1 Potentials

1.1 Laplace's Equation

1.1.1 Intro

In principle, electrostatis is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\mathbf{r}')\mathrm{d}\tau', \quad \mathscr{r} = \mathbf{r} - \mathbf{r}'$$

And simplifying with potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\mathrm{d}\tau'$$

So we often use Poisson's equation e.g.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Even better, Laplace's equation

$$\boxed{\nabla^2 V = 0}$$

or in Cartesian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

1.1.2 Start in 1D

$$\frac{\partial^2 V}{\partial x^2} = 0 \implies V = mx + b$$

where we have two undetermined constants m & b. We can determine these constants by *boundary conditions.*

- e.g. $V(1) = 4 \quad V(5) = 0$; we get a line $V = -\frac{4}{5}x + 4$

Two features:

1. $V(x)$ is average of $V(x+a)$ and $V(x-a)$

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$$

2. **NO** local minima or maxima (no curvature!)

1.1.3 On to 2D

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{no general solution}$$

and no requirement on the # of constants. But we can note common properties e.g. soap film on a wireframe assumes the same shape. The solutions are called *harmonic functions*:

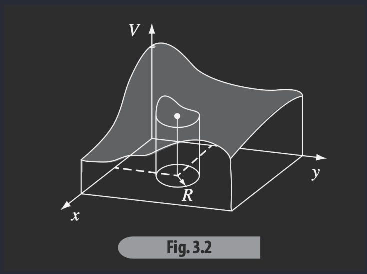

Figure 1.1

1. value $V(x, y)$ is average of nearby values; more precisely, for a circle of radius R (Fig. 1.1)

$$V(x, y) = \frac{1}{2\pi R} \oint V \, \mathrm{d}\ell$$

where $2\pi R$ is the circumference of the circle.

2. **NO** local minima or maxima

1.1.4 In 3D

$$\nabla^2 V = 0$$

Holds same properties as 2D:

1. Average over spherical surface of radius R centered at $\mathbf{r}$:

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_S V \, \mathrm{d}a$$

where $4\pi R^2$ is the surface area of the sphere.

Example: Point charge outside sphere; The potential at da

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}$$

and from law of cosines

$$\mathscr{r}^2 = z^2 + R^2 - 2rR\cos\theta$$

so the average potential is

$$\begin{aligned}
V_{\text{avg}} &= \frac{q}{4\pi R^2} \frac{1}{4\pi\epsilon_0} \int \frac{R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi}{\sqrt{z^2 + R^2 - 2zR\cos\theta}} \\
&= \frac{q}{2zR} \frac{1}{4\pi\epsilon_0} \left[z^2 + R^2 - 2zR\cos\theta\right]^{1/2} \Big|_0^\pi \\
&= \frac{q}{2zR} \frac{1}{4\pi\epsilon_0} [(z + R) - (z - R)] \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{z}
\end{aligned}$$

which is just the potential of a point charge q in the center of the sphere.

Question: Is it possible to stably trap a charged particle using electrostatic forces alone?

Answer Earnshaw's theorem: A charged particle cannot be held in stable equilibrium by electrostatic forces alone.

1.1.5 Boundary Conditions & Uniqueness Theorem

Laplace's eq requires boundary conditions (b.c.c)

1st uniqueness theorem: "Solutions to L's eq in volume V is uniquely determined if potential is specied in surface S bounding V."

How is the solution unique?

Have solution in V_1 s.l.

$$\nabla^2 V_1 = 0 \quad \text{also} \quad \nabla^2 V_2 = 0$$

Then

$$V_3 \equiv \Delta V = V_1 - V_2$$

which means

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0 - 0 = 0$$

thus

$$\begin{aligned} \implies \nabla^2 V_1 &= \nabla^2 V_2 \\ V_1 &= V_2 \end{aligned}$$

We should emphasize that V_1 defined on the boundary S is also the same b.c.s as V_2 while V_3 on the boundary equals zero. That is V_3 is zero everywhere in space.

1.1.6 2nd uniqueness theorem (conductors):

"In a volume V surrounded by conductors, and containing a specifed charge density ρ, then the $\mathbf{E}$ field is uniquely determined if the total charge on each conductor is given."

"This proof was not easy" - Griffiths

Example: Connecting two pairs of opposite charges with a conductor; what is the final charge config and E-field?

Answer: $\mathbf{E} = 0$ everywhere. The total charge of each conductor is zero.

1.1.7 Boundary conditions pt. II

(Griffiths 2.3.5 pg 85) Given a sheet of charge $\sigma = Q/A$ and using the Gaussian pillbox method

$$\begin{aligned} \oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{q_{\text{enc}}}{\epsilon_0} \\ E_a A - E_b A &= \sigma \frac{A}{\epsilon_0} \\ E_a - E_b &= \frac{\sigma}{\epsilon_0} \end{aligned}$$

For the same surface we know that

$$\boldsymbol{\nabla} \times \mathbf{E} = \oint \mathbf{E} \cdot \mathrm{d}\ell = 0$$

So going around the loop we have

$$\begin{aligned} E_a \ell - E_b \ell &= 0 \\ E_a &= E_b \end{aligned}$$

1.2 Method of Images

$\nabla^2 V = -\rho/\epsilon_0$ *is* electrostatics. AND when we have $\nabla^2 V = 0$, Uniqueness theorems tell us there is a solution, but doesn't tell us how to find it... thus we have a set of "easily" solvable problems.

1.2.1 Classic image problem:

"Ground" is infinite conducting plane $V = 0$ at $z = 0$. For a charge q at $z = d$ what is the potential in $z > 0$? We know that the point charge will induce a charge on the plane which will effect the electic potential in the region $z > 0$.

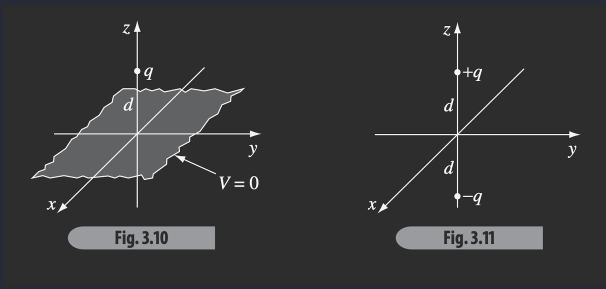

Figure 1.2

GOAL: solve $\nabla^2 V = -\rho/\epsilon_0$ for $z \geq 0$ with $+q$ at $(0, 0, d)$ subject to boundary conditions (b.c.s)

1. $V(z = 0) = 0$
2. $V \to 0$ for far away

The next step is to replace the conducting plane with an equivalent charge $-q$ at $z = -d$.

NOTE: We only care about $z \geq 0$ and ignore $z < 0$ region; furthermore, the potential below the plane should be zero as the boundary of the plane sort of "wraps" around the charge

Thus the solution is a superposition of charges

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

we can see that in the replacement problem, the region below the grounded plane is nonzero which is different from the real problem which is why we ignore it!

1.2.2 Induced surface charge

What is σ? Recall on the sheet σ we have a field

$$\mathbf{E}_{\text{ab}} - \mathbf{E}_{\text{be}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

where the $\hat{\mathbf{n}}$ tells us that we are dealing with the perpendicular planes; thus the same eq

$$\boldsymbol{\nabla} V_{\text{ab}} - \boldsymbol{\nabla} V_{\text{be}} = -\frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \implies \mathbf{E} = -\boldsymbol{\nabla} V$$

We can infer that

$$\frac{\partial V_a}{\partial n} - \cancel{\frac{\partial V_b}{\partial n}} = -\frac{\sigma}{\epsilon_0}$$

since anything below the plane is zero from the previous example. We know have

$$\left.\frac{\partial V_a}{\partial z}\right|_{z\to 0} = \frac{1}{4\pi\epsilon_0}\left[-\frac{q(z-d)}{(x^2 + y^2 + (z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2 + y^2 + (z+d)^2)^{3/2}}\right]\Bigg|_{z\to 0}$$

So

$$\sigma = -\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}}$$

We can check that the total induced charge in the plane is infact

$$Q = \int_0^{2\pi}\int_0^{\infty} \sigma r \mathrm{d}r \mathrm{d}\phi = -q$$

where $r^2 = x^2 + y^2$.

This charge comes from the reservoir of charge given by ground.

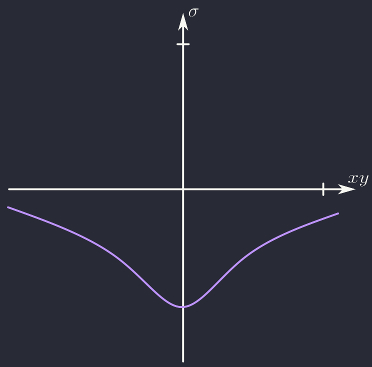

Figure 1.3: Just a cool graph of xy vs σ

1.2.3 Force and energy

Calcuating the force of attraction because of the negative induced charge:

$$F = qE = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}(-\hat{\mathbf{z}})$$

Naively, we calculate the energy/work done as

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d} \quad (= q\Delta V)$$

but we only have a single charge and the total work is half of this value

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

This is because the ideal conductor requires no work to build up a charge distribution σ.

1.3 Seperation of Variables

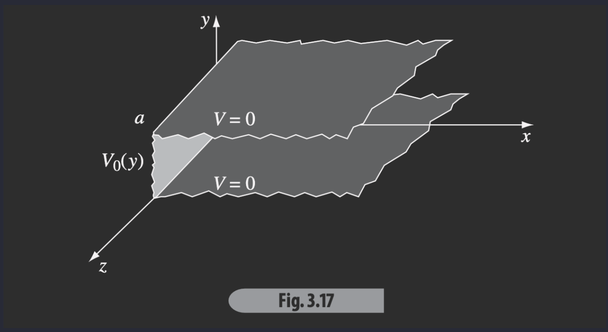

Figure 1.4: 3 planes where top and bottom are grounded and the middle is at $V_0(y)$

Find potential V in $x > 0$, $0 < y < a$, $-\infty < z < \infty$:
This is a 2D problem $V \to V(x, y)$
No change in $V \to \nabla^2 V = 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$.
From b.c.s

1. $V(y = 0) = 0$
2. $V(y = a) = 0$
3. $V(x = 0) = V_0(y)$
4. $V(x \to \infty) = 0$

PROPOSE: $V(x, y) = X(x)Y(y)$

$$\nabla^2 V = Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{\nabla^2 V}{V} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$