1.5 Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$
$$= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x\end{vmatrix}$$

where the x component is

$$A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

From the "BAC-CAB" rule,

$$\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)$$

where x component simplifes to

$$B_x(\cancel{A_xC_x} + A_yC_y + A_zC_z) - C_x(\cancel{A_xB_x} + A_yB_y + A_zB_z) = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

So,

$$[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_x = [\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})]_x$$

and similary for the y and z components. Therefore, the "BAC–CAB" rule holds true.

1.11 (a) Finding gradient of $f(x, y, z) = x^2 + y^3 + z^4$:

$$\begin{aligned}\boldsymbol{\nabla} f &= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\ &= 2x\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + 4z^3\hat{\mathbf{z}}\end{aligned}$$

(b) Gradient of $f(x, y, z) = x^2y^3z^4$:

$$\boldsymbol{\nabla} f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$$

(c) Gradient of $f(x, y, z) = e^x \sin(y)\ln(z)$:

$$\boldsymbol{\nabla} f = e^x\hat{\mathbf{x}} + e^x\cos(y)\ln(z)\hat{\mathbf{y}} + \frac{e^x\sin(y)}{z}\hat{\mathbf{z}}$$

1.13 Given the seperation vector

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad \mathscr{r} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

(a) Show that $\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$:

$$\mathscr{r}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

and each component of the gradient is

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\mathscr{r}^2\right) &= 2(x - x') \\
\frac{\partial}{\partial y}\left(\mathscr{r}^2\right) &= 2(y - y') \\
\frac{\partial}{\partial z}\left(\mathscr{r}^2\right) &= 2(z - z')
\end{aligned}$$

so

$$\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$$

(b) Show $\boldsymbol{\nabla}(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$:

$$\frac{1}{\mathscr{r}} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2} = [(\ )]^{-1/2}$$

looking at the x component of the gradient (using chain rule),

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right) &= -\frac{1}{2}[(\ )]^{-3/2} \cdot 2(x - x') \\
&= -\frac{(x - x')}{\mathscr{r}^3}
\end{aligned}$$

and similarly for the y and z components:

$$\begin{aligned}
\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{z}} \\
&= -\frac{1}{\mathscr{r}^3}[(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}] = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}
\end{aligned}$$

Finally, substituting the unit vector $\hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}/\mathscr{r}$ gives us

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(c) The general formula for $\boldsymbol{\nabla}(\mathscr{r}^n)$:

$$\boldsymbol{\nabla}(\mathscr{r}^n) = n\,\mathscr{r}^{n-1} \cdot \boldsymbol{\nabla}(\mathscr{r})$$

where

$$\begin{aligned}
\boldsymbol{\nabla}(\mathscr{r}) &= \frac{\partial}{\partial x}(\mathscr{r})\hat{\mathbf{x}} + \frac{\partial}{\partial y}(\mathscr{r})\hat{\mathbf{y}} + \frac{\partial}{\partial z}(\mathscr{r})\hat{\mathbf{z}} \\
&= \frac{1}{2}[(\ )]^{-1/2} \cdot 2(x - x')\hat{\mathbf{x}} + \dots \quad \text{[similar to part (b)]} \\
&= \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \hat{\boldsymbol{\mathscr{r}}}
\end{aligned}$$

So the general formula is

$$\boldsymbol{\nabla}(\mathscr{r}^n) = n\,\mathscr{r}^{n-1}\,\hat{\boldsymbol{\mathscr{r}}}$$

1.15 (a) Calculating divergence of $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v}_a &= \frac{\partial v_{ax}}{\partial x} + \frac{\partial v_{ay}}{\partial y} + \frac{\partial v_{az}}{\partial z} \\ &= 2x + 0 - 2x = 0\end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_b = y + 2z + 3x$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_c = 0 + 2x + 2y = 2(x + y)$$

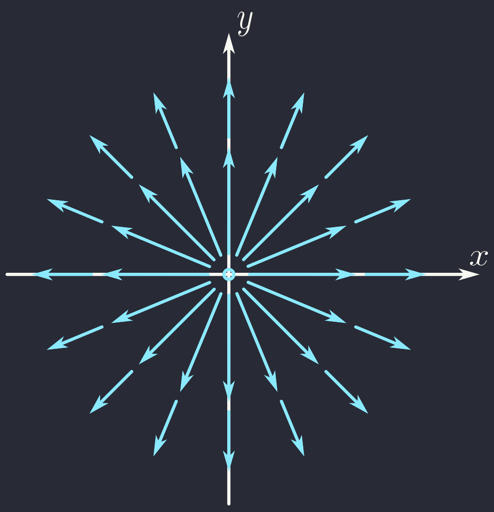

Figure 1.16: The vector field $\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$

1.16 Given

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector. The vector functions is

$$v = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

where the components are

$$v_x = \frac{x}{r^3}, \quad v_y = \frac{y}{r^3}, \quad \text{and} \quad v_z = \frac{z}{r^3}$$

Looking at the x component of the divergence,

$$\begin{aligned}
[\boldsymbol{\nabla} \cdot \mathbf{v}]_x &= \frac{\partial v_x}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) \quad \text{using chain rule...} \\
&= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{1}{r^3} - \frac{3x^2}{r^5}
\end{aligned}$$

therefore, the divergence of $\mathbf{v}$ is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v} &= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \\
&= \frac{3}{r^3} - 3\frac{x^2 + y^2 + z^2}{r^5} \\
&= \frac{3}{r^3} - 3\frac{r^2}{r^5} = 0
\end{aligned}$$

The divergence is zero everywhere except at the origin where $r = 0$ because division by r^3 tells us that the divergence is infinite at the origin.

1.18 Curl of vector functions from Problem 1.15: (a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_a &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0) \\
&= -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}
\end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_b &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x) \\
&= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}
\end{aligned}$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_c &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\
&= \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) \\
&= 0
\end{aligned}$$

1.32 Given $T = x^2 + 4xy + 2yz^3$,

$$\frac{\partial T}{\partial x} = 2x + 4y, \quad \frac{\partial T}{\partial y} = 4x + 2z^3, \quad \text{and} \quad \frac{\partial T}{\partial z} = 6yz^2$$

therefore

$$\nabla T = \hat{\mathbf{x}}(2x + 4y) + \hat{\mathbf{y}}(4x + 2z^3) + \hat{\mathbf{z}}(6yz^2)$$

Checking the fundamental theorem for gradients using the points $a = (0,0,0) \to b = (1,1,1)$:

$$\int_a^b \nabla T \cdot \mathrm{d}\mathbf{l} = T(b) - T(a) = 1^2 + 4(1)(1) + 2(1)(1)^3 - 0 = 7$$

For the three paths:

(a) $a \to (1,0,0) \to (1,1,0) \to b$;

(i) $a \to (1,0,0)$:

$$x : 0 \to 1; \quad y = z = \mathrm{d}y = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \nabla T \cdot \mathrm{d}\mathbf{l} = 2x\,\mathrm{d}x$$

and

$$\int_a^c \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2x\,\mathrm{d}x = 1$$

(ii) $(1,0,0) \to (1,1,0)$:

$$y : 0 \to 1; \quad x = 1, \quad z = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \nabla T \cdot \mathrm{d}\mathbf{l} = 4\,\mathrm{d}y$$

and

$$\int_c^d \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 4\,\mathrm{d}y = 4$$

(iii) $(1,1,0) \to b$:

$$z : 0 \to 1; \quad x = y = 1, \quad \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \nabla T \cdot \mathrm{d}\mathbf{l} = 6z^2\,\mathrm{d}z$$

and

$$\int_d^b \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 6z^2\,\mathrm{d}z = 2$$

therefore

$$\int_a^b \nabla T \cdot \mathrm{d}\mathbf{l} = 1 + 4 + 2 = 7$$

(b) $a \to (0,0,1) \to (0,1,1) \to b$;

(i) $a \to (0,0,1)$:

$$z : 0 \to 1; \quad x = y = \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \nabla T \cdot \mathrm{d}\mathbf{l} = 0$$

and

$$\int_a^c \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 0 = 0$$

(ii) $(0,0,1) \to (0,1,1)$:

$$z = 1, \quad x = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \boldsymbol{\nabla}T \cdot \mathrm{d}\mathbf{l} = 2\,\mathrm{d}y$$

and

$$\int_c^d \boldsymbol{\nabla}T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2\,\mathrm{d}y = 2$$

(iii) $(0,1,1) \to b$:

$$z : 0 \to 1; \quad y = z = 1, \quad \mathrm{d}y = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \boldsymbol{\nabla}T \cdot \mathrm{d}\mathbf{l} = (2x+4)\,\mathrm{d}x$$

and

$$\int_d^b \boldsymbol{\nabla}T \cdot \mathrm{d}\mathbf{l} = \int_0^1 (2x+4)\,\mathrm{d}x = 5$$

therefore

$$\int_a^b \boldsymbol{\nabla}T \cdot \mathrm{d}\mathbf{l} = 0 + 2 + 5 = 7$$

(c) the parabolic path $z = x^2$; $y = x$:

$$\mathrm{d}x = \mathrm{d}y\,, \quad \text{and} \quad \mathrm{d}z = 2x\,\mathrm{d}x\,; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}x\,\hat{\mathbf{y}} + 2x\,\mathrm{d}x\,\hat{\mathbf{z}}$$

and

$$\begin{aligned} \boldsymbol{\nabla}T \cdot \mathrm{d}\mathbf{l} &= (2x+4y)\,\mathrm{d}x + (4x+2z^3)\,\mathrm{d}x + (6yz^2)2x\,\mathrm{d}x \\ &= 6x\,\mathrm{d}x + (4x+2x^6)\,\mathrm{d}x + (12x^6)\,\mathrm{d}x \\ &= 10x\,\mathrm{d}x + 14x^6\,\mathrm{d}x \end{aligned}$$

therefore

$$\begin{aligned} \int_a^b \boldsymbol{\nabla}T \cdot \mathrm{d}\mathbf{l} &= \int_0^1 (10x + 14x^6)\,\mathrm{d}x \\ &= 5x^2 + 2x^7 \Big|_0^1 = 7 \end{aligned}$$

1.33 Testing the divergence theorem: For the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

the divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{v} = y + 2z + 3x$$

so the volume integral is

$$\begin{aligned}
\int_V \boldsymbol{\nabla} \cdot \mathbf{v} \, \mathrm{d}\tau &= \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_0^2 \int_0^2 (2y + 4z + 6) \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_0^2 (4 + 8z + 12) \, \mathrm{d}z \\
&= 8 + 16 + 24 \\
\int_V \boldsymbol{\nabla} \cdot \mathbf{v} \, \mathrm{d}\tau &= 48
\end{aligned}$$

The surface integral is evaluated over the six faces of the cube:

(i) $x = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}y \, \mathrm{d}z \, \hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 2y \, \mathrm{d}y \, \mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 2y \, \mathrm{d}y \, \mathrm{d}z = 8$$

(ii) $x = 0$, $\mathrm{d}\mathbf{A} = -\,\mathrm{d}y \, \mathrm{d}z \, \hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 0 \, \mathrm{d}y \, \mathrm{d}z = 0$$

(iii) $y = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x \, \mathrm{d}z \, \hat{\mathbf{y}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 4z \, \mathrm{d}x \, \mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 4z \, \mathrm{d}x \, \mathrm{d}z = 16$$

(iv) $y = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

(v) $z = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x \, \mathrm{d}y \, \hat{\mathbf{z}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 6x \, \mathrm{d}x \, \mathrm{d}y$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 6x \, \mathrm{d}x \, \mathrm{d}y = 24$$

(vi) $z = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

So the total flux is

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 8 + 0 + 16 + 0 + 24 + 0 = 48$$

therefore, the divergence theorem is verified.

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{v}) \, \mathrm{d}\tau = \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A}$$

1.34 Testing Stokes' theorem for the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

using the triangular shaded area bounded by the vertices $O = (0,0,0)$, $A = (0,2,0)$, and $B = (0,0,2)$:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{v} &= (0-2y)\hat{\mathbf{x}} - (3z-0)\hat{\mathbf{y}} + (0-x)\hat{\mathbf{z}} \quad \text{and} \quad \mathrm{d}\mathbf{A} = \mathrm{d}z\,\mathrm{d}y\,\hat{\mathbf{x}} \\ &= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}} \end{aligned}$$

$x = 0$ on this surface, and the limits of integration are $y : 0 \to 2$ and $z = 0 \to z = 2 - y$:

$$(\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = -2y\,\mathrm{d}z\,\mathrm{d}y$$

Thus, the flux of the curl through the surface is

$$\begin{aligned} \int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} &= \int_0^2 \int_0^{2-y} -2y\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_0^2 -2y(2-y)\,\mathrm{d}y \\ &= -2y^2 + \frac{2}{3}y^3 \Big|_0^2 = -8/3 \end{aligned}$$

The line integral is evaluated over the three sides of the triangle:

(i) On the path OA:
$x = z = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 2yz\,\mathrm{d}y = 0$;

$$\int_{OA} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

(ii) On the path AB:
$x = 0$, $y = 2 - z$; $\mathrm{d}y = -\,\mathrm{d}z$; $\mathrm{d}\mathbf{l} = -\,\mathrm{d}z\,(\hat{\mathbf{y}} - \hat{\mathbf{z}})$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = -2yz\,\mathrm{d}z = -2(2-z)z\,\mathrm{d}z = (2z^2 - 4z)\,\mathrm{d}z$;

$$\int_{AB} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^2 (2z^2 - 4z)\,\mathrm{d}z = -8/3$$

(iii) On the path BO:
$x = y = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$;

$$\int_{BO} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

So, the circulation of $\mathbf{v}$ around the triangle is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0 + -8/3 + 0 = -8/3$$

thus,

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$