1 Symmetries

Quiz review:

3. The Energy of the large mass is

$$Mc^2 = E_1 + E_2 = 2\gamma mc^2$$

where the energy of the smaller masses are

$$E = \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}$$

where $|\mathbf{p}| = \gamma mv$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Thus the mass $M > 2m$.

4. Using the same thought from 3. we know that the rest mass of M is greater.

Lorentz Invariant

$$p^2 = m^2 c^2$$

From Wikipedia: this is the lightlike vector. For the timelike $p^2 > 0$ and spacelike $p^2 < 0$.

Symmetries

Equilateral triangles are symmetric under 3 axes where we can flip the triangle and it is still the same. For the square, we have 4 axes, and so and so forth. All of these objects are studied in Group Theory.

Group Theory Group is a set of objects satisfying certain properies under an operation.

Properties

1. Closure: For $a, b \in G, \quad a \cdot b \in G$
2. Identity: For any $a \in G, \quad a \cdot I = I \cdot a = a$
3. Inverse: For each $a \in G, \quad a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. Associativity: For $a, b, c \in G, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (optional) Commutativity: For $a, b \in G, \quad a \cdot b = b \cdot a$ AKA Abelian Group. Not all groups are commutatiive and thus are called non-Abelian groups.

Two Types of Groups

1. Finite: Finite number of elements. e.g. $Z_2 = \{1, -1\} = \{I, r\}$ where $r^2 = I$
2. Infinite: Descrete or continuous. e.g. set of integers under addition (discrete), set of real numbers under multiplication (continuous), $U(1)$ (continuous)

Examples For an isoscale triangle $Z_2 = \{1, -1\}$ and for an equilateral triangle $Z_3 = \{0, 1, 2\}$ or the operation mod 3. Which is isomorphic to

$$\equiv \left\{1, \omega, \omega^2\right\}, \quad \omega = e^{2\pi i/3}$$

For the square

$$Z_4 = \{0, 1, 2, 3, \} \equiv \{1, i, -1, -i\} \quad \text{or} \quad \left\{1, \omega, \omega^2, \omega^3\right\}$$

Thus for n elements.

$$Z_n = \left\{e^{i2\pi j/n}\right\}, \quad j = 0, 1, \ldots, n-1$$

where all of these groups are Abelian.

For $n \to \infty$ We get a circle as it has an infinite number of symmetries.
In addition $j \to \infty$

$$\frac{2\pi j}{n} = \theta$$

we get

$$U = e^{i\theta} = \cos\theta + i\sin\theta$$

where $\theta \in [0, 2\pi]$, and we have the $U(1)$ group.

$$U^\dagger U = I \qquad U^\dagger = (U^*)^T$$

where the dagger is the transpose of the complex conjugate (conjugate transpose).

Standard Model

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_y$$

$U(N)$ set of unitary $N \times N$ matrices (non-Abelian in general except for $N > 1$). Taking the determinant of the matrix

$$\det\left(U^\dagger U\right) = \det I = 1$$

and

$$\det\left(U^\dagger\right)\det(U) = 1 \qquad \det\left(U^{*T}\right) = \det(U^*) = (\det U)$$

and

$$\begin{aligned} |\det U|^2 &= 1 \\ \det U &= e^{i\alpha} \quad \alpha \in [0, 2\pi] \end{aligned}$$

Choosing the phase angle $\alpha = 0$ we get

$$\det U = 1 \qquad SU(N) C U(N)$$

$\otimes$ is a direct product: Two groups F and G. For $f \in F$ and $g \in G$ we have

$$(f, g) \in F \otimes G$$

The $U(1)$ group is related to the photon γ, the $SU(2)$ group is related to the weak force $W^\pm, Z^0$, and the $SU(3)$ group is related to the strong force g (gluon).

SU(2) A set of 2×2 matrices with a determinant of 1.
Given the theorem

$$U = e^{iH}$$

for the hermitian matrix H where

$$U^\dagger U = 1 \to e^{-iH^\dagger} e^{iH} = 1$$

thus

$$H^\dagger = H$$

we take the determinant of U:

$$\det U = \det\left(e^{iH}\right) = e^{i \operatorname{Tr} H} = 1 = e^0$$

thus $\operatorname{Tr} H = 0$. This means that the Hermitian H is traceless.

Pauli Matrices

traceless matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus we can write the Hermitian matrix as

$$H = \frac{1}{2} \sum_i \theta_i \sigma_i = \frac{1}{2} \theta \cdot \sigma$$

where we have the group element of $SU(2)$

$$U = e^{i\theta \cdot \sigma/2}$$

From QM

$$\mathbf{S} = \frac{\hbar}{2} \sigma$$

$$[S_y, S_z] = iS_x$$
$$[S_z, S_x] = iS_y$$
$$[S_x, S_y] = iS_z [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

where ϵ_{ijk} is the Levi-Civita symbol.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ interchange any two indices } (3,2,1) \\ 0 & \text{otherwise any index is repeated} \end{cases}$$

thus

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

The Lie Algebra for $SU(2)$ is $SO(3)$ where both groups are isomorphic.

$$[L_i, L_j] = i\epsilon_{ijk} L_k \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

the generators of $SU(2)$ is $\sigma/2$. For $SU(3)$

$$U = e^{i\theta \cdot \lambda/2}$$

where we have the Gell-Mann matrices λ. In general for $SU(N)$

Addition of Angular Momenta

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and

$$[J^2, J_i] = 0 \qquad J^2 = J_x^2 + J_y^2 + J_z^2 T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_o} \frac{1}{\sqrt{\cos\theta - \cos\theta_o}} \, \mathrm{d}\theta$$

where J^2 is the Casimir operator. Since we have simultaneous eigenstates of J^2 and J_z we can write

$$|j, j_z\rangle$$

Symmetries Part 2: Spin & Isospin

Quiz 3 Review SU(2) is the group of 2x2 unitary matrices with determinant 1. Using the basisc vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ we can write the group element as

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

or the linear combination of the basis vectors. Thus the transformation is

$$\begin{pmatrix}a'\\b'\end{pmatrix} = U(\theta)\begin{pmatrix}a\\b\end{pmatrix} = e^{i\theta\cdot\sigma/2}\begin{pmatrix}a\\b\end{pmatrix}$$

THe Lie Algebra for SU(2) is

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

and

$$[J^2, J_i] = 0$$

for simultaneous eigenstates of $|j,m\rangle$.

$$J_z|j,m\rangle = m\hbar|j,m\rangle \qquad J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$$

from the ladder operators

$$J_\pm = J_x \pm iJ_y$$

where these are not Hermitian (does not commute). Thus

$$\begin{aligned} J^2 &= J_x^2 + J_y^2 + J_z^2 \\ &= J_+J_- + J_+J_-J_z^2 \end{aligned}$$

furthermore

$$J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m)}\,|j,m\pm 1\rangle$$

where going up the ladder $m \to m+1$ and going down the ladder $m \to m-1$. For fixed j there is a maximum and minimum m value

$$m_{max} = j \qquad m_{min} = -j$$

so for example

$$J_+|j,j\rangle = 0 \qquad J_-|j,-j\rangle = 0$$

Spin

$$j \equiv s = 1/2, \qquad m \equiv m_s = \pm 1/2$$

The basis states are

$$\begin{aligned} (1/2, 1/2) &= \begin{pmatrix}1\\0\end{pmatrix} = |\uparrow\rangle \qquad m_s = 1/2 \\ (1/2, -1/2) &= \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle\, m_s = -1/2 \end{aligned}$$

For the addition of spin

$$\frac{1}{2}\otimes\frac{1}{2} = ? \qquad \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \qquad S_{tot} = (S_1+S_2), ..., (S_1-S_2) = 1, 0 \qquad m_{s,tot} = 1, 0, -1, 0$$

General Addition of Angular Momentum

$$|1,1\rangle = |\uparrow\uparrow\rangle \qquad |1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle \qquad |1,-1\rangle = |\downarrow\downarrow\rangle$$

finding the linear combination through basis transformation by using the resolution of the identity

$$\begin{aligned}|j,m\rangle &\to |j_1,m_1\rangle \otimes |j_2,m_2\rangle \\ &= \sum_{m_1,m_2} |j_1,m_1,j_2,m_2\rangle \langle j_1,m_1,j_2,m_2|j,m\rangle\end{aligned}$$

where the bra-ket is the Clebsch-Gordan coefficient. thus

$$= \sum_{m_1,m_2} c^{j,j_1,j_2}_{m,m1,m2} |j_1,m_1,j_2,m_2\rangle$$

where $m = m_1 + m_2$ and $c^{j,j_1,j_2}_{m,m1,m2}$ is the Clebsch-Gordan coefficient.

Example For the $S = 1$ state $m = 1$

$$\begin{aligned}|1,1,\rangle &= |1/2,1/2\rangle \otimes |1/2,1/2\rangle \\ &= |1/2,1/2,1/2,1/2\rangle \\ &= |\uparrow\uparrow\rangle\end{aligned}$$

For $m = 0$ we have a linear combination of the basis states

$$\begin{aligned}J_- |1,1\rangle &= \hbar\sqrt{2}|1,0\rangle \\ \text{or} \quad |1,0\rangle &= \frac{1}{\hbar\sqrt{2}} J_- |1,1\rangle\end{aligned}$$

the sum of the basis states is

$$\begin{aligned}J_-(|1/2,1/2\rangle \otimes |1/2,1/2\rangle) &= \hbar\sqrt{(1/2+1/2)(1/2-1/2+1)}\,|1/2,-1/2\rangle \otimes |1/2,1/2\rangle \\ &+ |1/2,1/2\rangle \otimes |1/2,-1/2\rangle \\ &= \hbar(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\end{aligned}$$

or

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

for $m = -1$ we have

$$J_- |1,0\rangle = \hbar\sqrt{2}|1,-1\rangle$$

where

$$|1,-1\rangle = |1/2,-1/2\rangle \otimes |1/2,-1/2\rangle \qquad = |\downarrow\downarrow\rangle$$

Now for $S = 0$, $m = 0$ we have

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

since it is the way to make it orthogonal to $|1,0\rangle$. Therefore

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

Thus the there are 3 triplet states $m_s = 1,0,-1$ and 1 singlet state $m_s = 0$.

Isospin

$$m_p = 938.3\ \mathrm{MeV/c^2} \qquad m_n = 939.6\ \mathrm{MeV/c^2}$$

why are they so close? Heisenberg postulated an isospin state of a nucleon N as

$$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \left| p \right\rangle + \beta \left| n \right\rangle$$

with

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the isospin state of the proton and neutron are

$$\left| p \right\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \qquad \left| n \right\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

1. Strong interactions preserve isospin symmetry
2. EM & Weak interactions do not preserve isospin symmetry

Examples

Pions: π^+, π^0, π^- where the approximate symmetry is a triplet state

$$\begin{aligned} \pi^+ &= \left|1, 1\right\rangle \qquad I = 1, \quad I_3 = 1 \\ \pi^- &= \left|1, 0\right\rangle \qquad I = 1, \quad I_3 = 0 \\ \pi^0 &= \left|1, -1\right\rangle \qquad I = 1, \quad I_3 = -1 \end{aligned}$$

Δ-baryons:

$$\begin{aligned} \Delta^{++} &= \left|3/2, 3/2\right\rangle \qquad I = 3/2, \quad I_3 = 3/2 \\ \Delta^{+} &= \left|3/2, 1/2\right\rangle \qquad I = 3/2, \quad I_3 = 1/2 \\ \Delta^{0} &= \left|3/2, -1/2\right\rangle \qquad I = 3/2, \quad I_3 = -1/2 \\ \Delta^{-} &= \left|3/2, -3/2\right\rangle \qquad I = 3/2, \quad I_3 = -3/2 \end{aligned}$$

where Δ^{--} is an antiparticle of Δ^{++}. We write from the highest to lowest from the empirical Gellman-Nishijima formula

$$Q = I_3 + \frac{1}{2}(B + S)$$

where Q is the charge, I_3 is the third component of isospin, B is the baryon number, and S is the strangeness.

Pions

Since a Pion is a *meson* and not a baryon, it has a baryon number of 0. Thus with no strangeness

$$S = 0 \qquad B = 0$$

Nucleons

$$S = 0 \qquad B = 1$$

$$Q = \begin{cases} 1/2 + 1/2(1+0) = 1 & \text{proton} \\ -1/2 + 1/2(1+0) = 0 & \text{neutron} \end{cases}$$

For all elementary particles there is a general formula

$$Q = I_3 + \frac{Y}{2}$$

where Y is the hyper charge $U(1)_Y$.

Power of Symmetry: Applications

1. Deuteron (neutron of deuterium): Two-Nucleon system

$$I = 1 \quad \text{or} \quad 0 \qquad I_3 = 1, 0, -1 \quad \text{or} \quad 0 \text{ (singlet)}$$

$$\begin{aligned} |1,1\rangle &= |p,p\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle + |n,p\rangle) \\ |1,-1\rangle &= |n,n\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle - |n,p\rangle) \end{aligned}$$

experimentally, we only see the singlet state because we see only one deuteron state. Thus we can only see a isospin state of $I = 0$.

Two-nucleon potential $\propto \mathbf{I}_1 \cdot \mathbf{I}_2$ where we hae the total isospin

$$\mathbf{I}^2 = (\mathbf{I}_1 + \mathbf{I}_2)^2 = \mathbf{I}_1^2 + \mathbf{I}_2^2 + 2\mathbf{I}_1 \cdot \mathbf{I}_2$$

where the s^2 term is

$$s^2 = 1/2(1/2+1)\hbar^2 = \frac{3}{4}\hbar^2$$

Thus

$$\mathbf{I}_1^2 + \mathbf{I}_2^2 = \frac{3}{2}$$

and

$$\begin{aligned} \mathbf{I}_1 \cdot \mathbf{I}_2 &= \frac{1}{2}\left(\mathbf{I}^2 - 3/2\right)^{3/2} \\ &= \begin{cases} 1/2(1(1+1) - 3/2) & = 1/4 \quad \text{triplet} \\ 1/2(0(0+1) - 3/2) & = -3/4 \quad \text{singlet} \end{cases} \end{aligned}$$

Symmetries Part 3: Scattering

Quiz 5 Review For j

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

For $2j+1$

$$2 \otimes 2 = 3 \oplus 1$$

Isospins of particles

1. pion: 1
2. deuteron: 0
3. Δ-baryons: 3/2
4. nucleons: 1/2

The strong ineteraction preserves I and I_3, and the weak interactions do not preserve I and I_3 (e.g. in beta decay the iso spin of the neutron (-1/2) go to an iso spin of the proton (1/2)). In E&M the isospin preserves only I and not I_3 (e.g. π_o decay to two photons $\gamma\gamma$: $I_3 = 0$ for the π_o and $I_3 = 0$ for the two photons).

Applications of Isospin: Nucleon-nucleon Scattering

$$p + p \to D + \pi^+$$
$$p + n \to D + \pi^0$$
$$n + n \to D + \pi^-$$

The relative probabilities of these processes: we get this from the amplitude A where the probability $|A|^2$ is proportional to the cross section $\sigma = \pi r^2$ (the cross section of a sphere, but this is not a solid sphere and rather a 'fuzzy' sphere). With the fact that 'strong interactions preserve isospin' we have the the ratio of the cross sections

$$\sigma_a : \sigma_b : \sigma_c$$

For all three processes the RHS the isospin is

$$I_{tot} = 0 \otimes 1 = 1$$

on the left hand side

$$I_{tot} = \frac{1}{2} \otimes \frac{1}{2} = 0 \quad \text{or} \quad 1$$

(a) The ratio of getting an isospin of 1 on the left hand side for the first process

$$|pp\rangle = |11\rangle$$

(c) for the third proccess

$$|nn\rangle = |1, -1\rangle$$

(b) The second is the linear combination of $|10\rangle$ and $|00\rangle$

$$|pn\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |00\rangle)$$

the $|00\rangle$ does not contribute to the isospin of 1. Thus the ratio of the probability is

$$A_a : A_b : A_c = 1 : \frac{1}{\sqrt{2}} : 1$$

and the ratio of the cross sections is

$$\sigma_a : \sigma_b : \sigma_c = 1 : \frac{1}{2} : 1$$

Example 3 Pion-nucleon Scattering

$$\begin{pmatrix}\pi^+\\ \pi^0\\ \pi^-\end{pmatrix} \quad I=1, \qquad \begin{pmatrix}p\\ n\end{pmatrix} \quad I=1/2$$

So the total isospin j is

$$I_{tot} = 1/2 \otimes 1 = 3/2 \oplus 1/2$$

and for $2j+1$

$$3 \otimes 2 = 4 \oplus 2$$

The elastic processes are (from $a \to f$)

$$\begin{aligned}
\pi^+ + p &\to \pi^+ + p\\
\pi^0 + p &\to \pi^0 + p\\
\pi^- + p &\to \pi^- + p\\
\pi^+ + n &\to \pi^+ + n\\
\pi^0 + n &\to \pi^0 + n\\
\pi^- + n &\to \pi^- + n
\end{aligned}$$

and the charge-exchange processes are (from $g \to j$)

$$\begin{aligned}
\pi^+ + n &\to \pi^0 + p\\
\pi^0 + p &\to \pi^+ + n\\
\pi^- + p &\to \pi^0 + n\\
\pi^0 + n &\to \pi^- + p
\end{aligned}$$

The states of the 3/2 isospin are

$$|3/2, 3/2\rangle, |3/2, 1/2\rangle, |3/2, -1/2\rangle, |3/2, -3/2\rangle$$

and the states of the 1/2 isospin are

$$|1/2, 1/2\rangle, |1/2, -1/2\rangle$$

so we have the following states

$$\begin{aligned}
\left|\pi^+ p\right\rangle &= |1,1\rangle \otimes |1/2, 1/2\rangle = |3/2, 3/2\rangle\\
\left|\pi^- n\right\rangle &= |1,-1\rangle \otimes |1/2, -1/2\rangle = |3/2, -3/2\rangle
\end{aligned}$$

for the obvious highest and lowest isospin states. Carrying on...

$$\left|\pi^+ n\right\rangle = |1,1\rangle \otimes |1/2, -1/2\rangle$$

this is the linear combination of $|3/2, 1/2\rangle$ and $|1/2, 1/2\rangle$, and so on. To find the proportional cross sections we know that

$$\langle i|f\rangle \propto A \qquad |\langle i|f\rangle|^2 \propto \sigma$$

We can use the Clebsch-Gordan coefficients to find the linear combination of the states. For example

$$\left|\pi^+ n\right\rangle = |3/2, 1/2\rangle + |1/2, 1/2\rangle$$

where the Clebsch-Gordan coefficient is

$$\langle 3/2, 1/2, 1/2, 1/2 | 3/2, 1/2\rangle = \sqrt{\frac{2}{3}}$$

e.g. for the $\pi^+ p$ state

$$\left|\pi^+ p\right\rangle = |3/2, 3/2\rangle$$
$$\langle 3/2, 3/2, 1/2, 1/2 | 3/2, 3/2\rangle = 1$$

using the lowering operator

$$J_- |j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle$$

so

$$J_- |3/2, 3/2\rangle = \hbar\sqrt{3}\, |3/2, 1/2\rangle$$

applying the lower operator to $J_{1-} + J_{2-}$ we get

$$\begin{aligned} J_-(|11\rangle \otimes |1/2, 1/2\rangle) &= \hbar\sqrt{2}\, |10\rangle \otimes |1/2, 1/2\rangle + \hbar\sqrt{1}\, |11\rangle \otimes |1/2, -1/2\rangle \\ &= \hbar\sqrt{2}\, |10\rangle \otimes |1/2, 1/2\rangle + \hbar\, |11\rangle \otimes |1/2, -1/2\rangle \end{aligned}$$

we then get

$$\begin{aligned} |3/2, 1/2\rangle &= \sqrt{2/3}\, |11\rangle \otimes |1/2, 1/2\rangle + \sqrt{1/3}\, |10\rangle \otimes |1/2, 1/2\rangle \\ &= \sqrt{2/3}\, \left|\pi^+ p\right\rangle + \sqrt{1/3}\, \left|\pi^+ n\right\rangle \end{aligned}$$

and the orthogonal state is

$$|1/2, 1/2\rangle = \sqrt{2/3}\, \left|\pi^+ p\right\rangle - \sqrt{1/3}\, \left|\pi^+ n\right\rangle$$

and so on for the other states. At the end we will find that the ratio of the total cross sections (adding up the matching elastic and exchange processes) is 3.
The amplitude has a factor

$$\left\langle \pi^+ p \middle| \pi^+ p \right\rangle = \langle 3/2, 3/2 | 3/2, 3/2\rangle = M_3$$

where for example

$$\begin{aligned} (\sqrt{2}3\, \langle 3/2, 1/2| - 1/\sqrt{3}\, \langle 1/2, 1/2|)(\sqrt{2}3\, |3/2, 1/2\rangle - 1/\sqrt{3}\, |1/2, 1/2\rangle) &= \\ &= 2/3\, \langle 3/2, 1/2 | 3/2, 1/2\rangle - 1/3\, \langle 1/2, 1/2 | 1/2, 1/2\rangle \\ &= 2/3 M_3 - 1/3 M_1 \end{aligned}$$

for $M_3 >> M_1$ the ratio is 4/9, and for $M_3 << M_1$ the ratio is 1/3.

$SU(3)$

$$\begin{pmatrix} p \\ n \end{pmatrix} \quad SU(2)\text{doublet}$$

where the spins are

$$p : uud \quad Q_u = 2/3$$
$$n : udd \quad Q_d = -1/3$$

For the two spins

$$\begin{pmatrix} u \\ d \end{pmatrix}$$

the isospins are

$$I = 1/2, \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

for the up and down quarks respectively. In reality we have six quarks

- Light quarks: u, d, s
- Heavy quarks: c, b, t

For the light quaks we have a $SU(3)$ symmetry

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

the masses are all different:

$$m_u \approx 2\,\mathrm{MeV/c^2} \qquad m_d \approx 4\,\mathrm{MeV/c^2} \qquad m_s \approx 95\,\mathrm{MeV/c^2}$$

so we have to add a flavor symmetry to the $SU(2)$ isospin symmetry:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} \oplus s$$

or the $SU(3)$ symmetry

$$SU(3)_f \to SU(2)_I \otimes U(1)_y$$

From $SU(2)$ algebra:

$$[J_i, J_j] = i\epsilon_{ijk} I_k \qquad J_i = \sigma_i/2$$

For the three pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now for $SU(3)$: We know that the generators

$$U = e^{i\theta\cdot\lambda/2}$$

For $SU(N)$ we have $N^2 - 1$ generators. For $SU(3)$ we have 8 generators. The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Symmetries Part 4: Young's Tableaux & Eightfold Way

Quiz 6 Review

(1) With respect to the QCD scale (≈ 200 MeV) the masses of the quarks are divided into light and heavy quarks.

(2) (2) The effective mass mass is much larger than the mass of the light quark.

(3) $SU(2)$ is a subgroup of $SU(3)$.

For the $SU(3)$: a 3x3 unitary matrix with determinant 1. There are $n^2 - 1 = 8$ generators where

$$U = e^{iH}$$

where H is the Hermitian matrix:

$$\begin{aligned} U^\dagger U &= 1 \qquad H^\dagger = H \\ \det U &= 1 \qquad \operatorname{tr} H = 0 \\ \det M = e^{\operatorname{tr} \ln M} & \end{aligned}$$

or Hermitian matrices are traceless.

Gell-Mann Matrices Starting with the Pauli matrices but in 3x3 form

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and moving the sectors of the matrices we also get

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \cancel{\lambda_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

and

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

but λ_9 is not linearly independent since it can be written as a linear combination of $\lambda_3 + \lambda_8$.

Commutation Relation For $SU(2)$ we know

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and for $SU(3)$ we have

$$[J_i, J_j] = i f_{ijk} J_k$$

where f_{ijk} are the structure constants.

Subgroup We know that $SU(2) \leq SU(3)$ (where $\leq$ means 'is a subgroup of'). So $\{\lambda_1, \lambda_2, \lambda_3\}$ forms an $SU(2)$ sub-algebra. Also

$$\begin{gathered} \{\lambda_4, \lambda_5, a\lambda_3 + b\lambda_8\} \\ \{\lambda_6, \lambda_7, a\lambda_3 + b\lambda_8\} \end{gathered}$$

are also $SU(2)$ sub-algebras. NOTE that

$$SU(3) \neq SU(2) \otimes SU(2) \otimes SU(2)$$

Isospin and Strangeness

$$\lambda_3/2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I = 1/2$$

and the isospins are

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

and the strangeness is

$$S : I = 0$$

For λ_8 we define a hypercharge y such that

$$\lambda_8/2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}} \frac{2}{\sqrt{3}}$$

and

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad y = 1/3$$

and for the strangeness $S : y = -2/3$. This is because the Gell-Mann-Nishijima formula

$$Q = I_3 + \frac{y}{2}$$
$$2/3 = 1/2 + 1/2(1/3) \qquad -1/3 = 0 - 1/2(2/3)$$

For the triplet

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

and the anti-triplet

$$\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

Mesons $(q, \bar{q})$ in $SU(3)$ is

$$3 \otimes \bar{3} = 8 \oplus 1$$

where the 8 is the octet and the 1 is the singlet. We can do this using the Young Tableaux. For $SU(3)$ we have a 3 fundamental and $\bar{3}$ anti-fundamental.

$$\begin{array}{|c|c|} \hline N & N+1 \\ \hline N-1 & N \\ \hline N-2 & \\ \cline{1-1} \end{array}$$

Using Hook Law:

$$\dim = \frac{\Pi_i N_i}{\Pi_i h_i}$$

$$\frac{N(N+1)(N-1)N(N-2)}{1\cdot 3\cdot 4\cdot 1\cdot 2}$$

so

$$3\otimes\bar{3} = \boxed{3} \otimes \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} =$$

or

$$\boxed{a} \otimes \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} = \frac{3\cdot 4\cdot 2}{1\cdot 3\cdot 1} \oplus \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}$$

Goin from $SU(3)$ to $SU(2)$ we have

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} + s$$

or

$$3 \to 2_1 + 1_{-2}$$

where the hypercharges are subscripts. So the octet is

$$\begin{aligned} 3\otimes\bar{3} &= (2_1+1_{-2})\otimes(2_{-1}+1_2) \\ &= (2_1\otimes 2_{-1})\oplus(2_1\otimes 1_2)\oplus(1_{-2}\otimes 2_{-1})\oplus(1_{-2}\otimes 1_2) \\ &= (3_0\oplus 1_0)\oplus 2_3\oplus 2_{-3}\oplus 1_0 \\ &= 8\oplus 1 \end{aligned}$$

This is called the eightfold way.

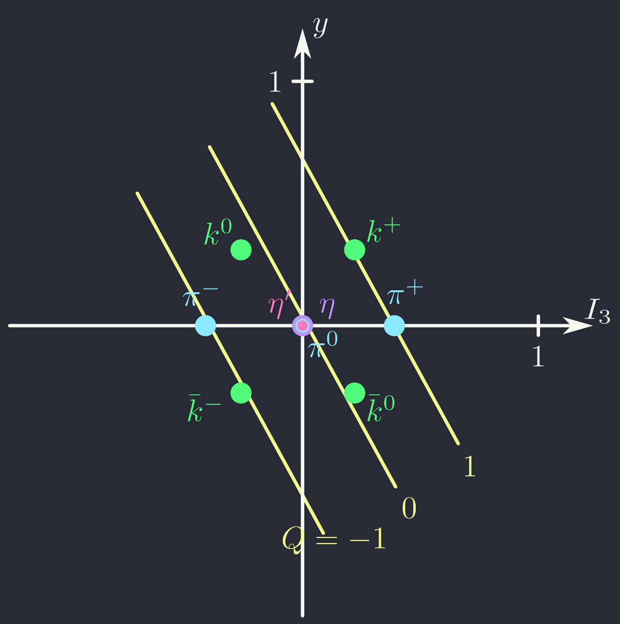

Figure 1.1: The eightfold way

Eightfold way Where η is a $SU(2)$ singlet but a $SU(3)$ octet. and η' is a $SU(3)$ singlet.
If $SU(3)_f$ was a good symmetry, expet all these 8 mesons to have similar mass. All of these obey up to a factor of 2.

Baryons (222) or $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$. The baryons are antisymmetric as each quark is a fermion. Using the Young Tablauex

$$3 \otimes 3 = \begin{array}{|c|}\hline a \\ \hline \end{array} \otimes \begin{array}{|c|}\hline b \\ \hline \end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array} = \frac{3 \cdot 4}{1 \cdot 3} \oplus \frac{3 \cdot 2}{1 \cdot 2} = 6$$

so

$$6 \otimes 3 = \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \otimes \begin{array}{|c|}\hline c \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} = \frac{3 \cdot 4 \cdot 5}{3 \cdot 2 \cdot 1} \oplus \frac{3 \cdot 4 \cdot 2}{1 \cdot 3 \cdot 1} = 10 \oplus 8$$

This is a 10-plet as shown in the figure.

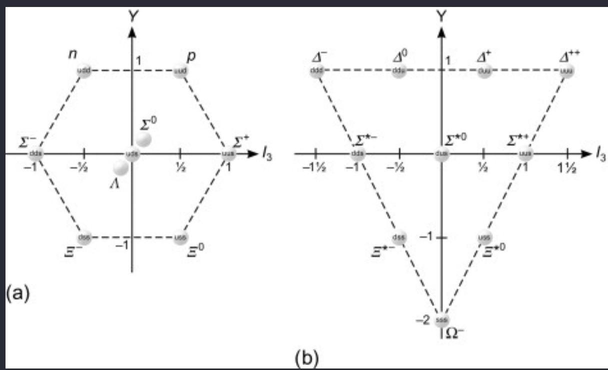

Figure 1.2: The 10-plet of baryons

Symmetries Part 5: Parity

Parity (Discrete Symmetry) For a simple reflection on a z axis the point $A = (x, y, z)$ goes to $A' = (x, -y, z)$. But for parity operation we go to $P(A) = (-x, -y, -z)$ or in the general form

$$P(\mathbf{a}) = -\mathbf{a}$$

also known as inversion. Taking the parity again we get

$$P^2(\mathbf{a}) = P(-\mathbf{a}) = \mathbf{a}$$

Thus it has a discrete Z_2 symmetry.

Pseudo-vector (axial vector) for a pseudo vector $\mathbf{c}$

$$P(\mathbf{c}) = \mathbf{c}$$

where cross products (of two vectors) are pseudo-vectors. For example

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

and

$$P(\mathbf{c}) = P(\mathbf{a} \times \mathbf{b}) = (-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{c}$$

e.g. of pseudo-vectors:

- Torque: $\tau = \mathbf{r} \times \mathbf{F}$
- Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- Magnetic field: $\mathbf{B} = \mathbf{E} \times \mathbf{v}$

But for the lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

the cross product of a vector and a pseudo-vector is a vector, so the lorentz force is a vector. Also from the general definition

$$P(\mathbf{F}) = \frac{q}{c}P(\mathbf{v}) \times P(\mathbf{B}) = \frac{q}{c}(-\mathbf{v}) \times (-\mathbf{B}) = -\mathbf{F}$$

The weak interaction violates parity...

Scalar For a scalar $s = \mathbf{a} \cdot \mathbf{b}$ is invariant under parity:

$$P(s) = P(\mathbf{a} \cdot \mathbf{b}) = (-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = s$$

for a pseudo-scalar p (a dot product of a vector and pseudo-vector):

$$P(p) = P(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})) = P(\mathbf{a})P(\mathbf{b} \times \mathbf{c}) = -\mathbf{a}(\mathbf{b} \times \mathbf{c}) = -p$$

So the partities of the four types of quantities:

- Scalar: $P(s) = s$
- Pseudo-scalar: $P(p) = -p$
- Vector: $P(\mathbf{v}) = -\mathbf{v}$
- Pseudo-vector: $P(\mathbf{c}) = \mathbf{c}$

Intrinsic Parity The parity of a fermion is

$$P(\text{fermion}) = -P(\text{anti-fermion})$$

for bosons

$$P(\text{boson}) = P(\text{anti-boson})$$

For composite particles i.e. mesons $q\bar{q}$ and baryons qqq:

$$P(\text{meson}) = -1 \quad \text{or} \quad (+1)(-1) = -1$$

Since mesons are two pairs (particle, antiparticle) the parity is always negative. For baryons we can only have a positive parity:

$$P(\text{baryon}) = (+1)^3 = +1$$

For spherical harmonics $Y_l^m(\theta, \phi)$ under parity of each term:

$$\mathbf{r} \to -\mathbf{r} \qquad \theta \to -\theta \qquad \phi \to \pi + \phi$$

so

$$P(Y_l^m(\theta, \phi)) = (-1)^l Y_l^m(\theta, \phi)$$

and for excited states

$$P = (-1)^l \times P(\text{ground state})$$

where l is the orbital angular momentum.

Parity Violation THe $\theta - \tau$ puzzle: given two particles

$$\begin{aligned} \theta^+ &\to \pi^+ + \pi^0 \qquad & P = +1 \\ \tau^+ &\to \pi^+ + \pi^0 + \pi^0 \qquad & P = -1 \end{aligned}$$

the same two particles are found to be the same particle K^+ having the same mass and lifetime, but this violates the parity. To solve this puzzle came from the Colmbia University known as Wu's experiment (wikipedia):

$$^{60}_{27}\text{Co} \to ^{60}_{28}\text{Ni} + e^- + \bar{\nu}_e$$

The Cobalt has a spin state of $J = 5$, the Nickel has spin $J = 4$ and the spin states of the electron-antielectron pair is $J = 1$. The electron is always emitted in the direction opposite of the Cobalt spin, and when the magnetic field was inverted, the electrons were emitted in the opposite direction of nuclear spin. This breaks parity, because in the mirror world, the spin of the electron would be in the same direction as the nuclear spin.

Helicity From spin $\mathbf{s}$ and momentum $\mathbf{p}$ we can define the helicity

$$\begin{aligned} \lambda &= \frac{\mathbf{s} \cdot \mathbf{p}}{|\mathbf{s}||\mathbf{p}|} \\ &= \begin{cases} +1 & \text{right-handed} \\ -1 & \text{left-handed} \end{cases} \end{aligned}$$

But this depends on the reference frame. e.g. for a case where $\mathbf{p}$ is faster and in the same direction of $\mathbf{s}$ the helicity is $\lambda = +1$, but in the reference frame faster that $\mathbf{p}$ the helicity is $\lambda = -1$.

Massless Particles For massless particles the helicity is the same in all reference frames because the speed is always c and thus the helicity is well-defined. For an electron, we can get a frame where the momentum is different for changes in reference frames.

Back to Wu's experiment The spin of the electron and neutrino are in the same direction as the spin of the Cobalt and Ni.

Note: In the SM

- neutrinos are always left-handed; $\lambda = -1$
- anti-neutrinos are always right-handed; $\lambda = +1$

Thus the electron momentum is in the opposite direction of spin as shown in Figure 1.3

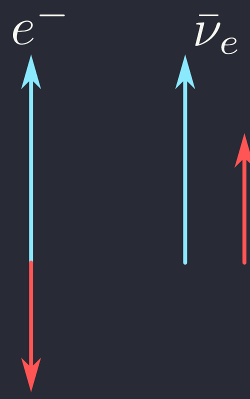

Figure 1.3: Wu's experiment

Another Example Pions and muons

$$\pi^+ \to \mu^+ + \nu_\mu \quad \text{or} \quad \to e^+ + \nu_e$$

From the comparison of masses

$$m_\pi = 140\,\text{MeV} \qquad m_\mu = 105\,\text{MeV} \qquad m_e = 0.511\,\text{MeV}$$

we would think the small mass reaction would be more likely due to the higher velocity, but this is not the case. For the pion the spin is 0, so the combination of spin will be in opposite directions. From Figure 1.4 we can see that the anti-leption(anti particle) must be left-handed and thus the lepton must also be left-handed. This is a parity violation. Thus the less favored reaction $u^+ + \nu_\mu$ is seen 99.7% of

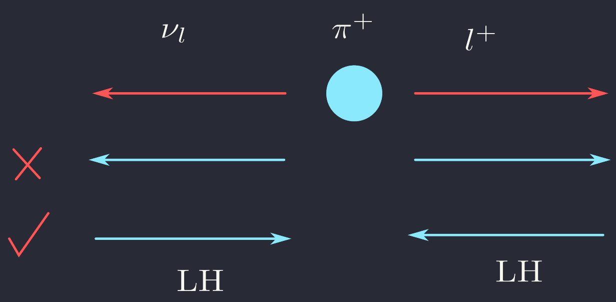

Figure 1.4: Pion decay

the time.

Anti-charged lepton has to be left-handed in this process of appoximate

$$\Gamma \propto m_\ell^\beta$$

where Γ is the decay rate.

Muon decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

This 3 body decay for a polarized muon (choosing the handedness of the muon) we have the following possiblities:

1. LH
2. RH

For maximum energy to the electron, the electron goes in one direction while the neutrinos go in the opposite direction as shown in Figure 1.5. From this the RH case is less favored than the LH case because of the helicity of the neutrinos.

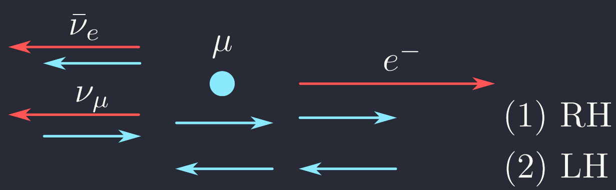

Figure 1.5: Muon decay

Symmetries Part 6: G-Parity and CP Violation

Quiz Review :

1. The neutrino is an eigenstate of partity (neutrino has handedness)

$$P\left|\nu\right\rangle_L = \pm\left|\nu\right\rangle_L x \qquad A\left|\psi\right\rangle = \lambda\left|\psi\right\rangle \qquad P^2 = 1$$

   but $\left|\nu\right\rangle_R$ does not exists!

2. Charge Conjugation (C)

$$C\left|m\right\rangle = \left|\bar{m}\right\rangle$$

   taking the charge conjugation twice

$$C^2\left|m\right\rangle = C(C\left|m\right\rangle) = C\left|\bar{m}\right\rangle = \left|m\right\rangle$$
$$\implies C^2 = 1 \implies C = \pm 1 \quad Z_z\text{-symmetry}$$

   So if $\left|m\right\rangle$ is an eigenstate of C then

$$C\left|m\right\rangle = \pm\left|m\right\rangle = \left|\bar{m}\right\rangle \implies \left|m\right\rangle = \pm\left|\bar{m}\right\rangle$$

   e.g. For charged particles:

$$C\left|\pi^+\right\rangle = \left|\pi^-\right\rangle$$

   and for some neutral particles the charge conjugation is the same as the particle. One exception is the neutron

$$C\left|n\right\rangle = \left|\bar{n}\right\rangle$$

   and for the violation of Charge conjugation: the neutrino

$$C\left|\nu\right\rangle_L = \left|\nu_L\right\rangle \times$$

   here C must be violated (in weak interactions) but preserved in strong & EM interactions.

3. G-parity: defined as

$$G = CR$$

   where R is a rotation. From the 8-fold way, we can get from π^+ to π^- by reflecting it about the hypercharge axis or rotation by 2π in the I_2 axis and then taking its charge conjugation:

$$G = Ce^{i\pi I_2}$$

   so

$$G\left|\pi^+\right\rangle = Ce^{i\pi I_2}\left|\pi^+\right\rangle = C\left|\pi^-\right\rangle = \left|\pi^+\right\rangle$$

   finding the G-parity of K^+: from the 8-fold way(Figure 1.1) we know that rotating K^+ actually gives us K^0 so

$$G\left|K^+\right\rangle = Ce^{i\pi I_2}\left|K^+\right\rangle = C\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle$$

Neutral Pion Decay $\pi^0 \to \gamma + \gamma$ is a EM interaction (particle antiparticle pair: $(u\bar{u} - d\bar{d})/\sqrt{2}$).

$$\pi^0 \to \gamma + \gamma$$

The partity on the LHS is $P = -1$ and on the RHS $P = (-1)^2 = +1$. For the photon of spin:

$$S = 1, \qquad S_z = +1, \cancel{0}, -1$$

there is no longitudinal polarization $S_z = 0$, and the transverse polatization of the EM wave $S_z = \pm 1$ is the helicity. So the helicity of the photons must be the same; either $\lambda = |{++}\rangle$ or $\lambda = |{--}\rangle$. This is not an eigenstate of partity. The two photons must have aligned polarizations. We also need to find p in

$$\lambda = \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{S}||\mathbf{p}|}$$

where we know

$$p\,|+\rangle = |-\rangle \quad p\,|-\rangle = |+\rangle \quad p\,|{++}\rangle = |{--}\rangle$$

we can write a linear combination of the two states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle)$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|{++}\rangle - |{--}\rangle)$$

for

$$P\,|\psi_1\rangle = |\psi_1\rangle \quad \text{(even parity)}$$
$$P\,|\psi_2\rangle = -\,|\psi_2\rangle \quad \text{(odd parity)}$$

This is similar to quantum optics where polarization is used:

$$|+\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle)$$

we know the G parity is

$$G = (-1)^I$$

4. Pion decay to muon and neutrino: For the mesons

$$C = (-1)^{l+s}$$

where we have two types: Pseudoscalars ($s = 0$) e.g π, K with $C = (-1)^l$ and vector mesons ($s = 1$) e.g. ρ, K^* with $C = (-1)^{l+1}$. For ρ we know that $I(\rho) = 1$ from the 8-fold way, as well as $I(\eta) = 0$ so $\rho \to 3\pi$ and $\eta \to 2\pi$ are allowed.

CP

$$P\,|\nu_L\rangle = |\nu_R\rangle \times$$
$$CP\,|\nu_L\rangle = C\,|\nu_L\rangle = |\bar{\nu}_R\rangle$$

where the P is violated due to the handedness of the neutrino, but the CP is conserved. applying CP on to the charged pion decay:

$$\pi^+ \to \mu^+ + \nu_\mu$$

CP Violation

For an oscillation of a neutral kaon $K^0(d\bar{s})$ and $\bar{K}^0(\bar{d}s)$. Because K's are pseudoscalars:

$$P\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad P\left|\bar{K}^0\right\rangle = \left|\bar{K}^0\right\rangle$$

and the charge conjugation is

$$C\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle \qquad C\left|\bar{K}^0\right\rangle = \left|K^0\right\rangle$$

and the CP is

$$CP\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad CP\left|\bar{K}^0\right\rangle = -\left|K^0\right\rangle$$

which are not eigenstates of CP, so taking a linear combination of the two states

$$|K_1\rangle = \frac{1}{\sqrt{2}}\big(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\big) \qquad |K_2\rangle = \frac{1}{\sqrt{2}}\big(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\big)$$

the CP of the two states are

$$CP\,|K_1\rangle = |K_1\rangle \qquad CP\,|K_2\rangle = -\,|K_2\rangle$$

which are eigenstates of CP. Karons decay to pions(lightest meson): to either 2 or 3 pions. From the phase space, the 2 pion decay is more likely than the 3 pion decay because faster particles are more likely to decay: A kaon with 490 MeV to 2×140 = 280 MeV is more likely than 3×140 = 420 MeV. The 2 pion decay has CP

$$CP\left|\pi^+\pi^-\right\rangle = (-1)^2\left|\pi^+\pi^-\right\rangle = \left|\pi^+\pi^-\right\rangle$$
$$CP\left|\pi^+\pi^-\pi^0\right\rangle = (-1)^3\left|\pi^+\pi^-\pi^0\right\rangle = -\left|\pi^+\pi^-\pi^0\right\rangle$$

so the 2 pion decay is CP even and the 3 pion decay is CP odd. If CP were conserved,

$$|K_1\rangle \to |2\pi\rangle \qquad |K_2\rangle \to |3\pi\rangle$$

so we call this fast decay K_S meaning short ($9\times 10^{-11} sec$) and the slow decay K_L meaning long ($5\times 10^{-8} sec$) or

$$|K_1\rangle \equiv \left|K_S^0\right\rangle \qquad |K_2\rangle \equiv \left|K_L^0\right\rangle$$

from a nobel prize winning experiment, a detector far away would be expected to see mostly K_L but the experiment showed that we saw some amount to the 2 pion decay:

$$\left|K_L^0\right\rangle = \frac{1}{\sqrt{1+\epsilon^2}}\big(\left|K_S^0\right\rangle + \epsilon\left|K_L^0\right\rangle\big)$$

where $\epsilon = 2.3\times 10^{-3}$ characterizes CP violation.

Symmetries Part 7: CP Violation

Quiz Reviw:

- matter-antimatter asymmetry: the ϵ parameter measures the CP violation in the neutral kaon (hadronic decay mode). For the semi leptonic decay mode:

$$K_L^0 \to \pi^+ + e^- + \bar{\nu}_e$$

The CP symmetry shows

$$\to \pi^- + e^+ + \nu_e$$

And the rates for the decay modes differ by ϵ. This relates to the matter-antimatter symmetry. (barygenesis) $\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} = 6 \times 10^{-10}$ where n is the number of baryons/antibaryons/photons. In the SM $\eta_{SM} \approx 10^{-20}$, so the SM does not explain the matter-antimatter asymmetry fully. Finding the change in the rate of the process:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} \propto \Gamma(X \to Y + B) - \Gamma(\bar{X} \to \bar{Y} + \bar{B})$$

where the first decay rate Γ violates B(baryon number) for the charge conjugation

$$C(X \to Y + B) = \bar{X} \to \bar{Y} + \bar{B}$$

if C is conserved then

$$\Gamma(X \to Y + B) = \Gamma(\bar{X} \to \bar{Y} + \bar{B})$$

For CP violation: Quarks are chiral e.g. $X \to q_L q_L, q_R q_R$ with (chirality:L and R). We consider X as a linear combination:

$$X \to qq = q_L q_L + q_R q_R$$

The CP operation is

$$CP(X \to q_L q_L) = \bar{X} \to \bar{q}_R \bar{q}_R$$
$$CP(X \to q_R q_R) = \bar{X} \to \bar{q}_L \bar{q}_L$$

so we can write this as a linear combination ($r = \Gamma$)

$$r(X \to qq) = r(X \to q_L q_L) + r(X \to q_R q_R)$$
$$r(\bar{X} \to \bar{q}\bar{q}) = r(\bar{X} \to \bar{q}_R \bar{q}_R) + r(\bar{X} \to \bar{q}_L \bar{q}_L)$$

these are known as the Sakharov conditions:

1. Baryon number violation
2. C and CP violation
3. Departure from thermal equilibrium (forward process $\neq$ backward process)

thus $r(X \to Y + B) \neq r(Y + B \to X)$

- Rotation angles needed to describe mixing n generations of quarks: Cahibbo angles…in 2D we have 1 angle, in 3D we have 3 angles, and in 4D we have 6 angles. This is related to the number of orthogonal places in the space so $n(n-1)/2$ angles are needed.

nxn **unitary matrix:** How many free parameters are there? n^2 complex parts and n^2 real parts thus $2n^2$ parts. But the unitary matrix has n^2 constraints (conditions) e.g. $UU^\dagger = 1$ so we are left with n^2 parameters. To find the number of physical parameters we add a phase which does not change the physical observables so we can have $2n-1$ phases e.g.

$$\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} e^{i\alpha}u \\ e^{i\beta}d \end{pmatrix} e^{i\gamma}$$

here we can always add an overall phase hence the -1. so the free parameters are

$$n^2 - (2n-1) = (n-1)^2$$

and out these free parameters we subtract the number of rotation angles to get the number of physical phase angles.

$$(n-1)^2 - \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2}$$

nxn unitary matrix is described by $n(n-1)/2$ rotation angles and $(n-1)(n-2)/2$ phase angles. e.g. for $n=2$: 1 rotation and 0 phase, for $n=3$: 3 rotation and 1 phase. The 1 phase angle in the 3D case is the CP violation angle. This is known as the CKM matrix.

For SU(3) The euler rotation is:

$$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$c_{12} \equiv \cos\theta_{12} \qquad s_{12} \equiv \sin\theta_{12}$$

If the V_{CKM} is close to the identity, the rotation angles would be small, so the two bases are roughly the same. Experimentally we find that the rotation angles

$$\theta_{12} \approx 13^\circ \qquad \theta_{23} \approx 2.4^\circ \qquad \theta_{13} \approx 0.2^\circ \qquad \delta \approx 70^\circ$$

which gives us the CKM matrix

$$V_{CKM} = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

or from the Feynman diagram

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

theses are called the Cabbibo-suppressed processes. There also is a Wolfenstein parameterization

$$V_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

for the mixing of the quarks of neutrinos: ν_e, ν_μ, ν_τ we have the PMNS matrix

$$\begin{pmatrix} \nu_e' \\ \nu_\mu' \\ \nu_\tau' \end{pmatrix} = V_{PMNS} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

parametrized by $(\theta_{12}, \theta_{23}, \theta_{13}, \delta)$ again. This is due to the angles of the rotations being much larger than the CKM matrix:

$$\theta_{12} \approx 34^\circ \qquad \theta_{23} \approx 45^\circ \qquad \theta_{13} \approx 8^\circ \qquad \delta \approx ?$$

this is known as the flavor puzzle.

Time reversal (T) $t \to^T -t$
In principle we can have a reversable process $n \leftrightarrow p + e^- + \bar{\nu}_e$ such as in stars, but we can't do this in lab.

T violation Dipole Moment:

Symmetries Part 8: Time Reversal

Time Reversal Symmetry

$$n \leftrightarrow p + e^- + \bar{\nu}_e$$

the reverse process is hard to observe, but in principle it is possible in dipole moments.

Dipole Moment Two opposite charges $\pm q$ separated by a distance d, we have electric field lines going from the positive to the negative charge which have a dipole moment

$$\mathbf{p} = q\mathbf{r}$$

for a uniform charge distribution we usually have a displacement vector

$$\mathbf{d} = \int \mathrm{d}^3 r \rho \mathbf{r}$$

Magnetic dipole Moment

$$\mu = \int \mathrm{d}^3 r \mathbf{r} \times \mathbf{j}_q$$

where $\mathbf{j}_q$ is the current density. A charged particle with a spin has a electric dipole moment

$$\mathbf{d} = d\frac{\mathbf{J}}{|\mathbf{J}|} \qquad \mathbf{J} = \mathbf{L} + \mathbf{S}$$

and a magnetic dipole moment

$$\mu = u\frac{\mathbf{J}}{|\mathbf{J}|}$$

from Wigner-Eckart theorem. The Hamiltonian is

$$H = -(\mu \cdot \mathbf{B} + \mathbf{d} \cdot \mathbf{E}) = -\frac{1}{|\mathbf{J}|}(\mu \mathbf{J} \cdot \mathbf{B} + d\mathbf{J} \cdot \mathbf{E})$$

This is T-even because from the Schrodinger Eq $H\psi = E\psi$ and invariant under T. Using Using Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ t \to -t : \qquad \mathbf{E} &\to \mathbf{E} \quad \text{T-even} \end{aligned}$$

and for second equation

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ t \to -t : \quad &\text{LHS doesn't change} \\ \text{RHS} \quad \mathbf{B} &\to -\mathbf{B} \quad \text{T-odd} \end{aligned}$$

To find if the angular momentum is T-even or T-odd we can use the definition of angular momentum:

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} \\ &= \mathbf{r} \times m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\ t \to -t : \mathbf{v} &\to -\mathbf{v} \\ \mathbf{L} &\to -\mathbf{L} \quad \text{T-odd} \end{aligned}$$

for spin angular momentum we can visualize that from the right hand rule, the spin is T-odd i.e. $\mathbf{S} \to -\mathbf{S}$ (T-odd). Adding the orbital and spin angular momentum we get the total angular momentum $\mathbf{J}$ as T-odd. Since $\mathbf{J}$ and $\mathbf{B}$ are T-odd, we can deduce that μ is T-even (odd times odd is even, and even times odd is even). By the same logic $\mathbf{d}$ is T-odd.

ACME Experiment The limit of the electron dipole moment is

$$d_e < 8 \times 10^{-30} e.cm$$

and for the neutron dipole moment

$$d_n < 10^{-27} e.cm$$

Axions(Not in Textbook) A hypothetical particle: The Langrangian for fundamental particle is

$$\mathcal{L} = \theta G^{\mu\nu} G_{\mu\nu}$$

the electromagnetic field tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

where the magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

and

$$A^0 = \rho = F^{\mu\nu} F_{\mu\nu}$$

the Gluon field tensor

$$G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

and θ is T-odd. Experimentally we find that $\theta < 10^{-10}$ (Strong CP problem). ADMX experiment is currently ongoing.

CPT Theorem All Observables must be CPT invariant in a Lorentz-invariant theory. Sometimes C, P, or T are violated, but CPT is always conserved. As a result, all anti-particles have the same mass as particles (Tested by LHC alpha experiment).