Lecture 24: **3/25/24**

Solid Body Rotation

Last Week: Non-inertial Frames

1. Just linear acceleration $\mathbf{A}$, N2L

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}$$

2. Rotating frame:

$$m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \Omega + m(\Omega \times \mathbf{r}) \times \Omega$$

Solid body

N particles on a continuous distribution

$$m_\alpha, \qquad \alpha = 1, 2, \ldots, N$$
$$\mathbf{r}_\alpha, \qquad \mathbf{r}_\alpha - \mathbf{r}_\beta = \text{constant}$$

With a center of mass (COM/CM)

$$\mathbf{R} = \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}_\alpha, \qquad M = \sum_\alpha m_\alpha$$
$$\mathbf{P} = \sum_\alpha m_\alpha \mathbf{v}_\alpha = \sum_\alpha m\alpha \dot{\mathbf{r}}_\alpha = M\dot{\mathbf{R}}$$
$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F}_{\text{ext}}$$

Angular Momentum

$$\begin{aligned}\ell_\alpha &= \mathbf{r}_\alpha \times \mathbf{p}_\alpha \\ &= \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha\end{aligned}$$

and the total angular momentum

$$\mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha$$

Defining a position $\mathbf{r}'_\alpha$ relative to the CM

$$\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}, \quad \mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$$

we can rewrite the total angular momentum as

$$\begin{aligned}\mathbf{L} &= \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}'_\alpha) \times (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha) \\ &= \sum_\alpha m_\alpha \mathbf{R} \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{R} \times \dot{\mathbf{r}}'_\alpha + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha\end{aligned}$$

but since we know that

$$\begin{aligned}\mathbf{R} &= \frac{1}{M}\sum_\alpha m_\alpha(\mathbf{R} + \mathbf{r}'_\alpha) \\ &= \frac{1}{M}\sum_\alpha m_\alpha \mathbf{R} + \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}'_\alpha \\ \implies \sum_\alpha m_\alpha \mathbf{r}'_\alpha &= 0 \\ \sum_\alpha m_\alpha \dot{\mathbf{r}}'_\alpha &= 0\end{aligned}$$

so the middle terms of the total angular momentum are zero:

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha$$

which can be re-expressed as

$$\begin{aligned}
\mathbf{L} &= \mathbf{L}_{\text{cm}} + \mathbf{L}_{\text{rel}} \\
\mathbf{L}_{\text{cm}} &= M\mathbf{R} \times \dot{\mathbf{R}} \\
\mathbf{L}_{\text{rel}} &= \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha
\end{aligned}$$

For example we can consider the earth as a rigid body with angular momentum

$$\mathbf{L}_E = \mathbf{L}_{\text{spin}} + \mathbf{L}_{\text{orb}}$$

Time derivative of angular momentum we have two parts

$$\begin{aligned}
\dot{\mathbf{L}}_{\text{cm}} &= M\dot{\mathbf{R}} \times \dot{\mathbf{R}} + M\mathbf{R} \times \ddot{\mathbf{R}} \\
&= M\mathbf{R} \times \mathbf{F}_{\text{ext}} = \mathbf{\Gamma}_{\text{cm}}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\mathbf{L}}_{\text{rel}} &= \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{r}}'_\alpha, \quad \ddot{\mathbf{r}}'_\alpha = \ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}} \\
&= \mathbf{\Gamma}_{\text{rel}}
\end{aligned}$$

Energy The kinetic energy of the system is

$$\begin{aligned}
T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha)^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha (\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{r}}'^2_\alpha) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}'^2_\alpha
\end{aligned}$$

and the potential energy is

$$U = U_{\text{ext}} + U_{\text{int}} = U_{\text{ext}}$$

where there is no relative motion between the particles, the internal potential energy is a constant which can be ignored.

Example: Rotating disk We consider a disk rotating about the z-axis with angular velocity

$$\omega = (0, 0, \omega)$$

with a particle with position and velocity

$$\begin{aligned}
\mathbf{r}_\alpha &= (x_\alpha, y_\alpha, z_\alpha) \\
\dot{\mathbf{r}}_\alpha &= (\dot{x}_\alpha, \dot{y}_\alpha, \dot{z}_\alpha)
\end{aligned}$$

the time derivative of the position vector is

$$\dot{\mathbf{r}}_\alpha = \omega \times \mathbf{r}_\alpha = (-\omega y_\alpha, \omega x_\alpha, 0)$$

and the angular momentum is

$$\begin{aligned}\ell_\alpha &= m_\alpha \mathbf{r}_\alpha \times \dot{\mathbf{r}}_\alpha = m_\alpha \mathbf{r}_\alpha \times (\omega \times \mathbf{r}_\alpha) \\ &= m_\alpha(-\omega x_\alpha z_\alpha, -\omega y_\alpha z_\alpha, \omega(x_\alpha^2 + y_\alpha^2))\end{aligned}$$

thus the z component of total angular momentum is

$$L_z = \sum_\alpha m_\alpha \ell_{\alpha,z} = \sum_\alpha m_\alpha \omega(x_\alpha^2 + y_\alpha^2) = \omega \sum_\alpha m_\alpha \rho_\alpha^2 = \omega I_z$$

where ρ is radius in cylindrical coordinates and I_z is the moment of inertia about the z-axis (parallel axis theorem). The other two components of angular momentum are

$$\begin{aligned}L_x &= -\sum_\alpha m_\alpha \omega x_\alpha z_\alpha \\ L_y &= -\sum_\alpha m_\alpha \omega y_\alpha z_\alpha\end{aligned}$$

and since L_x and L_y can be nonzero, that means that $\mathbf{L}$ can be in any direction! If we define the products of inertia

$$\begin{aligned}I_{xz} &= -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ I_{yz} &= -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ I_{zz} &= \sum_\alpha m_\alpha (x_\alpha + y_\alpha)^2\end{aligned}$$

we define the total angular momentum as

$$\begin{aligned}\mathbf{L} &= I \cdot \omega \\ &= (I_{xz} \cdot \omega_z, I_{yz} \cdot \omega_z, I_{zz} \cdot \omega_z)\end{aligned}$$