1 Newtons Laws

The Four Horsemen of the Apocalypse (In Physics)

- Classical Mechanics
- Electromagnetism
- Statistical Mechanics
- Quantum Mechanics

Before 1900, there was no relativity or QM and the world was a simple place . . .

Newton's 1st Law: The Law of Inertia
And object keeps going unless acted on by a force.
This only applies to and 'inertial frame'.

Newton's 2nd Law: $\mathbf{F} = m\mathbf{a}$
Sum notation: The position vector is

$$\mathbf{r} = (x, y, z) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$$

in the Cartesian coordinate system. The time derivative gives the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

and acceleration is the time derivative of velocity

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

Thus in vector notation, Newton's 2nd law is

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$$

where $\mathbf{r}(t)$ is and ordinary differential equation (ODE).
The basic idea of solving mechanics problems is writing down the ODEs and solving them.

What is mass? m is an 'inertial mass'.
In Newton's law of gravity

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

m is the 'gravitational mass' and $g \approx 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.
A larger mass has a larger inertia or 'resistance to being accelerated' (Taylor). Key fact: When acceleration is zero ($\mathbf{a} = 0$), the velocity is constant ($\mathbf{v} = \text{constant}$).

Momentum: $\mathbf{p} = m\mathbf{v}$
The third law of motion in terms of momentum is

$$\mathbf{F} = \dot{\mathbf{p}} = m\dot{\mathbf{v}}$$

Newton's Third Law: $\mathbf{F}_{12} = -\mathbf{F}_{21}$
In a two body system, the total force of the system is

$$\mathbf{F}_t = \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

From the second law,

$$\dot{\mathbf{p}}_1 = \mathbf{F}_{21} \qquad \dot{\mathbf{p}}_2 = \mathbf{F}_{12}$$

adding these two equations gives

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$$

thus the total momentum of the system is conserved.
For a system of N particles, the total momentum is

$$\mathrm{d}t \sum_i \mathbf{p}_i = \frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = \mathbf{F}_{ext}$$

sometimes $\mathbf{p}_{tot} = \mathbf{P}$ where the capital P denotes the total momentum of the system.

2 A pendulum

How to solve a problem:

1. Write down the eq
2. Solve it
3. Understand the solution

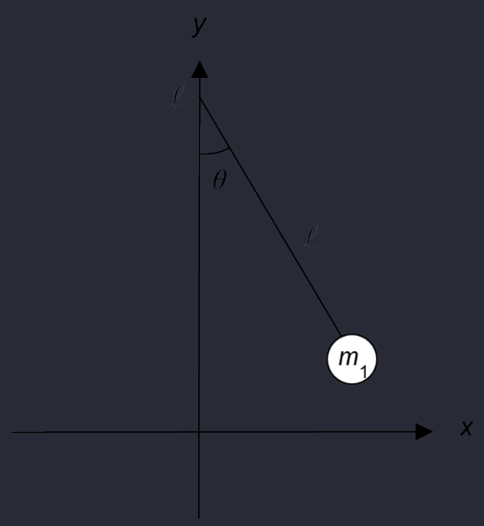

Figure 2.1: A pendulum with mass m and length l.

From Figure 2.1, we can write down Newton's 2nd law:

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} = m\ddot{\mathbf{r}} \\ F_x &= -mg\sin\theta = m\ddot{x} \\ F_y &= -mg\cos\theta + T\cos\theta = m\ddot{x} \end{aligned}$$

Using a right triangle we can find the angle using $\tan\theta = x/y$. Furthermore, we can use the constrain that the length of the pendulum is constant thus $x^2 + y^2 = l^2$. But solving this system of equations is difficult. Instead we now use a new coordinate system.

Quick Hack Using the arc length $l = L\theta$ and choosing a coordinate in the direction of the pendulums path, we can write the force equation as

$$F_l = -mg\sin\theta = m\ddot{l} = mL\ddot{\theta}$$

Thus the equation of motion is

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

which is a second order ODE. This can only be solved with two conditions. We can use the initial conditions (at $t = 0$) of the position $\theta(t = 0) = \theta_0$ and velocity $\dot{\theta}(0) = 0$.

Polar Coordinates From Taylor:

$$
\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi
\end{aligned}
$$

For an arbitrary vector $\mathbf{v}$ it has the Cartesian vector components

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

Where the magnitude of the unit vectors are equivalent:

$$|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = 1$$

and the magnitude of the vector is

$$
\begin{aligned}
|\mathbf{v}| &= \sqrt{\mathbf{v}\cdot\mathbf{v}} \\
&= \sqrt{v_x^2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + 2v_xv_y\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}v_y^2\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}} \\
&= \sqrt{v_x^2 + v_y^2}
\end{aligned}
$$

The vector $\mathbf{v}$ can be written in polar coordinates as

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\phi}$$

where radial vector is

$$\mathbf{r} = r\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

taking the time derivative of $\mathbf{r}$ gives the velocity

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}}$$

but how do we find $\dot{\hat{r}}$? We can look at the change in the direction of the radial unit vector for a small change in time Δt. Thus,

$$\Delta\hat{\mathbf{r}} \approx r\Delta\phi\hat{\boldsymbol{\phi}}$$

dividing both sides by Δt gives

$$\frac{\Delta\hat{\mathbf{r}}}{\Delta t} \approx r\frac{\Delta\phi}{\Delta t}\hat{\boldsymbol{\phi}} = r\dot{\phi}\hat{\boldsymbol{\phi}}$$

Therefore, the vector in polar coordinates is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}}$$

where the polar components v_r and v_ϕ are related to the radial and angular velocity respectively. Taking the time derivative of $\dot{\mathbf{r}}$ gives the acceleration

$$
\begin{aligned}
\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}} \\
&= \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\hat{r}} + \dot{v}_\phi\hat{\boldsymbol{\phi}} + v_\phi\dot{\hat{\phi}}
\end{aligned}
$$

3 Polar Coordinates

using the geometric relation $\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$, we can write the acceleration as

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}}\end{aligned}$$

where $r\dot{\phi}^2 = r\omega^2$ is the centripetal acceleration and $r\ddot{\phi} = r\dot{\omega}$ is the tangential acceleration.
From the Pendulum problem we know that the string is taut $r = L$ thus the radial velocity is zero $\dot{r} = 0$.
Thus the force equation in the $\hat{\boldsymbol{\phi}}$ direction is

$$\begin{aligned}F_\phi &= mL\ddot{\phi} = -mg\sin\theta \\ \ddot{\phi} &= -\frac{g}{L}\sin\theta\end{aligned}$$

which is the same equation of motion.

Projectile in 2D The initial conditions of a general projectile is usually

$$\begin{aligned}F_x &= 0 = m\ddot{x} \\ F_y &= -mg = m\ddot{y}\end{aligned}$$

thus the equations of motion are

$$\begin{aligned}\ddot{x} &= 0 \\ \ddot{y} &= -g\end{aligned}$$

And solving these equations gives the position of the projectile

$$\begin{aligned}x(t) &= v_{ox}t \\ y(t) &= y_o v_{oy}t - \frac{1}{2}gt^2\end{aligned}$$

This can be expanded on with the addition of air resistance $\mathbf{f}$. This drag force is proportional to the velocity:

$$\mathbf{f} \propto -\hat{\mathbf{v}}$$

and there are two types of air resistance: linear

$$\mathbf{f}_l = -bv\hat{\mathbf{v}} = -b\mathbf{v}$$

and quadratic

$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

where we compare the terms with

$$\frac{f_l}{f_q} = \frac{cv}{b}$$

Linear $\quad \mathbf{f}_l = -b\mathbf{v}$
From Newton's 2nd law

$$F_x = -bv_x = m\ddot{x} = m\dot{v}_x$$
$$F_y = -mg - bv_y = m\ddot{y} = m\dot{v}_y$$

For the case of uncoupled differential equations (such as pure horizontal motion), we can solve the horizontal equation

$$\dot{v}_x = -\frac{b}{m}v_x$$

which has a general solution

$$v_x = Ae^{-kt}$$

where

$$k = \frac{b}{m}, \quad A = v_{ox}$$

to find the position we have to integrate $\dot{x} = v_x$:

$$x = x_o + \int_0^t v_x(t')\,\mathrm{d}t'$$
$$= x_o + \left[-\frac{v_{xo}}{k}e^{-kt}\right]\Big|_0^t$$

where we have a limit of $t \to \infty$
For pure vertical motion, we solve the equation

$$\dot{v}_y = -g - \frac{b}{m}v_y \tag{3.1}$$

and with the initial condition $\dot{v}_y = 0$ we can solve for the velocity

$$v_y = -\frac{mg}{b} = v_{ter}$$

where v_{ter} is the terminal velocity. To get position, we use a trick by rewriting the equation as

$$m\dot{v}_y = -mg - bv_y = -mg - b(v_y - v_{ter})$$

and we can solve similar to the horizontal case using the general solution

$$v_y - v_{ter} = Ae^{-kt} = (v_{oy} - v_{ter})e^{-kt}bye$$

4 Air Resistance

Last time:

$$\mathbf{f}_l = -b\mathbf{v} \quad \dot{\mathbf{r}} = \mathbf{v}$$
$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

In the case of linear, x motino has a range, y velocity has a terminal velocity v_t.

Horizontal Quadratic Drag

$$\begin{aligned} F_y &= -mg - c|v_y|v_y \\ m\ddot{y} &= F_y \\ m\dot{v}_y &= -mg - c|v_y|v_y \end{aligned}$$

when $v_y = 0$ we have the terminal velocity

$$v_{ter} = \sqrt{\frac{mg}{c}} \quad \text{or} \quad c = \frac{mg}{v_{ter}^2}$$

thus the equation of motion is

$$\dot{v}_y = -g - \frac{c}{m}v_y^2 = -g(1 - \frac{v_y^2}{v_{ter}^2}) = \frac{\mathrm{d}v_y}{\mathrm{d}t}$$

using separation of variables

$$\frac{1}{1 - \frac{v_y^2}{v_t^2}} \,\mathrm{d}v_y = -g\,\mathrm{d}t$$

integrating both sides

$$\int_{v_{oy}}^{v_y} \frac{1}{1 - \frac{v_y^2}{v_t^2}} \,\mathrm{d}v_y = -g\int_0^t \mathrm{d}t$$

where we get the integral using the hyperbolic tangent

$$\begin{aligned} v_t \operatorname{arctanh} \frac{v_y}{v_t} &= -gt \\ v_y &= -v_t \tanh(gt) \end{aligned}$$

2D Motion For Quadratic

$$F_x = -cvv_x = -c\sqrt{v_x^2 + v_y^2}v_x = m\dot{v}_x$$
$$F_y = -mg - cvv_y = -mg - c\sqrt{v_x^2 + v_y^2}v_y = m\dot{v}_y$$

where $v = \sqrt{v_x^2 + v_y^2}$. For linear, it is simply

$$\begin{aligned} F_x &= -bv_x = m\dot{v}_x \\ F_y &= -mg - bv_y = m\dot{v}_y \end{aligned}$$

5 Center of Mass

For N particles, the center of mass is

$$\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i$$

where $M = \sum_i m_i$ is the total mass of the system. This is similar to the 'weighted' average! Taking the time derivative of $\mathbf{R}$ gives the total momentum

$$\dot{\mathbf{R}} = \frac{1}{M}\sum_i m_i \dot{\mathbf{r}}_i = \frac{1}{M}\sum_i \mathbf{p}_i = \mathbf{P}$$

From Newton's 3rd Law

$$\sum \dot{\mathbf{p}}_i = \mathbf{F}_{ext}$$

and from the second Law

$$M\ddot{\mathbf{R}} = \mathbf{F}_{ext}$$

in integral form

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,\mathrm{d}m$$

and using the mass density $\mathrm{d}m = \rho\,\mathrm{d}V$ we can write

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\rho\,\mathrm{d}V = \frac{1}{M}\int \mathbf{r}\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

For a uniform solid semisphere lying on the xy plane with radius $R = 1$ and mass $M = 2\pi/3$, the CM is

$$\begin{aligned} z &= \frac{1}{M}\int \rho z\,\mathrm{d}m \\ &= \frac{1}{M}\int z\pi r^2\,\mathrm{d}z \\ &= \frac{1}{M}\int_0^1 \pi z(1-z^2)\,\mathrm{d}z \\ &= \frac{\pi}{M}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{3}{8} \end{aligned}$$

Angular Momentum

For the singular particle, the angular momentum is

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}$$

and the total angular momentum of an multi particle system is

$$\mathbf{L} = \sum_i \boldsymbol{\ell}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

and the time derivative of $\mathbf{L}$ is

$$\dot{\mathbf{L}} = \sum_i \dot{\mathbf{r}}_i \times \mathbf{p}_i + \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \boldsymbol{\Gamma}_i$$

where $\dot{\mathbf{r}}_i \times \mathbf{p}_i = 0$ since $\dot{\mathbf{r}}_i$ is parallel to $\mathbf{p}_i$. Since $\mathbf{F}_i$ is the force on the ith particle,

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{ext}$$

Plugging into the time derivative of angular momentum

$$\dot{\mathbf{L}} = \sum_i \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{ij} + \sum_i \mathbf{F}_i^{ext}$$

In terms of a matrix, the double sum skips the diagonal elements and thus we can pair the indices that are reflected on the diagonal

$$\sum_i \sum_{j>i} (\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}) = \sum_i \sum_{j>i} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}$$

where we use the associativity of the cross product and N3L $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. In addition the force must be central along the line connecting the two particles. Thus we get

$$\dot{\mathbf{L}} = \sum_i \Gamma_i^{ext}$$

The direction of the angular momentum is along the axis of rotation.

A car To move a car forward, you exert a torque clockwise on the wheels, and from the conservation of angular momentum the car will typically want to rotate counter clockwise which feels like the weight is being pushed back. The torque on the car will increase the friction on the rear wheel (increasing traction) and thus RWD are better at high accelerations.

6 Energy

Review: There are two requirements for conservation of angular momentum

1. Force is central
2. External torque is zero

Kinetici Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$. Taking the time derivative

$$\begin{aligned}\dot{T} &= \frac{1}{2}(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) \\ &= m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}\end{aligned}$$

and integrating over time t_1 to t_2

$$\int_{t_1}^{t_2} \dot{T}\,\mathrm{d}t = \Delta T = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = W(1\to 2)$$

since $\mathbf{v}\cdot\mathrm{d}t = \mathrm{d}\mathbf{r}$ and $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ hints that this is a line integral.

Example:

$$\begin{aligned}\mathbf{F}(x,y) &= \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ \mathrm{d}vbr &= \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}}\end{aligned}$$

(a) $y = x$ from $a = (0,0)$ to $b = (1,1)$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + y\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 x\,\mathrm{d}x = 1\end{aligned}$$

(b) $y = x^2$ and $\mathrm{d}y = 2x\,\mathrm{d}x$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + x^2\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 2x^2\,\mathrm{d}x = 1\end{aligned}$$

thus the line integral is independent of the path.

Conservative force

1. Given $\mathbf{F}(\mathbf{r})$, there is no dependence on $\mathbf{v}$, t.
2. $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is independent of the path.

$$U(\mathbf{r}) = -W(\mathbf{r}_0\to\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}'$$

For gravity

$$U_g(\mathbf{r}) = -\int_a^b \mathbf{F}_g\cdot\mathrm{d}\mathbf{r} = -\int_a^b -mg\,\mathrm{d}y' = mg(y_a - y_b)$$

Work-Kinetic Energy Theorem:

$$W(1 \to 2) = W(1 \to \mathcal{O}) + W(\mathcal{O} \to 2) = U(1) - U(2) = \Delta T$$

From this we have

$$\Delta T = T_2 - T_1 = U_1 - U_2$$

rearranging terms give the mechanical energy

$$T_2 + U_2 = T_1 + U_1 = E$$

and for N conservative forces in a system

$$E = T + U_1 + U_2 + \cdots + U_N$$

7 Energy

Conservative Force: Potential Energy The mechanical energy

$$E = T + U$$

is made up of the sum of the kinetic and potential energy. This is useful for

- obtaining equations of motion (EOM)

e.g. finding the EOM for a simple pendulum of mass m, length L and initial angle θ. The kinetic energy is

$$T = \frac{1}{2}mL^2\dot{\theta}^2$$

where the magnitude of velocity is the tangential component $v = L\omega = L\dot{\theta}$. The potential energy is

$$U = -mgy = -mgL\cos\theta$$

and the conservation of energy tells us that the mechanical energy is constant

$$\begin{aligned} T + U &= \text{constant} = E \\ \frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta &= E \end{aligned}$$

and in the intial condition we know that the velocity is zero $\dot{\theta} = 0$ and thus

$$-mgL\cos\theta_{max} = E$$

taking the time derivative of the energy equation gives

$$\begin{aligned} mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\dot{\theta} &= 0 \\ \ddot{\theta} + \frac{g}{L}\sin\theta &= 0 \\ \ddot{\theta} &= -\frac{g}{L}\sin\theta \end{aligned}$$

taking the time derivative of the energy is a useful trick for finding the EOM when we are trying to solve for $\dot{v}^2$.

Last time we found the potential energy for a position $\mathbf{r}$ in a conservative force field $\mathbf{F}(\mathbf{r})$ is

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}'$$

from the fundamental theorem of calculus (derivatives and intergrals are inverses) we want to find a function where the derivative equals the conservative force: First we take an infinitesimal change in the position $\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}$ and the change in potential energy is

$$\begin{aligned} U(\mathbf{r} + \mathrm{d}\mathbf{r}) &= -\int_{\mathbf{r}_0}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{\mathbf{r}}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= U(\mathbf{r}) - F(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \end{aligned}$$

where is know that the force is constant over a small distance. Moving the terms gives

$$
\begin{aligned}
U(\mathbf{r} + \mathrm{d}\mathbf{r}) - U(\mathbf{r}) &= -\mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= -(F_x \,\mathrm{d}x + F_y \,\mathrm{d}y + F_z \,\mathrm{d}z)
\end{aligned}
$$

where we use Cartesian Coordinates, and we know that the gradient of potential is

$$
\begin{aligned}
\boldsymbol{\nabla} U &= \hat{\mathbf{x}} \frac{\partial U}{\partial x} + \hat{\mathbf{y}} \frac{\partial U}{\partial y} + \hat{\mathbf{z}} \frac{\partial U}{\partial z} \\
&= -(F_x \hat{\mathbf{x}} + F_y \hat{\mathbf{y}} + F_z \hat{\mathbf{z}}) = -\mathbf{F}
\end{aligned}
$$

Example 3: 1D motion If we know what U is as a function of x, we can find the force! At points where $E = U$ we call these classical turning points. At a region of a relative minimum, a particle below the threshold of the turning point will oscillate between the two turning points. And at $E > U_max$ the particle is unbound and will escape the forces that attracted it.

Example 4:

$$
\begin{aligned}
E &= T + U(x) \text{ is constant} \\
T &= \frac{1}{2} m \dot{x}^2 = E - U(x) \\
\dot{x}^2 &= \frac{2}{m}(E - U(x)) \\
\dot{x} &= \pm \sqrt{\frac{2}{m}(E - U(x))}
\end{aligned}
$$

using seperation of variables

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\frac{2}{m}(E - U(x))} \\
\sqrt{\frac{m}{2}} \,\mathrm{d}t &= \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\
\int_{t_1}^{t_2} \sqrt{\frac{m}{2}} \,\mathrm{d}t &= \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\
(t_2 - t_1) &= \sqrt{\frac{2}{m}} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{E - U(x)}}
\end{aligned}
$$

where the sign of the velocity is positive within the oscillating bounds of the turning points and changes sign as the particle moves past the turning point.

Energy cont'd

Last time: Conservative force as a negative gradient of potential:

$$\mathbf{F} = -\boldsymbol{\nabla} U$$

with classical turning points at $E = U$.

Conditions of a conservative force

- Only depends on position $\mathbf{r}$ (or just constant)
- Work done is path independent (this is sometimes hard to check) $\Leftrightarrow \boldsymbol{\nabla} \times \mathbf{F} = 0$

What is curl? In 3D Cartesian coordinates

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \hat{\mathbf{x}} + \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \hat{\mathbf{y}} + \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \hat{\mathbf{z}} () \right)\end{aligned}$$

Mathematically we know that if the force vector is a gradient of a scalar potential

$$\mathbf{F} = \boldsymbol{\nabla} \phi = -\boldsymbol{\nabla} U \quad \Leftrightarrow \quad \boldsymbol{\nabla} \times \mathbf{F} = 0$$

The curl of a gradient is always zero! Short 'proof':

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

and the curl

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial U}{\partial x} & -\frac{\partial U}{\partial y} & -\frac{\partial U}{\partial z} \end{vmatrix} = 0$$

Proving that the line integral is path independent is a little difficult, but in general we know that the two different paths a and b from points 1 to 2 we can write the work as

$$\int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_2 - \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_1 = \oint_{a-b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_A (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = 0$$

where we invoke Stokes' Theorem to find the integral of the curl over the surface A is zero.

Conservative Force: $\mathbf{F} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is conservative, but is

$$\mathbf{F}(\mathbf{r}) = F(\mathbf{r})\hat{\mathbf{r}}$$

a central force always conservative? We will need to check the curl of the force to find out. But first we define spherical coordinates (r, θ, ϕ)

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

and the central force in spherical coordinates is

$$\mathbf{F} = F(\mathbf{r})\hat{\mathbf{r}} + 0\hat{\theta} + 0\hat{\phi}$$

the curl in spherical coordinates is

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} = & \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial\phi}\right]\hat{\mathbf{r}} \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\hat{\theta} \\ & + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]\hat{\phi}\end{aligned}$$

And since

$$\frac{\partial F}{\partial\theta} = \frac{\partial F}{\partial\phi} = 0$$

the curl is zero $\boldsymbol{\nabla} \times \mathbf{F} = 0$ and thus $\mathbf{F}$ is a conservative central force.

Gravity Conservative? The force due to gravity

$$\mathbf{F}_g = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}$$

is a central force as it only depends on $\mathbf{r}$. e.g. for a a two mass system:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Using Translational Invariance, we can shift the origin to the center of m_2

$$\mathbf{r}_2 = 0 \quad \mathbf{F}_{12} = -\frac{Gm_1m_2}{\mathbf{r}_1^3}\mathbf{r}_1$$

or

$$F_{12} = -\boldsymbol{\nabla}_1 U = -(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial z_1})$$

where the potential energy due to the interaction between 1 and 2 is

$$U_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

from newtons 3rd law, the force on 2 due to 1 is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

so

$$\begin{gathered}-\boldsymbol{\nabla}_1 U_{12} \rightarrow \mathbf{F}_{21} = \boldsymbol{\nabla}_1 U_{12} \\ \boldsymbol{\nabla}_1 U_{12}(\mathbf{r}_1 0 \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U_{12}(\mathbf{r}_2 0 \mathbf{r}_1) \\ u_{12}(\mathbf{x}) \quad \mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2 \\ \boldsymbol{\nabla}_1 U_{12}(\mathbf{x}) = \boldsymbol{\nabla}_x U_{12}(\mathbf{x}) = -\boldsymbol{\nabla}_2 U_{12}(\mathbf{x})\end{gathered}$$

so

$$\mathbf{F}_{12} = -\boldsymbol{\nabla}_1 U_{12} \qquad \mathbf{F}_{21} = -\boldsymbol{\nabla}_2 U_{12}$$

and for N particles

$$\mathbf{F}_i = -\boldsymbol{\nabla}_i U \qquad U = \sum_{i,j} U_{ij} + \sum_i U_i^{\text{ext}}$$