1 Magnetic Fields in Matter

1.1 Magnetization

1.1.1 Paramagnets, Diamagnet, Ferromagnets

1.1.2 Forces and Torques on Magnetic Dipoles

Using $\mathbf{F} = I \int \mathrm{dl} \times \mathbf{B}$ we can find the directions of the forces on each section of the current loop

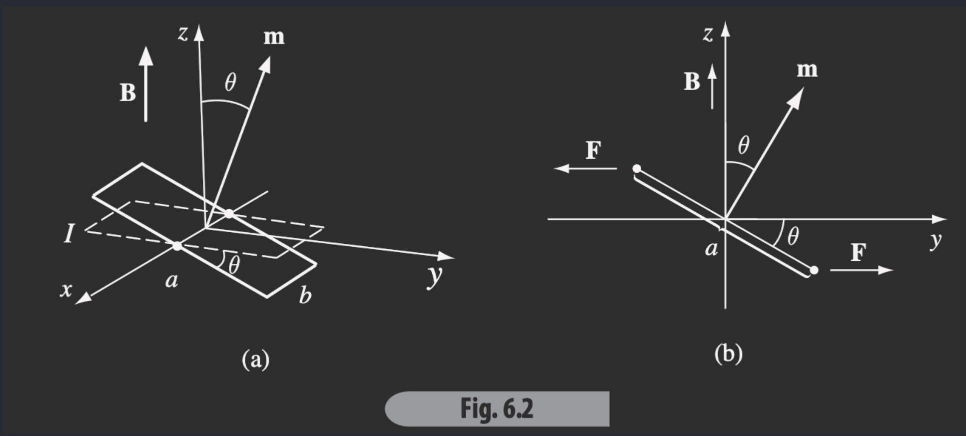

Figure 1.1: Magnetic field due to a current loop

Since the sum of the forces $\sum \mathbf{F} = 0$ so the magnitude of force on each segment is

$$F = IbB$$

and the torque on the loop is

$$\mathbf{N} = aF \sin\theta \hat{\mathbf{x}}$$

Or

$$\mathbf{N} = IabB \sin\theta \hat{\mathbf{x}} \quad \text{using} \quad m = Iab$$
$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

which encapsulates the defintion of "paramagnetism" i.e. the magnetic moments align with the magnetic field. This is similar to the electric form

$$\mathbf{N}_E = \mathbf{p} \times \mathbf{E}$$

. . . in a uniform field The net force → 0

$$\mathbf{F} = I \oint \mathrm{dl} \times \mathbf{B} = 0$$

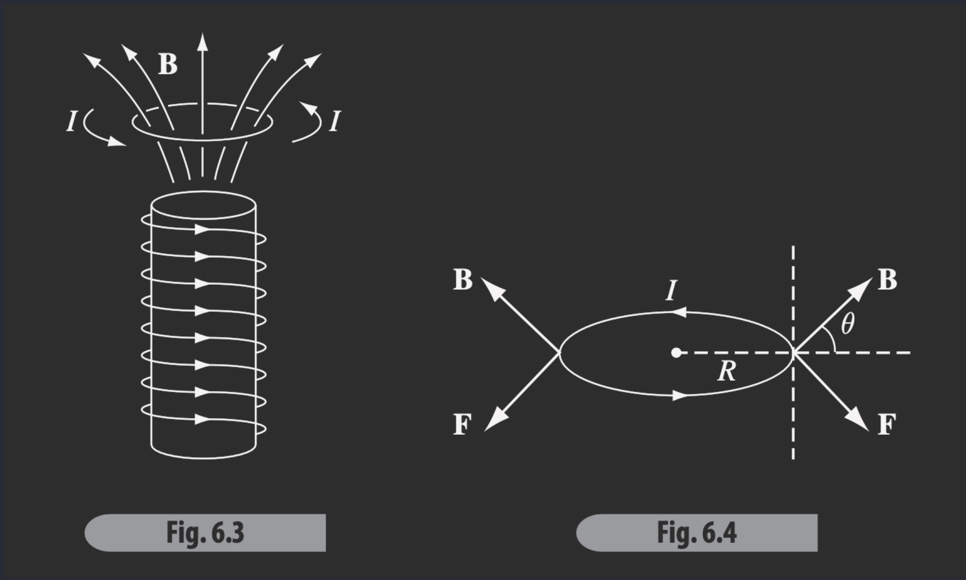

Figure 1.2: Solenoid

non-uniform field Using RHR we can see that the horizontal components cancel out from symmetry and we are left with a downward force on the dipole with magnitude

$$|\mathbf{F}| = 2\pi RIB \sin\theta$$

At the other end of the dipole, the force will be upwards; so for the pure dipole

$$\mathbf{F}_B = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$$
$$\mathbf{F}_E = \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E})$$

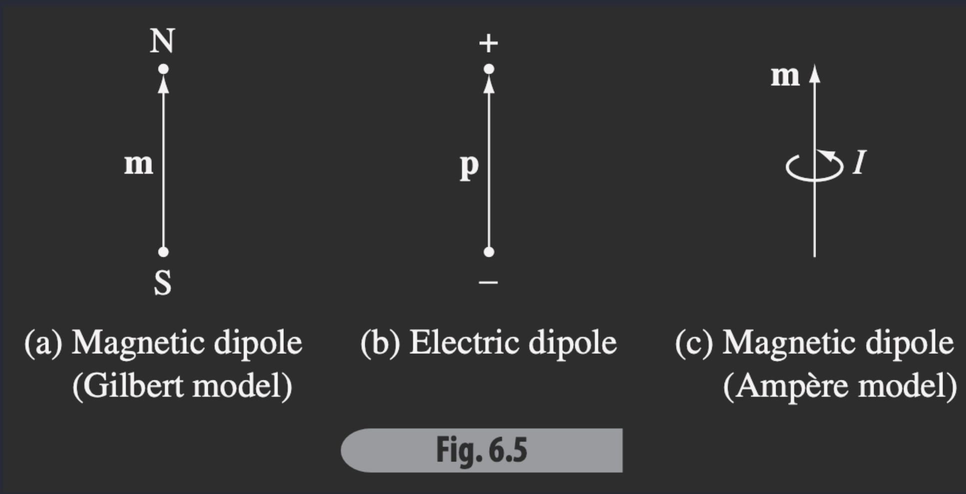

Figure 1.3: Early models of magnetism

From the electric dipole, we could easiliy infer a magnetic dipole or “Gilbert” model.

1.1.3 Effect of B on atomic orbitals

For an electron circulating a nucleus,

- Period $T = 2\pi R/v$
- $\sim$ stead current $I = -e/T = -ev/(2\pi R)$
- Thus a magnetic moment $\mathbf{m} = I\mathbf{a} = I\pi R^2(-\hat{\mathbf{z}}) = -\frac{1}{2}evR\hat{\mathbf{z}}$

The centripetal orbital force keeping the electron in orbit is entirely due to the coulomb force

$$\mathbf{F}_{\text{orb}} = \mathbf{F}_{\text{coul}}$$
$$m_e\frac{v^2}{R} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2}$$

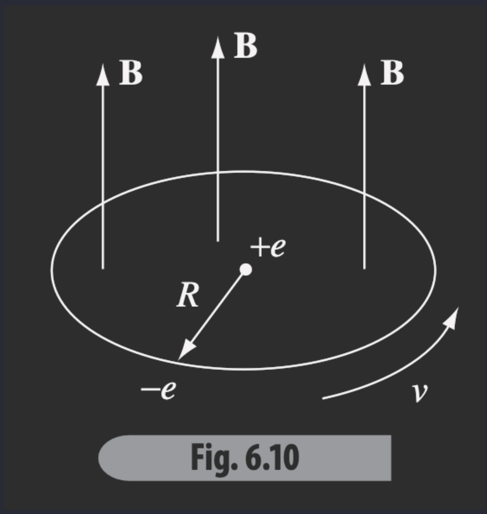

Figure 1.4: Magnetic field due to an electron

From the LHR (for electron) the force points inwards, so

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + ev'B = m_e\frac{v^2}{R}$$
$$\implies ev'B = \frac{mv'^2}{R} - \frac{mv^2}{R} = \frac{m}{R}(v'^2 - v^2)$$
$$= \frac{m}{R}(v + v')(v' - v) \quad \text{using} \quad v' - v = \delta v$$

where

$$\delta v = \frac{ev'B}{m/R}\frac{1}{v' + v} \approx \frac{eRB}{2} \quad v' \approx v$$

Using

$$\mathbf{m} = -\frac{1}{2}evR\hat{\mathbf{z}} \implies \delta m = -\frac{e}{2}R\delta v\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}$$

This opposite alignment of the magnetic moment is called “diamagnetism”. Andre Geim shared the Ig Nobel prize for levitating a frog in a magnetic field Wiki.

1.1.4 Magnetization

The magnetization is defined by the vector quantity

$$\mathbf{M} = \text{magnetic dipole moment/unit volume}$$

which is analagous to the electric polarization $\mathbf{P}$:

1.2 The field of a magnetized object

1.2.1 Bound Currents

The field produced by $\mathbf{M}$? For each tiny dipole moment

$$\mathbf{m} = \mathbf{M}\,\mathrm{d}\tau$$

we can use a vector potential

$$\mathbf{A}_m(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

So the total vector potential (using product rule 7 [insert ref]) and $\boldsymbol{\nabla}'\frac{1}{\mathscr{r}} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

$$\begin{aligned}
\mathbf{A}_{\text{tot}}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}\tau' \\
&= \frac{\mu_0}{4\pi}\int \mathbf{M}(\mathbf{r}')\times\boldsymbol{\nabla}'\frac{1}{\mathscr{r}}\,\mathrm{d}\tau' \\
&= \frac{\mu_0}{4\pi}\left(\int \frac{1}{\mathscr{r}}[\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')]\,\mathrm{d}\tau' - \int \boldsymbol{\nabla}'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]\mathrm{d}\tau'\right)
\end{aligned}$$

where the second term is

$$-\int \boldsymbol{\nabla}'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]\mathrm{d}\tau' = \oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times\mathrm{d}\mathbf{a}']$$

from Stokes' theorem of sorts. The two integrals represent the volume and surface current

$$\mathbf{J}_b \equiv \boldsymbol{\nabla}\times\mathbf{M} \qquad \mathbf{A}\sim\int\frac{\mathbf{J}_b}{\mathscr{r}}\,\mathrm{d}\tau$$

$$\mathbf{K}_b \equiv \mathbf{M}\times\hat{\mathbf{n}} \qquad \mathbf{A}\sim\oint\frac{\mathbf{K}_b}{\mathscr{r}}\,\mathrm{d}a$$

so

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_b}{\mathscr{r}}\,\mathrm{d}\tau' + \frac{\mu_0}{4\pi}\oint\frac{\mathbf{K}_b}{\mathscr{r}}\,\mathrm{d}a'$$

Example Find $\mathbf{B}$ of a uniformly magnetized sphere

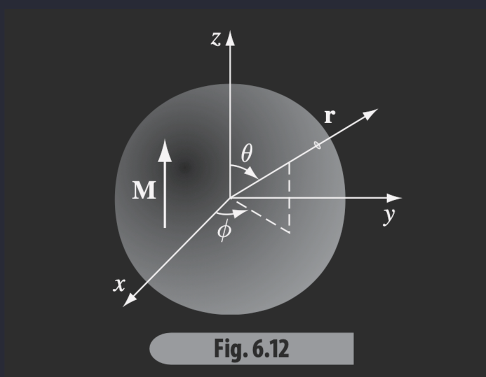

Figure 1.5: Magnetized sphere

Thre curl of a constant vector is zero so

$$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = 0$$

and for the surface current, we can see that the cross product is always in the azimuthal direction:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta \hat{\boldsymbol{\phi}}$$

Previously we found the field of a charged spinning sphere

$$\mathbf{K}_e = \sigma \mathbf{v} = \sigma \omega R \sin\theta \hat{\boldsymbol{\phi}}$$

so we can think of the magnetic counterpart as $\sigma\omega R = M$ where the solution inside the sphere is

$$\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M} \quad \text{inside}$$

Outside the sphere, we can think of this as a perfect dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$$

1.2.2 Physical Interpretation of Bound Currents

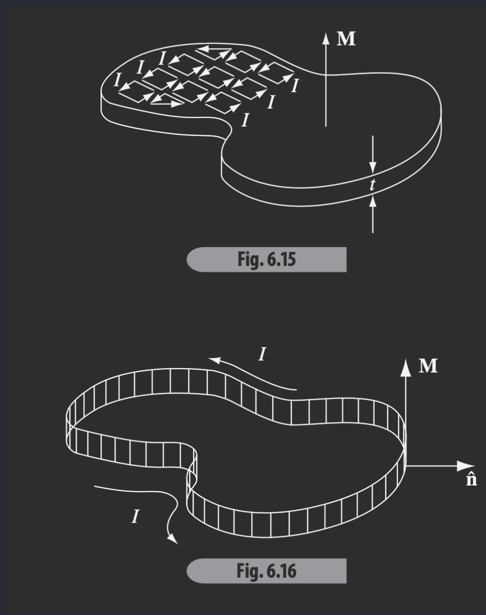

Figure 1.6: Physical interpretation of bound currents

For a magnetization $\mathbf{M} = M\hat{\mathbf{z}}$ the little loops of current (area a) cancel out in the bulk of the material (of thickness t) but at the edges, they line up and create a train of current. The magnitude

$$|\mathbf{m}| = Mat = Ia \implies I_b = Mt$$

Then the current density is

$$\mathbf{K} = \mathbf{M}(\hat{\mathbf{z}} \times \hat{\mathbf{n}}) = \frac{I_b}{t}(\hat{\mathbf{z}} \times \hat{\mathbf{n}})$$

So the current in the x direction of two adjacent loops is

$$I_x = M_z(y + \mathrm{d}y) - M_z(y) = \frac{\partial M_z}{\partial y}\,\mathrm{d}y$$

or

$$J_b\Big|_x = \frac{\partial M_z}{\partial y}$$

also adding the contribution from the y direction

$$J_b\Big|_x = \frac{\partial M_z}{\partial y}\hat{\mathbf{x}} - \frac{\partial M_y}{\partial x}\hat{\mathbf{z}} \implies \mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M}$$

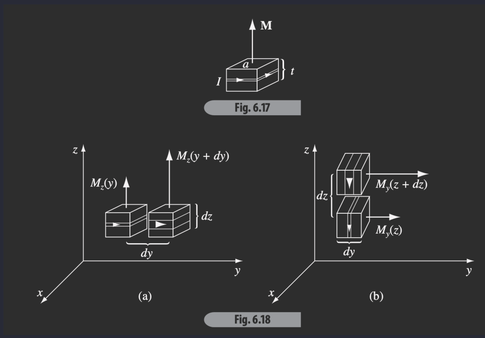

Figure 1.7: Physical interpretation of bound currents

Example A cylinder along the z axis with a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$: Find the magnetic field $\mathbf{B}$

THe volume current is

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} = 0$$

and the surface current is

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}$$

so

$$\mathbf{B} = \mu_0 n I \hat{\mathbf{z}} = \mu_0 M \hat{\mathbf{z}}$$

since the surface current density is $K \sim \frac{\mathrm{d}I}{\mathrm{d}l_\perp}$

Example Finding the magnetici field for a very long cylinder along z with a mamgetization

$$\mathbf{M} = ks^2\hat{\boldsymbol{\phi}}$$

The surface current density is

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = -kR^2\hat{\mathbf{z}}$$

and the volume current density is

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} = 3ks\hat{\mathbf{z}}$$

To find the current we can first find the total current due to the volume current density for each cross sectional area

$$\begin{aligned} I_J &= \int \mathbf{J}_b \cdot \mathrm{d}\mathbf{a} \\ &= \int 3ks(s\,\mathrm{d}s)\,\mathrm{d}\phi = 2\pi k R^3 \end{aligned}$$

then for the surface current density we integrate over the surface along the z axis

$$I_K = \int -kR^2 R\,\mathrm{d}\phi = -2\pi k R^3$$

Then using Ampere's law; an amperian circular loop far away

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0(I_J + I_K) = 0 \implies \mathbf{B} = 0$$

but inside the cyclinder it starts from zero and increases until we get to the surface where it drops to zero again.