Physics 411: Mechanics

Homework: Junseo Shin

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Homework 1

Due 1/24 9pm

1. Given: the 2D Cartesian relation to polar coordinates

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \quad \hat{\phi} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$
 (1)

We can write v as a linear combination of $\hat{\mathbf{r}}$ and $\hat{\phi}$

$$\mathbf{v} = v_r \hat{\mathbf{r}} + v_\phi \hat{\boldsymbol{\phi}}$$

$$= v_r (\hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi) + v_\phi (-\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi)$$

$$= (v_r \cos \phi - v_\phi \sin \phi) \hat{\mathbf{x}} + (v_r \sin \phi + v_\phi \cos \phi) \hat{\mathbf{y}}$$

and since we know the vector in Cartesian coordinates is

$$\mathbf{v} = v_x \mathbf{\hat{x}} + v_y \mathbf{\hat{y}}$$

we can equate the components to get

$$v_x = v_r \cos \phi - v_\phi \sin \phi$$
$$v_y = v_r \sin \phi + v_\phi \cos \phi$$

multiplying the first equation by $\cos \phi$ and the second by $\sin \phi$ and adding them together

$$v_x \cos \phi = v_r \cos^2 \phi - v_\phi \sin \phi \cos \phi$$
$$v_y \sin \phi = v_r \sin^2 \phi + v_\phi \sin \phi \cos \phi$$
$$v_x \cos \phi + v_y \sin \phi = v_r (\cos^2 \phi + \sin^2 \phi)$$

or simply

$$v_r = v_x \cos \phi + v_y \sin \phi$$

Likewise,

$$v_y \cos \phi = v_r \sin \phi \cos \phi + v_\phi \sin^2 \phi$$
$$v_x \sin \phi = v_r \sin \phi \cos \phi - v_\phi \cos^2 \phi$$

and subtracting the second equation from the first

$$v_y \cos \phi - v_x \sin \phi = v_\phi (\sin^2 \phi + \cos^2 \phi)$$

Therefore we get the components of ${\bf v}$ in polar coordinates

$$v_r = v_x \cos \phi + v_y \sin \phi$$

$$v_\phi = -v_x \sin \phi + v_y \cos \phi$$

Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are *constant*, the time derivatives of (1) are

$$\dot{\hat{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\phi) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\phi)$$
$$= (-\dot{\phi}\sin\phi)\hat{\mathbf{x}} + (\dot{\phi}\cos\phi)\hat{\mathbf{y}} = \dot{\phi}\hat{\phi}$$

and

$$\dot{\hat{\phi}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi} = -\hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\phi) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\phi)$$
$$= (-\dot{\phi}\cos\phi)\hat{\mathbf{x}} + (-\dot{\phi}\sin\phi)\hat{\mathbf{y}} = -\dot{\phi}\hat{\mathbf{r}}$$

2. From Taylor Problem 1.45: [*Hint:* Consider the derivative of \mathbf{v}^2]. Since the magnitude of $\mathbf{v}(t)$ is also $\sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$, the derivative of \mathbf{v}^2 is tells us if the magnitude is constant.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\cdot\mathbf{v}(t))$$
$$= 2\dot{\mathbf{v}}(t)\cdot\mathbf{v}(t)$$

The magnitude of $\mathbf{v}(t)$ is constant if $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2=0$. Since the dot product is zero, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$.

3. Using the product rule for the dot product $\frac{d}{dt}(\mathbf{x} \cdot \mathbf{y}) = \dot{\mathbf{x}} \cdot \mathbf{y} + \mathbf{x} \cdot \dot{\mathbf{y}}$

$$\begin{split} \frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] &= \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\frac{d\mathbf{r}}{dt} \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\mathbf{a}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\mathbf{v} \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) \end{split}$$

The cross product of a vector with itself is zero ($\mathbf{v} \times \mathbf{v} = 0$), and the dot product of orthogonal vectors are zero (acceleration and position are perpendicular based on the result from Problem 2) QED

4. Given

$$\ddot{\theta} = -\frac{g}{l}\sin\theta\tag{3}$$

(a) Moving everything to one side:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

multiplying by $\dot{\theta}$ and grouping terms:

$$\begin{split} \dot{\theta}\ddot{\theta} + \frac{g}{l}\dot{\theta}\sin\theta &= 0\\ \dot{\theta}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{\theta}\Big) + \frac{g}{l}\sin\theta\frac{\mathrm{d}}{\mathrm{d}t}(\theta) &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\dot{\theta}^2\Big) - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{g}{l}\cos\theta\Big) &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta\Big) &= 0 \end{split}$$

so the integral constant is

$$\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = C$$

Initially the pendulum starts at rest, $\dot{\theta}(t=0) = 0$ and $\theta(0) = \theta_o$ thus

$$C = -\frac{g}{l}\cos\theta_o$$

(b) Rewriting X = C

$$\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = -\frac{g}{l}\cos\theta_o$$

$$\dot{\theta} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_o)}$$

using separation of variables

$$d\theta = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_o)} dt$$

Integrating gives the analytic solution

$$\theta(t) = \sqrt{\frac{2g}{l}} \int_0^T \sqrt{\cos \theta - \cos \theta_o} \, dt$$

where T is the period of the pendulum.

(c) The period of the pendulum is the time it takes to complete one cycle. Since

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

using separation of variables

$$dt = \frac{1}{\dot{\theta}} d\theta$$

Integrating both sides gives the period

$$\int dt = \int \frac{1}{\dot{\theta}} d\theta$$

$$T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_o} \frac{1}{\sqrt{\cos \theta - \cos \theta_o}} d\theta$$

where the constant '4' comes from the fact that the total cycle is 4 times the period it takes to go from path from θ_o to 0.

Homework 2

Due 1/31

Check for the code. asdfasdf stuff here why is this not wokring here

Homework 3

Due 2/7 9pm