Physics 421: Intro to Electrodynamics

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Contents

1 Vector Analysis

1.1 What is a Vector?

In type we use boldface $\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}$, where we we can do some simple operations as such:

- Adding and Subtraction: $\mathbf{A} \pm \mathbf{B} = \mathbf{C}$ or aligning the head to the tail
- Multiplication:
  - Scalar: $\mathbf{A} \to 2\mathbf{A}$
  - Dot Product: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = AB\cos\theta$
  - Cross Product: $\mathbf{A} \times \mathbf{B} = AB\sin\theta$, and $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$

Components of a Vector In 3D space, we often use the familiar Cartesian coordinates, e.g.

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

and we can add components by adding the components:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

and likewise for subtraction. For the dot product:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

or more shortly

$$= \sum_{i,j} A_iB_j\delta_{ij}$$

where δ_{ij} is the Kronecker delta. The cross product is a bit trickier…

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{\mathbf{x}} - (A_xB_z - A_zB_x)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}$$

This can also be written in short form using the Levi-Civita symbol (look it up)

Scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

This is the also the volume of the parallelepiped formed by the three vectors. NOTE that

$$\cancel{(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}}$$

since you can't cross a scalar with a vector.

Vector triple product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Using the BAC-CAB rule.

Some important vectors We define a position vector

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = r\hat{\mathbf{r}}$$

where the unit vector is

$$\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}$$

and an infinitesimal displacement vector

$$\mathrm{d}l = \mathrm{d}x\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}} + \mathrm{d}z\hat{\mathbf{z}}$$

In EM we define a source point $\mathbf{r}'$ (e.g. a charge) and a field point $\mathbf{r}$ that give us the seperation vector

$$\boldsymbol{\mathscr{r}} = \mathbf{r} = \mathbf{r}'$$

with magnitude

$$|\boldsymbol{\mathscr{r}}| = |\mathbf{r} - \mathbf{r}'|$$

and unit vector

$$\hat{\mathbf{r}} = \frac{\mathbf{r} - \mathbf{r}'}{\mathbf{r} - \mathbf{r}'}$$

1.2 Differential Calculus

And ordinary derivative $\frac{\mathrm{d}F}{\mathrm{d}x}$ is a change in $F(x)$ in $\mathrm{d}x$

$$\mathrm{d}F = \left(\frac{\partial F}{\partial x}\right)\mathrm{d}x$$

...geometrically, it's the slope

Gradient for functions of 2 or more variables, generalize for $h(x, y)$

$$\mathrm{d}h = \left(\frac{\partial h}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial h}{\partial y}\right)\mathrm{d}y$$

it's a scalar so $\mathrm{d}h = (\boldsymbol{\nabla} h) \cdot (\mathrm{d}l)$ where

$$\boldsymbol{\nabla} h = \frac{\partial h}{\partial x}\hat{\mathbf{x}} + \frac{\partial h}{\partial y}\hat{\mathbf{y}}$$

In 3D

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

If $\boldsymbol{\nabla} u = 0$, we are at an extremum (max, min, or shoulder/saddle point) Rewriting:

$$\boldsymbol{\nabla} T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)T(x, y, z)$$

wehere we can assume the ∇ as an "operator" acting on T:

1. Scalars like T: $\boldsymbol{\nabla} T$, "grad T", generalized slope
2. Dot product on $\mathbf{V}$: $\boldsymbol{\nabla} \cdot \mathbf{V}$, "divergence" or "div"
3. Cross product : $\boldsymbol{\nabla} \times \mathbf{V}$, "curl" or "rotatation"

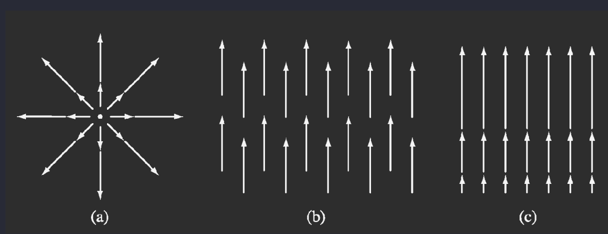

Figure 1.1: Divergence of field lines

Divergence From the Figure, we can see that (a) & (c) diverges, and (b) does not.

Geometrical Interpretation: Sources of positive divergence is a source or "faucet", and negative divergence is a sink or "drain".

Curl

$$\boldsymbol{\nabla} \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

E.g. for $\mathbf{V} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, $\boldsymbol{\nabla} \times \mathbf{V} = 2\hat{\mathbf{z}}$.

Combining Multiple Operations Two ways to get scalar from two functions:

$$fg \quad \text{or} \quad \mathbf{A} \cdot \mathbf{B}$$

Two ways to get vector from two functions:

$$f\mathbf{A} \quad \text{or} \quad \mathbf{A} \times \mathbf{B}$$

And we have 3 'derivatives': div, grad, and curl.

Product rule:

$$\partial_x(fg) = f\partial_x g + g\partial_x f$$

i $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f$

ii $\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + B \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}$

...(see Griffiths for more)

Second Derivatives Combining $\boldsymbol{\nabla}, \boldsymbol{\nabla}\cdot, \boldsymbol{\nabla}\times$

$\boldsymbol{\nabla} T$ is a vector

i

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} T) &= (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}} + \partial_y + \hat{\mathbf{z}}\partial_z)\cdot(\hat{\mathbf{x}}\partial_x T + \hat{\mathbf{y}}\partial_y T + \hat{\mathbf{z}}\partial_z T)\\ &= \partial_x^2 T + \partial_y^2 T + \partial_z^2 T\\ &= \nabla^2 T\end{aligned}$$

ii $\boldsymbol{\nabla}\times(\boldsymbol{\nabla} T) = 0$

iii $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \dots$ ignored

iv $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = 0$

v $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

1.3 Integral Calculus:

line, surface and volume integrals

"Fundamental theorem for gradients" Start with a scalar $T(x, y, z)$: from $a \to b$, in small steps $\mathrm{d}T = \boldsymbol{\nabla}\cdot T\,\mathrm{d}\boldsymbol{\ell}$

Total change in T:

$$\boxed{\int_a^b \mathrm{d}T = \int_a^b \boldsymbol{\nabla} T\cdot\mathrm{d}\boldsymbol{\ell} = T(b) - T(a)}$$

This line integral is path independent but $\int_a^b \mathbf{F}\cdot\mathrm{d}\ell$ is *not*!

Divergence Theorem, "Gauss' Theorem", or "Green's Theorem"

$$\boxed{\int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\mathrm{d}\tau = \oint_S v\cdot\mathrm{d}\mathbf{a}}$$

where V is the volume enclosed by the surface S. The divergence of a field in a volume is equivalent to the flux of the field at the boundary or surface.

Geometrical Interpretation: The "source"(or faucet) should present a flux (or flow) out through the surface.

Fundamental Theorem of Curls: Stokes' Theorem

$$\boxed{\oint_S (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{a} = \oint_P \mathbf{v}\cdot\mathrm{d}\boldsymbol{\ell}}$$

We have a 2D surfaces S bounded by a closed 1D perimeter P.

In 3D, imagine a soap bubble in a wire frame. We can change the surface of the bubble by moving the wire frame and making almost making a bubble, but the perimeter that bounds the bubble is still the wireframe.

Example:

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + 4yz^2\hat{\mathbf{z}}$$

On a surface S square on the $y - z$ plane:

$$\oint \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} =$$

RHS: We can split this into 4 integrals going counterclockwise around the square: First $x = 0, z = 0, y : 0 \to 1$: $dx = dz = 0$ $\int_0^1 3y^2 \mathrm{d}y = 1$

Second $\int_0^1 4z^2 \mathrm{d}z = 4/3$

Third: -1

Fourth: 0

Summing them all gives: $\oint \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} = 4/3$

LHS: The curl gives: $4z^2 - 2x, -(0 - 0), 2z$ so

$$\oint (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint$$

...

1.4 Dirac Delta Function

Considering a vector function

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}$$

[insert picture of field lines]

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = 0$$

The div theorem doesn't obviously tell what volume and surface to choose, but we can choose a sphere of radius R and its corresponding surface:

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \frac{1}{R^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = 4\pi$$

where the polar angle is $\theta : 0 \to \pi$ and the azimuthal angle is $\phi : 0 \to 2\pi$.

We think that the divergence is zero (thus the theorem does not hold), but we make a tiny mistake:

$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ everywhere except at the origin $r \to 0$

and we stumbled on the Dirac delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

where

$$\int f(x)\delta(x)\mathrm{d}x = f(0)$$

Shifting the delta function:

$$\delta(x - a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

so

$$\int f(x)\delta(x - a)\mathrm{d}x = f(a)$$

Multiplying by a constant:

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Generalizing to 3D:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

Examples:

$$\int_V (\nabla \cdot (\mathbf{v}))\mathrm{d}\tau = \int 4\pi\delta^3(\mathbf{r}) = 4\pi$$

and

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

2 Electrostatics

2.1 The Electric Field

given charge q: find force on Q , where $\mathbf{F}$ depends on $\boldsymbol{\mathscr{r}}, \mathbf{v}_i, \mathbf{a}_i$

2.1.1 Electrostatics:

Coulomb's Law empirically,

$$\mathbf{F}_Q = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

where $k = \frac{1}{4\pi\epsilon_0}$ and the permittivity of free space is $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{Nm}^2$

The force is attractive if $sgn(qQ) = -1$ and repulsive if $= +1$.

Principal of superposition:

$$\begin{aligned}\mathbf{F}_T &= \mathbf{F}_{Q1} + \mathbf{F}_{Q2} + \dots \\ &= \frac{1}{4\pi\epsilon_0}Q\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \dots\right) \\ &= Q\mathbf{E}_T\end{aligned}$$

where $\mathbf{E}_T$ is the total electric field due to all of the source (point) charges.

$\mathbf{E}$ doesn't depend on Q

- $\mathbf{E} \sim F/Q$

Example: $\mathbf{E}$ field midway above two charges q: The electric fields are zero in the x and y direction:

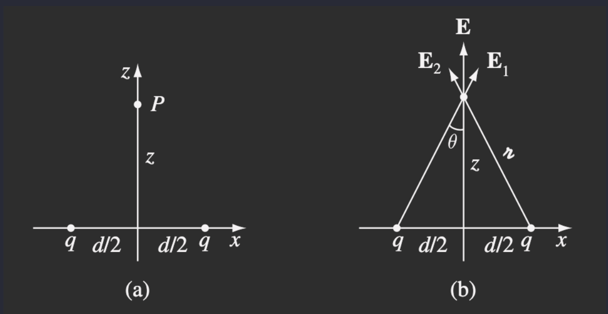

Figure 2.1: Griffiths Example 2.1

$$E_x = E_y = 0$$

But we can sum the fields in the z direction:

$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta$$

where

$$\mathscr{r} = \left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2} \qquad \cos\theta = \frac{z}{\mathscr{r}}$$

so

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}$$

Far away: $z \gg d$, so $d \to 0$ thus

$$E_z \approx \frac{1}{4\pi\epsilon_0}\frac{2qz}{z^3} = \frac{1}{4\pi\epsilon_0}\frac{2}{z^2}$$

Continuous Charge Distributions

- line: charge per unit length λ; $\quad \mathrm{d}q = \lambda \mathrm{d}\ell$
- surface: charge per unit area σ; $\quad \mathrm{d}q = \sigma \mathrm{d}a$
- volume: charge per unit volume ρ; $\quad \mathrm{d}q = \rho \mathrm{d}\tau$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\mathrm{d}q$$

e.g. for a volume charge:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\mathrm{d}\tau'$$

where $'$ denotes the source charge in (no $'$ is a field point)

Example: Find $\mathbf{E}$ at z above a straight line segment of length $2L$ with uniform line charge λ. If we

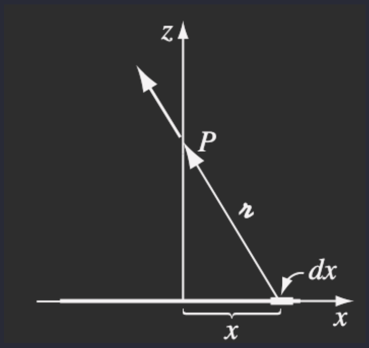

Figure 2.2: Griffiths Example 2.2

treat $\mathrm{d}q$ as a point particle, then we can use Ex 2.1 likewise but integrate over the line segment.

First we catalog what we know:

- Field point P is at $\mathbf{r} = z\hat{\mathbf{z}}$
- Sources at $\mathbf{r}' = x\hat{\mathbf{x}}$; $\quad \mathrm{d}\ell' = \mathrm{d}x$
- $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = z\hat{\mathbf{z}} - x\hat{\mathbf{x}}$
- $\mathscr{r} = \sqrt{x^2 + z^2}$
- $\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{x^2+z^2}}$

The electric field is then (line charge)

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{+L}\frac{\lambda}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\mathrm{d}x = \frac{1}{4\pi\epsilon_0}\lambda\int_{-L}^{+L}\frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{(z^2+x^2)^{3/2}}\mathrm{d}x \\
&= \frac{1}{4\pi\epsilon_0}\lambda\left[z\hat{\mathbf{z}}\int_{-L}^{L}\frac{\mathrm{d}x}{(z^2+x^2)^{3/2}} - \hat{\mathbf{x}}\int_{-L}^{L}\frac{x\mathrm{d}x}{(z^2+x^2)^{3/2}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\lambda\left[z\hat{\mathbf{z}}\frac{x}{z^2\sqrt{z^2+x^2}}\bigg|_{-L}^{L} - \hat{\mathbf{x}}\cancel{\frac{1}{\sqrt{z^2+x^2}}\bigg|_{-L}^{L}}\right]
\end{aligned}$$

we can easily see that the x component is zero through the geometrical symmetry of the line centered at the origin (like Ex 2.1). Simplifying gives us

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{\mathbf{z}}$$

Checks and balances:

- $\hat{\mathbf{z}}$ is expected!
- $$z \gg L \quad \sqrt{z^2+L^2} \approx z \quad E(P, z \gg L) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2}$$
  where we can treat this as a point charge $q = 2\lambda L$ when we are far away.

2.1.2 Divergence and curl of E: Gauss' Law

'flux' of field lines

$$\Phi = \oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{a}$$

What is Φ for point charge at origin surrounded by a spherical surface?

$$\begin{aligned}
\Phi &= \int \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{q_{enc}}{\epsilon_0}
\end{aligned}$$

A bunch of charges surrounded by a surface: $\mathbf{E}_T = \sum \mathbf{E}_i$

$$\Phi = \oint \mathbf{E}_T\cdot\mathrm{d}\mathbf{a} = \sum_i \oint \mathbf{E}_i\cdot\mathrm{d}\mathbf{a} = \sum_i \frac{q_i}{\epsilon_0}$$

Thus we have an integral form of Gauss's law:

$$\boxed{\oint \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

where $Q = \sum q_i$.

From the theorem of divergence:

$$\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a} = \int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\mathrm{d}\tau \quad \text{and} \quad Q = \int_V \rho\mathrm{d}\tau$$

so

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{E})\mathrm{d}\tau = \int_V \rho\mathrm{d}\tau \to \text{good for all volume}$$

therefore we have the differential form of Gauss' Law:

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}}$$

Three ways Gauss's law makes life nice: Gaussian surfaces

- spherical: gaussian sphere
- cylindrical: gaussian cylinder
- planar: gaussian pillbox

2.1.3 Applications of Gauss's Law

$$\int \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{q_{enc}}{\epsilon_0} \to \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

1. (Simple spherical) What is $\mathbf{E}$ outside a uniformly charged solid sphere of radius R and total charge Q? The spherical Gaussian surface implies a symmetry where we should *only have a radial component* $\mathbf{E} = E_r$.

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q}{\epsilon_0}$$
$$E \oint \mathrm{d}\mathbf{a} = E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$
$$\implies \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

where the integral is equivalent to the surface area of the sphere. This is also $\implies$ a field of a *point*.

2. (Simple cylindrical) A long cylinder (radius a) of charge density $\rho = ks$ ($\propto$ distance from axis) where k is a constant and s is the radial distance from the axis. What is $\mathbf{E}$ inside the cylinder? The cylindrical Gaussian surface has radius s and length ℓ:

$$\oint \mathbf{E}.\mathrm{d}\mathbf{a} = \frac{Q_enc}{\epsilon_0}; \quad Q_{enc} = \int \rho \mathrm{d}\tau = \int (ks')\mathrm{d}s'\mathrm{d}\phi\mathrm{d}z = \frac{2}{3}\pi k \ell s^3$$

When using the divergence theorem, note that only the curved part of the cylinder contributes to the flux. Thus,

$$\int \mathbf{E}\mathrm{d}\mathbf{a} \to E \int \mathrm{d}a = E(2\pi s\ell)$$
$$\implies \mathbf{E} = \frac{1}{3\epsilon_0}ks^2\hat{\mathbf{s}}$$

If we were to find the field outside the cylinder we would find that the enclosed charge is constant Q_{enc} thus the field is proportional to $1/s$.

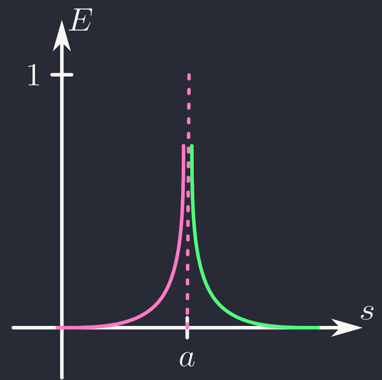

Figure 2.3: Electric field as a function of s

3. (Simple infinite plane) with uniform surface charge σ. Symmetry implies that $\mathbf{E}$ is perpendicular to the plane. The Gaussian pillbox (either box or cylinder) will have a field of

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

2.1.4 The curl of E

$$\boldsymbol{\nabla} \times \mathbf{E} = 0, \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

calculating

$$\int_a^b \mathbf{E} \cdot \mathrm{d}\ell, \quad \mathrm{d}\ell = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\theta\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\phi\hat{\boldsymbol{\phi}}$$

So the integral is

$$\frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a} - \frac{q}{b}\right)$$

This means:

- path independent!
- if $a = b$ then $\oint \mathbf{E} \cdot \mathrm{d}\ell = 0$ (ℓ is a vector but I don't know how to bold it)

We can now use Stokes' theorem: $\oint \mathbf{v} \cdot \mathrm{d}\ell = \int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a}$ or

$$\oint \mathbf{E} \cdot \mathrm{d}\ell = \int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathrm{d}\mathbf{a} = 0 \implies \boldsymbol{\nabla} \times \mathbf{E} = 0$$

2.1.5 Electric potential

Any function f with zero curl is the gradient of a scalar function: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$ (curl of gradient is always 0!)

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\ell$$

implies all paths give some value.

$V \sim$ "electric potential"

$$\begin{aligned} V(\mathbf{b}) - V(\mathbf{a}) &= -\left(\int_{\mathcal{O}}^b \mathbf{E} \cdot \mathrm{d}\ell\right) - \left(-\int_{\mathcal{O}}^a \mathbf{E} \cdot \mathrm{d}\ell\right) \\ &= -\int_{\mathcal{O}}^b - \int_{\dashv} O\mathbf{E} \cdot \mathrm{d}\ell \\ &= -\int_a^b \mathbf{E} \cdot \mathrm{d}\ell \end{aligned}$$

And from the fundamental theorem for gradients: $T(\mathbf{b}) - T(\mathbf{a}) = \int_a^b (\boldsymbol{\nabla} T) \cdot \mathrm{d}\ell$

$$\implies \mathbf{E} = -\boldsymbol{\nabla} V$$

i "potential" is a terrible name

ii $\mathbf{E} = (E_x, E_y, E_z)$ vs V with only *one* value at every point in space! Otherwise we would have to deal with

$$(\boldsymbol{\nabla} \times \mathbf{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

iii

$$V'(\mathbf{r}) = -\int_{O'}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\ell = -\int_{O'}^{O} \mathbf{E} \cdot \mathrm{d}\ell - \int_{O}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\ell = C + V(\mathbf{r})$$

$\implies \mathbf{E} = -\boldsymbol{\nabla} V$