

1.

$$\begin{aligned}
W(n) &= \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \cdots \sum_{m=1}^2 w_i w_j w_k \cdots w_m \\
&= \sum_{i=1}^2 w_i \sum_{j=1}^2 w_j \sum_{k=1}^2 w_k \cdots \sum_{m=1}^2 w_m \\
&= (w_1 + w_2)(w_1 + w_2)(w_1 + w_2) \cdots (w_1 + w_2) \\
&= (w_1 + w_2)^N
\end{aligned}$$

since there are N factors from $i \rightarrow m$. Using binomial theorem:

$$\begin{aligned}
(w_1 + w_2)^N &= \sum_{n=0}^N \binom{N}{n} w_1^n w_2^{N-n} \\
&= \sum_{n=0}^N \frac{N!}{n!(N-n)!} w_1^n w_2^{N-n}
\end{aligned}$$

where the term involving w_1^n is simply

$$W(n) = \frac{N!}{n!(N-n)!} w_1^n w_2^{N-n}$$

2. N_0 molecules in container of volume V_0 . N molecules in subvolume V .

(a) Probability that any given molecule is in the subvolume V :

The probability of *one single* molecule being in the subvolume is $p = \frac{V}{V_0}$, and $q = 1 - p$ for the opposite case. Then for any given molecule, we use the binomial theorem:

$$P(n) = \frac{N_0!}{N!(N_0 - N)!} \left(\frac{V}{V_0}\right)^N \left(1 - \frac{V}{V_0}\right)^{N_0 - N}$$

(b) the mean # of molecules in V for a binomial distribution is simply [From Information Theory, Mackay eq 1.7]

$$\bar{N} = N_0 p = N_0 \frac{V}{V_0}$$

(c) The relative dispersion:

$$\begin{aligned} \frac{\overline{(N - \bar{N})^2}}{\bar{N}^2} &= \frac{\overline{N^2 - 2N\bar{N} + \bar{N}^2}}{\bar{N}^2} \\ &= \frac{\overline{N^2} - 2\bar{N}^2 + \bar{N}^2}{\bar{N}^2} \\ &= \frac{\overline{N^2} - \bar{N}^2}{\bar{N}^2} \end{aligned}$$

where the top term is the variance/dispersion $\overline{N^2} - \bar{N}^2 = N_0 p q$, so

$$\begin{aligned} &= \frac{N_0 p q}{\bar{N}^2} \\ &= \frac{N_0 \frac{V}{V_0} \left(1 - \frac{V}{V_0}\right)}{(N_0 \frac{V}{V_0}) \bar{N}} \\ \text{relative dispersion} &= \frac{1 - \frac{V}{V_0}}{\bar{N}} \end{aligned}$$

(d) When $V \ll V_0$, $\frac{V}{V_0} \approx 0$, so

$$\text{relative dispersion} \approx \frac{1}{\bar{N}} \rightarrow \infty$$

(e) When $V \rightarrow V_0$:

$$\begin{aligned} \overline{(N - \bar{N})^2} &= N_0 p q \approx N_0 \frac{V_0}{V_0} \left(1 - \frac{V_0}{V_0}\right) \\ &= 0 \end{aligned}$$

which agrees with part (c) since

$$\text{relative dispersion} \rightarrow \frac{1 - \frac{V_0}{V_0}}{\bar{N}} = 0$$

3. N antennas with em radiation of wavelength λ and velocity c . Antennas are on the x -axis separated λ apart. Observer on x -axis measures intensity I from one antenna.

(a) Total intensity of all antennas:

All of the antennas are in phase, so the amplitudes add up i.e.

$$E_T = NE$$

and since intensity is proportional to the square of the amplitude $I \propto E^2$, the total intensity is

$$I_T = N^2 I$$

(b) For completely random phases (but same freq), the total amplitude as a vector is

$$E_T = \sum_{i=1}^N \mathbf{E}_i$$

so the mean square amplitude is [from Reif eq (1.9.9)]

$$\begin{aligned} \overline{E_T^2} &= \overline{\sum_{i=1}^N \mathbf{E}_i \cdot \sum_{j=1}^N \mathbf{E}_j} \\ &= \sum_{i=1}^N \overline{E_i^2} + \sum_{i \neq j} \sum \overline{\mathbf{E}_i \cdot \mathbf{E}_j} \\ &= NE^2 \end{aligned}$$

where the (second) cross terms add up to zero since the phases are random— there are just as many positive and negative values. So the mean intensity is

$$\bar{I}_T \propto \overline{E_T^2} = NI$$

4. N particles of spin $1/2$. Magnetic moment μ which points parallel or antiparallel in an applied field H . Energy E in the field is then $E = -(n_1 - n_2)\mu H$ where n_1 is parallel and n_2 is antiparallel.

(a) In the energy range $[E, E + \delta E]$ the total # of states $\Omega(E)$ in the range:

A single particle can have spin $\pm\mu H$, so in the range of δE there are $\delta E/2\mu H$ different states. So the total number of states for a large number N is

$$\Omega(E) = \binom{N}{n_1} \frac{\delta E}{2\mu H} = \frac{N!}{n_1!n_2!} \frac{\delta E}{2\mu H}$$

And using $n_1 + n_2 = N$ or $n_2 = N - n_1$ and $n_1 = N - n_2$ we can get

$$\begin{aligned} E &= -(n_1 - n_2)\mu H \\ \frac{E}{\mu H} &= -(n_1 - (N - n_1)) = -2n_1 + N \\ \Rightarrow n_1 &= \frac{1}{2} \left(N - \frac{E}{\mu H} \right), \quad n_2 = \frac{1}{2} \left(N + \frac{E}{\mu H} \right) \end{aligned}$$

So the total number of states is

$$\Omega(E) = \frac{N!}{\left[\frac{1}{2} \left(N - \frac{E}{\mu H} \right) \right]! \left[\frac{1}{2} \left(N + \frac{E}{\mu H} \right) \right]!} \frac{\delta E}{2\mu H}$$

(b) Using Stirling's approximation ($\ln N! \approx N \ln N - N$):

$$\ln \Omega(E) \approx N \ln N - N - [n_1 \ln n_1 - n_1] - [n_2 \ln n_2 - n_2] + \ln \frac{\delta E}{2\mu H}$$

simplifying some terms:

$$\begin{aligned} -[n_1 \ln n_1 - n_1] - [n_2 \ln n_2 - n_2] &= -n_1 \ln n_1 - n_2 \ln n_1 + n_1 + n_2 \\ \text{where } n_1 + n_2 &= \frac{1}{2} \left(N - \frac{E}{\mu H} \right) + \frac{1}{2} \left(N + \frac{E}{\mu H} \right) = N \end{aligned}$$

so we can cancel out a term:

$$\ln \Omega(E) = N \ln N - n_1 \ln n_1 - n_2 \ln n_2 + \ln \frac{\delta E}{2\mu H}$$

(c) A Gaussian approximation to part (a): From (a)

$$\begin{aligned} \Omega(E) &= \frac{N!}{n_1!n_2!} \frac{\delta E}{2\mu H} \\ &= \frac{N!}{n_1!(N - n_1)!} \frac{\delta E}{2\mu H} \\ &= W(n_1) \frac{\delta E}{2\mu H}, \quad W(n_1) = \frac{N!}{n_1!(N - n_1)!} \end{aligned}$$

Using $n_1 \equiv \bar{n}_1 + \xi$ the Taylor expansion gives [From lecture notes...]

$$\begin{aligned} \ln W(n_1) &\approx \ln W(\bar{n}_1) + \frac{1}{2} B_2 \xi^2 \\ \Rightarrow W(n_1) &= W(\bar{n}_1) e^{-\frac{1}{2} B_2 \xi^2} \end{aligned}$$

where

$$\begin{aligned} B_2 &= \frac{1}{N p q} \quad \text{using } p = \frac{1}{2}, q = \frac{1}{2} \\ B_2 &= \frac{4}{N} \end{aligned}$$

and using $\bar{n}_1 = N/2$

$$\begin{aligned}\xi &= n_1 - \bar{n}_1 = \frac{1}{2} \left(N - \frac{E}{\mu H} \right) - \frac{N}{2} \\ \Rightarrow \xi^2 &= \left(\frac{E}{2\mu H} \right)^2\end{aligned}$$

To find $W(\bar{n}_1)$ we must satisfy the normalization condition: the integral of $W(n_1)$ over all n_1 must equal the total number of possible spins 2^N (like N coin flips) i.e.

$$\begin{aligned}\int_{-\infty}^{\infty} W(n_1) \, dn_1 &= 2^N \\ \int_{-\infty}^{\infty} W(\bar{n}_1) e^{-\frac{1}{2} B_2 \xi^2} \, dn_1 &= 2^N\end{aligned}$$

and since [From Randy Harris Modern Physics Front Page]

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ax^2} &= \sqrt{\frac{\pi}{a}} \\ \Rightarrow W(\bar{n}_1) &= \frac{2^N}{\sqrt{2\pi/B_2}} = \frac{2^N}{\sqrt{\pi N/2}}\end{aligned}$$

So the Gaussian approximation is

$$\begin{aligned}W(n_1) &= \frac{2^N}{\sqrt{\pi N/2}} e^{-\frac{1}{2} B_2 \xi^2} \\ &= \frac{2^N}{\sqrt{\frac{\pi N}{2}}} e^{-\frac{2}{N} \left(\frac{E}{2\mu H} \right)^2}\end{aligned}$$

Finally, we get the total number of states from $\Omega(E) = W(n_1) \frac{\delta E}{2\mu H}$:

$$\boxed{\Omega(E) = \frac{2^N}{\sqrt{\frac{\pi N}{2}}} e^{-\frac{2}{N} \left(\frac{E}{2\mu H} \right)^2} \frac{\delta E}{2\mu H}}$$

5. $A dx + B dy \equiv dF$

(a) Show that $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$:

Since dF is an exact differential

$$\frac{\partial F}{\partial x} = A, \quad \frac{\partial F}{\partial y} = B$$

so

$$\frac{\partial A}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y} \right) = \frac{\partial B}{\partial x}$$

(b) Show that $\int dF$ on any closed path in xy plane is zero:

For an exact differential

$$\int_a^b dF = F(b) - F(a)$$

so for a closed path $a \rightarrow b$ then back $b \rightarrow a$:

$$\int_a^b dF + \int_b^a dF = F(b) - F(a) + F(a) - F(b) = 0$$

6. From $A \rightarrow B$ the mean pressure is

$$\bar{p} = \alpha V^{-5/3}$$

- (a) Work done when system expanded to final volume, heat added to maintain pressure ($V = 1 \rightarrow 8$), $\bar{p} = 32$. Heat extracted to reduce pressure to 10^6 dynes cm^{-2} :

First finding α at macrostate B :

$$\alpha = \bar{p} V^{5/3} = 1 * 8^{5/3} = 32$$

so the work done is

$$\begin{aligned} W_a &= \int dW = \int_{V_i}^{V_f} \bar{p} dV \\ &= 32V \Big|_1^8 \\ &= 224 \times 10^9 \text{ dynes cm} \end{aligned}$$

From wikipedia, $1 \text{ dynes} = 10^{-5} \text{ N}$, so the units of work is

$$10^6 \text{ dynes cm}^{-2} \times 10^3 \text{ cm}^3 = 10^9 \text{ dynes cm} * \frac{10^{-5} \text{ N}}{1 \text{ dynes}} * \frac{1 \text{ m}}{10^2 \text{ cm}} = 100 \text{ J}$$

so

$$W_a = 22\,400 \text{ J}$$

To find the net heat absorbed we use the first law of thermodynamics:

$$\Delta E = Q - W \implies Q = \Delta E + W$$

where from macro state A to B

$$\begin{aligned} \Delta E &= \int dE = - \int \bar{p} dV \\ &= -32 \int_1^8 V^{-5/3} dV \\ &= 32 \frac{3}{2} V^{-2/3} \Big|_1^8 \\ &= 48(8^{-2/3} - 1) \\ &= -36 \times 10^9 \text{ dynes cm} = -3600 \text{ J} \end{aligned}$$

Finally the net heat absorbed is

$$Q = \Delta E + W = -3600 \text{ J} + 22\,400 \text{ J} = 18\,800 \text{ J}$$

- (b) Volume increase and heat added to cause linear decrease in pressure:

New pressure equation is in the form $p = mV + b$, where the slope $m = \frac{-31}{7}$ and the intercept is at

$$32 = \frac{-31}{7} + b \implies b = \frac{255}{7}$$

thus

$$p = \frac{-31}{7} V + \frac{255}{7}$$

The work done is

$$\begin{aligned}W_b &= \int_1^8 p \, dV = \int_1^8 \left(\frac{-31}{7}V + \frac{255}{7} \right) dV \\&= -\frac{31}{14}V^2 + \frac{255}{7}V \Big|_1^8 \\&= 11\,550 \text{ J}\end{aligned}$$

and using the energy change found in part (a)

$$Q = \Delta E + W = -3600 \text{ J} + 11\,550 \text{ J} = 7950 \text{ J}$$

(c) Part (a) but in reverse:

First the pressure is reduced to $1 \times 10^6 \text{ dynes/cm}^2$, then expanding the volume from $V = 1 \rightarrow 8$ amounts to work

$$\begin{aligned}W_c &= \int_1^8 p \, dV = \int_1^8 1 \, dV \\&= 700 \text{ J}\end{aligned}$$

and the net heat absorbed is

$$Q = \Delta E + W = -3600 \text{ J} + 700 \text{ J} = -2900 \text{ J}$$

7. 3D particle in a box with energy level

$$E = \frac{\hbar^2}{2m} \pi^2 \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right)$$

(a) Force by particle on wall perpendicular to x axis:

$$dW = -dE = -\frac{\partial E}{\partial L_x} dL_x = F_x dL_x$$

where F_x is the generalized force on the wall perpendicular to the x axis. This holds true as long as $dQ = 0$.

(b) Pressure on the wall, and the mean pressure:

The unit area for the pressure on the wall perpendicular to the x axis is $a = L_y L_z$, so the pressure is

$$p_x = \frac{F_x}{a} = -\frac{1}{L_y L_z} \frac{\partial E}{\partial L_x}$$

where

$$\frac{\partial E}{\partial L_x} = \frac{\hbar^2}{2m} \pi^2 n_x^2 \frac{\partial}{\partial L_x} \left(\frac{1}{L_x^2} \right) = -\frac{\hbar^2}{m} \pi^2 \frac{n_x^2}{L_x^3}$$

so

$$p_x = \frac{\hbar^2}{mV} \pi^2 \frac{n_x^2}{L_x^2}$$

where $V = L_x L_y L_z$ is the volume. The mean pressure is then

$$\bar{p} = \frac{\hbar^2}{mV} \pi^2 \frac{\overline{n_x^2}}{L_x^2}$$

Since $\overline{n_x^2} = \overline{n_y^2} = \overline{n_z^2}$ and $L_x = L_y = L_z$ by *symmetry* we can rewrite the mean energy as

$$\begin{aligned} \bar{E} &= \frac{\hbar^2}{2m} \pi^2 \left(\frac{\overline{n_x^2}}{L_x^2} + \frac{\overline{n_y^2}}{L_y^2} + \frac{\overline{n_z^2}}{L_z^2} \right) = \frac{\hbar^2}{2m} \pi^2 \left(\frac{3\overline{n_x^2}}{L_x^2} \right) \\ \implies \frac{2}{3} \bar{E} &= \frac{\hbar^2}{m} \pi^2 \frac{\overline{n_x^2}}{L_x^2} \end{aligned}$$

Thus we can substitute \bar{E} into the mean pressure equation:

$$\bar{p} = \frac{2}{3} \frac{\bar{E}}{V}$$

