Homework 6

Due 2/28

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1. (a) Geometrically we find a constraint

$$\tan\alpha = \frac{r}{z} \quad \text{or} \quad z = r\cot\alpha; \qquad \dot{z} = \dot{r}\cot\alpha$$

where z is the vertical position. The position vector is a linear combination of this z position and polar position:

$$\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}; \quad \mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$$

so the kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{r}^2\cot^2\alpha) \\
\text{using} \quad & 1 + \tan^2\alpha = \sec^2\alpha \implies \cot^2\alpha = \csc^2\alpha - 1 \\
T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{r}^2(\csc^2\alpha - 1)) = \frac{1}{2}m(r^2\dot{\phi}^2 + \dot{r}^2\csc^2\alpha)
\end{aligned}$$

and the potential energy is

$$U = mgz = mgr\cot\alpha$$

so the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m(r^2\dot{\phi}^2 + \dot{r}^2\csc^2\alpha) - mgr\cot\alpha$$

(b) The EL eqn for ϕ is

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) \implies mr^2\dot{\phi} = \text{constant} = \ell
\end{aligned}$$

which states the conservation of angular momentum. The EL eqn for r is

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}} \\
mr\dot{\phi}^2 - mgr\cot\alpha &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{r}\csc^2\alpha\right) \\
&= m\ddot{r}\csc^2\alpha
\end{aligned}$$

where the mass cancels out so we can simplify to

$$\begin{aligned}
r\dot{\phi}^2 - gr\cot\alpha = \ddot{r}\csc^2\alpha \\
0 &= \ddot{r} - r\dot{\phi}^2\sin^2\alpha + g\frac{\cos\alpha}{\sin\alpha}\sin^2\alpha \\
0 &= \ddot{r} - r\dot{\phi}^2\sin^2\alpha + g\cos\alpha\sin\alpha
\end{aligned}$$

from the conservation of angular momentum

$$mr^2\dot{\phi} = \ell \implies \dot{\phi} = \frac{\ell}{mr^2}$$

so

$$0 = \ddot{r} - \frac{\ell^2}{m^2r^3}\sin^2\alpha + g\cos\alpha\sin\alpha$$

and solving for when $r = r_o \implies \ddot{r} = 0$:

$$\begin{aligned}
0 = 0 - \frac{\ell^2}{m^2r_o^3}\sin^2\alpha + g\cos\alpha\sin\alpha \\
\frac{\ell^2}{m^2r_o^3}\sin^2\alpha = g\cos\alpha\sin\alpha \\
r_o^3 = \frac{\ell^2}{m^2g}\frac{\sin\alpha}{\cos\alpha} = \frac{\ell^2}{m^2g}\tan\alpha \\
r_o = \left(\frac{\ell^2}{m^2g}\tan\alpha\right)^{1/3}
\end{aligned}$$

we can analyze the stability of this solution by assuming a small deviation from the eq point:

$$r = r_o + \eta$$

and looking at how the second derivative behaves: rewriting the EL eqn,

$$\ddot{r} = \frac{\ell^2}{m^2r^3}\sin^2\alpha - g\cos\alpha\sin\alpha$$

and we find a substitution to directly compare the two terms:

$$\frac{\ell^2}{m^2r_o^3}\sin^2\alpha = \frac{\ell^2}{m^2\frac{\ell^2}{m^2g}\tan\alpha}\sin^2\alpha = \frac{g}{\tan\alpha}\sin^2\alpha = g\cos\alpha\sin\alpha$$

so

$$\begin{aligned}
\ddot{r} &= \frac{\ell^2}{m^2r^3}\sin^2\alpha - \frac{\ell^2}{m^2r_o^3}\sin^2\alpha \\
&= \frac{\ell^2}{m^2}\sin^2\alpha\left(\frac{1}{(r_o+\eta)^3} - \frac{1}{r_o^3}\right)
\end{aligned}$$

where

$$\frac{1}{(r_o+\eta)^3} < \frac{1}{r_o^3}$$

so when r slightly increases ($\eta > 0$), the bead tends back toward the eq point ($\ddot{r} < 0$) and when r slightly decreases ($\eta < 0$), the bead tends back toward the eq point i.e.

$$\frac{1}{(r_o-\eta)^3} > \frac{1}{r_o^3}$$

thus r_o is a stable equilibrium point.

2. (a) Setting the origin at the circle of radius R, the position vector of the bead is

$$\begin{aligned}
\mathbf{r} &= (R\cos(\omega t) + r\cos(\theta + \omega t))\hat{\mathbf{x}} + (R\sin(\omega t) + r\sin(\theta + \omega t))\hat{\mathbf{y}} \\
\mathbf{v} &= (-R\omega\sin(\omega t) - r(\dot\theta + \omega)\sin(\theta + \omega t))\hat{\mathbf{x}} + (R\omega\cos(\omega t) + r(\dot\theta + \omega)\cos(\theta + \omega t))\hat{\mathbf{y}} \\
v^2 &= R^2\omega^2\sin^2(\omega t) + r^2(\dot\theta + \omega)^2\sin^2(\theta + \omega t) - 2Rr\omega(\dot\theta + \omega)\sin(\omega t)\sin(\theta + \omega t) \\
&\quad + R^2\omega^2\cos^2(\omega t) + r^2(\dot\theta + \omega)^2\cos^2(\theta + \omega t) + 2Rr\omega(\dot\theta + \omega)\cos(\omega t)\cos(\theta + \omega t) \\
&= R^2\omega^2 + r^2(\dot\theta + \omega)^2 + 2Rr\omega(\dot\theta + \omega)(\cos(\omega t)\cos(\theta + \omega t) + \sin(\omega t)\sin(\theta + \omega t))
\end{aligned}$$

where we use the sum identity:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

so

$$\cos(\omega t)\cos(\theta + \omega t) + \sin(\omega t)\sin(\theta + \omega t) = \cos(\omega t - (\theta + \omega t)) = \cos(-\theta) = \cos\theta$$

thus the velocity squared is

$$v^2 = R^2\omega^2 + r^2(\dot\theta + \omega)^2 + 2R\omega r(\dot\theta + \omega)\cos\theta$$

which is equivalent to the square of the sum of vectors:

$$(\mathbf{a} + \mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} = a^2 + b^2 + 2ab\cos\theta$$

where $\mathbf{a} = (R\omega)\hat{\mathbf{a}}$ and $\mathbf{b} = (r(\dot\theta + \omega))\hat{\mathbf{b}}$ is the velocity of the hoop and the bead respectively. The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(R^2\omega^2 + r^2(\dot\theta + \omega)^2 + 2R\omega r(\dot\theta + \omega)\cos\theta)$$

where we can assume there is no potential energy i.e. $U = 0$. So the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(R^2\omega^2 + r^2(\dot\theta + \omega)^2 + 2R\omega r(\dot\theta + \omega)\cos\theta)$$

(b) from which we can find the EL eqn for θ:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= -mR\omega r(\dot\theta + \omega)\sin\theta \\
\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) &= \frac{\mathrm{d}}{\mathrm{dt}}[mr^2(\dot\theta + \omega) + mR\omega r\cos\theta] \\
&= mr^2\ddot\theta - mR\omega r\dot\theta\sin\theta
\end{aligned}$$

so

$$\begin{aligned}
-mR\omega r(\dot\theta + \omega)\sin\theta &= mr^2\ddot\theta - mR\omega r\dot\theta\sin\theta \\
0 &= mr^2\ddot\theta + mR\omega^2 r\sin\theta
\end{aligned}$$

and dividing by mr^2:

$$\ddot\theta + \frac{R}{r}\omega^2\sin\theta = 0$$

which is the EOM for a simple pendulum when $R = r$. Finding the eq point(s):

$$\theta = \theta_0 \implies \ddot\theta = 0$$

so

$$0 + \omega^2\sin\theta_0 = 0 \implies \sin\theta_0 = 0 \implies \theta_0 = n\pi$$

which gives us two eq points: $\theta_0 = 0, \pi (2\pi \equiv 0$ in this context). We can analyze the stability of these eq points by assuming a small deviation from the eq point:

$$\theta = \theta_0 + \eta$$

For the case $\theta_0 = 0$: We can use the simple approximation

$$\sin(0 + \eta) \approx \eta$$

and we look at the characteristic of the second derivative:

$$\ddot{\theta} = -\frac{R}{r}\omega^2 \sin\theta \quad \text{using} \quad \frac{R}{r}\omega^2 = C_1$$
$$\ddot{\theta} = -C_1\eta$$

so when the bead moves slightly counter-clockwise(CCW) i.e. $\eta > 0$, it accelerates clockwise(CW) i.e. $\ddot{\theta} < 0$ and vice versa. When the bead deviates from the eq point $\theta_0 = 0$, it tends back toward equilibrium thus a stable equilibrium point.

For $\theta_0 = \pi$: We will be more careful with the approximation by using Taylor expansion:

$$\sin(\pi + \eta) \approx \sin\pi + \eta\cos\pi = -\eta$$

so the second derivative is

$$\ddot{\theta} = -C_1(-\eta) = C_1\eta$$

so when the bead moves slightly CCW($\eta > 0$), it accelerates CCW($\ddot{\theta} > 0$) and vice versa. Thus $\theta_0 = \pi$ is an unstable equilibrium point.

3. (a) The position and velocity of mass M is

$$\begin{aligned}\mathbf{r}_M &= (x + L\sin\phi)\hat{\mathbf{x}} + (L\cos\phi)\hat{\mathbf{y}} \\ \mathbf{v}_M &= (\dot{x} + L\dot{\phi}\cos\phi)\hat{\mathbf{x}} + (-L\dot{\phi}\sin\theta)\hat{\mathbf{y}}\end{aligned}$$

and the velocity squared is

$$\begin{aligned}v_M^2 &= \dot{x}^2 + 2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2\cos^2\phi + L^2\dot{\phi}^2\sin^2\phi \\ &= \dot{x}^2 + 2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2\end{aligned}$$

The kinetic energy is

$$\begin{aligned}T &= \frac{1}{2}(Mv_M^2 + mv_m^2) \qquad v_m^2 = \dot{x}^2 \\ &= \frac{1}{2}[M(\dot{x}^2 + 2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2) + m\dot{x}^2] \\ T &= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}M(2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2)\end{aligned}$$

and the potential energy is the gravitational potential energy + the spring potential energy:

$$U = Mgy + \frac{1}{2}kx^2 = -MgL\cos\phi + \frac{1}{2}kx^2$$

so the Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}M(2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2) + MgL\cos\phi - \frac{1}{2}kx^2\end{aligned}$$

The EL eqn for x:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) \\ -kx &= \frac{\mathrm{d}}{\mathrm{d}t}\Big((M+m)\dot{x} + ML\dot{\phi}\cos\phi\Big) \\ -kx &= (M+m)\ddot{x} + ML\ddot{\phi}\cos\phi - ML\dot{\phi}^2\sin\phi\end{aligned}$$

The EL eqn for ϕ:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) \\ -ML\dot{x}\dot{\phi}\sin\phi - MgL\sin\phi &= ML\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{x}\cos\phi + L\dot{\phi}\Big) \\ -\dot{x}\dot{\phi}\sin\phi - g\sin\phi &= \ddot{x}\cos\phi - \dot{x}\dot{\phi}\sin\phi + L\ddot{\phi} \\ -g\sin\phi &= \ddot{x}\cos\phi + L\ddot{\phi}\end{aligned}$$

(b) For small ϕ we can approximate

$$\cos\phi \approx 1, \qquad \sin\phi \approx \phi$$

which simplfies the two EL eqns to

$$\begin{aligned}-kx &= (M+m)\ddot{x} + ML(\ddot{\phi} - \dot{\phi}^2\phi) \\ -g\phi &= \ddot{x} + L\ddot{\phi}\end{aligned}$$

and throwing away some terms

$$\begin{aligned}-kx = (M+m)\ddot{x} &\implies \ddot{x} = -\frac{k}{M+m}x \\ -g\phi = L\ddot{\phi} &\implies \ddot{\phi} = -\frac{g}{L}\phi\end{aligned}$$

4. (a) From HW 3:

$$\mathbf{r} = (r\cos\phi)\hat{\mathbf{x}} + (r\sin\phi)\hat{\mathbf{y}}$$
$$\mathbf{v} = (\dot{r}\cos\phi - r\dot{\phi}\sin\phi)\hat{\mathbf{x}} + (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)\hat{\mathbf{y}}$$

so the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

and the potential energy is

$$U = \frac{1}{2}k(r-a)^2$$

So the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}k(r-a)^2$$

The EL eqn for r:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) \\ mr\dot{\phi}^2 - k(r-a) &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) \\ mr\dot{\phi}^2 - k(r-a) &= m\ddot{r} \\ \implies -k(r-a) &= m\ddot{r} - mr\dot{\phi}^2 \end{aligned}$$

The EL eqn for ϕ:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) = m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi}) \\ \implies 0 &= m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{aligned}$$

which is what we found in HW 3.

(b) The kinetic eqn is the same as before:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

but the potential energy is adds gravitational potential energy:

$$U = \frac{1}{2}k(r-a)^2 - mgr\cos\phi$$

so the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}k(r-a)^2 + mgr\cos\phi$$

(c) The EL eqn for r:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) \\ mr\dot{\phi}^2 - k(r-a) + mg\cos\phi &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) \\ r\dot{\phi}^2 - \frac{k}{m}(r-a) + g\cos\phi &= \ddot{r} \end{aligned}$$

so

$$\ddot{r} - r\dot{\phi}^2 + \frac{k}{m}(r-a) - g\cos\phi = 0$$

and the EL eqn for ϕ:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \\
-mgr\sin\phi &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) \\
&= m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi}) \\
-g\sin\phi &= 2\dot{r}\dot{\phi} + r\ddot{\phi}
\end{aligned}$$

so

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + \frac{g}{r}\sin\phi = 0$$

assuming ϕ and $(r-a) = \epsilon$ are small, we can use the approximation around the angular eq point $\phi = 0$:

$$\sin\phi \approx \phi, \qquad \cos\phi \approx 1$$

and getting rid of some higher order terms:

$$\begin{aligned}
\ddot{r} - \cancel{r\dot{\phi}^2} + \frac{k}{m}(r-a) - g(1) &= 0 \\
\ddot{r} &= g - \frac{k}{m}(r-a) \\
\ddot{\phi} + \cancel{\frac{2}{r}\dot{r}\dot{\phi}} + \frac{g}{r}\phi &= 0 \\
\ddot{\phi} &= -\frac{g}{r}\phi
\end{aligned}$$

the eq point for r is when $\ddot{r} = 0$:

$$0 = g - \frac{k}{m}(r-a) \implies r_o = a + \frac{mg}{k}$$

and expanding for small values around eqn point $r = r_o + \epsilon$ where we define $\epsilon = r - a$:

$$\begin{aligned}
\ddot{\epsilon} &= g - \frac{k}{m}(r_o + \epsilon - a) \\
&= g - \frac{k}{m}(\frac{mg}{k} + \epsilon) \\
&= g - g - \frac{k}{m}\epsilon
\end{aligned}$$

thus we get the two simple harmonics oscillators

$$\begin{aligned}
\ddot{\epsilon} &= -\frac{k}{m}\epsilon \qquad \omega_1^2 = \frac{k}{m} \\
\ddot{\phi} &= -\frac{g}{r_o}\phi \qquad \omega_2^2 = \frac{g}{r_o}
\end{aligned}$$