2.22 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

The electric field outside the sphere is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

and from Problem 2.8, the electric field inside the sphere is

$$\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$$

For points outside the sphere ($r > R$),

$$V(r) = -\int_{\infty}^{r}\mathbf{E}\cdot\mathrm{d}\mathbf{l} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\Big|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

For points inside the sphere ($r < R$),

$$\begin{aligned} V(r) &= -\int_{\infty}^{R}\mathbf{E}\cdot\mathrm{d}\mathbf{l} - \int_{R}^{r}\mathbf{E}\cdot\mathrm{d}\mathbf{l} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\int_{R}^{r} r'\,\mathrm{d}r' \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\left(\frac{r'^2}{2}\right)\Big|_{R}^{r} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right) \end{aligned}$$

The gradient of V for $r > R$:

$$-\boldsymbol{\nabla}V = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} = \mathbf{E}_{out}$$

and for $r < R$:

$$-\boldsymbol{\nabla}V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}} = \mathbf{E}_{in}$$

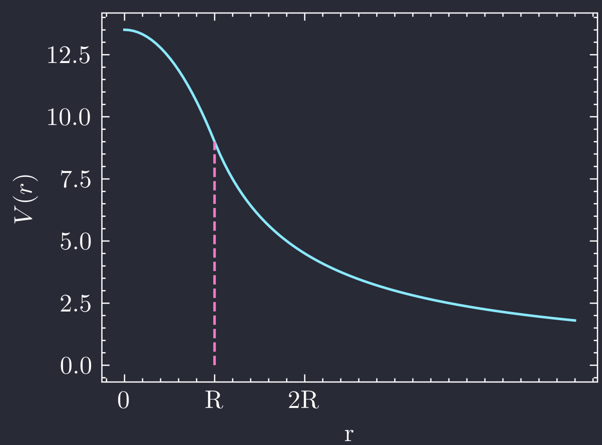

Figure 2.1: Plot of $V(r)$ as a function of r where $q = 1\,\mathrm{nC}$ and $R = 1\,\mathrm{m}$.

2.26 From Griffiths

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i} \tag{2.27}$$

and

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,d\ell' \quad \text{and} \quad V = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,da' \tag{2.30}$$

(a.1) Two point charges $+q$ a distance d apart: Find the potential a distance z above the center of the charges: Using Eq. (2.27), the potential is

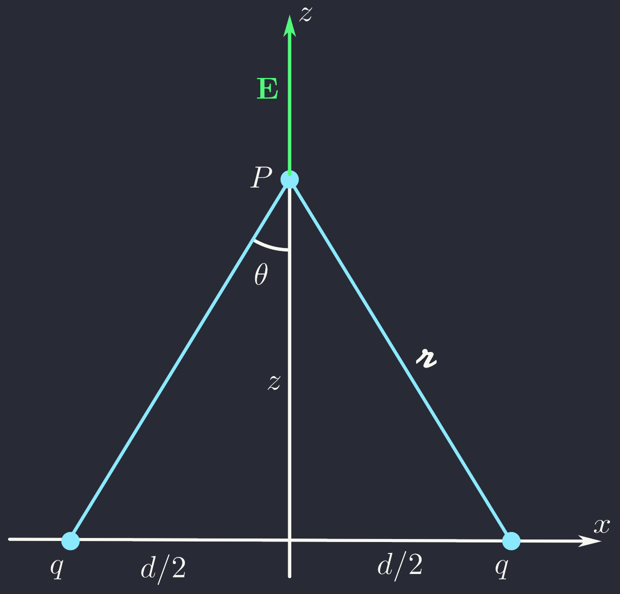

Figure 2.2: Two point charges $+q$ a distance d apart.

$$V_a = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{z^2+\frac{d^2}{4}}}+\frac{q}{\sqrt{z^2+\frac{d^2}{4}}}\right)$$

$$\boxed{V_a = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{z^2+\frac{d^2}{4}}}}$$

(a.2) Computing the electric field $\mathbf{E} = -\boldsymbol{\nabla}V$:

$$\begin{aligned}\mathbf{E}_a &= -\frac{\partial V_a}{\partial z}\hat{\mathbf{z}} \\ &= -\frac{1}{4\pi\epsilon_0}\frac{-1}{2}\frac{2q(2z)}{\left(z^2+\frac{d^2}{4}\right)^{3/2}}\hat{\mathbf{z}}\end{aligned}$$

simplifying to

$$\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2+\frac{d^2}{4}\right)^{3/2}}\hat{\mathbf{z}}}$$

which is the same as Ex. 2.1

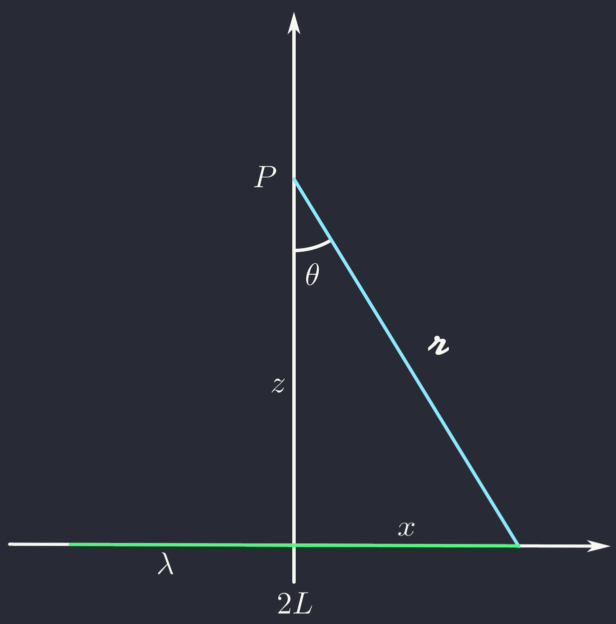

Figure 2.3: A line charge of density λ.

(b.1) Using Eq. (2.30), the potential is

$$V_b = \frac{1}{4\pi\epsilon_0}\lambda \int_{-L}^{L} \frac{1}{\sqrt{z^2+x^2}}\,\mathrm{d}x$$

To solve the integral, we can use the substitution from the trig identity

$$\begin{aligned}\cosh^2 u - \sinh^2 u &= 1\\ \implies z^2\cosh^2 u &= z^2 + z^2\sinh^2 u\\ &= z^2 + x^2\end{aligned}$$

where

$$\begin{aligned}x = z\sinh u &\implies u = \operatorname{arcsinh}\frac{x}{z}\\ \mathrm{d}x &= z\cosh u\,\mathrm{d}u\end{aligned}$$

Thus the integral becomes

$$\begin{aligned}V_b &= \frac{1}{4\pi\epsilon_0}\lambda\int \frac{\cancel{z\cosh u}}{\cancel{z\cosh u}}\,\mathrm{d}u\\ &= \frac{1}{4\pi\epsilon_0}\lambda u\Big|_{-L}^{L}\\ &= \frac{1}{4\pi\epsilon_0}\lambda\left[\operatorname{arcsinh}\frac{L}{z} - \operatorname{arcsinh}\frac{-L}{z}\right]\end{aligned}$$

Using $\operatorname{arcsinh}(a) = \ln\left|a+\sqrt{a^2+1}\right|$:

$$\begin{aligned}\implies \operatorname{arcsinh}(\frac{L}{z}) &= \ln\left|\frac{L}{z} + \sqrt{\left(\frac{L}{z}\right)^2+1}\right|\\ &= \ln\left|\frac{1}{z}(L+\sqrt{L^2+z^2})\right|\end{aligned}$$

so the potential is

$$\boxed{V_b = \frac{1}{4\pi\epsilon_0}\lambda \ln\left|\frac{L+\sqrt{L^2+z^2}}{-L+\sqrt{L^2+z^2}}\right|}$$

(b.2) The electric field is

$$\begin{aligned}
\mathbf{E}_b &= -\frac{\partial V_b}{\partial z}\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\left[\frac{1}{L+\sqrt{L^2+z^2}}\left(\frac{1}{2}\frac{2z}{\sqrt{L^2+z^2}}\right) - \frac{1}{-L+\sqrt{L^2+z^2}}\left(\frac{1}{2}\frac{2z}{\sqrt{L^2+z^2}}\right)\right]\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\frac{z}{\sqrt{L^2+z^2}}\left[\frac{-L+\sqrt{L^2+z^2}}{-L^2+(L^2+z^2)} - \frac{L+\sqrt{L^2+z^2}}{-L^2+(L^2+z^2)}\right]\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\frac{-2Lz}{z^2\sqrt{L^2+z^2}}\hat{\mathbf{z}}
\end{aligned}$$

simplifying to

$$\boxed{\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{L^2+z^2}}\hat{\mathbf{z}}}$$

which is the same as Ex. 2.2

(c.1) Using Eq. (2.30) and polar coordinates, the potential is

$$\begin{aligned}
V_c &= \frac{1}{4\pi\epsilon_0}\sigma\int_0^{2\pi}\int_0^R \frac{1}{\sqrt{z^2+r^2}}r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma\int_0^R \frac{r}{\sqrt{z^2+r^2}}\,\mathrm{d}r
\end{aligned}$$

substituting $u = z^2 + r^2$; $\mathrm{d}u = 2r\,\mathrm{d}r$:

$$\begin{aligned}
V_c &= \frac{1}{4\pi\epsilon_0}\pi\sigma\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{4\pi\epsilon_0}\pi\sigma 2\sqrt{z^2+r^2}\Big|_0^R
\end{aligned}$$

thus

$$\boxed{V_c = \frac{\sigma}{2\epsilon_0}\left[\sqrt{z^2+R^2} - z\right]}$$

(c.2) The electric field is

$$\begin{aligned}
\mathbf{E}_c &= -\frac{\partial V_c}{\partial z}\hat{\mathbf{z}} \\
&= -\frac{\sigma}{2\epsilon}\left[\frac{z}{\sqrt{z^2+R^2}} - 1\right]\hat{\mathbf{z}}
\end{aligned}$$

thus

$$\boxed{\mathbf{E}_c = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}}$$

which is the same as Problem 2.6:

2.6 The electric field is only in the z-direction where $\cos\theta = z/\mathscr{r}$:

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}^2}\cos\theta\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a}\\ &= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a}\end{aligned}$$

Using polar coordinates: since $\mathrm{d}\mathbf{a} = r\,\mathrm{d}r\,\mathrm{d}\theta$

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{\sigma z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \frac{1}{4\pi\epsilon_0}\sigma z(2\pi)\hat{\mathbf{z}}\int_0^R \frac{r}{(z^2+r^2)^{3/2}}\,\mathrm{d}r\\ &= \frac{\sigma}{2\epsilon_0}z\hat{\mathbf{z}}\left[-\frac{1}{\sqrt{z^2+r^2}}\right]_0^R\\ &= \frac{\sigma}{2\epsilon_0}z\left[\frac{1}{z}-\frac{1}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}\\ \mathbf{E} &= \frac{\sigma}{2\epsilon_0}\left[1-\frac{1}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}\end{aligned}$$

(d) if the right-hand charge of Fig. 2.2 is replaced by a charge $-q$, the potential at P using Eq. (2.27) is

$$V_d = 0 \implies \mathbf{E}_d = 0$$

which contradicts the result from Prob 2.2. This is because point P does not give us any information about the electric field which points in the x-direction. In fact any reference point on the z-axis will give us the same result.

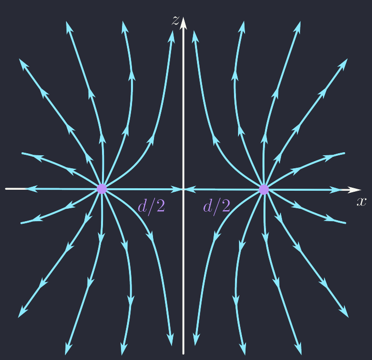

Figure 2.4: E-field for (a)

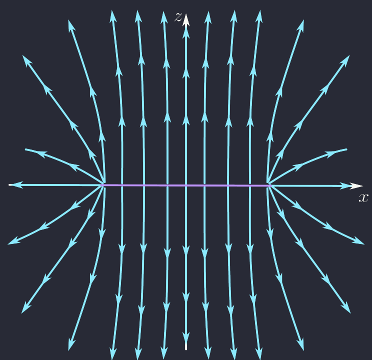

Figure 2.5: E-field for (b)

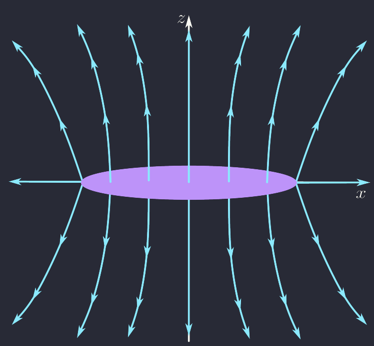

Figure 2.6: E-field for (c)

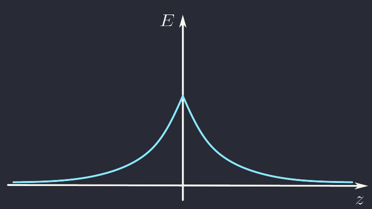

Figure 2.7: E-field for (c) E vs z

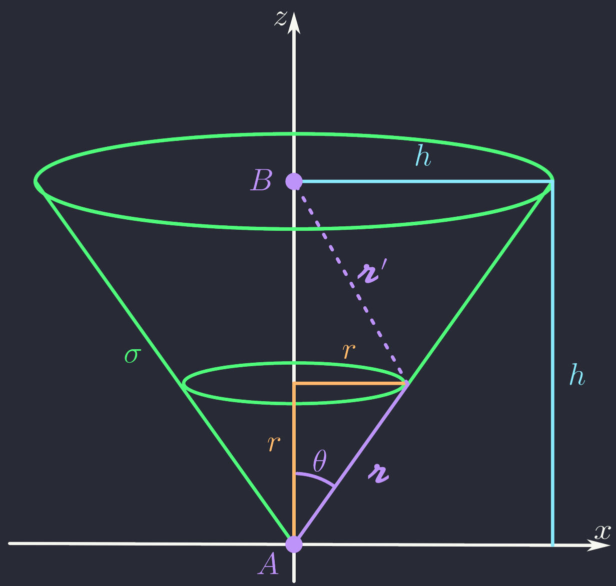

Figure 2.8: Empty ice cream cone with surface charge density σ.

2.27

(i) Potential at A: Geometrically, we can see from the large right triangle that

$$\begin{aligned} \mathscr{r}^2 &= h^2 + h^2 \\ \implies \mathscr{r} &= h\sqrt{2}, \quad h = \frac{\mathscr{r}}{\sqrt{2}} \end{aligned}$$

and from the smaller right triangle

$$\mathscr{r}^2 = 2r^2 \implies r = \frac{\mathscr{r}}{\sqrt{2}}$$

We can find the potential at A using Eq. (2.30) and integrate the rings of the cone along the slant length $0 \to h\sqrt{2}$ which gives us the area element $\mathrm{d}a = 2\pi r \, \mathrm{d}\mathscr{r}$:

$$\begin{aligned} V(A) &= \frac{1}{4\pi\epsilon_0} \int_0^{h\sqrt{2}} \frac{\sigma}{\mathscr{r}} 2\pi r \, \mathrm{d}\mathscr{r} \\ &= \frac{\sigma}{2\epsilon_0\sqrt{2}} \int_0^{h\sqrt{2}} \mathrm{d}\mathscr{r} \\ &= \frac{\sigma}{2\epsilon_0\sqrt{2}} \mathscr{r} \Big|_0^{h\sqrt{2}} \\ V(A) &= \frac{\sigma h}{2\epsilon_0} \end{aligned}$$

(ii) Potential at B: Using the law of cosines,

$$\mathscr{r}'^2 = h^2 + \mathscr{r}^2 - 2h\mathscr{r}\cos\theta$$

where

$$\begin{aligned} \cos\theta &= \frac{h}{\mathscr{r}} \\ &= \frac{h}{h\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \implies \mathscr{r}' &= \sqrt{h^2 + \mathscr{r}^2 - h\mathscr{r}\sqrt{2}} \end{aligned}$$

so the potential at B is

$$\begin{aligned} V(B) &= \frac{1}{4\pi\epsilon_0}\int_0^{h\sqrt{2}} \frac{\sigma}{\mathscr{r}'} 2\pi r \,\mathrm{d}\mathscr{r} \\ &= \frac{\sigma}{2\epsilon_0\sqrt{2}}\int_0^{h\sqrt{2}} \frac{\mathscr{r}}{\sqrt{h^2+\mathscr{r}^2-h\mathscr{r}\sqrt{2}}}\,\mathrm{d}\mathscr{r} \end{aligned}$$

I just used integral-calculator for this one...

$$\begin{aligned} V(B) &= \frac{\sigma}{2\epsilon_0\sqrt{2}}\left[h\sqrt{2}\ln\left(1+\sqrt{2}\right)\right] \\ V(B) &= \frac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right) \end{aligned}$$

Finally the potential difference between A and B is

$$V(B)-V(A) = \frac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right) - \frac{\sigma h}{2\epsilon_0}$$

$$\boxed{V(B)-V(A) = \frac{\sigma h}{2\epsilon_0}\left[\ln\left(1+\sqrt{2}\right)-1\right]}$$

2.35 For a solid sphere radius R and charge q

(a) From Problem 2.22

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

and

$$W = \frac{1}{2}\int \rho V \,\mathrm{d}\tau \tag{2.43}$$

So the energy is

$$\begin{aligned}
W &= \frac{\rho}{2}\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(3 - \frac{r^2}{R^2}\right)r^2\sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{\rho}{2}\frac{1}{4\pi\epsilon_0}\frac{q}{2R}4\pi\int_0^{R}\left(3r^2 - \frac{r^4}{R^2}\right)\mathrm{d}r \\
&= \frac{\rho q}{4R\epsilon_0}\left[r^3 - \frac{r^5}{5R^2}\right]_0^R \\
&= \frac{\rho q}{4R\epsilon_0}\left[R^3 - \frac{R^3}{5}\right] \\
&= \frac{\rho q}{5\epsilon_0}R^2
\end{aligned}$$

where the charge over the volume of the sphere is $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, thus

$$W = \frac{q}{5\epsilon_0}R^2\frac{q}{\frac{4}{3}\pi R^3}$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}}$$

(b) Integrating over all space using

$$W = \frac{\epsilon_0}{2}\int E^2 \,\mathrm{d}\tau \tag{2.45}$$

Where the electric field is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \quad \mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$$

so the energy is

$$\begin{aligned}
W &= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}4\pi q^2\left[\int_0^R \frac{r^2}{R^6}r^2\mathrm{d}r + \int_R^\infty \frac{1}{r^4}r^2\mathrm{d}r\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\int_0^R \frac{r^4}{R^6}\mathrm{d}r + \int_R^\infty \frac{1}{r^2}\mathrm{d}r\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{r^5}{5R^6}\Big|_0^R - \frac{1}{R}\Big|_R^\infty\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{R^5}{5R^6} + \frac{1}{R}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\frac{6}{5R} \\
W &= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}$$

checkmark.

(c) For a spherical volume of radius a and

$$W = \frac{\epsilon_0}{2}\left(\int_V E^2 \mathrm{d}\tau + \oint_S V\mathbf{E}\cdot \mathrm{d}\mathbf{a}\right) \tag{2.44}$$

we can assume the volume is outside the charged sphere so

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

From part (b), the first term is

$$\begin{aligned}\frac{\epsilon_0}{2}\int_V E^2 \mathrm{d}\tau &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{1}{5R} - \frac{1}{a} + \frac{1}{R}\right]\\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{6}{5R} - \frac{1}{a}\right]\end{aligned}$$

the second term is at $r = a$

$$\begin{aligned}\frac{\epsilon_0}{2}\oint_V V\mathbf{E}\cdot \mathrm{d}\mathbf{a} &= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}\int \frac{q}{r}\frac{q}{r^2} r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi\\ &= \frac{4\pi\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}\frac{1}{r}\bigg|_{r=a}\\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\end{aligned}$$

so the total energy is

$$\begin{aligned}W &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{6}{5R} - \frac{1}{a}\right] + \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\\ &= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}\end{aligned}$$

As $a \to \infty$ the $\int V\mathbf{E}\cdot \mathrm{d}\mathbf{a}$ term goes to zero.

2.40 Two cavities radii a and b in a conducting sphere of radius R with a point charge q_a and q_b respectively in each cavity.

(a) Surface charge densities:

On the surface of cavity a the charge density is simply

$$\sigma_a = \frac{-q_a}{4\pi a^2}$$

and

$$\sigma_b = \frac{-q_b}{4\pi b^2}$$

respectively. For the surface of the conducting sphere, the charge density is positive and equal to the superposition of the two charges:

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b) The field outside the conductor is equivalent to a point charge at the center of the sphere with the sum of the charges:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$$

(c) The field in cavity a with respect to the center of the cavity is

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{a^2}\hat{\mathbf{a}}$$

and in cavity b is

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{b^2}\hat{\mathbf{b}}$$

(d) The field due to to the cavity charge is zero in the exterior of the cavity, so there is no Force on q_a or q_b.

(e) If a charge q_c was brought near the conductor from outside, there would be a change in (a) σ_R and (b) $\mathbf{E}_{out}$.

2.48 Net force of of the southern hemisphere extering on the northern hemisphere (solid sphere) with an inside Electric field (Problem 2.8)

$$E_{in} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$$

where the total force is

$$\mathbf{F} = Q\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{R^3}\mathbf{r}$$

Finding the net force exerted by the southern hemisphere: integrate $\mathrm{d}F = \mathbf{F}/V$ over the southern hemisphere:

$$\begin{aligned}
\mathrm{d}F &= \frac{1}{\frac{4}{3}\pi R^3}\frac{1}{4\pi\epsilon_0}\frac{Q^2}{R^3}\mathbf{r}\mathrm{d}\tau \\
&= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}\mathbf{r}\mathrm{d}\tau
\end{aligned}$$

The symmetry of the sphere implies that the Force is only in the z-direction i.e. $F_z = F\cos\theta$, so integrating over the southern hemisphere:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/2}\int_0^R F_z \mathrm{d}\tau &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R r\cos\theta r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}(2\pi)\left(\frac{r^4}{4}\right)\Bigg|_0^R \int_0^{\pi/2} \sin\theta\cos\theta \mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{3Q^2}{32\pi\epsilon_0 R^2}\frac{\sin^2 x}{2}\Bigg|_0^{\pi/2} \\
&= \boxed{\frac{3Q^2}{64\pi\epsilon_0 R^2}}
\end{aligned}$$