $2d$, carrying a uniform volume charge density ρ. For the case $y > 2d$ The enclosed charge is

$$Q_{enc} = \rho(2d)A = 2\rho Ad$$

where A is the area of the Gaussian pillbox. Using Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc}$$
$$|\mathbf{E}| \int \mathrm{d}a = \frac{1}{\epsilon_o} 2\rho Ad$$
$$E(2A) = \frac{1}{\epsilon_o} 2\rho Ad$$
$$\mathbf{E} = \frac{\rho d}{\epsilon_o} \hat{\mathbf{y}}$$

For the case $0 < y < 2d$, the enclosed charge is

$$Q_{enc} = 2\rho y A$$

and the electric field is

$$E(2A) = \frac{1}{\epsilon_o} \rho y A$$
$$\mathbf{E} = \frac{\rho y}{\epsilon_o} \hat{\mathbf{y}}$$

In the $-y$ direction, E is negative as shown in Figure 2.11.

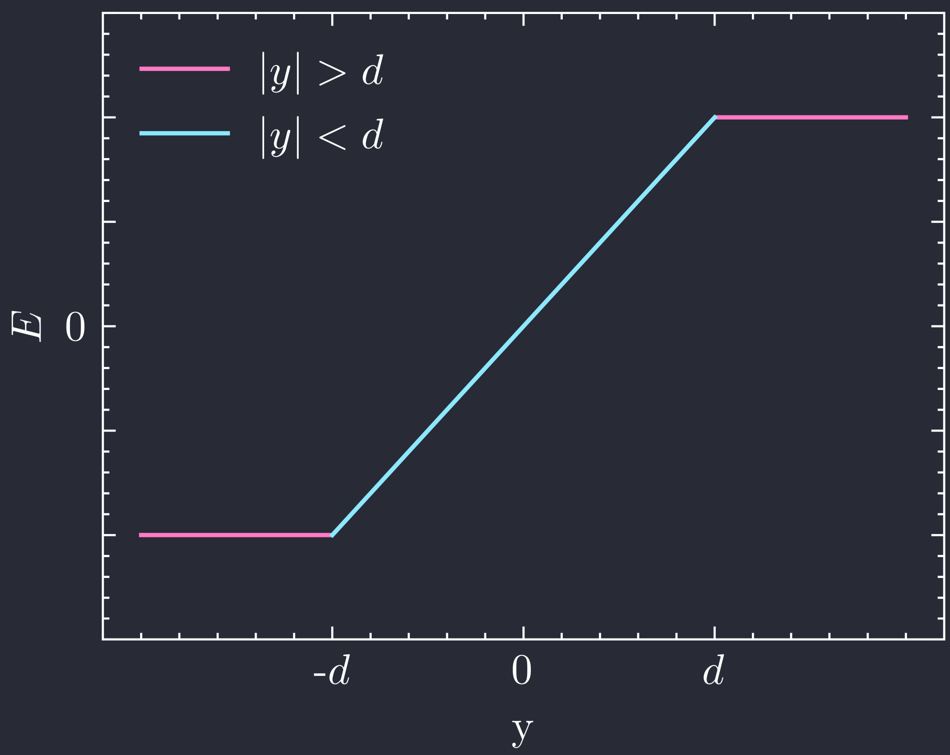

Figure 2.11: Plot of $|\mathbf{E}|$ as a function of y

2.22 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

The electric field outside the sphere is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

and from Problem 2.8, the electric field inside the sphere is

$$\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$$

For points outside the sphere ($r > R$),

$$V(r) = -\int_\infty^r \mathbf{E}\cdot\mathrm{d}\mathbf{l} = -\frac{1}{4\pi\epsilon_0}\int_\infty^r \frac{q}{r'^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_\infty^r = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

For points inside the sphere ($r < R$),

$$\begin{aligned} V(r) &= -\int_\infty^R \mathbf{E}\cdot\mathrm{d}\mathbf{l} - \int_R^r \mathbf{E}\cdot\mathrm{d}\mathbf{l} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\int_R^r r'\,\mathrm{d}r' \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\left(\frac{r'^2}{2}\right)\bigg|_R^r \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right) \end{aligned}$$

The gradient of V for $r > R$:

$$-\boldsymbol{\nabla}V = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} = \mathbf{E}_{out}$$

and for $r < R$:

$$-\boldsymbol{\nabla}V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}} = \mathbf{E}_{in}$$

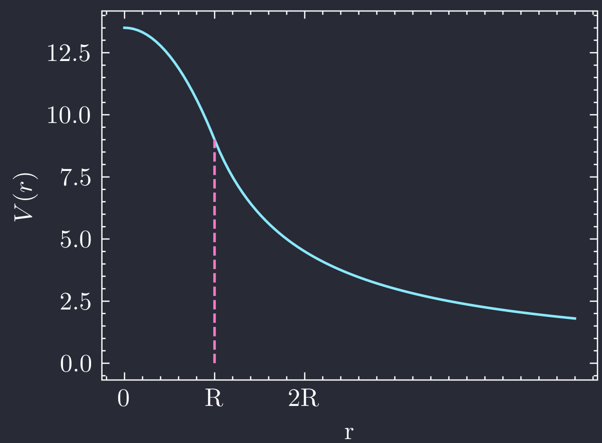

Figure 2.12: Plot of $V(r)$ as a function of r where $q = 1\,\mathrm{nC}$ and $R = 1\,\mathrm{m}$.

2.26 From Griffiths

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i} \tag{2.27}$$

and

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}\ell' \quad \text{and} \quad V = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}a' \tag{2.30}$$

(a.1) Two point charges $+q$ a distance d apart: Find the potential a distance z above the center of the charges: Using Eq. (2.27), the potential is

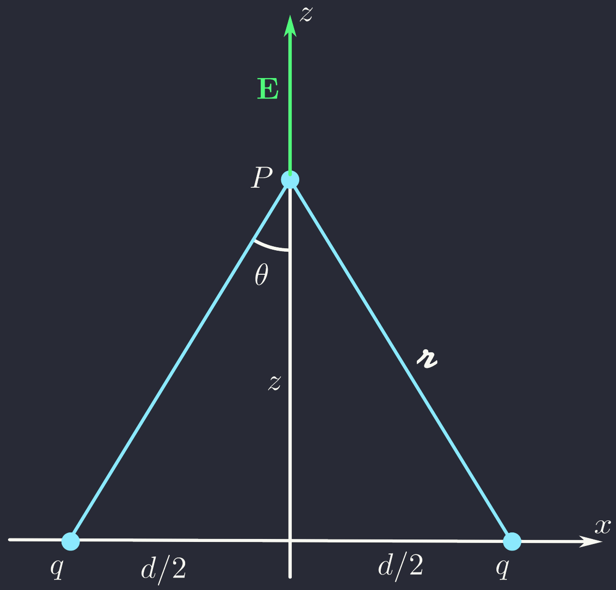

Figure 2.13: Two point charges $+q$ a distance d apart.

$$V_a = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{z^2+\frac{d^2}{4}}} + \frac{q}{\sqrt{z^2+\frac{d^2}{4}}}\right)$$

$$\boxed{V_a = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{z^2+\frac{d^2}{4}}}}$$

(a.2) Computing the electric field $\mathbf{E} = -\boldsymbol{\nabla}V$:

$$\begin{aligned}\mathbf{E}_a &= -\frac{\partial V_a}{\partial z}\hat{\mathbf{z}} \\ &= -\frac{1}{4\pi\epsilon_0}\frac{-1}{2}\frac{2q(2z)}{\left(z^2+\frac{d^2}{4}\right)^{3/2}}\hat{\mathbf{z}}\end{aligned}$$

simplifying to

$$\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2+\frac{d^2}{4}\right)^{3/2}}\hat{\mathbf{z}}}$$

which is the same as Ex. 2.1

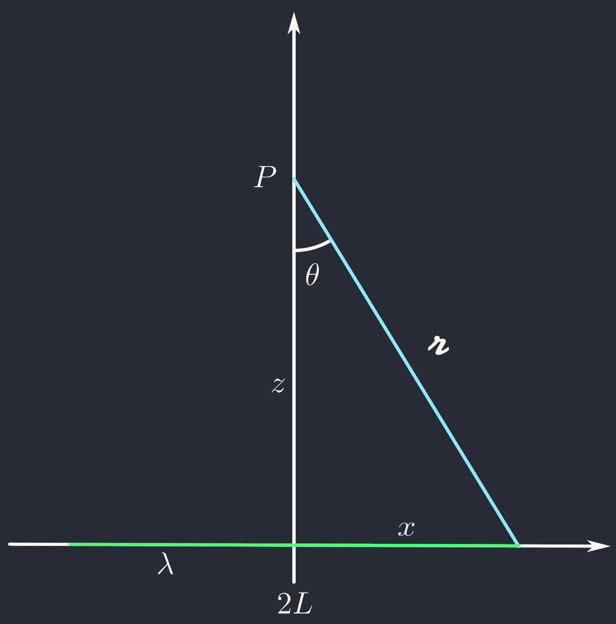

Figure 2.14: A line charge of density λ.

(b.1) Using Eq. (2.30), the potential is

$$V_b = \frac{1}{4\pi\epsilon_0}\lambda \int_{-L}^{L} \frac{1}{\sqrt{z^2+x^2}}\,\mathrm{d}x$$

To solve the integral, we can use the substitution from the trig identity

$$\begin{aligned} \cosh^2 u - \sinh^2 u &= 1 \\ \implies z^2\cosh^2 u &= z^2 + z^2\sinh^2 u \\ &= z^2 + x^2 \end{aligned}$$

where

$$\begin{aligned} x = z\sinh u &\implies u = \operatorname{arcsinh}\frac{x}{z} \\ \mathrm{d}x = z\cosh u\,\mathrm{d}u \end{aligned}$$

Thus the integral becomes

$$\begin{aligned} V_b &= \frac{1}{4\pi\epsilon_0}\lambda \int \frac{\cancel{z\cosh u}}{\cancel{z\cosh u}}\,\mathrm{d}u \\ &= \frac{1}{4\pi\epsilon_0}\lambda u\Big|_{-L}^{L} \\ &= \frac{1}{4\pi\epsilon_0}\lambda\left[\operatorname{arcsinh}\frac{L}{z} - \operatorname{arcsinh}\frac{-L}{z}\right] \end{aligned}$$

Using $\operatorname{arcsinh}(a) = \ln\left|a + \sqrt{a^2+1}\right|$:

$$\begin{aligned} \implies \operatorname{arcsinh}(\frac{L}{z}) &= \ln\left|\frac{L}{z} + \sqrt{\left(\frac{L}{z}\right)^2 + 1}\right| \\ &= \ln\left|\frac{1}{z}(L + \sqrt{L^2+z^2})\right| \end{aligned}$$

so the potential is

$$\boxed{V_b = \frac{1}{4\pi\epsilon_0}\lambda \ln\left|\frac{L+\sqrt{L^2+z^2}}{-L+\sqrt{L^2+z^2}}\right|}$$

(b.2) The electric field is

$$\begin{aligned}
\mathbf{E}_b &= -\frac{\partial V_b}{\partial z}\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\left[\frac{1}{L+\sqrt{L^2+z^2}}\left(\frac{1}{2}\frac{2z}{\sqrt{L^2+z^2}}\right) - \frac{1}{-L+\sqrt{L^2+z^2}}\left(\frac{1}{2}\frac{2z}{\sqrt{L^2+z^2}}\right)\right]\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\frac{z}{\sqrt{L^2+z^2}}\left[\frac{-L+\sqrt{L^2+z^2}}{-L^2+(L^2+z^2)} - \frac{L+\sqrt{L^2+z^2}}{-L^2+(L^2+z^2)}\right]\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\frac{-2Lz}{z^2\sqrt{L^2+z^2}}\hat{\mathbf{z}}
\end{aligned}$$

simplifying to

$$\boxed{\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{L^2+z^2}}\hat{\mathbf{z}}}$$

which is the same as Ex. 2.2

(c.1) Using Eq. (2.30) and polar coordinates, the potential is

$$\begin{aligned}
V_c &= \frac{1}{4\pi\epsilon_0}\sigma\int_0^{2\pi}\int_0^R \frac{1}{\sqrt{z^2+r^2}} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma\int_0^R \frac{r}{\sqrt{z^2+r^2}}\,\mathrm{d}r
\end{aligned}$$

substituting $u = z^2 + r^2$; $\mathrm{d}u = 2r\,\mathrm{d}r$:

$$\begin{aligned}
V_c &= \frac{1}{4\pi\epsilon_0}\pi\sigma\int\frac{1}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{4\pi\epsilon_0}\pi\sigma 2\sqrt{z^2+r^2}\Big|_0^R
\end{aligned}$$

thus

$$\boxed{V_c = \frac{\sigma}{2\epsilon_0}\left[\sqrt{z^2+R^2} - z\right]}$$

(c.2) The electric field is

$$\begin{aligned}
\mathbf{E}_c &= -\frac{\partial V_c}{\partial z}\hat{\mathbf{z}} \\
&= -\frac{\sigma}{2\epsilon}\left[\frac{z}{\sqrt{z^2+R^2}} - 1\right]\hat{\mathbf{z}}
\end{aligned}$$

thus

$$\boxed{\mathbf{E}_c = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}}$$

which is the same as Problem 2.6:

2.6 The electric field is only in the z-direction where $\cos\theta = z/\mathscr{r}$:

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}^2}\cos\theta\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a} \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a}\end{aligned}$$

Using polar coordinates: since $\mathrm{d}\mathbf{a} = r\,\mathrm{d}r\,\mathrm{d}\theta$

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{\sigma z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{1}{4\pi\epsilon_0}\sigma z(2\pi)\hat{\mathbf{z}}\int_0^R \frac{r}{(z^2+r^2)^{3/2}}\,\mathrm{d}r \\ &= \frac{\sigma}{2\epsilon_0}z\hat{\mathbf{z}}\left[-\frac{1}{\sqrt{z^2+r^2}}\right]_0^R \\ &= \frac{\sigma}{2\epsilon_0}z\left[\frac{1}{z}-\frac{1}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}} \\ \mathbf{E} &= \frac{\sigma}{2\epsilon_0}\left[1-\frac{1}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}\end{aligned}$$

(d) if the right-hand charge of Fig. 2.13 is replaced by a charge $-q$, the potential at P using Eq. (2.27) is

$$V_d = 0 \implies \mathbf{E}_d = 0$$

which contradicts the result from Prob 2.2. This is because point P does not give us any information about the electric field which points in the x-direction. In fact any reference point on the z-axis will give us the same result.

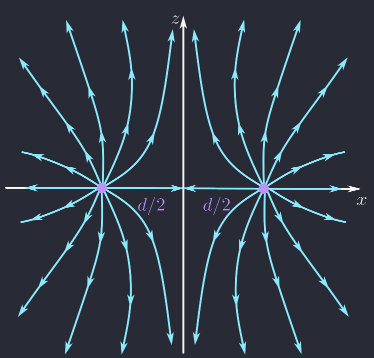

Figure 2.15: E-field for (a)

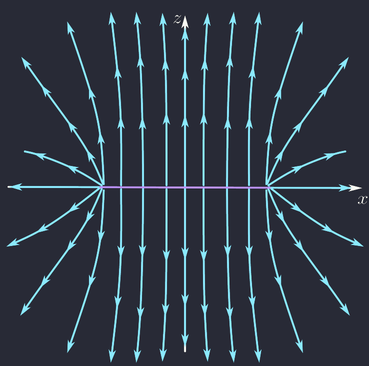

Figure 2.16: E-field for (b)

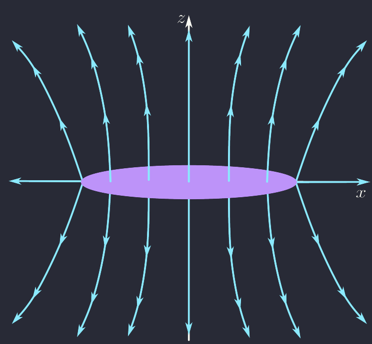

Figure 2.17: E-field for (c)

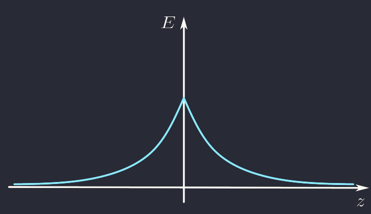

Figure 2.18: E-field for (c) E vs z

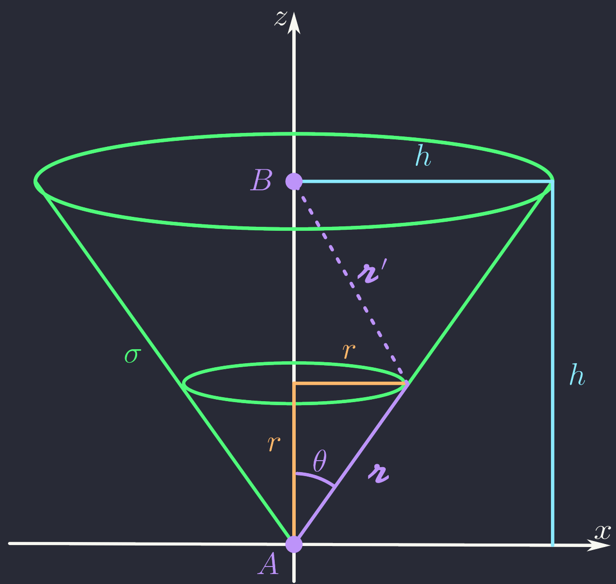

Figure 2.19: Empty ice cream cone with surface charge density σ.

2.27

(i) Potential at A: Geometrically, we can see from the large right triangle that

$$\mathscr{r}^2 = h^2 + h^2$$
$$\implies \mathscr{r} = h\sqrt{2}, \quad h = \frac{\mathscr{r}}{\sqrt{2}}$$

and from the smaller right triangle

$$\mathscr{r}^2 = 2r^2 \implies r = \frac{\mathscr{r}}{\sqrt{2}}$$

We can find the potential at A using Eq. (2.30) and integrate the rings of the cone along the slant length $0 \to h\sqrt{2}$ which gives us the area element $\mathrm{d}a = 2\pi r \, \mathrm{d}\mathscr{r}$:

$$\begin{aligned} V(A) &= \frac{1}{4\pi\epsilon_0} \int_0^{h\sqrt{2}} \frac{\sigma}{\mathscr{r}} 2\pi r \, \mathrm{d}\mathscr{r} \\ &= \frac{\sigma}{2\epsilon_0\sqrt{2}} \int_0^{h\sqrt{2}} \mathrm{d}\mathscr{r} \\ &= \frac{\sigma}{2\epsilon_0\sqrt{2}} \mathscr{r} \Big|_0^{h\sqrt{2}} \\ V(A) &= \frac{\sigma h}{2\epsilon_0} \end{aligned}$$

(ii) Potential at B: Using the law of cosines,

$$\mathscr{r}'^2 = h^2 + \mathscr{r}^2 - 2h\mathscr{r}\cos\theta$$

where

$$\begin{aligned} \cos\theta &= \frac{h}{\mathscr{r}} \\ &= \frac{h}{h\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}$$
$$\implies \mathscr{r}' = \sqrt{h^2 + \mathscr{r}^2 - h\mathscr{r}\sqrt{2}}$$

so the potential at B is

$$
\begin{aligned}
V(B) &= \frac{1}{4\pi\epsilon_0}\int_0^{h\sqrt{2}} \frac{\sigma}{\mathscr{r}'} 2\pi r \,\mathrm{d}\mathscr{r} \\
&= \frac{\sigma}{2\epsilon_0\sqrt{2}}\int_0^{h\sqrt{2}} \frac{\mathscr{r}}{\sqrt{h^2 + \mathscr{r}^2 - h\mathscr{r}\sqrt{2}}} \,\mathrm{d}\mathscr{r}
\end{aligned}
$$

I just used integral-calculator for this one...

$$
\begin{aligned}
V(B) &= \frac{\sigma}{2\epsilon_0\sqrt{2}}\left[h\sqrt{2}\ln\left(1+\sqrt{2}\right)\right] \\
V(B) &= \frac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right)
\end{aligned}
$$

Finally the potential difference between A and B is

$$
V(B) - V(A) = \frac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right) - \frac{\sigma h}{2\epsilon_0}
$$

$$
\boxed{V(B) - V(A) = \frac{\sigma h}{2\epsilon_0}\left[\ln\left(1+\sqrt{2}\right) - 1\right]}
$$

2.35 For a solid sphere radius R and charge q

(a) From Problem 2.22

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

and

$$W = \frac{1}{2}\int \rho V \,\mathrm{d}\tau \tag{2.43}$$

So the energy is

$$\begin{aligned}
W &= \frac{\rho}{2}\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(3 - \frac{r^2}{R^2}\right)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{\rho}{2}\frac{1}{4\pi\epsilon_0}\frac{q}{2R}4\pi\int_0^{R}\left(3r^2 - \frac{r^4}{R^2}\right)\mathrm{d}r \\
&= \frac{\rho q}{4R\epsilon_0}\left[r^3 - \frac{r^5}{5R^2}\right]_0^R \\
&= \frac{\rho q}{4R\epsilon_0}\left[R^3 - \frac{R^3}{5}\right] \\
&= \frac{\rho q}{5\epsilon_0}R^2
\end{aligned}$$

where the charge over the volume of the sphere is $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, thus

$$W = \frac{q}{5\epsilon_0}R^2\frac{q}{\frac{4}{3}\pi R^3}$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}}$$

(b) Integrating over all space using

$$W = \frac{\epsilon_0}{2}\int E^2\,\mathrm{d}\tau \tag{2.45}$$

Where the electric field is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \quad \mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$$

so the energy is

$$\begin{aligned}
W &= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}4\pi q^2\left[\int_0^R \frac{r^2}{R^6}r^2\mathrm{d}r + \int_R^\infty \frac{1}{r^4}r^2\mathrm{d}r\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\int_0^R \frac{r^4}{R^6}\mathrm{d}r + \int_R^\infty \frac{1}{r^2}\mathrm{d}r\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{r^5}{5R^6}\bigg|_0^R - \frac{1}{R}\bigg|_R^\infty\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{R^5}{5R^6} + \frac{1}{R}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\frac{6}{5R} \\
W &= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}$$

checkmark.

(c) For a spherical volume of radius a and

$$W = \frac{\epsilon_0}{2}\left(\int_V E^2 \mathrm{d}\tau + \oint_S V\mathbf{E}\cdot \mathrm{d}\mathbf{a}\right) \tag{2.44}$$

we can assume the volume is outside the charged sphere so

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

From part (b), the first term is

$$\begin{aligned}\frac{\epsilon_0}{2}\int_V E^2 \mathrm{d}\tau &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{1}{5R} - \frac{1}{a} + \frac{1}{R}\right]\\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{6}{5R} - \frac{1}{a}\right]\end{aligned}$$

the second term is at $r = a$

$$\begin{aligned}\frac{\epsilon_0}{2}\oint_V V\mathbf{E}\cdot \mathrm{d}\mathbf{a} &= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}\int \frac{q}{r}\frac{q}{r^2}r^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi\\ &= \frac{4\pi\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}\frac{1}{r}\bigg|_{r=a}\\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\end{aligned}$$

so the total energy is

$$\begin{aligned}W &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{6}{5R} - \frac{1}{a}\right] + \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\\ &= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}\end{aligned}$$

As $a \to \infty$ the $\int V\mathbf{E}\cdot \mathrm{d}\mathbf{a}$ term goes to zero.

2.40 Two cavities radii a and b in a conducting sphere of radius R with a point charge q_a and q_b respectively in each cavity.

(a) Surface charge densities:

On the surface of cavity a the charge density is simply

$$\sigma_a = \frac{-q_a}{4\pi a^2}$$

and

$$\sigma_b = \frac{-q_b}{4\pi b^2}$$

respectively. For the surface of the conducting sphere, the charge density is positive and equal to the superposition of the two charges:

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b) The field outside the conductor is equivalent to a point charge at the center of the sphere with the sum of the charges:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$$

(c) The field in cavity a with respect to the center of the cavity is

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{a^2}\hat{\mathbf{a}}$$

and in cavity b is

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{b^2}\hat{\mathbf{b}}$$

(d) The field due to to the cavity charge is zero in the exterior of the cavity, so there is no Force on q_a or q_b.

(e) If a charge q_c was brought near the conductor from outside, there would be a change in (a) σ_R and (b) $\mathbf{E}_{out}$.

2.48 Net force of of the southern hemisphere extering on the northern hemisphere (solid sphere) with an inside Electric field (Problem 2.8)

$$E_{in} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$$

where the total force is

$$\mathbf{F} = Q\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{R^3}\mathbf{r}$$

Finding the net force exerted by the southern hemisphere: integrate $\mathrm{d}F = \mathbf{F}/V$ over the southern hemisphere:

$$\begin{aligned}\mathrm{d}F &= \frac{1}{\frac{4}{3}\pi R^3}\frac{1}{4\pi\epsilon_0}\frac{Q^2}{R^3}\mathbf{r}\mathrm{d}\tau \\ &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}\mathbf{r}\mathrm{d}\tau\end{aligned}$$

The symmetry of the sphere implies that the Force is only in the z-direction i.e. $F_z = F\cos\theta$, so integrating over the southern hemisphere:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R F_z \mathrm{d}\tau &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R r\cos\theta r^2\sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}(2\pi)\left(\frac{r^4}{4}\right)\Bigg|_0^R \int_0^{\pi/2} \sin\theta\cos\theta \mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{3Q^2}{32\pi\epsilon_0 R^2}\frac{\sin^2 x}{2}\Bigg|_0^{\pi/2} \\ &= \boxed{\frac{3Q^2}{64\pi\epsilon_0 R^2}}\end{aligned}$$

3.4

(a) Average field over a spherical surface due to charges outside the sphere is the same at the center:

For a charge q a distance z above the center of the sphere, we can use the same geometrical argument from HW 2 Problem 2.7: The average field at over the surface will be in the negative z direction

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\psi(-\hat{\mathbf{z}})$$

where (using law of cosines)

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta \qquad \cos\phi = \frac{z - R\cos\theta}{\mathscr{r}}$$

The surface element is $\mathrm{d}a = R^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$, so the average field is

$$\begin{aligned}
\mathbf{E}_{\text{avg}} &= \frac{1}{4\pi R^2}\frac{1}{4\pi\epsilon_0}(-qR^2)\hat{\mathbf{z}}\int_0^{2\pi}\int_0^{\pi}\frac{z - R\cos\theta}{(z^2+R^2-2zR\cos\theta)^{3/2}}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{4\pi}\frac{1}{4\pi\epsilon_0}(-q)\hat{\mathbf{z}}(2\pi)\int_0^{\pi}\frac{z - R\cos\theta}{(z^2+R^2-2zR\cos\theta)^{3/2}}\sin\theta\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{-q}{2}\hat{\mathbf{z}}f
\end{aligned}$$

The integral evaluates to (from Problem 2.7): Using the substitution $u = \cos\theta$: $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$, and the limits of integration are $[\cos 0, \cos\pi]$. So,

$$\begin{aligned}
f(\theta) &= -\int_1^{-1}\frac{z - Ru}{(z^2+R^2-2zRu)^{3/2}}\,\mathrm{d}u \\
&= \int_{-1}^{1}\frac{z - Ru}{(z^2+R^2-2zRu)^{3/2}}\,\mathrm{d}u
\end{aligned}$$

substituting again with $v = \sqrt{z^2+R^2-2zRu}$; $\mathrm{d}v = \dfrac{zR}{v}\,\mathrm{d}u$; and $u = \dfrac{1}{2zR}(z^2+R^2-v^2)$:

$$\begin{aligned}
f(v) &= -\frac{1}{zR}\int\frac{z - \frac{1}{2z}(z^2+R^2-v^2)}{v^3}v\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{2z^2-(z^2+R^2-v^2)}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{v^2+z^2-R^2}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\left(1+\frac{z^2-R^2}{v^2}\right)\mathrm{d}v \\
&= -\frac{1}{2z^2R}\left(v - \frac{z^2-R^2}{v}\right)
\end{aligned}$$

substituting back in $v = \sqrt{z^2 + R^2 - 2zRu}$,

$$f(u) = -\frac{1}{2z^2R}\left(\frac{z^2 + R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}} - \frac{z^2 - R^2}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1}$$
$$= -\frac{1}{2z^2R}\left(\frac{2R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1}$$
$$= \frac{1}{z^2}\left(\frac{zu - R}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1}$$
$$= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} - \frac{-z - R}{\sqrt{z^2 + R^2 + 2zR}}\right)$$
$$= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} + \frac{z + R}{z + R}\right)$$
$$= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} + 1\right)$$

where the positive root $\sqrt{z^2 + R^2 - 2zR} = (z - R)$ for $z > R$, so

$$\mathbf{E}_{\text{avg}} = \frac{1}{4\pi\epsilon_0}\left(-\frac{q}{2z^2}\right)\left(\frac{z - R}{z - R} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}}$$

Simplfying to

$$\boxed{\mathbf{E}_{\text{avg}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}}$$

which is the same as the field at the center of the sphere. For a collection of particles, we can use superposition and find the net field as the sum of the fields at the center from each charge.

(b) For charges inside the sphere we can use the result from before: for one charge

$$\mathbf{E}_{\text{avg}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}}$$

but now the positive root is $\sqrt{z^2 + R^2 - 2zR} = (R - z)$ for $z < R$, so

$$\mathbf{E}_{\text{avg}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left(\frac{z - R}{R - z} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}}$$
$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}(-1 + 1)\hat{\mathbf{z}}$$
$$\mathbf{E}_{\text{avg}} = 0$$

And we can superimpose the fields from a collection of charges

$$\boxed{\mathbf{E}_{\text{avg}} = 0 + 0 + \cdots = 0}$$

3.7 Charges $+q$ & $-2q$ are respectively $z = 3d$ & $z = d$ above the xy plane (grounded conductor). Find the force of the charge $+q$:

We can use the method of images and replace the grounded conductor with two charges $-q$ at $z = -3d$ and $+2q$ at $z = -d$. Thus the force on $+q$ is the superposition of the forces from the three charges: The seperation vectors are

$$
\begin{aligned}
\boldsymbol{\mathscr{r}}_{-2q} &= 2d\hat{\mathbf{z}} \\
\boldsymbol{\mathscr{r}}_{+2q} &= 4d\hat{\mathbf{z}} \\
\boldsymbol{\mathscr{r}}_{-q} &= 6d\hat{\mathbf{z}}
\end{aligned}
$$

Finally, the force on charge $+q$ is

$$
\begin{aligned}
\mathbf{F} &= \mathbf{F}_{-2q} + \mathbf{F}_{+2q} + \mathbf{F}_{-q} \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{-2q^2}{(2d)^2} + \frac{2q^2}{(4d)^2} + \frac{-q^2}{(6d)^2}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{36}\right)\hat{\mathbf{z}}
\end{aligned}
$$

which simplifies to

$$
\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{29q^2}{72d^2}\hat{\mathbf{z}}}
$$

3.8 From Griffiths, where the configuration has another point charge

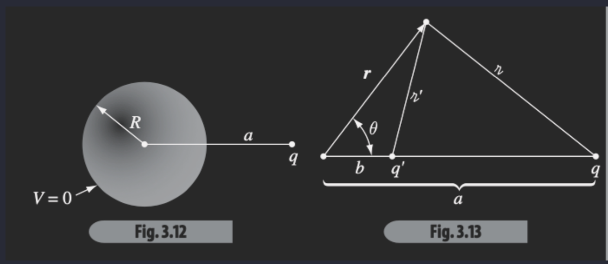

Figure 3.20: From Griffiths Example 3.2

$$q' = -\frac{R}{a}q \tag{3.15}$$

placed a distance

$$b = \frac{R^2}{a} \tag{3.16}$$

thus the potential of the config

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) \tag{3.17}$$

(a) Using law of cosines, show that Eq. (3.17) can be written as

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right]$$

From Fig. 3.20, we can see that

$$\mathscr{r} = \sqrt{r^2 + a^2 - 2ra\cos\theta} \qquad \mathscr{r}' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$$

so using Eq. (3.15) and Eq. (3.16) we can rewrite

$$\begin{aligned}
\frac{q'}{\mathscr{r}'} &= \frac{-R}{a}\frac{q}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}} \\
&= -\frac{1}{\sqrt{\frac{a^2}{R^2}}}\frac{q}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}} \\
&= -\frac{q}{\sqrt{r^2a/R^2 + (R^2/a)^2a^2/R^2 - 2r(R^2/a)\cos\theta a^2/R^2}} \\
\frac{q'}{\mathscr{r}'} &= -\frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}
\end{aligned}$$

Now we can rewrite Eq. (3.17) as

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right]$$

(b) Finding the induced charge on the sphere & integrating to get total induced charge: The normal component of the potential is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so

$$\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} \\
&= -\epsilon_0 \frac{1}{4\pi\epsilon_0} q\left(-\frac{1}{2}\right)\left[\frac{2r - 2a\cos\theta}{(r^2+a^2-2ra\cos\theta)^{3/2}} - \frac{2ra^2/R^2 - 2a\cos\theta}{(R^2+(ra/R)^2-2ra\cos\theta)^{3/2}}\right]\bigg|_{r=R} \\
&= \frac{q}{4\pi}\left[\frac{R - a\cos\theta}{(R^2+a^2-2Ra\cos\theta)^{3/2}} - \frac{a^2/R - a\cos\theta}{(R^2+a^2-2Ra\cos\theta)^{3/2}}\right] \\
&= \frac{q}{4\pi}\left[\frac{R - a^2/R}{(R^2+a^2-2Ra\cos\theta)^{3/2}}\right]
\end{aligned}$$

which simplifies to

$$\boxed{\sigma(\theta) = \frac{q}{4\pi R}\frac{R^2-a^2}{(R^2+a^2-2Ra\cos\theta)^{3/2}}}$$

Integrating to get the total induced charge using the surface element $\mathrm{d}a = R^2\sin\theta \mathrm{d}\theta\mathrm{d}\phi$:

$$\begin{aligned}
Q &= \int \sigma \mathrm{d}a \\
Q &= \frac{q}{4\pi R}(R^2-a^2)(2\pi R^2)\int_0^\pi \frac{\sin\theta}{(R^2+a^2-2Ra\cos\theta)^{3/2}}\mathrm{d}\theta \\
\text{using} \quad & u = R^2+a^2-2Ra\cos\theta; \quad \mathrm{d}u = 2Ra\sin\theta\,\mathrm{d}\theta \\
&= \frac{qR}{2}(R^2-a^2)\frac{1}{2Ra}\int\frac{1}{u^{3/2}}\,\mathrm{d}u \\
&= \frac{qR}{2}(R^2-a^2)\frac{1}{2Ra}\frac{-2}{\sqrt{u}} \\
&= -\frac{q}{2a}(R^2-a^2)\frac{1}{\sqrt{R^2+a^2-2Ra\cos\theta}}\bigg|_0^\pi \\
&= -\frac{q}{2a}(R^2-a^2)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}} - \frac{1}{\sqrt{R^2+a^2-2Ra}}\right]
\end{aligned}$$

From Fig. 3.20, we can see that $R < a$ so the positive root is $\sqrt{R^2+a^2-2Ra} = (a-R)$. Now the total induced charge is

$$\begin{aligned}
Q &= -\frac{q}{2a}(R^2-a^2)\left[\frac{1}{a+R} - \frac{1}{a-R}\right] \\
&= -\frac{q}{2a}(R^2-a^2)\left[\frac{a-R}{(a+R)(a-R)} - \frac{a+R}{(a-R)(a+R)}\right] \\
&= -\frac{q}{2a}(R^2-a^2)\left[\frac{a-R-(a+R)}{a^2-R^2}\right] \\
&= -\frac{q}{2a}(R^2-a^2)\left[\frac{-2R}{-(R^2-a^2)}\right]
\end{aligned}$$

Thus the total induced charge is

$$\boxed{Q = -\frac{R}{a}q = q'}$$

(c) The energy of the config:

First we find the force on q due the induced charge q' which are separated by a distance $a-b$:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2}\hat{\mathbf{n}}$$

Using Eq. (3.15) and Eq. (3.16)

$$\begin{aligned}\mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{R}{a}\frac{q^2}{(a-R^2/a)^2}\hat{\mathbf{a}}\\ &= -\frac{1}{4\pi\epsilon_0}\frac{R}{a}\frac{q^2}{\frac{1}{a^2}(a^2-R^2)^2}\hat{\mathbf{a}}\\ \mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{Raq^2}{(a^2-R^2)^2}\hat{\mathbf{a}}\end{aligned}$$

Now we can determine the energy by calculating the work it takes to bring q from infinity: The line element is $\mathrm{d}\ell = \mathrm{d}a\hat{\mathbf{a}}$ since the force is in the negative a direction; so the work required to *oppose* the force is

$$\begin{aligned}W &= -\int_\infty^a \mathbf{F}\cdot \mathrm{d}\ell\\ &= -\frac{1}{4\pi\epsilon_0}Rq^2\int_\infty^a \frac{a'}{(a'^2-R^2)^2}(-\,\mathrm{d}a')\\ \text{using}\quad & u = a'^2 - R^2; \quad \mathrm{d}u = 2a'\,\mathrm{d}a'\\ &= \frac{1}{4\pi\epsilon_0}Rq^2\int\frac{1}{2u^2}\,\mathrm{d}u\\ &= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2u}\right]\\ &= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2(a^2-R^2)}\right]\Bigg|_\infty^a\\ &= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2(a^2-R^2)}-0\right]\end{aligned}$$

which simplifies to

$$\boxed{W = -\frac{1}{4\pi\epsilon_0}\frac{Rq^2}{2(a^2-R^2)}}$$

3.10 For a second image charge q'' inside the center of the sphere (it must not be outside the sphere) with potential

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathcal{r}} + \frac{q'}{\mathcal{r}'}\right) + V_0$$

where

$$V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{\mathcal{r}''} = \frac{1}{4\pi\epsilon_0}\frac{q''}{R} \implies q'' = 4\pi\epsilon_0 R V_0$$

So for a neutral conducting sphere the potential should be zero at the surface, i.e. the magnitude of the image charges q' and q'' are equal and opposite:

$$q' = -q''$$

The distance from the second image charge and q is a, so

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{qq''}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0} q\left[\frac{q'a^2}{(a^2-R^2)^2} - \frac{q'}{a^2}\frac{(a^2-R^2)^2}{(a^2-R^2)^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a^2-R^2)^2}\left[a^2 - a^2 + 2R^2 - R^4/a^2\right] \end{aligned}$$

Using Eq. (3.15) $q' = -\frac{R}{a}q$

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{(a^2-R^2)^2}\left(\frac{-R}{a}\right)\left[2R^2 - R^4/a^2\right] \\ &= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(a^2-R^2)^2}\left(\frac{R^3}{a^3}\right)\left[2a^2 - R^2\right] \end{aligned}$$

So the force of attraction has magnitude

$$\boxed{F_{\text{att}} = \frac{1}{4\pi\epsilon_0}\frac{q^2 R^3}{a^3(a^2-R^2)^2}\left[2a^2 - R^2\right]}$$

3.11 Force between point charge q and spherical conductor of total charge q:
We can use a second image charge (at the center of the sphere) where

$$q'' + q' = q$$

So the force between q and the conductor is

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{qq''}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}q\left[\frac{q'a^2}{(a^2-R^2)^2} + \frac{q-q'}{a^2}\right] \end{aligned}$$

Using Eq. (3.15) $q' = -\frac{R}{a}q$:

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}q\left[\frac{-Rqa}{(a^2-R^2)^2} + \frac{q+Rq/a}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}q^2\left[-\frac{Ra}{(a^2-R^2)^2} + \frac{a+R}{a^3}\right] \end{aligned}$$

So the force is attractive when $[\,] < 0$, or if we define the critical value a_c where

$$\begin{aligned} \frac{Ra_c}{(a_c^2-R^2)^2} &= \frac{a_c+R}{a_c^3} \\ Ra_c^4 &= (a_c^2-R^2)^2(a_c+R) \\ Ra_c^4 &= (a_c^4 - 2a_c^2R^2 + R^4)(a_c+R) \\ Ra_c^4 &= a_c^5 - 2a_c^3R^2 + R^4a_c + Ra_c^4 - 2a_c^2R^3 + R^5 \\ 0 &= a_c^5 - 2a_c^3R^2 - 2a_c^2R^3 + R^4a_c + R^5 \end{aligned}$$

From the hint, the solution must be in the form

$$a_c = R\frac{1+\sqrt{5}}{2}$$

which is the golden ratio i.e. in quadratic form

$$\phi^2 - \phi - 1 = 0 \implies \phi = \frac{1+\sqrt{5}}{2}$$

So

$$a_c = R\phi \implies \phi = \frac{a_c}{R}$$

With this intuition, we can divide our quintic equation by R^5:

$$\begin{aligned} 0 &= \frac{a_c^5}{R^5} - 2\frac{a_c^3}{R^3} - 2\frac{a_c^2}{R^2} + \frac{a_c^4}{R^4} + 1 \\ &= \phi^5 - 2\phi^3 - 2\phi^2 + \phi + 1 \end{aligned}$$

then we can factor it by dividing by the golden ratio equation $\phi^2 - \phi - 1 = 0$ (using polynomial long division):

$$\begin{array}{r|rrrrrr}
 & & & x^3 & +x^2 & & -1 \\ \hline
x^2-x-1 & x^5 & & -2x^3 & -2x^2 & +x & +1 \\
 & -x^5 & +x^4 & +x^3 & & & \\ \hline
 & & x^4 & -x^3 & -2x^2 & & \\
 & & -x^4 & +x^3 & +x^2 & & \\ \hline
 & & & & -x^2 & +x & +1 \\
 & & & & x^2 & -x & -1 \\ \hline
 & & & & & & 0
\end{array}$$

Thus

$$0 = (\phi^2 - \phi - 1)(\phi^3 + \phi^2 - 1)$$

Where the quadratic equation factors to the golden ratio (positive root),

$$\phi^2 - \phi - 1 = 0$$
$$\implies \phi = \frac{1+\sqrt{5}}{2} = \frac{a_c}{R}$$
$$\implies a_c = R\frac{1+\sqrt{5}}{2}$$

So when $a < a_c$, $[\,] < 0$ i.e.

$$F = \frac{1}{4\pi\epsilon_0} q^2 [\,]$$

thus the magnitude of the force $F < 0$ which implies that the force is attractive.

Furthermore, in the cubic equation, there is a real root at approximately $\phi = 0.75488$ (using desmos/root-finding calculator)

$$\implies a_{c2} \approx 0.75488R$$

So, at $a < 0.75488R$ (pretending $\phi = a/R$)

$$(\phi^2 - \phi - 1)(\phi^3 + \phi^2 - 1) = (-C_1)(-C_2) = +C_3 = [\,] > 0$$

where C_1, C_2, C_3 are constants. So the force becomes repulsive again when $a < 0.75488R$.

3.13 Two semi-infinite grounded conducting planes meeting as shown in Fig. 3.21 To set up the

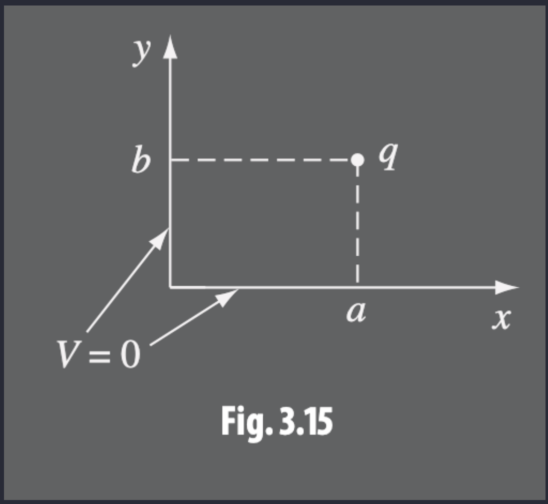

Figure 3.21: From Griffiths

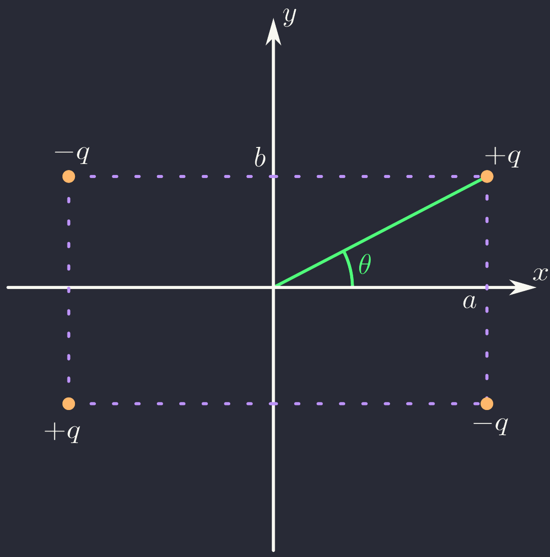

Figure 3.22: Three image charges

problem so we can have a potential of zero at the planes, we can place image charges $-q$ at $(a, -b)$ and $(-a, b)$, and place an image charge $+q$ at $(-a, -b)$ to balance the potentials at the axes.

The potential in the region $x > 0, y > 0$ is:

$$V = \frac{1}{4\pi\epsilon_0} q \left[\frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2 + z^2}} \right.$$
$$\left. - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} \right]$$

The force on q is (using Fig. 3.22):

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} q^2 \left[-\frac{1}{(2a)^2}\hat{\mathbf{x}} - \frac{1}{(2b)^2}\hat{\mathbf{y}} \right.$$
$$\left. + \frac{1}{(2\sqrt{a^2+b^2})^2} (\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}) \right]$$

where

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

So

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4} \left[-\frac{1}{a^2}\hat{\mathbf{x}} - \frac{1}{b^2}\hat{\mathbf{y}} + \left(\frac{a}{(a^2+b^2)^{3/2}}\hat{\mathbf{x}} + \frac{b}{(a^2+b^2)^{3/2}}\hat{\mathbf{y}} \right) \right]$$

$$\boxed{\mathbf{F} = \frac{q^2}{16\pi\epsilon_0} \left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2} \right)\hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2} \right)\hat{\mathbf{y}} \right]}$$

The work done to bring q from infinity to the origin:
Integrating the opposing force from $\infty \to (a, b)$

$$\begin{aligned} W &= -\int_{\infty}^{(a,b)} \mathbf{F} \cdot \mathrm{d}\ell \\ &= -\left[\int_{(\infty,\infty)}^{(a,\infty)} \mathbf{F} \cdot \mathrm{d}\ell_a + \int_{(a,\infty)}^{(a,b)} \mathbf{F} \cdot \mathrm{d}\ell_b\right] \end{aligned}$$

where

$$\begin{aligned} \mathrm{d}\ell_a = \mathrm{d}x\hat{\mathbf{x}} + (0)\hat{\mathbf{y}} = \mathrm{d}x\hat{\mathbf{x}}, \quad \mathrm{d}\ell_b = (0)\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}} = \mathrm{d}y\hat{\mathbf{y}} \\ \implies \int_{\infty,\infty}^{(a,\infty)} \mathbf{F} \cdot \mathrm{d}\ell_a = \frac{q^2}{16\pi\epsilon_0}\int_{\infty}^{a}\left[\left(\frac{x}{(x^2+b^2)^{3/2}} - \frac{1}{x^2}\right)\mathrm{d}x\right]\Bigg|_{b=\infty} \\ = \frac{q^2}{16\pi\epsilon_0}\int_{\infty}^{a}\left[-\frac{1}{x^2}\mathrm{d}x\right] \\ \text{and} \quad \mathbf{F} \cdot \mathrm{d}\ell_b = \frac{q^2}{16\pi\epsilon_0}\int_{\infty}^{b}\left[\left(\frac{y}{(a^2+y^2)^{3/2}} - \frac{1}{y^2}\right)\mathrm{d}y\right] \end{aligned}$$

So the work done is

$$W = -\frac{q^2}{16\pi\epsilon_0}\left[\int_{\infty}^{a}\left(-\frac{1}{a^2}\right)\mathrm{d}x + \int_{\infty}^{b}\left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\mathrm{d}y\right]$$

Evaluating the two integrals using

$$\begin{aligned} \int -\frac{1}{x^2}\mathrm{d}x = \frac{1}{x} \\ \int \frac{y}{(a^2+y^2)^{3/2}}\mathrm{d}y = -\frac{1}{\sqrt{a^2+y^2}} \end{aligned}$$

gives the work done is

$$W = -\frac{q^2}{16\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{\sqrt{a^2+b^2}} + \frac{1}{b}\right]$$

or

$$\boxed{W = \frac{q^2}{16\pi\epsilon_0}\left[\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right]}$$

We can solve the problem with the method of images, as long as the angle ϕ divides $180°$ into an integer, e.g., $\phi = 180, 90, 60, 45, 36, 30, 20, 18, 15, 12, 10, 9, 6, 5, 4, 3, 2, 1, 0.5, \ldots$

We would place a 'mirror' at each ϕ division and place an image charge $-q$ that mirrors the point charge q and repeat the process with the next image charge q (making sure to flip charges each time) until we have a symmetric configuration of charges as shown in Fig. 3.23.

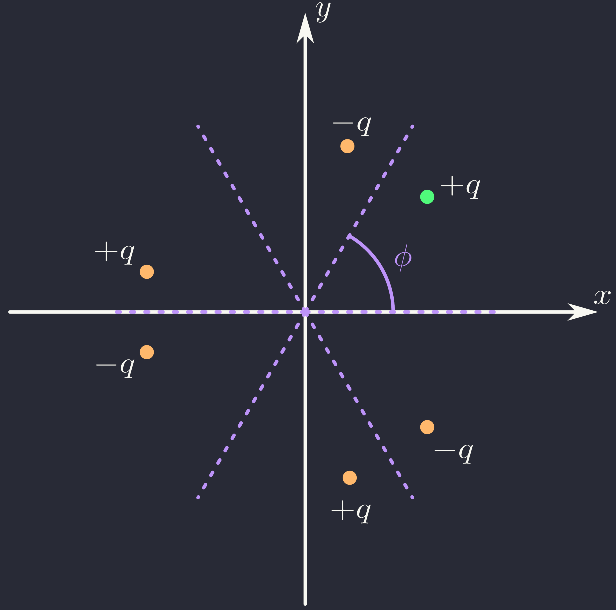

Figure 3.23: Method of images for $\phi = 60°$