Physics 421: Intro to Electrodynamics

Lectures: Eric Henriksen
Notes and Homework: Junseo Shin

Contents

1 Vector Analysis

What is a Vector? In type we use boldface $\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}$, where we we can do some simple operations as such:

- Adding and Subtraction: $\mathbf{A} \pm \mathbf{B} = \mathbf{C}$ or aligning the head to the tail
- Multiplication:
  - Scalar: $\mathbf{A} \to 2\mathbf{A}$
  - Dot Product: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = AB\cos\theta$
  - Cross Product: $\mathbf{A} \times \mathbf{B} = AB\sin\theta$, and $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$

Components of a Vector In 3D space, we often use the familiar Cartesian coordinates, e.g.

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

and we can add components by adding the components:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

and likewise for subtraction. For the dot product:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

or more shortly

$$= \sum_{i,j} A_iB_j\delta_{ij}$$

where δ_{ij} is the Kronecker delta. The cross product is a bit trickier…

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{\mathbf{x}} - (A_xB_z - A_zB_x)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}$$

This can also be written in short form using the Levi-Civita symbol (look it up)

Scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

This is the also the volume of the parallelepiped formed by the three vectors. NOTE that

$$\cancel{(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}}$$

since you can't cross a scalar with a vector.

Vector triple product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Using the BAC-CAB rule.

Some important vectors We define a position vector

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = r\hat{\mathbf{r}}$$

where the unit vector is

$$\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}$$

and an infinitesimal displacement vector

$$\mathrm{d}l = \mathrm{d}x\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}} + \mathrm{d}z\hat{\mathbf{z}}$$

In EM we define a source point $\mathbf{r}'$ (e.g. a charge) and a field point $\mathbf{r}$ that give us the seperation vector

$$\boldsymbol{\mathscr{r}} = \mathbf{r} = \mathbf{r}'$$

with magnitude

$$|\boldsymbol{\mathscr{r}}| = |\mathbf{r} - \mathbf{r}'|$$

and unit vector

$$\hat{\mathbf{r}} = \frac{\mathbf{r} - \mathbf{r}'}{\mathbf{r} - \mathbf{r}'}$$

Differential Calculus And ordinary derivative $\frac{\mathrm{d}F}{\mathrm{d}x}$ is a change in $F(x)$ in $\mathrm{d}x$

$$\mathrm{d}F = \left(\frac{\partial F}{\partial x}\right)\mathrm{d}x$$

. . . geometrically, it's the slope

Gradient for functions of 2 or more variables, generalize for $h(x, y)$

$$\mathrm{d}h = \left(\frac{\partial h}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial h}{\partial y}\right)\mathrm{d}y$$

it's a scalar so $\mathrm{d}h = (\boldsymbol{\nabla} h) \cdot (\mathrm{d}l)$ where

$$\boldsymbol{\nabla} h = \frac{\partial h}{\partial x}\hat{\mathbf{x}} + \frac{\partial h}{\partial y}\hat{\mathbf{y}}$$

In 3D

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

If $\boldsymbol{\nabla} u = 0$, we are at an extremum (max, min, or shoulder/saddle point) Rewriting:

$$\boldsymbol{\nabla} T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)T(x, y, z)$$

wehere we can assume the ∇ as an "operator" acting on T:

1. Scalars like T: $\boldsymbol{\nabla} T$, "grad T", generalized slope
2. Dot product on $\mathbf{V}$: $\boldsymbol{\nabla} \cdot \mathbf{V}$, "divergence" or "div"
3. Cross product : $\boldsymbol{\nabla} \times \mathbf{V}$, "curl" or "rotatation"

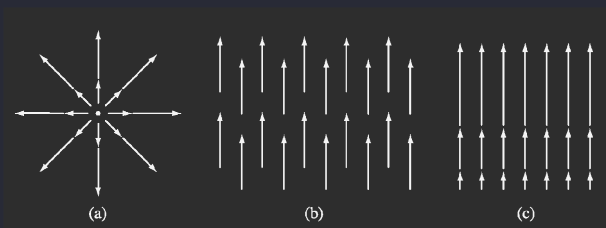

Figure 1.1: Divergence of field lines

Divergence From the Figure, we can see that (a) & (c) diverges, and (b) does not.

Geometrical Interpretation: Sources of positive divergence is a source or "faucet", and negative divergence is a sink or "drain".

Curl

$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

E.g. for $\mathbf{V} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, $\nabla \times \mathbf{V} = 2\hat{\mathbf{z}}$.

Combining Multiple Operations Two ways to get scalar from two functions:

$$fg \quad \text{or} \quad \mathbf{A} \cdot \mathbf{B}$$

Two ways to get vector from two functions:

$$f\mathbf{A} \quad \text{or} \quad \mathbf{A} \times \mathbf{B}$$

And we have 3 'derivatives': div, grad, and curl.
Product rule:

$$\partial_x(fg) = f\partial_x g + g\partial_x f$$

i $\nabla(fg) = f\nabla g + g\nabla f$

ii $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + B \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$