1 Calculus of Variations

Why do we care?

- What is the shortest distance between two points in a 2D plane?
- What is the shortest path between two points on a sphere?
- What is the fastest path for a ball to roll down a hill?
- For a car driving on a flat path $A \to B$, what shape of a pot hole will minimize the time it takes to get from $A \to B$?

For some path $a \to b$, we have a path defined as an integral

$$S = \int_a^b f(x, y, y')\mathrm{d}x$$

with a *Goal*: find $y(x)$ that minimizes S (path).

Path Length:

$$l = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{1 + y'^2}\mathrm{d}x$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. To minimize $y = f(x)$ it is equivalent to finding where

$$f'(x) = 0$$

where we note that this could be a maximium point, but it is usually a minimum in these cases. Another look at this function:

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

we can define a small change in the path $y(x)$ as

$$y(x) + \delta y(x)$$

where

$$\delta y(x_2) = 0 \quad \delta y(x_2) = 0$$

so the change in the path is

$$\delta S = \int_a^b \delta f \mathrm{d}x$$

and from the change of variables

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \qquad \delta y' = \frac{\mathrm{d}}{\mathrm{d}x}\delta y$$

thus we have

$$\delta S = \int_a^b \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x$$

this is the line integral of the change in the new path

$$\delta S = S_{new} - S_{old}$$

looking at the second term: using integration by parts

$$\int_a^b \left(\frac{\partial f}{\partial y}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x = \left[\frac{\partial f}{\partial y'}\delta y\right]\Bigg|_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y \mathrm{d}x$$

the first term is zero because $\delta y(a) = \delta y(b) = 0$. Thus we have

$$\delta S = \int_a^b \left[\frac{\partial f}{\partial y} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\right]\delta y \mathrm{d}x$$

Near a minimum, $\delta S = 0$ for any small δy. So the the terms in the brackets must be zero as well! This gives us the **Euler-Lagrange Equation**:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0$$

NOTE: δS is the variation of S(some number) under δy(a function).

Example: Shortest path between two points $a \to b$ in a 2D cartesian plane.

Goal: find $y(x)$ tha minimzes the path length $l = \int_a^b \sqrt{1+y'^2}\mathrm{d}x$ where $f(x, y, y') = \sqrt{1+y'^2}$.

$$\frac{\partial f}{\partial y} = 0$$
$$\frac{\partial f}{\partial y'} = \frac{2y'}{2\sqrt{1+y'^2}} = \frac{y'}{\sqrt{1+y'^2}}$$

From the EL:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{\partial f}{\partial y} = 0$$

and

$$\frac{y'}{\sqrt{1+y'^2}} = Const = C$$
$$y'^2 = C(1+y'^2)$$
$$y'^2 = \frac{C}{1-C}$$
$$y' = \pm\sqrt{\frac{C}{1-C}} = \pm k$$
$$y = \pm kx + b$$

which is just a straight line as we expected.

Example: The Brachistochrone.

Goal: Find $y(x)$ that minimizes $t = \int_a^b \mathrm{d}t$ where

$$t = \frac{s}{v} \to \mathrm{d}t = \frac{\mathrm{d}s}{v}$$

where v is the velocity which can be found using the conservation of energy:

$$\frac{1}{2}mv^2 = mg(y_o - y) \to v = \sqrt{2g(y_o - y)}$$

thus we have

$$\mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o - y)}}\mathrm{d}x$$

where $f(x, y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o-y)}}$. Using EL:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}\sqrt{2g(y_o - y)}}$$
$$\frac{\partial f}{\partial y} = \frac{\sqrt{1+y^2}\sqrt{2g}}{(2g(y_o - y))^{3/2}}$$