1 Quantum Electrodynamics

Quiz Review

Schrodinger Equation

$$E = \frac{\mathbf{p}^2}{2m} + v \to \left(i\hbar\frac{\partial}{\partial t}\right)\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + v\right)\psi$$

where we apply

$$\mathbf{p} \to -i\hbar\boldsymbol{\nabla}$$
$$E \to i\hbar\frac{\partial}{\partial t}$$

and

$$p_\mu \to i\hbar\partial_\mu$$
$$p_0 = \frac{E}{c} \to i\hbar\frac{\partial}{\partial t}$$

Relativistic Equation

$$E^2 = \mathbf{p}^2c^2 + m^2c^4$$

so

$$\begin{aligned}
p_\mu p^\mu &= m^2c^2 \\
[(i\hbar\partial^\mu)(i\hbar\partial_\mu)]\psi &= m^2c^2\psi \\
\implies (-\hbar^2\partial^\mu\partial_\mu)\psi &= m^2c^2\psi \\
\implies -\partial^\mu\partial_\mu\psi &= \frac{m^2c^2}{\hbar^2}\psi \\
\implies -\Box\psi &= \frac{m^2c^2}{\hbar^2}\psi
\end{aligned}$$

Which is the *Klein-Gordan* equation where the box operator is the d'Alembertian operator. This only describes spin-0 particles as a 2nd order in time derivative.

Dirac Equation To describe spin-1/2 particles, we need a relativistic wave eqn in the 1st order in time. Setting $\mathbf{p} = 0$ or at rest,

$$\begin{aligned}
\partial^\mu\partial_\mu \to \partial^0\partial_0 &= \frac{\partial^2}{\partial t^2} \\
p_\mu &= (p_0, \mathbf{0}) \\
p^\mu p_\mu &= m^2c^2 \\
\text{or} \quad p^0p_0 = m^2c^2 \quad \text{if} \quad \mathbf{p} &= 0 \\
\text{or} \quad p_0^2 - m^2c^2 &= 0 \\
\text{or} \quad (p^0 + mc)(p^0 - mc) &= 0 \\
\text{or} \quad p^0 &= \pm mc \\
\text{or} \quad i\hbar\frac{\partial}{\partial t}\psi &= \pm mc\psi
\end{aligned}$$

which gives us the Plane wave solution

$$\psi(t) = \psi(0)e^{\pm i \frac{mc^2}{\hbar} t}$$

If $\mathbf{p} \neq 0$, then from the dirac equation

$$p^\mu p_\mu - m^2c^2 = 0$$

and writing into the form

$$(\beta_K p^K + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

expanding out the terms

$$\beta_K \gamma^\lambda p^K p_\lambda - mc(\beta_K p^K - \gamma^\lambda p_l ambda) - m^2c^2 = 0$$

and since we are using dummy indices:

$$\gamma^\lambda p_\lambda = \gamma_\lambda p^\lambda = \gamma_k p^K$$

or

$$\beta_K \gamma^\lambda p^K p_\lambda - mc(\beta_K - \gamma_k)p^K - m^2c^2 = 0$$

and comparing with

$$p^\mu p_\mu - m^2c^2 = 0$$

which implies that the linear term in p^μ must be zero

$$\implies \beta_K = \gamma_K$$

so the equation becomes

$$\gamma_K \gamma^\lambda p^K p_\lambda - m^2c^2 = 0$$

So we have the terms

$$\begin{aligned}\implies \gamma_0\gamma^0 p^0 p^0 + \gamma_1\gamma^1 p^1 p_1 + \gamma_2\gamma^2 p^2 p_2 + \gamma_3\gamma^3 p^3 p^3 \\ + \gamma_0\gamma^1 p^0 p_1 + \cdots - m^2c^2 = 0\end{aligned}$$

The LHS should be equal to

$$p^0 p_0 + p^1 p_1 + p^2 p_2 + p^3 p_3 - m^2c^2 = 0$$

so

$$\gamma_0\gamma^0 = \gamma_1\gamma^1 = \gamma_2\gamma^2 = \gamma_3\gamma^3 = 1$$

and the cross terms should be zero

$$\gamma_0\gamma^1 = \gamma_1\gamma^0 = \gamma_2\gamma^3 = \gamma_3\gamma^2 = 0$$

and

$$\begin{aligned}\gamma_0 &= g_{0\mu}\gamma^\mu \\ &= g_{00}\gamma^0 = \gamma^0 \\ \gamma_1 &= g_{11}\gamma^1 = -\gamma^1\end{aligned}$$

so

$$(\gamma^0)^2 = -(\gamma^j)^2 = 1 \quad (j = 1, 2, 3)$$
$$\gamma^\mu \gamma^\nu = 0 \quad \text{if} \quad \mu \neq \nu$$
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

We can try $\gamma^0 = 1$ and $\gamma^j = i$ but it does not satisfy the third anticommutator relation. so γs have to be matrices, or more specifcally a 4×4 matrix (Dirac matrices). We obtain the Dirac equation we take out one of the terms

$$(\gamma_K p^K + mc)(\gamma^\lambda p_\lambda - mc) = 0$$
$$\implies \gamma_K p^K \pm mc = 0$$

and from the relativistic relation

$$p^K \to i\hbar \partial^K$$

we get the Dirac equation

$$(i\hbar \gamma^K \partial_K \pm mc)\psi = 0$$

where we can interchange the indices, i.e.,

$$\gamma^K \partial_K = \gamma^\lambda \partial_\lambda = \gamma_\mu \partial^\mu$$

Since γ^μ is a 4×4 matrix, we have a Dirac spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Using the Dirac basis

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

where the matrices are in spinor space and not in Lorentz space.

Solution to the Dirac Equation for

$$(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0$$

First consider the rest case $\mathbf{p} = 0$ so

$$(i\hbar \gamma^0 \partial_0 - mc)\psi = 0$$
$$\implies \left(i\hbar \gamma^0 \frac{1}{c}\frac{\partial}{\partial t} - mc\right)\psi = 0$$
$$\implies \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\psi = -\frac{imc^2}{\hbar}\psi$$

and we write psi as a two components

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \quad \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

so

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\psi_A \\ \frac{\mathrm{d}}{\mathrm{d}t}\psi_B \end{pmatrix} = -\frac{imc^2}{\hbar}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

which splits into two equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_A = -\frac{imc^2}{\hbar}\psi_A$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_B = \frac{imc^2}{\hbar}\psi_B$$

so the solutions are

$$\psi_A(t) = \psi_A(0)e^{-\frac{imc^2}{\hbar}t}$$
$$\psi_B(t) = \psi_B(0)e^{\frac{imc^2}{\hbar}t}$$

Which is similar to the plane wave solution in QM that has the factor

$$\psi(t) = \psi(0)e^{-i\frac{E}{\hbar}t}$$

but here the energy is $E = mc^2$. psi_A is the particle solution and psi_B is the antiparticle solution. For the vectors

$$\psi_A = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad \psi_B = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$$

we have two basis vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

so

$$\psi_A(t) = e^{-imc^2t/\hbar}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e^{-imc^2t/\hbar}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

which corresponds to the electron spin up and spin down states. We also have

$$\psi_B(t) = e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

For the positron spin up and spin down states respectively.

General Case $\mathbf{p} \neq 0$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

We would expect a plane wave solution of the form

$$\psi(x) = e^{-ip\cdot x/\hbar}\psi(0)$$

or

$$\psi = ae^{-ik\cdot x/\hbar}u(k)$$

where $u(k)$ is the spinor part, and a is the normalization constant. The derivative of the exponent will give

$$\partial_\mu\psi = -ik_\mu\psi$$

and back into the dirac equation

$$(i\hbar\gamma^\mu(-ik_\mu) - mc)\psi = 0$$
$$\text{or} \quad \left(\gamma^\mu k_m - \frac{mc}{\hbar}\right)u = 0$$

where we know that

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

so using

$$\begin{aligned}
\gamma^\mu k_\mu &= \gamma^0 k_0 - \gamma^j k^j \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} k_0 - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} k^j \\
&= \begin{pmatrix} k_0 1 & -\sigma \cdot \mathbf{k} \\ \sigma \cdot \mathbf{k} & -k_0 1 \end{pmatrix}
\end{aligned}$$

where we define a Weyl spinor

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

So we get

$$\begin{pmatrix} k_0 - \frac{mc}{\hbar} & -\sigma \cdot \mathbf{k} \\ \sigma \cdot \mathbf{k} & -k_0 - \frac{mc}{\hbar} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

which gives us the coupled equations

$$(k_0 - \frac{mc}{\hbar})u_A - \sigma \cdot \mathbf{k} u_B = 0$$
$$\sigma \cdot \mathbf{k} u_A - (k_0 + \frac{mc}{\hbar})u_B = 0$$

and we can solve this by solving for u_A in the first equation and substituting into the second equation

$$u_A = \frac{\sigma \cdot \mathbf{k}}{k_0 - \frac{mc}{\hbar}} u_B$$

and substituting the second eq to the first

$$\begin{aligned}
u_B &= \frac{\sigma \cdot \mathbf{k}}{k_0 + \frac{mc}{\hbar}} u_A \\
&= \frac{\sigma \cdot \mathbf{k}}{k_0 + \frac{mc}{\hbar}} \frac{\sigma \cdot \mathbf{k}}{k_0 - \frac{mc}{\hbar}} u_B \\
&= \frac{(\sigma \cdot \mathbf{k})^2}{(k^0)^2 - (\frac{mc}{\hbar})2} u_A
\end{aligned}$$

where

$$(\sigma \cdot \mathbf{k})^2 = (k^0)^2 - \left(\frac{mc}{\hbar}\right)^2$$
$$\implies \mathbf{k}^2 = (k^0)^2 - \left(\frac{mc}{\hbar}\right)^2$$

and from the relativistic relation

$$k^2 = k^\mu k_\mu = (k^0)^2 - \mathbf{k}^2 = \left(\frac{mc}{\hbar}\right)^2$$

this tells us that $\hbar k_\mu$ must be the momentum p_μ

Quiz Review

- From the Dirac Spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

  we have the particles represented by the wavefunction ψ_A and the antiparticles represented by the wavefunction ψ_B.

$$\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

- Weyl Spinors describe either the particle or antiparticle

$$\psi = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

  When the particle is the same as the antiparticle, then the two spinors are related by

$$u_A = i\sigma^2 u_B^*$$

  so the dirac spinor becomes

$$\psi = C\psi^*$$

  where C is the charge conjugation (Majorana fermion).

- Dirac matrices must be atleast 4 dimensional.

Solutions to the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

General Solution

$$\psi = ae^{-ik\cdot x}u(k)$$

where $u(k)$ is the spinor part. Since

$$\begin{aligned} k^2 = \left(\frac{mc}{\hbar}\right)^2 \implies (\hbar k)^2 = (mc)^2 = p^2 \\ \implies \hbar k = \pm p \\ \text{or} \quad k_\mu = \pm\frac{p_\mu}{\hbar} \end{aligned}$$

where $+$ is the particle solution and $-$ is the antiparticle solution. The zeroth component would be

$$k^0 = \pm\frac{p^0}{\hbar} = \pm\frac{E^0}{\hbar}$$

so

$$\begin{aligned} \psi &\propto e^{-ip\cdot x/\hbar} \\ &= e^{\mp ip\cdot x/\hbar} \\ &\to \begin{cases} e^{-ip\cdot x/\hbar}\psi_A & \text{Particle} \\ e^{ip\cdot x/\hbar}\psi_B & \text{Antiparticle} \end{cases} \end{aligned}$$

From the solutions

$$u_A = \frac{\sigma\cdot\mathbf{k}}{k^0 - \frac{mc}{\hbar}}u_B$$

$$u_B = \frac{\sigma\cdot\mathbf{k}}{k^0 + \frac{mc}{\hbar}}u_A$$

Solution 1 $u^{(1)}$ If we choose a solution $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (Choosing u_A to be the particle solution), then

$$\begin{aligned} u_B &= \frac{\sigma \cdot \mathbf{p}/\hbar}{p^0/\hbar + mc/\hbar} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\sigma \cdot \mathbf{p}}{p^0 + mc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

and using

$$\begin{aligned} \sigma \cdot \mathbf{p} &= \sigma^1 p_x + \sigma^2 p_y + \sigma^3 p_z \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z \\ &= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \end{aligned}$$

we get

$$u_B = \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \frac{c}{E + mc^2}$$

Solution 2 $u^{(2)}$ If we choose $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$u_B = \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \frac{c}{E + mc^2}$$

Solution 3 $v^{(1)}$ The third solution is to choose $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then (choosing the minus sign for k)

$$\begin{aligned} u_A &= \frac{-\sigma \cdot \mathbf{p}}{-p^0 - mc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\sigma \cdot \mathbf{p}}{p^0 + mc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \frac{c}{E + mc^2} \end{aligned}$$

Solution 4 $v^{(1)}$ and similarly for $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_A = \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \frac{c}{E + mc^2}$$

So with the 4 solutions for $u(k)$

$$u^{(1)} = Ne^{-ip\cdot x\hbar}\begin{pmatrix} 1 \\ 0 \\ \frac{c}{E+mc^2}p_z \\ \frac{c}{E+mc^2}(p_x+ip_y) \end{pmatrix}$$
$$u^{(2)} = Ne^{-ip\cdot x\hbar}\begin{pmatrix} 0 \\ 1 \\ \frac{c}{E+mc^2}(p_x-ip_y) \\ -\frac{c}{E+mc^2}p_z \end{pmatrix}$$
$$v^{(2)} = Ne^{ip\cdot x\hbar}\begin{pmatrix} \frac{c}{E+mc^2}p_z \\ \frac{c}{E+mc^2}(p_x+ip_y) \\ 1 \\ 0 \end{pmatrix}$$
$$v^{(1)} = Ne^{ip\cdot x\hbar}\begin{pmatrix} \frac{c}{E+mc^2}(p_x-ip_y) \\ -\frac{c}{E+mc^2}p_z \\ 0 \\ 1 \end{pmatrix}$$

where $u^{(1)}$ is the particle with spin up, $u^{(2)}$ is the particle with spin down, $v^{(1)}$ is the antiparticle with spin down, and $v^{(2)}$ is the antiparticle with spin up. So

$$\psi = \begin{cases} ae^{-ip\cdot x/\hbar}u^{(1)} \quad \text{or} \quad u^{(2)} \quad \text{Particle} \\ ae^{ip\cdot x/\hbar}v^{(1)} \quad \text{or} \quad v^{(2)} \quad \text{Antiparticle} \end{cases}$$

where $\psi^\dagger\psi = 1$ or

$$u^\dagger u = \frac{2E}{c} \quad \text{or} \quad v^\dagger v = \frac{2E}{c}$$

Nonrelativistic Limit

$$\mathbf{p} = m\mathbf{v}, \quad E \approx mc^2$$

so

$$\frac{c}{E+mc^2}p_z \approx \frac{c}{2mc^2}mv_z = \frac{v_z}{2c} \to 0$$

and

$$\frac{c}{E+mc^2}(p_x+ip_y) \approx \frac{v_x+iv_y}{2c} \to 0$$

for $v \ll c$.

Dirac Equation in momentum space

$$\begin{aligned} (i\hbar\gamma^\mu\partial_\mu - mc)\psi &= 0 \\ p_\mu &\to i\hbar\partial_\mu \\ (\gamma^\mu p_\mu - mc)u &= 0 \quad \text{Particle} \\ (\gamma^\mu p_\mu + mc)v &= 0 \quad \text{Antiparticle} \end{aligned}$$

we usually use the notation $\not{p} = \gamma^\mu p_\mu$

Spin Operator

$$\mathbf{S} = \frac{\hbar}{2}\mathbf{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$
$$S^2 = \frac{\hbar^2}{4}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

since $\sigma_i^2 = I$, $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3I$ so

$$\begin{aligned} S^2 &= \frac{3\hbar^2}{4}I \\ &= \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 I \\ &= S(S+1)\hbar^2 I \\ \implies S &= \frac{1}{2}\hbar \end{aligned}$$

So for the total spin

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

then

$$[H, \mathbf{J}] = 0$$

from the Heisenberg equation of motion.

$$\frac{\mathrm{d}0}{\mathrm{d}t} = i\hbar[H, 0] = 0$$

We find that

$$[H, \mathbf{L}] \neq 0$$
$$[H, \mathbf{S}] \neq 0$$

or simply

$$[H, \mathbf{L}] = -[H, \mathbf{S}]$$

The Hamiltonion or total energy is

$$H = cp^0 = c\sqrt{\mathbf{p}^2c^2 + m^2c^4}$$

from the Dirac equation in momentum space, we know that

$$\begin{aligned} \gamma^\mu p_\mu - mc &= 0 \\ \text{or} \quad \gamma^0 p_0 + \gamma^j p_j - mc &= 0 \\ \text{or} \quad \gamma^0 p_0 &= -\gamma^j p_j + mc \end{aligned}$$

We know that

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

squaring γ^0 gives the identity matrix so

$$\begin{aligned} (\gamma^0)^2 p_0 &= -\gamma^0\gamma^j p_j + mc \\ p_0 &= -\gamma^0\gamma^j p_j + mc \\ &= \gamma^0\gamma^j p^j - mc \end{aligned}$$

Thus

$$H = c(\gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + mc)$$

Lorentz Transformation Properties Starting again from the Dirac Equation

$$(i\hbar\gamma^v\partial_v - mc)\psi = 0$$

Under Lorentz transformation we let

$$\psi \to \psi' = S\psi$$
$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi' = 0$$

We denote

$$\partial'_\mu = \frac{\partial x^v}{\partial x'_\mu}\partial x^v \equiv \frac{\partial x^v}{\partial x'_\mu}\partial_v$$
$$x^\mu \to^\Lambda x'^\mu$$

We also have axioms for the Dirac matrices (they don't transform) so

$$\left(i\hbar\gamma^\mu\frac{\partial x^v}{\partial x'_\mu}\partial_v - mc\right)S\psi' = 0$$

Mutliplying by S^{-1} on the left

$$S^{-1}i\hbar\gamma^\mu\frac{\partial x^v}{\partial x'_\mu}S(\partial_v\psi) - mc\psi = 0$$

so matching this to the original Dirac equation

$$S^{-1}\gamma^\mu\frac{\partial x^v}{\partial x'_\mu}S = \gamma^v$$

Since the S commutes we can write

$$S^{-1}\gamma^\mu S\frac{\partial x^v}{\partial x'_\mu} = \gamma^v$$

If the frame moves along the x-axis,

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where we have the lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

we can show that

$$S = a_+ + a_-\gamma^0\gamma^1 \qquad a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$$

In matrix form

$$S = a_+ I + a_-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}$$

where I is a 4x4 identity matrix and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & -a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}$$

a symmetric real matrix

Invariant

$$\psi \to |\psi|^2 = \psi^\dagger \psi = 1$$

For invariant quantities in non relativistic QM. For relativistic transformations

$$\begin{aligned} \psi^\dagger \psi \to^\Lambda &(S\psi)^\dagger (S\psi) \\ &= \psi^\dagger S^\dagger S \psi \end{aligned}$$

But since S is real and symmetric $S^\dagger = S$ So

$$= \psi^\dagger S^2 \psi$$

We find that

$$S^2 = \gamma \begin{pmatrix} I & -\frac{v}{c}\sigma_1 \\ -\frac{v}{c}\sigma_1 & I \end{pmatrix} \neq I$$

So we need to find the lorentz invariant quanities (Bilinears)

Bilinears Under Lorentz transformations

$$\begin{aligned} \psi^\dagger \psi &\to \psi'^\dagger \psi' \\ &= (S\psi)^\dagger (S\psi) = \psi^\dagger S^\dagger S \psi \end{aligned}$$

Since S is real and symmetric, $S^\dagger = S$ or $S^\dagger S = S^2$ where

$$S^2 = \gamma \begin{pmatrix} I & -\frac{v}{c}\sigma_1 \\ -\frac{v}{c}\sigma_1 & I \end{pmatrix} \neq I$$

For $v \ll c$, $\quad S^2 \to I$. So we need to find a Lorentz invariant quantity, or the adjoint spinor, $\bar{\psi} = \psi^\dagger \gamma^0$:

$$\begin{aligned} \bar{\psi}\psi \to \bar{\psi}'\psi' &= (\psi'^\dagger \gamma^0)\psi' = (S\psi)^\dagger \gamma^0 S \psi \\ &= \psi^\dagger S^\dagger \gamma^0 S \psi = \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi \end{aligned}$$

This is one of the bilinears ($\bar{\psi}\psi$) that is lorentz invariant.

Discrete Symmetry Operators

$$\begin{aligned} P &= \gamma^0 \qquad \psi(t,\mathbf{x}) \to \gamma^0 \psi(t, -\mathbf{x}) \\ C &= i\gamma^2 \qquad \psi(t,\mathbf{x}) \to i\gamma^2 \psi^*(t, \mathbf{x}) \\ T &= \gamma^1\gamma^3 \qquad \psi(t,\mathbf{x}) \to \gamma^1\gamma^3 \psi(-t, \mathbf{x}) \end{aligned}$$

So under parity

$$\begin{aligned} \bar{\psi}\psi \to^P \bar{\psi}'\psi' &= \psi'^\dagger \gamma^0 \psi' \\ &= (\gamma^0\psi)^\dagger \gamma^0 (\gamma^0 \psi) \\ &= \psi^\dagger \gamma^{0\dagger} (\gamma^0\gamma^0)\psi \qquad \gamma^0\gamma^0 = I \\ &= (\psi^\dagger \gamma^0)\psi = \bar{\psi}\psi \end{aligned}$$

which is P-even. For The pseudoscalar

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

where

$$\begin{aligned} &\{\gamma^\mu, \gamma^v\} = 2g^{\mu v} \\ \implies &\{\gamma^\mu, \gamma^v\} = 0 \quad \text{if} \quad \mu \neq v \end{aligned}$$

and

$$\{\gamma^\mu, i\gamma^0\gamma^1\gamma^2\gamma^3\} = i(\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu)$$

And at $\mu = 0$:

$$\begin{aligned} &= i(\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0) \\ &= i(\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3) = 0 \qquad \gamma^\mu\gamma^v = -\gamma^v\gamma^\mu \end{aligned}$$

So

$$\{\gamma^\mu, \gamma^5\} = 0$$

So the Parity operator

$$\begin{aligned} \bar{\psi}\gamma^5\psi \to^P \bar{\psi}'\gamma^5\psi' &= \psi'^\dagger \gamma^0\gamma^5\psi' \\ &= (\gamma^0\psi)^\dagger \gamma^0\gamma^5(\gamma^0\psi) \\ &= \psi^\dagger \gamma^{0\dagger}\gamma^0\gamma^5\gamma^0\psi \\ &= \psi^\dagger \gamma^5\gamma^0\psi = -\bar{\psi}\gamma^5\psi \end{aligned}$$

which is P-odd. Thus the Bilinears are

1. $\bar{\psi}\psi$: P-even (scalar)
2. $\bar{\psi}\gamma^5\psi$: P-odd (pseudoscalar)
3. $\bar{\psi}\gamma^\mu\psi$: Vector
4. $\bar{\psi}\gamma^\mu\gamma^5\psi$: Axial Vector
5. $\bar{\psi}\sigma^{\mu v}\psi$: Tensor where $\sigma^{\mu v} = \frac{i}{2}[\gamma^\mu, \gamma^v]$

Clifford Algebra Any 4×4 matrix can be written in the basis of the five bilnears

$$\{I, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu v}\}$$

So there are

$$1 + 1 + 4 + 4 + 6 = 16$$

independent 4×4 matrices.

Momentum Space

$$\begin{aligned}
(\gamma^\mu p_\mu - mc)u^{(s)} &= 0 \quad \text{(Particle)} \\
(\gamma^\mu p_\mu + mc)v^{(s)} &= 0 \quad \text{(Antiparticle)} \\
\bar{u}(\gamma^\mu p_\mu - mc) &= 0 \\
\bar{v}(\gamma^\mu p_\mu + mc) &= 0
\end{aligned}$$

where

$$\begin{aligned}
u^\dagger u = v^\dagger v &= \frac{2E}{c} \\
\implies \bar{u}u = -\bar{v}v &= 2mc
\end{aligned}$$

We also get the completeness relation

$$\begin{aligned}
\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} &= \gamma^\mu p_\mu + mc \\
\sum_{s=1,2} v^{(s)}\bar{v}^{(s)} &= \gamma^\mu p_\mu - mc
\end{aligned}$$

In the basis states $|a_i\rangle$ where

$$I = \sum_i |a_i\rangle \langle a_i|$$

so

$$|\psi\rangle = \sum_i |a_i\rangle \langle a_i|\psi\rangle = \sum_i c_i |a_i\rangle$$

Photon

In QED we have just an electron/position and a photon. We found electron/positron but now to find the photon. From the maxwell equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

THe EM field strength tensor $F^{\mu\nu}$ is defined as

$$F^{\mu v} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and the current vector $J^\mu = (\rho c, \mathbf{J})$.

Inhomogenous Maxwell Equations For the First and Fourth equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

where

$$F^{\mu\nu} = -F^{\nu\mu}$$

so

$$\begin{aligned} \partial_\nu \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} \partial_\nu J^\nu \\ 0 &= \partial_\nu J^\nu \end{aligned}$$

Which is equivalent to the charge conservation

$$\begin{aligned} \frac{1}{c}\frac{\mathrm{d}\rho c}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \\ \frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \end{aligned}$$

and integrating over a volume

$$\begin{aligned} \int \mathrm{d}V \left(\frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot \mathbf{J} \right) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}V\, \rho + \int \mathrm{d}V\, \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \frac{\mathrm{d}Q}{\mathrm{d}t} + \oint \mathbf{J} \cdot \mathrm{d}\mathbf{S} &= 0 \end{aligned}$$

From Gauss' Law so we do indead find that

$$\mathbf{J} = 0 \implies Q = \text{constant}$$

From the Hemholtz potential we know that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, so the second Maxwell eq is

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla} \times \mathbf{A})$$

or

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} \right) = 0$$

or

$$\mathbf{E} + \frac{1}{c}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = -\boldsymbol{\nabla} V$$

So we have an EM field

$$A^\mu = (V, \mathbf{A})$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Which gives us the general Maxwell eq

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c} J^\nu$$

From the Gauge transformation

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \lambda$$

then

$$F^{\mu\nu} \to F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$$
$$= \partial^\mu(A^\nu + \partial^\nu\lambda) - \partial^\nu(A^\mu + \partial^\mu\lambda) = F^{\mu\nu}$$

THe Gauge fixing condition $\partial_\mu A^\mu = 0$ so the maxwell eq becomes

$$\partial_\mu\partial^\nu A^\nu = \frac{4\pi}{c} J^\nu$$

or

$$\Box A^\nu = \frac{4\pi}{c} J^\nu$$

If $\Box\lambda = 0$, choose a particular gauge, or the COulomb gauge

$$A^0 = 0$$

in free space. For $\partial_\mu A^\mu = 0$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0$$

In free space there is no current density, $J^\mu = 0$

$$\Box A^\mu = 0$$

which is the Klein-Gordan eq with mass $m = 0$. So the solution is a plane wave

$$A^\mu = ae^{-ipx/\hbar}\epsilon^\mu$$

where ϵ^μ is the polarization vector. So

$$\partial_\mu A^\mu = 0 \implies p_\mu \epsilon^\mu = 0$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \implies \mathbf{p} \cdot \boldsymbol{\epsilon} = 0$$

so $\boldsymbol{\epsilon}$ is perpendicular to $\mathbf{p}$. The components of the vector are

$$\mathbf{p} = p\hat{\mathbf{z}}$$

and

$$\epsilon^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

so

$$\epsilon_\pm = \frac{1}{\sqrt{2}}(\epsilon^{(1)} \pm i\epsilon^{(2)})$$

or circular polarization.

Photon We have

$$A^\mu = ae^{-ipx/\hbar}\epsilon^\mu_{(s)}$$

where $s = 1, 2$ for two polarizations. The lorentz condition tells us $p_\mu \epsilon^\mu = 0$. And the Coulomb gauge $\mathbf{p} \cdot \boldsymbol{\epsilon} = 0$. The completeness relation

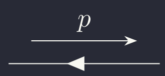
$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s_1)*} = \delta_{ij} - \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j$$

Feynman Rules

1. - External Momental: p_i
   - Internal Momenta: q_i
2. - Fermions: (straight line) Fermion and Momentum flow in same direction: $u^{(s)}(p)$ for incoming, $\bar{u}^{(s)}(p)$ for outgoing

   - Anti-fermion: Fermion and Momentum flow in opposite direction: $\bar{v}^{(s)}(p)$ for incoming, $v^{(s)}(p)$ for outgoing

   - Photon: (wave line) $\epsilon_\mu(p)$ for incoming, $\epsilon^*_\mu(p)$ for outgoing
3. Vertex: $ig_e\gamma^\mu$ for fermion-photon vertex

$$g_e = \sqrt{4\pi\alpha} = \sqrt{4\pi\frac{e^2}{\hbar c}} = e\sqrt{\frac{4\pi}{\hbar c}}$$

4. Propogator: For the particle/antiparticle

$$\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$$

   For the photon

$$\frac{-ig_{\mu v}}{q^2}$$

5. Conservation of four-momenta at each vertex

$$(2\pi)^4\delta^4(k_1 + k_2 + k_3)$$

   where k's are incoming momenta
6. Integrate over internal momenta

$$\int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\frac{\mathrm{d}^4 q_2}{(2\pi)^4}\cdots$$

7. Drop $(2\pi)^4\delta^4(p_1 + p_2 + \dots - p_n - p_{n+1} - \dots)$
8. Multiply the answer by i:
9. Anti-symmetrization: Diagrams differing only by the inter change of identical fermions have a relative minus sign

Example: e-μ scattering From the matrix multiplication we need $(1\times 4)(4\times 4)(4\times 1)$ so

$$\bar{u}(p_3)(ig_e\gamma^\mu)u(p_1)$$

or in the opposite direction of the fermion flow. The amplitude is

$$\mathcal{M} = i\int \frac{\mathrm{d}^4 q}{(2\pi)^4}[\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)]\left(\frac{-ig_{\mu v}}{q^2}\right)[\bar{u}^{(s_4)}(p_4)(ig_e\gamma^\mu)u^{(s_2)}(p_2)]$$
$$\times(2\pi)^4\delta^4(p_1 - q - p_3)\delta^4(p_2 + q - p_4)$$

and since $q = p_4 - p_2$ substituting in to the first delta function

$$(2\pi)^4\delta^4(p_1 - (p_4 - p_2) - p_4) = (2\pi)^4\cancel{\delta^4(p_1 + p_2 - p_3 - p_4)}$$

Which cancels out from rule 7. Thus we ge the amplitude

$$\mathcal{M} = \frac{-g_e^2}{(p_4 - p_2)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]$$
$$g_{\mu v}\gamma^v = \gamma_\mu$$

Where

$$\begin{aligned} p_1 + p_2 &= p_3 + p_4 \\ \implies p_4 - p_2 &= p_1 - p_3 \end{aligned}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto |\mathcal{M}|^2$$

Example: e^-e^- Scattering (Møeller Scattering) If we have the diagram set horizontally, we actually have $e^-e^+ \to e^-e^+$ scattering (Bhabha Scattering). The first diagram is the same as the electron-muon scattering:

$$\begin{aligned} \mathcal{M}_1 &= \frac{-g_e^2}{(p_1 - p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)] \\ \mathcal{M}_2 &= \frac{-g_e^2}{(p_1 - p_4)^2}[\bar{u}^{(s_4)}(p_4)\gamma^\mu u^{(s_2)}(p_2)][\bar{u}^{(s_3)}(p_3)\gamma_\mu u^{(s_1)}(p_1)] \end{aligned}$$

since the momentum of p_3 and p_4 are interchanged so the total amplitude is

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$$

where the minus sign comes from rule 9, or the interchange of identical fermions.

Example: e^-e^+ Scattering (Bhabha Scattering) The bottom part of the diagram is the same as the electron-muon scattering:

$$\begin{aligned} \mathcal{M}_1 = i\int \frac{\mathrm{d}^4q}{(2\pi)^4}&[\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)] \\ &\times\left(\frac{-ig_{\mu v}}{q^2}\right)[\bar{v}^{(s_2)}(p_2)(ig_e\gamma^\mu)v^{(s_4)}(p_4)] \\ &\times(2\pi)^4\delta^4(p_1 - q - p_3)\delta^4(p_2 + q - p_4) \end{aligned}$$

and again using $q = p_4 - p_2$ we can cancel out the delta functions

$$= -\frac{g_e^2}{(p_4 - p_2)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{v}^{(s_2)}(p_2)\gamma_\mu v^{(s_4)}(p_4)]$$

for the first diagram, and the second diagram is a combination:

$$\begin{aligned} \mathcal{M}_2 = i\int \frac{\mathrm{d}^4q}{(2\pi)^4}&[\bar{v}^{(s_2)}(p_2)(ig_e\gamma^\mu)u^{(s_1)}(p_1)] \\ &\times\left(\frac{-ig_{\mu v}}{q^2}\right)[\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)v^{(s_4)}(p_4)] \\ &\times(2\pi)^4\delta^4(p_1 + p_2 - q)\delta^4(q - p_3 - p_4) \end{aligned}$$

using the second delta function again $q = p_3 + p_4$ so

$$p_1 + p_2 - q = p_1 + p_2 - p_3 - p_4$$

so the amplitude is

$$\mathcal{M}_2 = -\frac{g_e^2}{(p_3 + p_4)^2}[\bar{v}^{(s_2)}(p_2)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_3)}(p_3)\gamma_\mu v^{(s_4)}(p_4)]$$

and the total amplitude is

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$$

where the plus is because we have two different fermions... this is wrong. Interchanging the 2nd and 4th fermions on the second diagram gives the first diagram.

Matrix Elements

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S}{(E_1 + E_2)^2}|\mathcal{M}|^2 \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

so the squared factor

$$\begin{aligned}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}&[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)] \\ &\times [\bar{u}^{(s_3)}(p_3)\gamma^\nu u^{(s_1)}(p_1)]^\dagger[\bar{u}^{(s_4)}(p_4)\gamma_\nu u^{(s_2)}(p_2)]^\dagger\end{aligned}$$

where the Hermitian conjugate of the two terms are

$$u^\dagger(p_1)\gamma^{\nu\dagger}\gamma^{0\dagger}u(p_3)$$

multiplying by $\gamma^0\gamma^0$:

$$\begin{aligned}(u^\dagger(p_1)\gamma^0)\gamma^0\gamma^{\nu\dagger}\gamma^{0\dagger}u(p_3) \\ = \bar{u}(p_1)\gamma^0\gamma^{\nu\dagger}\gamma^0 u(p_3) \\ = \bar{u}(p_1)\gamma^\nu u(p_3)\end{aligned}$$

and similarly for the second term:

$$\bar{u}(p_2)\gamma_\nu u(p_4)$$

For the unpolarized cross sections we sum over the final spins and average over the initial spins:

$$\left|\bar{\mathcal{M}}\right|^2 = \frac{1}{4}\sum\sum$$

Quiz Review:

- $$\begin{aligned}\gamma^\mu\gamma_\mu &= \gamma^0\gamma_0 + \gamma^1\gamma_1 + \gamma^2\gamma_2 + \gamma^3\gamma_3\\ &= \gamma^0 g_{00}\gamma^0 + 3\gamma^i\gamma^i g_{ii}\\ &= 1 + 3(-1)(-1) = 4\end{aligned}$$

- Since $\not{a}\not{b} = a_\mu\gamma^\mu b_\nu\gamma^\nu$ we have
$$\operatorname{Tr}\big(\not{a}\not{b}\big) = a_\mu b_\nu \operatorname{Tr}(\gamma^\mu\gamma^\nu)$$
and since the Trace is cyclic
$$\begin{aligned}&= \frac{1}{2}a_\mu b_\nu \operatorname{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\\ &= a_\mu b_\nu g^{\mu\nu}\operatorname{Tr}(I_{4x4}) = 4a_\mu b^\mu = 4a\cdot b\end{aligned}$$

- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and from the cyclicity of the trace
$$\operatorname{Tr}\big(\gamma^\mu\gamma^5\big) = \operatorname{Tr}\big(\gamma^5\gamma^\mu\big) = -\operatorname{Tr}\big(\gamma^5\gamma^\mu\big)$$
which is only true if the trace is zero.

- From the anticommutator $\{\gamma^\mu, \gamma^\nu = 2g^{\mu\nu}\}$ so
$$\begin{aligned}\operatorname{Tr}\big(\{\gamma^\mu,\gamma^\nu\},\gamma^5\big) &= 2g^{\mu\nu}\operatorname{Tr}\big(\gamma^5\big) = 0\\ &= \operatorname{Tr}\big(\gamma^\mu\gamma^\nu\gamma^5 + \gamma^\nu\gamma^\mu\gamma^5\big) = 0\end{aligned}$$
So the quantity is an antisymmetric tensor or rank 2, which does not exist.

From the completeness relation
$$\begin{aligned}\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} &= \gamma^\mu p_\mu + mc = \not{p} + mc\\ \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} &= \not{p} - mc\end{aligned}$$

So from the squared amplitude:
$$\begin{aligned}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1-p_3)^4}&[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]\\ &\times[\bar{u}_1\gamma^\nu u_3][\bar{u}_2\gamma_\nu u_4]\end{aligned}$$

we can rearrange the terms by writing the 3rd between the first and second terms and the 4th between the 2nd and 3rd terms:
$$\begin{aligned}\sum_{s_1,s_2}|\mathcal{M}|^2 &= \frac{g_e^4}{(p_1-p_3)^4}[\bar{u}_3\gamma^\mu(\sum_{s_1} u^{(s_1)}(p_1)\bar{u}^{(s_1)}(p_1))\gamma^\nu u_3]\\ &\qquad[\bar{u}_4\gamma_\mu(\sum_{s_2} u^{(s_2)}(p_2)\bar{u}^{(s_2)}(p_2))\gamma_\nu u_4]\\ &= \frac{g_e^4}{(p_1-p_3)^4}[\bar{u}_3[_i\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu]_{ij}u_3]_{jk}\\ &\qquad[\bar{u}_4[_k\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu]_{kl}u_4]_l\end{aligned}$$

so the sum over the spins (sub indices) gives us

$$= \frac{g_e^4}{(p_1 - p_3)^4}[\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu][\sum_{s_3} u(p_3)\bar{u}(p_3)][\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu]$$
$$[\sum_{s_4} u(p_4)\bar{u}(p_4)]$$
$$= \frac{g_e^4}{(p_1 - p_3)^4}[\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu]_{ij}(\not{p}_3 + m_e c)_{ji}$$
$$[\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu]_{lm}(\not{p}_4 + m_\mu c)_{ml}$$

and the sum over the indices

$$\sum_{ij} A_{ij}B_{ji} = \mathrm{Tr}(AB)$$

so the amplitude in terms of the traces is

$$= \frac{g_e^4}{(p_1 - p_3)^4}\,\mathrm{Tr}(\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu(\not{p}_3 + m_e c))$$
$$\mathrm{Tr}(\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu(\not{p}_4 + m_\mu c))$$

where the first Trace is the first fermion flow, and the second Trace is the second "disconnected" fermion flow. The electron flow is disconnected from the muon flow, so the Traces are separated out.

Expanding out the first trace

$$\mathrm{Tr}\big[\gamma^\mu p_{1k}\gamma^k + \gamma^\mu m_e c(\gamma^\nu p_{3b}\gamma^b + \gamma^\nu m_e c)\big]$$
$$= \mathrm{Tr}\big(p_{1k}p_{3b}\gamma^\mu\gamma^k\gamma^\nu\gamma^b\big) + \mathrm{Tr}\big((m_e c)^2\gamma^\mu\gamma^\nu\big)$$

the second term is

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu)$$
$$\implies = \frac{1}{2}\,\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$
$$= \frac{1}{2}\,\mathrm{Tr}(2\gamma^{\mu\nu}) = g^{\mu\nu}\,\mathrm{Tr}\{I\} = 4g^{\mu\nu}$$

and the first term is

$$\mathrm{Tr}\big(p_{1k}p_{3b}\gamma^\mu\gamma^k\gamma^\nu\gamma^b\big) = p_{1k}p_{3b}4\,\mathrm{Tr}\big(g^{\mu k}g^{\nu b} - g^{\mu\nu}g^{kb} + g^{\mu b}g^{k\nu}\big)$$

and after finding the second trace we know

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}$$
$$[(p_1p_2)(p_3p_4) + (p_1p_4)(p_3p_2)$$
$$- (p_1p_3)m_\mu^2c^2 - (p_2p_4)m_e^2c^2 + 2(m_e m_\mu c^2)^2]$$

Mott Scattering Using the assumption that the the muon mass M is much larger than the electron mass m, i.e., $M \gg m$. In the lab frame $\mathbf{p}_2 = 0$: Before the collision

$$p_1 = (E_1, \mathbf{p}_1), \quad p_2 = (Mc, \mathbf{0})$$

and after

$$p_3 = (E_3, \mathbf{p}_3), \quad p_4 \approx (Mc, 0)$$

Where $|\mathbf{p}_1| = |\mathbf{p}_3| = p$ and $\mathbf{p}_1 \cdot \mathbf{p}_3 = p^2 \cos\theta$. We also know

$$\begin{aligned} p_1 \cdot p_3 &= E_1 E_3 - \mathbf{p}_1 \cdot \mathbf{p}_3 \\ &= E^2 - p^2 \cos\theta \end{aligned}$$

or

$$= p^2(1 - \cos\theta) = 2p^2 \sin^2(\theta/2)$$

Finally the sum of the final spin and average of the initial spin is

$$\left\langle |\mathcal{M}|^2 \middle| \right\rangle = \frac{g_e^2 Mc}{p^2 \sin^2 \theta/2} p^2 \cos^2 \theta/2$$

and the scattering differential

$$\frac{\partial\sigma}{\partial\Omega} = (\frac{\alpha\hbar}{2p^2 \sin^2 \theta/2})^2 p^2 \cos^2 \theta/2$$

Which approaches infinity as $\theta \to 0$ or $\theta \to \pi$.

Bhabha Scattering For $e^-e^+ \to e^-e^+$ we have two diagrams, one with electron flow & position flow, and another with electron to position flow. For $e^-e^- \to \mu^-\mu^+$ we have only one diagram with the matrix element:

$$\begin{aligned}\mathcal{M} = i \int \cancel{\mathrm{d}^4 q \frac{1}{(2\pi)^4}} &[\bar{v}(p_2)(ig_e\gamma^\mu)u(p_1)] \\ \left(\frac{-ig_{\mu\nu}}{q^2}\right) &[\bar{u}(p_3)(ig_e\gamma^\nu)v(p_4)]\end{aligned}$$

And taking the limit of

$$E \gg (Mc)^2 \gg (mc)^2$$

The differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\Pi\alpha^2}{2E_{\mathrm{cm}}^2}(1+\cos^2\theta)$$
$$\sigma = \frac{\Pi\alpha^2}{3E^2} \quad E_{\mathrm{cm}} = 2E$$

Within the snapshots of time, the electron positron pair transfer all the energy into a real photon, and the photon transforms to the muon pair.

$e^+e^- \to q\bar{q}$ This is the same but we have a quark charge Q:

$$\begin{aligned}\mathcal{M} = i \int \cancel{\mathrm{d}^4 q \frac{1}{(2\pi)^4}} &[\bar{v}(p_2)(ig_e\gamma^\mu)u(p_1)] \\ \left(\frac{-ig_{\mu\nu}}{q^2}\right) &[\bar{u}(p_3)(iQg_e\gamma^\nu)v(p_4)]\end{aligned}$$

and the cross section is

$$\sigma = \frac{\Pi Q^2\alpha^2}{3E^2}$$

In experiment we have the quarks hadronize into two mesons. So the ratio of the cross section is

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum Q_i^2$$

For $E < m_c^2$ the ratio is

$$R = 3(Q_u^2 + Q_d^2 + Q_s^2) = 3\left((\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{1}{3})\right)^2 = 2$$

For $m_c c^2 < E < m_b c^2$ We have to introduce the charm quark

$$R = 2 + 3Q_c^2 = 2 + 3\frac{4}{9} = 3.333$$

For $m_b c^2 < E < m_t c^2$ we have to introduce the bottom quark

$$R = 3.333 + 3Q_b^2 = 3.333 + 3\frac{1}{9} = 3.67$$

For $E > m_t c^2$ we have to introduce the top quark

$$R = 3.67 + 3Q_t^2 = 3.67 + 3\frac{4}{9} = 5$$

$e^-e^- \to ss$ Or the ϕ-meson which results in a peak at each of the resonances.

Up down quarks The pions are the lightest mesons, and we don't have scalar meson peaks because the photon has spin 1, so the meson of spin 0 can not be produced.

Electron-Proton Scattering It must be a vertical diagram, with electron p_1, proton p_2, and electron p_3 and proton p_4. The matrix element squared is

$$\begin{aligned}\langle |\,|\mathcal{M}|^2\,|\rangle = \frac{4g_e^4}{q^4}[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}((mc)^2 - (p_1 \cdot p_3))] \\ [p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}((Mc)^2 - (p_2 \cdot p_4))] \\ = \frac{4g_e^4}{q^4}(L_{\text{electron}^{\mu\nu}})(L_{\mu\nu})_{\text{proton}}\end{aligned}$$

but the photon interacts arbitrarily with the proton quarks uud so

$$\langle |\,\mathcal{M}^2\,|\rangle = \frac{4g_e^4}{q^4}(L_{\text{electron}})_{\mu\nu}(K_{\mu\nu})_{\text{proton}}$$

where $K_{\mu\nu}(p_2, q, p_4)$ is an unknown describing the vertex of the photon with the proton. From momentum conservation

$$p_2 + q = p_4 \to K_{\mu\nu}(p, q) \quad p \equiv p_2, q = p_4 - p_2$$

So to construct a 2nd rank tensor

$$\begin{aligned}K_{\mu\nu} = -K_1 g_{\mu\nu} + \frac{K_2}{(Mc)^2}p_\mu\nu \\ + \frac{K_4}{(Mc)^2}q_\mu q_\nu \\ + \frac{K_5}{(Mc)^2}(p_\mu q_\nu + q_\mu p_\nu) \\ +(\frac{K_6}{(Mc)^2}p_\mu q_\nu - q_\mu p_\nu)\end{aligned}$$

where the last term has a form factor of zero because the matrix element is symmetric and summing with the antisymmetric part of the $K_{\mu\nu}$ gives zero.

$$q^\mu K_{\mu\nu} = 0$$

since

$$\begin{aligned}q^\mu L_{\mu\nu} &= 0 \\ &= q^\mu(p_{1\mu}p_{3\nu} + p_{1\nu}p_{3\mu} + g_{\mu\nu}((mc)^2 - p_1 \cdot p_3))\end{aligned}$$

...

$$\begin{aligned}K_4 = \frac{(Mc)^2}{q^2}K_1 + \frac{1}{4}K_2 \\ K_5 = \frac{1}{2}K_2 \\ K_{\mu\nu} = K_1\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{(Mc)^2}\right) + \frac{K_2}{(Mc)^2}(p_\mu + \frac{1}{2}q_\mu)(p_\nu + \frac{1}{2}q_\nu)\end{aligned}$$

where (K_1, K_2) are the proton form factors. The differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\alpha\hbar}{4ME\sin^2\theta/2}\right)^2 \times \frac{2K_1\sin^2\theta/2 + K_2\cos^2\theta/2}{1 + \frac{2E}{Mc^2}\sin^2\theta/2}$$

If $E \ll Mc^2$ then we can neglect the second term in the denominator, and

$$K_1 \approx -q^2$$
$$K_2 \approx (2Mc)^2$$

also known as the Dirac Limit. The cross section is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \left(\frac{\alpha\hbar c}{2E\sin^2\theta/2}\right)^2 \cos^2\theta/2$$

which is the Mott formula.

Quark-Quark Scattering This is an uninteresting process becuase the strong interaction (or exchange of gluons) that occurs (QCD).

Feynman Rules for QCD

1. Fermions: Incoming $u^{(s)}(p)c$, Outgoing $\bar{u}^{(s)}(p)c^\dagger$ where we have a color matrix of basis

$$c = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{red} \quad , c = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{blue} \quad , c = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{green}$$

   Antifermions: Incoming $\bar{v}^{(s)}(p)c^\dagger$, Outgoing $v^{(s)}(p)c$

2. Vertex: $-\frac{ig_s}{2}\gamma^\mu$ where the strong charge is

$$g_s = \sqrt{4\pi\alpha_s}, \quad \alpha_s = \frac{g_s^2}{\hbar c}$$

   Propogator: gluons(spring) $\frac{-g_{\mu\nu}}{q^2}\delta_{ab}$

   Additional Vertices (due to gluon color charge): e.g. 3 & 4 gluon vertex (glueball)

Exam Overview

- 5 Multiple Choice
- 2 Short
- 2 Long
- 1 Bonus

Quiz Review

- Gluon ($c\bar{c}$) where the color can be $c, \bar{c} = r, g, b$. Thus like mesons we have $8 \oplus 1$ states (octet and singlet). In SU(3) we have 8 generators which can give us the gluon states

  $$|\alpha_i\rangle = \begin{pmatrix} r & g & b \end{pmatrix} \lambda_i \begin{pmatrix} \bar{r} \\ \bar{g} \\ \bar{b} \end{pmatrix}$$

  so the first gluon state is

  $$\begin{aligned} |1\rangle &\propto \begin{pmatrix} r & g & b \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{r} \\ \bar{g} \\ \bar{b} \end{pmatrix} \\ &= (r\bar{g} + g\bar{r}) \frac{1}{\sqrt{2}} \end{aligned}$$

  and

  $$\begin{aligned} |2\rangle &= -\frac{1}{\sqrt{2}}(r\bar{g} - g\bar{r}) \\ |3\rangle &= \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}) \\ &\dots \\ |8\rangle &= \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b}) \end{aligned}$$

  The singlet state

  $$|9\rangle = \frac{1}{3}(r\bar{r} + g\bar{g} + b\bar{b})$$

  does not exist. Color singlets are always colorless, but the reverse is not true. But since the gluon is massless, this singlet state would act like a photon with a long range force.

$q\bar{q}$ **Scattering** For Different flavors, we only need one vertical diagram of

$$p_2, c_2 \to p_4, c_4 \quad p_3, c_3 \to p_1, c_1$$

The matrix element is

$$\begin{aligned} \mathcal{M} &= i[\bar{u}(3)c_3^\dagger(-ig_s\gamma^\mu \frac{\lambda^\alpha}{2})u(1)c_1]\left(\frac{ig_{\mu\nu}}{q^2}\delta_{\alpha\beta}\right)[\bar{v}(2)c_2^\dagger(-ig_s\gamma^\nu \frac{\lambda^\beta}{2})v(4)c_4] \\ &= -\frac{g_s^2}{q^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{v}(2)\gamma_\mu v(4)]\frac{1}{4}(c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\beta c_4) \end{aligned}$$

where we have a color factor added to the QED matrix element

$$f = \frac{1}{4}(c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\beta c_4)$$

Octet Examples $r\bar{g}$: The initial states

$$c_1 = r = \begin{pmatrix}1\\0\\0\end{pmatrix} = c_3, \quad c_2 = g = \begin{pmatrix}0\\1\\0\end{pmatrix} = c_4$$

so the color must stay the same in the fermion flow. The color factor is

$$\begin{aligned} f &= \frac{1}{4}(\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix})(\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^\beta \begin{pmatrix}0\\1\\0\end{pmatrix}) \\ &= \frac{1}{4}\lambda^\alpha_{11}\lambda^\beta_{22} \end{aligned}$$

The only non-zero term is given by λ^3, λ^8 so

$$\begin{aligned} f &= \frac{1}{4}(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22}) \\ &= -\frac{1}{6} \end{aligned}$$

Color Singlet: Color factor

$$\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

This means that the incoming quarks and outcoming quarks are

$$c_1 = c_3 = rgb, \quad c_2 = c_4 = rgb$$

so the color factor is

$$\begin{aligned} f &= \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_2^\dagger \lambda^\beta c_4) \\ &= \frac{1}{4}\frac{1}{\sqrt{3}^2}\lambda^\alpha_{ij}\lambda^\alpha_{ji} \\ &= \frac{1}{12}\operatorname{Tr}(\lambda^\alpha\lambda^\alpha) = \frac{1}{12}2\delta_{\alpha\alpha} = \frac{1}{12}2(8) = \frac{4}{3} \end{aligned}$$

For the Hydogen atom the potential is

$$V = -\frac{e^2}{r} = -\frac{\alpha\hbar c}{r}$$

so the potentials for the quarks are

$$V = -f\frac{\alpha_s\hbar c}{r} = \begin{cases} \frac{1}{6}\frac{\alpha_s\hbar c}{r} & \text{Octet} \\ -\frac{4}{3}\frac{\alpha_s\hbar c}{r} & \text{Singlet} \end{cases}$$

This lower potential for the singlet state tells us that mesons bind in the singlet state.

qq **Scattering** (Different flavors for simplicity) The color factor is

$$f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_4^\dagger \lambda^\beta c_2)$$

since the flow is reversed for the second flow. We have a sextet(symmetric) and triplet(antisymmetric) configurations for the gluon:

$$3 \otimes 3 = 6 \oplus 3$$

and the color factos is

$$f = \begin{cases} \frac{1}{3} & \text{Sextet} \\ -\frac{1}{3} & \text{Triplet} \end{cases}$$

Weak Interaction

- Charged-current ($W^{\pm}$)
- Neutral-current (Z^0)

For the charge current we have $e^- \to W^- \nu_e$ for leptons and $d \to W^- u$ for quarks.

Neutral Current $\nu_e \to Z\nu_e$, $e \to Ze$ etc.

Feynman rule changes The vertex facor

$$-\frac{g_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

is in the form $V - A$ (Parity Violation). For the neutral current we have

$$(c_v\gamma^\mu - c_A\gamma^\mu\gamma^5)$$

For the partial vector current interaction.

Propogator

$$\frac{-i(g^{\mu\nu} - q^\mu q^\nu/(Mc)^2)}{q^2 - (Mc)^2}$$

where if $q^2 \ll (Mc)^2$ we have

$$\frac{ig^{\mu\nu}}{(Mc)^2}$$

β-decay (Neutron)

$$n(udd) \to p(uud) + e^- + \bar{\nu}_e$$

Where $M_W = 80.4$ GeV$/c^2$ and $M_Z = 91.2$ GeV$/c^2$. So the decay is mediated by the W^- boson

$$\Gamma \propto \frac{g_w^4}{M_W^4}$$

Fermi β-decay theory The is a constant $G_F \sim \frac{g_w^2}{M_W^2}$ which is the Fermi constant (Effective Field Theory). This contracts the W boson to a point-like interaction.