Homework 12

1. (a) The canonical momenta are

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$
$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$
$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2 \sin^2\theta\dot{\phi}$$

(b) So the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \sum_i p_i \dot{q}_i - \mathcal{L} \\ &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + U\end{aligned}$$

and substituting

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

we get

$$\begin{aligned}\mathcal{H} &= \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U \\ &= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r,\theta,\phi)\end{aligned}$$

Finally, the three sets of Hamilton's equations are

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m}$$
$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{\partial U}{\partial r}$$
$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$
$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = \frac{1}{mr^2}\left(\frac{p_\phi^2\cos\theta}{\sin^3\theta}\right) - \frac{\partial U}{\partial \theta}$$
$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$
$$\dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = -\frac{\partial U}{\partial \phi}$$

2. (a) The coordinates of the spinning pendulum is

$$r = L, \quad \phi = \omega t$$

where the azimithal component is time-dependent, and the polar coordinate is found by the EL equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \\ \implies \ddot{\theta} = \omega^2 \sin\theta\cos\theta - \frac{g}{L}\sin\theta\end{aligned}$$

which is also time-dependent. (b) The canonical momenta are

$$\begin{aligned}p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \\ p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} \\ p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0\end{aligned}$$

and the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \sum_i p_i \dot{q}_i - \mathcal{L} \\ &= p_\theta \dot{\theta} - \mathcal{L} \\ &= mL^2\dot{\theta}^2 - \frac{1}{2}mL^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mgL(1-\cos\theta) \\ &= \frac{1}{2}mL^2\left(\dot{\theta}^2 - \omega^2\sin^2\theta\right) + mgL(1-\cos\theta) \\ &\neq T + U\end{aligned}$$

(c)

$$\begin{aligned}\mathcal{H} = T' - \frac{1}{2}mL^2\omega^2\sin^2\theta + mgL(1-\cos\theta) \\ \implies U' = -\frac{1}{2}mL^2\omega^2\sin^2\theta + mgL(1-\cos\theta)\end{aligned}$$

Finding the minimum of U', we have

$$\begin{aligned}\frac{\partial U'}{\partial \theta} = 0 = -mL^2\omega^2\sin\theta\cos\theta + mgL\sin\theta \\ 0 = -L\omega^2\cos\theta + g \implies \theta = \arccos\frac{g}{L\omega^2}\end{aligned}$$

so theta is not always zero. Plotting U', we have a mexican hat cross section:

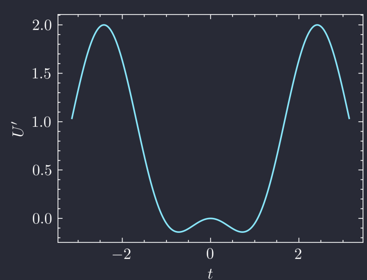

Figure 12.1: Plot of U' where $L = g = m = 1$ and $\omega^2 = 1.3^2 > g/L$

3. (a) Hamilton's equations for the particle:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \left[\frac{mc^2}{\sqrt{1+(\mathbf{p}/mc)^2}}\frac{1}{2}2(\mathbf{p}/mc)\frac{1}{mc}\right]_x = \frac{p_x}{m\sqrt{1+(p_x/mc)^2}}$$
$$\dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial U}{\partial x}$$

and similarly for y and z. (b) For small $\mathbf{p}/mc$, $\sqrt{1+(\mathbf{p}/mc)^2} \approx 1$ so

$$\dot{x} = \frac{p_x}{m} = \frac{mv_x}{m} = v_x$$

which reduces to the Newtonian case.

4. (a) The canonical momenta are

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{A}$$

And the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} \\ &= m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\mathbf{A}\cdot\dot{\mathbf{r}} - \frac{1}{2}m\dot{\mathbf{r}}^2 + q(\phi_e - \dot{\mathbf{r}}\cdot\mathbf{A}) \\ &= \frac{1}{2}m\dot{\mathbf{r}}^2 + q\phi_e\end{aligned}$$

or using $\dot{\mathbf{r}} = \frac{1}{m}(\mathbf{p} - q\mathbf{A})$,

$$\mathcal{H} = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi_e$$

(b) Using Hamilton's equation:

$$\dot{\mathbf{r}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \frac{1}{m}(\mathbf{p} - q\mathbf{A}) = \mathbf{v}$$

and taking the derivative gives,

$$m\ddot{\mathbf{r}} = \dot{\mathbf{p}} - q\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$$

where

$$\begin{aligned}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} &= \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \mathbf{A}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \mathbf{A}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}\end{aligned}$$

To find the first term, we use the second part of Hamilton's equation:

$$\begin{aligned}\dot{p}_x &= -\frac{\partial \mathcal{H}}{\partial x} \\ &= \frac{q}{m}(\mathbf{p} - q\mathbf{A})\frac{\partial \mathbf{A}}{\partial x} - q\frac{\partial \phi_e}{\partial x} \\ &= q\mathbf{v}\frac{\partial \mathbf{A}}{\partial x} - q\frac{\partial \phi_e}{\partial x}\end{aligned}$$

so

$$\dot{\mathbf{p}} = q\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) - q\boldsymbol{\nabla}\phi_e$$

This gives us

$$
\begin{aligned}
m\ddot{\mathbf{r}} &= q[\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) - \boldsymbol{\nabla}\phi_e - \frac{\partial \mathbf{A}}{\partial t} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}] \\
&= q[\mathbf{E} + \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}]
\end{aligned}
$$

where we can use the B(AC)-C(AB) rule to simplify

$$
\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}
$$

so

$$
\begin{aligned}
m\ddot{\mathbf{r}} &= q[\mathbf{E} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})] \\
&= q(\mathbf{E} + \mathbf{v}\times\mathbf{B})
\end{aligned}
$$

5. The components of the angular momentum are

$$
\mathbf{r}\times\mathbf{p} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}
$$

so

$$
\begin{aligned}
\ell_x &= yp_z - zp_y \\
\ell_y &= zp_x - xp_z \\
\ell_z &= xp_y - yp_x
\end{aligned}
$$

Poisson brakets of angular momenta:

$$
\begin{aligned}
[\ell_x, \ell_y] &= \sum_i \left(\frac{\partial \ell_x}{\partial r_i}\frac{\partial \ell_y}{\partial p_i} - \frac{\partial \ell_y}{\partial r_i}\frac{\partial \ell_x}{\partial p_i} \right) \\
&= \left(\frac{\partial \ell_x}{\partial x}\frac{\partial \ell_y}{\partial p_x} - \frac{\partial \ell_y}{\partial x}\frac{\partial \ell_x}{\partial p_x} \right) + \left(\frac{\partial \ell_x}{\partial y}\frac{\partial \ell_y}{\partial p_y} - \frac{\partial \ell_y}{\partial y}\frac{\partial \ell_x}{\partial p_y} \right) + \left(\frac{\partial \ell_x}{\partial z}\frac{\partial \ell_y}{\partial p_z} - \frac{\partial \ell_y}{\partial z}\frac{\partial \ell_x}{\partial p_z} \right) \\
&= 0 + 0 + [(-p_y)(-x) - (p_x)(y)] \\
&= xp_y - yp_x = \ell_z
\end{aligned}
$$

and similarly for the other components:

$$
\begin{aligned}
[\ell_y, \ell_z] &= \left(\frac{\partial \ell_y}{\partial x}\frac{\partial \ell_z}{\partial p_x} - \frac{\partial \ell_z}{\partial x}\frac{\partial \ell_y}{\partial p_x} \right) + \left(\frac{\partial \ell_y}{\partial y}\frac{\partial \ell_z}{\partial p_y} - \frac{\partial \ell_z}{\partial y}\frac{\partial \ell_y}{\partial p_y} \right) + \left(\frac{\partial \ell_y}{\partial z}\frac{\partial \ell_z}{\partial p_z} - \frac{\partial \ell_z}{\partial z}\frac{\partial \ell_y}{\partial p_z} \right) \\
&= [(-p_z)(-y) - (p_y)(z)] + 0 + 0 \\
&= yp_z - zp_y = \ell_x
\end{aligned}
$$

there is a pattern here...

$$
\begin{aligned}
[\ell_z, \ell_x] &= \left(\frac{\partial \ell_z}{\partial y}\frac{\partial \ell_x}{\partial p_y} - \frac{\partial \ell_x}{\partial y}\frac{\partial \ell_z}{\partial p_y} \right) \\
&= (-p_x)(-z) - (p_z)(x) \\
&= zp_x - xp_z = \ell_y
\end{aligned}
$$