Title

Contents

1 Lecture (1/17/24)

Four Fundamental Forces

- Strong (gluon)
- Weak (W, Z)
- Electromagnetic (photon)
- Gravity (graviton?)

The ‘Standard Model’ describe the first three forces and unifies the Strong and Weak Forces known as the ‘Electroweak’ force. So, the Standard Model does not include gravity.

The Standard Model (SM)

- Basic building blocks: spin 1/2 particles (fermions)
- Interaction between then are mediated by force carriers: spin 1 particles (vector bosons)
- How particles get mass? $\rightarrow$ Higgs Boson (spin 0)

The Range of Forces:

- Strong: 10^{-15} m
- Weak: $10^{-18} \quad 10^{-16}$ m
- EM: $1/r^2$
- Gravity: $1/r^2$

The ranges of forces are related by

$$R \quad \frac{e^{-r/a}}{r^2}$$

where $a \approx 10^{-15}$ m for the Strong and Weak forces.

The Rise of Quantum Field Theory (QFT) Relativity + Quantum Mechanics $\rightarrow$ QFT

	Macroscopic	Micro
SLOW	CM	Quantum Mechanics
FAST	Special Relativity	QFT

QFT Discoveries

- Existence of anti-particles
- Spin-statistics theorem
- CPT Theorem (Charge conjugation, Parity, Time reversal)

Units!

- Mass: (kg) $\rightarrow$ (eV) from $E = mc^2$

$$m_e = 0.5 \times 10^6 \,\mathrm{eV/c^2} \quad E_n = \frac{-13.6\,\mathrm{eV}}{n^2}$$
$$m_p = 1\,\mathrm{GeV/c^2} \quad 1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$$

- Momentum: $\frac{eV}{C} \rightarrow p = \frac{E}{c}$
- Energy: eV

Matter Fermions are divided into two groups:

- Leptons (electrons, muon, tau, neutrinos): Doesn't have the strong force
- Quarks (up, down, charm, strange, top, bottom): Feels the strong force

e.g. the proton is made of 2 up quarks and 1 down quark (uud) and the Neurtron is (udd).

Quarks make up composite subparticles (Hadrons) are held together by the strong force.

- Mesons: 1 quark + 1 anti-quark ($q\bar{q}$) e.g. pion, kaon...
- Baryons: 3 quarks (qqq) e.g proton, neutron

Quark charges:

- $Q = +2/3$ (up, charm, top)
- $Q = -1/3$ (down, strange, bottom)

Leptons are fundamental particles

- Charged electrically (-1)
  - electron (0.5 MeV)
  - muon (105 MeV)
  - tau (1.8 GeV)
- Neutral (neutrinos)
  - electron neutrino ν_e
  - mueon neutrino ν_μ
  - tau neutrino ν_τ

Crossing Symmetry

$$\begin{aligned} A + B &\to C + D \quad \text{Scattering} \\ A &\to B + C + D \quad \text{Decay} \\ A + \bar{C} &\to \bar{B} + D \end{aligned}$$

e.g. Neutron Decay

$$n \to p + e^- + \bar{\nu}_e$$

Sum rules to think about:

- Baryon Number Conservation
- Lepton Number Conservation
- Electric Charge Conservation

another example:

$$\begin{aligned} n + e^+ &\to p + \bar{\nu}_e \\ p + e^- &\to n + \nu_e \end{aligned}$$

Particle Conservation Laws

2 Relativistic Kinematics

Quiz 2 Review

1. The Baryon, Lepton, and Electric Charge are conserved in the Standard Model.
2. The Baryon and Lepton number ensure the stability of the proton.
3. In Neutron Decay $n \to p + e^- + \bar{\nu}_e$, the weak force is responsible for the decay.

	Strong	EM	Weak	Gravity
Strength	1	10^{-2}	10^{-7}	10^{-40}
Time scale	10^{-23} sec	10^{-16}	10^{-10}	$>$ yr

   The decay rate is proportional to the coupling strength of the force $\Gamma \propto \alpha^2$. For the time scale is τ it is inversely proportional:

$$\tau \propto \frac{1}{\Gamma}$$

4. The strong force is responsible for holding the nucleus together.
5.

Experimental Discoveries To discover and observe particles, there are typically three ways:

1. Scattering (cross section)
2. Decay (decay rate or lifetime)
3. Bound states (binding energy/mass)

Relativistic Kinematics 4-vectors

$$x^\mu = (ct, x, y, z) \quad \text{space-time}$$
$$p^\mu = (E/c, p_x, p_y, p_z) \quad \text{momentum}$$

where x^μ and p^μ are the space-time (position) four-vector and energy-momentum four-vector.

NOTE: Totak four-momentum is conserved in all interactions.
Starting with the lorentz invariant

$$p^\mu p_\mu = p^2$$

using the Einstein-summation convention

$$p^\mu p_\mu = \sum_{\mu=0}^{3} p^\mu p_\mu = p^2$$

and the metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we can write the lower momentum vector as

$$p_\mu = p^\nu g_{\mu\nu}$$

thus

$$\begin{aligned} p^\mu p_\mu &= p^\mu p^\nu g_{\mu\nu} \\ &= \left(\frac{E}{c}\right)^2 + \mathbf{p}\cdot\mathbf{p}(-1) \\ &= \left(\frac{E}{c}\right)^2 - |\mathbf{p}|^2 \\ &= m^2c^2 \end{aligned}$$

Using

$$E = \sqrt{|\mathbf{p}|^2 + m^2c^4} \tag{2.1}$$

Lorentz Transformation At rest $\mathbf{p} = 0$ and $E = mc^2$.
In the Galilean transformation in the x direction:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

where we assume absolute time, but in the Lorentz transformation:

$$\begin{aligned} x' &= \gamma(\beta ct + x) \quad \beta = \frac{v}{c} \\ ct' &= \gamma(t - \beta x) \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \\ y' &= y \\ z' &= z \end{aligned}$$

In matrix form:

$$\Lambda = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

and thus $p^\mu p_\mu$ is invariant under Lorentz transformation.

Massless particle: From the energy momentum relation

$$E^2 = |\mathbf{p}|^2 c^2 + m^2c^4$$

The massless particle has energy $E = |\mathbf{p}|c$. But we have to include the frequency (Planck) relation from quantum mechanics as well:

$$E = h\nu = \hbar\omega$$

And in the SM photons and neutrinos are massless thus

$$p^2 = p^\mu p_\mu = m^2c^2 = 0$$

Collisions Non-relativistic vs. Relativistic
Non-relativistic:

- Elastic (KE conserved)
- Inelastic (KE not conserved)

Relativistic:

- Elastic (KE conserved) e.g. particle splitting into two
- Inelastic (KE not conserved) or Rest energy and mass e.g. colliding two particles to form a new particle
  - KE increases (Explosive)
  - KE decreases (Sticky)

In the extreme case:

$$\begin{aligned} A + B &\to C \quad \text{inverse decay} \\ A &\to B + C \quad \text{decay} \end{aligned}$$

Example $\pi^+ \to \mu^+ + \nu_\mu$ (decay)
The Rest energies are $m_{\pi^+} = 135\,\mathrm{MeV/c^2}$, $m_{\mu^+} = 105\,\mathrm{MeV/c^2}$, and $m_{\nu_\mu} = 0$. But this energy is lost through the kinetic energy of the muon and muon-neutrino.
The momentum before is just the momentum of the pion

$$p_i = p_\pi = 0$$

since it is startionary. Afterward the momentum is split between the muon and neutrino

$$p_f = p_\mu + p_{\nu_\mu}$$

where energy and momentum is conserved:

$$\begin{aligned} \mathbf{p}_\mu &= -\mathbf{p}_\nu \\ m_\pi c^2 &= E_\mu + E_{\nu_\mu} \end{aligned}$$

4-momentum conservation

$$\begin{aligned} p_{before} &= p_{after} \\ p_\pi &= p_\mu + p_{\nu_\mu} \end{aligned}$$

since the massless particle has no momentum from the energy momentum relation

$$\begin{aligned} p_\nu &= p_\pi - p_\mu \\ p_\nu^2 &= (p_\pi - p_\mu)^2 \\ &= p_\pi^2 - 2p_\pi p_\mu + p_\mu^2 \\ 0 &= m_\pi^2 c^2 + m_\mu^2 c^2 - 2\frac{m_\pi c^2}{c}\frac{E_\mu}{c} \\ 2E_\mu m_\pi &= (m_\pi^2 + m_\mu^2)c^2 \\ E_\mu &= \frac{m_\pi^2 + m_\mu^2 - m_\nu^2}{2m_\pi}c^2 \end{aligned}$$

Another way is finding

$$p_\pi = p_\mu + p_\nu$$

rewritten as

$$p_\mu = p_\pi - p_\nu$$

squaring both sides gives

$$p_\mu^2 = p_\pi^2 - 2p_\pi p_\nu + p_\nu^2$$

and since $p_\nu^2 = 0$ we have

$$p_\mu^2 = p_\pi^2 - 2p_\pi p_\nu$$

which implies

$$m_\mu^2 c^2 = m_\pi^2 c^2 - 2m_\pi E_\nu$$

the Planck relation tells us

$$E_\nu = |\mathbf{p}_\nu| c = |\mathbf{p}_\mu| c$$

thus

$$2m_\pi |\mathbf{p}_\mu| c = (m_\pi^2 - m_\mu^2)c^2$$

and

$$|\mathbf{p}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c$$

Scattering experiments

- Head-on collision: (LHC)
- Fixed target collision: Beam of protons hitting a target (e.g. Carbon) (SLAC)

From momentum conservation, the head-on collision is more energy efficient as it loses the minimum amount of energy. The created particle is at rest, thus the energy is the rest energy. But the Fixed target collision has a higher energy loss since the particle loses energy since the created particle has kinetic energy.
e.g. The Anti-proton Discovery is due to the Bevatron colliding two protons to create an anti-proton

$$p + p \to p + p + p + \bar{p}$$

HW HINT: $E_{cm} < E_{fixed}$

3 Symmetries

Quiz review:

3. The Energy of the large mass is

$$Mc^2 = E_1 + E_2 = 2\gamma mc^2$$

where the energy of the smaller masses are

$$E = \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}$$

where $|\mathbf{p}| = \gamma mv$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Thus the mass $M > 2m$.

4. Using the same thought from 3. we know that the rest mass of M is greater.

Lorentz Invariant

$$p^2 = m^2 c^2$$

From Wikipedia: this is the lightlike vector. For the timelike $p^2 > 0$ and spacelike $p^2 < 0$.

Symmetries

Equilateral triangles are symmetric under 3 axes where we can flip the triangle and it is still the same. For the square, we have 4 axes, and so and so forth. All of these objects are studied in Group Theory.

Group Theory Group is a set of objects satisfying certain properies under an operation.

Properties

1. Closure: For $a, b \in G, \quad a \cdot b \in G$
2. Identity: For any $a \in G, \quad a \cdot I = I \cdot a = a$
3. Inverse: For each $a \in G, \quad a \cdot a^{-1} = a^{-1} \cdot a = 1$
4. Associativity: For $a, b, c \in G, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (optional) Commutativity: For $a, b \in G, \quad a \cdot b = b \cdot a$ AKA Abelian Group. Not all groups are commutatiive and thus are called non-Abelian groups.

Two Types of Groups

1. Finite: Finite number of elements. e.g. $Z_2 = \{1, -1\} = \{I, r\}$ where $r^2 = I$
2. Infinite: Descrete or continuous. e.g. set of integers under addition (discrete), set of real numbers under multiplication (continuous), $U(1)$ (continuous)

Examples For an isoscale triangle $Z_2 = \{1, -1\}$ and for an equilateral triangle $Z_3 = \{0, 1, 2\}$ or the operation mod 3. Which is isomorphic to

$$\equiv \left\{1, \omega, \omega^2\right\}, \quad \omega = e^{2\pi i/3}$$

For the square

$$Z_4 = \{0, 1, 2, 3, \} \equiv \{1, i, -1, -i\} \quad \text{or} \quad \left\{1, \omega, \omega^2, \omega^3\right\}$$

Thus for n elements.

$$Z_n = \left\{e^{i2\pi j/n}\right\}, \quad j = 0, 1, \dots, n-1$$

where all of these groups are Abelian.

For $n \to \infty$ We get a circle as it has an infinite number of symmetries.
In addition $j \to \infty$

$$\frac{2\pi j}{n} = \theta$$

we get

$$U = e^{i\theta} = \cos\theta + i\sin\theta$$

where $\theta \in [0, 2\pi]$, and we have the $U(1)$ group.

$$U^\dagger U = I \qquad U^\dagger = (U^*)^T$$

where the dagger is the transpose of the complex conjugate (conjugate transpose).

Standard Model

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_y$$

$U(N)$ set of unitary $N \times N$ matrices (non-Abelian in general except for $N > 1$). Taking the determinant of the matrix

$$\det\left(U^\dagger U\right) = \det I = 1$$

and

$$\det\left(U^\dagger\right)\det(U) = 1 \qquad \det\left(U^{*T}\right) = \det(U^*) = (\det U)$$

and

$$\begin{aligned} |\det U|^2 &= 1 \\ \det U &= e^{i\alpha} \quad \alpha \in [0, 2\pi] \end{aligned}$$

Choosing the phase angle $\alpha = 0$ we get

$$\det U = 1 \qquad SU(N)CU(N)$$

$\otimes$ is a direct product: Two groups F and G. For $f \in F$ and $g \in G$ we have

$$(f, g) \in F \otimes G$$

The $U(1)$ group is related to the photon γ, the $SU(2)$ group is related to the weak force $W^\pm, Z^0$, and the $SU(3)$ group is related to the strong force g (gluon).

SU(2) A set of 2×2 matrices with a determinant of 1.
Given the theorem

$$U = e^{iH}$$

for the hermitian matrix H where

$$U^\dagger U = 1 \to e^{-iH^\dagger} e^{iH} = 1$$

thus

$$H^\dagger = H$$

we take the determinant of U:

$$\det U = \det\left(e^{iH}\right) = e^{i\,\mathrm{Tr}\,H} = 1 = e^0$$

thus $\mathrm{Tr}\,H = 0$. This means that the Hermitian H is traceless.

Pauli Matrices

traceless matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus we can write the Hermitian matrix as

$$H = \frac{1}{2}\sum_i \theta_i \sigma_i = \frac{1}{2}\theta \cdot \sigma$$

where we have the group element of $SU(2)$

$$U = e^{i\theta \cdot \sigma/2}$$

From QM

$$\mathbf{S} = \frac{\hbar}{2}\sigma$$

$$\begin{aligned}
[S_y, S_z] &= iS_x \\
[S_z, S_x] &= iS_y \\
[S_x, S_y] &= iS_z [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k
\end{aligned}$$

where ϵ_{ijk} is the Levi-Civita symbol.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ interchange any two indices } (3,2,1) \\ 0 & \text{otherwise any index is repeated} \end{cases}$$

thus

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$

The Lie Algebra for $SU(2)$ is $SO(3)$ where both groups are isomorphic.

$$[L_i, L_j] = i\epsilon_{ijk}L_k \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

the generators of $SU(2)$ is $\sigma/2$. For $SU(3)$

$$U = e^{i\theta \cdot \lambda/2}$$

where we have the Gell-Mann matrices λ. In general for $SU(N)$

Addition of Angular Momenta

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and

$$[J^2, J_i] = 0 \qquad J^2 = J_x^2 + J_y^2 + J_z^2 T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_o} \frac{1}{\sqrt{\cos\theta - \cos\theta_o}} \, \mathrm{d}\theta$$

where J^2 is the Casimir operator. Since we have simultaneous eigenstates of J^2 and J_z we can write

$$|j, j_z\rangle$$

4 Symmetries

Quiz 3 Review SU(2) is the group of 2x2 unitary matrices with determinant 1. Using the basisc vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ we can write the group element as

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

or the linear combination of the basis vectors. Thus the transformation is

$$\begin{pmatrix}a'\\b'\end{pmatrix} = U(\theta)\begin{pmatrix}a\\b\end{pmatrix} = e^{i\theta\cdot\sigma/2}\begin{pmatrix}a\\b\end{pmatrix}$$

THe Lie Algebra for SU(2) is

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

and

$$[J^2, J_i] = 0$$

for simultaneous eigenstates of $|j,m\rangle$.

$$J_z|j,m\rangle = m\hbar|j,m\rangle \qquad J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$$

from the ladder operators

$$J_\pm = J_x \pm iJ_y$$

where these are not Hermitian (does not commute). Thus

$$\begin{aligned} J^2 &= J_x^2 + J_y^2 + J_z^2 \\ &= J_+J_- + J_+J_-J_z^2 \end{aligned}$$

furthermore

$$J_\pm|j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m)}\,|j, m \pm 1\rangle$$

where going up the ladder $m \to m+1$ and going down the ladder $m \to m-1$. For fixed j there is a maximum and minimum m value

$$m_{max} = j \qquad m_{min} = -j$$

so for example

$$J_+|j,j\rangle = 0 \qquad J_-|j,-j\rangle = 0$$

Spin

$$j \equiv s = 1/2, \qquad m \equiv m_s = \pm 1/2$$

The basis states are

$$(1/2, 1/2) = \begin{pmatrix}1\\0\end{pmatrix} = |\uparrow\rangle \qquad m_s = 1/2$$

$$(1/2, -1/2) = \begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle\, m_s = -1/2$$

For the addition of spin

$$\frac{1}{2}\otimes\frac{1}{2} = ? \qquad \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \qquad S_{tot} = (S_1 + S_2), ..., (S_1 - S_2) = 1, 0 \qquad m_{s,tot} = 1, 0, -1, 0$$

General Addition of Angular Momentum

$$|1,1\rangle = |\uparrow\uparrow\rangle \qquad |1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle \qquad |1,-1\rangle = |\downarrow\downarrow\rangle$$

finding the linear combination through basis transformation by using the resolution of the identity

$$\begin{aligned}|j,m\rangle &\to |j_1,m_1\rangle \otimes |j_2,m_2\rangle \\ &= \sum_{m_1,m_2} |j_1,m_1,j_2,m_2\rangle \langle j_1,m_1,j_2,m_2|j,m\rangle\end{aligned}$$

where the bra-ket is the Clebsch-Gordan coefficient. thus

$$= \sum_{m_1,m_2} c^{j,j_1,j_2}_{m,m1,m2} |j_1,m_1,j_2,m_2\rangle$$

where $m = m_1 + m_2$ and $c^{j,j_1,j_2}_{m,m1,m2}$ is the Clebsch-Gordan coefficient.

Example For the $S = 1$ state $m = 1$

$$\begin{aligned}|1,1,\rangle &= |1/2,1/2\rangle \otimes |1/2,1/2\rangle \\ &= |1/2,1/2,1/2,1/2\rangle \\ &= |\uparrow\uparrow\rangle\end{aligned}$$

For $m = 0$ we have a linear combination of the basis states

$$\begin{aligned}J_-|1,1\rangle &= \hbar\sqrt{2}|1,0\rangle \\ \text{or} \quad |1,0\rangle &= \frac{1}{\hbar\sqrt{2}}J_-|1,1\rangle\end{aligned}$$

the sum of the basis states is

$$\begin{aligned}J_-(|1/2,1/2\rangle \otimes |1/2,1/2\rangle) &= \hbar\sqrt{(1/2+1/2)(1/2-1/2+1)}\,|1/2,-1/2\rangle \otimes |1/2,1/2\rangle \\ &+ |1/2,1/2\rangle \otimes |1/2,-1/2\rangle \\ &= \hbar(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\end{aligned}$$

or

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

for $m = -1$ we have

$$J_-|1,0\rangle = \hbar\sqrt{2}|1,-1\rangle$$

where

$$|1,-1\rangle = |1/2,-1/2\rangle \otimes |1/2,-1/2\rangle \qquad = |\downarrow\downarrow\rangle$$

Now for $S = 0$, $m = 0$ we have

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

since it is the way to make it orthogonal to $|1,0\rangle$. Therefore

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

Thus the there are 3 triplet states $m_s = 1,0,-1$ and 1 singlet state $m_s = 0$.

Isospin

$$m_p = 938.3 \text{ MeV/c}^2 \qquad m_n = 939.6 \text{ MeV/c}^2$$

why are they so close? Heisenberg postulated an isospin state of a nucleon N as

$$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \left| p \right\rangle + \beta \left| n \right\rangle$$

with

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the isospin state of the proton and neutron are

$$\left| p \right\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \qquad \left| n \right\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

1. Strong interactions preserve isospin symmetry
2. EM & Weak interactions do not preserve isospin symmetry

Examples

Pions: π^+, π^0, π^- where the approximate symmetry is a triplet state

$$\begin{aligned} \pi^+ &= |1, 1\rangle \qquad I = 1, \quad I_3 = 1 \\ \pi^- &= |1, 0\rangle \qquad I = 1, \quad I_3 = 0 \\ \pi^0 &= |1, -1\rangle \qquad I = 1, \quad I_3 = -1 \end{aligned}$$

Δ-baryons:

$$\begin{aligned} \Delta^{++} &= |3/2, 3/2\rangle \qquad I = 3/2, \quad I_3 = 3/2 \\ \Delta^{+} &= |3/2, 1/2\rangle \qquad I = 3/2, \quad I_3 = 1/2 \\ \Delta^{0} &= |3/2, -1/2\rangle \qquad I = 3/2, \quad I_3 = -1/2 \\ \Delta^{-} &= |3/2, -3/2\rangle \qquad I = 3/2, \quad I_3 = -3/2 \end{aligned}$$

where Δ^{--} is an antiparticle of Δ^{++}. We write from the highest to lowest from the empirical Gellman-Nishijima formula

$$Q = I_3 + \frac{1}{2}(B + S)$$

where Q is the charge, I_3 is the third component of isospin, B is the baryon number, and S is the strangeness.

Pions

Since a Pion is a *meson* and not a baryon, it has a baryon number of 0. Thus with no strangeness

$$S = 0 \qquad B = 0$$

Nucleons

$$S = 0 \qquad B = 1$$

$$Q = \begin{cases} 1/2 + 1/2(1+0) = 1 & \text{proton} \\ -1/2 + 1/2(1+0) = 0 & \text{neutron} \end{cases}$$

For all elementary particles there is a general formula

$$Q = I_3 + \frac{Y}{2}$$

where Y is the hyper charge $U(1)_Y$.

Power of Symmetry: Applications

1. Deuteron (neutron of deuterium): Two-Nucleon system

$$I = 1 \quad \text{or} \quad 0 \qquad I_3 = 1, 0, -1 \quad \text{or} \quad 0 \text{ (singlet)}$$

$$\begin{aligned} |1,1\rangle &= |p,p\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle + |n,p\rangle) \\ |1,-1\rangle &= |n,n\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}(|p,n\rangle - |n,p\rangle) \end{aligned}$$

experimentally, we only see the singlet state because we see only one deuteron state. Thus we can only see a isospin state of $I = 0$.

Two-nucleon potential $\propto \mathbf{I}_1 \cdot \mathbf{I}_2$ where we hae the total isospin

$$\mathbf{I}^2 = (\mathbf{I}_1 + \mathbf{I}_2)^2 = \mathbf{I}_1^2 + \mathbf{I}_2^2 + 2\mathbf{I}_1 \cdot \mathbf{I}_2$$

where the s^2 term is

$$s^2 = 1/2(1/2+1)\hbar^2 = \frac{3}{4}\hbar^2$$

Thus

$$\mathbf{I}_1^2 + \mathbf{I}_2^2 = \frac{3}{2}$$

and

$$\begin{aligned} \mathbf{I}_1 \cdot \mathbf{I}_2 &= \frac{1}{2}\left(\mathbf{I}^2 - 3/2\right)^{3/2} \\ &= \begin{cases} 1/2(1(1+1) - 3/2) & = 1/4 \quad \text{triplet} \\ 1/2(0(0+1) - 3/2) & = -3/4 \quad \text{singlet} \end{cases} \end{aligned}$$

5 Symmetries

Quiz 5 Review For j

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

For $2j+1$

$$2 \otimes 2 = 3 \oplus 1$$

Isospins of particles

1. pion: 1
2. deuteron: 0
3. Δ-baryons: 3/2
4. nucleons: 1/2

The strong ineteraction preserves I and I_3, and the weak interactions do not preserve I and I_3 (e.g. in beta decay the iso spin of the neutron (-1/2) go to an iso spin of the proton (1/2)). In E&M the isospin preserves only I and not I_3 (e.g. π_o decay to two photons $\gamma\gamma$: $I_3 = 0$ for the π_o and $I_3 = 0$ for the two photons).

Applications of Isospin: Nucleon-nucleon Scattering

$$p + p \to D + \pi^+$$
$$p + n \to D + \pi^0$$
$$n + n \to D + \pi^-$$

The relative probabilities of these processes: we get this from the amplitude A where the probability $|A|^2$ is proportional to the cross section $\sigma = \pi r^2$ (the cross section of a sphere, but this is not a solid sphere and rather a 'fuzzy' sphere). With the fact that 'strong interactions preserve isospin' we have the the ratio of the cross sections

$$\sigma_a : \sigma_b : \sigma_c$$

For all three processes the RHS the isospin is

$$I_{tot} = 0 \otimes 1 = 1$$

on the left hand side

$$I_{tot} = \frac{1}{2} \otimes \frac{1}{2} = 0 \quad \text{or} \quad 1$$

(a) The ratio of getting an isospin of 1 on the left hand side for the first process

$$|pp\rangle = |11\rangle$$

(c) for the third proccess

$$|nn\rangle = |1, -1\rangle$$

(b) The second is the linear combination of $|10\rangle$ and $|00\rangle$

$$|pn\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |00\rangle)$$

the $|00\rangle$ does not contribute to the isospin of 1. Thus the ratio of the probability is

$$A_a : A_b : A_c = 1 : \frac{1}{\sqrt{2}} : 1$$

and the ratio of the cross sections is

$$\sigma_a : \sigma_b : \sigma_c = 1 : \frac{1}{2} : 1$$

Example 3 Pion-nucleon Scattering

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \quad I = 1, \qquad \begin{pmatrix} p \\ n \end{pmatrix} \quad I = 1/2$$

So the total isospin j is

$$I_{tot} = 1/2 \otimes 1 = 3/2 \oplus 1/2$$

and for $2j + 1$

$$3 \otimes 2 = 4 \oplus 2$$

The elastic processes are (from $a \to f$)

$$\begin{aligned} \pi^+ + p &\to \pi^+ + p \\ \pi^0 + p &\to \pi^0 + p \\ \pi^- + p &\to \pi^- + p \\ \pi^+ + n &\to \pi^+ + n \\ \pi^0 + n &\to \pi^0 + n \\ \pi^- + n &\to \pi^- + n \end{aligned}$$

and the charge-exchange processes are (from $g \to j$)

$$\begin{aligned} \pi^+ + n &\to \pi^0 + p \\ \pi^0 + p &\to \pi^+ + n \\ \pi^- + p &\to \pi^0 + n \\ \pi^0 + n &\to \pi^- + p \end{aligned}$$

The states of the 3/2 isospin are

$$|3/2, 3/2\rangle, |3/2, 1/2\rangle, |3/2, -1/2\rangle, |3/2, -3/2\rangle$$

and the states of the 1/2 isospin are

$$|1/2, 1/2\rangle, |1/2, -1/2\rangle$$

so we have the following states

$$\begin{aligned} \left|\pi^+ p\right\rangle &= |1, 1\rangle \otimes |1/2, 1/2\rangle = |3/2, 3/2\rangle \\ \left|\pi^- n\right\rangle &= |1, -1\rangle \otimes |1/2, -1/2\rangle = |3/2, -3/2\rangle \end{aligned}$$

for the obvious highest and lowest isospin states. Carrying on...

$$\left|\pi^+ n\right\rangle = |1, 1\rangle \otimes |1/2, -1/2\rangle$$

this is the linear combination of $|3/2, 1/2\rangle$ and $|1/2, 1/2\rangle$, and so on. To find the proportional cross sections we know that

$$\langle i|f\rangle \propto A \qquad |\langle i|f\rangle|^2 \propto \sigma$$

We can use the Clebsch-Gordan coefficients to find the linear combination of the states. For example

$$\left|\pi^+ n\right\rangle = |3/2, 1/2\rangle + |1/2, 1/2\rangle$$

where the Clebsch-Gordan coefficient is

$$\langle 3/2, 1/2, 1/2, 1/2|3/2, 1/2\rangle = \sqrt{\frac{2}{3}}$$

e.g. for the $\pi^+ p$ state

$$\left|\pi^+ p\right\rangle = |3/2, 3/2\rangle$$
$$\langle 3/2, 3/2, 1/2, 1/2|3/2, 3/2\rangle = 1$$

using the lowering operator

$$J_- |j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle$$

so

$$J_- |3/2, 3/2\rangle = \hbar\sqrt{3}\, |3/2, 1/2\rangle$$

applying the lower operator to $J_{1-} + J_{2-}$ we get

$$\begin{aligned} J_-(|11\rangle \otimes |1/2, 1/2\rangle) &= \hbar\sqrt{2}\, |10\rangle \otimes |1/2, 1/2\rangle + \hbar\sqrt{1}\, |11\rangle \otimes |1/2, -1/2\rangle \\ &= \hbar\sqrt{2}\, |10\rangle \otimes |1/2, 1/2\rangle + \hbar\, |11\rangle \otimes |1/2, -1/2\rangle \end{aligned}$$

we then get

$$\begin{aligned} |3/2, 1/2\rangle &= \sqrt{2/3}\, |11\rangle \otimes |1/2, 1/2\rangle + \sqrt{1/3}\, |10\rangle \otimes |1/2, 1/2\rangle \\ &= \sqrt{2/3}\, \left|\pi^+ p\right\rangle + \sqrt{1/3}\, \left|\pi^+ n\right\rangle \end{aligned}$$

and the orthogonal state is

$$|1/2, 1/2\rangle = \sqrt{2/3}\, \left|\pi^+ p\right\rangle - \sqrt{1/3}\, \left|\pi^+ n\right\rangle$$

and so on for the other states. At the end we will find that the ratio of the total cross sections (adding up the matching elastic and exchange processes) is 3.
The amplitude has a factor

$$\left\langle \pi^+ p \middle| \pi^+ p \right\rangle = \langle 3/2, 3/2|3/2, 3/2\rangle = M_3$$

where for example

$$\begin{aligned} (\sqrt{2}3\, \langle 3/2, 1/2| - 1/\sqrt{3}\, \langle 1/2, 1/2|)(\sqrt{2}3\, |3/2, 1/2\rangle - 1/\sqrt{3}\, |1/2, 1/2\rangle) = \\ = 2/3\, \langle 3/2, 1/2|3/2, 1/2\rangle - 1/3\, \langle 1/2, 1/2|1/2, 1/2\rangle \\ = 2/3 M_3 - 1/3 M_1 \end{aligned}$$

for $M_3 >> M_1$ the ratio is 4/9, and for $M_3 << M_1$ the ratio is 1/3.

$SU(3)$

$$\begin{pmatrix} p \\ n \end{pmatrix} \quad SU(2)\text{doublet}$$

where the spins are

$$p : uud \quad Q_u = 2/3$$
$$n : udd \quad Q_d = -1/3$$

For the two spins

$$\begin{pmatrix} u \\ d \end{pmatrix}$$

the isospins are

$$I = 1/2, \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

for the up and down quarks respectively. In reality we have six quarks

- Light quarks: u, d, s
- Heavy quarks: c, b, t

For the light quaks we have a $SU(3)$ symmetry

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

the masses are all different:

$$m_u \approx 2\,\mathrm{MeV/c^2} \qquad m_d \approx 4\,\mathrm{MeV/c^2} \qquad m_s \approx 95\,\mathrm{MeV/c^2}$$

so we have to add a flavor symmetry to the $SU(2)$ isospin symmetry:

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u \\ d \end{pmatrix} \oplus s$$

or the $SU(3)$ symmetry

$$SU(3)_f \to SU(2)_I \otimes U(1)_y$$

From $SU(2)$ algebra:

$$[J_i, J_j] = i\epsilon_{ijk} I_k \qquad J_i = \sigma_i/2$$

For the three pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now for $SU(3)$: We know that the generators

$$U = e^{i\theta\cdot\lambda/2}$$

For $SU(N)$ we have $N^2 - 1$ generators. For $SU(3)$ we have 8 generators. The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Lecture 6: **2/3/24**

Symmetries??

Quiz 6 Review

(1) With respect to the QCD scale (≈ 200 MeV) the masses of the quarks are divided into light and heavy quarks.

(2) (2) The effective mass mass is much larger than the mass of the light quark.

(3) $SU(2)$ is a subgroup of $SU(3)$.

For the $SU(3)$: a 3x3 unitary matrix with determinant 1. There are $n^2 - 1 = 8$ generators where

$$U = e^{iH}$$

where H is the Hermitian matrix:

$$U^\dagger U = 1 \qquad H^\dagger = H$$
$$\det U = 1 \qquad \operatorname{tr} H = 0$$
$$\det M = e^{\operatorname{tr} \ln M}$$

or Hermitian matrices are traceless.

Gell-Mann Matrices Starting with the Pauli matrices but in 3x3 form

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and moving the sectors of the matrices we also get

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \cancel{\lambda_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

and

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

but λ_9 is not linearly independent since it can be written as a linear combination of $\lambda_3 + \lambda_8$.

Commutation Relation For $SU(2)$ we know

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

and for $SU(3)$ we have

$$[J_i, J_j] = i f_{ijk} J_k$$

where f_{ijk} are the structure constants.

Subgroup We know that $SU(2) \leq SU(3)$ (where $\leq$ means 'is a subgroup of'). So $\{\lambda_1, \lambda_2, \lambda_3\}$ forms an $SU(2)$ sub-algebra. Also

$$\{\lambda_4, \lambda_5, a\lambda_3 + b\lambda_8\}$$
$$\{\lambda_6, \lambda_7, a\lambda_3 + b\lambda_8\}$$

are also $SU(2)$ sub-algebras. NOTE that

$$SU(3) \neq SU(2) \otimes SU(2) \otimes SU(2)$$

Isospin and Strangeness

$$\lambda_3/2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad I = 1/2$$

and the isospins are

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad I_3 = 1/2 \quad \text{or} \quad -1/2$$

and the strangeness is

$$S : I = 0$$

For λ_8 we define a hypercharge y such that

$$\lambda_8/2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{3}} \frac{2}{\sqrt{3}}$$

and

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad y = 1/3$$

and for the strangeness $S : y = -2/3$. This is because the Gell-Mann-Nishijima formula

$$Q = I_3 + \frac{y}{2}$$

$$2/3 = 1/2 + 1/2(1/3) \qquad -1/3 = 0 - 1/2(2/3)$$

For the triplet

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

and the anti-triplet

$$\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \quad I = 1/2, \quad I_3 = 1/2, -1/2, 0$$

Mesons $(q, \bar{q})$ in $SU(3)$ is

$$3 \otimes \bar{3} = 8 \oplus 1$$

where the 8 is the octet and the 1 is the singlet. We can do this using the Young Tableaux. For $SU(3)$ we have a 3 fundamental and $\bar{3}$ anti-fundamental.

$$\begin{array}{|c|c|} \hline N & N+1 \\ \hline N-1 & N \\ \hline N-2 \\ \cline{1-1} \end{array}$$

Using Hook Law:

$$\dim = \frac{\Pi_i N_i}{\Pi_i h_i}$$

$$\frac{N(N+1)(N-1)N(N-2)}{1\cdot 3\cdot 4\cdot 1\cdot 2}$$

so

$$3\otimes\bar{3} = \boxed{3}\otimes\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} =$$

or

$$\boxed{a}\otimes\begin{array}{|c|}\hline b\\\hline c\\\hline\end{array} = \begin{array}{|c|c|}\hline 3 & 4\\\hline 2 \\\cline{1-1}\end{array}\oplus\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} = \frac{3\cdot 4\cdot 2}{1\cdot 3\cdot 1}\oplus\frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}$$

Goin from $SU(3)$ to $SU(2)$ we have

$$\begin{pmatrix} u\\ d\\ s\end{pmatrix}\to\begin{pmatrix} u\\ d\end{pmatrix}+s$$

or

$$3\to 2_1+1_{-2}$$

where the hypercharges are subscripts. So the octet is

$$\begin{aligned}3\otimes\bar{3} &= (2_1+1_{-2})\otimes(2_{-1}+1_2)\\ &= (2_1\otimes 2_{-1})\oplus(2_1\otimes 1_2)\oplus(1_{-2}\otimes 2_{-1})\oplus(1_{-2}\otimes 1_2)\\ &= (3_0\oplus 1_0)\oplus 2_3\oplus 2_{-3}\oplus 1_0\\ &= 8\oplus 1\end{aligned}$$

This is called the eightfold way.

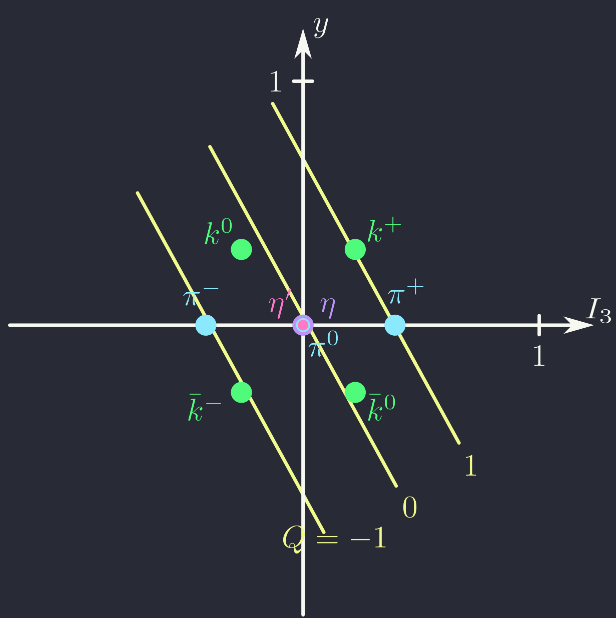

Figure 5.1: The eightfold way

Eightfold way Where η is a $SU(2)$ singlet but a $SU(3)$ octet. and η' is a $SU(3)$ singlet. If $SU(3)_f$ was a good symmetry, expet all these 8 mesons to have similar mass. All of these obey up to a factor of 2.

Baryons (222) or $3\otimes 3\otimes 3 = 10\oplus 8\oplus 8\oplus 1$. The baryons are antisymmetric as each quark is a fermion. Using the Young Tablauex

$$3\otimes 3 = \boxed{a}\otimes\boxed{b} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \oplus \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} = \frac{3\cdot 4}{1\cdot 3}\oplus\frac{3\cdot 2}{1\cdot 2} = 6$$

so

$$6\otimes 3 = \begin{array}{|c|c|}\hline a & b \\ \hline\end{array}\otimes\boxed{c} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} = \frac{3\cdot 4\cdot 5}{3\cdot 2\cdot 1}\oplus\frac{3\cdot 4\cdot 2}{1\cdot 3\cdot 1} = 10\oplus 8$$

This is a 10-plet as shown in the figure.

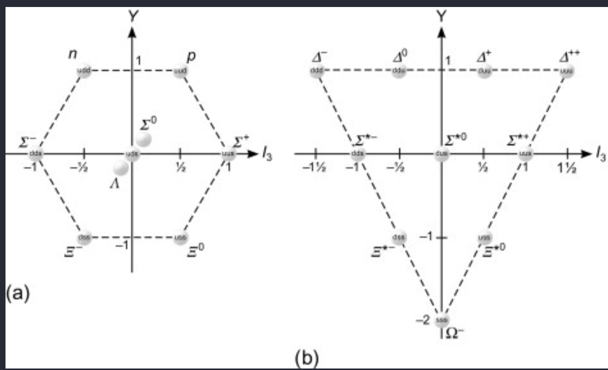

Figure 5.2: The 10-plet of baryons

Parity (Discrete Symmetry) For a simple reflection on a z axis the point $A = (x, y, z)$ goes to $A' = (x, -y, z)$. But for parity operation we go to $P(A) = (-x, -y, -z)$ or in the general form

$$P(\mathbf{a}) = -\mathbf{a}$$

also known as inversion. Taking the parity again we get

$$P^2(\mathbf{a}) = P(-\mathbf{a}) = \mathbf{a}$$

Thus it has a discrete Z_2 symmetry.

Pseudo-vector (axial vector) for a pseudo vector $\mathbf{c}$

$$P(\mathbf{c}) = \mathbf{c}$$

where cross products (of two vectors) are pseudo-vectors. For example

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

and

$$P(\mathbf{c}) = P(\mathbf{a} \times \mathbf{b}) = (-\mathbf{a}) \times (-\mathbf{b}) = \mathbf{c}$$

e.g. of pseudo-vectors:

- Torque: $\tau = \mathbf{r} \times \mathbf{F}$
- Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
- Magnetic field: $\mathbf{B} = \mathbf{E} \times \mathbf{v}$

But for the lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

the cross product of a vector and a pseudo-vector is a vector, so the lorentz force is a vector. Also from the general definition

$$P(\mathbf{F}) = \frac{q}{c} P(\mathbf{v}) \times P(\mathbf{B}) = \frac{q}{c}(-\mathbf{v}) \times (-\mathbf{B}) = -\mathbf{F}$$

The weak interaction violates parity...

Scalar For a scalar $s = \mathbf{a} \cdot \mathbf{b}$ is invariant under parity:

$$P(s) = P(\mathbf{a} \cdot \mathbf{b}) = (-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} = s$$

for a pseudo-scalar p (a dot product of a vector and pseudo-vector):

$$P(p) = P(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})) = P(\mathbf{a})P(\mathbf{b} \times \mathbf{c}) = -\mathbf{a}(\mathbf{b} \times \mathbf{c}) = -p$$

So the partities of the four types of quantities:

- Scalar: $P(s) = s$
- Pseudo-scalar: $P(p) = -p$
- Vector: $P(\mathbf{v}) = -\mathbf{v}$
- Pseudo-vector: $P(\mathbf{c}) = \mathbf{c}$

Intrinsic Parity The parity of a fermion is

$$P(\text{fermion}) = -P(\text{anti-fermion})$$

for bosons

$$P(\text{boson}) = P(\text{anti-boson})$$

For composite particles i.e. mesons $q\bar{q}$ and baryons qqq:

$$P(\text{meson}) = -1 \quad \text{or} \quad (+1)(-1) = -1$$

Since mesons are two pairs (particle, antiparticle) the parity is always negative. For baryons we can only have a positive parity:

$$P(\text{baryon}) = (+1)^3 = +1$$

For spherical harmonics $Y_l^m(\theta, \phi)$ under parity of each term:

$$\mathbf{r} \to -\mathbf{r} \qquad \theta \to -\theta \qquad \phi \to \pi + \phi$$

so

$$P(Y_l^m(\theta, \phi)) = (-1)^l Y_l^m(\theta, \phi)$$

and for excited states

$$P = (-1)^l \times P(\text{ground state})$$

where l is the orbital angular momentum.

Parity Violation THe $\theta - \tau$ puzzle: given two particles

$$\begin{aligned} \theta^+ &\to \pi^+ + \pi^0 \qquad & P = +1 \\ \tau^+ &\to \pi^+ + \pi^0 + \pi^0 \qquad & P = -1 \end{aligned}$$

the same two particles are found to be the same particle K^+ having the same mass and lifetime, but this violates the parity. To solve this puzzle came from the Colmbia University known as Wu's experiment (wikipedia):

$${}^{60}_{27}\text{Co} \to {}^{60}_{28}\text{Ni} + e^- + \bar{\nu}_e$$

The Cobalt has a spin state of $J = 5$, the Nickel has spin $J = 4$ and the spin states of the electron-antielectron pair is $J = 1$. The electron is always emitted in the direction opposite of the Cobalt spin, and when the magnetic field was inverted, the electrons were emitted in the opposite direction of nuclear spin. This breaks parity, because in the mirror world, the spin of the electron would be in the same direction as the nuclear spin.

Helicity From spin $\mathbf{s}$ and momentum $\mathbf{p}$ we can define the helicity

$$\begin{aligned} \lambda &= \frac{\mathbf{s} \cdot \mathbf{p}}{|\mathbf{s}||\mathbf{p}|} \\ &= \begin{cases} +1 & \text{right-handed} \\ -1 & \text{left-handed} \end{cases} \end{aligned}$$

But this depends on the reference frame. e.g. for a case where $\mathbf{p}$ is faster and in the same direction of $\mathbf{s}$ the helicity is $\lambda = +1$, but in the reference frame faster that $\mathbf{p}$ the helicity is $\lambda = -1$.

Massless Particles For massless particles the helicity is the same in all reference frames because the speed is always c and thus the helicity is well-defined. For an electron, we can get a frame where the momentum is different for changes in reference frames.

Back to Wu's experiment The spin of the electron and neutrino are in the same direction as the spin of the Cobalt and Ni.

Note: In the SM

- neutrinos are always left-handed; $\lambda = -1$
- anti-neutrinos are always right-handed; $\lambda = +1$

Thus the electron momentum is in the opposite direction of spin as shown in Figure 5.3

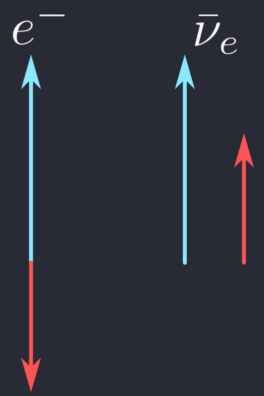

Figure 5.3: Wu's experiment

Another Example Pions and muons

$$\pi^+ \to \mu^+ + \nu_\mu \quad \text{or} \quad \to e^+ + \nu_e$$

From the comparison of masses

$$m_\pi = 140\,\text{MeV} \qquad m_\mu = 105\,\text{MeV} \qquad m_e = 0.511\,\text{MeV}$$

we would think the small mass reaction would be more likely due to the higher velocity, but this is not the case. For the pion the spin is 0, so the combination of spin will be in opposite directions. From Figure 5.4 we can see that the anti-leption(anti particle) must be left-handed and thus the lepton must also be left-handed. This is a parity violation. Thus the less favored reaction $u^+ + \nu_\mu$ is seen 99.7% of

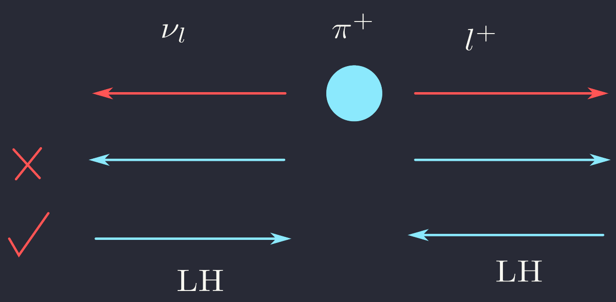

Figure 5.4: Pion decay

the time.

Anti-charged lepton has to be left-handed in this process of appoximate

$$\Gamma \propto m_\ell^\beta$$

where Γ is the decay rate.

Muon decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

This 3 body decay for a polarized muon (choosing the handedness of the muon) we have the following possiblities:

1. LH
2. RH

For maximum energy to the electron, the electron goes in one direction while the neutrinos go in the opposite direction as shown in Figure 5.5. From this the RH case is less favored than the LH case because of the helicity of the neutrinos.

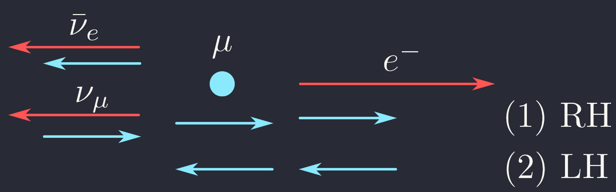

Figure 5.5: Muon decay

Homework 1

Due 1/24

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1. Gravity vs. E&M
Given the force of gravitational attraction

$$\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$$

and the force of electrostatic repulsion

$$\mathbf{F}_e = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\hat{\mathbf{r}}$$

The ratio of the two forces between two electrons is

$$\frac{F_g}{F_e} = \frac{4\pi\epsilon_0 G m_e^2}{q_e^2}$$

Using the values for the constants

$$G = 6.67\times10^{-11}\,\mathrm{N\,m^2/kg^2},\quad \epsilon_0 = 8.85\times10^{-12}\,\mathrm{C^2/N\,m^2}$$
$$m_e = 9.11\times10^{-31}\,\mathrm{kg},\quad q_e = 1.60\times10^{-19}\,\mathrm{C}$$

we find

$$\boxed{\frac{F_g}{F_e} = 2.40\times10^{-43}}$$

this tells us that the denominator (the electrostatic force) is *much* larger than the numerator (the gravitational force), which convinces us that gravitaional forces are negligible for elementary particles.

2. Mesons and Baryons

(a) For mesons, you can have n possible quarks and n possible antiquarks, thus there are $\boxed{n^2}$ combinations.

For baryons order does not matter we have to make sure not to double count instances such as (uud) and (udu): This is essentially a combination problem with the solution

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \boxed{\frac{n(n-1)(n-2)}{6}}$$

(b) Given the 6 flavors of quarks, we would expect $6^2 = 36$ mesons and $\binom{6}{3} = 20$ baryons.

(c) We haven't found all of them because of energy required to observe the heavier particles. In the Particle Data Group website, the heaviest baryon is in the order of 6000 MeV and the LHC has a beam energy of 6.5 TeV and the energy consumption is about 1.3 TWh per year compared to the global energy consumption of 20 000 TWh per year (source). In addition to the enourmous energy required to produce these particles, they are also very unstable and decay very quickly thus detecting them require us to measure at very small time scales.

3. Global Conservation Laws

(a) $n \to \bar{p} + e^+ + \nu_e$

not valid: violates Baryon number conservation

(b) $\nu_e + n \to p + e^-$
valid

(c) $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$
valid

(d) $\mu^- \to e^- + \gamma$
not valid because it violates electron and muon lepton number conservation

(e) $e^+ + e^- \to \gamma$
valid

4. Nuclear β-decay

(a)

$$^A_Z X \to ^A_{Z+1} Y + e^-$$

From the conservation of momentum

$$p^\mu_X = p^\mu_Y + p^\mu_e \quad \text{or} \quad p^\mu_Y = p^\mu_X - p^\mu_e$$

squaring both sides

$$p^2_Y = p^2_X + p^2_e - 2p_X \cdot p_e$$

and since

$$p^2_X = m^2_X c^2, \quad p^2_Y = m^2_Y c^2, \quad p^2_e = m^2_e c^2$$

and

$$p_X p_e = \frac{E_X}{c}\frac{E_e}{c} - \mathbf{p}_X \cdot \mathbf{p}_e$$

but we know that particle X has momentum $\mathbf{p}_X = 0$ and rest mass $E_X = m_X c^2$ so

$$m^2_Y c^2 = m^2_X c^2 + m^2_e c^2 - 2m_X E_e$$

solving for the Energy of the outgoing particle is

$$\boxed{E_e = \frac{m^2_X + m^2_e - m^2_Y}{2m_X} c^2}$$

to find the momentum of the outgoing electron we use energy-momentum relation

$$E^2_e = |\mathbf{p}_e|^2 c^2 + m^2_e c^4$$

or

$$|\mathbf{p}_e|^2 = \frac{E^2_e}{c^2} - m^2_e c^2$$

using the energy of the outgoing electron we found earlier:

$$\begin{aligned}
|\mathbf{p}_e|^2 &= c^2 \left(\frac{m^2_X + m^2_e - m^2_Y}{2m_X} \right)^2 - m^2_e c^2 \\
&= \frac{c^2}{4m^2_X}(m^2_X + m^2_e - m^2_Y)^2 - \left(\frac{c^2}{4m^2_X} \right) 4m^2_X m^2_e \\
&= \frac{c^2}{4m^2_X}(m^4_X + m^4_e + m^4_Y + 2m^2_X m^2_e - 2m^2_X m^2_Y - 2m^2_e m^2_Y - 4m^2_X m^2_e) \\
&= \frac{c^2}{4m^2_X}(m^4_X + m^4_e + m^4_Y - 2m^2_X m^2_e - 2m^2_X m^2_Y - 2m^2_e m^2_Y)
\end{aligned}$$

and therefore the momentum of the outgoing electron is

$$\boxed{|\mathbf{p}_e| = \frac{c}{2m_X}\sqrt{m_X^4 + m_e^4 + m_Y^4 - 2m_X^2 m_e^2 - 2m_X^2 m_Y^2 - 2m_e^2 m_Y^2}}$$

(b) For the decay including an anti-neutrino

$$^A_Z X \rightarrow ^A_{Z+1} Y + e^- + \bar{\nu}_e$$

For the massless neutrino, the energy is

$$E_\nu = |\mathbf{p}_\nu| c$$

or from planck's relation

$$E_\nu = h\nu = \frac{hc}{\lambda}$$

so the energy of the neutrino is

$$\boxed{E_\nu = \frac{6.63 \times 10^{-34}\,\mathrm{J\,s} \cdot 3 \times 10^8\,\mathrm{m\,s^{-1}}}{10^{-15}\,\mathrm{m}} = 1.99 \times 10^{-10}\,\mathrm{J} = 1240\,\mathrm{MeV}}$$

and the momentum of the neutrino is

$$\boxed{|\mathbf{p}_\nu| = \frac{E_\nu}{c} = \frac{1.99 \times 10^{-10}\,\mathrm{J}}{3 \times 10^8\,\mathrm{m\,s^{-1}}} = 6.63 \times 10^{-19}\,\mathrm{kg\,m\,s^{-1}} \quad \text{or} \quad 1240\,\frac{\mathrm{MeV}}{\mathrm{c}}}$$

This is much larger compared to the typical neutrino energy (keV). This means that the neutrino could not have come from inside the nucleus.
Using the Heisenberg uncertainty principle

$$\Delta p \Delta x \geq \frac{h}{4\pi} \quad \text{or} \quad \Delta p \geq \frac{h}{4\pi \Delta x}$$

and the typical size of a nucleus is $\Delta x \approx 1 \times 10^{-15}\,\mathrm{m}$ so

$$\Delta p \geq \frac{6.63 \times 10^{-34}\,\mathrm{J\,s}}{4\pi \cdot 10^{-15}\,\mathrm{m}} = 0.53\,\frac{\mathrm{J}}{\mathrm{m/s}}$$

or in more convenient units $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$ and $\mathrm{c} = 3 \times 10^8\,\mathrm{m/s}$:

$$\Delta p \geq 0.53\,\frac{\mathrm{J}}{\mathrm{m/s}}\,\frac{1\,\mathrm{eV}}{1.6 \times 10^{-19}\,\mathrm{J}}\,\frac{3 \times 10^8\,\mathrm{m/s}}{\mathrm{c}}$$

$$\boxed{\Delta p \geq 99\,\frac{\mathrm{MeV}}{\mathrm{c}}}$$

and the energy of the neutrino (a massless particle) is

$$\boxed{E_\nu = |\mathbf{p}_\nu| c \geq 99\,\mathrm{MeV}}$$

Compared to the typical neutrino energy of 1 keV, this is much larger and thus the neutrino could not have come from inside the nucleus.

Homework 2

Due 1/31

1. Muon Decay
The minimum energy of the electron would be equivalent to the rest mass

$$E_{\text{min}} = m_e c^2 = 9.11 \times 10^{-31}\,\text{kg} \cdot (3 \times 10^8\,\text{m/s})^2 = 8.2 \times 10^{-14}\,\text{J} * \frac{1\text{eV}}{1.6 \times 10^{-19}\,\text{J}} = \boxed{0.512\,\text{MeV}}$$

The maximum energy of the electron would be when the electron moves in one direction and both neutrinos move together in the opposing direction. Thus we can treat this like a two body decay with the energy of the electron:

$$E_e = \frac{m_\mu^2 + m_e^2 - M^2}{2m_u} c^2$$

where M is the sum of the masses of the neutrinos $M = 0$ (neutrinos are massless lol). Using the mass of the muon and electron:

$$\begin{aligned} m_e &= 9.11 \times 10^{-31}\,\text{kg} \cdot \frac{(3 \times 10^8\,\text{m/s})^2}{1.6 \times 10^{-19}\,\text{J/eV}} = 0.51\,\text{MeV} \\ m_\mu &= 1.88 \times 10^{-28}\,\text{kg} = 106\,\text{MeV} \end{aligned}$$

The maximum energy of the electron is then

$$E_{\text{max}} = \frac{m_\mu^2 + m_e^2}{2m_\mu} c^2 = \boxed{53\,\text{MeV}}$$

2. (a) For the head-on collision, we know that the initial energy is the sum of the rest masses of the protons and the minimum energy required to produce the antiproton. Thus

$$\begin{aligned} E_i &= E_f \\ 2m_p c^2 + E_{\text{min}} &= 4m_p c^2 \\ E_{\text{min}} &= 2m_p c^2 = \boxed{1.88\,\text{GeV}} \end{aligned}$$

(b) For the fixed target, we first look at the total momentum four vector before the collision: The zeroth component of the total momentum is

$$p^0 = \frac{E}{c} + \frac{E_{rest}}{c} = \frac{E}{c} + m_p c$$

thus the four-vector before the collision is

$$p^\mu = \left(\frac{E}{c} + m_p c, |\mathbf{p}|\right)$$

to find the four-vector after the collision, we can use the center of momentum frame (CM) where the two protons are viewed as going towards each other at the same speed like part (a), so the total three-vector momentum is zero. Thus the zeroth component of the four-vector is jus the sum of the rest masses of the protons and the antiproton

$$p^{\mu'} = (4mc, 0)$$

we can then exploit the invariant dot product of the four-vectors to find the minimum energy

$$\begin{aligned} p^\mu p_\mu &= p^{\mu'} p'_\mu \\ \left(\frac{E}{c} + m_p c\right)^2 - |\mathbf{p}|^2 &= (4mc)^2 \end{aligned}$$

where the can use the energy momentum relation

$$E^2 = |\mathbf{p}|^2 c^2 + m^2 c^4$$
$$|\mathbf{p}|^2 = \frac{1}{c^2}(E^2 - m^2 c^4)$$

to solve for the minimum energy

$$\begin{aligned}
\left(\frac{E}{c} + m_p c\right)^2 - \frac{1}{c^2}(E^2 - m_p^2 c^4) &= (4mc)^2 \\
\frac{E^2}{c^2} + 2m_p E + m_p^2 c^2 - \frac{E^2}{c^2} + m_p^2 c^2 &= 16m^2 c^2 \\
2m_p E &= 14 m_p c^2 \\
E = 7m_p c^2 &= \boxed{6.6\,\mathrm{GeV}}
\end{aligned}$$

Thus the head-on collision requires less energy to produce the antiproton.

3. (a) Given

$$s \equiv \frac{(p_A + p_B)^2}{c^2}, \quad t \equiv \frac{(p_A - p_C)^2}{c^2}, \quad u \equiv \frac{(p_A - p_D)^2}{c^2}$$

the sums of the Mandelstroms variables are

$$s + t + u = \frac{1}{c^2}\left[(p_A + p_B)^2 + (p_A - p_C)^2 + (p_A - p_D)^2\right]$$

where we expand the squares

$$\begin{aligned}
[\,] &= p_A^2 + p_B^2 + 2p_A \cdot p_B + p_A^2 + p_C^2 - 2p_A \cdot p_C + p_A^2 + p_D^2 - 2p_A \cdot p_D \\
&= 3p_A^2 + p_B^2 + p_C^2 + p_D^2 - 2p_A \cdot p_B - 2p_A \cdot p_C - 2p_A \cdot p_D
\end{aligned}$$

spliting the $3p_A^2$ into $p_A^2 + 2p_A^2$ we can factor out the dot products

$$[\,] = p_A^2 + p_B^2 + p_C^2 + p_D^2 - 2p_A \cdot (p_A + p_B - p_C - p_D)$$

and from the conservation of momentum

$$p_A + p_B = p_C + p_D$$

so we are left with

$$\begin{aligned}
[\,] &= p_A^2 + p_B^2 + p_C^2 + p_D^2 \\
&= c^2(m_A^2 + m_B^2 + m_C^2 + m_D^2)
\end{aligned}$$

thus plugging $[\,]$ back into the sum of the Mandelstroms variables gives

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

(b) In the CM frame, the total momentum is zero, so the four-vector of the total momentum is and the total energy is

$$E_T = E_A + E_B$$

we know that the momentum of the two particles are zero:

$$\mathbf{p}_A + \mathbf{p}_B = 0$$

so from the first Mandelstroms variable

$$s = \frac{(p_A + p_B)^2}{c^2} = \frac{1}{c^2}\left[\left(\frac{E_A}{c} + \frac{E_b}{c}\right)^2 + (\mathbf{p}_A + \mathbf{p}_B)^2\right]$$

$$s = \frac{E_T^2}{c^4}$$

so

$$E_T = c^2\sqrt{s}$$

(c) Since $\mathbf{p}_A + \mathbf{p}_B = 0$ and $E_A = E_B = E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$, the Mandelstroms variable s is

$$\begin{aligned}
\frac{1}{c^2}(p+p)^2 &= \frac{1}{c^2}\left[\frac{(E_A + E_B)^2}{c^2} - (\mathbf{p}_A + \mathbf{p}_B)^2\right] \\
&= \frac{1}{c^2}\left[\frac{4E^2}{c^2}\right] \\
&= \frac{1}{c^2}\left[\frac{\mathbf{p}^2c^2 + m^2c^4}{c^2}\right] \\
s &= \frac{\mathbf{p}^2c^2 + m^2c^4}{c^2}
\end{aligned}$$

for t we assume the angle between $\mathbf{p}_A$ and $\mathbf{p}_C$ is θ so

$$\begin{aligned}
t &= \frac{1}{c^2}(p_A - p_C)^2 \\
&= \frac{1}{c^2}\left[\left(\frac{E_A}{c} - \frac{E_C}{c}\right)^2 - (\mathbf{p}_A - \mathbf{p}_C)^2\right]
\end{aligned}$$

and we know that $E_A = E_C$ so the first term is zero and the second term is

$$(\mathbf{p}_A - \mathbf{p}_C)^2 = \mathbf{p}_A^2 + \mathbf{p}_C^2 - 2\mathbf{p}_A \cdot \mathbf{p}_C$$

where $\mathbf{p}_A^2 = \mathbf{p}_C^2 = \mathbf{p}^2$ and $\mathbf{p}_A \cdot \mathbf{p}_C = \mathbf{p}^2 \cos\theta$ so

$$(\mathbf{p}_A - \mathbf{p}_C)^2 = 2\mathbf{p}^2 - \mathbf{p}^2\cos\theta = 2\mathbf{p}^2(1 - \cos\theta)$$

thus

$$t = \frac{1}{c^2}[0 - 2\mathbf{p}^2(1 - \cos\theta)] = \frac{-2\mathbf{p}^2(1 - \cos\theta)}{c^2}$$

for u everything is the same but $\mathbf{p}_A \cdot \mathbf{p}_D = -\mathbf{p}^2\cos\theta$ so

$$(\mathbf{p}_A - \mathbf{p}_D)^2 = 2\mathbf{p}^2 + \mathbf{p}^2\cos\theta = 2\mathbf{p}^2(1 + \cos\theta)$$

and thus

$$u = \frac{-2\mathbf{p}^2(1 + \cos\theta)}{c^2}$$

4. First we know that the energy and momentum of a photon γ are

$$E_\gamma = h\nu = pc \quad p = \frac{h\nu}{c}$$

from the planck relation and energy-mass relation. Using conservation of energy we know that the before the collision it is the energy of the photon plus the rest mass of the electron:

$$\begin{aligned}
E_i &= E_f \\
E_\gamma + m_ec^2 &= E'_\gamma + E_e
\end{aligned}$$

and using the energy momentum relation for the electron

$$h\nu + m_e c^2 = h\nu' + \sqrt{p_e^2 c^2 + (m_e c^2)^2} \tag{1}$$

and from the conservation of momentum we know that

$$\mathbf{p} = \mathbf{p}' + \mathbf{p}_e \quad \text{or} \quad \mathbf{p}_e = \mathbf{p} - \mathbf{p}'$$

where the momentum of the electron is initially zero, squaring both sides

$$\begin{aligned} \mathbf{p}_e^2 &= (\mathbf{p} - \mathbf{p}')^2 \\ p_e^2 &= p^2 + p'^2 - 2\mathbf{p} \cdot \mathbf{p}' \end{aligned}$$

where we know that the dot product of the two momenta is

$$\mathbf{p} \cdot \mathbf{p}' = pp' \cos\theta$$

so

$$p_e^2 = p^2 + p^2 - 2pp' \cos\theta \tag{2}$$

now we relate the two equations by first solving (1) for $p_e^2 c^2$

$$p_e^2 c^2 = (h\nu^2 - h\nu'^2 + m_e c^2)^2 - (m_e c^2)^2$$

and then multiplying (2) by c^2

$$p_e^2 c^2 = (pc)^2 + (p'c)^2 - 2pp'c^2 \cos\theta$$

and substituting the momentum of the photon from the energy relation $p = \frac{h\nu}{c}$

$$p_e^2 c^2 = (h\nu)^2 + (h\nu')^2 - 2h^2\nu\nu' \cos\theta$$

thus we can set the two equations equal to each other

$$\begin{aligned} (h\nu^2 - h\nu'^2 + m_e c^2)^2 - (m_e c^2)^2 &= (h\nu)^2 + (h\nu')^2 - 2h^2\nu\nu' \cos\theta \\ (h\nu)^2 + (h\nu')^2 + (m_e c^2)^2 + 2(h^2\nu\nu' - h\nu' m_e c^2 + h\nu m_e c^2) - (m_e c^2)^2 &= (h\nu)^2 + (h\nu')^2 - 2h^2\nu\nu' \cos\theta \end{aligned}$$

where the first three terms cancel out

$$2(h^2\nu\nu' - h\nu' m_e c^2 + h\nu m_e c^2) = -2h^2\nu\nu' \cos\theta$$

dividing both sides by $2h$ and rearranging terms gives

$$m_e c^2(\nu - \nu') = h\nu\nu'(1 - \cos\theta)$$

dividing both sides again but by $m_e c\nu\nu'$ gives

$$\left(\frac{c}{\nu'} - \frac{c}{\nu}\right) = \frac{h}{m_e c}(1 - \cos\theta)$$

and since the wavelength is $\lambda = \frac{c}{\nu}$ we can solve for the outgoing wavelength

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

$$\boxed{\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\theta)}$$

Homework 2

Due 2/7

1