1. P and Q are T, U and V are F, and W is unknown:

   (a) Since $(P \vee Q) \equiv (\mathrm{T} \vee \mathrm{T}) \equiv \mathrm{T}$ and $(U \wedge V) \equiv (\mathrm{F} \wedge \mathrm{F}) \equiv \mathrm{F}$, we have

   $$\begin{aligned}(P \vee Q) \vee (U \wedge V) &\equiv T \vee F \\ &\equiv T\end{aligned}$$

   (b) Since $(\neg P \vee \neg U) \equiv (\mathrm{F} \vee \mathrm{T}) \equiv \mathrm{T}$, and $(Q \vee \neg V) \equiv (\mathrm{T} \vee \mathrm{T}) \equiv \mathrm{T}$, we have

   $$\begin{aligned}(\neg P \vee \neg U) \wedge (Q \vee \neg V) &\equiv T \wedge T \\ &\equiv T\end{aligned}$$

   (c) We know that $(P \wedge \neg V) \equiv (T \wedge T) \equiv T$, but $(U \vee W)$ can either be T or F depending on what W is and one of the two truth values must be T for $U \vee W \equiv T$. Therefore, the statement is an unknown truth.

   Citations: NONE

2. Let x be a real number:

   (a) If $x = 3$, then $x^2 = 9$: This statement is T because we can simply see that $3^2 = 9$.

   (b) If $x^2 = 9$, then $x = 3$: This statement is F because x can also be -3 since $(-3)^2 = 9$.

   (c) If $x^2 \neq 9$, then $x \neq 3$: Looking at the contrapositive, if $x = 3$, then $x^2 = 9$ which is logically equivalent to (a), so the statement is T.

   (d) If $x \neq 3$, then $x^2 \neq 9$: This is the contrapositive of (b) which is logically equivalent, so the statement is F.

   Citations: BOP Section 2.6 Contrapositive Law (2.1) $P \Rightarrow Q \equiv (\neg Q) \Rightarrow (\neg P)$

3. (a) Let N, O, P, Q, and R be mathematical statements. Constructing truth tables:

   i. Truth table for $\neg(P \Rightarrow \neg Q)$:

P	Q	$\neg Q$	$P \Rightarrow \neg Q$	$\neg(P \Rightarrow Q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

   ii. Truth table for $(P \wedge Q) \vee R$:

P	Q	R	$P \wedge Q$	$(P \wedge Q) \vee R$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

   (b) $N \wedge O \wedge P \wedge Q \wedge R$ will have $2^5 = 32$ rows in the truth table since we have 5 statements and each statement can be either T or F. All of the statements must be T for the entire statement to be T, so there is *only one* row where the entire statement is T, and 31 rows where the entire statement is F.

   Citations: NONE

4. Prove that $(\neg P \wedge Q) \Rightarrow (Q \vee R)$ is a tautology: From clase we showed that $A \Rightarrow B \equiv A \wedge (\neg B)$ or from De Morgan's Law $A \Rightarrow B \equiv (\neg A) \vee B$. Therefore,

$$\begin{aligned}(\neg P \wedge Q) \Rightarrow (Q \vee R) &\equiv \neg(\neg P \wedge Q) \vee (Q \vee R)\\ &\equiv (P \vee \neg Q) \vee (Q \vee R) \quad \text{De Morgan's Law}\\ &\equiv P \vee (\neg Q \vee Q) \vee R \quad \text{Associative Law}\\ &\equiv P \vee T \vee R\\ &\equiv T\end{aligned}$$

because having one T in a disjunction makes the entire statement T i.e $P \vee T \equiv T$ which carries over to $T \vee R \equiv T$.

For example, let's say

P: Penguins have wings
Q: Penguins swim in water
R: Penguins fly in the sky

In English: if penguins do not have wings and penguins swim in water, then penguins swim in water or penguins fly in the sky.

Citations: Lecture 3 1/17/25 (Jesus Sanchez!) & Discrete Mathematics and its Applications by Kenneth Rosen—Section 1.3, Table 7

5. Prove that n and m are odd integers if and only if nm is odd:

   P: n and m are odd integers
   Q: nm is odd

   From lecture an odd integer can be expressed as $2k + 1$ for some integer k, and an even integer can be expressed as $2l$ for some integer l. Also $(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$, so we need to show both $P \Rightarrow Q$ and $Q \Rightarrow P$:

   - $P \Rightarrow Q$: Assume n and m are odd integers. Then $n = 2k + 1$ and $m = 2l + 1$ for some integers k and l. Multiplying the two odd integers gives

     $$\begin{aligned} nm &= (2k+1)(2l+1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

     Since $2kl + k + l$ is an integer, nm is odd. So n and m being odd integers implies nm is odd $\square$

   - $Q \Rightarrow P$: The contrapositive is $\neg P \Rightarrow \neg Q$. Assume n and m are even integers. Then $n = 2k$ and $m = 2l$ for some integers k and l. Multiplying the two even integers gives $nm = 4kl = 2(2kl)$. Since $2kl$ is an integer, nm is even. So the contrapositive is true, thus $Q \Rightarrow P$ $\square$

   Citations: Lecture 2 1/15/25

6. For each statement write it out in English, write the negation in symbolic form, and write the negation in English:

   (a) $\exists x \in \mathbb{Q}, x > \sqrt{2}$:

   English: There exists a rational number x such that x is greater than $\sqrt{2}$
   Negation: $\forall x \in \mathbb{Q}, x \leq \sqrt{2}$
   Negation in English: For all rational numbers x, x is less than or equal to $\sqrt{2}$

   (b) $\forall x \in \mathbb{Q}, x^2 - 2 \neq 0$:

   English: For all rational numbers x, $x^2 - 2$ does not equal 0
   Negation: $\exists x \in \mathbb{Q}, x^2 - 2 = 0$
   Negation in English: There exists a rational number x such that $x^2 - 2$ equals 0

   (c) $\forall x \in \mathbb{Z}, x^2$ is odd $\Rightarrow x$ is odd:

   English: For all integers x, if x^2 is odd, then x is odd
   Negation: $\exists x \in \mathbb{Z}, x^2$ is odd $\wedge\, x$ is even
   Negation in English: There exists an integer x such that x^2 is odd and x is even

   (d) $\exists x \in \mathbb{R}, \cos(2x) = 2\cos(x)$:

   English: There exists a real number x such that $\cos(2x) = 2\cos(x)$
   Negation: $\forall x \in \mathbb{R}, \cos(2x) \neq 2\cos(x)$
   Negation in English: For all real numbers x, $\cos(2x)$ does not equal $2\cos(x)$

   Citations: Lecture 4 1/22/25