1 Energy

Review: There are two requirements for conservation of angular momentum

1. Force is central
2. External torque is zero

Kinetici Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$. Taking the time derivative

$$\begin{aligned}\dot{T} &= \frac{1}{2}(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) \\ &= m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}\end{aligned}$$

and integrating over time t_1 to t_2

$$\int_{t_1}^{t_2} \dot{T}\,\mathrm{d}t = \Delta T = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = W(1\to 2)$$

since $\mathbf{v}\cdot\mathrm{d}t = \mathrm{d}\mathbf{r}$ and $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ hints that this is a line integral.

Example:

$$\begin{aligned}\mathbf{F}(x,y) &= \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ \mathrm{d}vbr &= \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}}\end{aligned}$$

(a) $y = x$ from $a = (0,0)$ to $b = (1,1)$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + y\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 x\,\mathrm{d}x = 1\end{aligned}$$

(b) $y = x^2$ and $\mathrm{d}y = 2x\,\mathrm{d}x$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + x^2\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 2x^2\,\mathrm{d}x = 1\end{aligned}$$

thus the line integral is independent of the path.

Conservative force

1. Given $\mathbf{F}(\mathbf{r})$, there is no dependence on $\mathbf{v}$, t.
2. $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is independent of the path.

$$U(\mathbf{r}) = -W(\mathbf{r}_0\to\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}'$$

For gravity

$$U_g(\mathbf{r}) = -\int_a^b \mathbf{F}_g\cdot\mathrm{d}\mathbf{r} = -\int_a^b -mg\,\mathrm{d}y' = mg(y_a - y_b)$$

Work-Kinetic Energy Theorem:

$$W(1 \to 2) = W(1 \to \mathcal{O}) + W(\mathcal{O} \to 2) = U(1) - U(2) = \Delta T$$

From this we have

$$\Delta T = T_2 - T_1 = U_1 - U_2$$

rearranging terms give the mechanical energy

$$T_2 + U_2 = T_1 + U_1 = E$$

and for N conservative forces in a system

$$E = T + U_1 + U_2 + \cdots + U_N$$

Energy: Part 2

Conservative Force: Potential Energy The mechanical energy

$$E = T + U$$

is made up of the sum of the kinetic and potential energy. This is useful for

- obtaining equations of motion (EOM)

e.g. finding the EOM for a simple pendulum of mass m, length L and initial angle θ. The kinetic energy is

$$T = \frac{1}{2}mL^2\dot{\theta}^2$$

where the magnitude of velocity is the tangential component $v = L\omega = L\dot{\theta}$. The potential energy is

$$U = -mgy = -mgL\cos\theta$$

and the conservation of energy tells us that the mechanical energy is constant

$$\begin{aligned} T + U &= \text{constant} = E \\ \frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta &= E \end{aligned}$$

and in the intial condition we know that the velocity is zero $\dot{\theta} = 0$ and thus

$$-mgL\cos\theta_{max} = E$$

taking the time derivative of the energy equation gives

$$\begin{aligned} mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\dot{\theta} &= 0 \\ \ddot{\theta} + \frac{g}{L}\sin\theta &= 0 \\ \ddot{\theta} &= -\frac{g}{L}\sin\theta \end{aligned}$$

taking the time derivative of the energy is a useful trick for finding the EOM when we are trying to solve for $\dot{v}^2$.

Last time we found the potential energy for a position $\mathbf{r}$ in a conservative force field $\mathbf{F}(\mathbf{r})$ is

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}'$$

from the fundamental theorem of calculus (derivatives and intergrals are inverses) we want to find a function where the derivative equals the conservative force: First we take an infinitesimal change in the position $\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}$ and the change in potential energy is

$$\begin{aligned} U(\mathbf{r} + \mathrm{d}\mathbf{r}) &= -\int_{\mathbf{r}_0}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{\mathbf{r}}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= U(\mathbf{r}) - F(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \end{aligned}$$

where is know that the force is constant over a small distance. Moving the terms gives

$$\begin{aligned}U(\mathbf{r}+\mathrm{d}\mathbf{r})-U(\mathbf{r}) &= -\mathbf{F}\cdot\mathrm{d}\mathbf{r}\\ &= -(F_x\,\mathrm{d}x+F_y\,\mathrm{d}y+F_z\,\mathrm{d}z)\end{aligned}$$

where we use Cartesian Coordinates, and we know that the gradient of potential is

$$\begin{aligned}\boldsymbol{\nabla}U &= \hat{\mathbf{x}}\frac{\partial U}{\partial x}+\hat{\mathbf{y}}\frac{\partial U}{\partial y}+\hat{\mathbf{z}}\frac{\partial U}{\partial z}\\ &= -(F_x\hat{\mathbf{x}}+F_y\hat{\mathbf{y}}+F_z\hat{\mathbf{z}}) = -\mathbf{F}\end{aligned}$$

Example 3: 1D motion If we know what U is as a function of x, we can find the force! At points where $E=U$ we call these classical turning points. At a region of a relative minimum, a particle below the threshold of the turning point will oscillate between the two turning points. And at $E>U_max$ the particle is unbound and will escape the forces that attracted it.

Example 4:

$$\begin{aligned}E &= T+U(x)\text{ is constant}\\ T &= \frac{1}{2}m\dot{x}^2 = E-U(x)\\ \dot{x}^2 &= \frac{2}{m}(E-U(x))\\ \dot{x} &= \pm\sqrt{\frac{2}{m}(E-U(x))}\end{aligned}$$

using seperation of variables

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\frac{2}{m}(E-U(x))}\\ \sqrt{\frac{m}{2}}\,\mathrm{d}t &= \frac{\mathrm{d}x}{\sqrt{E-U(x)}}\\ \int_{t_1}^{t_2}\sqrt{\frac{m}{2}}\,\mathrm{d}t &= \int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E-U(x)}}\\ (t_2-t_1) &= \sqrt{\frac{2}{m}}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E-U(x)}}\end{aligned}$$

where the sign of the velocity is positive within the oscillating bounds of the turning points and changes sign as the particle moves past the turning point.

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Energy: Part 3

Last time: Conservative force as a negative gradient of potential:

$$\mathbf{F} = -\boldsymbol{\nabla} U$$

with classical turning points at $E = U$.

Conditions of a conservative force

- Only depends on position $\mathbf{r}$ (or just constant)
- Work done is path independent (this is sometimes hard to check) $\Leftrightarrow \boldsymbol{\nabla} \times \mathbf{F} = 0$

What is curl? In 3D Cartesian coordinates

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\hat{\mathbf{x}} + \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\hat{\mathbf{y}} + \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\hat{\mathbf{z}}() \right)\end{aligned}$$

Mathematically we know that if the force vector is a gradient of a scalar potential

$$\mathbf{F} = \boldsymbol{\nabla}\phi = -\boldsymbol{\nabla} U \quad \Leftrightarrow \quad \boldsymbol{\nabla} \times \mathbf{F} = 0$$

The curl of a gradient is always zero! Short 'proof':

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

and the curl

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial U}{\partial x} & -\frac{\partial U}{\partial y} & -\frac{\partial U}{\partial z} \end{vmatrix} = 0$$

Proving that the line integral is path independent is a little difficult, but in general we know that the two different paths a and b from points 1 to 2 we can write the work as

$$\int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_2 - \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_1 = \oint_{a-b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_A (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = 0$$

where we invoke Stokes' Theorem to find the integral of the curl over the surface A is zero.

Conservative Force: $\mathbf{F} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is conservative, but is

$$\mathbf{F}(\mathbf{r}) = F(\mathbf{r})\hat{\mathbf{r}}$$

a central force always conservative? We will need to check the curl of the force to find out. But first we define spherical coordinates (r, θ, ϕ)

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

and the central force in spherical coordinates is

$$\mathbf{F} = F(\mathbf{r})\hat{\mathbf{r}} + 0\hat{\theta} + 0\hat{\phi}$$

the curl in spherical coordinates is

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial\phi}\right]\hat{\mathbf{r}} \\ + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\hat{\theta} \\ + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]\hat{\phi}\end{aligned}$$

And since

$$\frac{\partial F}{\partial\theta} = \frac{\partial F}{\partial\phi} = 0$$

the curl is zero $\boldsymbol{\nabla} \times \mathbf{F} = 0$ and thus $\mathbf{F}$ is a conservative central force.

Gravity Conservative? The force due to gravity

$$\mathbf{F}_g = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}$$

is a central force as it only depends on $\mathbf{r}$. e.g. for a a two mass system:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Using Translational Invariance, we can shift the origin to the center of m_2

$$\mathbf{r}_2 = 0 \quad \mathbf{F}_{12} = -\frac{Gm_1m_2}{\mathbf{r}_1^3}\mathbf{r}_1$$

or

$$F_{12} = -\boldsymbol{\nabla}_1 U = -(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial z_1})$$

where the potential energy due to the interaction between 1 and 2 is

$$U_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

from newtons 3rd law, the force on 2 due to 1 is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

so

$$\begin{aligned}-\boldsymbol{\nabla}_1 U_{12} \to \mathbf{F}_{21} = \boldsymbol{\nabla}_1 U_{12} \\ \boldsymbol{\nabla}_1 U_{12}(\mathbf{r}_1 0 \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U_{12}(\mathbf{r}_2 0 \mathbf{r}_1) \\ u_{12}(\mathbf{x}) \quad \mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2 \\ \boldsymbol{\nabla}_1 U_{12}(\mathbf{x}) = \boldsymbol{\nabla}_x U_{12}(\mathbf{x}) = -\boldsymbol{\nabla}_2 U_{12}(\mathbf{x})\end{aligned}$$

so

$$\mathbf{F}_{12} = -\boldsymbol{\nabla}_1 U_{12} \qquad \mathbf{F}_{21} = -\boldsymbol{\nabla}_2 U_{12}$$

and for N particles

$$\mathbf{F}_i = -\boldsymbol{\nabla}_i U \qquad U = \sum_{i,j} U_{ij} + \sum_i U_i^{\text{ext}}$$