Lecture 18: **3/4/24**

1 Central Force Problems

Two-Body Considering a two-body system of masses m_1, m_2 we know that under the influence of gravitational potential

$$U = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = -\frac{Gm_1m_2}{\mathscr{r}}$$

so the force on each mass is

$$\begin{aligned}
\mathbf{F}_{12} &= -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_1 U \\
\mathbf{F}_{21} &= +\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U
\end{aligned}$$

computing the Lagrangian:

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 \\
U &= -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}
\end{aligned}$$

in 3D we have 6 degrees of freedom, so we have 6 generalized coordinates:

$$\mathbf{r}_1 = (x_1, y_1, z_2) \quad \mathbf{r}_2 = (x_2, y_2, z_2)$$

and from the separation vector

$$\boldsymbol{\mathscr{r}} = \mathbf{r}_1 - \mathbf{r}_2$$

the center of mass is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \qquad M = m_1 + m_2$$

we can rewrite the position vectors in terms of the COM and separation vector:

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\boldsymbol{\mathscr{r}} \\
\mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\boldsymbol{\mathscr{r}}
\end{aligned}$$

and thus the derivatives are

$$\begin{aligned}
\dot{\mathbf{r}}_1 &= \dot{\mathbf{R}} + \frac{m_2}{M}\dot{\boldsymbol{\mathscr{r}}} \\
\dot{\mathbf{r}}_2 &= \dot{\mathbf{R}} - \frac{m_1}{M}\dot{\boldsymbol{\mathscr{r}}}
\end{aligned}$$

so the Lagrangian is rewritten as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\boldsymbol{\mathscr{r}}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\boldsymbol{\mathscr{r}}}\right)^2 - U \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1m_2}{M}\dot{\boldsymbol{\mathscr{r}}}^2 - U
\end{aligned}$$

where we have the reduced mass

$$\mu = \frac{m_1m_2}{M} = \frac{m_1m_2}{m_1 + m_2}$$

here we can see that $\mathcal{L}$ does not depend on $\mathbf{R}$

$$\implies \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{R}}_i} = \text{const} \implies M\dot{\mathbf{R}} = \text{const} \quad \text{or} \quad M\ddot{\mathbf{R}} = 0$$

this is the igorable coordinate, so Transforming into the COM frame

$$\begin{aligned} \mathbf{r}_1' &= \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\boldsymbol{r} \\ \mathbf{r}_2' &= \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\boldsymbol{r} \end{aligned}$$

and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(\boldsymbol{r})$$

which is basically a single particle leaving us with 3 coordinates(Degrees of freedom).

Angular momentum in the COM frame is

$$\begin{aligned} L &= \sum_i \boldsymbol{r}_i' \times \mathbf{p}_i' \\ &= \boldsymbol{r}' \times m_i\dot{\boldsymbol{r}}' \\ &= m_1\boldsymbol{r}_1' \times \dot{\boldsymbol{r}}_1' + m_2\boldsymbol{r}_2' \times \dot{\boldsymbol{r}}_2' \\ &= \frac{m_1 m_2^2}{M}\boldsymbol{r} \times \dot{\boldsymbol{r}} + \frac{m_1^2 m_2}{M}\boldsymbol{r} \times \dot{\boldsymbol{r}} \\ &= \frac{m_1 m_2}{M}\boldsymbol{r} \times \dot{\boldsymbol{r}} = \mu\boldsymbol{r} \times \dot{\boldsymbol{r}} \end{aligned}$$

which is the same as the angular momentum of a single particle with reduced mass μ.

- If $m_2 \gg m_1$ then $\mathbf{R} \approx \mathbf{r}_2$ and $\mu \approx m_2$.
- If $m_1 \gg m_2$ then $\mathbf{R} \approx \mathbf{r}_1$ and $\mu \approx m_1$.
- If $m_1 = m_2$ then $\mathbf{R}$ is directly in the middle of the two particles and $\mu = \frac{m_1}{2} = \frac{m_2}{2}$.

We can see that for two vectors, any linear combination will result in a vector on a plane, so we can turn this into a 2D problem. Using polar coordinates we can write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\mu(\dot{\boldsymbol{r}}^2 + \boldsymbol{r}^2\dot{\phi}^2) - U(r)$$

where we can see that it does not depend on ϕ, so we have the conserved quantity

$$\implies \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{const} = \mu\boldsymbol{r}\dot{\phi} = \ell$$

and the EL equation is only needed for r:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} &= \mu\boldsymbol{r}\dot{\phi}^2 - \frac{\partial U}{\partial \boldsymbol{r}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{r}}}\right) &= \mu\ddot{\boldsymbol{r}} \\ \implies \mu\ddot{\boldsymbol{r}} &= \mu\boldsymbol{r}\dot{\phi}^2 - \frac{\partial U}{\partial \boldsymbol{r}} \qquad U = -\frac{Gm_1m_2}{\boldsymbol{r}} \\ &= \frac{l^2}{\mu\boldsymbol{r}^3} - \frac{\partial U}{\partial \boldsymbol{r}} \\ &= \frac{l^2}{\mu\boldsymbol{r}^3} - \frac{Gm_1m_2}{\boldsymbol{r}^2} \end{aligned}$$

Solving the 1D Problem First we note the first term is equivalent to the minus gradient of the centrifugal potential:

$$\begin{aligned} m\ddot{\mathscr{r}} &= -\frac{\partial U_{cf}}{\partial \mathscr{r}} - \frac{\partial U}{\partial \mathscr{r}} \\ &= -\frac{\partial}{\partial \mathscr{r}}\left(\frac{\ell^2}{2\mu \mathscr{r}^2}\right) - \frac{\partial U}{\partial \mathscr{r}} \\ &= -\frac{\partial}{\partial \mathscr{r}}(U_{cf} + U) \end{aligned}$$

where $U_{cf} = \frac{\ell^2}{2\mu \mathscr{r}^2}$ is the centrifugal potential. We can see that the effective potential is

$$U_{eff} = \frac{\ell^2}{2\mu \mathscr{r}^2} - \frac{Gm_1 m_2}{\mathscr{r}}$$

From Last Time For a 2-Body problem where $M = m_1 + m_2$ and the COM

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)$$
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mu = \frac{m_1 m_2}{M}$$

we found the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\\ &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)\end{aligned}$$

where $\mathcal{L}$ is independent of ϕ, so we have the conserved quantity

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2\dot{\phi} = \ell \implies \dot{\phi} = \frac{l}{mr^2}$$

so the EL equation for r is

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

where the centrifugal force is

$$F_{cf} = \frac{\ell^2}{\mu r^3}$$

and the effective potential is

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = U_{cf} + U$$

From the graph of this effective potential, there is a centrifugal barrier for finite ℓ for $\ell = \mathbf{r} \times \mathbf{p}$ and for $r \to 0$ the potential is dominated by the centrifugal term.

Conservation of Energy If this problem is independent of time we know that

$$\begin{aligned}E &= \sum_i \dot{q}_i p_i - \mathcal{L}\\ &= \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} + \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L}\\ &= \mu\dot{r}^2 + \frac{\ell^2}{\mu r^2} - \frac{1}{2}\mu\dot{r}^2 - \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)\\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = T + U\end{aligned}$$

we can find the equilibrium point at

$$\begin{aligned}\frac{\partial U_{eff}}{\partial r} &= 0\\ &= -\frac{\ell^2}{\mu r^3} + \frac{\gamma}{r^2} \qquad \gamma = Gm_1m_2\\ \implies r_o &= \frac{\ell^2}{\gamma\mu}\end{aligned}$$

this radius is related to a perfectly circular *orbit.* and at

$$r = r_o, \qquad \dot{\phi} = \frac{\ell\mu^2\gamma^2}{\mu\ell^4} = \frac{\mu\gamma^2}{\ell^3}$$

so

$$\phi(t) = \int_0^t \dot{\phi}(t')\,\mathrm{d}t'$$

For $E < 0$ we have a bound (bounded) orbit, and for $E > 0$ we have an unbounded orbit. For $E = 0$ we also have an unbounded orbit.

What does the orbit look like? Find $r(\phi)$ using a differential equation (For a circular orbit we know $r = r_o$). First we introduce a variable transformation

$$q = \frac{1}{r}, \qquad r = \frac{1}{q}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{q}\right) = -\frac{1}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\phi}, \qquad q' = \frac{\mathrm{d}q}{\mathrm{d}\phi}$$

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\cdot\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\ell}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi} = -\frac{\ell}{\mu r^2}\frac{1}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\phi} = -\frac{\ell}{\mu}\frac{\mathrm{d}q}{\mathrm{d}\phi}$$

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}\dot{r}}{\mathrm{d}\phi} = -\dot{\phi}\frac{\ell}{\mu}q'' = -\frac{\ell^2 q^2}{\mu^2}q''$$

and the central force is

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} + F$$

$$-\mu\frac{\ell^2 q^2}{\mu^2}q'' = \frac{\ell^2 q^3}{\mu r^3} + F$$

$$q'' = -q - \frac{\mu}{q^2\ell^2}F$$

and since the force is

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{\gamma}{r^2} = -\gamma q^2$$

so the differential equation is just

$$q'' = -q + \frac{\gamma\mu}{\ell^2}$$

and the RHS vanishes when

$$q = \frac{\gamma\mu}{\ell^2} \quad \text{or} \quad r_o = \frac{\ell^2}{\gamma\mu}$$

we can redefine the constant

$$\omega = q - \frac{\gamma\mu}{\ell^2} \implies \omega'' = q'' = -\omega$$

so

$$\omega(\phi) = A\cos(\phi - \delta)$$

and choosing initial conditions so that $\delta = 0$

$$\omega(\phi) = A\cos(\phi) \implies q(\phi) = A\cos(\phi) + \frac{\gamma\mu}{\ell^2} = \frac{1}{r(\phi)}$$

and thus

$$r(\phi) = \frac{\ell^2/\gamma\mu}{1 + \epsilon\cos(\phi)} = \frac{C}{1 + \epsilon\cos(\phi)} \qquad \epsilon = \frac{A}{C}$$

we can check and see that r has the unit of length and the denominator is unitless, so C has the unit of length. We can see that ϵ only depends on the initial conditions, and at

$$\epsilon = 0 \implies r(\phi) = C = r_o$$

so

$$r(\phi) = \frac{r_o}{1 + \epsilon \cos(\phi)}$$

and ϵ is the eccentricity of the orbit.

- If $\epsilon = 0$ then $r = r_o$ and we have a circular orbit.
- If $\epsilon > 1$ then the denominator can $\to 0$ and we have $r \to \infty$ or hyperbolic orbit.
- If $0 < \epsilon < 1$ then we have a bounded orbit or ellipse.
- IF $\epsilon = 1$ then we have a parabolic orbit.

Lecture 20: 3/8/24

Missed Lecture: