Title

Contents

1 Intro to Particle Physics

Four Fundamental Forces

- Strong (gluon)
- Weak (W, Z)
- Electromagnetic (photon)
- Gravity (graviton?)

The 'Standard Model' describe the first three forces and unifies the Strong and Weak Forces known as the 'Electroweak' force. So, the Standard Model does not include gravity.

The Standard Model (SM)

- Basic building blocks: spin 1/2 particles (fermions)
- Interaction between then are mediated by force carriers: spin 1 particles (vector bosons)
- How particles get mass? → Higgs Boson (spin 0)

The Range of Forces:

- Strong: 10^{-15} m
- Weak: 10^{-18} 10^{-16} m
- EM: $1/r^2$
- Gravity: $1/r^2$

The ranges of forces are related by

$$R \ \frac{e^{-r/a}}{r^2}$$

where $a \approx 10^{-15}$ m for the Strong and Weak forces.

The Rise of Quantum Field Theory (QFT) Relativity + Quantum Mechanics → QFT

	Macroscopic	Micro
SLOW	CM	Quantum Mechanics
FAST	Special Relativity	QFT

QFT Discoveries

- Existence of anti-particles
- Spin-statistics theorem
- CPT Theorem (Charge conjugation, Parity, Time reversal)

Units!

- Mass: (kg) $\rightarrow$ (eV) from $E = mc^2$

$$m_e = 0.5 \times 10^6\,\mathrm{eV/c^2} \quad E_n = \frac{-13.6\,\mathrm{eV}}{n^2}$$
$$m_p = 1\,\mathrm{GeV/c^2} \quad 1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$$

- Momentum: $\frac{eV}{C} \rightarrow p = \frac{E}{c}$
- Energy: eV

Matter Fermions are divided into two groups:

- Leptons (electrons, muon, tau, neutrinos): Doesn't have the strong force
- Quarks (up, down, charm, strange, top, bottom): Feels the strong force

e.g. the proton is made of 2 up quarks and 1 down quark (uud) and the Neurtron is (udd).

Quarks make up composite subparticles (Hadrons) are held together by the strong force.

- Mesons: 1 quark + 1 anti-quark ($q\bar{q}$) e.g. pion, kaon...
- Baryons: 3 quarks (qqq) e.g proton, neutron

Quark charges:

- $Q = +2/3$ (up, charm, top)
- $Q = -1/3$ (down, strange, bottom)

Leptons are fundamental particles

- Charged electrically (-1)
  - electron (0.5 MeV)
  - muon (105 MeV)
  - tau (1.8 GeV)
- Neutral (neutrinos)
  - electron neutrino ν_e
  - mueon neutrino ν_μ
  - tau neutrino ν_τ

Crossing Symmetry

$$A + B \rightarrow C + D \quad \text{Scattering}$$
$$A \rightarrow B + C + D \quad \text{Decay}$$
$$A + \bar{C} \rightarrow \bar{B} + D$$

e.g. Neutron Decay

$$n \rightarrow p + e^- + \bar{\nu}_e$$

Sum rules to think about:

- Baryon Number Conservation

- Lepton Number Conservation
- Electric Charge Conservation

another example:

$$n + e^+ \to p + \bar{\nu}_e$$
$$p + e^- \to n + \nu_e$$

Particle Conservation Laws

2 Relativistic Kinematics

Quiz 2 Review

1. The Baryon, Lepton, and Electric Charge are conserved in the Standard Model.
2. The Baryon and Lepton number ensure the stability of the proton.
3. In Neutron Decay $n \to p + e^- + \bar{\nu}_e$, the weak force is responsible for the decay.

	Strong	EM	Weak	Gravity
Strength	1	10^{-2}	10^{-7}	10^{-40}
Time scale	10^{-23} sec	10^{-16}	10^{-10}	$>$ yr

The decay rate is proportional to the coupling strength of the force $\Gamma \propto \alpha^2$. For the time scale is τ it is inversely proportional:

$$\tau \propto \frac{1}{\Gamma}$$

4. The strong force is responsible for holding the nucleus together.
5.

Experimental Discoveries To discover and observe particles, there are typically three ways:

1. Scattering (cross section)
2. Decay (decay rate or lifetime)
3. Bound states (binding energy/mass)

Relativistic Kinematics 4-vectors

$$x^\mu = (ct, x, y, z) \quad \text{space-time}$$
$$p^\mu = (E/c, p_x, p_y, p_z) \quad \text{momentum}$$

where x^μ and p^μ are the space-time (position) four-vector and energy-momentum four-vector.

NOTE: Totak four-momentum is conserved in all interactions.
Starting with the lorentz invariant

$$p^\mu p_\mu = p^2$$

using the Einstein-summation convention

$$p^\mu p_\mu = \sum_{\mu=0}^{3} p^\mu p_\mu = p^2$$

and the metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we can write the lower momentum vector as

$$p_\mu = p^\nu g_{\mu\nu}$$

thus

$$\begin{aligned} p^\mu p_\mu &= p^\mu p^\nu g_{\mu\nu} \\ &= \left(\frac{E}{c}\right)^2 + \mathbf{p}\cdot\mathbf{p}(-1) \\ &= \left(\frac{E}{c}\right)^2 - |\mathbf{p}|^2 \\ &= m^2c^2 \end{aligned}$$

Using

$$E = \sqrt{|\mathbf{p}|^2 + m^2c^4} \tag{2.1}$$

Lorentz Transformation At rest $\mathbf{p} = 0$ and $E = mc^2$.
In the Galilean transformation in the x direction:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

where we assume absolute time, but in the Lorentz transformation:

$$\begin{aligned} x' &= \gamma(\beta ct + x) \quad \beta = \frac{v}{c} \\ ct' &= \gamma(t - \beta x) \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \\ y' &= y \\ z' &= z \end{aligned}$$

In matrix form:

$$\Lambda = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

and thus $p^\mu p_\mu$ is invariant under Lorentz transformation.

Massless particle: From the energy momentum relation

$$E^2 = |\mathbf{p}|^2 c^2 + m^2c^4$$

The massless particle has energy $E = |\mathbf{p}|c$. But we have to include the frequency (Planck) relation from quantum mechanics as well:

$$E = h\nu = \hbar\omega$$

And in the SM photons and neutrinos are massless thus

$$p^2 = p^\mu p_\mu = m^2c^2 = 0$$

Collisions Non-relativistic vs. Relativistic
Non-relativistic:

- Elastic (KE conserved)
- Inelastic (KE not conserved)

Relativistic:

- Elastic (KE conserved) e.g. particle splitting into two
- Inelastic (KE not conserved) or Rest energy and mass e.g. colliding two particles to form a new particle
  - KE increases (Explosive)
  - KE decreases (Sticky)

In the extreme case:

$$\begin{aligned} A + B &\to C \quad \text{inverse decay} \\ A &\to B + C \quad \text{decay} \end{aligned}$$

Example $\pi^+ \to \mu^+ + \nu_\mu$ (decay)
The Rest energies are $m_{\pi^+} = 135\,\mathrm{MeV/c^2}$, $m_{\mu^+} = 105\,\mathrm{MeV/c^2}$, and $m_{\nu_\mu} = 0$. But this energy is lost through the kinetic energy of the muon and muon-neutrino.
The momentum before is just the momentum of the pion

$$p_i = p_\pi = 0$$

since it is startionary. Afterward the momentum is split between the muon and neutrino

$$p_f = p_\mu + p_{\nu_\mu}$$

where energy and momentum is conserved:

$$\begin{aligned} \mathbf{p}_\mu &= -\mathbf{p}_\nu \\ m_\pi c^2 &= E_\mu + E_{\nu_\mu} \end{aligned}$$

4-momentum conservation

$$\begin{aligned} p_{before} &= p_{after} \\ p_\pi &= p_\mu + p_{\nu_\mu} \end{aligned}$$

since the massless particle has no momentum from the energy momentum relation

$$\begin{aligned} p_\nu &= p_\pi - p_\mu \\ p_\nu^2 &= (p_\pi - p_\mu)^2 \\ &= p_\pi^2 - 2p_\pi p_\mu + p_\mu^2 \\ 0 &= m_\pi^2 c^2 + m_\mu^2 c^2 - 2\frac{m_\pi c^2}{c}\frac{E_\mu}{c} \\ 2E_\mu m_\pi &= (m_\pi^2 + m_\mu^2)c^2 \\ E_\mu &= \frac{m_\pi^2 + m_\mu^2 - m_\nu^2}{2m_\pi}c^2 \end{aligned}$$

Another way is finding

$$p_\pi = p_\mu + p_\nu$$

rewritten as

$$p_\mu = p_\pi - p_\nu$$

squaring both sides gives

$$p_\mu^2 = p_\pi^2 - 2p_\pi p_\nu + p_\nu^2$$

and since $p_\nu^2 = 0$ we have

$$p_\mu^2 = p_\pi^2 - 2p_\pi p_\nu$$

which implies

$$m_\mu^2 c^2 = m_\pi^2 c^2 - 2m_\pi E_\nu$$

the Planck relation tells us

$$E_\nu = |\mathbf{p}_\nu| c = |\mathbf{p}_\mu| c$$

thus

$$2m_\pi |\mathbf{p}_\mu| c = (m_\pi^2 - m_\mu^2)c^2$$

and

$$|\mathbf{p}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c$$

Scattering experiments

- Head-on collision: (LHC)
- Fixed target collision: Beam of protons hitting a target (e.g. Carbon) (SLAC)

From momentum conservation, the head-on collision is more energy efficient as it loses the minimum amount of energy. The created particle is at rest, thus the energy is the rest energy. But the Fixed target collision has a higher energy loss since the particle loses energy since the created particle has kinetic energy.
e.g. The Anti-proton Discovery is due to the Bevatron colliding two protons to create an anti-proton

$$p + p \to p + p + p + \bar{p}$$

HW HINT: $E_{cm} < E_{fixed}$

3 Symmetries

Quiz review:

3. The Energy of the large mass is

$$Mc^2 = E_1 + E_2 = 2\gamma mc^2$$

where the energy of the smaller masses are

$$E = \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}$$

where $|\mathbf{p}| = \gamma mv$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Thus the mass $M > 2m$.

4. Using the same thought from 3. we know that the rest mass of M is greater.

Lorentz Invariant

$$p^2 = m^2 c^2$$

From Wikipedia: this is the lightlike vector. For the timelike $p^2 > 0$ and spacelike $p^2 < 0$.

Symmetries

Equilateral triangles are symmetric under 3 axes where we can flip the triangle and it is still the same. For the square, we have 4 axes, and so and so forth. All of these objects are studied in Group Theory.

Group Theory Group is a set of objects satisfying certain properies under an operation.

Properties

1. Closure: For $a, b \in G, \quad a \cdot b \in G$
2. Identity: For any $a \in G, \quad a \cdot I = I \cdot a = a$
3. Inverse: For each $a \in G, \quad a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. Associativity: For $a, b, c \in G, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (optional) Commutativity: For $a, b \in G, \quad a \cdot b = b \cdot a$ AKA Abelian Group. Not all groups are commutatiive and thus are called non-Abelian groups.

Two Types of Groups

1. Finite: Finite number of elements. e.g. $Z_2 = \{1, -1\} = \{I, r\}$ where $r^2 = I$
2. Infinite: Descrete or continuous. e.g. set of integers under addition (discrete), set of real numbers under multiplication (continuous), $U(1)$ (continuous)

Examples For an isoscale triangle $Z_2 = \{1, -1\}$ and for an equilateral triangle $Z_3 = \{0, 1, 2\}$ or the operation mod 3. Which is isomorphic to

$$\equiv \left\{1, \omega, \omega^2\right\}, \quad \omega = e^{2\pi i/3}$$

For the square

$$Z_4 = \{0, 1, 2, 3, \} \equiv \{1, i, -1, -i\} \quad \text{or} \quad \left\{1, \omega, \omega^2, \omega^3\right\}$$

Thus for n elements.

$$Z_n = \left\{e^{i2\pi j/n}\right\}, \quad j = 0, 1, \ldots, n-1$$

where all of these groups are Abelian.

For $n \to \infty$ We get a circle as it has an infinite number of symmetries.
In addition $j \to \infty$

$$\frac{2\pi j}{n} = \theta$$

we get

$$U = e^{i\theta} = \cos\theta + i\sin\theta$$

where $\theta \in [0, 2\pi]$, and we have the $U(1)$ group.

$$U^\dagger U = I \qquad U^\dagger = (U^*)^T$$

where the dagger is the transpose of the complex conjugate (conjugate transpose).

Standard Model

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_y$$

$U(N)$ set of unitary $N \times N$ matrices (non-Abelian in general except for $N > 1$). Taking the determinant of the matrix

$$\det\left(U^\dagger U\right) = \det I = 1$$

and

$$\det\left(U^\dagger\right)\det(U) = 1 \qquad \det\left(U^{*T}\right) = \det(U^*) = (\det U)$$

and

$$|\det U|^2 = 1$$
$$\det U = e^{i\alpha} \quad \alpha \in [0, 2\pi]$$

Choosing the phase angle $\alpha = 0$ we get

$$\det U = 1 \qquad SU(N)CU(N)$$

$\otimes$ is a direct product: Two groups F and G. For $f \in F$ and $g \in G$ we have

$$(f, g) \in F \otimes G$$

The $U(1)$ group is related to the photon γ, the $SU(2)$ group is related to the weak force $W^\pm, Z^0$, and the $SU(3)$ group is related to the strong force g (gluon).

SU(2) A set of 2×2 matrices with a determinant of 1.
Given the theorem

$$U = e^{iH}$$

for the hermitian matrix H where

$$U^\dagger U = 1 \to e^{-iH^\dagger} e^{iH} = 1$$

thus

$$H^\dagger = H$$

we take the determinant of U:

$$\det U = \det\left(e^{iH}\right) = e^{i\,\mathrm{Tr}\,H} = 1 = e^0$$

thus $\mathrm{Tr}\,H = 0$. This means that the Hermitian H is traceless.

Pauli Matrices

traceless matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus we can write the Hermitian matrix as

$$H = \frac{1}{2}\sum_i \theta_i \sigma_i = \frac{1}{2}\theta \cdot \sigma$$

where we have the group element of $SU(2)$

$$U = e^{i\theta\cdot\sigma/2}$$

From QM

$$\mathbf{S} = \frac{\hbar}{2}\sigma$$

$$\begin{aligned} [S_y, S_z] &= iS_x \\ [S_z, S_x] &= iS_y \\ [S_x, S_y] &= iS_z[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \end{aligned}$$

where ϵ_{ijk} is the Levi-Civita symbol.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ interchange any two indices } (3,2,1) \\ 0 & \text{otherwise any index is repeated} \end{cases}$$

thus

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$

The Lie Algebra for $SU(2)$ is $SO(3)$ where both groups are isomorphic.

$$[L_i, L_j] = i\epsilon_{ijk}L_k \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

the generators of $SU(2)$ is $\sigma/2$. For $SU(3)$

$$U = e^{i\theta\cdot\lambda/2}$$

where we have the Gell-Mann matrices λ. In general for $SU(N)$

Addition of Angular Momenta

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and

$$[J^2, J_i] = 0 \qquad J^2 = J_x^2 + J_y^2 + J_z^2 T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_o} \frac{1}{\sqrt{\cos\theta - \cos\theta_o}} \, \mathrm{d}\theta$$

where J^2 is the Casimir operator. Since we have simultaneous eigenstates of J^2 and J_z we can write

$$|j, j_z\rangle$$

Symmetries Part 2: Spin & Isospin

Quiz 3 Review SU(2) is the group of 2x2 unitary matrices with determinant 1. Using the basisc vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ we can write the group element as

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

or the linear combination of the basis vectors. Thus the transformation is

$$\begin{pmatrix}a'\\b'\end{pmatrix} = U(\theta)\begin{pmatrix}a\\b\end{pmatrix} = e^{i\theta\cdot\sigma/2}\begin{pmatrix}a\\b\end{pmatrix}$$

THe Lie Algebra for SU(2) is

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

and

$$[J^2, J_i] = 0$$

for simultaneous eigenstates of $|j,m\rangle$.

$$J_z\,|j,m\rangle = m\hbar\,|j,m\rangle \qquad J^2\,|j,m\rangle = j(j+1)\hbar^2\,|j,m\rangle$$

from the ladder operators

$$J_\pm = J_x \pm iJ_y$$

where these are not Hermitian (does not commute). Thus

$$\begin{aligned} J^2 &= J_x^2 + J_y^2 + J_z^2 \\ &= J_+J_- + J_+J_-J_z^2 \end{aligned}$$

furthermore

$$J_\pm\,|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m)}\,|j,m\pm 1\rangle$$

where going up the ladder $m \to m+1$ and going down the ladder $m \to m-1$. For fixed j there is a maximum and minimum m value

$$m_{max} = j \qquad m_{min} = -j$$

so for example

$$J_+\,|j,j\rangle = 0 \qquad J_-\,|j,-j\rangle = 0$$

Spin

$$j \equiv s = 1/2, \qquad m \equiv m_s = \pm 1/2$$

The basis states are

$$\begin{aligned} (1/2, 1/2) &= \begin{pmatrix}1\\0\end{pmatrix} = |\uparrow\rangle \qquad m_s = 1/2 \\ (1/2, -1/2) &= \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle\, m_s = -1/2 \end{aligned}$$

For the addition of spin

$$\frac{1}{2}\otimes\frac{1}{2} = ? \qquad \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \qquad S_{tot} = (S_1+S_2), ..., (S_1 - S_2) = 1, 0 \qquad m_{s,tot} = 1, 0, -1, 0$$

General Addition of Angular Momentum

$$|1,1\rangle = |\uparrow\uparrow\rangle \qquad |1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle \qquad |1,-1\rangle = |\downarrow\downarrow\rangle$$

finding the linear combination through basis transformation by using the resolution of the identity

$$\begin{aligned}|j,m\rangle &\to |j_1,m_1\rangle \otimes |j_2,m_2\rangle \\ &= \sum_{m_1,m_2} |j_1,m_1,j_2,m_2\rangle \langle j_1,m_1,j_2,m_2|j,m\rangle\end{aligned}$$

where the bra-ket is the Clebsch-Gordan coefficient. thus

$$= \sum_{m_1,m_2} c^{j,j_1,j_2}_{m,m1,m2} |j_1,m_1,j_2,m_2\rangle$$

where $m = m_1 + m_2$ and $c^{j,j_1,j_2}_{m,m1,m2}$ is the Clebsch-Gordan coefficient.

Example For the $S = 1$ state $m = 1$

$$\begin{aligned}|1,1,\rangle &= |1/2,1/2\rangle \otimes |1/2,1/2\rangle \\ &= |1/2,1/2,1/2,1/2\rangle \\ &= |\uparrow\uparrow\rangle\end{aligned}$$

For $m = 0$ we have a linear combination of the basis states

$$\begin{aligned}J_- |1,1\rangle &= \hbar\sqrt{2}\,|1,0\rangle \\ \text{or} \quad |1,0\rangle &= \frac{1}{\hbar\sqrt{2}} J_- |1,1\rangle\end{aligned}$$

the sum of the basis states is

$$\begin{aligned}J_-(|1/2,1/2\rangle \otimes |1/2,1/2\rangle) &= \hbar\sqrt{(1/2+1/2)(1/2-1/2+1)}\,|1/2,-1/2\rangle \otimes |1/2,1/2\rangle \\ &+ |1/2,1/2\rangle \otimes |1/2,-1/2\rangle \\ &= \hbar(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\end{aligned}$$

or

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

for $m = -1$ we have

$$J_- |1,0\rangle = \hbar\sqrt{2}\,|1,-1\rangle$$

where

$$|1,-1\rangle = |1/2,-1/2\rangle \otimes |1/2,-1/2\rangle \qquad = |\downarrow\downarrow\rangle$$

Now for $S = 0$, $m = 0$ we have

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

since it is the way to make it orthogonal to $|1,0\rangle$. Therefore

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

Thus the there are 3 triplet states $m_s = 1, 0, -1$ and 1 singlet state $m_s = 0$.

Isospin

$$m_p = 938.3 \text{ MeV/c}^2 \qquad m_n = 939.6 \text{ MeV/c}^2$$

why are they so close? Heisenberg postulated an isospin state of a nucleon N as

$$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha |p\rangle + \beta |n\rangle$$

with

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the isospin state of the proton and neutron are

$$|p\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \qquad |n\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

1. Strong interactions preserve isospin symmetry
2. EM & Weak interactions do not preserve isospin symmetry

Examples

Pions: π^+, π^0, π^- where the approximate symmetry is a triplet state

$$\begin{aligned} \pi^+ &= |1,1\rangle \qquad I = 1, \quad I_3 = 1 \\ \pi^- &= |1,0\rangle \qquad I = 1, \quad I_3 = 0 \\ \pi^0 &= |1,-1\rangle \qquad I = 1, \quad I_3 = -1 \end{aligned}$$

Δ-baryons:

$$\begin{aligned} \Delta^{++} &= |3/2, 3/2\rangle \qquad I = 3/2, \quad I_3 = 3/2 \\ \Delta^{+} &= |3/2, 1/2\rangle \qquad I = 3/2, \quad I_3 = 1/2 \\ \Delta^{0} &= |3/2, -1/2\rangle \qquad I = 3/2, \quad I_3 = -1/2 \\ \Delta^{-} &= |3/2, -3/2\rangle \qquad I = 3/2, \quad I_3 = -3/2 \end{aligned}$$

where Δ^{--} is an antiparticle of Δ^{++}. We write from the highest to lowest from the empirical Gellman-Nishijima formula

$$Q = I_3 + \frac{1}{2}(B + S)$$

where Q is the charge, I_3 is the third component of isospin, B is the baryon number, and S is the strangeness.

Pions

Since a Pion is a *meson* and not a baryon, it has a baryon number of 0. Thus with no strangeness

$$S = 0 \qquad B = 0$$

Nucleons

$$S = 0 \qquad B = 1$$

$$Q = \begin{cases} 1/2 + 1/2(1+0) = 1 & \text{proton} \\ -1/2 + 1/2(1+0) = 0 & \text{neutron} \end{cases}$$

For all elementary particles there is a general formula

$$Q = I_3 + \frac{Y}{2}$$

where Y is the hyper charge $U(1)_Y$.

Power of Symmetry: Applications

1. Deuteron (neutron of deuterium): Two-Nucleon system

$$I = 1 \quad \text{or} \quad 0 \qquad I_3 = 1, 0, -1 \quad \text{or} \quad 0 \text{ (singlet)}$$

$$\begin{aligned} |1,1\rangle &= |p,p\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle + |n,p\rangle) \\ |1,-1\rangle &= |n,n\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle - |n,p\rangle) \end{aligned}$$

experimentally, we only see the singlet state because we see only one deuteron state. Thus we can only see a isospin state of $I = 0$.

Two-nucleon potential $\propto \mathbf{I}_1 \cdot \mathbf{I}_2$ where we hae the total isospin

$$\mathbf{I}^2 = (\mathbf{I}_1 + \mathbf{I}_2)^2 = \mathbf{I}_1^2 + \mathbf{I}_2^2 + 2\mathbf{I}_1 \cdot \mathbf{I}_2$$

where the s^2 term is

$$s^2 = 1/2(1/2+1)\hbar^2 = \frac{3}{4}\hbar^2$$

Thus

$$\mathbf{I}_1^2 + \mathbf{I}_2^2 = \frac{3}{2}$$

and

$$\begin{aligned} \mathbf{I}_1 \cdot \mathbf{I}_2 &= \frac{1}{2}\left(\mathbf{I}^2 - 3/2\right)^{3/2} \\ &= \begin{cases} 1/2(1(1+1) - 3/2) & = 1/4 \quad \text{triplet} \\ 1/2(0(0+1) - 3/2) & = -3/4 \quad \text{singlet} \end{cases} \end{aligned}$$

Symmetries Part 3: Scattering

Quiz 5 Review For j

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

For $2j + 1$

$$2 \otimes 2 = 3 \oplus 1$$

Isospins of particles

1. pion: 1
2. deuteron: 0
3. Δ-baryons: 3/2
4. nucleons: 1/2

The strong ineteraction preserves I and I_3, and the weak interactions do not preserve I and I_3 (e.g. in beta decay the iso spin of the neutron (-1/2) go to an iso spin of the proton (1/2)). In E&M the isospin preserves only I and not I_3 (e.g. π_o decay to two photons $\gamma\gamma$: $I_3 = 0$ for the π_o and $I_3 = 0$ for the two photons).

Applications of Isospin: Nucleon-nucleon Scattering

$$p + p \to D + \pi^+$$
$$p + n \to D + \pi^0$$
$$n + n \to D + \pi^-$$

The relative probabilities of these processes: we get this from the amplitude A where the probability $|A|^2$ is proportional to the cross section $\sigma = \pi r^2$ (the cross section of a sphere, but this is not a solid sphere and rather a 'fuzzy' sphere). With the fact that 'strong interactions preserve isospin' we have the the ratio of the cross sections

$$\sigma_a : \sigma_b : \sigma_c$$

For all three processes the RHS the isospin is

$$I_{tot} = 0 \otimes 1 = 1$$

on the left hand side

$$I_{tot} = \frac{1}{2} \otimes \frac{1}{2} = 0 \quad \text{or} \quad 1$$

(a) The ratio of getting an isospin of 1 on the left hand side for the first process

$$|pp\rangle = |11\rangle$$

(c) for the third proccess

$$|nn\rangle = |1, -1\rangle$$

(b) The second is the linear combination of $|10\rangle$ and $|00\rangle$

$$|pn\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |00\rangle)$$

the $|00\rangle$ does not contribute to the isospin of 1. Thus the ratio of the probability is

$$A_a : A_b : A_c = 1 : \frac{1}{\sqrt{2}} : 1$$

and the ratio of the cross sections is

$$\sigma_a : \sigma_b : \sigma_c = 1 : \frac{1}{2} : 1$$

Example 3 Pion-nucleon Scattering

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \quad I = 1, \qquad \begin{pmatrix} p \\ n \end{pmatrix} \quad I = 1/2$$

So the total isospin j is

$$I_{tot} = 1/2 \otimes 1 = 3/2 \oplus 1/2$$

and for $2j + 1$

$$3 \otimes 2 = 4 \oplus 2$$

The elastic processes are (from $a \to f$)

$$\begin{aligned} \pi^+ + p &\to \pi^+ + p \\ \pi^0 + p &\to \pi^0 + p \\ \pi^- + p &\to \pi^- + p \\ \pi^+ + n &\to \pi^+ + n \\ \pi^0 + n &\to \pi^0 + n \\ \pi^- + n &\to \pi^- + n \end{aligned}$$

and the charge-exchange processes are (from $g \to j$)

$$\begin{aligned} \pi^+ + n &\to \pi^0 + p \\ \pi^0 + p &\to \pi^+ + n \\ \pi^- + p &\to \pi^0 + n \\ \pi^0 + n &\to \pi^- + p \end{aligned}$$

The states of the 3/2 isospin are

$$|3/2, 3/2\rangle, |3/2, 1/2\rangle, |3/2, -1/2\rangle, |3/2, -3/2\rangle$$

and the states of the 1/2 isospin are

$$|1/2, 1/2\rangle, |1/2, -1/2\rangle$$

so we have the following states

$$\begin{aligned} \left|\pi^+ p\right\rangle &= |1, 1\rangle \otimes |1/2, 1/2\rangle = |3/2, 3/2\rangle \\ \left|\pi^- n\right\rangle &= |1, -1\rangle \otimes |1/2, -1/2\rangle = |3/2, -3/2\rangle \end{aligned}$$

for the obvious highest and lowest isospin states. Carrying on...

$$\left|\pi^+ n\right\rangle = |1, 1\rangle \otimes |1/2, -1/2\rangle$$

this is the linear combination of $|3/2, 1/2\rangle$ and $|1/2, 1/2\rangle$, and so on. To find the proportional cross sections we know that

$$\langle i|f\rangle \propto A \qquad |\langle i|f\rangle|^2 \propto \sigma$$

We can use the Clebsch-Gordan coefficients to find the linear combination of the states. For example

$$\left|\pi^+ n\right\rangle = |3/2, 1/2\rangle + |1/2, 1/2\rangle$$

where the Clebsch-Gordan coefficient is

$$\langle 3/2, 1/2, 1/2, 1/2 | 3/2, 1/2 \rangle = \sqrt{\frac{2}{3}}$$

e.g. for the $\pi^+ p$ state

$$\left|\pi^+ p\right\rangle = |3/2, 3/2\rangle$$
$$\langle 3/2, 3/2, 1/2, 1/2 | 3/2, 3/2 \rangle = 1$$

using the lowering operator

$$J_- |j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|j, m-1\rangle$$

so

$$J_- |3/2, 3/2\rangle = \hbar\sqrt{3}\,|3/2, 1/2\rangle$$

applying the lower operator to $J_{1-} + J_{2-}$ we get

$$\begin{aligned} J_-(|11\rangle \otimes |1/2, 1/2\rangle) &= \hbar\sqrt{2}\,|10\rangle \otimes |1/2, 1/2\rangle + \hbar\sqrt{1}\,|11\rangle \otimes |1/2, -1/2\rangle \\ &= \hbar\sqrt{2}\,|10\rangle \otimes |1/2, 1/2\rangle + \hbar\,|11\rangle \otimes |1/2, -1/2\rangle \end{aligned}$$

we then get

$$\begin{aligned} |3/2, 1/2\rangle &= \sqrt{2/3}\,|11\rangle \otimes |1/2, 1/2\rangle + \sqrt{1/3}\,|10\rangle \otimes |1/2, 1/2\rangle \\ &= \sqrt{2/3}\,\left|\pi^+ p\right\rangle + \sqrt{1/3}\,\left|\pi^+ n\right\rangle \end{aligned}$$

and the orthogonal state is

$$|1/2, 1/2\rangle = \sqrt{2/3}\,\left|\pi^+ p\right\rangle - \sqrt{1/3}\,\left|\pi^+ n\right\rangle$$

and so on for the other states. At the end we will find that the ratio of the total cross sections (adding up the matching elastic and exchange processes) is 3.
The amplitude has a factor

$$\left\langle \pi^+ p \middle| \pi^+ p \right\rangle = \langle 3/2, 3/2 | 3/2, 3/2 \rangle = M_3$$

where for example

$$\begin{aligned} (\sqrt{2}3\,\langle 3/2, 1/2| - 1/\sqrt{3}\,\langle 1/2, 1/2|)(\sqrt{2}3\,|3/2, 1/2\rangle - 1/\sqrt{3}\,|1/2, 1/2\rangle) &= \\ &= 2/3\,\langle 3/2, 1/2 | 3/2, 1/2 \rangle - 1/3\,\langle 1/2, 1/2 | 1/2, 1/2 \rangle \\ &= 2/3 M_3 - 1/3 M_1 \end{aligned}$$

for $M_3 >> M_1$ the ratio is 4/9, and for $M_3 << M_1$ the ratio is 1/3.

$SU(3)$

$$\begin{pmatrix} p \\ n \end{pmatrix} \quad SU(2)\text{doublet}$$

where the spins are

$$p : uud \quad Q_u = 2/3$$
$$n : udd \quad Q_d = -1/3$$

For the two spins

$$\begin{pmatrix} u \\ d \end{pmatrix}$$

the isospins are

$$I = 1/2, \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

for the up and down quarks respectively. In reality we have six quarks

- Light quarks: u, d, s
- Heavy quarks: c, b, t

For the light quaks we have a $SU(3)$ symmetry

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

the masses are all different:

$$m_u \approx 2\,\mathrm{MeV/c^2} \qquad m_d \approx 4\,\mathrm{MeV/c^2} \qquad m_s \approx 95\,\mathrm{MeV/c^2}$$

so we have to add a flavor symmetry to the $SU(2)$ isospin symmetry:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} \oplus s$$

or the $SU(3)$ symmetry

$$SU(3)_f \to SU(2)_I \otimes U(1)_y$$

From $SU(2)$ algebra:

$$[J_i, J_j] = i\epsilon_{ijk} I_k \qquad J_i = \sigma_i/2$$

For the three pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now for $SU(3)$: We know that the generators

$$U = e^{i\theta \cdot \lambda/2}$$

For $SU(N)$ we have $N^2 - 1$ generators. For $SU(3)$ we have 8 generators. The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Symmetries Part 4: Young's Tableaux & Eightfold Way

Quiz 6 Review

(1) With respect to the QCD scale ($\approx$ 200 MeV) the masses of the quarks are divided into light and heavy quarks.

(2) (2) The effective mass mass is much larger than the mass of the light quark.

(3) $SU(2)$ is a subgroup of $SU(3)$.

For the $SU(3)$: a 3x3 unitary matrix with determinant 1. There are $n^2 - 1 = 8$ generators where

$$U = e^{iH}$$

where H is the Hermitian matrix:

$$U^\dagger U = 1 \qquad H^\dagger = H$$
$$\det U = 1 \qquad \operatorname{tr} H = 0$$
$$\det M = e^{\operatorname{tr} \ln M}$$

or Hermitian matrices are traceless.

Gell-Mann Matrices Starting with the Pauli matrices but in 3x3 form

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and moving the sectors of the matrices we also get

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \cancel{\lambda_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

and

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

but λ_9 is not linearly independent since it can be written as a linear combination of $\lambda_3 + \lambda_8$.

Commutation Relation For $SU(2)$ we know

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and for $SU(3)$ we have

$$[J_i, J_j] = i f_{ijk} J_k$$

where f_{ijk} are the structure constants.

Subgroup We know that $SU(2) \leq SU(3)$ (where $\leq$ means 'is a subgroup of'). So $\{\lambda_1, \lambda_2, \lambda_3\}$ forms an $SU(2)$ sub-algebra. Also

$$\{\lambda_4, \lambda_5, a\lambda_3 + b\lambda_8\}$$
$$\{\lambda_6, \lambda_7, a\lambda_3 + b\lambda_8\}$$

are also $SU(2)$ sub-algebras. NOTE that

$$SU(3) \neq SU(2) \otimes SU(2) \otimes SU(2)$$

Isospin and Strangeness

$$\lambda_3/2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I = 1/2$$

and the isospins are

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

and the strangeness is

$$S : I = 0$$

For λ_8 we define a hypercharge y such that

$$\lambda_8/2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}} \frac{2}{\sqrt{3}}$$

and

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad y = 1/3$$

and for the strangeness $S : y = -2/3$. This is because the Gell-Mann-Nishijima formula

$$Q = I_3 + \frac{y}{2}$$

$$2/3 = 1/2 + 1/2(1/3) \qquad -1/3 = 0 - 1/2(2/3)$$

For the triplet

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

and the anti-triplet

$$\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

Mesons $(q, \bar{q})$ in $SU(3)$ is

$$3 \otimes \bar{3} = 8 \oplus 1$$

where the 8 is the octet and the 1 is the singlet. We can do this using the Young Tableaux. For $SU(3)$ we have a 3 fundamental and $\bar{3}$ anti-fundamental.

$$\begin{array}{|c|c|} \hline N & N+1 \\ \hline N-1 & N \\ \hline N-2 & \\ \cline{1-1} \end{array}$$

Using Hook Law:

$$\dim = \frac{\Pi_i N_i}{\Pi_i h_i}$$

$$\frac{N(N+1)(N-1)N(N-2)}{1\cdot 3\cdot 4\cdot 1\cdot 2}$$

so

$$3\otimes\bar{3} = \boxed{3} \otimes \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} =$$

or

$$\boxed{a} \otimes \begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} = \frac{3\cdot 4\cdot 2}{1\cdot 3\cdot 1} \oplus \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}$$

Goin from $SU(3)$ to $SU(2)$ we have

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} + s$$

or

$$3 \to 2_1 + 1_{-2}$$

where the hypercharges are subscripts. So the octet is

$$\begin{aligned} 3\otimes\bar{3} &= (2_1 + 1_{-2}) \otimes (2_{-1} + 1_2) \\ &= (2_1\otimes 2_{-1}) \oplus (2_1 \otimes 1_2) \oplus (1_{-2}\otimes 2_{-1}) \oplus (1_{-2}\otimes 1_2) \\ &= (3_0 \oplus 1_0) \oplus 2_3 \oplus 2_{-3} \oplus 1_0 \\ &= 8 \oplus 1 \end{aligned}$$

This is called the eightfold way.

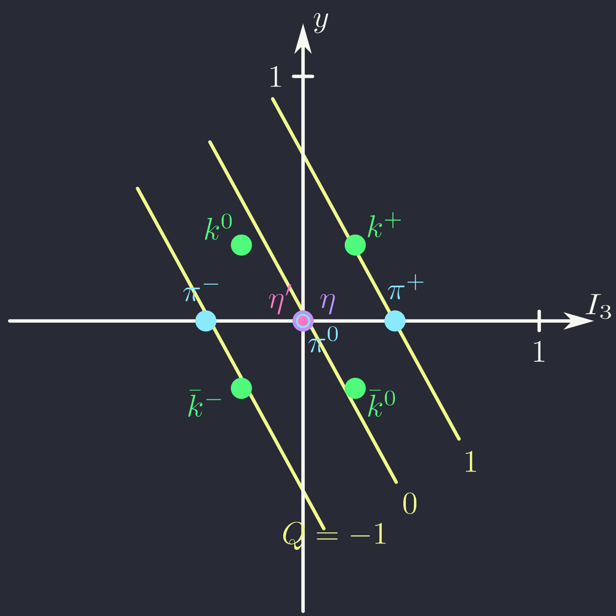


Figure 3.1: The eightfold way

Eightfold way Where η is a $SU(2)$ singlet but a $SU(3)$ octet. and η' is a $SU(3)$ singlet.
If $SU(3)_f$ was a good symmetry, expet all these 8 mesons to have similar mass. All of these obey up to a factor of 2.

Baryons (222) or $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$. The baryons are antisymmetric as each quark is a fermion. Using the Young Tablauex

$$3 \otimes 3 = \boxed{a} \otimes \boxed{b} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array} = \frac{3 \cdot 4}{1 \cdot 3} \oplus \frac{3 \cdot 2}{1 \cdot 2} = 6$$

so

$$6 \otimes 3 = \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \otimes \boxed{c} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline \end{array} \oplus \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c|}{4} \\ \hline 2 & \\ \cline{1-1} \end{array} = \frac{3 \cdot 4 \cdot 5}{3 \cdot 2 \cdot 1} \oplus \frac{3 \cdot 4 \cdot 2}{1 \cdot 3 \cdot 1} = 10 \oplus 8$$

This is a 10-plet as shown in the figure.

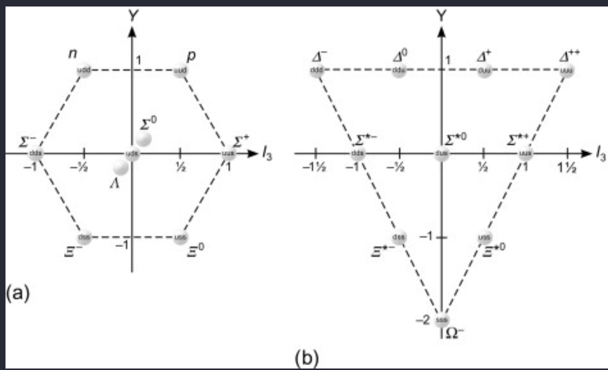


Figure 3.2: The 10-plet of baryons

Symmetries Part 5: Parity

Parity (Discrete Symmetry) For a simple reflection on a z axis the point $A = (x, y, z)$ goes to $A' = (x, -y, z)$. But for parity operation we go to $P(A) = (-x, -y, -z)$ or in the general form

$$P(\mathbf{a}) = -\mathbf{a}$$

also known as inversion. Taking the parity again we get

$$P^2(\mathbf{a}) = P(-\mathbf{a}) = \mathbf{a}$$

Thus it has a discrete Z_2 symmetry.

Pseudo-vector (axial vector) for a pseudo vector $\mathbf{c}$

$$P(\mathbf{c}) = \mathbf{c}$$

where cross products (of two vectors) are pseudo-vectors. For example

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

and

$$P(\mathbf{c}) = P(\mathbf{a} \times \mathbf{b}) = (-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{c}$$

e.g. of pseudo-vectors:

- Torque: $\tau = \mathbf{r} \times \mathbf{F}$
- Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- Magnetic field: $\mathbf{B} = \mathbf{E} \times \mathbf{v}$

But for the lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

the cross product of a vector and a pseudo-vector is a vector, so the lorentz force is a vector. Also from the general definition

$$P(\mathbf{F}) = \frac{q}{c}P(\mathbf{v}) \times P(\mathbf{B}) = \frac{q}{c}(-\mathbf{v}) \times (-\mathbf{B}) = -\mathbf{F}$$

The weak interaction violates parity...

Scalar For a scalar $s = \mathbf{a} \cdot \mathbf{b}$ is invariant under parity:

$$P(s) = P(\mathbf{a} \cdot \mathbf{b}) = (-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = s$$

for a pseudo-scalar p (a dot product of a vector and pseudo-vector):

$$P(p) = P(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})) = P(\mathbf{a})P(\mathbf{b} \times \mathbf{c}) = -\mathbf{a}(\mathbf{b} \times \mathbf{c}) = -p$$

So the partities of the four types of quantities:

- Scalar: $P(s) = s$
- Pseudo-scalar: $P(p) = -p$
- Vector: $P(\mathbf{v}) = -\mathbf{v}$
- Pseudo-vector: $P(\mathbf{c}) = \mathbf{c}$

Intrinsic Parity The parity of a fermion is

$$P(\text{fermion}) = -P(\text{anti-fermion})$$

for bosons

$$P(\text{boson}) = P(\text{anti-boson})$$

For composite particles i.e. mesons $q\bar{q}$ and baryons qqq:

$$P(\text{meson}) = -1 \quad \text{or} \quad (+1)(-1) = -1$$

Since mesons are two pairs (particle, antiparticle) the parity is always negative. For baryons we can only have a positive parity:

$$P(\text{baryon}) = (+1)^3 = +1$$

For spherical harmonics $Y_l^m(\theta, \phi)$ under parity of each term:

$$\mathbf{r} \to -\mathbf{r} \qquad \theta \to -\theta \qquad \phi \to \pi + \phi$$

so

$$P(Y_l^m(\theta, \phi)) = (-1)^l Y_l^m(\theta, \phi)$$

and for excited states

$$P = (-1)^l \times P(\text{ground state})$$

where l is the orbital angular momentum.

Parity Violation THe $\theta - \tau$ puzzle: given two particles

$$\begin{aligned} \theta^+ &\to \pi^+ + \pi^0 & P = +1 \\ \tau^+ &\to \pi^+ + \pi^0 + \pi^0 & P = -1 \end{aligned}$$

the same two particles are found to be the same particle K^+ having the same mass and lifetime, but this violates the parity. To solve this puzzle came from the Colmbia University known as Wu's experiment (wikipedia):

$${}^{60}_{27}\text{Co} \to^{60}_{28} \text{Ni} + e^- + \bar{\nu}_e$$

The Cobalt has a spin state of $J = 5$, the Nickel has spin $J = 4$ and the spin states of the electron-antielectron pair is $J = 1$. The electron is always emitted in the direction opposite of the Cobalt spin, and when the magnetic field was inverted, the electrons were emitted in the opposite direction of nuclear spin. This breaks parity, because in the mirror world, the spin of the electron would be in the same direction as the nuclear spin.

Helicity From spin $\mathbf{s}$ and momentum $\mathbf{p}$ we can define the helicity

$$\begin{aligned} \lambda &= \frac{\mathbf{s} \cdot \mathbf{p}}{|\mathbf{s}||\mathbf{p}|} \\ &= \begin{cases} +1 & \text{right-handed} \\ -1 & \text{left-handed} \end{cases} \end{aligned}$$

But this depends on the reference frame. e.g. for a case where $\mathbf{p}$ is faster and in the same direction of $\mathbf{s}$ the helicity is $\lambda = +1$, but in the reference frame faster that $\mathbf{p}$ the helicity is $\lambda = -1$.

Massless Particles For massless particles the helicity is the same in all reference frames because the speed is always c and thus the helicity is well-defined. For an electron, we can get a frame where the momentum is different for changes in reference frames.

Back to Wu's experiment The spin of the electron and neutrino are in the same direction as the spin of the Cobalt and Ni.

Note: In the SM

- neutrinos are always left-handed; $\lambda = -1$
- anti-neutrinos are always right-handed; $\lambda = +1$

Thus the electron momentum is in the opposite direction of spin as shown in Figure 3.3

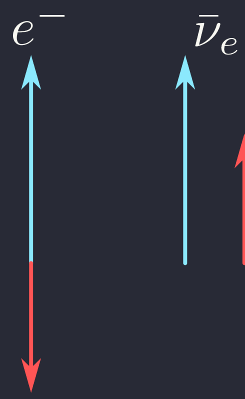


Figure 3.3: Wu's experiment

Another Example Pions and muons

$$\pi^+ \to \mu^+ + \nu_\mu \quad \text{or} \quad \to e^+ + \nu_e$$

From the comparison of masses

$$m_\pi = 140\,\text{MeV} \qquad m_\mu = 105\,\text{MeV} \qquad m_e = 0.511\,\text{MeV}$$

we would think the small mass reaction would be more likely due to the higher velocity, but this is not the case. For the pion the spin is 0, so the combination of spin will be in opposite directions. From Figure 3.4 we can see that the anti-leption(anti particle) must be left-handed and thus the lepton must also be left-handed. This is a parity violation. Thus the less favored reaction $u^+ + \nu_\mu$ is seen 99.7% of

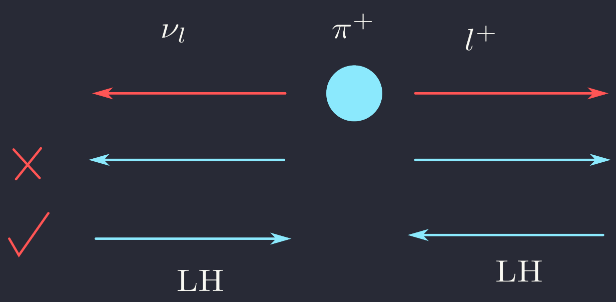


Figure 3.4: Pion decay

the time.

Anti-charged lepton has to be left-handed in this process of appoximate

$$\Gamma \propto m_\ell^\beta$$

where Γ is the decay rate.

Muon decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

This 3 body decay for a polarized muon (choosing the handedness of the muon) we have the following possiblities:

1. LH
2. RH

For maximum energy to the electron, the electron goes in one direction while the neutrinos go in the opposite direction as shown in Figure 3.5. From this the RH case is less favored than the LH case because of the helicity of the neutrinos.

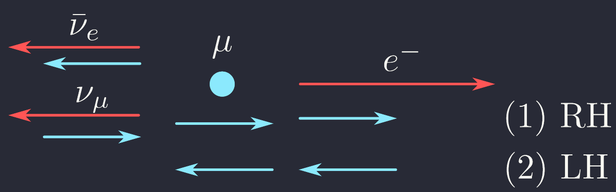


Figure 3.5: Muon decay

Symmetries Part 6: G-Parity and CP Violation

Quiz Review :

1. The neutrino is an eigenstate of partity (neutrino has handedness)

$$P\left|\nu\right\rangle_L = \pm\left|\nu\right\rangle_L x \qquad A\left|\psi\right\rangle = \lambda\left|\psi\right\rangle \qquad P^2 = 1$$

but $\left|\nu\right\rangle_R$ does not exists!

2. Charge Conjugation (C)

$$C\left|m\right\rangle = \left|\bar{m}\right\rangle$$

taking the charge conjugation twice

$$C^2\left|m\right\rangle = C(C\left|m\right\rangle) = C\left|\bar{m}\right\rangle = \left|m\right\rangle$$
$$\implies C^2 = 1 \implies C = \pm 1 \quad Z_z\text{-symmetry}$$

So if $\left|m\right\rangle$ is an eigenstate of C then

$$C\left|m\right\rangle = \pm\left|m\right\rangle = \left|\bar{m}\right\rangle \implies \left|m\right\rangle = \pm\left|\bar{m}\right\rangle$$

e.g. For charged particles:

$$C\left|\pi^+\right\rangle = \left|\pi^-\right\rangle$$

and for some neutral particles the charge conjugation is the same as the particle. One exception is the neutron

$$C\left|n\right\rangle = \left|\bar{n}\right\rangle$$

and for the violation of Charge conjugation: the neutrino

$$C\left|\nu\right\rangle_L = \left|\nu_L\right\rangle \times$$

here C must be violated (in weak interactions) but preserved in strong & EM interactions.

3. G-parity: defined as

$$G = CR$$

where R is a rotation. From the 8-fold way, we can get from π^+ to π^- by reflecting it about the hypercharge axis or rotation by 2π in the I_2 axis and then taking its charge conjugation:

$$G = Ce^{i\pi I_2}$$

so

$$G\left|\pi^+\right\rangle = Ce^{i\pi I_2}\left|\pi^+\right\rangle = C\left|\pi^-\right\rangle = \left|\pi^+\right\rangle$$

finding the G-parity of K^+: from the 8-fold way(Figure 3.1) we know that rotating K^+ actually gives us K^0 so

$$G\left|K^+\right\rangle = Ce^{i\pi I_2}\left|K^+\right\rangle = C\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle$$

Neutral Pion Decay $\pi^0 \to \gamma + \gamma$ is a EM interaction (particle antiparticle pair: $(u\bar{u} - d\bar{d})/\sqrt{2}$).

$$\pi^0 \to \gamma + \gamma$$

The partity on the LHS is $P = -1$ and on the RHS $P = (-1)^2 = +1$. For the photon of spin:

$$S = 1, \qquad S_z = +1, \not{0}, -1$$

there is no longitudinal polarization $S_z = 0$, and the transverse polatization of the EM wave $S_z = \pm 1$ is the helicity. So the helicity of the photons must be the same; either $\lambda = |{++}\rangle$ or $\lambda = |{--}\rangle$. This is not an eigenstate of partity. The two photons must have aligned polarizations. We also need to find p in

$$\lambda = \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{S}||\mathbf{p}|}$$

where we know

$$p\,|+\rangle = |-\rangle \quad p\,|-\rangle = |+\rangle \quad p\,|{++}\rangle = |{--}\rangle$$

we can write a linear combination of the two states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle)$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|{++}\rangle - |{--}\rangle)$$

for

$$P\,|\psi_1\rangle = |\psi_1\rangle \quad \text{(even parity)}$$
$$P\,|\psi_2\rangle = -\,|\psi_2\rangle \quad \text{(odd parity)}$$

This is similar to quantum optics where polarization is used:

$$|+\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle)$$

we know the G parity is

$$G = (-1)^I$$

4. Pion decay to muon and neutrino: For the mesons

$$C = (-1)^{l+s}$$

where we have two types: Pseudoscalars ($s = 0$) e.g π, K with $C = (-1)^l$ and vector mesons ($s = 1$) e.g. ρ, K^* with $C = (-1)^{l+1}$. For ρ we know that $I(\rho) = 1$ from the 8-fold way, as well as $I(\eta) = 0$ so $\rho \to 3\pi$ and $\eta \to 2\pi$ are allowed.

CP

$$P\,|\nu_L\rangle = |\nu_R\rangle \times$$
$$CP\,|\nu_L\rangle = C\,|\nu_L\rangle = |\bar{\nu}_R\rangle$$

where the P is violated due to the handedness of the neutrino, but the CP is conserved. applying CP on to the charged pion decay:

$$\pi^+ \to \mu^+ + \nu_\mu$$

CP Violation

For an oscillation of a neutral kaon $K^0(d\bar{s})$ and $\bar{K}^0(\bar{d}s)$. Because K's are pseudoscalars:

$$P\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad P\left|\bar{K}^0\right\rangle = \left|\bar{K}^0\right\rangle$$

and the charge conjugation is

$$C\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle \qquad C\left|\bar{K}^0\right\rangle = \left|K^0\right\rangle$$

and the CP is

$$CP\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle \qquad CP\left|\bar{K}^0\right\rangle = -\left|K^0\right\rangle$$

which are not eigenstates of CP, so taking a linear combination of the two states

$$\left|K_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) \qquad \left|K_2\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right)$$

the CP of the two states are

$$CP\left|K_1\right\rangle = \left|K_1\right\rangle \qquad CP\left|K_2\right\rangle = -\left|K_2\right\rangle$$

which are eigenstates of CP. Karons decay to pions(lightest meson): to either 2 or 3 pions. From the phase space, the 2 pion decay is more likely than the 3 pion decay because faster particles are more likely to decay: A kaon with 490 MeV to 2×140 = 280 MeV is more likely than 3×140 = 420 MeV. The 2 pion decay has CP

$$CP\left|\pi^+\pi^-\right\rangle = (-1)^2\left|\pi^+\pi^-\right\rangle = \left|\pi^+\pi^-\right\rangle$$
$$CP\left|\pi^+\pi^-\pi^0\right\rangle = (-1)^3\left|\pi^+\pi^-\pi^0\right\rangle = -\left|\pi^+\pi^-\pi^0\right\rangle$$

so the 2 pion decay is CP even and the 3 pion decay is CP odd. If CP were conserved,

$$\left|K_1\right\rangle \to \left|2\pi\right\rangle \qquad \left|K_2\right\rangle \to \left|3\pi\right\rangle$$

so we call this fast decay K_S meaning short ($9\times10^{-11}sec$) and the slow decay K_L meaning long ($5\times10^{-8}sec$) or

$$\left|K_1\right\rangle \equiv \left|K_S^0\right\rangle \qquad \left|K_2\right\rangle \equiv \left|K_L^0\right\rangle$$

from a nobel prize winning experiment, a detector far away would be expected to see mostly K_L but the experiment showed that we saw some amount to the 2 pion decay:

$$\left|K_L^0\right\rangle = \frac{1}{\sqrt{1+\epsilon^2}}\left(\left|K_S^0\right\rangle + \epsilon\left|K_L^0\right\rangle\right)$$

where $\epsilon = 2.3\times10^{-3}$ characterizes CP violation.

Symmetries Part 7: CP Violation

Quiz Reviw:

- matter-antimatter asymmetry: the ϵ parameter measures the CP violation in the neutral kaon (hadronic decay mode). For the semi leptonic decay mode:

  $$K_L^0 \to \pi^+ + e^- + \bar{\nu}_e$$

  The CP symmetry shows

  $$\to \pi^- + e^+ + \nu_e$$

  And the rates for the decay modes differ by ϵ. This relates to the matter-antimatter symmetry. (barygenesis) $\eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} = 6 \times 10^{-10}$ where n is the number of baryons/antibaryons/photons. In the SM $\eta_{SM} \approx 10^{-20}$, so the SM does not explain the matter-antimatter asymmetry fully. Finding the change in the rate of the process:

  $$\frac{\mathrm{d}\eta}{\mathrm{d}t} \propto \Gamma(X \to Y + B) - \Gamma(\bar{X} \to \bar{Y} + \bar{B})$$

  where the first decay rate Γ violates B(baryon number) for the charge conjugation

  $$C(X \to Y + B) = \bar{X} \to \bar{Y} + \bar{B}$$

  if C is conserved then

  $$\Gamma(X \to Y + B) = \Gamma(\bar{X} \to \bar{Y} + \bar{B})$$

  For CP violation: Quarks are chiral e.g. $X \to q_L q_L, q_R q_R$ with (chirality:L and R). We consider X as a linear combination:

  $$X \to qq = q_L q_L + q_R q_R$$

  The CP operation is

  $$CP(X \to q_L q_L) = \bar{X} \to \bar{q}_R \bar{q}_R$$
  $$CP(X \to q_R q_R) = \bar{X} \to \bar{q}_L \bar{q}_L$$

  so we can write this as a linear combination ($r = \Gamma$)

  $$r(X \to qq) = r(X \to q_L q_L) + r(X \to q_R q_R)$$
  $$r(\bar{X} \to \bar{q}\bar{q}) = r(\bar{X} \to \bar{q}_R \bar{q}_R) + r(\bar{X} \to \bar{q}_L \bar{q}_L)$$

  these are known as the Sakharov conditions:

  1. Baryon number violation
  2. C and CP violation
  3. Departure from thermal equilibrium (forward process $\neq$ backward process)

  thus $r(X \to Y + B) \neq r(Y + B \to X)$

- Rotation angles needed to describe mixing n generations of quarks: Cahibbo angles...in 2D we have 1 angle, in 3D we have 3 angles, and in 4D we have 6 angles. This is related to the number of orthogonal places in the space so $n(n-1)/2$ angles are needed.

nxn **unitary matrix:** How many free parameters are there? n^2 complex parts and n^2 real parts thus $2n^2$ parts. But the unitary matrix has n^2 constraints (conditions) e.g. $UU^\dagger = 1$ so we are left with n^2 parameters. To find the number of physical parameters we add a phase which does not change the physical observables so we can have $2n-1$ phases e.g.

$$\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} e^{i\alpha}u \\ e^{i\beta}d \end{pmatrix} e^{i\gamma}$$

here we can always add an overall phase hence the -1. so the free parameters are

$$n^2 - (2n-1) = (n-1)^2$$

and out these free parameters we subtract the number of rotation angles to get the number of physical phase angles.

$$(n-1)^2 - \frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2}$$

nxn unitary matrix is described by $n(n-1)/2$ rotation angles and $(n-1)(n-2)/2$ phase angles. e.g. for $n = 2$: 1 rotation and 0 phase, for $n = 3$: 3 rotation and 1 phase. The 1 phase angle in the 3D case is the CP violation angle. This is known as the CKM matrix.

For SU(3) The euler rotation is:

$$V_{CKM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$c_{12} \equiv \cos\theta_{12} \qquad s_{12} \equiv \sin\theta_{12}$$

If the V_{CKM} is close to the identity, the rotation angles would be small, so the two bases are roughly the same. Experimentally we find that the rotation angles

$$\theta_{12} \approx 13^\circ \qquad \theta_{23} \approx 2.4^\circ \qquad \theta_{13} \approx 0.2^\circ \qquad \delta \approx 70^\circ$$

which gives us the CKM matrix

$$V_{CKM} = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

or from the Feynman diagram

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

theses are called the Cabbibo-suppressed processes. There also is a Wolfenstein parameterization

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

for the mixing of the quarks of neutrinos: ν_e, ν_μ, ν_τ we have the PMNS matrix

$$\begin{pmatrix} \nu_e' \\ \nu_\mu' \\ \nu_\tau' \end{pmatrix} = V_{PMNS} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

parametrized by $(\theta_{12}, \theta_{23}, \theta_{13}, \delta)$ again. This is due to the angles of the rotations being much larger than the CKM matrix:

$$\theta_{12} \approx 34^\circ \qquad \theta_{23} \approx 45^\circ \qquad \theta_{13} \approx 8^\circ \qquad \delta \approx ?$$

this is known as the flavor puzzle.

Time reversal (T) $t \to^T -t$

In principle we can have a reversable process $n \leftrightarrow p + e^- + \bar{\nu}_e$ such as in stars, but we can't do this in lab.

T violation Dipole Moment:

Symmetries Part 8: Time Reversal

Time Reversal Symmetry

$$n \leftrightarrow p + e^- + \bar{\nu}_e$$

the reverse process is hard to observe, but in principle it is possible in dipole moments.

Dipole Moment Two opposite charges $\pm q$ separated by a distance d, we have electric field lines going from the positive to the negative charge which have a dipole moment

$$\mathbf{p} = q\mathbf{r}$$

for a uniform charge distribution we usually have a displacement vector

$$\mathbf{d} = \int \mathrm{d}^3 r \rho \mathbf{r}$$

Magnetic dipole Moment

$$\mu = \int \mathrm{d}^3 r \mathbf{r} \times \mathbf{j}_q$$

where $\mathbf{j}_q$ is the current density. A charged particle with a spin has a electric dipole moment

$$\mathbf{d} = d\frac{\mathbf{J}}{|\mathbf{J}|} \qquad \mathbf{J} = \mathbf{L} + \mathbf{S}$$

and a magnetic dipole moment

$$\mu = u\frac{\mathbf{J}}{|\mathbf{J}|}$$

from Wigner-Eckart theorem. The Hamiltonian is

$$H = -(\mu \cdot \mathbf{B} + \mathbf{d} \cdot \mathbf{E}) = -\frac{1}{|\mathbf{J}|}(\mu \mathbf{J} \cdot \mathbf{B} + d\mathbf{J} \cdot \mathbf{E})$$

This is T-even because from the Schrodinger Eq $H\psi = E\psi$ and invariant under T. Using Using Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ t \to -t: \qquad \mathbf{E} &\to \mathbf{E} \quad \text{T-even} \end{aligned}$$

and for second equation

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ t \to -t: \quad &\text{LHS doesn't change} \\ \text{RHS} \quad \mathbf{B} &\to -\mathbf{B} \quad \text{T-odd} \end{aligned}$$

To find if the angular momentum is T-even or T-odd we can use the definition of angular momentum:

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{p} \\ &= \mathbf{r} \times m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\ t \to -t: \mathbf{v} &\to -\mathbf{v} \\ \mathbf{L} &\to -\mathbf{L} \quad \text{T-odd} \end{aligned}$$

for spin angular momentum we can visualize that from the right hand rule, the spin is T-odd i.e. $\mathbf{S} \to -\mathbf{S}$ (T-odd). Adding the orbital and spin angular momentum we get the total angular momentum $\mathbf{J}$ as T-odd. Since $\mathbf{J}$ and $\mathbf{B}$ are T-odd, we can deduce that μ is T-even (odd times odd is even, and even times odd is even). By the same logic $\mathbf{d}$ is T-odd.

ACME Experiment The limit of the electron dipole moment is

$$d_e < 8 \times 10^{-30} e.cm$$

and for the neutron dipole moment

$$d_n < 10^{-27} e.cm$$

Axions(Not in Textbook) A hypothetical particle: The Langrangian for fundamental particle is

$$\mathcal{L} = \theta G^{\mu\nu} G_{\mu\nu}$$

the electromagnetic field tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

where the magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

and

$$A^0 = \rho = F^{\mu\nu} F_{\mu\nu}$$

the Gluon field tensor

$$G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

and θ is T-odd. Experimentally we find that $\theta < 10^{-10}$ (Strong CP problem). ADMX experiment is currently ongoing.

CPT Theorem All Observables must be CPT invariant in a Lorentz-invariant theory. Sometimes C, P, or T are violated, but CPT is always conserved. As a result, all anti-particles have the same mass as particles (Tested by LHC alpha experiment).

4 Bound States

Two Types:

- Binding Energy < Rest mass energy: Nonrelativistic bound state e.g. Hydrogen atom (-13.6 eV < 1GeV rest mass of proton).
- Binding Energy > Rest mass energy: Relativistic bound state e.g. light meson.

Hydrogen Atom: The potential energy is given by

$$V(r) = -\frac{e^2}{r}$$

or the coulomb potential. The Hamiltonian is given by the Schrödinger equation

$$H\psi = -\frac{\hbar}{2m}\nabla^2\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

where $V(r)$ is the central potential with spherical symmetry SO(3). But there also is an enhanced symmetry.

Noether's Theorem: Symmetry $\leftrightarrow$ Conservation Law. e.g.

- SO(3) $\leftrightarrow$ Conservation of Angular momentum.
- SO(1,3) $\leftrightarrow$ linear momentum (Poincare symmetry)
- T-reversal $\leftrightarrow$ energy
- $\text{U}(1)_{em}$ $\leftrightarrow$ electric charge

so from the central potential, we know that angular momentum $\mathbf{L}$ is conserved. But for $1/r$ there is a SO(4) symmetry from the LRL (Laplace-Runge-Lenz) vector

$$\mathcal{L} = \frac{1}{m}\mathbf{L} \times \mathbf{p} + \frac{\kappa\mathbf{r}}{r}$$

where

$$V(r) = -\frac{\kappa}{r}$$

the energy eigenvalues of the hydrogen atom are given by

$$E_n = -\frac{13.6\,\text{eV}}{n^2} = -\frac{me^4}{2\hbar^2 n^2} = -\frac{1}{2}\frac{\alpha m_e c^2}{n^2}$$

where $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine structure constant.

Degeneracy n^2 e.g. For SO(3), $(2l+1)$ degeneracy

$$\sum_{l=0}^{n-1}(2l+1) = 2\sum_{l=0}^{n-1} l + n = 2\frac{(n-1)(n)}{2} + n = n^2$$

n	l	m	degeneracy
1	0	0	1
2	0	0	1
	1	-1,0,1	3
3	0	0	1
	1	-1,0,1	3
	2	-2,-1,0,1,2	5

Positronium (e^+e^- bound state) has the same energy levels as the hydrogen atom the energy eigen value is given by first looking at the reduced mass

$$\mu = \frac{1}{\frac{1}{m_e} + \frac{1}{m_e}} = \frac{m_1 m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_1 \ll m_2$$

but here $m_1 = m_2 = m_e$ so $\mu = \frac{m_e}{2}$. The energy eigenvalues are given by

$$E_n = \frac{1}{2} - \frac{13.6\,\text{eV}}{n^2} = -\frac{6.8\,\text{eV}}{n^2}$$

We can do this for Muonium (μ^+e^- bound state) and Pionic Hydrogen(π^+e^- bound state).

Fine Structure

1. Relativistic Correction

$$T = E - m_e c^2$$

2. Spin-Orbit Coupling
3. Lambd Shift (QED)
4. Hyperfine Splitting aka zeeman effect

Quiz Review

- For the Positronium:

$$C : (-1)^{l+s} = (-1)^n$$

where $l + s = n$ (the selection rule for Positronium decay). *for n photons, $C = (-1)^n$. FOr the ground states $l = 0$ so the spin is

$$S : \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

where we have a triplet state $S = 1$ and a singlet state $S = 0$. For this singlet:

$$S = 0 \implies (-1)^0 = 1 = (-1)^2$$

or two photons can be emitted (para-positronium). For the triplet state:

$$S = 1 \implies (-1)^1 = -1 = (-1)^3$$

or three photons can be emitted (ortho-positronium). The mass of each photon for two photons is roughtly a half of the mass of the positronium $E_\gamma = 511\,\text{keV}$. For three photons $E_\gamma < 511$ keV.

- Binding Energy vs. Rest Mass Energy: Quarkonium ($q\bar{q}$): *uds* light quarks, *cbt* heavy quarks.
  - Heavy Quarkonium: $c\bar{c}$: Charmonium (J/ψ), $b\bar{b}$: Bottomonium (Υ), $t\bar{t}$: Toponium *does not exist* (very heavy so it decays really fast $\sim 10^{-25}$s vs $\tau_{\text{bound state}} \sim 10^{-23}$ sec).

For Charmoniun, the reduced mass is

$$\mu = \frac{m_c m_c}{m_c + m_c} \approx \frac{m_c}{2}$$

and the energy of the Hydrogen atom is

$$E_n = -\frac{me^4}{2\hbar^2 n^2} = -\frac{1}{2}\frac{\alpha mc^2}{n^2}$$

and for the Charmonium:

$$E_n = -\frac{4}{9}\frac{1}{2}\frac{\alpha m_c c^2}{n^2} \quad \text{incorrect}$$

where we have to adjust for the charge of the quark $e \to \frac{2}{3}e$ and the potential: For electron coulomb potential we know that

$$V = -\frac{e^2}{r} = -\frac{e^2}{\hbar r}\frac{\hbar c}{r} = -\frac{\alpha \hbar c}{r}$$

but for quarks there is a different potential from the strong interaction (gluon)

$$V(r) = -\frac{\alpha_s \hbar c}{r} - \frac{4}{9}\frac{\alpha \hbar c}{r} \qquad \alpha_s = \frac{g_s^2}{\hbar c} \gg \alpha$$

which is much larger than the coulomb potential (suppressed second term), but there is a transition to a linear potential as the distance get very large.

$$V(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} + F_o r \quad \text{QCD Potential}$$

there also is a color factor $\frac{4}{3}$ based on the three colors of the quarks. So the energy is given by

$$E_n = -\frac{4}{3}\frac{1}{2}\frac{\alpha_s m_c c^2}{n^2}$$

- Decay of Charmonium:

$$J/\psi \to \pi^+\pi^-\pi^0 \quad \text{or} \quad D^+D^-$$

For the ground state, $m_{J/\psi} = 3.1$ GeV. And the total rest mass of D^+D^- is kinematically forbidden $m_{D^+} + m_{D^-} = 3.7$ GeV. We have a decay to 3 pions due to the G-parity conservation $(-1)^I C$ or $(-1)^n$.

OZI rule (Okubo, Zweig, Iizuka) Cutting a hard gluon line in the Feynman diagram separates the quarks and the decay is suppressed. For soft gluon lines, cutting a line does not separate the quarks and the decay is not suppressed.

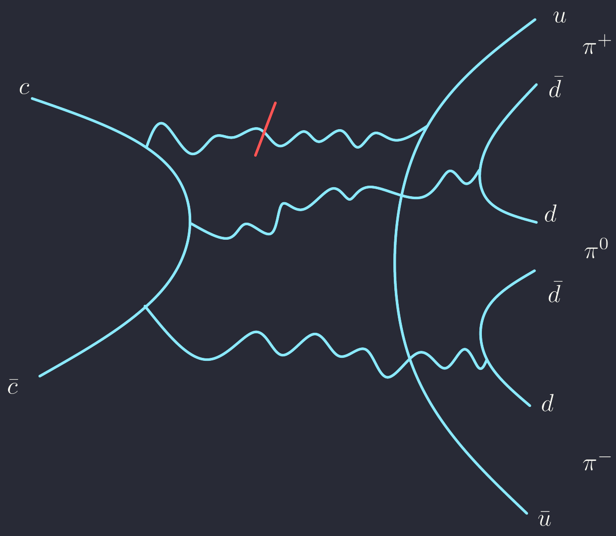


Figure 4.1: OZI Rule

- Light Mesons: $q\bar{q}$ where $q = u, d, s$. There are nine spin-0 (pseudo scalar) mesons and nine spin-1 (vector) mesons. (insert figure 5.11 from Griffiths). From the lie algebra of the spin-0 nonet

$$3 \otimes \bar{3} = 8 \oplus 1$$

where the 1 is the η' meson. and we break down the 8 into

$$8 \to 2 \oplus 3 \oplus 2 \oplus 1$$

where they refer to the top row, middle row pions, bottom row and η meson. For the vector mesons. For the isospin doublet:

$$u = \left|\frac{1}{2}\frac{1}{2}\right\rangle \qquad d = \left|\frac{1}{2} - \frac{1}{2}\right\rangle$$

and for the antiquarks:

$$\bar{u} = \left|\frac{1}{2} - \frac{1}{2}\right\rangle \qquad \bar{d} = -\left|\frac{1}{2}\frac{1}{2}\right\rangle$$

and the pions are given by

$$\begin{aligned}
\pi^+ &= \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle = -u\bar{d} \\
\pi^- &= \left|\frac{1}{2} - \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} - \frac{1}{2}\right\rangle = d\bar{u} \\
\pi^0 &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2} - \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle + \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} - \frac{1}{2}\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})
\end{aligned}$$

for the corner mesons:

$$K^0 = d\bar{s} \qquad \bar{K}^0 = \bar{d}s \qquad K^+ = u\bar{s} \qquad K^- = \bar{u}s$$

and the η mesons are

$$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$
$$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

For spin-1 mesons, in terms of the flavor

$$\rho^+, \rho^0, \rho^- = \pi^+, \pi^0, \pi^-$$

and the same for the K^* mesons. The difference is in the center

$$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \qquad \phi = s\bar{s}$$

Quiz Review: For the kinetic energy of a particle

$$\begin{aligned} T &= E - mc^2 \\ &= \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4} - mc^2 \\ &= mc^2 \left(1 + \frac{\mathbf{p}^2}{m^2 c^2}\right)^{1/2} - mc^2 \end{aligned}$$

and using the binomial expansion

$$(1+x)^n \approx 1 + nx + \frac{1}{2}\frac{n(n-1)}{2}x^2 \dots \quad \text{for} \quad x \ll 1$$

so

$$\begin{aligned} T &= mc^2 \left(\cancel{1} + \frac{\mathbf{p}^2}{2m^2c^2} - \frac{\mathbf{p}^4}{8m^4c^4} + \dots \right) - \cancel{mc^2} \\ &= \frac{\mathbf{p}^2}{2m} - \frac{1}{8}\frac{\mathbf{p}^4}{m^3c^2} + \dots \end{aligned}$$

From last time: Light mesons (u, d, s) with $q\bar{q}$ bound states and spin

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

where we for the spin-0 pseudoscalar mesons: π, K, η we have 9 states. And for the spin-1 vector mesons: ρ, K^*, ω, ϕ we have 9 states. The flavor states of K mesons:

$$K^+ : (u\bar{s}) \quad K^- : (\bar{u}s) \quad K^0 : (d\bar{s}) \quad \bar{K}^0 : (\bar{d}s)$$

and for vector K mesons

$$K^{*+} : (u\bar{s}) \quad \text{etc.}$$

so the Isospin states

$$\begin{pmatrix} u \\ d \end{pmatrix} \otimes \begin{pmatrix} \bar{d} \\ \bar{u} \end{pmatrix} = 1 \oplus 0$$

so

- spin-0: $1 \sim \pi^+, \pi^-, \pi^0$, $0 \sim \eta$
- spin-1: $1 \sim \rho^+, \rho^0, \rho^-$, $0 \sim \omega$

and in $SU(3)$:

$$3 \otimes \bar{3} = 8 \oplus 1$$

where η' is the spin-0 singlet (1). We can break down the spin algebra to isospin

$$\begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \otimes \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix} \otimes \begin{pmatrix} \bar{d} \\ \bar{u} \end{pmatrix} = 1 \oplus 0$$

where

$$\begin{aligned} |1,1\rangle &= |\uparrow\uparrow\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1,-1\rangle &= |\downarrow\downarrow\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{aligned}$$

and the charge conjugation operator

$$Cu \to \bar{u} \qquad Cd \to \bar{d}$$
$$C = i\sigma^2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$C\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

so

$$\begin{aligned} |1,1\rangle &= -u\bar{d} = \left|\pi^+\right\rangle, \left|\rho^+\right\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) = \left|\pi^0\right\rangle, \left|\rho^0\right\rangle \\ |1,-1\rangle &= \bar{u}d = \left|\pi^-\right\rangle, \left|\rho^-\right\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) = \cancel{|\eta\rangle}, |\omega\rangle \end{aligned}$$

where the η is not correct. We know that ψ is orthgonal to ω so

$$|\psi\rangle = |s\bar{s}\rangle$$

we know that η' is an $SU(3)$ singlet so

$$|\eta'\rangle = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

and since η is orthogonal to η' and all other states:

$$|\eta\rangle = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

this is a special case for the pseudoscalar light mesons. Even though the spin-0 and spin-1 particles are made of the same quarks, their masses are very different! So the true meson mass is

$$M = m_1 + m_2 + A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}$$

this third term is the spin-spin interaction from the Hydrogen Atom (n^2 degeneracry) hyperfine splitting. The spin-spin interaction breaks the degeneracy of the n^2 states. Finding

$$\begin{aligned} \mathbf{S}_T &= \mathbf{S}_1 + \mathbf{S}_2 \\ \mathbf{S}^2 &= (\mathbf{S}_1 + \mathbf{S}_2)^2 \\ &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \end{aligned}$$

Where from the operator

$$\begin{aligned} [J^2, J_z] = 0 &\qquad |j,m\rangle \\ J_z\,|j,m\rangle &+ \hbar m\,|j,m\rangle \\ J^2\,|j,m\rangle &= \hbar^2 j(j+1)\,|j,m\rangle \end{aligned}$$

so the eigenvalues of $\mathbf{S}$ are $\frac{1}{2}(\frac{1}{2}+1)\hbar^2$:

$$\begin{aligned} \mathbf{S}^2 &= \frac{3}{2}\hbar^2 + \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \\ &= \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \end{aligned}$$

so for the scalar case $s = 0$, $\mathbf{S}^2 = 0$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = -\frac{3}{4}\hbar^2$$

and for the vector case $s = 1$, $\mathbf{S}^2 = 2\hbar^2$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(2 - \frac{3}{2}\hbar^2\right)$$

So

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \begin{cases} -\frac{3}{4}\hbar^2 & \text{spin-0} \\ \frac{1}{4}\hbar^2 & \text{spin-1} \end{cases}$$

e.g. we have meson masses

- $(u, d) : m_\rho = 775\,\mathrm{MeV/c^2}$
- $(u, d) : m_\pi = 140\,\mathrm{MeV/c^2}$

And we have an effective mass (MIT bag model):

$$m_{eff} \geq \Lambda_{QCD} \sim 200\,\mathrm{MeV/c^2}$$

Quarks are always in bound states, and do not feel the strong interaction. And the bare mass is different from the effective mass. Some bare masses:

$$\begin{aligned} m_{u,d} &\sim 300\,\mathrm{MeV/c^2} \\ m_s &\sim 400\,\mathrm{MeV/c^2} \end{aligned}$$

Baryons... (light): qqq bound states. We can treat the 3 body system as a 2 body system(CM of 2 quarks plus the third quark). For the ground state, $l = l' = 0$ for simplicity.

Spin configurations

$$\begin{aligned} \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= (1 \oplus 0) \otimes \frac{1}{2} \\ &= (1 \otimes \frac{1}{2}) \oplus (0 \otimes \frac{1}{2}) \\ &= (\frac{3}{2} \oplus \frac{1}{2}) \oplus (\frac{1}{2}) \\ &= 4 \oplus 2 \oplus 2 \end{aligned}$$

(the 3/2 spin has 4 states: $3/2, 1/2, -1/2, -3/2$) where

$$j_1 \otimes j_2 = (j_1 + j_2), \ldots, |j_1 - j_2|$$

we also have 8 spin states for the 3 quarks:

$$\begin{aligned} &\uparrow\uparrow\uparrow \\ \uparrow\uparrow\downarrow, \uparrow\downarrow\uparrow, &\downarrow\uparrow\uparrow \\ \uparrow\downarrow\downarrow, \downarrow\uparrow\downarrow, &\downarrow\downarrow\uparrow \\ &\downarrow\downarrow\downarrow \end{aligned}$$

so

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = |{\uparrow\uparrow\uparrow}\rangle$$
$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|{\uparrow\uparrow\downarrow}\rangle + |{\uparrow\downarrow\uparrow}\rangle + |{\downarrow\uparrow\uparrow}\rangle)$$
$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}(|{\uparrow\downarrow\downarrow}\rangle + |{\downarrow\uparrow\downarrow}\rangle + |{\downarrow\downarrow\uparrow}\rangle)$$
$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = |{\downarrow\downarrow\downarrow}\rangle$$

which are all symmetric states! But Baryons are fermions so the wavefunction must be antisymmetric! And from 3 spins, we cant make an antisymmetric state, but we can make a partically antisymmetric state

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\downarrow$$

where the subscript 12 denotes that the first two spins are antisymmetric and the third spin is free. So we can also get

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{23} =\uparrow \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{23} =\downarrow \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

but we cant get an antisymmetric state for the first and third spins:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_{13} = \frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{13} = \frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)$$

this is just a linear combination of the other states:

$$|\,\rangle_{13} = |\,\rangle_{12} + |\,\rangle_{23}$$

Pauli Exclusion Principle: For probability of wavefunction to be the same

$$\psi(1,2) \to \psi(2,1)$$
$$|\psi(1,2)|^2 = |\psi(2,1)|^2$$
$$\implies \psi(1,2) = e^{i\phi}\psi(2,1)$$
$$\psi(1,2) \to \psi(2,1)e^{i\phi} \to \psi(1,2)e^{2i\phi}$$
$$\implies e^{2i\phi} = 1 \implies e^{i\phi} = \pm 1$$

so

$$\psi(1,2) = \begin{cases} +\psi(2,1) & \text{even} \\ -\psi(2,1) & \text{odd} \end{cases}$$

And distinguishable particles

$$\psi_\alpha(1)\psi_\beta(2)$$

and for indistinguishable particles

$$\psi(1,2) = \frac{1}{\sqrt{2}}[\psi_\alpha(1)\psi_\beta(2) \pm \psi_\alpha(2)\psi_\beta(1)]$$

and from the *Spin-Statistics Theorem*:

- Bosons $\rightarrow$ Even wavefunctions(+)
- Fermions $\rightarrow$ Odd wavefunctions(-)

where if $\alpha = \beta$ we then know $\psi(1,2) = 0$ for fermions.

Quiz Review:

- For the baryon wavefunction its a little more complicated:
  - Fermion → Pauli Exclusion Principle
  - Three body system
  - Color Quantum Number

  *The baryon wavefunction must be antisymmetric under the inerchange of any two constituent quarks. So the baryon wavefunction is a combination of 4 parts:

$$\psi(baryon) = \psi(space)\psi(spin)\psi(flavor)\psi(color)$$

  So for total antisymmetry

  - For the space part, we can assume a ground state where $\ell = 0$ thus sphericaly symmetric.
  - The spin part can be symmetric (3/2) or partially antisymmetric (1/2).
  - For the flavor part we have $n^3 = 27$ states, or from group theory we have a SU(3):

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

    or from Young Tableux for mesons

$$\begin{array}{|c|}\hline \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline & \\ \hline \end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$= \frac{3*4*2}{1*3*1} \oplus \frac{3*2*1}{1*2*3} = 8 \oplus 1$$

    but for the Bosons

$$\begin{array}{|c|}\hline \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \\ \hline \end{array} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array}$$

$$= \frac{3*4}{1*2} \oplus \frac{3*2}{1*2} = 6 \oplus \bar{3}$$

    where the $\bar{3}$ comes from the antisymmetric part of now expanding again

$$6 \otimes 3 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline \end{array} = 10 \oplus 8$$

    thus

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

    How do we arrange this 10 states? From the mesons we know that the octet splits like

$$8 \rightarrow 2 + 3 + 2 + 1$$

    and we are told that the decuplet splits to

$$10 \rightarrow 4 + 3 + 2 + 1$$

    The figure has isospin I_3 on the x-axis and spin S on the y-axis. We know that the quark charges are

$$u = \frac{2}{3} \qquad d = -\frac{1}{3} \qquad s = -\frac{1}{3}$$

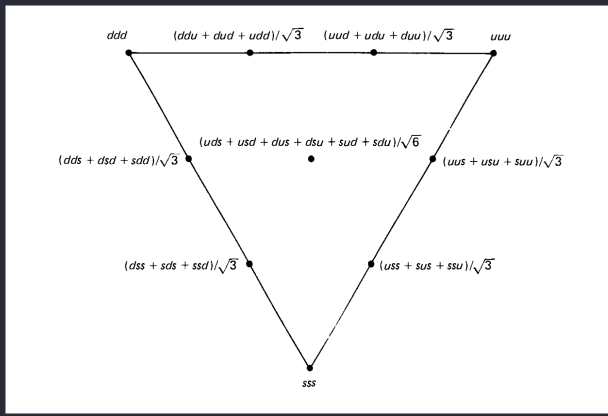


Figure 4.2: Decuplet

So the top right is (uuu) and etc. for the other corners. But these states are symmetric and the spin is also symmetric so we need something else that needs to be antisymmetric. Thus we have colors in $SU(3)_c$:

$$q = \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

this also by $SU(3)$ algebra has 27 states of color:

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

but....

Every Naturally Occuring Particle is a Color Singlet - Griffiths

AKA the *Color Confinement* principle. So the singlet is always antisymmetric:

$$\psi = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)$$

So since everything except color is must be symmetric for the total to be antisymmetric i.e.

$$\psi(spin)\psi(flavor) = \text{symmetric}$$

So how do we get from Δ^{++} to Δ^{+}? We use the I_- operator: From the lowering operator

$$J_- \left| j, m \right\rangle = \hbar\sqrt{(j+m)(j-m+1)} \left| j, m-1 \right\rangle$$

so in isospin-space

$$u = \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

thus

$$I_- u = I_- \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \hbar\sqrt{1} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = d$$

and we also get

$$I_- d = 0 \quad \text{lowest } I_3 \text{ state}$$
$$I_- s = 0 \quad \text{singlet}$$

SO

$$
\begin{aligned}
I_-\Delta^{++} &= I_-(uuu) = (uud + udu + duu) \\
I_-\left|\frac{3}{2},\frac{3}{2}\right\rangle &= \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
&= \sqrt{3}\Delta^+ \\
\implies \Delta^+ &= \frac{1}{\sqrt{3}}(uud + udu + duu)
\end{aligned}
$$

applying the lowering operator again we get

$$\Delta^0 = \frac{1}{\sqrt{3}}(udd + dud + ddu)$$

and again

$$\Delta^- = ddd$$

and for the Σ (strangeness $S = -1$) states we have the following:

$$
\begin{aligned}
\Sigma^{*+} &= \frac{1}{\sqrt{3}}(uus + usu + suu) \\
\Sigma^{*0} &= \frac{1}{\sqrt{3}}(uds + dus + dsu + sud + sdu + uds) \\
\Sigma^{*-} &= \frac{1}{\sqrt{3}}(dds + dsd + sdd)
\end{aligned}
$$

for $S = -2$ we have the 2 Ξ states:

$$
\begin{aligned}
\Xi^{*0} &= \frac{1}{\sqrt{3}}(uss + sus + ssu) \\
\Xi^{*-} &= \frac{1}{\sqrt{3}}(dss + sds + ssd)
\end{aligned}
$$

and a singlet

$$\Omega^- = sss$$

so the flavor is symmetric for the decuplet states

Octet States In the first octet we have 8 states

- n, p
- $\Sigma^+, \Sigma^0, \Sigma^-$
- Ξ^0, Ξ^-
- Λ^0

now we need to do the same partial antisymmetry thing we did for spins onto the flavor states:

$$\psi(flavor) = |\,\rangle_{12} \quad \text{or} \quad |\,\rangle_{23} \quad \text{or} \quad |\,\rangle_{13}$$

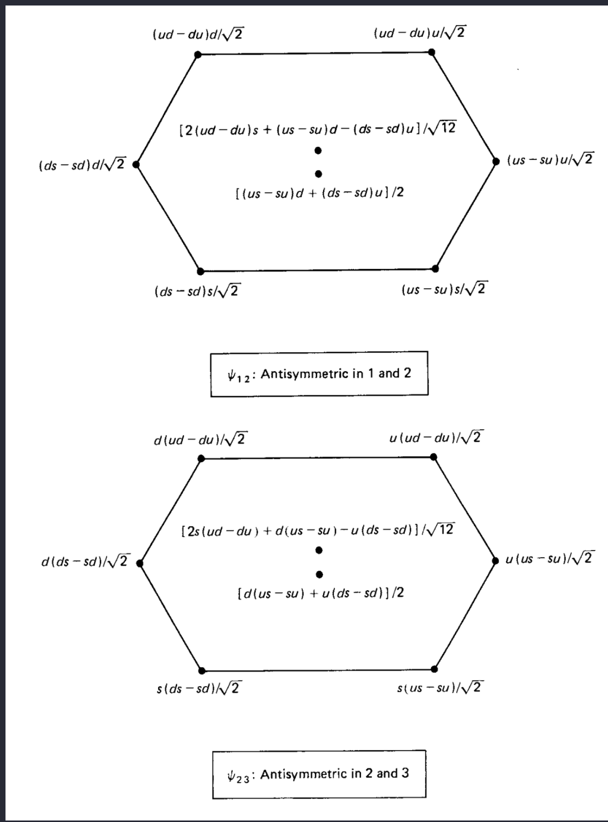


Figure 4.3: 1st and 2nd Octets

since the proton is a quark made of uud we know that

$$
\begin{aligned}
|\ \rangle_{12} &= \frac{1}{\sqrt{2}}(ud - du)u \\
\left|p(s_z = \frac{1}{2})\right\rangle &= \frac{1}{\sqrt{2}}(ud - du)u \otimes \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow \\
&= \frac{1}{2}[u(\uparrow)d(\downarrow)u(\uparrow) \\
&\quad - u(\downarrow)d(\uparrow)u(\uparrow) \\
&\quad - d(\uparrow)u(\downarrow)u(\uparrow) \\
&\quad + d(\downarrow)u(\uparrow)u(\uparrow)]
\end{aligned}
$$

and we can do this for the flavor and spin wavefunctions:

$$
\begin{aligned}
\psi(flavor)\psi(spin) = \frac{\sqrt{2}}{3}&[\psi_{12}(f)\psi_{12}(s) \\
&+ \psi_{23}(f)\psi_{23}(s) \\
&+ \psi_{13}(f)\psi_{13}(s)]
\end{aligned}
$$

where the will have to find this constant in HW. Thus the proton is

$$\begin{aligned}
|p(\text{spin up})\rangle &= \frac{1}{2}\sqrt{2}3[(ud-du)u(\uparrow\downarrow-\downarrow\uparrow)\uparrow \\
&\quad + u(ud-da)\uparrow(\uparrow\downarrow-\downarrow\uparrow) \\
&\quad + (uud-duu)(\uparrow\uparrow\downarrow-\downarrow\uparrow\uparrow)] \\
&= \frac{1}{3\sqrt{2}}[uud(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow) \\
&\quad + udu(2\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow-\uparrow\uparrow\downarrow) \\
&\quad + duu(-\uparrow\downarrow\uparrow+2\downarrow\uparrow\uparrow-\uparrow\uparrow\downarrow)]
\end{aligned}$$

so for n we have

$$|n\rangle_{12} = \frac{1}{\sqrt{2}}(ud-du)d$$

and for sigma

$$\begin{aligned}
\left|\Sigma^{+}\right\rangle_{12} &= \frac{1}{\sqrt{2}}(us-su)u \\
\left|\Xi^{0}\right\rangle_{12} &= \frac{1}{\sqrt{2}}(us-su)s
\end{aligned}$$

Quiz Review (From last time) We found for the baryon wavefunction that the space part is symmetric ($\ell = 0$), the spin & flavor parts are symmetric, and the color part is antisymmetric. For the spin up proton:

$$\psi = \frac{\sqrt{2}}{3}[\psi_{12}(spin)\psi_{12}(flavor) + \psi_{23}(spin)\psi_{23}(flavor) + \psi_{13}(spin)\psi_{13}(flavor)]$$

where

$$\langle\psi_{12}|\psi_{23}\rangle \neq 0 \qquad \langle\psi_{12}|\psi_{13}\rangle \neq 0$$

and

$$\psi_{12}(spin) = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$$
$$\psi_{12}(flavor) = \frac{1}{\sqrt{2}}(ud - du)u$$

so the wavefunction of the two parts is

$$\psi = \frac{\sqrt{2}}{3}[2u(\uparrow)u(\uparrow)d(\downarrow) - u(\uparrow)u(\downarrow)d(\uparrow) - u(\downarrow)u(\uparrow)d(\uparrow) + \text{permuations}]$$

Magnetic Moment The experimental test, the magnetic moment

$$\boldsymbol{\mu} = \frac{q}{mc}\mathbf{S}$$

where the Hamiltonian is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$

and the magnetic moment of a baryon is a vector sum of the three quarks

$$\boldsymbol{\mu}(\text{baryon}) = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3$$

so the z-component of the magnetic moment is

$$\mu_z(\text{baryon}) = \sum_i \mu_{i_z} = \sum_i \frac{q_i}{mc} S_{i_z}$$

where

$$\mu_{i_z} = \frac{q_i}{mc}\frac{\hbar}{2}$$

so for each quark

$$\mu_u = \frac{2}{3}\frac{\hbar}{2m_u c} \qquad \mu_d = -\frac{1}{3}\frac{\hbar}{2m_d c} \qquad \mu_s = -\frac{1}{3}\frac{\hbar}{2m_s c}$$

so the expectation value of the magnetic moment operator μ_z is

$$\begin{aligned}\mu_{\text{baryon},\uparrow} &= \langle B(\uparrow)|\, \mu_z \,|B(\uparrow)\rangle \\ &= \sum_i \langle B(\uparrow)|\, \mu_{i_z} \,|B(\uparrow)\rangle \\ &= \frac{2}{\hbar}\sum_i \langle B(\uparrow)|\, \mu_i S_{i_z} \,|B(\uparrow)\rangle\end{aligned}$$

So calculating the expectation value for the proton

$$\sum_i \mu_i S_{i_z}\,|u(\uparrow)u(\uparrow)d(\downarrow)\rangle = \left(\mu_u\frac{\hbar}{2} + \mu_u\frac{\hbar}{2} + \mu_d\frac{-\hbar}{2}\right)|u(\uparrow)u(\uparrow)d(\downarrow)\rangle$$

the first term is

$$\begin{aligned}\mu_1 &= \sum_i \langle p(\uparrow)| \mu_i S_{i_z} |p(\uparrow)\rangle \\ &= \left(\frac{2}{3\sqrt{2}}\right)^2 \frac{\hbar}{2}(2\mu_u - \mu_d)\frac{2}{\hbar} \langle u(\uparrow)u(\uparrow)d(\downarrow)|u(\uparrow)u(\uparrow)d(\downarrow)\rangle \\ &= \frac{2}{9}(2\mu_u - \mu_d)\end{aligned}$$

the second term $-u(\uparrow)u(\downarrow)d(\uparrow)$ is

$$\frac{\hbar}{2}(\mu_u - \mu_u + \mu_d) = \frac{\hbar}{2}\mu_d$$

so

$$\left(\frac{1}{3\sqrt{2}}\right)^2 \mu_d = \frac{1}{18}\mu_d$$

and same for the third term. We get the magnetic moment as

$$\begin{aligned}\mu_{p(\uparrow)} &= 3\left[\frac{2}{9}(2\mu_u - \mu_d) + \frac{1}{18}\mu_d + \frac{1}{18}\mu_d\right] \\ &= \frac{3}{18}(4(2\mu_u - \mu_d) + 2\mu_d) \\ &= \frac{1}{6}(8\mu_u - 2\mu_d) \\ &= \frac{1}{3}(4\mu_u - \mu_d)\end{aligned}$$

and from the experimental values:

$$m_p = 2.79 m_u = 2.79 m_d$$

we get

$$\mu_p = 2.79\frac{e\hbar}{2m_p c} = 2.79\mu_B$$

where $\mu_B = \frac{e\hbar}{2m_p c}$ is the Bohr magneton. And for the neutron we know

$$p : uud \qquad n : udd$$

so all we have to do is replace the u with d and thus:

$$\begin{aligned}\mu_n &= \frac{1}{3}(4\mu_d - \mu_u) \\ &= \frac{1}{3}\left(4(-1/3)\frac{\hbar}{2m_d c} - 2/3\frac{\hbar}{2m_u c}\right) \\ &= -\frac{2}{3}(2.79)\frac{e\hbar}{2m_p c} = -1.86\mu_B\end{aligned}$$

experimentally, we got the proton and neutron magnetic moments to be

$$\mu_p = 2.793\mu_B \qquad \mu_n = -1.913\mu_B$$

we can also take the ratio of the magnetic moments

$$\frac{\mu_n}{\mu_p} = -\frac{2}{3}$$

which is independent of the quark masses! This is a (robust) prediction of the quark model

If there was no color Then the wavefunction would be

$$\psi = \psi_{space}\psi_{spin}\psi_{flavor}$$

so

$$\psi(spin)\psi(flavor) = \text{anti-symmetric}$$

and we would get the ratio of the magnetic moments to be

$$\frac{\mu_n}{\mu_p} = -2$$

Baryon Masses For the meson masses we had the mass as the sum of the masses and the spin-spin interaction:

$$M = m_1 + m_2 + A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}$$

But for the baryons we have 3 quarks so we have

$$M = m_1 + m_2 + m_3 + A'\left(\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2} + \frac{\mathbf{S}_1 \cdot \mathbf{S}_3}{m_1 m_3} + \frac{\mathbf{S}_2 \cdot \mathbf{S}_3}{m_2 m_3}\right)$$

For the decuplet we have the inverted triangle from Figure 4.2 and the octet from Figure 4.3. Taking $m_u = m_d$

- For Baryons with no S,

  $$M = 3\mu_u + \frac{A'}{m_\mu^2}(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3)$$

  so the total spin is

  $$\begin{aligned}\mathbf{S}^2 &= (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3)^2 \\ &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 + 2(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3) \\ \implies \mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3 &= \frac{1}{2}(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2 - \mathbf{S}_3^2) = S'\end{aligned}$$

  where the eigen values are

  $$\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$$

  i.e.

  $$\begin{aligned}s = 3/2 : \mathbf{S}^2 &= \frac{3}{2}(\frac{3}{2}+1)\hbar^2 = \frac{15}{4}\hbar^2 \\ s = 1/2 : \mathbf{S}^2 &= \frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2\end{aligned}$$

  so

  $$\begin{aligned}&= \frac{1}{2}\left(\mathbf{S}^2 - 3\frac{3}{4}\hbar^2\right) \\ &= \frac{1}{2}\begin{cases}\frac{3}{2}\hbar^2 & S = \frac{3}{2} \quad \text{(decuplet)} \\ -\frac{3}{2}\hbar^2 & S = \frac{1}{2} \quad \text{(octet)}\end{cases}\end{aligned}$$

  so the masses are

  $$M_N = 3m_u - \frac{3}{4}\frac{\hbar^2}{m_u^2}A'$$
  $$M_\Delta = 3m_u + \frac{3}{4}\frac{\hbar^2}{m_u^2}A' M_\Omega = 3m_s + \frac{3}{4}\frac{\hbar^2}{m_s^2}A'$$

For the decuplet case all spin are parallel so for $s = 3/2$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \mathbf{S}_1 \cdot \mathbf{S}_3 = \mathbf{S}_2 \cdot \mathbf{S}_3 = \frac{1}{4}\hbar^2$$

and

$$\begin{aligned}(\mathbf{S}_1 + \mathbf{S}_2)^2 &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \\ \mathbf{S}_1 \cdot \mathbf{S}_2 &= \frac{1}{2}(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2) = \frac{3}{4}\hbar^2 & &= \frac{1}{2}\left(1(1+1)\hbar^2 - 2\frac{3}{4}\hbar^2\right) = \frac{1}{4}\hbar^2\end{aligned}$$

so the maasses are calculated by:

$$M = m_1 + m_2 + m_3 + \frac{A'}{4}\hbar^2\left(\frac{1}{m_1 m_3} + \frac{1}{m_2 m_3} + \frac{1}{m_2 m_3}\right)$$

For the octet case

$$\begin{aligned}&M_\Sigma : (ud) \quad I = 1 \\ &M_\Lambda : (ud) \quad I = 0\end{aligned}$$

so for example the Σ^+ case:

$$M_{\Sigma^+} = 2m_u + m_s + A'\left(\frac{\mathbf{S}_u \cdot \mathbf{S}_u}{m_u^2} + \frac{\mathbf{S}_u \cdot \mathbf{S}_s}{m_u m_s} + \frac{\mathbf{S}_u \cdot \mathbf{S}_s}{m_u m_s}\right)$$

and since we have already calculated

$$\mathbf{S}_u \cdot \mathbf{S}_u = \frac{1}{4}\hbar^2$$

and we know the spin for the octet is

$$\mathbf{S}_u \cdot \mathbf{S}_u + \mathbf{S}_{u1} \cdot \mathbf{S}_s + \mathbf{S}_{u2} \cdot \mathbf{S}_s = -\frac{3}{4}\hbar^2 \mathbf{S}_{u1} \cdot \mathbf{S}_s + \mathbf{S}_{u2} \cdot \mathbf{S}_s = -\hbar^2$$

and from anti symmetry we know that

$$\mathbf{S}_{u1} = \mathbf{S}_{u2} = -\frac{3}{2}\hbar^2$$

and we can do the same for the other octet states.

Midterm: Multiple choice 25pt, bound states(general structure why $1/n^2$) heavy masses, wavefunctions, relativistic kinematics.

5 Decay and Scattering

Decay rate Γ

- Probability per unit time for the decay to happen

For a decay process the change in the number of particles (amount of stuff that decayed)

$$-N(t)\Gamma \mathrm{d}t = \mathrm{d}N$$

we can solve this differential equation to find

$$\begin{aligned}\int \frac{\mathrm{d}N}{N} &= -\int \Gamma \mathrm{d}t \\ \ln N &= -\Gamma t + C \\ \implies N(t) &= N_0 e^{-\Gamma t}\end{aligned}$$

we can find the mean lifetime $\tau = \frac{1}{\Gamma}$ so

$$N(t) = N_0 e^{-t/\tau}$$

Half time and the half-life is when

$$\begin{aligned}N(t_{1/2}) &= \frac{N_0}{2} = N(0)e^{-\Gamma t_{1/2}} \\ \implies e^{\Gamma t_{1/2}} &= 2 \\ \Gamma t_{1/2} &= \ln 2\end{aligned}$$

or

$$t_{1/2} = \tau \ln 2$$

Example

$$\begin{aligned}\pi^+ \to & \mu^+ + \nu_\mu \qquad \Gamma_1 \gg \Gamma_2 \\ & e^+ + \nu_e \qquad \Gamma_2\end{aligned}$$

and

$$\Gamma_{tot} = \sum_i \Gamma_i \qquad \tau_{tot} = \frac{1}{\Gamma_{tot}}$$

we have a branching ratio (or fraction)

$$\mathrm{Br}_i = \frac{\Gamma_i}{\Gamma_{tot}} \qquad [0, 1]$$

and we find the branching ratio of the pion decay is experimentally

$$\begin{aligned}\mathrm{Br}_1 &= 0.999877 \\ \mathrm{Br}_2 &= 0.000123\end{aligned}$$

Insert Griffiths Figure 6.1 here

Scattering From the impact parameter b and scattering angle θ we can find the cross section, or the probability of scattering. We have an infinitesimal area of

$$\mathrm{d}\sigma = |\mathrm{d}b \cdot b\mathrm{d}\phi|$$

which is like the area of a rectangle made by the differential impact parameter. The solid angle is

$$\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

like the theta and phi part of spherical coordinates. The differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{b}{\sin\theta} \cdot \frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

Hard Sphere Scattering We have a hard sphere of radius R and we send a particle toward the sphere and it scatters on the surface. Thus the cross section is expected to be

$$\sigma = \pi R^2$$

or the area of a circle that cuts the sphere. From the law of inflection we have an inflection

$$2\alpha + \theta = \pi$$

and the trigonometry shows that the impact parameter is

$$b = R\sin\alpha$$

or

$$\begin{aligned} b &= R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\ &= R\cos\left(\frac{\theta}{2}\right) \end{aligned}$$

so the differential cross section is

$$\begin{aligned} \frac{\mathrm{d}b}{\mathrm{d}\theta} &= -\frac{R}{2}\sin\frac{\theta}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \left|\frac{R\cos\frac{\theta}{2}}{\sin\theta} \cdot \frac{R}{2}\sin\frac{\theta}{2}\right| \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= \frac{R^2}{4} \end{aligned}$$

and

$$\begin{aligned} \int \mathrm{d}\sigma &= \int \frac{R^2}{4}\,\mathrm{d}\Omega \\ \sigma &= \frac{R^2}{4} \cdot 4\pi = \pi R^2 \end{aligned}$$

Rutherford Scattering In the experiment we can find the impact parameter

$$b = \frac{q_1 q_2}{2E}\cot\frac{\theta}{2}$$

so

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{q_1 q_2}{4E}\csc^2\frac{\theta}{2}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \left|\frac{b}{\sin\theta}\cdot\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \\
&= \left|\frac{q_1 q_2}{2E}\cot\frac{\theta}{2}\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\cdot -\frac{q_1 q_2}{4E}\csc^2\frac{\theta}{2}\right| \\
&= \frac{q_1^2 q_2^2}{16E^2}\csc^4\frac{\theta}{2}
\end{aligned}$$

so the cross section is

$$\begin{aligned}
\sigma &= \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega \\
&= \frac{q_1^2 q_2^2}{16E^2}\int \csc^4\frac{\theta}{2}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= 2\pi\frac{q_1^2 q_2^2}{16E^2}\int \frac{\sin\theta}{\sin^4\frac{\theta}{2}}\mathrm{d}\theta \\
&= 2\pi\frac{q_1^2 q_2^2}{16E^2}\int \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}\mathrm{d}\theta
\end{aligned}$$

and substituting

$$x = \sin\frac{\theta}{2} \implies \mathrm{d}x = \frac{1}{2}\cos\frac{\theta}{2}\mathrm{d}\theta$$

so

$$2\pi\int_0^1 \frac{2x}{x^4}\mathrm{d}x = 2\pi\left(\frac{1}{2x^2}\right)\Big|_0^1 \to \infty$$

Fermi Golden Rule For nonrelativistic system

$$\text{Transition probability} = \text{phase space} \times |\text{amplitude}|^2$$

or

$$\rho\cdot|\langle f|\,0\,|i\rangle|^2$$

where ρ is the density of states.

Relativistic System

$$\begin{aligned}
\mathrm{d}\Gamma &\propto |\mathcal{M}|^2\mathrm{d}\Pi \\
\mathrm{d}\sigma &\propto |\mathcal{M}|^2\mathrm{d}\Pi
\end{aligned}$$

where $\mathrm{d}\Pi$ is the phase space. For the two body decay

$$\begin{aligned}
1 &\to 2+3 \\
m_1 &> m_2 + m_3
\end{aligned}$$

Wigner-Eckart Theorem For spherically symmetric systems we can split the amplitude into two parts: the symmetric and dynamic parts.

$$\langle f|\,0\,|\propto\rangle\,\text{symmetric}\times\text{dynamic}$$

Quiz Review

- The decay formula gives us
$$N(t) = N_0 e^{-t/\tau} = 10^6 e^{-10} \approx 45$$
- The probability of 1 particle still being there after 10 average lifetimes is directly equal to
$$e^{-t/\tau} = e^{-10} \approx 4.5 \times 10^{-5}$$
- Dirac Delta Function
$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$
or
$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$
We can also think of a rectangle with area 1 at $x = 0$ and we keep shortening the width and increasing the height to keep the area 1. As the width gets infinitesimally small, the height gets infinitely large.
- From the heaviside step function
$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\theta(x)\mathrm{d}x = \theta(x)\Big|_{-\infty}^{\infty} &= 1 \\ &= \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x \\ \implies \delta(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\theta(x) \end{aligned}$$

Fermi Golden Rule (again) We know that the phase space is dependent of the kinematics i.e. it only depends on the number of paritcles involved. The amplitude $\mathcal{M}$ is dependent on the dynamics or the type of interaction.

Decay $1 \to 2 + 3 + \cdots + n$

$$\begin{aligned} \Gamma = \frac{S}{2m_1\hbar} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - p_3 - \cdots - p_n) \\ \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2c^2)\theta(p_j^0) \cdot \frac{\mathrm{d}^4 p_j}{(2\pi)^4} \end{aligned}$$

Is the decay rate where S is the symmetry factor

$$S = \frac{1}{\prod_i k_i!}$$

e.g. $a \to b + b + c + c + c$

$$S = \frac{1}{2!3!} = \frac{1}{12}$$

and we also have the phase space part which is in a 4-dimensional component i.e.

$$\begin{aligned} \delta^3(\mathbf{r}) &= \delta(x)\delta(y)\delta(z) \\ \delta^4(p) &= \delta(p^0)\delta^3(\mathbf{p}) \end{aligned}$$

Phase space parts

1. In the first part
$$\delta^4(p_1 - p_2 - p_3 - \cdots - p_n)$$
we have a non-zero value *only* when
$$\begin{aligned} &p_1 - p_2 - p_3 - \cdots - p_n = 0 \\ \implies &\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3 + \cdots + \mathbf{p}_n \end{aligned}$$
or the Energy-momentum conservation.

2. In the second part
$$\delta(p_j^2 - m_j^2c^2)$$
we have a non-zero value *only* when
$$\begin{aligned} &p_j^2 - m_j^2c^2 = 0 \\ \implies &p_j^2 = m_j^2c^2 \qquad \forall j = 2, 3, \ldots, n \end{aligned}$$
which is true for all real particles (on-shell condition). If this is not true i.e. $p_j^2 \neq m_j^2c^2$ we have a virtual particle.

3. In the third part
$$\theta(p_j^0)$$
is non-zero *only* when $p_j^0 > 0$ or $E_j > 0$ (positivity of energy). So from the energy momentum relation
$$\begin{aligned} &E_j^2 = \mathbf{p}_j^2c^2 + m_j^2c^4 \\ \implies &E_j = \pm\sqrt{\mathbf{p}_j^2c^2 + m_j^2c^4} > 0 \end{aligned}$$

Evaluating the integral From the delta function
$$\int \mathrm{d}x\, \delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)]$$
so
$$\begin{aligned} \delta(p_j^2 - \mathbf{p}_j^2 - m_j^2c^2) &= \delta(p_j^0 - a^2) \quad a = \sqrt{\mathbf{p}_j^2 + m_j^2c^2} \\ &= \frac{1}{2a}[\delta(p_j^0 - a) + \delta(p_j^0 + a)] \\ &= \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right)\right] \end{aligned}$$
the second term does not contribute so we are left with
$$\int \mathrm{d}p_j^0\, \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\delta\left(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right) = \frac{\mathrm{d}^3\mathbf{p}_j}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}$$
so we have removed one of the integrals. Now we are left with the integral
$$\begin{aligned} \Gamma = \frac{S}{2m_1\hbar}\int &|\mathcal{M}|^2(2\pi)^4\delta(m_1c - p_2^0 - p_3^0 - \cdots - p_n^0) \\ &\delta^3(\mathbf{0} - \mathbf{p}_2 - \mathbf{p}_3 - \cdots - \mathbf{p}_n) \\ &\times \prod_{j=2}^{n}\frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}} \end{aligned}$$
and from the energy-momentum relation
$$\frac{E_j}{c} = \sqrt{\mathbf{p}_j^2 + m_j^2c^2}$$

Example Two-body decay $1 \to 2+3$

Sidenote: we cannot have $1 \to 2$ as it would violate the conservation of 4-momentum Since the delta function is even, $\delta(\mathbf{x}) = \delta(-\mathbf{x})$, so

$$\Gamma = \frac{S}{2m_1\hbar}\int |\mathcal{M}|^2(2\pi)^4\delta(m_1c - E_2/c - E_3/c)\delta^3(\mathbf{p}_2+\mathbf{p}_3) \times \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_2^2+m_2^2c^2}}\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3}\frac{1}{2\sqrt{\mathbf{p}_3^2+m_3^2c^2}}$$

We have nonzero values when $\mathbf{p}_2 = -\mathbf{p}_3$ and $E_2 = E_3 = \frac{m_1c}{2}$. We can use the delta function to remove the integral over $\mathbf{p}_3$ and we are left with

$$= \frac{S}{2m_1\hbar}\int |\mathcal{M}|^2(2\pi)^4\delta(m_1c - (E_2+E_3)/c)\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^6}\frac{1}{2\sqrt{\mathbf{p}_2^2+m_2^2c^2}}\frac{1}{2\sqrt{\mathbf{p}_2^2+m_3^2c^2}}$$

and now we can remove one more integral using

$$\mathrm{d}^3\mathbf{p}_2 = |\mathbf{p}_2|^2\,\mathrm{d}p_2\,\mathrm{d}\Omega \qquad \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and we also know that

$$E_2 = c\sqrt{|\mathbf{p}_2|^2+m_2^2c^2} \qquad E_3 = c\sqrt{|\mathbf{p}_2|^2+m_3^2c^2}$$

so

$$\Gamma = \frac{S}{2m_1\hbar}\frac{1}{4(2\pi)^2}\int |\mathcal{M}|^2\delta(m_1c - (E_2+E_3)/c)\frac{|\mathbf{p}_2|^2\,\mathrm{d}|\mathbf{p}_2|\,\mathrm{d}\Omega}{\sqrt{|\mathbf{p}_2|^2+m_2^2c^2}\sqrt{|\mathbf{p}_2|^2+m_3^2c^2}}$$

we know the momentums are

$$p_1 = (m_1c, \mathbf{0}) \qquad p_2 = (E_2/c, \mathbf{p}_2) \qquad p_3 = (E_3/c, -\mathbf{p}_2)$$

we can construct a scalar out of two vectors using the dot product which is always dependent on $|\mathbf{p}_2|^2$ (there is no angular dependence) so

$$|\mathcal{M}|^2(\mathbf{p}_2) = f(|\mathbf{p}_2|^2)$$

so we are left with one integral and one delta function

$$\Gamma = \frac{S}{2m_1\hbar}\frac{1}{4\pi^2 4}(4\pi)\int_0^\infty |\mathcal{M}|^2\delta(m_1c-(E_2+E_3)/c)|\mathbf{p}_2|^2\frac{\mathrm{d}|\mathbf{p}_2|}{\sqrt{|\mathbf{p}_2|^2+m_2^2c^2}\sqrt{|\mathbf{p}_2|^2+m_3^2c^2}}$$

using a change of variables we can use

$$u = \sqrt{|\mathbf{p}_2|^2+m_2^2c^2}+\sqrt{|\mathbf{p}_2|^2+m_3^2c^2}$$
$$\mathrm{d}u = \frac{2|\mathbf{p}_2|\,\mathrm{d}|\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_2|^2+m_2^2c^2}}+\frac{2|\mathbf{p}_2|\,\mathrm{d}|\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_2|^2+m_3^2c^2}}$$

and thus we get

$$\Gamma = \frac{S}{8m_1\pi\hbar}\int_{(m_2+m_3)c}^\infty |\mathcal{M}|^2\delta(m_1c-u)\,\mathrm{d}u\,\frac{|\mathbf{p}_2|^2}{u}$$
$$= \frac{S|\mathbf{p}|}{8\pi\hbar m_1^2c}|\mathcal{M}|^2$$

Quiz review

- A simple delta function integral tells us

$$\int_{a-e}^{a+e} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

- If the the non zero term is out of bounds of the integral, then the integral is zero!
- From the theta function (step function) we know that

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

  and thus

$$\theta(2x-4) = \begin{cases} 1 & x > 2 \\ 0 & x < 2 \end{cases}$$

  so we can split the integral from $-1 \to 2$ and $2 \to 5$ and we get

$$\int_{-1}^{2} 0e^{-3x}\,\mathrm{d}x = 0$$
$$\begin{aligned} \int_{2}^{5} \theta(2x-4)e^{-3x}\,\mathrm{d}x &= \int_{2}^{5} e^{-3x}\,\mathrm{d}x \\ &= -\frac{1}{3}e^{-3x}\bigg|_{2}^{5} \end{aligned}$$

- For integration over a sphere we can just find if the magnitude of distance is less than the radius of the sphere 1.5:

$$|(2,2,2)-(3,2,1,)| = \sqrt{2} \approx 1.4 < 1.5$$

  so we find the function

$$\begin{aligned} \oint \mathrm{d}V\,\mathbf{r}\cdot(\mathbf{a}-\mathbf{r})\delta^3(\mathbf{r}-\mathbf{b}) &= \int \mathrm{d}V\,f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{b}) \\ &= f(\mathbf{b}) \end{aligned}$$

  which is

$$\begin{aligned} f(\mathbf{b}) &= \mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) \\ &= (3,2,1)\cdot[(1,2,3)-(3,2,1)] \\ &= -4 \end{aligned}$$

- The decay rate using dimensional analysis from last time

$$\Gamma = \frac{1}{[\mathrm{J\,s\,kg^2\,m/s}]}\cdot \mathrm{kg\,m/s}\cdot\mathcal{M}$$

  and since $\mathrm{J} = \mathrm{kg\,m^2/s^2}$ we can see that the amplitude has units of $\mathrm{kg\,m/s}$ or momentum. Thus the number of particles involved is the only thing that is dependent on the number of particles involved.

Scattering

($2 \to n$ Scattering)

$$1 + 2 \to 3 + 4 + \cdots + n$$

the cross section is given by

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \cdots - p_n) \times \prod_{j=3}^{n} \frac{\mathrm{d}^4\mathbf{p}_j}{(2\pi)^4} (2\pi)\delta(p_j^2 - m_j^2c^2)\theta(p_j^0)$$

From momentum conservation we have

$$p^2 = (p^0)^2 - \mathbf{p}^2$$

so the delta function can be rewritten as

$$\delta(p_j^2 - m_j^2c^2) = \delta((p_j^0)^2 - \mathbf{p}_j^2 - m_j^2c^2)$$

and using the same trick as last time we can split

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)]$$

or in the general form

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

so defining =

$$x = p_j^0 \qquad a = \sqrt{\mathbf{p}_j^2 + m_j^2c^2}$$

we can rewrite the delta function as

$$\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\left[\delta\left(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right) + \delta\left(p_j^0 + \sqrt{\mathbf{p}_j^2 + m_j^2c^2}\right)\right]$$

and we can remove the second term becase the theta function removes negative energies! So we are left with

$$\frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}\delta(p_j^0 - \sqrt{\mathbf{p}_j^2 + m_j^2c^2})$$

Now we we are left with an integral

$$\int \frac{\mathrm{d}p_j^0}{\cancel{(2\pi)}}\cancel{(2\pi)}\delta(p_j^2 - m_j^2c^2)\theta(p_j^0)f(p_j^0) = f(\sqrt{\mathbf{p}_j^2 + m_j^2c^2})$$

which removes the zeroth component of the 4-momentum in the original integral which leaves us with

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - \cdots - p_n) \prod_{j=3}^{n} \frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2c^2}}$$

with

$$p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2c^2} = \frac{E_j}{c}$$

2 - 2 Scattering $1 + 2 \to 3 + 4$
In the center of mass frame the total 3-momentum is zero (HW In the lab frame with one particle at rest initially i.e. $p_2 = (m_2 c, \mathbf{0})$). We have two momenta of the *beam* of particles (LHC)

$$p_1 = (E_1/c, \mathbf{p}_1) \qquad p_2 = (m_2 c, \mathbf{p}_2)$$

where

$$p_1 + p_2 = \mathbf{0} \implies \mathbf{p}_1 = -\mathbf{p}_2$$

which means

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = \frac{|\mathbf{p}_1|^2}{c}\sqrt{S} \qquad S = (E_1 + E_2)^2$$

where S is the Mandelstam variable. So the cross section is

$$\sigma = \frac{S\hbar^2}{4\frac{|\mathbf{p}_1|^2}{c}\sqrt{S}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$
$$\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{p}_4}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} \frac{1}{2\sqrt{\mathbf{p}_4^2 + m_4^2 c^2}}$$

and we can remove the delta function by using the energy-momentum relation

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta\left(\frac{E_1 + E_2}{c} - \frac{E_3 + E_4}{c}\right)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

but since the total momentum is zero i.e.

$$\mathbf{p}_1 + \mathbf{p}_2 = 0$$

we can replace the $\mathrm{d}^3\mathbf{p}_4$ with the $\mathrm{d}^3\mathbf{p}_3 + \mathbf{p}_4$ and we are left with

$$\sigma = \frac{S\hbar^2}{4\frac{|\mathbf{p}_1|^2}{c}\sqrt{S}} \int |\mathcal{M}|^2 (2\pi)^4 \delta\left(\frac{E_1 + E_2}{c} - \frac{E_3 + E_4}{c}\right)$$
$$\frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3} \frac{1}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}} \frac{1}{2\sqrt{\mathbf{p}_3^2 + m_4^2 c^2}}$$

and since

$$\mathrm{d}^3\mathbf{p}_3 = |\mathbf{p}_3|^2 \,\mathrm{d}|\mathbf{p}_3| \,\mathrm{d}\Omega \qquad \mathrm{d}\Omega = \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$$

We know that

$$\begin{aligned} \mathbf{p}_4 &= -\mathbf{p}_3 \\ \implies E_4 = \sqrt{\mathbf{p}_4^2 c^2 + m_4^2 c^4} &= \sqrt{\mathbf{p}_3^2 c^2 + m_4^2 c^4} \end{aligned}$$

so we can represent

$$\begin{aligned} |\mathcal{M}|^2(p_1, p_2, p_3, p_4) &= |\mathcal{M}|^2(p_3, p_4) \\ &= |\mathcal{M}|^2(\mathbf{p}_3, \theta, \phi) \end{aligned}$$

which can't be written as a function of $|\mathbf{p}_3|$ so we must use the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{S\hbar^2}{4|\mathbf{p}_1^0|\sqrt{S}} \frac{1}{(2\pi)^4} \frac{1}{4} \int |\mathcal{M}|^2 \delta\left(\frac{E_1 + E_2}{c} - \frac{E_3 + E_4}{c}\right)$$
$$|\mathbf{p}_3|^2 \,\mathrm{d}|\mathbf{p}_3| \frac{1}{\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}\sqrt{\mathbf{p}_3^2 + m_4^2 c^2}}$$

using the change of variables we can use

$$\begin{aligned}
u &= \frac{E_3 + E_4}{c} \\
&= \sqrt{\mathbf{p}_3^2 + m_3^2c^2} + \sqrt{\mathbf{p}_3^2 + m_4^2c^2} \\
\mathrm{d}u &= \frac{2|\mathbf{p}_3|\,\mathrm{d}|\mathbf{p}_3|}{2\sqrt{\mathbf{p}_3^2 + m_3^2c^2}} + \frac{2|\mathbf{p}_3|\,\mathrm{d}|\mathbf{p}_3|}{2\sqrt{\mathbf{p}_3^2 + m_4^2c^2}} \\
&= |\mathbf{p}_3|\,\mathrm{d}|\mathbf{p}_3|\,\frac{u}{\sqrt{\mathbf{p}_3^2 + m_3^2c^2}\sqrt{\mathbf{p}_3^2 + m_4^2c^2}}
\end{aligned}$$

which is the last part of the integral So

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{S\hbar^2}{4|\mathbf{p}_1^0|\sqrt{S}}\frac{1}{(2\pi)^4}\frac{1}{4}\int |\mathcal{M}|^2\delta\left(\frac{E_1+E_2}{c} - \frac{E_3+E_4}{c}\right)\mathrm{d}u\,\frac{1}{u}|\mathbf{p}_3| \\
&= \frac{S\hbar^2 c}{64\pi^2|\mathbf{p}_1|(E_1+E_2)}\frac{|\mathcal{M}|^2|\mathbf{p}_3|}{\frac{E_1+E_2}{c}} \\
&= \left(\frac{hc}{8\pi}\right)^2\frac{S|\mathcal{M}|^2}{(E_1+E_2)^2}\frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}
\end{aligned}$$

We find that this cross section is proportional to many things:

$$\sigma \propto \frac{1}{S}, \qquad \sigma \propto \frac{|p_f|}{|p_i|}$$

But why use the collider like this?

- In the past we used $\sqrt{S} = 91\,\mathrm{GeV}$ (LEP)
- $\sqrt{S} = 1.96\,\mathrm{TeV}$ (Tevatron)
- $\sqrt{S} = 13.6\,\mathrm{TeV}$ (LHC)
- $\sqrt{S} = 100\,\mathrm{TeV}$ (FCC/SPPC)

But we can only find the cross section to grow with S if $|\mathcal{M}|^2$ is independent of S.‘’

Quiz Review

Feynman Rules

QED: $e^{\pm}, \gamma$

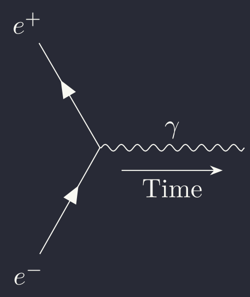


Figure 5.1: Not allowed

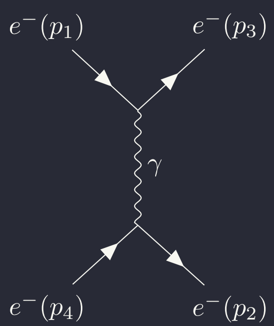


Figure 5.2: Allowed

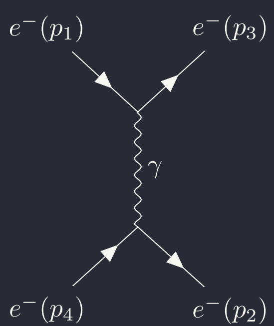


Figure 5.3: Allowed

is not allowed, but (diagram 3) is allowed only if the initial electron has some KE. But also there is (diagram 4) due to the symmetry.

Notation

- Fermion: solid line with a forward arrow for the direction of the particle, and a backward arrow for the antiparticle.
- Photon: wavy line

- Gluon: springy line
- W/Z boson: triangle wave
- Higgs: dashed line

Rules

- Label the external momenta as p_i and interntal momenta as q_i
- For the Vertex, insert a factor of $-ig$
- Propogator: For each internal line write a factor of

$$\frac{i}{q_j^2 - m_j^2 c^2}$$

  (For Virtual particles $q_j^2 \neq m_j^2 c^2$)

- 4-momentum conservation: For each vertex, write a factor of $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$, where k_i's are momenta flowing into the vertex. (Total momentum is zero)
- Integrate over all interal momenta

$$\prod_i \int \frac{\mathrm{d}^4 q_i}{(2\pi)^4}$$

- Drop $(2\pi)^4 \delta^4(p_1 + p_2 + \cdots - p_m - \cdots - p_n)$
- Multiply the final result by i

Example Decay $A \to B + C$

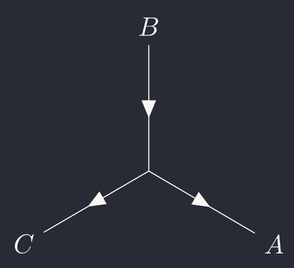


Figure 5.4: Decay

$$\mathcal{M} = i(-ig)\cancel{(2\pi)^4 \delta^4(p_1 - p_2 - p_3)} = g$$

The decay rate is

$$\begin{aligned}
\Gamma &= \frac{S}{8\pi m_1^2 \hbar c} |\mathcal{M}|^2 |\mathbf{p}_B| \\
&= \frac{S}{8\pi m_A^2 \hbar c} g^2 |\mathbf{p}_B|
\end{aligned}$$

$$|\mathbf{p}_B| = \frac{c}{2m_A} \lambda^{1/2}(m_A^2, m_B^2, m_C^2)$$

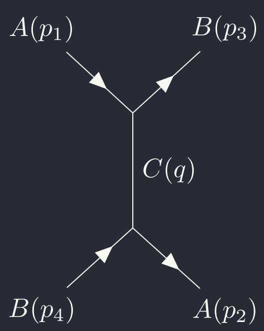


Figure 5.5: Diagram 1

Example: 2-2 Scattering $A + A \to B + B$ For the first diagram we can take the virtual particle direction to be upwards, we get the amplitude

$$\mathcal{M}_1 = i \int (-ig)^2 \frac{i}{q^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + q - q_3)(2\pi)^4 \delta^4(p_2 - q - p_4) \frac{\mathrm{d}^4 q}{(2\pi)^4}$$

getting rid of the integral with the first delta function

$$= g^2 \frac{1}{(p_3 - p_1)^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_2 - (p_3 - p_1) - p_4)$$

we can drop the factors using rule 6:

$$= \frac{g^2}{(p_3 - p_1)^2 - m_C^2 c^2}$$

From the second diagram we get

$$\mathcal{M}_2 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2 c^2}$$

and the total amplitude is

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \frac{g^2}{t^2 - m_C^2 c^2} + \frac{g^2}{u^2 - m_C^2 c^2}$$

where

$$t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2, \qquad s = (p_1 + p_2)^2$$

And to find the cross section we use

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

and in the center of mass frame we have

$$m_A = m_B = m, \quad m_C = 0$$

and from the conservation of momentum and energy momentum relation

$$\begin{aligned}(p_1 - p_3)^2 &= p_1^2 + p_3^2 - 2p_1 p_3 \\ &= m_A^2 c^2 + m_B^2 c^2 - 2\left(\frac{E_1 E_3}{c^2} - \mathbf{p}_1 \cdot \mathbf{p}_3\right) \\ &= (m_A^2 + m_B^2)c^2 - 2\left(\frac{E_1 E_3}{c^2} - |\mathbf{p}_1||\mathbf{p}_3| \cos\theta\right)\end{aligned}$$

where

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0} \implies |\mathbf{p}_1| = |\mathbf{p}_2|$$
$$\mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0} \implies |\mathbf{p}_3| = |\mathbf{p}_4|$$

so from the energy conservation

$$E_1 + E_2 = E_3 + E_4$$

$$\implies \sqrt{m^2c^4 + |\mathbf{p}_1|^2 c^2} + \sqrt{m^2c^4 + |\mathbf{p}_2|^2 c^2} = \sqrt{m^2c^4 + |\mathbf{p}_3|^2 c^2} + \sqrt{m^2c^4 + |\mathbf{p}_4|^2 c^2}$$

so the energies are equivalent and thus we can simplify

$$(p_1 - p_3)^2 = 2m^2c^2 - 2\left(\frac{E^2}{c^2} - |\mathbf{p}|^2 \cos\theta\right)$$

and using

$$E^2 = m^2c^4 + |\mathbf{p}|^2 c^2$$
$$\implies \frac{E^2}{c^2} = m^2c^2 + |\mathbf{p}|^2$$

we finally get

$$(p_1 - p_3)^2 = -2|\mathbf{p}|^2(1 - \cos\theta)$$

and for the second diagram we get

$$(p_1 - p_4)^2 = -2|\mathbf{p}|^2(1 + \cos\theta)$$

Back to the total amplitude

$$\mathcal{M} = \frac{g^2}{-2|\mathbf{p}|^2}\left(\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\right)$$
$$= -\frac{g^2}{|\mathbf{p}|^2 \sin^2\theta}$$

and now we can find the cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{2}\frac{1}{4E^2}\frac{g^4}{|\mathbf{p}|^4 \sin^4\theta}$$

integrating

$$\sigma \propto \int_0^\pi \frac{\sin\theta \,\mathrm{d}\theta}{\sin^4\theta} \to \infty$$

The mediator of the force is the photon C, a massless mediator, which is why the cross section is infinite. In the homework we will see that this will go to $\propto \frac{1}{m_C^4}$.

Vacuum Polarization We can have multiple loops in the diagrams and we would get

$$\mathcal{M} = \int \frac{\mathrm{d}^4 q}{q^4} = \int_0^\infty \frac{q^3 \,\mathrm{d}q}{q^4} = \infty$$

and to get rid of this we use *Regularization* i.e.

$$\int^M \frac{\mathrm{d}q}{q} = \ln(M)$$

known as the Cut-off scale. We will get finite quantities

$$m = m_0 + \delta m$$
$$g = g_0 + \delta g$$

6 Quantum Electrodynamics

Quiz Review

Schrodinger Equation

$$E = \frac{\mathbf{p}^2}{2m} + v \to \left(i\hbar\frac{\partial}{\partial t}\right)\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + v\right)\psi$$

where we apply

$$\mathbf{p} \to -i\hbar\boldsymbol{\nabla}$$
$$E \to i\hbar\frac{\partial}{\partial t}$$

and

$$p_\mu \to i\hbar\partial_\mu$$
$$p_0 = \frac{E}{c} \to i\hbar\frac{\partial}{\partial t}$$

Relativistic Equation

$$E^2 = \mathbf{p}^2c^2 + m^2c^4$$

so

$$\begin{aligned} p_\mu p^\mu &= m^2c^2 \\ [(i\hbar\partial^\mu)(i\hbar\partial_\mu)]\psi &= m^2c^2\psi \\ \implies (-\hbar^2\partial^\mu\partial_\mu)\psi &= m^2c^2\psi \\ \implies -\partial^\mu\partial_\mu\psi &= \frac{m^2c^2}{\hbar^2}\psi \\ \implies -\Box\psi &= \frac{m^2c^2}{\hbar^2}\psi \end{aligned}$$

Which is the *Klein-Gordan* equation where the box operator is the d'Alembertian operator. This only describes spin-0 particles as a 2nd order in time derivative.

Dirac Equation To describe spin-1/2 particles, we need a relativistic wave eqn in the 1st order in time. Setting $\mathbf{p} = 0$ or at rest,

$$\begin{aligned} \partial^\mu\partial_\mu \to \partial^0\partial_0 &= \frac{\partial^2}{\partial t^2} \\ p_\mu &= (p_0, \mathbf{0}) \\ p^\mu p_\mu &= m^2c^2 \\ \text{or} \quad p^0p_0 = m^2c^2 \quad \text{if} \quad \mathbf{p} &= 0 \\ \text{or} \quad p_0^2 - m^2c^2 &= 0 \\ \text{or} \quad (p^0 + mc)(p^0 - mc) &= 0 \\ \text{or} \quad p^0 &= \pm mc \\ \text{or} \quad i\hbar\frac{\partial}{\partial t}\psi &= \pm mc\psi \end{aligned}$$

which gives us the Plane wave solution

$$\psi(t) = \psi(0)e^{\pm i\frac{mc^2}{\hbar}t}$$

If $\mathbf{p} \neq 0$, then from the dirac equation

$$p^\mu p_\mu - m^2c^2 = 0$$

and writing into the form

$$(\beta_K p^K + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

expanding out the terms

$$\beta_K \gamma^\lambda p^K p_\lambda - mc(\beta_K p^K - \gamma^\lambda p_lambda) - m^2c^2 = 0$$

and since we are using dummy indices:

$$\gamma^\lambda p_\lambda = \gamma_\lambda p^\lambda = \gamma_k p^K$$

or

$$\beta_K \gamma^\lambda p^K p_\lambda - mc(\beta_K - \gamma_k)p^K - m^2c^2 = 0$$

and comparing with

$$p^\mu p_\mu - m^2c^2 = 0$$

which implies that the linear term in p^μ must be zero

$$\implies \beta_K = \gamma_K$$

so the equation becomes

$$\gamma_K \gamma^\lambda p^K p_\lambda - m^2c^2 = 0$$

So we have the terms

$$\begin{aligned}\implies \gamma_0\gamma^0 p^0 p^0 + \gamma_1\gamma^1 p^1 p_1 + \gamma_2\gamma^2 p^2 p_2 + \gamma_3\gamma^3 p^3 p^3 \\ +\gamma_0\gamma^1 p^0 p_1 + \cdots - m^2c^2 = 0\end{aligned}$$

The LHS should be equal to

$$p^0p_0 + p^1p_1 + p^2p_2 + p^3p_3 - m^2c^2 = 0$$

so

$$\gamma_0\gamma^0 = \gamma_1\gamma^1 = \gamma_2\gamma^2 = \gamma_3\gamma^3 = 1$$

and the cross terms should be zero

$$\gamma_0\gamma^1 = \gamma_1\gamma^0 = \gamma_2\gamma^3 = \gamma_3\gamma^2 = 0$$

and

$$\begin{aligned}\gamma_0 &= g_{0\mu}\gamma^\mu \\ &= g_{00}\gamma^0 = \gamma^0 \\ \gamma_1 &= g_{11}\gamma^1 = -\gamma^1\end{aligned}$$

so

$$
\begin{aligned}
(\gamma^0)^2 = -(\gamma^j)^2 = 1 \quad (j = 1, 2, 3) \\
\gamma^\mu \gamma^\nu = 0 \quad \text{if} \quad \mu \neq \nu \\
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}
\end{aligned}
$$

We can try $\gamma^0 = 1$ and $\gamma^j = i$ but it does not satisfy the third anticommutator relation. so γs have to be matrices, or more specifcally a 4×4 matrix (Dirac matrices). We obtain the Dirac equation we take out one of the terms

$$
\begin{aligned}
(\gamma_K p^K + mc)(\gamma^\lambda p_\lambda - mc) = 0 \\
\implies \gamma_K p^K \pm mc = 0
\end{aligned}
$$

and from the relativistic relation

$$
p^K \to i\hbar \partial^K
$$

we get the Dirac equation

$$
(i\hbar \gamma^K \partial_K \pm mc)\psi = 0
$$

where we can interchange the indices, i.e.,

$$
\gamma^K \partial_K = \gamma^\lambda \partial_\lambda = \gamma_\mu \partial^\mu
$$

Since γ^μ is a 4×4 matrix, we have a Dirac spinor

$$
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}
$$

Using the Dirac basis

$$
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}
$$

where the matrices are in spinor space and not in Lorentz space.

Solution to the Dirac Equation for

$$
(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0
$$

First consider the rest case $\mathbf{p} = 0$ so

$$
\begin{aligned}
(i\hbar \gamma^0 \partial_0 - mc)\psi = 0 \\
\implies \left(i\hbar \gamma^0 \frac{1}{c} \frac{\partial}{\partial t} - mc \right) \psi = 0 \\
\implies \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \psi = -\frac{imc^2}{\hbar} \psi
\end{aligned}
$$

and we write *psi* as a two components

$$
\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \quad \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}
$$

so

$$
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \psi_A \\ \frac{\mathrm{d}}{\mathrm{d}t} \psi_B \end{pmatrix} = -\frac{imc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}
$$

which splits into two equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_A = -\frac{imc^2}{\hbar}\psi_A$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_B = \frac{imc^2}{\hbar}\psi_B$$

so the solutions are

$$\psi_A(t) = \psi_A(0)e^{-\frac{imc^2}{\hbar}t}$$
$$\psi_B(t) = \psi_B(0)e^{\frac{imc^2}{\hbar}t}$$

Which is similar to the plane wave solution in QM that has the factor

$$\psi(t) = \psi(0)e^{-i\frac{E}{\hbar}t}$$

but here the energy is $E = mc^2$. psi_A is the particle solution and psi_B is the antiparticle solution. For the vectors

$$\psi_A = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad \psi_B = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$$

we have two basis vectors

$$\begin{pmatrix}1\\ 0\end{pmatrix} \quad \begin{pmatrix}0\\ 1\end{pmatrix}$$

so

$$\psi_A(t) = e^{-imc^2t/\hbar}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e^{-imc^2t/\hbar}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

which corresponds to the electron spin up and spin down states. We also have

$$\psi_B(t) = e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e^{imc^2t/\hbar}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

For the positron spin up and spin down states respectively.

General Case $\mathbf{p} \neq 0$

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

We would expect a plane wave solution of the form

$$\psi(x) = e^{-ip\cdot x/\hbar}\psi(0)$$

or

$$\psi = ae^{-ik\cdot x/\hbar}u(k)$$

where $u(k)$ is the spinor part, and a is the normalization constant. The derivative of the exponent will give

$$\partial_\mu\psi = -ik_\mu\psi$$

and back into the dirac equation

$$(i\hbar\gamma^\mu(-ik_\mu) - mc)\psi = 0$$
$$\text{or} \quad \left(\gamma^\mu k_m - \frac{mc}{\hbar}\right)u = 0$$

where we know that

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

so using

$$\begin{aligned}\gamma^\mu k_\mu &= \gamma^0 k_0 - \gamma^j k^j \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} k_0 - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} k^j \\ &= \begin{pmatrix} k_0 1 & -\sigma \cdot \mathbf{k} \\ \sigma \cdot \mathbf{k} & -k_0 1 \end{pmatrix}\end{aligned}$$

where we define a Weyl spinor

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

So we get

$$\begin{pmatrix} k_0 - \frac{mc}{\hbar} & -\sigma \cdot \mathbf{k} \\ \sigma \cdot \mathbf{k} & -k_0 - \frac{mc}{\hbar} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

which gives us the coupled equations

$$(k_0 - \frac{mc}{\hbar})u_A - \sigma \cdot \mathbf{k} u_B = 0$$
$$\sigma \cdot \mathbf{k} u_A - (k_0 + \frac{mc}{\hbar})u_B = 0$$

and we can solve this by solving for u_A in the first equation and substituting into the second equation

$$u_A = \frac{\sigma \cdot \mathbf{k}}{k_0 - \frac{mc}{\hbar}} u_B$$

and substituting the second eq to the first

$$\begin{aligned}u_B &= \frac{\sigma \cdot \mathbf{k}}{k_0 + \frac{mc}{\hbar}} u_A \\ &= \frac{\sigma \cdot \mathbf{k}}{k_0 + \frac{mc}{\hbar}} \frac{\sigma \cdot \mathbf{k}}{k_0 - \frac{mc}{\hbar}} u_B \\ &= \frac{(\sigma \cdot \mathbf{k})^2}{(k^0)^2 - (\frac{mc}{\hbar})2} u_A\end{aligned}$$

where

$$(\sigma \cdot \mathbf{k})^2 = (k^0)^2 - \left(\frac{mc}{\hbar}\right)^2$$
$$\implies \mathbf{k}^2 = (k^0)^2 - \left(\frac{mc}{\hbar}\right)^2$$

and from the relativistic relation

$$k^2 = k^\mu k_\mu = (k^0)^2 - \mathbf{k}^2 = \left(\frac{mc}{\hbar}\right)^2$$

this tells us that $\hbar k_\mu$ must be the momentum p_μ

Quiz Review

- From the Dirac Spinor

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

we have the particles represented by the wavefunction ψ_A and the antiparticles represented by the wavefunction ψ_B.

$$\psi_A = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad \psi_B = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$$

- Weyl Spinors describe either the particle or antiparticle

$$\psi = \begin{pmatrix}u_A\\ u_B\end{pmatrix}$$

When the particle is the same as the antiparticle, then the two spinors are related by

$$u_A = i\sigma^2 u_B^*$$

so the dirac spinor becomes

$$\psi = C\psi^*$$

where C is the charge conjugation (Majorana fermion).

- Dirac matrices must be atleast 4 dimensional.

Solutions to the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

General Solution

$$\psi = ae^{-ik\cdot x}u(k)$$

where $u(k)$ is the spinor part. Since

$$\begin{aligned} k^2 = \left(\frac{mc}{\hbar}\right)^2 \implies (\hbar k)^2 = (mc)^2 = p^2 \\ \implies \hbar k = \pm p \\ \text{or} \quad k_\mu = \pm\frac{p_\mu}{\hbar} \end{aligned}$$

where $+$ is the particle solution and $-$ is the antiparticle solution. The zeroth component would be

$$k^0 = \pm\frac{p^0}{\hbar} = \pm\frac{E^0}{\hbar}$$

so

$$\begin{aligned} \psi &\propto e^{-ip\cdot x/\hbar} \\ &= e^{\mp ip\cdot x/\hbar} \\ &\to \begin{cases} e^{-ip\cdot x/\hbar}\psi_A & \text{Particle} \\ e^{ip\cdot x/\hbar}\psi_B & \text{Antiparticle} \end{cases} \end{aligned}$$

From the solutions

$$u_A = \frac{\sigma\cdot\mathbf{k}}{k^0 - \frac{mc}{\hbar}}u_B$$
$$u_B = \frac{\sigma\cdot\mathbf{k}}{k^0 + \frac{mc}{\hbar}}u_A$$

Solution 1 $u^{(1)}$ If we choose a solution $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (Choosing u_A to be the particle solution), then

$$\begin{aligned} u_B &= \frac{\sigma \cdot \mathbf{p}/\hbar}{p^0/\hbar + mc/\hbar} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\sigma \cdot \mathbf{p}}{p^0 + mc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

and using

$$\begin{aligned} \sigma \cdot \mathbf{p} &= \sigma^1 p_x + \sigma^2 p_y + \sigma^3 p_z \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z \\ &= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \end{aligned}$$

we get

$$u_B = \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \frac{c}{E + mc^2}$$

Solution 2 $u^{(2)}$ If we choose $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$u_B = \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \frac{c}{E + mc^2}$$

Solution 3 $v^{(1)}$ The third solution is to choose $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then (choosing the minus sign for k)

$$\begin{aligned} u_A &= \frac{-\sigma \cdot \mathbf{p}}{-p^0 - mc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\sigma \cdot \mathbf{p}}{p^0 + mc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \frac{c}{E + mc^2} \end{aligned}$$

Solution 4 $v^{(1)}$ and similarly for $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_A = \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \frac{c}{E + mc^2}$$

So with the 4 solutions for $u(k)$

$$u^{(1)} = Ne^{-ip\cdot x\hbar}\begin{pmatrix} 1 \\ 0 \\ \frac{c}{E+mc^2}p_z \\ \frac{c}{E+mc^2}(p_x+ip_y) \end{pmatrix}$$
$$u^{(2)} = Ne^{-ip\cdot x\hbar}\begin{pmatrix} 0 \\ 1 \\ \frac{c}{E+mc^2}(p_x-ip_y) \\ -\frac{c}{E+mc^2}p_z \end{pmatrix}$$
$$v^{(2)} = Ne^{ip\cdot x\hbar}\begin{pmatrix} \frac{c}{E+mc^2}p_z \\ \frac{c}{E+mc^2}(p_x+ip_y) \\ 1 \\ 0 \end{pmatrix}$$
$$v^{(1)} = Ne^{ip\cdot x\hbar}\begin{pmatrix} \frac{c}{E+mc^2}(p_x-ip_y) \\ -\frac{c}{E+mc^2}p_z \\ 0 \\ 1 \end{pmatrix}$$

where $u^{(1)}$ is the particle with spin up, $u^{(2)}$ is the particle with spin down, $v^{(1)}$ is the antiparticle with spin down, and $v^{(2)}$ is the antiparticle with spin up. So

$$\psi = \begin{cases} ae^{-ip\cdot x/\hbar}u^{(1)} \quad \text{or} \quad u^{(2)} \quad \text{Particle} \\ ae^{ip\cdot x/\hbar}v^{(1)} \quad \text{or} \quad v^{(2)} \quad \text{Antiparticle} \end{cases}$$

where $\psi^\dagger\psi = 1$ or

$$u^\dagger u = \frac{2E}{c} \quad \text{or} \quad v^\dagger v = \frac{2E}{c}$$

Nonrelativistic Limit

$$\mathbf{p} = m\mathbf{v}, \quad E \approx mc^2$$

so

$$\frac{c}{E+mc^2}p_z \approx \frac{c}{2mc^2}mv_z = \frac{v_z}{2c} \to 0$$

and

$$\frac{c}{E+mc^2}(p_x+ip_y) \approx \frac{v_x+iv_y}{2c} \to 0$$

for $v \ll c$.

Dirac Equation in momentum space

$$\begin{aligned} (i\hbar\gamma^\mu\partial_\mu - mc)\psi &= 0 \\ p_\mu &\to i\hbar\partial_\mu \\ (\gamma^\mu p_\mu - mc)u &= 0 \quad \text{Particle} \\ (\gamma^\mu p_\mu + mc)v &= 0 \quad \text{Antiparticle} \end{aligned}$$

we usually use the notation $\not{p} = \gamma^\mu p_\mu$

Spin Operator

$$\mathbf{S} = \frac{\hbar}{2}\mathbf{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$
$$S^2 = \frac{\hbar^2}{4}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

since $\sigma_i^2 = I$, $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3I$ so

$$\begin{aligned} S^2 &= \frac{3\hbar^2}{4} I \\ &= \frac{1}{2}\left(\frac{1}{2} + 1\right)\hbar^2 I \\ &= S(S+1)\hbar^2 I \\ \implies S &= \frac{1}{2}\hbar \end{aligned}$$

So for the total spin

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

then

$$[H, \mathbf{J}] = 0$$

from the Heisenberg equation of motion.

$$\frac{\mathrm{d}0}{\mathrm{d}t} = i\hbar[H, 0] = 0$$

We find that

$$[H, \mathbf{L}] \neq 0$$
$$[H, \mathbf{S}] \neq 0$$

or simply

$$[H, \mathbf{L}] = -[H, \mathbf{S}]$$

The Hamiltonion or total energy is

$$H = cp^0 = c\sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$$

from the Dirac equation in momentum space, we know that

$$\begin{aligned} \gamma^\mu p_\mu - mc &= 0 \\ \text{or} \quad \gamma^0 p_0 + \gamma^j p_j - mc &= 0 \\ \text{or} \quad \gamma^0 p_0 &= -\gamma^j p_j + mc \end{aligned}$$

We know that

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

squaring γ^0 gives the identity matrix so

$$\begin{aligned} (\gamma^0)^2 p_0 &= -\gamma^0\gamma^j p_j + mc \\ p_0 &= -\gamma^0\gamma^j p_j + mc \\ &= \gamma^0\gamma^j p^j - mc \end{aligned}$$

Thus

$$H = c(\gamma^0 \boldsymbol{\gamma} \cdot \mathbf{p} + mc)$$

Lorentz Transformation Properties Starting again from the Dirac Equation

$$(i\hbar\gamma^v\partial_v - mc)\psi = 0$$

Under Lorentz transformation we let

$$\psi \to \psi' = S\psi$$
$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi' = 0$$

We denote

$$\partial'_\mu = \frac{\partial x^v}{\partial x'_\mu}\partial x^v \equiv \frac{\partial x^v}{\partial x'_\mu}\partial_v$$
$$x^\mu \to^\Lambda x'^\mu$$

We also have axioms for the Dirac matrices (they don't transform) so

$$\left(i\hbar\gamma^\mu\frac{\partial x^v}{\partial x'_\mu}\partial_v - mc\right)S\psi' = 0$$

Mutliplying by S^{-1} on the left

$$S^{-1}i\hbar\gamma^\mu\frac{\partial x^v}{\partial x'_\mu}S(\partial_v\psi) - mc\psi = 0$$

so matching this to the original Dirac equation

$$S^{-1}\gamma^\mu\frac{\partial x^v}{\partial x'_\mu}S = \gamma^v$$

Since the S commutes we can write

$$S^{-1}\gamma^\mu S\frac{\partial x^v}{\partial x'_\mu} = \gamma^v$$

If the frame moves along the x-axis,

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where we have the lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

we can show that

$$S = a_+ + a_-\gamma^0\gamma^1 \qquad a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$$

In matrix form

$$S = a_+I + a_-\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}$$

where I is a 4x4 identity matrix and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & -a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{pmatrix}$$

a symmetric real matrix

Invariant

$$\psi \to |\psi|^2 = \psi^\dagger \psi = 1$$

For invariant quantities in non relativistic QM. For relativistic transformations

$$\begin{aligned} \psi^\dagger \psi \to^\Lambda (S\psi)^\dagger (S\psi) \\ = \psi^\dagger S^\dagger S \psi \end{aligned}$$

But since S is real and symmetric $S^\dagger = S$ So

$$= \psi^\dagger S^2 \psi$$

We find that

$$S^2 = \gamma \begin{pmatrix} I & -\frac{v}{c}\sigma_1 \\ -\frac{v}{c}\sigma_1 & I \end{pmatrix} \neq I$$

So we need to find the lorentz invariant quanities (Bilinears)

Bilinears Under Lorentz transformations

$$\begin{aligned}\psi^\dagger\psi &\to \psi'^\dagger\psi' \\ &= (S\psi)^\dagger(S\psi) = \psi^\dagger S^\dagger S\psi\end{aligned}$$

Since S is real and symmetric, $S^\dagger = S$ or $S^\dagger S = S^2$ where

$$S^2 = \gamma\begin{pmatrix} I & -\frac{v}{c}\sigma_1 \\ -\frac{v}{c}\sigma_1 & I\end{pmatrix} \neq I$$

For $v \ll c$, $\quad S^2 \to I$. So we need to find a Lorentz invariant quantity, or the adjoint spinor, $\bar{\psi} = \psi^\dagger\gamma^0$:

$$\begin{aligned}\bar{\psi}\psi \to \bar{\psi}'\psi' &= (\psi'^\dagger\gamma^0)\psi' = (S\psi)^\dagger\gamma^0 S\psi \\ &= \psi^\dagger S^\dagger\gamma^0 S\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi\end{aligned}$$

This is one of the bilinears ($\bar{\psi}\psi$) that is lorentz invariant.

Discrete Symmetry Operators

$$\begin{aligned}P &= \gamma^0 \qquad \psi(t,\mathbf{x}) \to \gamma^0\psi(t,-\mathbf{x}) \\ C &= i\gamma^2 \qquad \psi(t,\mathbf{x}) \to i\gamma^2\psi^*(t,\mathbf{x}) \\ T &= \gamma^1\gamma^3 \qquad \psi(t,\mathbf{x}) \to \gamma^1\gamma^3\psi(-t,\mathbf{x})\end{aligned}$$

So under parity

$$\begin{aligned}\bar{\psi}\psi \to^P \bar{\psi}'\psi' &= \psi'^\dagger\gamma^0\psi' \\ &= (\gamma^0\psi)^\dagger\gamma^0(\gamma^0\psi) \\ &= \psi^\dagger\gamma^{0\dagger}(\gamma^0\gamma^0)\psi \qquad \gamma^0\gamma^0 = I \\ &= (\psi^\dagger\gamma^0)\psi = \bar{\psi}\psi\end{aligned}$$

which is P-even. For The pseudoscalar

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

where

$$\begin{aligned}\{\gamma^\mu,\gamma^v\} &= 2g^{\mu v} \\ \implies \{\gamma^\mu,\gamma^v\} &= 0 \quad \text{if} \quad \mu \neq v\end{aligned}$$

and

$$\{\gamma^\mu, i\gamma^0\gamma^1\gamma^2\gamma^3\} = i(\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu)$$

And at $\mu = 0$:

$$\begin{aligned}&= i(\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0) \\ &= i(\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3) = 0 \qquad \gamma^\mu\gamma^v = -\gamma^v\gamma^\mu\end{aligned}$$

So

$$\{\gamma^\mu,\gamma^5\} = 0$$

So the Parity operator

$$\begin{aligned}\bar{\psi}\gamma^5\psi \to^P \bar{\psi}'\gamma^5\psi' &= \psi'^\dagger\gamma^0\gamma^5\psi' \\ &= (\gamma^0\psi)^\dagger\gamma^0\gamma^5(\gamma^0\psi) \\ &= \psi^\dagger\gamma^{0\dagger}\gamma^0\gamma^5\gamma^0\psi \\ &= \psi^\dagger\gamma^5\gamma^0\psi = -\bar{\psi}\gamma^5\psi\end{aligned}$$

which is P-odd. Thus the Bilinears are

1. $\bar{\psi}\psi$: P-even (scalar)
2. $\bar{\psi}\gamma^5\psi$: P-odd (pseudoscalar)
3. $\bar{\psi}\gamma^\mu\psi$: Vector
4. $\bar{\psi}\gamma^\mu\gamma^5\psi$: Axial Vector
5. $\bar{\psi}\sigma^{\mu v}\psi$: Tensor where $\sigma^{\mu v} = \frac{i}{2}[\gamma^\mu, \gamma^v]$

Clifford Algebra Any 4×4 matrix can be written in the basis of the five bilnears

$$\{I, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu v}\}$$

So there are

$$1 + 1 + 4 + 4 + 6 = 16$$

independent 4×4 matrices.

Momentum Space

$$\begin{aligned}
(\gamma^\mu p_\mu - mc)u^{(s)} &= 0 \quad \text{(Particle)}\\
(\gamma^\mu p_\mu + mc)v^{(s)} &= 0 \quad \text{(Antiparticle)}\\
\bar{u}(\gamma^\mu p_\mu - mc) &= 0\\
\bar{v}(\gamma^\mu p_\mu + mc) &= 0
\end{aligned}$$

where

$$\begin{aligned}
u^\dagger u = v^\dagger v &= \frac{2E}{c}\\
\implies \bar{u}u = -\bar{v}v &= 2mc
\end{aligned}$$

We also get the completeness relation

$$\begin{aligned}
\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} &= \gamma^\mu p_\mu + mc\\
\sum_{s=1,2} v^{(s)}\bar{v}^{(s)} &= \gamma^\mu p_\mu - mc
\end{aligned}$$

In the basis states $|a_i\rangle$ where

$$I = \sum_i |a_i\rangle\langle a_i|$$

so

$$|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle = \sum_i c_i|a_i\rangle$$

Photon

In QED we have just an electron/position and a photon. We found electron/positron but now to find the photon. From the maxwell equations

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho\\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0\\
\boldsymbol{\nabla} \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

THe EM field strength tensor $F^{\mu\nu}$ is defined as

$$F^{\mu v} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and the current vector $J^\mu = (\rho c, \mathbf{J})$.

Inhomogenous Maxwell Equations For the First and Fourth equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

where

$$F^{\mu\nu} = -F^{\nu\mu}$$

so

$$\begin{aligned} \partial_\nu \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} \partial_\nu J^\nu \\ 0 &= \partial_\nu J^\nu \end{aligned}$$

Which is equivalent to the charge conservation

$$\begin{aligned} \frac{1}{c}\frac{\mathrm{d}\rho c}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \\ \frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \end{aligned}$$

and integrating over a volume

$$\begin{aligned} \int \mathrm{d}V \left(\frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot \mathbf{J} \right) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}V \, \rho + \int \mathrm{d}V \, \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \frac{\mathrm{d}Q}{\mathrm{d}t} + \oint \mathbf{J} \cdot \mathrm{d}\mathbf{S} &= 0 \end{aligned}$$

From Gauss' Law so we do indead find that

$$\mathbf{J} = 0 \implies Q = \text{constant}$$

From the Hemholtz potential we know that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, so the second Maxwell eq is

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla} \times \mathbf{A})$$

or

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} \right) = 0$$

or

$$\mathbf{E} + \frac{1}{c}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = -\boldsymbol{\nabla} V$$

So we have an EM field

$$A^\mu = (V, \mathbf{A})$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Which gives us the general Maxwell eq

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c}J^\nu$$

From the Gauge transformation

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \lambda$$

then

$$F^{\mu\nu} \to F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$$
$$= \partial^\mu(A^\nu + \partial^\nu \lambda) - \partial^\nu(A^\mu + \partial^\mu \lambda) = F^{\mu\nu}$$

THe Gauge fixing condition $\partial_\mu A^\mu = 0$ so the maxwell eq becomes

$$\partial_\mu \partial^\nu A^\nu = \frac{4\pi}{c}J^\nu$$

or

$$\Box A^\nu = \frac{4\pi}{c}J^\nu$$

If $\Box\lambda = 0$, choose a particular gauge, or the COulomb gauge

$$A^0 = 0$$

in free space. For $\partial_\mu A^\mu = 0$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0$$

In free space there is no current density, $J^\mu = 0$

$$\Box A^\mu = 0$$

which is the Klein-Gordan eq with mass $m = 0$. So the solution is a plane wave

$$A^\mu = ae^{-ipx/\hbar}\epsilon^\mu$$

where ϵ^μ is the polarization vector. So

$$\partial_\mu A^\mu = 0 \implies p_\mu \epsilon^\mu = 0$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \implies \mathbf{p} \cdot \boldsymbol{\epsilon} = 0$$

so $\boldsymbol{\epsilon}$ is perpendicular to $\mathbf{p}$. The components of the vector are

$$\mathbf{p} = p\hat{\mathbf{z}}$$

and

$$\epsilon^{(1)} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \epsilon^{(2)} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

so

$$\epsilon_\pm = \frac{1}{\sqrt{2}}(\epsilon^{(1)} \pm i\epsilon^{(2)})$$

or circular polarization.

Photon We have

$$A^\mu = ae^{-ipx/\hbar}\epsilon^\mu_{(s)}$$

where $s = 1, 2$ for two polarizations. The lorentz condition tells us $p_\mu \epsilon^\mu = 0$. And the Coulomb gauge $\mathbf{p} \cdot \boldsymbol{\epsilon} = 0$. The completeness relation

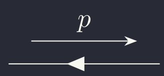
$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s_1)*} = \delta_{ij} - \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j$$

Feynman Rules

1. - External Momental: p_i
   - Internal Momenta: q_i
2. - Fermions: (straight line) Fermion and Momentum flow in same direction: $u^{(s)}(p)$ for incoming, $\bar{u}^{(s)}(p)$ for outgoing

   - Anti-fermion: Fermion and Momentum flow in opposite direction: $\bar{v}^{(s)}(p)$ for incoming, $v^{(s)}(p)$ for outgoing

   - Photon: (wave line) $\epsilon_\mu(p)$ for incoming, $\epsilon^*_\mu(p)$ for outgoing
3. Vertex: $ig_e\gamma^\mu$ for fermion-photon vertex
$$g_e = \sqrt{4\pi\alpha} = \sqrt{4\pi\frac{e^2}{\hbar c}} = e\sqrt{\frac{4\pi}{\hbar c}}$$
4. Propogator: For the particle/antiparticle
$$\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$$
   For the photon
$$\frac{-ig_{\mu v}}{q^2}$$
5. Conservation of four-momenta at each vertex
$$(2\pi)^4\delta^4(k_1 + k_2 + k_3)$$
   where k's are incoming momenta
6. Integrate over internal momenta
$$\int \frac{\mathrm{d}^4q_1}{(2\pi)^4}\frac{\mathrm{d}^4q_2}{(2\pi)^4}\cdots$$
7. Drop $(2\pi)^4\delta^4(p_1 + p_2 + \cdots - p_n - p_{n+1} - \dots)$
8. Multiply the answer by i:
9. Anti-symmetrization: Diagrams differing only by the inter change of identical fermions have a relative minus sign

Example: e-μ scattering From the matrix multiplication we need $(1\times 4)(4\times 4)(4\times 1)$ so
$$\bar{u}(p_3)(ig_e\gamma^\mu)u(p_1)$$
or in the opposite direction of the fermion flow. The amplitude is
$$\mathcal{M} = i\int \frac{\mathrm{d}^4q}{(2\pi)^4}[\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)]\left(\frac{-ig_{\mu v}}{q^2}\right)[\bar{u}^{(s_4)}(p_4)(ig_e\gamma^\mu)u^{(s_2)}(p_2)]$$
$$\times(2\pi)^4\delta^4(p_1 - q - p_3)\delta^4(p_2 + q - p_4)$$

and since $q = p_4 - p_2$ substituting in to the first delta function

$$(2\pi)^4\delta^4(p_1 - (p_4 - p_2) - p_4) = (2\pi)^4\cancel{\delta^4(p_1 + p_2 - p_3 - p_4)}$$

Which cancels out from rule 7. Thus we ge the amplitude

$$\mathcal{M} = \frac{-g_e^2}{(p_4 - p_2)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)]$$
$$g_{\mu v}\gamma^v = \gamma_\mu$$

Where

$$\begin{aligned} & p_1 + p_2 = p_3 + p_4 \\ \implies & p_4 - p_2 = p_1 - p_3 \end{aligned}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto |\mathcal{M}|^2$$

Example: e^-e^- Scattering (Møeller Scattering) If we have the diagram set horizontally, we actually have $e^-e^+ \to e^-e^+$ scattering (Bhabha Scattering). The first diagram is the same as the electron-muon scattering:

$$\begin{aligned} \mathcal{M}_1 &= \frac{-g_e^2}{(p_1 - p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)] \\ \mathcal{M}_2 &= \frac{-g_e^2}{(p_1 - p_4)^2}[\bar{u}^{(s_4)}(p_4)\gamma^\mu u^{(s_2)}(p_2)][\bar{u}^{(s_3)}(p_3)\gamma_\mu u^{(s_1)}(p_1)] \end{aligned}$$

since the momentum of p_3 and p_4 are interchanged so the total amplitude is

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$$

where the minus sign comes from rule 9, or the interchange of identical fermions.

Example: e^-e^+ Scattering (Bhabha Scattering) The bottom part of the diagram is the same as the electron-muon scattering:

$$\begin{aligned} \mathcal{M}_1 = i\int \frac{\mathrm{d}^4 q}{(2\pi)^4} & [\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)u^{(s_1)}(p_1)] \\ & \times\left(\frac{-ig_{\mu v}}{q^2}\right)[\bar{v}^{(s_2)}(p_2)(ig_e\gamma^\mu)v^{(s_4)}(p_4)] \\ & \times(2\pi)^4\delta^4(p_1 - q - p_3)\delta^4(p_2 + q - p_4) \end{aligned}$$

and again using $q = p_4 - p_2$ we can cancel out the delta functions

$$= -\frac{g_e^2}{(p_4 - p_2)^2}[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{v}^{(s_2)}(p_2)\gamma_\mu v^{(s_4)}(p_4)]$$

for the first diagram, and the second diagram is a combination:

$$\begin{aligned} \mathcal{M}_2 = i\int \frac{\mathrm{d}^4 q}{(2\pi)^4} & [\bar{v}^{(s_2)}(p_2)(ig_e\gamma^\mu)u^{(s_1)}(p_1)] \\ & \times\left(\frac{-ig_{\mu v}}{q^2}\right)[\bar{u}^{(s_3)}(p_3)(ig_e\gamma^\mu)v^{(s_4)}(p_4)] \\ & \times(2\pi)^4\delta^4(p_1 + p_2 - q)\delta^4(q - p_3 - p_4) \end{aligned}$$

using the second delta function again $q = p_3 + p_4$ so

$$p_1 + p_2 - q = p_1 + p_2 - p_3 - p_4$$

so the amplitude is

$$\mathcal{M}_2 = -\frac{g_e^2}{(p_3 + p_4)^2}[\bar{v}^{(s_2)}(p_2)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_3)}(p_3)\gamma_\mu v^{(s_4)}(p_4)]$$

and the total amplitude is

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$$

where the plus is because we have two different fermions...this is wrong. Interchanging the 2nd and 4th fermions on the second diagram gives the first diagram.

Matrix Elements

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S}{(E_1 + E_2)^2}|\mathcal{M}|^2 \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

so the squared factor

$$\begin{aligned}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}&[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)] \\ &\times [\bar{u}^{(s_3)}(p_3)\gamma^\nu u^{(s_1)}(p_1)]^\dagger[\bar{u}^{(s_4)}(p_4)\gamma_\nu u^{(s_2)}(p_2)]^\dagger\end{aligned}$$

where the Hermitian conjugate of the two terms are

$$u^\dagger(p_1)\gamma^{\nu\dagger}\gamma^{0\dagger}u(p_3)$$

multiplying by $\gamma^0\gamma^0$:

$$\begin{aligned}&(u^\dagger(p_1)\gamma^0)\gamma^0\gamma^{\nu\dagger}\gamma^{0\dagger}u(p_3) \\ &= \bar{u}(p_1)\gamma^0\gamma^{\nu\dagger}\gamma^0 u(p_3) \\ &= \bar{u}(p_1)\gamma^\nu u(p_3)\end{aligned}$$

and similarly for the second term:

$$\bar{u}(p_2)\gamma_\nu u(p_4)$$

For the unpolarized cross sections we sum over the final spins and average over the initial spins:

$$|\bar{\mathcal{M}}|^2 = \frac{1}{4}\sum\sum$$

Quiz Review:

- $$\begin{aligned}\gamma^\mu\gamma_\mu &= \gamma^0\gamma_0 + \gamma^1\gamma_1 + \gamma^2\gamma_2 + \gamma^3\gamma_3 \\ &= \gamma^0 g_{00}\gamma^0 + 3\gamma^i\gamma^i g_{ii} \\ &= 1 + 3(-1)(-1) = 4\end{aligned}$$

- Since $\not{a}\not{b} = a_\mu\gamma^\mu b_\nu\gamma^\nu$ we have
$$\mathrm{Tr}\big(\not{a}\not{b}\big) = a_\mu b_\nu \,\mathrm{Tr}(\gamma^\mu\gamma^\nu)$$
and since the Trace is cyclic
$$\begin{aligned}&= \frac{1}{2}a_\mu b_\nu \,\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \\ &= a_\mu b_\nu g^{\mu\nu}\,\mathrm{Tr}(I_{4x4}) = 4a_\mu b^\mu = 4a\cdot b\end{aligned}$$

- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and from the cyclicity of the trace
$$\mathrm{Tr}\big(\gamma^\mu\gamma^5\big) = \mathrm{Tr}\big(\gamma^5\gamma^\mu\big) = -\,\mathrm{Tr}\big(\gamma^5\gamma^\mu\big)$$
which is only true if the trace is zero.

- From the anticommutator $\{\gamma^\mu, \gamma^\nu = 2g^{\mu\nu}\}$ so
$$\begin{aligned}\mathrm{Tr}\big(\{\gamma^\mu,\gamma^\nu\},\gamma^5\big) &= 2g^{\mu\nu}\,\mathrm{Tr}\big(\gamma^5\big) = 0 \\ &= \mathrm{Tr}\big(\gamma^\mu\gamma^\nu\gamma^5 + \gamma^\nu\gamma^\mu\gamma^5\big) = 0\end{aligned}$$
So the quantity is an antisymmetric tensor or rank 2, which does not exist.

From the completeness relation
$$\begin{aligned}\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} &= \gamma^\mu p_\mu + mc = \not{p} + mc \\ \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} &= \not{p} - mc\end{aligned}$$

So from the squared amplitude:
$$\begin{aligned}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1-p_3)^4}&[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)] \\ &\times[\bar{u}_1\gamma^\nu u_3][\bar{u}_2\gamma_\nu u_4]\end{aligned}$$

we can rearrange the terms by writing the 3rd between the first and second terms and the 4th between the 2nd and 3rd terms:
$$\begin{aligned}\sum_{s_1,s_2}|\mathcal{M}|^2 = \frac{g_e^4}{(p_1-p_3)^4}&[\bar{u}_3\gamma^\mu(\sum_{s_1} u^{(s_1)}(p_1)\bar{u}^{(s_1)}(p_1))\gamma^\nu u_3] \\ &[\bar{u}_4\gamma_\mu(\sum_{s_2} u^{(s_2)}(p_2)\bar{u}^{(s_2)}(p_2))\gamma_\nu u_4] \\ = \frac{g_e^4}{(p_1-p_3)^4}&[\bar{u}_3[_i\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu]_{ij}u_3]_{jk} \\ &[\bar{u}_4[_k\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu]_{kl}u_4]_l\end{aligned}$$

so the sum over the spins (sub indices) gives us

$$= \frac{g_e^4}{(p_1 - p_3)^4} [\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu][\sum_{s_3} u(p_3)\bar{u}(p_3)][\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu]$$
$$[\sum_{s_4} u(p_4)\bar{u}(p_4)]$$
$$= \frac{g_e^4}{(p_1 - p_3)^4} [\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu]_{ij}(\not{p}_3 + m_e c)_{ji}$$
$$[\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu]_{lm}(\not{p}_4 + m_\mu c)_{ml}$$

and the sum over the indices

$$\sum_{ij} A_{ij}B_{ji} = \mathrm{Tr}(AB)$$

so the amplitude in terms of the traces is

$$= \frac{g_e^4}{(p_1 - p_3)^4} \mathrm{Tr}(\gamma^\mu(\not{p}_1 + m_e c)\gamma^\nu(\not{p}_3 + m_e c))$$
$$\mathrm{Tr}(\gamma_\mu(\not{p}_2 + m_\mu c)\gamma_\nu(\not{p}_4 + m_\mu c))$$

where the first Trace is the first fermion flow, and the second Trace is the second "disconnected" fermion flow. The electron flow is disconnected from the muon flow, so the Traces are separated out.

Expanding out the first trace

$$\mathrm{Tr}\big[\gamma^\mu p_{1k}\gamma^k + \gamma^\mu m_e c(\gamma^\nu p_{3b}\gamma^b + \gamma^\nu m_e c)\big]$$
$$= \mathrm{Tr}\big(p_{1k}p_{3b}\gamma^\mu\gamma^k\gamma^\nu\gamma^b\big) + \mathrm{Tr}\big((m_e c)^2\gamma^\mu\gamma^\nu\big)$$

the second term is

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu)$$
$$\implies = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$
$$= \frac{1}{2}\mathrm{Tr}(2\gamma^{\mu\nu}) = g^{\mu\nu}\,\mathrm{Tr}\{I\} = 4g^{\mu\nu}$$

and the first term is

$$\mathrm{Tr}\big(p_{1k}p_{3b}\gamma^\mu\gamma^k\gamma^\nu\gamma^b\big) = p_{1k}p_{3b}4\,\mathrm{Tr}\big(g^{\mu k}g^{\nu b} - g^{\mu\nu}g^{kb} + g^{\mu b}g^{k\nu}\big)$$

and after finding the second trace we know

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}$$
$$[(p_1p_2)(p_3p_4) + (p_1p_4)(p_3p_2)$$
$$- (p_1p_3)m_\mu^2c^2 - (p_2p_4)m_e^2c^2 + 2(m_e m_\mu c^2)^2]$$

Mott Scattering Using the assumption that the the muon mass M is much larger than the electron mass m, i.e., $M \gg m$. In the lab frame $\mathbf{p}_2 = 0$: Before the collision

$$p_1 = (E_1, \mathbf{p}_1), \quad p_2 = (Mc, \mathbf{0})$$

and after

$$p_3 = (E_3, \mathbf{p}_3), \quad p_4 \approx (Mc, 0)$$

Where $|\mathbf{p}_1| = |\mathbf{p}_3| = p$ and $\mathbf{p}_1 \cdot \mathbf{p}_3 = p^2 \cos\theta$. We also know

$$\begin{aligned} p_1 \cdot p_3 &= E_1 E_3 - \mathbf{p}_1 \cdot \mathbf{p}_3 \\ &= E^2 - p^2 \cos\theta \end{aligned}$$

or

$$= p^2(1 - \cos\theta) = 2p^2 \sin^2(\theta/2)$$

Finally the sum of the final spin and average of the initial spin is

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^2 Mc}{p^2 \sin^2 \theta/2} p^2 \cos^2 \theta/2$$

and the scattering differential

$$\frac{\partial\sigma}{\partial\Omega} = (\frac{\alpha\hbar}{2p^2 \sin^2 \theta/2})^2 p^2 \cos^2 \theta/2$$

Which approaches infinity as $\theta \to 0$ or $\theta \to \pi$.

Bhabha Scattering For $e^-e^+ \to e^-e^+$ we have two diagrams, one with electron flow & position flow, and another with electron to position flow. For $e^-e^- \to \mu^-\mu^+$ we have only one diagram with the matrix element:

$$\mathcal{M} = i\int \cancel{\mathrm{d}^4q\frac{1}{(2\pi)^4}}[\bar{v}(p_2)(ig_e\gamma^\mu)u(p_1)] \left(\frac{-ig_{\mu\nu}}{q^2}\right)[\bar{u}(p_3)(ig_e\gamma^\nu)v(p_4)]$$

And taking the limit of

$$E \gg (Mc)^2 \gg (mc)^2$$

The differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\Pi\alpha^2}{2E_{\mathrm{cm}}^2}(1+\cos^2\theta)$$
$$\sigma = \frac{\Pi\alpha^2}{3E^2} \quad E_{\mathrm{cm}} = 2E$$

Within the snapshots of time, the electron positron pair transfer all the energy into a real photon, and the photon transforms to the muon pair.

$e^+e^- \to q\bar{q}$ This is the same but we have a quark charge Q:

$$\mathcal{M} = i\int \cancel{\mathrm{d}^4q\frac{1}{(2\pi)^4}}[\bar{v}(p_2)(ig_e\gamma^\mu)u(p_1)] \left(\frac{-ig_{\mu\nu}}{q^2}\right)[\bar{u}(p_3)(iQg_e\gamma^\nu)v(p_4)]$$

and the cross section is

$$\sigma = \frac{\Pi Q^2\alpha^2}{3E^2}$$

In experiment we have the quarks hadronize into two mesons. So the ratio of the cross section is

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum Q_i^2$$

For $E < m_c^2$ the ratio is

$$R = 3(Q_u^2 + Q_d^2 + Q_s^2) = 3\left((\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{1}{3})\right)^2 = 2$$

For $m_cc^2 < E < m_bc^2$ We have to introduce the charm quark

$$R = 2 + 3Q_c^2 = 2 + 3\frac{4}{9} = 3.333$$

For $m_bc^2 < E < m_tc^2$ we have to introduce the bottom quark

$$R = 3.333 + 3Q_b^2 = 3.333 + 3\frac{1}{9} = 3.67$$

For $E > m_tc^2$ we have to introduce the top quark

$$R = 3.67 + 3Q_t^2 = 3.67 + 3\frac{4}{9} = 5$$

$e^-e^- \to ss$ Or the ϕ-meson which results in a peak at each of the resonances.

Up down quarks The pions are the lightest mesons, and we don't have scalar meson peaks because the photon has spin 1, so the meson of spin 0 can not be produced.

Electron-Proton Scattering It must be a vertical diagram, with electron p_1, proton p_2, and electron p_3 and proton p_4. The matrix element squared is

$$\begin{aligned}\langle| |\mathcal{M}|^2 |\rangle = \frac{4g_e^4}{q^4}[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}((mc)^2 - (p_1 \cdot p_3))] \\ [p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}((Mc)^2 - (p_2 \cdot p_4))] \\ = \frac{4g_e^4}{q^4}(L_{\text{electron}^{\mu\nu}})(L_{\mu\nu})_{\text{proton}}\end{aligned}$$

but the photon interacts arbitrarily with the proton quarks uud so

$$\langle| \mathcal{M}^2 |\rangle = \frac{4g_e^4}{q^4}(L_{\text{electron}})_{\mu\nu}(K_{\mu\nu})_{\text{proton}}$$

where $K_{\mu\nu}(p_2, q, p_4)$ is an unknown describing the vertex of the photon with the proton. From momentum conservation

$$p_2 + q = p_4 \to K_{\mu\nu}(p, q) \quad p \equiv p_2, q = p_4 - p_2$$

So to construct a 2nd rank tensor

$$\begin{aligned}K_{\mu\nu} = -K_1 g_{\mu\nu} + \frac{K_2}{(Mc)^2}p_\mu\nu \\ + \frac{K_4}{(Mc)^2}q_\mu q_\nu \\ + \frac{K_5}{(Mc)^2}(p_\mu q_\nu + q_\mu p_\nu) \\ +(\frac{K_6}{(Mc)^2}p_\mu q_\nu - q_\mu p_\nu)\end{aligned}$$

where the last term has a form factor of zero because the matrix element is symmetric and summing with the antisymmetric part of the $K_{\mu\nu}$ gives zero.

$$q^\mu K_{\mu\nu} = 0$$

since

$$\begin{aligned}q^\mu L_{\mu\nu} = 0 \\ = q^\mu(p_{1\mu}p_{3\nu} + p_{1\nu}p_{3\mu} + g_{\mu\nu}((mc)^2 - p_1 \cdot p_3))\end{aligned}$$

...

$$\begin{aligned}K_4 = \frac{(Mc)^2}{q^2}K_1 + \frac{1}{4}K_2 \\ K_5 = \frac{1}{2}K_2 \\ K_{\mu\nu} = K_1\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{(Mc)^2}\right) + \frac{K_2}{(Mc)^2}(p_\mu + \frac{1}{2}q_\mu)(p_\nu + \frac{1}{2}q_\nu)\end{aligned}$$

where (K_1, K_2) are the proton form factors. The differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\alpha\hbar}{4ME\sin^2\theta/2}\right)^2 \times \frac{2K_1\sin^2\theta/2 + K_2\cos^2\theta/2}{1 + \frac{2E}{Mc^2}\sin^2\theta/2}$$

If $E \ll Mc^2$ then we can neglect the second term in the denominator, and

$$K_1 \approx -q^2$$
$$K_2 \approx (2Mc)^2$$

also known as the Dirac Limit. The cross section is then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \left(\frac{\alpha\hbar c}{2E\sin^2\theta/2}\right)^2 \cos^2\theta/2$$

which is the Mott formula.

Quark-Quark Scattering This is an uninteresting process becuase the strong interaction (or exchange of gluons) that occurs (QCD).

Feynman Rules for QCD

1. Fermions: Incoming $u^{(s)}(p)c$, Outgoing $\bar{u}^{(s)}(p)c^\dagger$ where we have a color matrix of basis

$$c = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{red} \quad , c = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{blue} \quad , c = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{green}$$

   Antifermions: Incoming $\bar{v}^{(s)}(p)c^\dagger$, Outgoing $v^{(s)}(p)c$

2. Vertex: $-\frac{ig_s}{2}\gamma^\mu$ where the strong charge is

$$g_s = \sqrt{4\pi\alpha_s}, \quad \alpha_s = \frac{g_s^2}{\hbar c}$$

   Propogator: gluons(spring) $\frac{-g_{\mu\nu}}{q^2}\delta_{ab}$

   Additional Vertices (due to gluon color charge): e.g. 3 & 4 gluon vertex (glueball)

Exam Overview

- 5 Multiple Choice
- 2 Short
- 2 Long
- 1 Bonus

Quiz Review

- Gluon ($c\bar{c}$) where the color can be $c, \bar{c} = r, g, b$. Thus like mesons we have $8 \oplus 1$ states (octet and singlet). In SU(3) we have 8 generators which can give us the gluon states

$$|\alpha_i\rangle = \begin{pmatrix} r & g & b \end{pmatrix} \lambda_i \begin{pmatrix} \bar{r} \\ \bar{g} \\ \bar{b} \end{pmatrix}$$

so the first gluon state is

$$\begin{aligned} |1\rangle &\propto \begin{pmatrix} r & g & b \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{r} \\ \bar{g} \\ \bar{b} \end{pmatrix} \\ &= (r\bar{g} + g\bar{r}) \frac{1}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} |2\rangle &= -\frac{1}{\sqrt{2}}(r\bar{g} - g\bar{r}) \\ |3\rangle &= \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}) \\ &\dots \\ |8\rangle &= \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b}) \end{aligned}$$

The singlet state

$$|9\rangle = \frac{1}{3}(r\bar{r} + g\bar{g} + b\bar{b})$$

does not exist. Color singlets are always colorless, but the reverse is not true. But since the gluon is massless, this singlet state would act like a photon with a long range force.

$q\bar{q}$ **Scattering** For Different flavors, we only need one vertical diagram of

$$p_2, c_2 \to p_4, c_4 \quad p_3, c_3 \to p_1, c_1$$

The matrix element is

$$\begin{aligned} \mathcal{M} &= i[\bar{u}(3)c_3^\dagger(-ig_s\gamma^\mu \frac{\lambda^\alpha}{2})u(1)c_1] \left(\frac{ig_{\mu\nu}}{q^2}\delta_{\alpha\beta}\right) [\bar{v}(2)c_2^\dagger(-ig_s\gamma^\nu \frac{\lambda^\beta}{2})v(4)c_4] \\ &= -\frac{g_s^2}{q^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{v}(2)\gamma_\mu v(4)]\frac{1}{4}(c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\beta c_4) \end{aligned}$$

where we have a color factor added to the QED matrix element

$$f = \frac{1}{4}(c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\beta c_4)$$

Octet Examples $r\bar{g}$: The initial states

$$c_1 = r = \begin{pmatrix}1\\0\\0\end{pmatrix} = c_3, \quad c_2 = g = \begin{pmatrix}0\\1\\0\end{pmatrix} = c_4$$

so the color must stay the same in the fermion flow. The color factor is

$$\begin{aligned} f &= \frac{1}{4}(\begin{pmatrix}1 & 0 & 0\end{pmatrix}\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix})(\begin{pmatrix}0 & 1 & 0\end{pmatrix}\lambda^\beta \begin{pmatrix}0\\1\\0\end{pmatrix}) \\ &= \frac{1}{4}\lambda^\alpha_{11}\lambda^\beta_{22} \end{aligned}$$

The only non-zero term is given by λ^3, λ^8 so

$$\begin{aligned} f &= \frac{1}{4}(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22}) \\ &= -\frac{1}{6} \end{aligned}$$

Color Singlet: Color factor

$$\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

This means that the incoming quarks and outcoming quarks are

$$c_1 = c_3 = rgb, \quad c_2 = c_4 = rgb$$

so the color factor is

$$\begin{aligned} f &= \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_2^\dagger \lambda^\beta c_4) \\ &= \frac{1}{4}\frac{1}{\sqrt{3}^2}\lambda^\alpha_{ij}\lambda^\alpha_{ji} \\ &= \frac{1}{12}\,\mathrm{Tr}(\lambda^\alpha\lambda^\alpha) = \frac{1}{12}2\delta_{\alpha\alpha} = \frac{1}{12}2(8) = \frac{4}{3} \end{aligned}$$

For the Hydogen atom the potential is

$$V = -\frac{e^2}{r} = -\frac{\alpha\hbar c}{r}$$

so the potentials for the quarks are

$$V = -f\frac{\alpha_s\hbar c}{r} = \begin{cases} \frac{1}{6}\frac{\alpha_s\hbar c}{r} & \text{Octet} \\ -\frac{4}{3}\frac{\alpha_s\hbar c}{r} & \text{Singlet} \end{cases}$$

This lower potential for the singlet state tells us that mesons bind in the singlet state.

qq **Scattering** (Different flavors for simplicity) The color factor is

$$f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_4^\dagger \lambda^\beta c_2)$$

since the flow is reversed for the second flow. We have a sextet(symmetric) and triplet(antisymmetric) configurations for the gluon:

$$3 \otimes 3 = 6 \oplus 3$$

and the color factos is

$$f = \begin{cases} \frac{1}{3} & \text{Sextet} \\ -\frac{1}{3} & \text{Triplet} \end{cases}$$

Weak Interaction

- Charged-current ($W^{\pm}$)
- Neutral-current (Z^0)

For the charge current we have $e^- \to W^- \nu_e$ for leptons and $d \to W^- u$ for quarks.

Neutral Current $\nu_e \to Z\nu_e$, $e \to Ze$ etc.

Feynman rule changes The vertex facor

$$-\frac{g_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

is in the form $V - A$ (Parity Violation). For the neutral current we have

$$(c_v\gamma^\mu - c_A\gamma^\mu\gamma^5)$$

For the partial vector current interaction.

Propogator

$$\frac{-i(g^{\mu\nu} - q^\mu q^\nu/(Mc)^2)}{q^2 - (Mc)^2}$$

where if $q^2 \ll (Mc)^2$ we have

$$\frac{ig^{\mu\nu}}{(Mc)^2}$$

β-decay (Neutron)

$$n(udd) \to p(uud) + e^- + \bar{\nu}_e$$

Where $M_W = 80.4$ GeV/c^2 and $M_Z = 91.2$ GeV/c^2. So the decay is mediated by the W^- boson

$$\Gamma \propto \frac{g_w^4}{M_W^4}$$

Fermi β-decay theory The is a constant $G_F \sim \frac{g_w^2}{M_W^2}$ which is the Fermi constant (Effective Field Theory). This contracts the W boson to a point-like interaction.