Physics 411: Mechanics

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Contents

1 Newtons Laws

The Four Horsemen of the Apocalypse (In Physics)

- Classical Mechanics
- Electromagnetism
- Statistical Mechanics
- Quantum Mechanics

Before 1900, there was no relativity or QM and the world was a simple place ...

Newton's 1st Law: The Law of Inertia
And object keeps going unless acted on by a force.
This only applies to and 'inertial frame'.

Newton's 2nd Law: $\mathbf{F} = m\mathbf{a}$
Sum notation: The position vector is

$$\mathbf{r} = (x, y, z) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$$

in the Cartesian coordinate system. The time derivative gives the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

and acceleration is the time derivative of velocity

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

Thus in vector notation, Newton's 2nd law is

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$$

where $\mathbf{r}(t)$ is and ordinary differential equation (ODE).
The basic idea of solving mechanics problems is writing down the ODEs and solving them.

What is mass? m is an 'inertial mass'.
In Newton's law of gravity

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

m is the 'gravitational mass' and $g \approx 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.
A larger mass has a larger inertia or 'resistance to being accelerated' (Taylor). Key fact: When acceleration is zero ($\mathbf{a} = 0$), the velocity is constant ($\mathbf{v} = \text{constant}$).

Momentum: $\mathbf{p} = m\mathbf{v}$
The third law of motion in terms of momentum is

$$\mathbf{F} = \dot{\mathbf{p}} = m\dot{\mathbf{v}}$$

Newton's Third Law: $\mathbf{F}_{12} = -\mathbf{F}_{21}$
In a two body system, the total force of the system is

$$\mathbf{F}_t = \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

From the second law,

$$\dot{\mathbf{p}}_1 = \mathbf{F}_{21} \qquad \dot{\mathbf{p}}_2 = \mathbf{F}_{12}$$

adding these two equations gives

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$$

thus the total momentum of the system is conserved.
For a system of N particles, the total momentum is

$$\mathrm{d}t \sum_i \mathbf{p}_i = \frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = \mathbf{F}_{ext}$$

sometimes $\mathbf{p}_{tot} = \mathbf{P}$ where the capital P denotes the total momentum of the system.

2 A pendulum

How to solve a problem:

1. Write down the eq
2. Solve it
3. Understand the solution

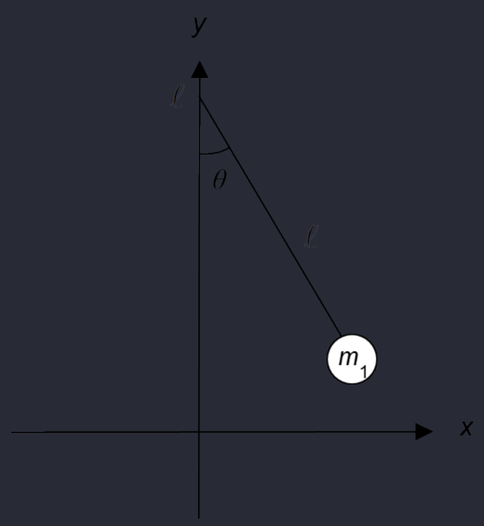

Figure 2.1: A pendulum with mass m and length l.

From Figure 2.1, we can write down Newton's 2nd law:

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} = m\ddot{\mathbf{r}} \\ F_x &= -mg\sin\theta = m\ddot{x} \\ F_y &= -mg\cos\theta + T\cos\theta = m\ddot{x} \end{aligned}$$

Using a right triangle we can find the angle using $\tan\theta = x/y$. Furthermore, we can use the constrain that the length of the pendulum is constant thus $x^2 + y^2 = l^2$. But solving this system of equations is difficult. Instead we now use a new coordinate system.

Quick Hack Using the arc length $l = L\theta$ and choosing a coordinate in the direction of the pendulums path, we can write the force equation as

$$F_l = -mg\sin\theta = m\ddot{l} = mL\ddot{\theta}$$

Thus the equation of motion is

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

which is a second order ODE. This can only be solved with two conditions. We can use the initial conditions (at $t = 0$) of the position $\theta(t = 0) = \theta_0$ and velocity $\dot{\theta}(0) = 0$.

Polar Coordinates From Taylor:

$$x = r\cos\phi$$
$$y = r\sin\phi$$

For an arbitrary vector $\mathbf{v}$ it has the Cartesian vector components

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

Where the magnitude of the unit vectors are equivalent:

$$|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = 1$$

and the magnitude of the vector is

$$\begin{aligned}|\mathbf{v}| &= \sqrt{\mathbf{v}\cdot\mathbf{v}} \\ &= \sqrt{v_x^2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + 2v_xv_y\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}v_y^2\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}} \\ &= \sqrt{v_x^2 + v_y^2}\end{aligned}$$

The vector $\mathbf{v}$ can be written in polar coordinates as

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\phi}$$

where radial vector is

$$\mathbf{r} = r\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

taking the time derivative of $\mathbf{r}$ gives the velocity

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}}$$

but how do we find $\dot{\hat{r}}$? We can look at the change in the direction of the radial unit vector for a small change in time Δt. Thus,

$$\Delta\hat{\mathbf{r}} \approx r\Delta\phi\hat{\boldsymbol{\phi}}$$

dividing both sides by Δt gives

$$\frac{\Delta\hat{\mathbf{r}}}{\Delta t} \approx r\frac{\Delta\phi}{\Delta t}\hat{\boldsymbol{\phi}} = r\dot{\phi}\hat{\boldsymbol{\phi}}$$

Therefore, the vector in polar coordinates is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}}$$

where the polar components v_r and v_ϕ are related to the radial and angular velocity respectively. Taking the time derivative of $\dot{\mathbf{r}}$ gives the acceleration

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}} \\ &= \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\hat{r}} + \dot{v}_\phi\hat{\boldsymbol{\phi}} + v_\phi\dot{\hat{\phi}}\end{aligned}$$

3 Polar Coordinates

using the geometric relation $\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$, we can write the acceleration as

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}}\end{aligned}$$

where $r\dot{\phi}^2 = r\omega^2$ is the centripetal acceleration and $r\ddot{\phi} = r\dot{\omega}$ is the tangential acceleration.
From the Pendulum problem we know that the string is taut $r = L$ thus the radial velocity is zero $\dot{r} = 0$.
Thus the force equation in the $\hat{\boldsymbol{\phi}}$ direction is

$$\begin{aligned}F_\phi &= mL\ddot{\phi} = -mg\sin\theta \\ \ddot{\phi} &= -\frac{g}{L}\sin\theta\end{aligned}$$

which is the same equation of motion.

Projectile in 2D The initial conditions of a general projectile is usually

$$\begin{aligned}F_x &= 0 = m\ddot{x} \\ F_y &= -mg = m\ddot{y}\end{aligned}$$

thus the equations of motion are

$$\begin{aligned}\ddot{x} &= 0 \\ \ddot{y} &= -g\end{aligned}$$

And solving these equations gives the position of the projectile

$$\begin{aligned}x(t) &= v_{ox}t \\ y(t) &= y_o v_{oy}t - \frac{1}{2}gt^2\end{aligned}$$

This can be expanded on with the addition of air resistance $\mathbf{f}$. This drag force is proportional to the velocity:

$$\mathbf{f} \propto -\hat{\mathbf{v}}$$

and there are two types of air resistance: linear

$$\mathbf{f}_l = -bv\hat{\mathbf{v}} = -b\mathbf{v}$$

and quadratic

$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

where we compare the terms with

$$\frac{f_l}{f_q} = \frac{cv}{b}$$

Linear $\mathbf{f}_l = -b\mathbf{v}$
From Newton's 2nd law

$$
\begin{aligned}
F_x &= -bv_x = m\ddot{x} = m\dot{v}_x \\
F_y &= -mg - bv_y = m\ddot{y} = m\dot{v}_y
\end{aligned}
$$

For the case of uncoupled differential equations (such as pure horizontal motion), we can solve the horizontal equation

$$\dot{v}_x = -\frac{b}{m}v_x$$

which has a general solution

$$v_x = Ae^{-kt}$$

where

$$k = \frac{b}{m}, \quad A = v_{ox}$$

to find the position we have to integrate $\dot{x} = v_x$:

$$
\begin{aligned}
x &= x_o + \int_0^t v_x(t')\,\mathrm{d}t' \\
&= x_o + \left[-\frac{v_{xo}}{k}e^{-kt}\right]\Big|_0^t
\end{aligned}
$$

where we have a limit of $t \to \infty$
For pure vertical motion, we solve the equation

$$\dot{v}_y = -g - \frac{b}{m}v_y \tag{3.1}$$

and with the initial condition $\dot{v}_y = 0$ we can solve for the velocity

$$v_y = -\frac{mg}{b} = v_{ter}$$

where v_{ter} is the terminal velocity. To get position, we use a trick by rewriting the equation as

$$m\dot{v}_y = -mg - bv_y = -mg - b(v_y - v_{ter})$$

and we can solve similar to the horizontal case using the general solution

$$v_y - v_{ter} = Ae^{-kt} = (v_{oy} - v_{ter})e^{-kt}bye$$

4 Air Resistance

Last time:

$$\mathbf{f}_l = -b\mathbf{v} \quad \dot{\mathbf{r}} = \mathbf{v}$$
$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

In the case of linear, x motino has a range, y velocity has a terminal velocity v_t.

Horizontal Quadratic Drag

$$\begin{aligned} F_y &= -mg - c|v_y|v_y \\ m\ddot{y} &= F_y \\ m\dot{v}_y &= -mg - c|v_y|v_y \end{aligned}$$

when $v_y = 0$ we have the terminal velocity

$$v_{ter} = \sqrt{\frac{mg}{c}} \quad \text{or} \quad c = \frac{mg}{v_{ter}^2}$$

thus the equation of motion is

$$\dot{v}_y = -g - \frac{c}{m}v_y^2 = -g(1 - \frac{v_y^2}{v_{ter}^2}) = \frac{\mathrm{d}v_y}{\mathrm{d}t}$$

using separation of variables

$$\frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\,\mathrm{d}t$$

integrating both sides

$$\int_{v_{oy}}^{v_y} \frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\int_0^t \mathrm{d}t$$

where we get the integral using the hyperbolic tangent

$$\begin{aligned} v_t \operatorname{arctanh}\frac{v_y}{v_t} &= -gt \\ v_y &= -v_t\tanh(gt) \end{aligned}$$

2D Motion For Quadratic

$$F_x = -cvv_x = -c\sqrt{v_x^2 + v_y^2}v_x = m\dot{v}_x$$
$$F_y = -mg - cvv_y = -mg - c\sqrt{v_x^2 + v_y^2}v_y = m\dot{v}_y$$

where $v = \sqrt{v_x^2 + v_y^2}$. For linear, it is simply

$$\begin{aligned} F_x &= -bv_x = m\dot{v}_x \\ F_y &= -mg - bv_y = m\dot{v}_y \end{aligned}$$

5 Energy

Review: There are two requirements for conservation of angular momentum

1. Force is central
2. External torque is zero

Kinetici Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$. Taking the time derivative

$$\begin{aligned}\dot{T} &= \frac{1}{2}(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) \\ &= m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}\end{aligned}$$

and integrating over time t_1 to t_2

$$\int_{t_1}^{t_2} \dot{T}\,\mathrm{d}t = \Delta T = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = W(1\to 2)$$

since $\mathbf{v}\cdot\mathrm{d}t = \mathrm{d}\mathbf{r}$ and $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ hints that this is a line integral.

Example:

$$\begin{aligned}\mathbf{F}(x,y) &= \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ \mathrm{d}vbr &= \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}}\end{aligned}$$

(a) $y = x$ from $a = (0,0)$ to $b = (1,1)$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + y\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 x\,\mathrm{d}x = 1\end{aligned}$$

(b) $y = x^2$ and $\mathrm{d}y = 2x\,\mathrm{d}x$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + x^2\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 2x^2\,\mathrm{d}x = 1\end{aligned}$$

thus the line integral is independent of the path.

Conservative force

1. Given $\mathbf{F}(\mathbf{r})$, there is no dependence on $\mathbf{v}$, t.
2. $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is independent of the path.

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}'$$

For gravity

$$U_g(\mathbf{r}) = -\int_a^b \mathbf{F}_g\cdot\mathrm{d}\mathbf{r} = -\int_a^b -mg\,\mathrm{d}y' = mg(y_a - y_b)$$

Work-Kinetic Energy Theorem:

$$W(1 \to 2) = W(1 \to \mathcal{O}) + W(\mathcal{O} \to 2) = U(1) - U(2) = \Delta T$$

From this we have

$$\Delta T = T_2 - T_1 = U_1 - U_2$$

rearranging terms give the mechanical energy

$$T_2 + U_2 = T_1 + U_1 = E$$

and for N conservative forces in a system

$$E = T + U_1 + U_2 + \cdots + U_N$$

Lecture 7: **1/31/24**

Energy: Part 2

Conservative Force: Potential Energy The mechanical energy

$$E = T + U$$

is made up of the sum of the kinetic and potential energy. This is useful for

- obtaining equations of motion (EOM)

e.g. finding the EOM for a simple pendulum of mass m, length L and initial angle θ. The kinetic energy is

$$T = \frac{1}{2}mL^2\dot{\theta}^2$$

where the magnitude of velocity is the tangential component $v = L\omega = L\dot{\theta}$. The potential energy is

$$U = -mgy = -mgL\cos\theta$$

and the conservation of energy tells us that the mechanical energy is constant

$$\begin{aligned} T + U &= \text{constant} = E \\ \frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta &= E \end{aligned}$$

and in the intial condition we know that the velocity is zero $\dot{\theta} = 0$ and thus

$$-mgL\cos\theta_{max} = E$$

taking the time derivative of the energy equation gives

$$\begin{aligned} mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\dot{\theta} &= 0 \\ \ddot{\theta} + \frac{g}{L}\sin\theta &= 0 \\ \ddot{\theta} &= -\frac{g}{L}\sin\theta \end{aligned}$$

taking the time derivative of the energy is a useful trick for finding the EOM when we are trying to solve for $\dot{v}^2$.

Last time we found the potential energy for a position $\mathbf{r}$ in a conservative force field $\mathbf{F}(\mathbf{r})$ is

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}'$$

from the fundamental theorem of calculus (derivatives and intergrals are inverses) we want to find a function where the derivative equals the conservative force: First we take an infinitesimal change in the position $\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}$ and the change in potential energy is

$$\begin{aligned} U(\mathbf{r} + \mathrm{d}\mathbf{r}) &= -\int_{\mathbf{r}_0}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{\mathbf{r}}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= U(\mathbf{r}) - F(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \end{aligned}$$

where is know that the force is constant over a small distance. Moving the terms gives

$$\begin{aligned} U(\mathbf{r} + \mathrm{d}\mathbf{r}) - U(\mathbf{r}) &= -\mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= -(F_x \, \mathrm{d}x + F_y \, \mathrm{d}y + F_z \, \mathrm{d}z) \end{aligned}$$

where we use Cartesian Coordinates, and we know that the gradient of potential is

$$\begin{aligned} \boldsymbol{\nabla} U &= \hat{\mathbf{x}} \frac{\partial U}{\partial x} + \hat{\mathbf{y}} \frac{\partial U}{\partial y} + \hat{\mathbf{z}} \frac{\partial U}{\partial z} \\ &= -(F_x \hat{\mathbf{x}} + F_y \hat{\mathbf{y}} + F_z \hat{\mathbf{z}}) = -\mathbf{F} \end{aligned}$$

Example 3: 1D motion If we know what U is as a function of x, we can find the force! At points where $E = U$ we call these classical turning points. At a region of a relative minimum, a particle below the threshold of the turning point will oscillate between the two turning points. And at $E > U_max$ the particle is unbound and will escape the forces that attracted it.

Example 4:

$$\begin{aligned} E &= T + U(x) \text{ is constant} \\ T &= \frac{1}{2} m \dot{x}^2 = E - U(x) \\ \dot{x}^2 &= \frac{2}{m}(E - U(x)) \\ \dot{x} &= \pm \sqrt{\frac{2}{m}(E - U(x))} \end{aligned}$$

using seperation of variables

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\frac{2}{m}(E - U(x))} \\ \sqrt{\frac{m}{2}} \, \mathrm{d}t &= \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\ \int_{t_1}^{t_2} \sqrt{\frac{m}{2}} \, \mathrm{d}t &= \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\ (t_2 - t_1) &= \sqrt{\frac{2}{m}} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \end{aligned}$$

where the sign of the velocity is positive within the oscillating bounds of the turning points and changes sign as the particle moves past the turning point.

Energy: Part 3

Last time: Conservative force as a negative gradient of potential:

$$\mathbf{F} = -\boldsymbol{\nabla} U$$

with classical turning points at $E = U$.

Conditions of a conservative force

- Only depends on position $\mathbf{r}$ (or just constant)
- Work done is path independent (this is sometimes hard to check) $\Leftrightarrow \boldsymbol{\nabla} \times \mathbf{F} = 0$

What is curl? In 3D Cartesian coordinates

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \hat{\mathbf{x}} + \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \hat{\mathbf{y}} + \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \hat{\mathbf{z}}() \right)\end{aligned}$$

Mathematically we know that if the force vector is a gradient of a scalar potential

$$\mathbf{F} = \boldsymbol{\nabla} \phi = -\boldsymbol{\nabla} U \quad \Leftrightarrow \quad \boldsymbol{\nabla} \times \mathbf{F} = 0$$

The curl of a gradient is always zero! Short 'proof':

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

and the curl

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial U}{\partial x} & -\frac{\partial U}{\partial y} & -\frac{\partial U}{\partial z} \end{vmatrix} = 0$$

Proving that the line integral is path independent is a little difficult, but in general we know that the two different paths a and b from points 1 to 2 we can write the work as

$$\int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_2 - \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_1 = \oint_{a-b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_A (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = 0$$

where we invoke Stokes' Theorem to find the integral of the curl over the surface A is zero.

Conservative Force: $\mathbf{F} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is conservative, but is

$$\mathbf{F}(\mathbf{r}) = F(\mathbf{r})\hat{\mathbf{r}}$$

a central force always conservative? We will need to check the curl of the force to find out. But first we define spherical coordinates (r, θ, ϕ)

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

and the central force in spherical coordinates is

$$\mathbf{F} = F(\mathbf{r})\hat{\mathbf{r}} + 0\hat{\theta} + 0\hat{\phi}$$

the curl in spherical coordinates is

$$\begin{aligned}\nabla \times \mathbf{F} = &\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial \theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right]\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\hat{\theta} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\hat{\phi}\end{aligned}$$

And since

$$\frac{\partial F}{\partial \theta} = \frac{\partial F}{\partial \phi} = 0$$

the curl is zero $\nabla \times \mathbf{F} = 0$ and thus $\mathbf{F}$ is a conservative central force.

Gravity Conservative? The force due to gravity

$$\mathbf{F}_g = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}$$

is a central force as it only depends on $\mathbf{r}$. e.g. for a a two mass system:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Using Translational Invariance, we can shift the origin to the center of m_2

$$\mathbf{r}_2 = 0 \quad \mathbf{F}_{12} = -\frac{Gm_1m_2}{\mathbf{r}_1^3}\mathbf{r}_1$$

or

$$F_{12} = -\nabla_1 U = -(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial z_1})$$

where the potential energy due to the interaction between 1 and 2 is

$$U_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

from newtons 3rd law, the force on 2 due to 1 is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

so

$$\begin{aligned}-\nabla_1 U_{12} &\to \mathbf{F}_{21} = \nabla_1 U_{12} \\ \nabla_1 U_{12}(\mathbf{r}_1 0 \mathbf{r}_2) &= -\nabla_2 U_{12}(\mathbf{r}_2 0 \mathbf{r}_1) \\ u_{12}(\mathbf{x}) &\quad \mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2 \\ \nabla_1 U_{12}(\mathbf{x}) &= \nabla_x U_{12}(\mathbf{x}) = -\nabla_2 U_{12}(\mathbf{x})\end{aligned}$$

so

$$\mathbf{F}_{12} = -\nabla_1 U_{12} \qquad \mathbf{F}_{21} = -\nabla_2 U_{12}$$

and for N particles

$$\mathbf{F}_i = -\nabla_i U \qquad U = \sum_{i,j} U_{ij} + \sum_i U_i^{\text{ext}}$$

6 Oscillations

& Simple Harmonic Oscillators For the simple case of a mas on a spring, the spring force is $F_s = -k(x-x_o)$ where the force is conservative and the (elastic) potential energy is $U_s = \frac{1}{2}k(x-x_o)^2$.

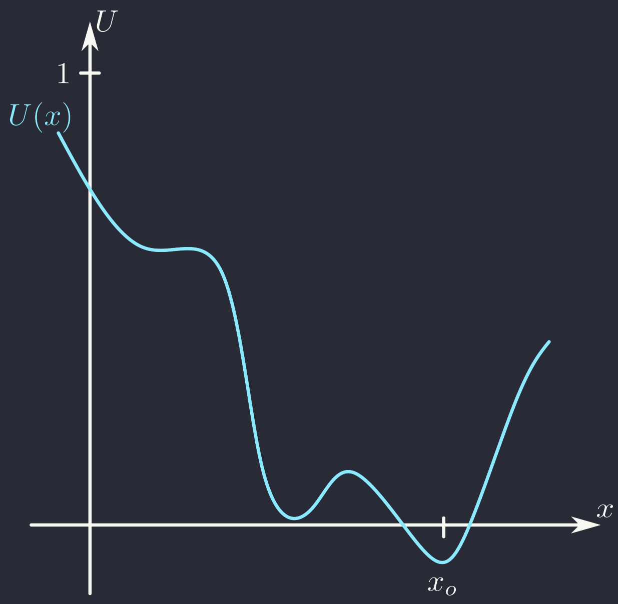

Figure 6.1: Arbitrary Potential Energy: $\mathbf{F} = -\boldsymbol{\nabla} U$

Aribtrary Potential energy For a equilibrium positon x_o we can take the taylor expansion of the potential energy

$$U(x) = U(x_o) + U'(x_o)\Delta x + \frac{1}{2}U''(x_o)\Delta x^2 + \dots$$

where $\Delta x = x - x_o$. Setting x_o to the reference point of U cancels the first term and the conservative nature tells us that the second term is also zero thus we are left with the third term where the spring constant is

$$k = U''(x_o)$$

To find the equations of motion, using N2L

$$m\ddot{x} = F = -k(x-x_o)$$
$$\ddot{x} = -\frac{k}{m}(x-x_o)$$

where we have a constant of angular frequency

$$\omega_o = \sqrt{\frac{k}{m}}$$

the solution could be a sinusoidal function

$$x(t) \approx \sin\omega_o t$$

but we are missing the initial value, so

$$x(t) \approx \sin\omega_o t + x_o$$

the general solution is linear combinations of the sine and cosine functions

$$x(t) = A\sin\omega_o t + B\cos\omega_o t + x_o$$
$$\dot{x}(t) = \omega_o A\cos\omega_o t - \omega_o B\sin\omega_o t$$

where we need 2 initial conditions to solve for A and B. e.g. $x(0)$ and $\dot{x}(0)$.

$$B = x(0) - x_o = \Delta x(0), \qquad A = \frac{\dot{x}(0)}{\omega_o}$$

Euler's Solution We can also use a general solution of the form

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where

$$\left|e^{i\theta}\right| = \cos^2\theta + \sin^2\theta = 1$$

taking the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{i\omega_o t} = i\omega_o e^{i\omega_o t}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{i\omega_o t} = -\omega_o^2 e^{i\omega_o t}$$

and the general solution is

$$x(t) = Ae^{i\omega_o t} + Be^{-i\omega_o t} + x_o$$

this does not mean that we have an imaginary solution, but rather we are using the geomtric nature of the solution.

Third Way We can also use a method where we introduce the phase

$$x(t) = A\cos(\omega_o t - \delta) + x_o$$
$$\dot{x}(t) = -A\omega_o\sin(\omega_o t - \delta)$$

for $t = 0$ we have

$$x(0) = A\cos(-\delta) + x_o = A\cos\delta + x_o$$
$$\dot{x}(0) = -A\omega_o\sin(-\delta) = A\omega_o\sin\delta$$

and the constants are found by squaring and adding the two equations

$$A^2 = (x(0) - x_o)^2 + \frac{\dot{x}(0)^2}{\omega_o^2} = \Delta x(0)^2 + \frac{\dot{x}(0)^2}{\omega_o^2}$$
$$\delta = \arctan\frac{\dot{x}(0)}{\omega_o(x(0) - x_o)} = \arctan\frac{\dot{x}(0)}{\omega_o\Delta x(0)}$$

Energy of the Oscillator The mechanical energy is $E = T + U$

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_o^2 A^2\sin^2(\omega_o t - \delta)$$
$$U = \frac{1}{2}k(x - x_o)^2 = \frac{1}{2}kA^2\cos^2(\omega_o t - \delta)$$

setting $x_o = 0$ we can work with a much simple case

$$U = \frac{1}{2}kx^2$$
$$T = \frac{1}{2}kx^2$$

using the third way where

$$x(t) = A\cos(\omega_o t - \delta)$$
$$\dot{x}(t) = -A\omega_o \sin(\omega_o t - \delta)$$

we have

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_o t - \delta)$$
$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_o^2\sin^2(\omega_o t - \delta) \qquad = \frac{1}{2}kA^2\sin^2(\omega_o t - \delta)$$

thus the total mechanical energy is

$$E = T + U = \frac{1}{2}kA^2$$

where this is the maximum potential energy of the system, or the potential energy at the maximum amplitude. This is also the classical turning point $E = U$. As time goes on, we can see that the energy oscillates betweeen being completely kinetic (T) and completely potential (U).

2D Oscillator We can have two cases of oscillation:

$$\mathbf{F} = -k(\mathbf{r} - \mathbf{r}_o) \quad \text{isotropic oscillator}$$

this is where each component share the same frequency, but different amplitudes and/or initial conditions

$$x(t) = A_x\cos(\omega_o t - \delta_x) + x_o$$
$$y(t) = A_y\cos(\omega_o t - \delta_y) + y_o$$

for the anisotropic oscillator

$$F_x = -k_x(x - x_o) \quad F_y = -k_y(y - y_o)$$

the frequency is decoupled thus

$$x(t) = A_x\cos(\omega_{ox} t - \delta_x) + x_o$$
$$y(t) = A_y\cos(\omega_{oy} t - \delta_y) + y_o$$

and if the ratio between the angular frequencies ω_{ox}/ω_{oy} are rational, the motion is periodic and the figure will be closed. But for irrational ratios, the motion is *quasiperiodic* and the figure is not closed (chaotic).

Oscillations: Damping

Damped Oscillator From last time the simple EOM for a spring is

$$m\ddot{x} = -k(x - x_o)$$

where the equilibrium position is x_o and the spring constant is k. When we add air resistance e.g. linear drag:

$$\mathbf{f} = -b\mathbf{v} \qquad m\ddot{x} = -kx - b\dot{x}$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

where we have two constants

$$\omega_o = \sqrt{\frac{k}{m}} \quad \beta = \frac{b}{2m}$$

where ω_o is the natural frequency and β is the damping coefficient. Rewriting in terms of the constants we get

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

where a general solution is

$$x = e^{rt}; \quad \dot{x} = re^{rt}; \quad \dot{x} = r^2 e^{rt}$$

plugging into the EOM gives

$$\begin{aligned} r^2 e^{rt} + 2\beta r e^{rt} + \omega_o^2 e^{rt} &= 0 \\ r^2 + 2\beta r + \omega_o^2 &= 0 \end{aligned}$$

which is the characteristic (or auxiliary) equation, and the solution is in the form of the quadratic formula

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$$

thus the position is a linear combination of the two solutions

$$x(t) = C_1 e^{(-\beta + \sqrt{\beta^2 - \omega_o^2})t} + C_2 e^{(-\beta - \sqrt{\beta^2 - \omega_o^2})t}$$

At $\beta = 0$ (no damping)

$$\sqrt{\beta^2 - \omega_o^2} = \sqrt{-\omega_o^2} = i\omega$$

thus the solution of a SHO

$$x(t) = C_1 \exp(i\omega t) + C_2 \exp(-i\omega t)$$

Weak Damping For the case $\beta < \omega_o$ (underdamping)

$$\sqrt{\beta^2 - \omega_o^2} = i\sqrt{\omega_o^2 - \beta^2}$$

thus the solution is

$$\begin{aligned} x(t) &= C_1 e^{(-\beta + i\sqrt{\omega_o^2 - \beta^2})t} + C_2 e^{(-\beta - i\sqrt{\omega_o^2 - \beta^2})t} \\ &= e^{-\beta t}(C_1 \cos\left(\sqrt{\omega_o^2 - \beta^2}t\right) + C_2 \sin\left(\sqrt{\omega_o^2 - \beta^2}t\right)) \end{aligned}$$

we can simplify this with a new frequency term $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ and therefore

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$

this is called underdamping because the amplitude oscillates and decays slowly.

Strong damping For the case $\beta > \omega_o$ we have the solution

$$x(t) = e^{-(\beta-\sqrt{\beta^2-\omega_o^2})t}\left(C_1 + C_2 e^{-2\sqrt{\beta^2-\omega_o^2}t}\right)$$

this called overdamping because the system cannot cannot complete a full oscillation, and decays exponentially to the equilibrium position. Thus we call the term

$$\text{decay parameter} = \beta - \sqrt{\beta^2 - \omega_o^2}$$

where the decaying tail is described by the decay parameter whereas the second term $-2\sqrt{\beta^2 - \omega_o^2}$ describes the fast initial damping of the system.

Large β For the case of $\beta \to \infty$ the decay parameter goes to zero:

$$\gamma = \beta - \sqrt{\beta^2 - \omega_o^2} \to 0$$

which is counter intuitive as the high damping coefficient results in a very slow exponential decay where it looks like a constant almost zero.

Critical Damping For the case $\beta = \omega_o$ we get

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t} \to x = e^{-\beta t}(C_1 + C_2 t)$$

where the extra factor of t comes from solving for a function $f(t)e^{-\beta t}$ to get the Constants. Pluggin this back into the initial EOM: $\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$

$$\dot{x} = e^{-\beta t} - te^{-\beta t} \qquad \ddot{x} = -2\beta e^{-\beta t} + te^{-\beta t}$$

so

$$-2\beta e^{-\beta t} + \beta^2 te^{-\beta t} + 2\beta e^{-\beta t} - 2\beta^2 te^{-\beta t} + \beta^2 te^{-\beta t} = 0$$

condition	γ
$\beta < \omega_o$	β
$\beta = \omega_o$	β
$\beta > \omega_o$	$\beta - \sqrt{\beta^2 - \omega_o^2}$

the critical damping will have the fastest decay of the system. The quickest way to stop an oscillating system is to apply a damping force at the natural frequency of the system.

NOTE: This all goes away when the magnitude of the damping force is not linear (e.g. quadratic drag). The linear EOM gives us something that can be easily analyzed, but for terms with higher powers (e.g. $\dot{x}^2$) the EOM becomes non-linear and the solutions are chaotic.

Driven Damped Oscillations

From last time: Note that the two parameters ω_o and β have the same units (rad/s) where we treat radians as a unitless quantity.

Time dependent force For the SHO we have a new EOM

$$m\ddot{x} + b\dot{x} + kx = F\cos(\omega t)$$

or in terms of the constants ω_o and β

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \frac{F(t)}{m} = f(t)$$

where $f(t)$ has the same units as acceleration/ force per unit mass. This is a inhomogeneous differential equation, but we can consider this as a combination of a homogeneous solution x_h and a particular solution x_p:

$$x_p(t) + x_h(t) = x(t)$$

denoting a differential operator

$$D = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_o^2$$

we know that

$$Dx_h(t) = 0 \qquad Dx_p(t) = f(t)$$

where from last time we know that the homogeneous solution is

$$x_h(t) = e^{-\beta t}(C_1\exp\left(t\sqrt{\beta^2 - \omega_o^2}\right) + C_2\exp\left(-t\sqrt{\beta^2 - \omega_o^2}\right))$$

and for the particular solution we can define the driving force as a sinusoidal function

$$f(t) = f_o\cos(\omega t) \quad \text{driving force}$$

where

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o\cos\omega t$$

or using Euler's formula we can define the EOM as the real part of the complex function

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o e^{i\omega t}$$

the particular solution is then

$$\begin{aligned} x &= Ce^{i\omega t} \\ \dot{x} = i\omega Ce^{i\omega t}, \qquad \ddot{x} &= -\omega^2 Ce^{i\omega t} \end{aligned}$$

subbing this back in to the EOM

$$-\omega^2 Ce^{i\omega t} + 2\beta i\omega Ce^{i\omega t} + \omega_o^2 Ce^{i\omega t} = f_o e^{i\omega t}$$

or

$$C = \frac{f_o}{\omega_o^2 - \omega^2 + 2i\beta\omega}$$

which is a complex number. Before taking the real part of the solution,

$$x(t) = \mathrm{Re}(Ce^{i\omega t})$$

we have to consider the complex part as well, so we rewrite C as

$$C = Ae^{-i\delta} = A\cos\delta - iA\sin\delta$$

so the real part of the solution is

$$x(t) = \mathrm{Re}(Ae^{i(\omega t - \delta)}) = A\cos(\omega t - \delta)$$

to find the amplitude A we have to multiply C by its complex conjugate

$$C^*C = |C|^2 = A^2$$

where we get the modulus squared. Thus we get

$$A^2 = \frac{f_o^2}{(\omega_o^2 - \omega^2)^2 + (2\beta\omega)^2}$$

to find δ we move some terms around

$$\begin{aligned} \frac{f_o}{\omega_o^2 - \omega^2 + 2i\beta\omega} &= Ae^{-i\delta} \\ f_o e^{i\delta} &= A(\omega_o^2 - \omega^2 + 2i\beta\omega) \\ f_o\cos\delta + if_o\sin\delta &= A(\omega_o^2 - \omega^2 + 2i\beta\omega) \end{aligned}$$

thus we have

$$\sin\delta = \frac{2A\beta\omega}{f_o} \qquad \cos\delta = \frac{A(\omega_o^2 - \omega^2)}{f_o}$$

dividing the two equations gives us the phase

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right)$$

The full solution is now

$$
\begin{aligned}
x(t) &= x_p(t) + x_h(t) \\
&= A\cos(\omega t - \delta) + C_1 e^{\left(-\beta + \sqrt{\beta^2 - \omega_o^2}\right)t} + C_2 e^{\left(-\beta - \sqrt{\beta^2 - \omega_o^2}\right)t}
\end{aligned}
$$

where ω is the driving frequency, and ω_o is the natural frequency. The last two exponential terms are known as the transient solution which decays very quick (exponentially) as shown in Figure 6.2 Finding

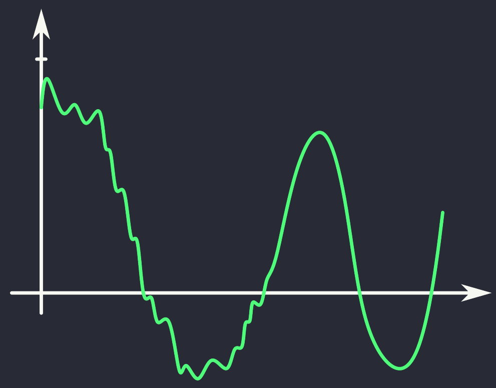

Figure 6.2: Driven Damped Oscillations

the maximum we look for where the derivative of A^2 is zero, or roughly

$$
\frac{\mathrm{d}}{\mathrm{d}\omega}\left((\omega_o^2 - \omega^2)^2 + (2\beta\omega)^2\right) = 0
$$

which gives us

$$
\omega = \omega_2 = \sqrt{\omega_o^2 - 2\beta^2}
$$

where at $B \ll \omega_o \rightarrow \omega \approx \omega_o$. Figure 6.3 shows that ω_2 is a resonant frequency where the amplitude is maximized.

$$
A_{max} = \frac{f_o}{\sqrt{4\beta^2(\omega_o^2 - \omega^2)}} \approx \frac{f_o}{2\beta\omega} \quad \text{for} \quad \beta \ll \omega_o
$$

What is δ? From the general solution, we can see that δ is a shift with respect to the driving force. This lag we can graph as a function of ω as shown in Figure 6.4

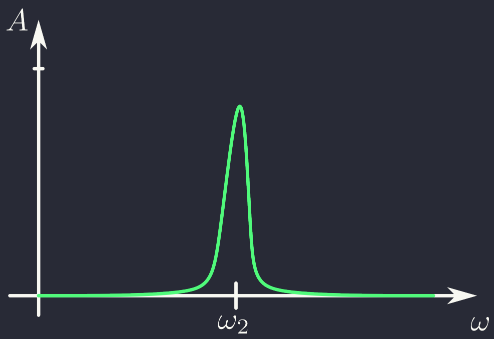

Figure 6.3: Resonance at ω_2

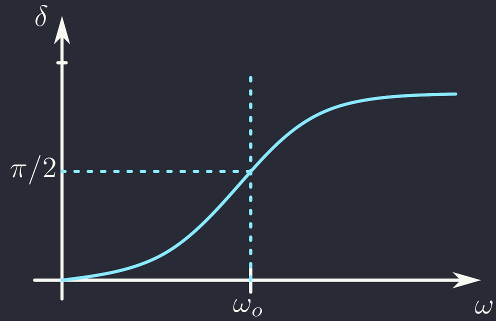

Figure 6.4: Phase shift δ

7 Calculus of Variations

Why do we care?

- What is the shortest distance between two points in a 2D plane?
- What is the shortest path between two points on a sphere?
- What is the fastest path for a ball to roll down a hill?
- For a car driving on a flat path $A \to B$, what shape of a pot hole will minimize the time it takes to get from $A \to B$?

For some path $a \to b$, we have a path defined as an integral

$$S = \int_a^b f(x, y, y')\mathrm{d}x$$

with a *Goal*: find $y(x)$ that minimizes S (path).

Path Length:

$$l = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{1 + y'^2}\mathrm{d}x$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. To minimize $y = f(x)$ it is equivalent to finding where

$$f'(x) = 0$$

where we note that this could be a maximium point, but it is usually a minimum in these cases. Another look at this function:

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

we can define a small change in the path $y(x)$ as

$$y(x) + \delta y(x)$$

where

$$\delta y(x_2) = 0 \quad \delta y(x_2) = 0$$

so the change in the path is

$$\delta S = \int_a^b \delta f \mathrm{d}x$$

and from the change of variables

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \qquad \delta y' = \frac{\mathrm{d}}{\mathrm{d}x}\delta y$$

thus we have

$$\delta S = \int_a^b \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x$$

this is the line integral of the change in the new path

$$\delta S = S_{new} - S_{old}$$

looking at the second term: using integration by parts

$$\int_a^b \left(\frac{\partial f}{\partial y}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x = \left[\frac{\partial f}{\partial y'}\delta y\right]\Big|_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y \mathrm{d}x$$

the first term is zero because $\delta y(a) = \delta y(b) = 0$. Thus we have

$$\delta S = \int_a^b \left[\frac{\partial f}{\partial y} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\right]\delta y \mathrm{d}x$$

Near a minimum, $\delta S = 0$ for any small δy. So the the terms in the brackets must be zero as well! This gives us the **Euler-Lagrange Equation**:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0$$

NOTE: δS is the variation of S(some number) under δy(a function).

Example: Shortest path between two points $a \to b$ in a 2D cartesian plane.

Goal: find $y(x)$ tha minimzes the path length $l = \int_a^b \sqrt{1+y'^2}\mathrm{d}x$ where $f(x,y,y') = \sqrt{1+y'^2}$.

$$\frac{\partial f}{\partial y} = 0$$
$$\frac{\partial f}{\partial y'} = \frac{2y'}{2\sqrt{1+y'^2}} = \frac{y'}{\sqrt{1+y'^2}}$$

From the EL:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{\partial f}{\partial y} = 0$$

and

$$\begin{aligned}\frac{y'}{\sqrt{1+y'^2}} &= Const = C \\ y'^2 &= C(1+y'^2) \\ y'^2 &= \frac{C}{1-C} \\ y' &= \pm\sqrt{\frac{C}{1-C}} = \pm k \\ y &= \pm kx + b\end{aligned}$$

which is just a straight line as we expected.

Example: The Brachistochrone.

Goal: Find $y(x)$ that minimizes $t = \int_a^b \mathrm{d}t$ where

$$t = \frac{s}{v} \to \mathrm{d}t = \frac{\mathrm{d}s}{v}$$

and v is the velocity which can be found using the conservation of energy:

$$\frac{1}{2}mv^2 = mg(y_o - y) \to v = \sqrt{2g(y_o - y)}$$

thus we have

$$\mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o - y)}}\mathrm{d}x$$

where $f(x, y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o-y)}}$. Using EL:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}\sqrt{2g(y_o - y)}}$$
$$\frac{\partial f}{\partial y} = \frac{\sqrt{1+y^2}\sqrt{2g}}{(2g(y_o - y))^{3/2}}$$

From the initial conditions: $x = y = 0, \quad \dot{x} = \dot{y} = 0$ (assuming no friction). So

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \to \frac{\partial f}{\partial y} = \frac{\sqrt{1+y'^2}}{\sqrt{2gy^3}}$$

and if $y(x)$ satisfies EL, then we have a conserved quantity (prove in HW):

$$f - y'\frac{\partial f}{\partial y'} = const$$

we will soon find out that this is the energy of the system... we can solve for $y(x)$ using the conserved quantity:

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gy}}\frac{y'}{\sqrt{1+y'^2}}$$

so

$$f - y'\frac{\partial f}{\partial y'} = C$$
$$\frac{1}{\sqrt{2gy}}\left(\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right) = C$$
$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}}(1 + y'^2 - y'^2) = \sqrt{\frac{1}{4ga}}$$

using

$$\frac{1}{y(1+y'^2)} = \frac{1}{2a} \implies 1 + y'^2 = \frac{2a}{y} \implies y'^2 = \frac{2a}{y} - 1$$

this can be solved using separation of variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{2a}{y} - 1} \implies \mathrm{d}x = \mathrm{d}y\sqrt{\frac{y}{2a - y}}$$

and integration both sides:

$$\int_a^b \mathrm{d}y\sqrt{\frac{y}{2a-y}} = x - x_a = x$$

using a change of variables $y = a(1-\cos\theta)$; $\quad \mathrm{d}y = a\sin\theta\mathrm{d}\theta$ and a substitution $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{(1-\cos\theta)(1+\cos\theta)}$:

$$\int_a^b a\sin\theta\mathrm{d}\theta\sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}} = \int_a^b a(1-\cos\theta)\mathrm{d}\theta = a\theta - a\sin\theta$$

this is a parametric equation:

$$x = a(\theta - \sin\theta) = x(\theta)$$
$$y = a(1-\cos\theta) = y(\theta)$$

where $\theta = \omega t$. This is a curve traced by a point on a wheel AKA cycloid. When we choose a variable time we get

$$x(t) = a(\omega t - \sin\omega t)$$
$$y(t) = a(1-\cos\omega t)$$

and thus we get $\omega = \sqrt{\frac{g}{a}}$. To find a we use the coordiate of the lower second point to find the curve that goes through the two points.

Example: Find two functions $x(u)$, $y(u)$ where the path

$$S = \int_a^b f(x, x', y, y', u)\mathrm{d}u$$

is minimized/stationary. We will get two EL equations:

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial x'} = 0$$
$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial y'} = 0$$

e.g. for a distance between two points:

$$L = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{x'^2 + y'^2}\mathrm{d}u \quad \text{using} \quad \mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}u}\mathrm{d}u = y'\mathrm{d}u$$

and from the EL equations:

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial x'} = 0 = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{x'}{\sqrt{x'^2 + y'^2}}\right)$$
$$\implies C_1 = \frac{x'}{\sqrt{x'^2 + y'^2}} \quad C_2 = \frac{y'}{\sqrt{x'^2 + y'^2}}$$

this also tells us that

$$\frac{y'}{x'} = const = \frac{\mathrm{d}y}{\mathrm{d}x}$$

For N unknown functions in time t:

$$S = \int_a^b f(x_1, x_1', \ldots, x_N, x_N', u)\mathrm{d}u$$

where f has $2N+1$ variables.

Generalized Coordinates: $q_1, q_2, \ldots, q_N$ we would define the Lagrangian

$$\mathcal{L}(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t)$$

and minimize the action

$$S = \int \mathcal{L}\mathrm{d}t$$

and N EL equations gives the trajectory for the path of minimal action.

8 Lagrange's Equations

From last time: we defined the path

$$S = \int_a^b f(x, y(x), y'(x))\mathrm{d}x$$

Goal: find $y(x)$ that minimizes S using EL

$$\text{EL:} \quad \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

where near the minimum $\delta S = 0$. From the EL, $y(x)$ is a stationary point of S(could also be a maximum!).

Lagrangian In Classical Mechanics, we use a specific form

$$\mathcal{L} = T - V$$

this has the units of energy and the action S has the units $[S] = [E \cdot T]$ similar to planck's constant $\hbar$.

3D Cartesian $x, y, z = q_1, q_2, q_3$

$$\begin{aligned} T &= \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ U &= U(x, y, z) \end{aligned}$$

where the potential energy only depends on the position and T only depends on the velocity, so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

and the EL equation for the Lagrangian is

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

For the 3D case, we have 3 equations of motion: For x we have

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x}, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

and using the EL equation, we get

$$-\frac{\partial U}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x}$$

which is Newton's second law $F_x = ma_x$ where $\mathbf{F} = -\boldsymbol{\nabla} U$. We can now get the general form

$$\mathbf{F} = m\mathbf{a}$$

Polar Coordinates $q : (r, \phi)$ we know that

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

and

$$U = U(r, \phi), \qquad T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$$

first we find the parts EL equation for r

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 - \frac{\partial U}{\partial r}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

and the EL equation is

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r})$$
$$m(\ddot{r} - r\dot{\phi}^2) = -\frac{\partial U}{\partial r}$$

which gives us N2L for r. For ϕ we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

and from the EL equation we get

$$-\frac{\partial U}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) = m(2r\dot{r}\dot{\phi} + r^2\ddot{\phi})$$

dividing both sides by r

$$-\frac{1}{r}\frac{\partial U}{\partial \phi} = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = -(\boldsymbol{\nabla} U)_\phi$$

from both forms we know that the two parts of the EL represent the momentum and force:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \quad \text{generalized momentum}$$
$$\frac{\partial \mathcal{L}}{\partial q_i} = F_i \quad \text{generalized force}$$

where $F_i = \frac{\mathrm{d}}{\mathrm{d}t}p_i$ is the generalized N2L.

Example: Mass m sliding down a frictionless *moving* ramp M. First we choose the coordinates x moving along withe the ramp and y down in the perpendicular direction. For the ramp M:

$$T_M = \frac{1}{2}M\dot{q}_2^2, \quad U_M = 0$$

and for the mass m: First we decompose the velocity of m into the x and y components

$$\mathbf{v}_m = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} = \hat{\mathbf{y}}(\dot{q}_1 \sin\alpha) + \hat{\mathbf{x}}(\dot{q}_1\cos\alpha + \dot{q}_2)$$

and the kinetic and potential energies are

$$T_m = \frac{1}{2}mv_m^2 = \frac{1}{2}m(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha + \dot{q}_2^2)$$
$$U_m = mgy = -mg(\dot{q}_1\sin\alpha)$$

using the Lagrangian $\mathcal{L} = T - U = T_M + T_m - U_M - U_m$ we get

$$\frac{\partial \mathcal{L}}{\partial q_2} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = M\dot{q}_2 + m\dot{q}_2 + m\dot{q}_1\cos\alpha$$

and the EL equation gives us

$$
\begin{aligned}
(M+m)\ddot{q}_2 + m\ddot{q}_1 \cos\alpha = 0 \\
a_2 = \ddot{q}_2 = -\frac{m\ddot{q}_1 \cos\alpha}{M+m}
\end{aligned}
$$

and for q_1 we have

$$
\frac{\partial \mathcal{L}}{\partial q_1} = mg\sin\alpha, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m(\dot{q}_1 + \dot{q}_2 \cos\alpha)
$$

and the EL equation gives us

$$
mg\sin\alpha = m(\ddot{q}_1 + \ddot{q}_2 \cos\alpha)
$$

and since we have two equations and two unknowns, we can solve for $\ddot{q}_1$ and $\ddot{q}_2$.

$$
\begin{aligned}
\ddot{q}_1 &= \frac{g\sin\alpha}{1 - \frac{m\cos^2\alpha}{m+M}} = const \\
\ddot{q}_2 &= -\frac{m\ddot{q}_1 \cos\alpha}{M+m} = const
\end{aligned}
$$

for $\alpha = 90°$, we get $\ddot{q}_1 = g$ and $\ddot{q}_2 = 0$ which is the same as a free falling. For and infinitely heavy ramp $M \to \infty$, we get $\ddot{q}_1 = g\sin\alpha$. For $M \to 0$ we get $\ddot{q}_1 = g/\sin\alpha$ which doesn't make sense because the force on the mass would be infinite. The normal force $N \to 0$ as $M \to 0$ and the mass would be in free fall.

Review Lagrangian: For a general integral

$$S\int f(x,y,y')\mathrm{d}x$$

find $y(x)$ minimizing S using the EL equation

$$\frac{\partial f}{\partial y}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

For Classical Mechanics, we use the Lagrangian in the generalized coordinate system q_i we define the action S as

$$S=\int\mathcal{L}(q_i,\dot{q}_i,t)\mathrm{d}t \quad \text{find} \quad q(t)$$

and from the EL equation we get

$$\frac{\partial\mathcal{L}}{\partial q_i}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}$$

for each degree of freedom. We define the Lagrangian in CM as the quantity $\mathcal{L}=T-U$

Examples, Examples, and more Examples: A pendulum but its spining on its axis. We first find the energies:

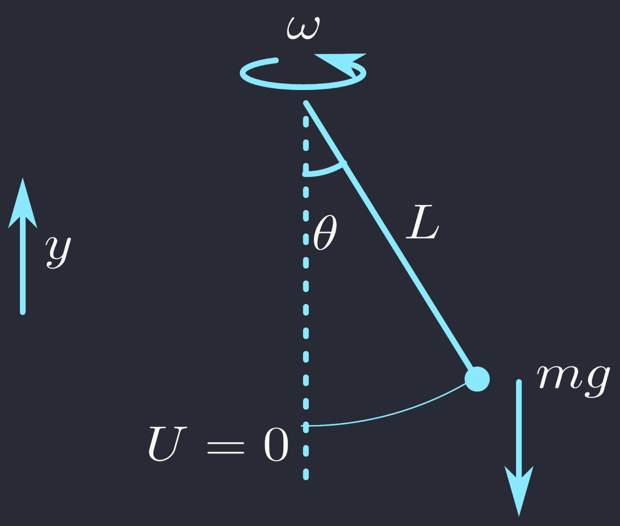

Figure 8.1: Pendulum

$$T=\frac{1}{2}mv^2=\frac{1}{2}m((\omega L\sin\theta)^2+(L\dot{\theta})^2)$$
$$U=mgy=mgL(1-\cos\theta)$$

from EL equation we get

$$\frac{\partial\mathcal{L}}{\partial\theta}=\frac{1}{2}m\omega^2L^2(2\sin\theta\cos\theta)-mgL\sin\theta=m\omega^2L^2\cos\theta\sin\theta-mgL\sin\theta$$
$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=mL^2\dot{\theta}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=mL^2\ddot{\theta}$$

so

$$
\begin{aligned}
mL^2\ddot{\theta} &= m\omega^2 L^2 \cos\theta \sin\theta - mgL\sin\theta \\
\ddot{\theta} &= \omega^2 \cos\theta\sin\theta - \frac{g}{L}\sin\theta
\end{aligned}
$$

when $\omega = 0$ we get the simple pendulum $\ddot{\theta} = -\frac{g}{L}\sin\theta$. Identifying the the equilibrium points where $\ddot{\theta} = 0 \implies$

$$\sin\theta = 0 \implies \theta = 0, \pi$$

at $\theta = 0$ the pendulum is just hanging vertically down which we can physically deduce as a stable equilibrium point. To check this analytically we can assume a small deviation from the equilibrium point:

$$
\begin{aligned}
\theta &= 0 + \epsilon \\
\cos(0+\epsilon) &= 1 - \frac{\epsilon^2}{2} \approx 1 \\
\sin(0+\epsilon) &= \epsilon - \frac{\epsilon^3}{6} \approx \epsilon
\end{aligned}
$$

and we get

$$
\begin{aligned}
\ddot{\theta} &= (\omega^2 - \frac{g}{L})\theta \\
\ddot{\theta} &= -\Omega^2\theta \implies \text{Stable} \\
\ddot{\theta} &= \Omega^2\theta \implies \text{Unstable}
\end{aligned}
$$

where

$$
\begin{aligned}
\omega^2 &< \frac{g}{L} \implies \text{Stable} \\
\omega^2 &> \frac{g}{L} \implies \text{Unstable}
\end{aligned}
$$

when they are equal $\omega^2 = \frac{g}{L}$ we get a simple pendulum. Finding another equilibrium point at

$$
\begin{aligned}
\omega^2\cos\theta - \frac{g}{L} &= 0 \\
\cos\theta &= \frac{g}{L\omega^2}, \qquad \theta = \pm\arccos\left(\frac{g}{L\omega^2}\right)
\end{aligned}
$$

where there only exists a solution when

$$\omega^2 > \frac{g}{L}$$

since $\cos\theta \leq 1$. For this case, we can also look at the radial force in polar:

$$F_r = m\ddot{r} - mr\omega^2 \quad \text{or} \quad m\ddot{r} = F_r + mr\omega^2$$

where in the second equation we can see that the sum of the centrifugal force and F_r sums to zero so

$$
\begin{aligned}
\tan\theta &= \frac{F_r}{mg} = \frac{mL\sin\theta\omega^2}{mg} \\
\implies \frac{L\omega^2}{g} &= \frac{1}{\cos\theta} \\
\cos\theta &= \frac{g}{L\omega^2}
\end{aligned}
$$

from the force analysis we can see that the centrifugal force is balanced by the radial force. Substituing the equilibium position back into the EOM

$$\cos\theta_o = \frac{g}{L\omega^2} \to \theta = \theta_o + \epsilon$$

Using Taylor Expansion, $f(x) = f(x_o) + f'(x_o)(x - x_o)$, the sine and cosine terms are

$$\cos\theta = \cos(\theta_o + \epsilon) = \cos\theta_o - \sin\theta_o\epsilon$$
$$\sin\theta = \sin(\theta_o + \epsilon) = \sin\theta_o + \cos\theta_o\epsilon$$

so the EOM becomes

$$\ddot{\theta} = (-\omega^2\sin\theta_o\epsilon)(\sin\theta_o + \cos\theta_o\epsilon)$$
$$\ddot{\epsilon} = -\omega^2\sin^2(\theta_o)\epsilon$$

where we have Bifurcation at $\omega^2 = \frac{g}{L}$. We can see that the EOM for ϵ is similar to the harmonic oscillator so:

$$\epsilon = A\cos(\Omega t - \delta) \qquad \Omega = \omega\sin\theta_o$$

HW 5 Given $f(x, y, y')$. Independence of y means:

$$f(x, y') \implies \frac{\partial f}{\partial y'} = constant$$
$$\mathcal{L}(t, \dot{q}) \implies \frac{\partial \mathcal{L}}{\partial \dot{q}} = constant$$

so for the Lagrangian

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - U(x) \qquad \frac{\partial \mathcal{L}}{\partial q_i} = F_i \quad \text{generalized force}$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = conserved, \qquad \frac{\partial U}{\partial x} = 0$$

if $\mathcal{L}$ doesn't depend on x, then p_x (momentum) is conserved. So for the generalized Lagrangian

$$\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$
$$\text{If} \quad \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{conserved}$$

So symmetry $\implies$ conservation (Noether's Theorem).

Conservation The two types:

- If $f(x, y')$ is independent of y, then

$$\frac{\partial f}{\partial y'} = \text{constant over } x$$

  or if $\mathcal{L}$ is independent of q_i, then

$$\frac{\partial \mathcal{L}}{\partial q_i} = \text{constant over } t = p_i$$

- If $f(y, y')$ is independent of x, then

$$f - y'\frac{\partial f}{\partial y'} = \text{constant over } x$$

  or if $\mathcal{L}$ is independent of t, then

$$\mathcal{L} - \dot{q}_i\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant over } t$$

looking at this more closely:

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{q}^2 - U(q)$$

where

$$\begin{aligned} \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} &= m\dot{q}^2; \\ \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} &= m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + U \\ &= \frac{1}{2}m\dot{q}^2 + U = T + U = E \end{aligned}$$

this is this Hamiltonian

$$\sum_i p_i\dot{q}_i - \mathcal{L} = \mathcal{H} = E$$

Noether's Theorem For a system independent of $t \leftrightarrow$ the system has time-translation symmetry $\implies$ conservation of energy

Dependence on t $U = U(q,t)$ e.g. Mass of sun is increasing over time, the potential energy is dependent on time, so the system is not conservative.

Pendulum thoughts: In our pendulum example, we chose $q = \theta$, but we could also choose $q_1 = x$ and $q_2 = y$. The truth lies in the fact that we intuitively chose $q_1 = r$ and $q_2 = \theta$. So in transforming from Cartesian coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = L$$

where we have a 'constraint' $r = L\ldots$

Legal Terms: Formal Definition of Constraints In the beginning, we defined the first defined position with

$$\mathbf{r} = (x, y, z)$$

for the generalized coordinates we have

$$\mathbf{r} = \mathbf{r}(q_1, \ldots, q_n, t)$$

where we decided that in a 3D system $n = 3$. A constraint is an equation

$$f(q_1, \ldots, q_n) = 0$$

where this is a *holonomic* (whole) constraint and to find the number of generalized coordinates:

$$\begin{aligned} \#\text{ of generalized coordinates we need} &= \#\text{ of dimensions} - \#\text{ of constraints} \\ &= \#\text{ of degrees of freedom} \end{aligned}$$

this is only true for holonomic constraints. For *nonholonomic* constraints, it is more complicated e.g. A ball on a horizontal table: We can see that # of generalized coordinates = 2, but to describe the position of the ball i.e. a dot on the ball, we need 3 more coordinates (Euler angles). So the configuration of the ball is described by 5 coordinates $(x, y, \alpha, \beta, \gamma)$. In other words, the configuration is path dependent and we see a nonholonomic constraint.

Example: What are the constraints for the mass sliding down a moving mass? The holonomic constraints are the vertical position of the ramp $y_M = 0$, and from x_m, y_m, x_M we know the $x_{COM} =$ constant.

Fact! A constraint is enforced by a constraint force $\mathbf{F}_c \perp$ path(in the pendulum example, the normal force N). Finding this force where $f(q_i) = 0$ can be found by taking the gradient of the function $\boldsymbol{\nabla} f$. So

$$\mathbf{F}_c = \lambda \boldsymbol{\nabla} f$$

Lecture 15: 2/23/24

Review

- Convservation: Lagrangian is independent of time $\implies$ conservation of energy

Lagrange Multiplier Want to find $q_i(t)$ by minimizing $S = \int \mathcal{L}(q_i, \dot{q}_i, t)\mathrm{d}t$.

$\star$ Under holonomic constraints,

$$f(q_i) = 0$$

So we introduce a new unknown $\lambda(t)$ and the new minimizing integral becomes

$$I = \int (\mathcal{L} - \lambda f)\mathrm{d}t$$

The EL eqn for $\lambda(t)$: $\quad f = 0$

$$\frac{\partial(\mathcal{L} - \lambda f)}{\partial \lambda} = -f \qquad \frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{\lambda}} = 0$$

The EL eqn for $q_i(t)$:

$$F_i = \frac{\partial(\mathcal{L} - \lambda f)}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L} - \lambda f)}{\partial \dot{q}_i}$$

or

$$F_i = \frac{\partial \mathcal{L}}{\partial q_i} - \lambda \frac{\partial f}{\partial q_i} \quad \text{where} \quad \frac{\partial \mathcal{L}}{\partial q_i} = p_i$$

So we are given $N+1$ unknowns and $N+1$ EL eqns with the addition of the lagrange multiplier.

Simple Pendulum (revisited) We have the Lagrangian $\mathcal{L}(x, y, \dot{x}, \dot{y}) = T - U$ where

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$U = -mgy$$

so

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

and using the constraint of the fixed length; $f(x, y) = x^2 + y^2 - L^2 = 0$ we get

$$\ell = \mathcal{L} - \lambda f = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy - \lambda(x^2 + y^2 - L^2)$$

and the EL eqns are

- x:

$$-2\lambda x = m\ddot{x}$$

- y:

$$mg - 2\lambda y = m\ddot{y}$$

- λ: Left as an exercise

We can see from force analysis of the pendulum:

$$m\ddot{x} = F_x = -2\lambda x \qquad m\ddot{y} = F_y = mg - 2\lambda y$$

so the lagrange multiplier quantities are equivalent to the tension

$$T_x = 2\lambda x \qquad T_y = 2\lambda y$$

where the negative sign indicates the correct direction of Tension.

Pendulum in Polar (r, ϕ)

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - mgr\cos\theta$$

where

$$f = r - L = 0$$

so we get the EL eqns

$$-\lambda + mg\cos\theta = m\ddot{r} \qquad \lambda = mg\cos\theta$$

Use cases of Lagrange Multipliers Although the previous example seems trivial, we consider its use in the example of a heavy chain hanging from two poles: The linear mass density is given by

$$M = \rho L$$

to find the shape, we need to minimize the potential energy

$$S = \int \mathrm{d}mgy$$

where $\mathrm{d}m = \rho\mathrm{d}s$ is the mass of a segment and under the constraint of chain length:

$$L = \int \mathrm{d}s = \int \mathrm{d}x\sqrt{1 + y'^2}$$

so

$$S = \int \rho g y\sqrt{1 + y'^2}\mathrm{d}x$$

and introducing λ we minimize

$$\int (\rho g y - \lambda)\sqrt{1 + y'^2}\mathrm{d}x = S - \lambda L$$

we can see that it is independent of x so

$$f = (\rho g y - \lambda)\sqrt{1 + y'^2}$$

and

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

so the EL eqn is:

$$\frac{\partial f}{\partial y'} = (\rho g y - \lambda)\frac{y'}{\sqrt{1 + y'^2}}$$

and therefore

$$f - y'\frac{\partial f}{\partial y'} = (\rho g y - \lambda)\left[\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}}\right] = \text{constant}$$

and quantity in brackets is

$$[\,] = \frac{1}{\sqrt{1 + y'^2}}$$

so

$$\left(\frac{\rho g y-\lambda}{\sqrt{1+y'^2}}\right)^2=C^2$$

$$1+y'^2=\frac{(\rho g y-\lambda)^2}{C^2}$$

for an easier solution we choose a change of variables

$$\tilde{y}=\frac{\rho g y-\lambda}{C}\implies \tilde{y}'=\frac{\rho g}{C}y'$$

and redifining the x

$$\tilde{y}'^2=1+y'^2\begin{cases}\tilde{x}=\frac{\rho g}{C}x\\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}}=\frac{\mathrm{d}y}{\mathrm{d}x}\end{cases}$$

and we get

$$\tilde{y}=\pm\cosh(\tilde{x}-\tilde{x}_0)$$

we could have also used the Lagrange Multiplier for the Maximum Area Fixed Perimeter problem.

Review Constraint – holonomic $f(q, \dots, q_n) = 0 \to$ Lagrange Multiplier

Example Simple pendulum spinning on its vertical axis. We have the Lagrangian

$$\begin{aligned} T &= \frac{1}{2}m(L^2\dot{\theta}^2 + L^2\sin^2\theta\omega^2) \\ U &= mgy = mgL(1-\cos\theta) \\ \mathcal{L} &= T - U \\ &= \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mL^2\sin^2\theta\omega^2 + mgL(1-\cos\theta) \end{aligned}$$

but we can see the derivatives are also conserved:

$$\begin{aligned} T' &= \frac{1}{2}mL^2\dot{\theta}^2, \quad U' = -\frac{1}{2}mL^2\omega^2\sin^2\theta + mg(1-\cos\theta) \\ \mathcal{L} &= T' - U' \end{aligned}$$

where U' is called the effective potential. E' is conserved, so

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= ml^2\dot{\theta} \quad \text{angular momentum} \\ \dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \mathcal{L} &= E' \\ &= \frac{1}{2}mL^2\dot{\theta}^2 - \frac{1}{2}mL^2\omega^2\sin^2\theta + mgL(1-\cos\theta) = T' + U' \\ T + U &= \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mL^2\omega^2\sin^2\theta + mgL(1-\cos\theta) \end{aligned}$$

we can see that the conserved quantity is different the mechanical energy $E = T + U$. We should be careful with finding what is the conserved quantity in noninertial frames (mechanical energy is not always conserved). In order to study the the function, we should look at the E' term.

Equilibrium points:

$$\begin{aligned} \frac{\mathrm{d}U'}{\mathrm{d}\theta} &= 0 \\ (g - L\omega^2\cos\theta)\sin\theta &= 0 \end{aligned}$$

- if $\omega^2 < \frac{g}{L}$ then only 1 equilibrium point
- if $\omega^2 > \frac{g}{L}$ then 2 equilibrium points

Figure 8.2 shows the effective potential $U'(\theta)$ for the two cases. Sidenote: spinning the green curve around the E' axis gives rise to the Mexican Hat potential in particle physics(Higgs Boson!).

Lagrangian for a charged particle $\mathbf{E}, \mathbf{B}$: we have a Lorentz force

$$m\mathbf{a} = F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where we can define a vector potential $\mathbf{A}$ such that

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Goal: to find $\mathcal{L}$ that gives

$$\frac{\partial\mathcal{L}}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)$$

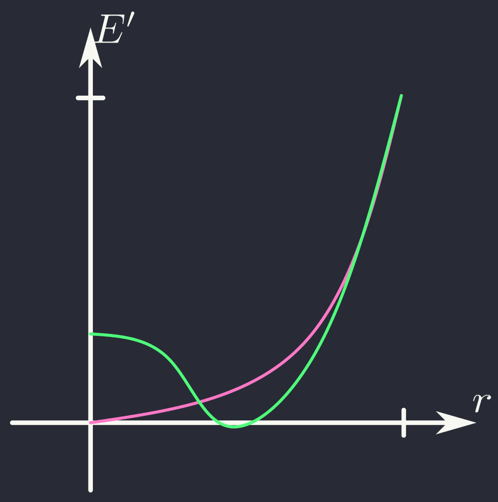

Figure 8.2: Red shows 1 eq point, green shows 2 eq points

so

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^2 - q(\phi - \mathbf{v}\cdot\mathbf{A})$$

the generalized coordinate is $q(x,y,z)$ and we just look at x:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x} &= -q\frac{\mathrm{d}\phi}{\mathrm{d}x} + q\mathbf{v}\cdot\frac{\partial\mathbf{A}}{\partial x} \\
&= -q\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) \\
\frac{\partial\mathcal{L}}{\partial\dot{x}} &= mv_x + qA_x
\end{aligned}$$

so from the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = m\dot{v}_x + q(\cancel{v_x\frac{\partial A_x}{\partial x}} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t})$$

so

$$\begin{aligned}
m\dot{v}_x = ma_x = -q\left(\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) & \qquad \equiv qE_x \\
+ qv_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) & \qquad \equiv B_z \\
- qv_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \qquad \equiv B_y \\
= qE_x + qv_yB_z - qv_zB_y &
\end{aligned}$$

which is the Lorentz force. We can think of the momentum as

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = p_i \qquad \mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

and from QM we know

$$\mathbf{p} \to -i\hbar\boldsymbol{\nabla} - q\mathbf{A}$$

And reviewing from Chapter 2, the linear drag force

$$\mathbf{f} = -b\mathbf{v}$$

is obviously nonconservative. So can get the EL eqn quickly by adding this force to the generalized force

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) &= \frac{\partial \mathcal{L}}{\partial q_i} + f \\ p_i &= F + f \end{aligned}$$

this comes from Rayliegh dissipation function.

$$R(\dot{q}) = \frac{1}{2}b\dot{q}^2 \qquad f = -\frac{\partial R}{\partial \dot{q}}$$

so the EL eqn is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = -\frac{\partial R}{\partial \dot{q}}$$

where we have functions

$$T(\dot{q}), U(q), R(\dot{q})$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial R}{\partial \dot{q}} + \frac{\partial U}{\partial q} = 0$$

Midterm Review

- Newton's Laws
  - 1. Inertial: Keep on going and it won't stop coming, so much to do so much to see.
  - 2. $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}$
  - 3. $\mathbf{F}_{12} = -\mathbf{F}_{21}$
- Polar Coordinates (r, ϕ)

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

  and unit vectors are orthogonal

$$\begin{cases} \hat{r} = \cos\phi\hat{x} + \sin\phi\hat{y} \\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \end{cases} \quad \rightarrow \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$$

  such that the unit vector time derivatives are

$$\dot{\hat{r}} = \dot{\phi}\hat{\boldsymbol{\phi}}$$
$$\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$$

  so the velocity and acceleration is actually

$$\begin{aligned} \mathbf{v} = \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \\ &= v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}} \\ \mathbf{a} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}} \end{aligned}$$

  where the forces are

$$F_r = ma_r \qquad F_\phi = ma_\phi$$

- Momentum $\mathbf{p} = m\mathbf{v}$ and in relation to force $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$. For a collection of particles, the total external force is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{p}_i = \mathbf{F}_{ext}$$

- Angular momentum

$$\ell = \mathbf{r} \times \mathbf{p}$$

- Center of mass

$$\mathbf{R} = \frac{1}{M}\sum m_i\mathbf{r}_i \qquad M = \sum m_i$$
$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\mathrm{d}m$$

- Energy: Kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$$

  and for the two coordinate systems:

$$v^2 = v_x^2 + v_y^2 = v_r^2 + v_\phi^2$$

- Work-KE Theorem:

$$T_2 - T_1 = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = W(1 \to 2)$$

and if $\mathbf{F}$ is conservative—only depends on position: $\mathbf{F}(\mathbf{r})$ & $\boldsymbol{\nabla} \times \mathbf{F} = 0$ thus $\mathbf{F} = -\boldsymbol{\nabla} U$—then

$$W(1 \to 2) = -\Delta U = U_1 - U_2$$
$$E = T_1 + U_1 = T_2 + U_2$$

and more closely finding the critical points of U i.e.

$$\frac{\partial U}{\partial x} = 0 \quad \text{or} \quad \boldsymbol{\nabla} U = 0$$

we also have classical turning points when $E = U$. (Not too important) For the case

$$\frac{1}{2} m \dot{x}^2 = E - U(x)$$
$$\implies \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

- Oscillators

$$\ddot{x} = -\frac{k}{m} x = -\omega_o^2 x \qquad \omega_o = \sqrt{\frac{k}{m}}$$

the solution is written in several forms:

$$\begin{aligned} x(t) &= B_1 \cos(\omega_o t) + B_2 \sin(\omega_o t) \\ &= \mathrm{Re}[C_1 e^{i\omega_o t} + C_2 e^{-i\omega_o t}] \\ &= A\cos(\omega_o t - \delta) \end{aligned}$$

where we solve for the constants using the initial conditions $x(0) = x_o, \quad \dot{x}(0) = v_o$

- Damped Oscillators:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

where we have a homogeneous solution for the three cases:

  - $\beta = \omega_o$: Critical damping

$$x_h(t) = e^{-\beta t}(C_1 + C_2 t)$$

  - $\beta > \omega_o$: Overdamping

$$x_h(t) = e^{-(\beta - \sqrt{\beta^2 - \omega_o^2})t}\left(C_1 + C_2 e^{-2\sqrt{\beta^2 - \omega_o^2} t}\right)$$

  - $\beta < \omega_o$: Underdamping (weak damping)

$$x_h(t) = e^{-\beta t} A\cos(\omega t - \delta) \qquad \omega = \sqrt{\omega_o^2 - \beta^2}$$

- Driven Damped Oscillators:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_0 \cos(\omega t)$$

where f_o has units of acceleration and β has units of frequency. The solution is always

$$x(t) = A\cos(\omega t - \delta) + x_h(t)$$

where $x_h(t)$ is the transient solution and the constants are

$$A^2 = \frac{f_0^2}{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2} \qquad \delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right)$$

where we have a resonance frequency around $\omega = \omega_o$.

- Calculus of Variations: minimizing the action

$$S = \int f(x, y, y')\mathrm{d}x$$

to find $y(x)$ from the EL eqn

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

- Lagrangian (Application of CoV)

$$\mathcal{L} = T - U$$

where the EL eqns are

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

- Conservation: Two special cases
  - If $\mathcal{L}$ is independent of $q_i \Leftrightarrow \frac{\partial \mathcal{L}}{\partial q_i} = 0$

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{is conserved}$$

  - If $\mathcal{L}$ is independent of t

$$\mathcal{H} = \sum_i \dot{q}_i p_i - \mathcal{L} = \text{constant}$$

9 Central Force Problems

Two-Body Considering a two-body system of masses m_1, m_2 we know that under the influence of gravitational potential

$$U = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = -\frac{Gm_1m_2}{\boldsymbol{\mathscr{r}}}$$

so the force on each mass is

$$\begin{aligned}
\mathbf{F}_{12} &= -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_1 U \\
\mathbf{F}_{21} &= +\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(\mathbf{r}_1 - \mathbf{r}_2) = -\boldsymbol{\nabla}_2 U
\end{aligned}$$

computing the Lagrangian:

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 \\
U &= -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}
\end{aligned}$$

in 3D we have 6 degrees of freedom, so we have 6 generalized coordinates:

$$\mathbf{r}_1 = (x_1, y_1, z_2) \quad \mathbf{r}_2 = (x_2, y_2, z_2)$$

and from the separation vector

$$\boldsymbol{\mathscr{r}} = \mathbf{r}_1 - \mathbf{r}_2$$

the center of mass is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \qquad M = m_1 + m_2$$

we can rewrite the position vectors in terms of the COM and separation vector:

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\boldsymbol{\mathscr{r}} \\
\mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\boldsymbol{\mathscr{r}}
\end{aligned}$$

and thus the derivatives are

$$\begin{aligned}
\dot{\mathbf{r}}_1 &= \dot{\mathbf{R}} + \frac{m_2}{M}\dot{\boldsymbol{\mathscr{r}}} \\
\dot{\mathbf{r}}_2 &= \dot{\mathbf{R}} - \frac{m_1}{M}\dot{\boldsymbol{\mathscr{r}}}
\end{aligned}$$

so the Lagrangian is rewritten as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\boldsymbol{\mathscr{r}}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\boldsymbol{\mathscr{r}}}\right)^2 - U \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1m_2}{M}\dot{\boldsymbol{\mathscr{r}}}^2 - U
\end{aligned}$$

where we have the reduced mass

$$\mu = \frac{m_1m_2}{M} = \frac{m_1m_2}{m_1 + m_2}$$

here we can see that $\mathcal{L}$ does not depend on $\mathbf{R}$

$$\implies \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{R}}_i} = \text{const} \implies M\dot{\mathbf{R}} = \text{const} \quad \text{or} \quad M\ddot{\mathbf{R}} = 0$$

this is the igorable coordinate, so Transforming into the COM frame

$$\begin{aligned}
\mathbf{r}_1' &= \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\boldsymbol{r} \\
\mathbf{r}_2' &= \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\boldsymbol{r}
\end{aligned}$$

and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(\boldsymbol{r})$$

which is basically a single particle leaving us with 3 coordinates(Degrees of freedom).

Angular momentum in the COM frame is

$$\begin{aligned}
L &= \sum_i \boldsymbol{r}_i' \times \mathbf{p}_i' \\
&= \boldsymbol{r}' \times m_i \dot{\boldsymbol{r}}' \\
&= m_1 \boldsymbol{r}_1' \times \dot{\boldsymbol{r}}_1' + m_2 \boldsymbol{r}_2' \times \dot{\boldsymbol{r}}_2' \\
&= \frac{m_1 m_2^2}{M}\boldsymbol{r} \times \dot{\boldsymbol{r}} + \frac{m_1^2 m_2}{M}\boldsymbol{r} \times \dot{\boldsymbol{r}} \\
&= \frac{m_1 m_2}{M}\boldsymbol{r} \times \dot{\boldsymbol{r}} = \mu\boldsymbol{r} \times \dot{\boldsymbol{r}}
\end{aligned}$$

which is the same as the angular momentum of a single particle with reduced mass μ.

- If $m_2 \gg m_1$ then $\mathbf{R} \approx \mathbf{r}_2$ and $\mu \approx m_2$.
- If $m_1 \gg m_2$ then $\mathbf{R} \approx \mathbf{r}_1$ and $\mu \approx m_1$.
- If $m_1 = m_2$ then $\mathbf{R}$ is directly in the middle of the two particles and $\mu = \frac{m_1}{2} = \frac{m_2}{2}$.

We can see that for two vectors, any linear combination will result in a vector on a plane, so we can turn this into a 2D problem. Using polar coordinates we can write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\mu(\dot{\boldsymbol{r}}^2 + \boldsymbol{r}^2\dot{\phi}^2) - U(r)$$

where we can see that it does not depend on ϕ, so we have the conserved quantity

$$\implies \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{const} = \mu\boldsymbol{r}\dot{\phi} = \ell$$

and the EL equation is only needed for r:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \mu\boldsymbol{r}\dot{\phi}^2 - \frac{\partial U}{\partial \boldsymbol{r}} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{r}}}\right) &= \mu\ddot{\boldsymbol{r}} \\
\implies \mu\ddot{\boldsymbol{r}} &= \mu\boldsymbol{r}\dot{\phi}^2 - \frac{\partial U}{\partial \boldsymbol{r}} \qquad U = -\frac{Gm_1m_2}{\boldsymbol{r}} \\
&= \frac{l^2}{\mu\boldsymbol{r}^3} - \frac{\partial U}{\partial \boldsymbol{r}} \\
&= \frac{l^2}{\mu\boldsymbol{r}^3} - \frac{Gm_1m_2}{\boldsymbol{r}^2}
\end{aligned}$$

Solving the 1D Problem First we note the first term is equivalent to the minus gradient of the centrifugal potential:

$$\begin{aligned} m\ddot{\mathscr{r}} &= -\frac{\partial U_{cf}}{\partial \mathscr{r}} - \frac{\partial U}{\partial \mathscr{r}} \\ &= -\frac{\partial}{\partial \mathscr{r}}\left(\frac{\ell^2}{2\mu \mathscr{r}^2}\right) - \frac{\partial U}{\partial \mathscr{r}} \\ &= -\frac{\partial}{\partial \mathscr{r}}(U_{cf} + U) \end{aligned}$$

where $U_{cf} = \frac{\ell^2}{2\mu \mathscr{r}^2}$ is the centrifugal potential. We can see that the effective potential is

$$U_{eff} = \frac{\ell^2}{2\mu \mathscr{r}^2} - \frac{Gm_1 m_2}{\mathscr{r}}$$

From Last Time For a 2-Body problem where $M = m_1 + m_2$ and the COM

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)$$
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mu = \frac{m_1 m_2}{M}$$

we found the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)\\ &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)\end{aligned}$$

where $\mathcal{L}$ is independent of ϕ, so we have the conserved quantity

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu r^2\dot{\phi} = \ell \implies \dot{\phi} = \frac{l}{mr^2}$$

so the EL equation for r is

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

where the centrifugal force is

$$F_{cf} = \frac{\ell^2}{\mu r^3}$$

and the effective potential is

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = U_{cf} + U$$

From the graph of this effective potential, there is a centrifugal barrier for finite ℓ for $\ell = \mathbf{r}\times\mathbf{p}$ and for $r \to 0$ the potential is dominated by the centrifugal term.

Conservation of Energy If this problem is independent of time we know that

$$\begin{aligned}E &= \sum_i \dot{q}_i p_i - \mathcal{L}\\ &= \dot{r}\frac{\partial\mathcal{L}}{\partial\dot{r}} + \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \mathcal{L}\\ &= \mu\dot{r}^2 + \frac{\ell^2}{\mu r^2} - \frac{1}{2}\mu\dot{r}^2 - \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)\\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = T + U\end{aligned}$$

we can find the equilibrium point at

$$\begin{aligned}\frac{\partial U_{eff}}{\partial r} &= 0\\ &= -\frac{\ell^2}{\mu r^3} + \frac{\gamma}{r^2} \qquad \gamma = Gm_1m_2\\ \implies r_o &= \frac{\ell^2}{\gamma\mu}\end{aligned}$$

this radius is related to a perfectly circular *orbit.* and at

$$r = r_o, \qquad \dot{\phi} = \frac{\ell\mu^2\gamma^2}{\mu\ell^4} = \frac{\mu\gamma^2}{\ell^3}$$

so

$$\phi(t) = \int_0^t \dot{\phi}(t')\,\mathrm{d}t'$$

For $E < 0$ we have a bound (bounded) orbit, and for $E > 0$ we have an unbounded orbit. For $E = 0$ we also have an unbounded orbit.

What does the orbit look like? Find $r(\phi)$ using a differential equation (For a circular orbit we know $r = r_o$). First we introduce a variable transformation

$$\begin{aligned}
q &= \frac{1}{r}, \qquad r = \frac{1}{q}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{q}\right) = -\frac{1}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\phi}, \qquad q' = \frac{\mathrm{d}q}{\mathrm{d}\phi} \\
\dot{r} &= \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\cdot\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\ell}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\phi} = -\frac{\ell}{\mu r^2}\frac{1}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\phi} = -\frac{\ell}{\mu}\frac{\mathrm{d}q}{\mathrm{d}\phi} \\
\ddot{r} &= \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}\dot{r}}{\mathrm{d}\phi} = -\dot{\phi}\frac{\ell}{\mu}q'' = -\frac{\ell^2 q^2}{\mu^2}q''
\end{aligned}$$

and the central force is

$$\begin{aligned}
\mu\ddot{r} &= \frac{\ell^2}{\mu r^3} + F \\
-\mu\frac{\ell^2 q^2}{\mu^2}q'' &= \frac{\ell^2 q^3}{\mu r^3} + F \\
q'' &= -q - \frac{\mu}{q^2\ell^2}F
\end{aligned}$$

and since the force is

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r} = -\frac{\gamma}{r^2} = -\gamma q^2$$

so the differential equation is just

$$q'' = -q + \frac{\gamma\mu}{\ell^2}$$

and the RHS vanishes when

$$q = \frac{\gamma\mu}{\ell^2} \quad \text{or} \quad r_o = \frac{\ell^2}{\gamma\mu}$$

we can redefine the constant

$$\omega = q - \frac{\gamma\mu}{\ell^2} \implies \omega'' = q'' = -\omega$$

so

$$\omega(\phi) = A\cos(\phi - \delta)$$

and choosing initial conditions so that $\delta = 0$

$$\omega(\phi) = A\cos(\phi) \implies q(\phi) = A\cos(\phi) + \frac{\gamma\mu}{\ell^2} = \frac{1}{r(\phi)}$$

and thus

$$r(\phi) = \frac{\ell^2/\gamma\mu}{1+\epsilon\cos(\phi)} = \frac{C}{1+\epsilon\cos(\phi)} \qquad \epsilon = \frac{A}{C}$$

we can check and see that r has the unit of length and the denominator is unitless, so C has the unit of length. We can see that ϵ only depends on the initial conditions, and at

$$\epsilon = 0 \implies r(\phi) = C = r_o$$

so

$$r(\phi) = \frac{r_o}{1 + \epsilon \cos(\phi)}$$

and ϵ is the eccentricity of the orbit.

- If $\epsilon = 0$ then $r = r_o$ and we have a circular orbit.
- If $\epsilon > 1$ then the denominator can $\to 0$ and we have $r \to \infty$ or hyperbolic orbit.
- If $0 < \epsilon < 1$ then we have a bounded orbit or ellipse.
- IF $\epsilon = 1$ then we have a parabolic orbit.

Lecture 20: **3/8/24**

Missed Lecture:

10 Non-inertial Frames

In the *noninertial* frame, N1L and N2L do not hold true. In the simplest example of an accelerating frame, we have simple pendulum in a car moving in a straight line with velocity $\mathbf{V}$ and acceleration $\dot{\mathbf{V}} = \mathbf{A}$. In the lab frame (inertial),$\mathcal{S}_0$, we can relate the position of the moving frame $\mathcal{S}$ by

$$\begin{aligned}
\dot{\mathbf{r}}_0 &= \dot{\mathbf{r}} + \mathbf{V} \\
\ddot{\mathbf{r}}_0 &= \ddot{\mathbf{r}} + \mathbf{A} \\
\implies \ddot{\mathbf{r}}_0 &= \ddot{\mathbf{r}} - \mathbf{A}
\end{aligned}$$

so N2L in $\mathcal{S}_0$ is always

$$\mathbf{F} = m\ddot{\mathbf{r}}_0 = m(\ddot{\mathbf{r}} - \mathbf{A})$$

so in the noninertial frame, $\mathcal{S}$, we have

$$m\ddot{\mathbf{r}} = F - m\mathbf{A}$$

The procedure of mechanics in noninertial frames is as follows:

1. write down equations in lab frame using $\mathbf{r}_0$
2. identify coordinates in noninertial frame

So in $\mathcal{S}$ we have an effective gravity

$$\begin{aligned}
m\ddot{r} &= m\mathbf{g} - m\mathbf{A} \\
&= m\mathbf{g}_{\text{eff}}
\end{aligned}$$

where the trigonometry tells us

$$\tan\phi_0 = \frac{A}{g}$$

so we have a new equation of motion

$$\ddot{\theta} = -\frac{g_{\text{eff}}}{L}\sin(\theta - \theta_0)$$

where

$$g_{\text{eff}} = \sqrt{g^2 - A^2}$$

with frequency

$$\omega = \sqrt{\frac{g_{\text{eff}}}{L}}$$

the force in the noninertial frame

$$F_{NI} = -m\mathbf{A}$$

is known as the inertial or *fictitious* force. From before we know that the gravitational force is actually

$$\begin{aligned}
F_g &= -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \\
&= m\mathbf{g} \qquad \mathbf{g} = -\frac{GM_E}{R_E^2}\hat{\mathbf{r}}
\end{aligned}$$

where we assume a constant gravity near the surface of the Earth.

Einsteins Equivalence Principle Einstein thought that gravitational mass equals inertial mass. Thus Gravity is a fictitious force.

Tidal Forces Without considering the earth rotating on its axis or orbiting the sun, we can see that it is in a noninertial frame due to the gravitational attraction of the moon! So this lab frame (or God Frame as we must consider ourselves external to the motion of the earth) the N2L tells us

$$m\ddot{\mathbf{r}}_0 = m\mathbf{g} - \frac{GM_m m}{d^2}\hat{\mathbf{d}} + \mathbf{F}_{ext}$$

and in the noninertial frame we have

$$\mathbf{A} = -\frac{GM_m}{d_0^2}\hat{\mathbf{d}}_0$$

so N2L is

$$m\ddot{r} = m\mathbf{g} - \frac{GM_m m}{d^2}\hat{\mathbf{d}} + \mathbf{F}_{ext} + \frac{GM_m m}{d_0^2}\hat{\mathbf{d}}_0$$

we call this difference of forces due to the moon the *tidal force*:

$$m\ddot{r} = m\mathbf{g} + \mathbf{F}_{ext} + \mathbf{F}_{\text{tid}}$$
$$\mathbf{F}_{\text{tid}} = -GM_m m\left(\frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{d}}_0}{d_0^2}\right)$$

so on the far side of the earth (furthest from the moon) we have $d^2 < d_0^2$ so the tidal force points away from the moon. On the near side of the earth, the tidal force points towards the moon. we can also see that the tidal force is equivalent to the difference in gravitational force of the moon on the object minus the force of the moon at the center of mass of the earth. Earth as an extended object is subject to tidal forces, and applies to all extended objects such as black holes.
For example the tidal forces of a dinosaur near a black hole would strech and compress the dinosaur.

Lecture 22: 3/20/24

Rotating Reference Frame What is rotation? u.r.t an axis i.e. angular velocity ω where the direction of the vector is the axis and magnitude is the speed of rotation ω. To determine the direction of rotation we use the Right Hand Rule.

Euler's Theorem The most general motion of a body about a fixed point is also a rotation about an axis that passes through the fixed point.
Common Notation:

- $\boldsymbol{\Omega}$ is a fixed angular velocity
- ω is a specific (unknown) angular velocity.

From Taylor Figure 9.7, we can see that the distance from the point from the rotating axis ρ gives the relation $\rho = r\sin\theta$. And the magnitude of velocity is

$$\nu = \omega\rho = \omega r\sin\theta$$

and hence the velocity

$$\nu = \omega \times \mathbf{r} = \dot{\mathbf{r}}$$

So in general, the time derivative of a vector in a rotating frame is always in the form

$$\dot{\mathbf{u}} = \omega \times \mathbf{u}$$

Rotating Frame From last time, we have the to frames of reference:

- $\mathcal{S}_0$ the lab frame (God frame)
- $\mathcal{S}$ the rotating frame

Where Ω is the angular velocity of $\mathcal{S}$ in $\mathcal{S}_0$.

For a rotation about the Earth we have

$$\Omega = \frac{2\pi}{24\times 3600} \sim 7.3\times 10^{-5}\,\text{rad/s}$$

In $\mathcal{S}$ we define coordinate basis vectors

$$\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3$$

Which are orthogonal, and in $\mathcal{S}_0$, $\hat{\mathbf{e}}_i$ rotates with ang. velocity $\boldsymbol{\Omega}$.

Goal: Write N2L in $\mathcal{S}$.
In $\mathcal{S}_0$ we have N2L

$$m\ddot{\mathbf{r}}_0 = \mathbf{F}$$

For a general vector

$$\mathbf{Q} = Q_1\hat{\mathbf{e}}_1 + Q_2\hat{\mathbf{e}}_2 + Q_3\hat{\mathbf{e}}_3 = \sum_i Q_i\hat{\mathbf{e}}_i$$

In the rotating frame the change in unit vectors are constant

$$\begin{aligned} \dot{\hat{e}}_i &= 0 \quad \text{in} \quad \mathcal{S} \\ \implies \dot{\hat{e}}_i &= \boldsymbol{\Omega}\times\hat{\mathbf{e}}_i \quad \text{in} \quad \mathcal{S}_0 \end{aligned}$$

So

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{\mathcal{S}} = \sum_i \frac{\mathrm{d}Q_i}{\mathrm{d}t}\hat{\mathbf{e}}_i$$
$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{\mathcal{S}} = \sum_i \frac{\mathrm{d}Q_i}{\mathrm{d}t}\hat{\mathbf{e}}_i + \sum Q_i\dot{\hat{e}}_i$$

And in the lab frame

$$\begin{aligned}\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{\mathcal{S}_0} &= \sum_i \frac{\mathrm{d}Q_i}{\mathrm{d}t}\hat{\mathbf{e}}_i + \sum Q_i\boldsymbol{\Omega}\times\hat{\mathbf{e}}_i \\ &= \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\Omega}\times\mathbf{Q}\end{aligned}$$

So N2L in the lab frame is

$$\begin{aligned} m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\mathcal{S}_0} &= \mathbf{F} \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}_0}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}_0} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}_0}\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\Omega}\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}}\right] \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}}\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \Omega\times\mathbf{r}\right] + \Omega\times\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \Omega\times\mathbf{r}\right] \\ &= \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\mathcal{S}} + 2\Omega\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \Omega\times(\Omega\times\mathbf{r})\end{aligned}$$

where we let

$$\dot{\mathbf{r}} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}}$$

which we can rewrite the N2L as

$$\mathbf{a}_{\mathcal{S}_0} = \ddot{\mathbf{r}} + 2\boldsymbol{\Omega}\times\dot{\mathbf{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

SO the N2L in the lab frame is simply

$$m\mathbf{a}_{\mathcal{S}_0} = \mathbf{F}$$

and in the rotating frame we have

$$\begin{aligned} m(\ddot{\mathbf{r}} + 2\boldsymbol{\Omega}\times\dot{\mathbf{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})) &= \mathbf{F} \\ m\ddot{\mathbf{r}} &= \mathbf{F} - 2m\boldsymbol{\Omega}\times\dot{\mathbf{r}} - m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) \\ &= \mathbf{F} + 2m\dot{\mathbf{r}}\times\boldsymbol{\Omega} + m(\boldsymbol{\Omega}\times\mathbf{r})\times\boldsymbol{\Omega} \\ &= \mathbf{F} + \mathbf{F}_{\mathrm{cor}} + \mathbf{F}_{\mathrm{cf}}\end{aligned}$$

where we have two extra terms: the Coriolis force and a Centrifugal force.

Coriolis Force From the Centrifugal force

$$\begin{aligned}\mathbf{F}_{\mathrm{cf}} &= m(\boldsymbol{\Omega}\times\mathbf{r})\times\boldsymbol{\Omega} \\ \mathbf{F}_{\mathrm{cf}} &= m\mathbf{v}\times\boldsymbol{\Omega} \\ F_{\mathrm{cf}} &= mv\Omega = m\rho\Omega^2 \\ &= \frac{m\rho^2\Omega^2}{\rho} = \frac{mv^2}{r}\end{aligned}$$

Example: St Louis The colatitude of St Louis is $\theta = 51.4°$ so using

$$\Omega_E = 7.3 \times 10^{-5}\,\text{rad/s} \quad R_E \approx 6400\,\text{km} = 6.4 \times 10^6\,\text{m}$$

we have the acceleration

$$\begin{aligned} a_{\text{cf}} = (\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega} = \rho\Omega^2 &= r\sin\theta\Omega^2 \\ &\approx 2.66 \times 10^{-2}\,\text{m/s}^2 \end{aligned}$$

and since we know that the acceleration due to Earth

$$a_g = \frac{GM_E}{r^2}$$

and using

$$\begin{aligned} G = 6.67 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2} \quad M_E &= 5.9 \times 10^{24}\,\text{kg} \\ r_{eq} \approx 6378\,\text{km} \quad r_{pole} &\approx 6356\,\text{km} \end{aligned}$$

the change in acceleration is

$$\begin{aligned} \Delta a_g &= GM_E\left(\frac{1}{r_{eq}^2} - \frac{1}{r_{pole}^2}\right) \\ &\approx 6.8 \times 10^{-2}\,\text{m/s}^2 \end{aligned}$$

So we have a nonzero tangential component of acceleration due to gravity due to the rotation of the Earth (it isn't just radial). When we look at the acceleration of the Earth orbiting the sun we find

$$a_{\text{orbit}} \sim 6.0 \times 10^{-3}\,\text{m/s}^2$$

which is only a third of the centrifugal force due to the Earth's axial rotation.

Lecture 23: **3/22/24**

Review: N2L in a rotating frame:

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega} \\ &= \mathbf{F} + \mathbf{F}_{\text{cor}} + \mathbf{F}_{\text{cf}} \end{aligned}$$

we have an additional Coriolis force and a Centrifugal force.

Figure 10.1: Turntable with frictionless puck. In the lab frame, the puck moves in a straight line, but in the noninertial frame—rotating with the turntable—the puck moves in a curved path.

Coriolis Force

$$\mathbf{F}_{\text{cor}} = 2m\dot{\mathbf{r}} \times \mathbf{\Omega}$$

where

$$\mathbf{F}_{\text{cor}} \cdot \mathbf{v} = 0$$

when we look at the direction of the Coriolis force in the Earth's frame, at the North side the Coriolis force points clockwise, and at the South side the Coriolis force points counterclockwise—or the origin of cyclones.

Free Fall & Coriolis Force From Taylor Figure 9.15 we have the N2L in the rotating frame

$$\ddot{\mathbf{r}} = \mathbf{g} + 2\dot{\mathbf{r}} \times \mathbf{\Omega}$$

where we omit the centrifugal force. We can see the vectors are

$$\begin{aligned} \mathbf{\Omega} &= (0, \Omega\sin\theta, \Omega\cos\theta) \\ \mathbf{r} &= (x, y, z) \\ \dot{\mathbf{r}} &= (\dot{x}, \dot{y}, \dot{z}) \end{aligned}$$

so

$$\dot{\mathbf{r}} \times \Omega = (\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta, -\dot{x}\Omega\cos\theta, \dot{x}\Omega\sin\theta)$$

and therefore using the N2L equation we have the equations of motion

$$\ddot{x} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$
$$\ddot{y} = -2\Omega\dot{x}\cos\theta$$
$$\ddot{z} = -g + 2\Omega\dot{x}\sin\theta$$

For the 0th-order approximation we take $\Omega = 0$ and we have the simple free fall equations

$$x = 0, \quad y = 0, \quad z = -\frac{1}{2}gt^2$$

For the 1st order of rotation we substitute the 0th-order solution into the original equations:

$$\ddot{x} = 2\Omega\dot{z}\sin\theta = 2gt\Omega\sin\theta$$
$$\ddot{y} = 0 \implies y = 0$$
$$\ddot{z} = -g \implies z = z_0 - \frac{1}{2}gt^2$$

where the y and z components are the same as the 0th-order approximation, but the x equation gives the solution

$$x = \frac{1}{3}gt^3\Omega\sin\theta$$

Example 100m Fall from St. Louis From the vertical component we know it takes

$$t = \sqrt{\frac{2z_0}{g}} \approx 4.5\,\mathrm{s}$$

and we find that the object will deviate by

$$\Delta x \approx \frac{1}{3}\Omega g t^3 \approx 2.2\,\mathrm{cm}$$

Foucault Pendulum The equations of motion have an added Tension:

$$\ddot{x} = \frac{T_x}{m} + 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$
$$\ddot{y} = \frac{T_y}{m} - 2\Omega\dot{x}\cos\theta$$
$$\ddot{z} = \frac{T_z}{m} - g + 2\Omega\dot{x}\sin\theta$$

With the y pointing north and x pointing east, we can see that the components of tension are

$$T_z = T\cos\beta$$
$$T_x = T\sin\beta\cos\phi = -T\frac{x}{L}$$
$$T_y = T\sin\beta\sin\phi = -T\frac{y}{L}$$

where $\beta \approx 0 \qquad \cos\beta \approx 1$ and $T_z \approx mg \approx T$. We can also assume that the pendulum is very long so $z \approx 0$ and the arc is more like a line:

$$\ddot{x} = -\frac{g}{L}x + 2\dot{y}\Omega_z \qquad \Omega_z = \Omega\cos\theta$$
$$\ddot{y} = -\frac{g}{L}y - 2\dot{x}\Omega_z \qquad \omega_0 = \frac{g}{L}$$

this is a coupled system of equations

$$\ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 x = 0$$
$$\ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y = 0$$

which is solved using complex exponentials (first see Taylor 2.5):

$$+i(\ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y) = 0$$

where

$$\begin{aligned}
\eta &= x + iy \\
\dot{\eta} &= \dot{x} + i\dot{y} \\
\ddot{\eta} &= \ddot{x} + i\ddot{y}
\end{aligned}$$

so adding the two equations we have

$$\begin{aligned}
\ddot{x} + i\ddot{y} + 2\Omega_z(\dot{x} + i\dot{y}) + \omega_0^2(x + iy) &= 0 \\
\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2\eta &= 0
\end{aligned}$$

which is a ODE from the damped harmonic oscillator with a solution $\eta \sim e^{-i\alpha t}$ where

$$\begin{aligned}
-\alpha^2 + 2\Omega_z\alpha + \omega_0^2 &= 0 \\
\implies \alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2} &\approx \Omega_z \pm \omega_0
\end{aligned}$$

so we can write η as

$$\begin{aligned}
\eta(t) &= C_1 e^{-(\Omega_z+\omega_0)t} + C_2 e^{-(\Omega_z-\omega_0)t} \\
&= e^{-i\Omega_z t}(C_1 e^{-\omega_0 t} + C_2 e^{\omega_0 t}) \\
&= e^{-i\Omega_z t} A \cos\omega_0 t
\end{aligned}$$

we can see that the Foucault pendulum will rotate around the flat plane by Ω_z which is slower than the Earth's rotation.

11 Solid Body Rotation

Last Week: Non-inertial Frames

1. Just linear acceleration $\mathbf{A}$, N2L

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}$$

2. Rotating frame:

$$m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \Omega + m(\Omega \times \mathbf{r}) \times \Omega$$

Solid body

N particles on a continuous distribution

$$\begin{aligned} m_\alpha, &\qquad \alpha = 1, 2, \ldots, N \\ \mathbf{r}_\alpha, &\qquad \mathbf{r}_\alpha - \mathbf{r}_\beta = \text{constant} \end{aligned}$$

With a center of mass (COM/CM)

$$\begin{aligned} \mathbf{R} = \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}_\alpha, &\qquad M = \sum_\alpha m_\alpha \\ \mathbf{P} = \sum_\alpha m_\alpha \mathbf{v}_\alpha = \sum_\alpha m\alpha \dot{\mathbf{r}}_\alpha &= M\dot{\mathbf{R}} \\ \dot{\mathbf{P}} = M\ddot{\mathbf{R}} &= \mathbf{F}_{\text{ext}} \end{aligned}$$

Angular Momentum

$$\begin{aligned} \ell_\alpha &= \mathbf{r}_\alpha \times \mathbf{p}_\alpha \\ &= \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha \end{aligned}$$

and the total angular momentum

$$\mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha$$

Defining a position $\mathbf{r}'_\alpha$ relative to the CM

$$\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}, \quad \mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$$

we can rewrite the total angular momentum as

$$\begin{aligned} \mathbf{L} &= \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}'_\alpha) \times (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha) \\ &= \sum_\alpha m_\alpha \mathbf{R} \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{R} \times \dot{\mathbf{r}}'_\alpha + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha \end{aligned}$$

but since we know that

$$\begin{aligned} \mathbf{R} &= \frac{1}{M}\sum_\alpha m_\alpha(\mathbf{R} + \mathbf{r}'_\alpha) \\ &= \frac{1}{M}\sum_\alpha m_\alpha \mathbf{R} + \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}'_\alpha \\ \implies &\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0 \\ &\sum_\alpha m_\alpha \dot{\mathbf{r}}'_\alpha = 0 \end{aligned}$$

so the middle terms of the total angular momentum are zero:

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha$$

which can be re-expressed as

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_{\text{cm}} + \mathbf{L}_{\text{rel}} \\ \mathbf{L}_{\text{cm}} &= M\mathbf{R} \times \dot{\mathbf{R}} \\ \mathbf{L}_{\text{rel}} &= \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha \end{aligned}$$

For example we can consider the earth as a rigid body with angular momentum

$$\mathbf{L}_E = \mathbf{L}_{\text{spin}} + \mathbf{L}_{\text{orb}}$$

Time derivative of angular momentum we have two parts

$$\begin{aligned} \dot{\mathbf{L}}_{\text{cm}} &= M\dot{\mathbf{R}} \times \dot{\mathbf{R}} + M\mathbf{R} \times \ddot{\mathbf{R}} \\ &= M\mathbf{R} \times \mathbf{F}_{\text{ext}} = \mathbf{\Gamma}_{\text{cm}} \end{aligned}$$

and

$$\begin{aligned} \dot{\mathbf{L}}_{\text{rel}} &= \sum_\alpha m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{r}}'_\alpha, \quad \ddot{\mathbf{r}}'_\alpha = \ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}} \\ &= \mathbf{\Gamma}_{\text{rel}} \end{aligned}$$

Energy The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha (\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha)^2 \\ &= \frac{1}{2}\sum_\alpha m_\alpha (\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{r}}'^2_\alpha) \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}'^2_\alpha \end{aligned}$$

and the potential energy is

$$U = U_{\text{ext}} + U_{\text{int}} = U_{\text{ext}}$$

where there is no relative motion between the particles, the internal potential energy is a constant which can be ignored.

Example: Rotating disk We consider a disk rotating about the z-axis with angular velocity

$$\omega = (0, 0, \omega)$$

with a particle with position and velocity

$$\begin{aligned} \mathbf{r}_\alpha &= (x_\alpha, y_\alpha, z_\alpha) \\ \dot{\mathbf{r}}_\alpha &= (\dot{x}_\alpha, \dot{y}_\alpha, \dot{z}_\alpha) \end{aligned}$$

the time derivative of the position vector is

$$\dot{\mathbf{r}}_\alpha = \omega \times \mathbf{r}_\alpha = (-\omega y_\alpha, \omega x_\alpha, 0)$$

and the angular momentum is

$$\begin{aligned}\ell_\alpha &= m_\alpha \mathbf{r}_\alpha \times \dot{\mathbf{r}}_\alpha = m_\alpha \mathbf{r}_\alpha \times (\omega \times \mathbf{r}_\alpha) \\ &= m_\alpha(-\omega x_\alpha z_\alpha, -\omega y_\alpha z_\alpha, \omega(x_\alpha^2 + y_\alpha^2))\end{aligned}$$

thus the z component of total angular momentum is

$$L_z = \sum_\alpha m_\alpha \ell_{\alpha,z} = \sum_\alpha m_\alpha \omega(x_\alpha^2 + y_\alpha^2) = \omega \sum_\alpha m_\alpha \rho_\alpha^2 = \omega I_z$$

where ρ is radius in cylindrical coordinates and I_z is the moment of inertia about the z-axis (parallel axis theorem). The other two components of angular momentum are

$$\begin{aligned}L_x &= -\sum_\alpha m_\alpha \omega x_\alpha z_\alpha \\ L_y &= -\sum_\alpha m_\alpha \omega y_\alpha z_\alpha\end{aligned}$$

and since L_x and L_y can be nonzero, that means that $\mathbf{L}$ can be in any direction! If we define the products of inertia

$$\begin{aligned}I_{xz} &= -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ I_{yz} &= -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ I_{zz} &= \sum_\alpha m_\alpha (x_\alpha + y_\alpha)^2\end{aligned}$$

we define the total angular momentum as

$$\begin{aligned}\mathbf{L} &= I \cdot \omega \\ &= (I_{xz} \cdot \omega_z, I_{yz} \cdot \omega_z, I_{zz} \cdot \omega_z)\end{aligned}$$

Lecture 25: **3/27/24**

HW 8 Hint For a puck on a rotating table

$$\ddot{\mathbf{r}} = 2\dot{\mathbf{r}} \times \mathbf{\Omega} + (\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}$$

where the first term points inward, and the second term points outward. Since

$$\dot{\mathbf{r}} = -\mathbf{\Omega} \times \mathbf{r}$$
$$\implies \ddot{\mathbf{r}} = -r\Omega^2$$

or the centripetal acceleration.

Inertia Tensor

For a general rigid body, we define the angular velocity

$$\omega = (\omega_x, \omega_y, \omega_z)$$

the angular momentum

$$\mathbf{L} = \sum_\alpha \ell_\alpha = \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \dot{\mathbf{r}}_\alpha$$

where $\dot{\mathbf{r}}_\alpha = \omega \times \mathbf{r}_\alpha$. Using the BAC-CAB rule we can write

$$\mathbf{r} \times (\omega \times \mathbf{r}) = \omega(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \omega)$$

so

$$\begin{aligned} L_x &= \sum_\alpha m_\alpha(\omega_x(x_\alpha^2 + y_\alpha^2 + z_\alpha^2) - x(\cancel{\omega_x x_\alpha} + \omega_y y_\alpha + \omega_z z_\alpha)) \\ &= \sum_\alpha m_\alpha(\omega_x(y^2 + z^2)) - m_\alpha x_\alpha y_\alpha \omega_y - m_\alpha x_\alpha z_\alpha \omega_z \end{aligned}$$

where we define the products of inertia for $\mathbf{L} = I\omega$:

$$L_i = \sum_j^3 I_{ij}\omega_j$$
$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

where

$$\begin{aligned} I_{xx} &= \sum_\alpha m_\alpha(y_\alpha^2 + z_\alpha^2) \\ I_{xy} &= -\sum_\alpha m_\alpha x_\alpha y_\alpha = I_{yx} \\ I_{xz} &= -\sum_\alpha m_\alpha x_\alpha z_\alpha = I_{zx} \end{aligned}$$

for the the first row, and

$$\begin{aligned} I_{yy} &= \sum_\alpha m_\alpha(x_\alpha^2 + z_\alpha^2) \\ I_{yx} &= -\sum_\alpha m_\alpha y_\alpha x_\alpha \\ I_{yz} &= -\sum_\alpha m_\alpha y_\alpha z_\alpha = I_{zy} \end{aligned}$$

similarly for the second row and third row. These creates a real 3×3 matrix that is symmetric or

$$I = I^T$$

Example: Dumbell in the yz plane The masses are placed at $(0, \pm y_0, z)$, so the products of inertia are

$$I_{zz} = 2my_0^2, \quad I_{xx} = 2m(y_0^2 + z_0^2), \quad I_{yy} = 2mz_0^2$$

and for the nondiagonal terms

$$I_{xy} = 0 = I_{xz}, \quad I_{yz} = 0 \dots$$

are all zero. Thus we create the inertia tensor

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Another example: A disk on the xy plane The disk has radius R and mass M lying on the xy plane at $z = z_0$. The mass is distributed evenly, so

$$\begin{aligned} m_\alpha &= \mathrm{d}m = \rho R \,\mathrm{d}\theta \\ M &= \int \mathrm{d}m = \int_0^{2\pi} \rho R \,\mathrm{d}\theta = 2\pi\rho R^2 \\ \implies \rho &= \frac{M}{2\pi R^2} \end{aligned}$$

The products of inertia are now calcaulable:

$$\begin{aligned} I_{zz} &= \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = 0 \\ &= \int_0^{2\pi} \rho R \,\mathrm{d}\theta \,(x^2 + y^2) \qquad R = x^2 + y^2 \\ &= \int_0^{2\pi} \rho R^3 \,\mathrm{d}\theta = 2\pi\rho R^3 = 2\pi \frac{M}{2\pi R^2} R^3 = MR \end{aligned}$$

and the other products

$$\begin{aligned} I_{xx} &= \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) \\ &= \int_0^{2\pi} \rho R \,\mathrm{d}\theta \,(y^2 + z_0^2) \\ &= \int_0^{2\pi} \rho R \,\mathrm{d}\theta \,(R^2 \cos^2\theta + z_0^2) \\ &= Mz_0^2 + \int_0^{2\pi} \rho R^3 \cos^2\theta \,\mathrm{d}\theta \\ &= Mz_0^2 + \pi\rho R^3 \\ &= Mz_0^2 + \pi \frac{M}{2\pi R^2} R^3 = Mz_0^2 + \frac{M}{2} R^2 \end{aligned}$$

we can see the familar term for the moment of inertia of a disk $I = \frac{1}{2}MR^2$ which is shifted. The cross terms are

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_0 = 0 = I_{xz}$$
$$I_{xy} = 0 \dots$$

where can see that the average of y is zero for the first term, and we see that all the cross terms are zero as well.

Final Example: A cube with corner at the origin Given the side length a we know that the mass is simply

$$M = \rho a^3, \quad \rho = \frac{M}{a^3}$$

and the products of inertia are

$$\begin{aligned} I_{xy} &= -\int_0^a \mathrm{d}x \int_0^a \mathrm{d}y \int_0^a \mathrm{d}z\, \rho xy \\ &= -\frac{1}{2}a^2 \cdot \frac{1}{2}a^2 \cdot a \cdot \rho = -\frac{1}{4}Ma^2 \end{aligned}$$

and the other cross terms are the same as well

$$I_{yz} = I_{xz} = -\frac{1}{4}Ma^2$$

The diagonal terms

$$\begin{aligned} I_{zz} &= \iiint_0^a \rho(x^2 + y^2)\,\mathrm{d}V \\ &= \frac{1}{3}a^3 \cdot a \cdot a \cdot \rho + a \cdot \frac{1}{3}a^3 \cdot a \cdot \rho \\ &= \frac{2}{3}Ma^2 = I_{xx} = I_{yy} \end{aligned}$$

this gives us the inertia tensor

$$I = Ma^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$$

The symmetry of the axes tell us how lopsided or asymmetrical the object is.

Inertial Tensor We can define the inertial product

$$I_{ij} = \sum_\alpha m_\alpha (r_\alpha^2 \delta_{ij} - r_{\alpha,i} r_{\alpha,j})$$

where δ_{ij} is the kronicker delta. We know that I in 3×3 is real and symmetric. I is diagonalizeable

$$\exists \quad 3 \text{ axes} \quad \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$$

such that I is diagonal i.e.

$$I = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

Where

$$\lambda_1, \quad \lambda_2, \quad \lambda_3$$

are the principle moments of inertia and the principle axes are

$$\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$$

Thus the matrix rows are in the form

$$I\hat{\mathbf{e}}_i = \lambda_i \hat{\mathbf{e}}_i$$

And I is diagonalized by rotation. To solve a diagonization problem we have to solve for λ by using the angular momentum equation

$$\begin{aligned} \mathbf{L} = I\omega &= \lambda\omega \\ I\omega - \lambda\omega &= 0 \\ (I - \lambda \mathbb{1})\omega &= 0 \\ \implies \det(I - \lambda \mathbb{1}) &= 0 \end{aligned}$$

which gives us the matrix

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{vmatrix} = 0$$

so from the cube example we have

$$I = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \qquad \mu = \frac{Ma^2}{12}$$

thus

$$\det(I - \lambda \mathbb{1}) = \begin{vmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{vmatrix}$$

To solve this, we can use some tricks:

- Circular matrix: Since the sum of every row is the same

$$\sum_j I_{ij} = \text{constant} = \lambda_1$$

- Trace of a matrix (sum of diagonal entries)

$$\begin{aligned}\mathrm{Tr}(I) &= I_{xx} + I_{yy} + I_{zz} \\ &= \sum_i \lambda_i\end{aligned}$$

- Determinant of a matrix

$$\det(I) = \lambda_1\lambda_2\lambda_3$$

so

$$\lambda_1 = 2\mu, \quad \mathrm{Tr}(I) = 24\mu, \quad \det(I) = 242\mu^3$$

so form the other two equations

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= 24\mu \\ \lambda_2\lambda_3 &= 22\mu^2\end{aligned}$$

and

$$\begin{aligned}\lambda_1\lambda_2\lambda_3 &= 242\mu^3 \\ \lambda_2\lambda_3 &= 121\mu^2\end{aligned}$$

which gives us

$$\lambda_2 = \lambda_3 = 11\mu$$

And to get the principle axes we can use the eigenvectors of the matrix

$$\begin{aligned}\lambda\hat{\mathbf{e}}_1 &= I\hat{\mathbf{e}}_1 \\ \begin{pmatrix} 8\mu & -3\mu & -3\mu \\ -3\mu & 8\mu & -3\mu \\ -3\mu & -3\mu & 8\mu \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= 2\mu\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ \begin{pmatrix} 6\mu & -3\mu & -3\mu \\ -3\mu & 6\mu & -3\mu \\ -3\mu & -3\mu & 6\mu \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= 0\end{aligned}$$

and we can guess a solution from the first two rows

$$\begin{aligned}6\mu w_1 - 3\mu w_2 - 3\mu w_3 = 0 \\ 6\mu w_2 - 3\mu w_1 - 3\mu w_3 = 0\end{aligned}$$

summing these two equations gives us

$$9\mu w_1 - 9\mu w_2 = 0 \implies w_1 = w_2 = w_3$$

and we can normalize the vector to get

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for the second unit vector

$$\begin{aligned}I\hat{\mathbf{e}}^2 &= \lambda_2\hat{\mathbf{e}}^2 \\ \begin{pmatrix} -3\mu & -3\mu & -3\mu \\ -3\mu & -3\mu & -3\mu \\ -3\mu & -3\mu & -3\mu \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= 0\end{aligned}$$

where we can guess that

$$\frac{1}{\sqrt{3}}(w_1 + w_2 + w_3) = 0 \leftrightarrow \omega \cdot \hat{\mathbf{e}}_1 = 0$$

We call the new basis vectors of the principle axes the *Body Frame*: $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. The *Space Frame* is the original frame of reference: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. And in the body frame

$$\mathbf{L}_{\text{body}} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

Note that this may point in a different direction than the angular velocity vector.

If $\omega = \omega e_3$

$$\mathbf{L} = I\omega = \lambda_3\omega_3\hat{\mathbf{e}}_1$$

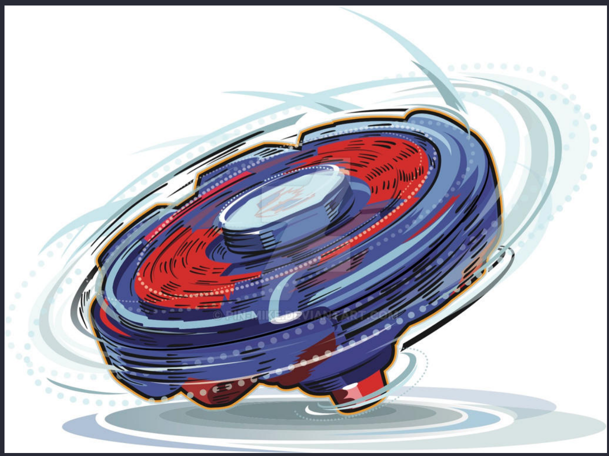

Figure 11.1: The body frame has $\hat{\mathbf{e}}_3$ point in the direction of the spinning beyblade.

Since $\dot{\mathbf{L}} = \mathbf{\Gamma} = \mathbf{R} \times M\mathbf{g}$ where a nonzero torque allows the body to precess. If we look at the torque more carefully given

$$\mathbf{R} = R\hat{\mathbf{e}}_1, \quad \mathbf{g} = -g\hat{\mathbf{z}}$$

$$\begin{aligned}\lambda_3\omega_3\dot{\hat{e}}_1 &= MgR\hat{\mathbf{z}} \times \hat{\mathbf{e}}_3 \\ \dot{\hat{e}}_3 &= \frac{MgR}{\lambda_3\omega_3}\hat{\mathbf{z}} \times \hat{\mathbf{e}}_3 = \Omega \times \hat{\mathbf{e}}_3\end{aligned}$$

Euler's Equations

- Body Frame $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ $\quad S$
- Space Frame $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ $\quad S_0$

$$\begin{aligned}\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{space}} &= \left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{\text{body}} + \omega \times \mathbf{L} \\ &= \dot{\mathbf{L}} + \omega \times \mathbf{L} = \mathbf{\Gamma}\end{aligned}$$

where

$$\begin{aligned}\mathbf{L} &= \lambda_1\omega_1\hat{\mathbf{e}}_1 + \lambda_2\omega_2\hat{\mathbf{e}}_2 + \lambda_3\omega_3\hat{\mathbf{e}}_3 \\ \omega &= \omega_1\hat{\mathbf{e}}_1 + \omega_2\hat{\mathbf{e}}_2 + \omega_3\hat{\mathbf{e}}_3\end{aligned}$$

Keep in mind that the dot product is not always zero(only if $\mathbf{L} = \lambda\omega$ i.e. a sphere).

$$\begin{aligned}\omega \times \mathbf{L} &= \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1\omega_1 & \lambda_2\omega_2 & \lambda_3\omega_3 \end{pmatrix} \\ &= \hat{\mathbf{e}}_1[\omega_2\omega_3(\lambda_3 - \lambda_2)] \\ &+ \hat{\mathbf{e}}_2[\omega_3\omega_1(\lambda_1 - \lambda_3)] \\ &+ \hat{\mathbf{e}}_3[\omega_1\omega_2(\lambda_2 - \lambda_1)]\end{aligned}$$

so the three components of the torque(or Euler's equations) are

$$\Gamma_1 = \lambda\dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_2\omega_3$$
$$\Gamma_2 = \lambda\dot{\omega}_2 + (\lambda_1 - \lambda_3)\omega_3\omega_1$$
$$\Gamma_3 = \lambda\dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1\omega_2$$

Zero Torque Case Setting the RHS to zero and moving the lambda terms

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3$$
$$\lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1$$
$$\lambda_3\dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2$$

Example: Let $\omega = \omega_3\hat{\mathbf{e}}_3, \quad \mathbf{L} = \lambda_3\omega = \lambda_3\omega_3\hat{\mathbf{e}}_3$ thus

$$\omega_1 = \omega_2 = 0$$

and the RHS for all three equations are zero. Thus the body keeps rotating in the same direction. If initially $\omega = \sum_i^3 \omega_i\hat{\mathbf{e}}_i$ we have a lot of motion in any direction.

Small Deviation $\mathbf{L} = \lambda_3\omega_3\hat{\mathbf{e}}_3$ add small ω_1, ω_2: The third equation would be approximately zero, i.e., $\lambda_3\dot{\omega}_3 = 0$ is constant. We are then left with two equations

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3$$
$$\lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1$$

where we have cross terms in the coupled equations. Taking the time derivative of the first equation

$$\ddot{\omega}_1 = -\left[\frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)\omega_3^2}{\lambda_2\lambda_1}\right]\omega_1$$
$$\ddot{x} = -\frac{k}{m}x = -\omega_0^2 x$$

where this resembles a harmonic oscillator so

$$\omega_0^2 = \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_2\lambda_1}\omega_3^2$$

but the caveat is that the term must be positive or

$$\lambda_3 > \lambda_2, \lambda_1$$

or both negative

$$\lambda_3 < \lambda_2, \lambda_1$$

for $\omega_0^2 > 0$ if

$$\lambda_1 < \lambda_3 < \lambda_2 \qquad \omega_0^2 < 0$$

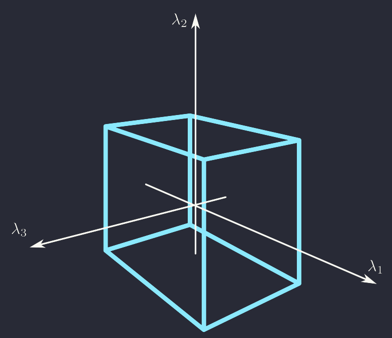

Figure 11.2: Book and its three principal moments of inertia.

Looking at a book, we can see that from the 3 principal moments of inertia, rotating around the largest moment (pointing out of the page) is stable, and rotating around the two smaller moments are unstable.

For a book.

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \quad \text{initially} \quad \omega = \omega_3 \hat{\mathbf{e}}_3$$

Adding a small ω_1, ω_2 where

$$\ddot{\omega}_1 = -\left[\frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2} \omega_3^2\right] \omega_1$$

We can see in the largest moment of inertia, a rotation is stable, but rotation in λ_1 in the figure leads to an unstable rotation, i.e., the book when tossed will rotate around the other axes.

Symmetric top $\lambda_1 = \lambda_2 \neq \lambda_3$ Then from the Euler's equations

$$\lambda_3 \dot{\omega}_3 = 0$$

and using $\lambda_1 = \lambda_2$ the other two equations are

$$\begin{aligned} \lambda_1 \dot{\omega}_1 &= (\lambda_1 - \lambda_3) \omega_3 \omega_2 \\ \lambda_1 \dot{\omega}_2 &= -(\lambda_1 - \lambda_3) \omega_3 \omega_1 \end{aligned}$$

and using

$$\Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3$$

the equations are

$$\begin{aligned} \dot{\omega}_1 &= \Omega_b \omega_2 \\ \dot{\omega}_2 &= -\Omega_b \omega_1 \end{aligned}$$

Which can be solved using the solution

$$\begin{aligned} \eta &= \omega_1 + i\omega_2 \\ \dot{\omega}_1 + i\dot{\omega}_2 &= \Omega_b \omega_2 - i\Omega_b \omega_1 \\ &= \Omega_b(\omega_2 - i\omega_1) \\ &= i\Omega_b(i\omega_2 - \omega_1) \\ \dot{\eta} &= -\Omega_b \eta \implies \eta = \eta_0 e^{-i\Omega_b t} \end{aligned}$$

so

$$\begin{aligned} \omega_1 &= \omega_0 \cos(\Omega_b t) \\ \omega_2 &= \omega_0 \sin(\Omega_b t) \end{aligned}$$

where Ω_b is the free precession frequency (zero torque still results in precession).

Euler Angles

Goal: To find the Lagrangian for a rotating body. We can describe the orientation of the body axes (principal moments) within a space frame: The three angles ϕ, θ, ψ

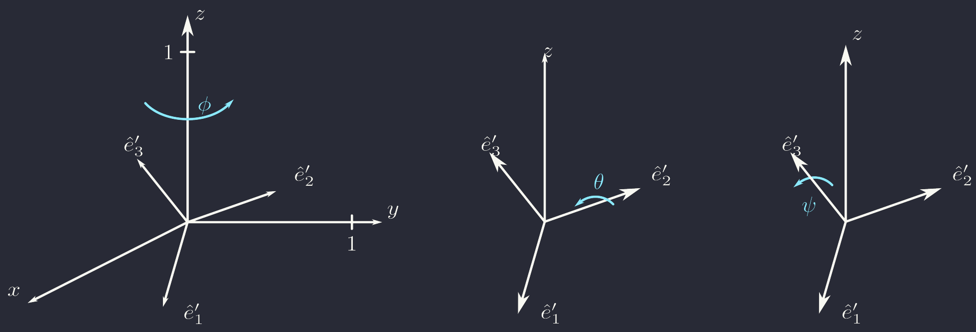

Figure 11.3: Euler angles ϕ, θ, ψ. We can relate $\cos\theta = \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{z}}$

First you rotate around $\hat{\mathbf{z}}$ by ψ, then around $\hat{\mathbf{e}}_2'$ by θ, and finally around $\hat{\mathbf{e}}_3'$ by ϕ.
The three operations can be defined as the angular velocity vector sum

$$\begin{aligned}
\omega &= \omega_a + \omega_b + \omega_c \\
\omega_a &= \dot{\phi}\hat{\mathbf{z}} \\
\omega_b &= \dot{\theta}\hat{\mathbf{e}}_2' \\
\omega_c &= \dot{\psi}\hat{\mathbf{e}}_3' \\
\omega &= \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{e}}_2' + \dot{\psi}\hat{\mathbf{e}}_3'
\end{aligned}$$

For the symmetric top we can discount the third step, and if $\lambda_1 = \lambda_2$ then

$$\begin{aligned}
\hat{\mathbf{e}}_1 &= \hat{\mathbf{e}}_1' \\
\hat{\mathbf{e}}_2 &= \hat{\mathbf{e}}_2'
\end{aligned}$$

so

$$\hat{\mathbf{z}} = \hat{\mathbf{e}}_3 \cos\theta - \hat{\mathbf{e}}_1' \sin\theta$$

which gives the angular velocity vector

$$\begin{aligned}
\omega &= -\dot{\phi}\sin\theta\hat{\mathbf{e}}_1' + \dot{\theta}\hat{\mathbf{e}}_1' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{e}}_3' \\
&= \omega_1\hat{\mathbf{e}}_1' + \omega_2\hat{\mathbf{e}}_2' + \omega_3\hat{\mathbf{e}}_3'
\end{aligned}$$

The angular momentum vector is then

$$\begin{aligned}
\mathbf{L} &= I\mathbf{o}mega \\
&= \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3
\end{aligned}$$

and the Kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}\omega \cdot \mathbf{L} \\
&= \frac{1}{2}[\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \lambda_3(\dot{\psi} + \phi\cos\theta)^2]
\end{aligned}$$

To find the potential energy we use the CM R and gravity $\mathbf{g} = -g\hat{\mathbf{z}}$:

$$U = Mgh = MgR\cos\theta$$

where we can find the Lagrangian $\mathcal{L} = T - U$

Euler Angles cont'd For a rotating rigid body where $\lambda_1 = \lambda_2$:

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}\lambda_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \phi\cos\theta)^2 - MgR\cos\theta\end{aligned}$$

And the two conserved quantities are ψ and ϕ: Since $\mathcal{L}$ is independent of ψ, ϕ

$$\begin{aligned}p_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \text{constant} = L_z \\ p_\psi &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{constant} = L_3\end{aligned}$$

So the momentum is conserved in the z direction and the 3 direction. Two get the third equation we can use the Euler-Lagrange equations for θ:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) \\ \lambda_1\ddot{\theta} &= \lambda_1\dot{\phi}^2\sin\theta\cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + MgR\sin\theta\end{aligned}$$

or

$$\frac{L_z - L_3\cos\theta}{\lambda_1\sin^2\theta} = \dot{\theta}$$

Assuming that $\theta = \text{constant}$, $\quad\dot{\phi} = \Omega$. We can also see that $(\dot{\psi} + \dot{\phi}\cos\theta) = \omega_3$, so

$$\begin{aligned}0 &= \lambda_1\Omega^2\sin\theta\cos\theta - \lambda_3\omega_3\Omega\sin\theta + MgR\sin\theta \\ &= \lambda_1\Omega^2\cos\theta - \lambda_3\omega_3\Omega + MgR\end{aligned}$$

and since everything except Ω is constant we can solve using the quadratic formula:

$$\Omega = \frac{\lambda_3\omega_3 \pm \sqrt{(\lambda_3\omega_3)^2 - 4\lambda_1 MgR\cos\theta}}{2\lambda_1\cos\theta}$$

and for $\omega_3 \gg 1$ we can find the free precession frequency Ω_1:

$$\begin{aligned}\Omega_1 &= \frac{\lambda_3\omega_3}{\lambda_1\cos\theta} \\ \Omega_2 &= \frac{MgR}{\lambda_3\omega_3}\end{aligned}$$

where Ω_2 is the precession due to gravity. Looking at the Lagrangian but rewriting as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\lambda_1\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta \\ &= \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{\text{eff}}\end{aligned}$$

where U_{eff} is the effective potential energy. Theta ranges from $0 \to \pi$ (From the $\sin\theta$ term), and as $\theta \to 0, \pi$ the potential energy goes to infinity!

12 Coupled Oscillators

Three springs in series and two carts We define equilibrium at $x_1 = 0, x_2 = 0$ and given the Lagrangian

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$
$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2$$

For the x_1 equation the EL equation gives

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k_1x_1 - k_2(x_1 - x_2)$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1\ddot{x}_1m_1\ddot{x}_1 \qquad = -(k_1 + k_2)x_1 + k_2x_2$$

and for x_2 we have

$$\frac{\partial \mathcal{L}}{\partial x_2} = k_2(x_1 - x_2) - k_3x_2$$
$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2\ddot{x}_2$$
$$m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2$$

We can rewrite these equations in matrix form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$M\ddot{\mathbf{x}} = -K\mathbf{x}$$

where we must find $\mathbf{x}(t)$. From oscillators we know that the solution is in the form

$$m\ddot{x} = -kx \implies x = x_0e^{\pm i\omega t}$$

so we can write the solution as

$$\mathbf{x}(t) = \mathbf{a}e^{i\omega t}$$

where we must find $\mathbf{A}$ and ω separately. Since

$$\ddot{\mathbf{x}} = -\omega^2\mathbf{A}e^{i\omega t} = -\omega^2\mathbf{x}$$

then we know that

$$-\omega^2M\mathbf{x} = -K\mathbf{x}$$
$$\implies (K - \omega^2M)\mathbf{x} = 0$$

so ω^2 is an eigenvalue of KM^{-1}

$$\implies \det\big(K - \omega^2M\big) = 0$$

With the assumption

$$m_1 = m_2 = m, \quad k_1 = k_2 = k_3 = k$$

we have

$$
\begin{aligned}
\det \begin{pmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{pmatrix} &= 0 \\
(2k - \omega^2 m)^2 - k^2 &= 0
\end{aligned}
$$

which is in the form $a^2 - b^2 = (a+b)(a-b) = 0$. So we have

$$
(2k - \omega^2 m + k)(2k - \omega^2 m - k) = 0
$$

which gives us the two solutions

$$
\omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{3k}{m}
$$

Question: Why not 4 solutions, i.e., $\pm\omega_1, \pm\omega_2$?
Plug in ω_1 into $K\mathbf{a} = \omega_1^2 M\mathbf{a}$ to find $\mathbf{a}$:

$$
\begin{pmatrix} 2k - \omega_1^2 m & -k \\ -k & 2k - \omega_1^2 m \end{pmatrix} = \begin{pmatrix} k & -k \\ k & -k \end{pmatrix} \mathbf{a} = 0
$$

with $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the solution is

$$
\begin{aligned}
\mathbf{x}_1 &= \mathbf{a}(C_1 e^{i\omega t} + C_2 e^{i\omega t}) \\
&= A\mathbf{a}\cos(\omega_1 t - \delta)
\end{aligned}
$$

Which gives us the first normal mode

$$
\begin{cases} x_1 = A\cos(\omega_1 t - \delta) \\ x_2 = A\cos(\omega_1 t - \delta) \end{cases}
$$

This describes when the two carts are moving in phase. For the second normal mode:

$$
\begin{aligned}
(K - \omega_2 M)\mathbf{a} &= 0 \\
\begin{pmatrix} 2k - \omega_2^2 m & -k \\ -k & 2k - \omega_2^2 m \end{pmatrix} \mathbf{a} &= 0 \\
\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \mathbf{a} &= 0
\end{aligned}
$$

where $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so the solution is

$$
\begin{aligned}
\mathbf{x} &= A\mathbf{a}\cos(\omega_2 t - \delta) \\
\implies &\begin{cases} x_1 = A\cos(\omega_2 t - \delta) \\ x_2 = -A\cos(\omega_2 t - \delta) \end{cases}
\end{aligned}
$$

This describes when the two carts are moving in opposite directions. The generalized solution is a linear combination of the normal modes

$$
\mathbf{x}(t) = A_1\mathbf{a}_1\cos(\omega_1 t - \delta_1) + A_2\mathbf{a}_2\cos(\omega_2 t - \delta_2)
$$

which can describe the complicated motion of the two carts when they are not completely in or out of phase.

Normal Coordinates

$$\xi_1 = \frac{1}{2}(x_1 + x_2)$$
$$\xi_2 = \frac{1}{2}(x_1 - x_2)$$

$\implies \xi_1, \xi_2$ into the EOM:

$$\ddot{\xi}_1 = f(\xi_1)$$
$$\ddot{\xi}_2 = f(\xi_2)$$

decouples the equations.

Last Time:

$$M\ddot{\mathbf{x}} = -K\mathbf{x}$$

For an undiagonalized matrix K we have to solve

$$\ddot{\mathbf{x}} = M^{-1}K\mathbf{x}$$

where

$$\mathbf{x} = \mathbf{a}e^{\pm i\omega t} \implies \omega^2\mathbf{a} = M^{-1}K\mathbf{a}$$

where the general solution is a linear combination of the normal modes

$$\mathbf{x}(t) = A_1\mathbf{a}_1\cos(\omega_1 t - \delta_1) + A_2\mathbf{a}_2\cos(\omega_2 t - \delta_2)$$

Double Pendulum

The Potential energy is made up of two parts

$$\begin{aligned} U_1 &= m_1 g L_1(1-\cos\phi_1) \\ U_2 &= m_2 g L_1(1-\cos\phi_1) + m_2 g L_2(1-\cos\phi_2) \end{aligned}$$

And the two kinetic energies are

$$\begin{aligned} T_1 &= \frac{1}{2}m_1 L_1^2\dot{\phi}_1^2 \\ T_2 &= \frac{1}{2}m_2(L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)) \end{aligned}$$

where we use the Law of Cosines (or dot product), so the Lagrangian is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}(m_1+m_2)L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2) \\ &- (m_1+m_2)gL_1(1-\cos\phi_1) - m_2gL_2(1-\cos\phi_2) \end{aligned}$$

Using a small angle approximation where both ϕ_1, ϕ_2 is small:

$$\cos\phi \approx 1 - \frac{\phi^2}{2}$$

we can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L} = \frac{1}{2}(m_1+m_2)L_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2L_2^2\dot{\phi}_2^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2 \\ - (m_1+m_2)gL_1\phi_1^2 - m_2gL_2\phi_2^2 \end{aligned}$$

where we use the second order terms in the potential energy, i.e.

$$T(\dot{\phi}_1, \dot{\phi}_2) \quad U(\phi_1, \phi_2)$$

So for the EL equations:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} &= (m_1+m_2)L_1^2\dot{\phi}_1 + m_2L_1L_2\dot{\phi}_2 \\ \frac{\partial\mathcal{L}}{\partial\phi_1} &= -(m_1+m_2)gL_1\phi_1 \\ \implies (m_1+m_2)L_1^2\ddot{\phi}_1 &+ m_2L_1L_2\ddot{\phi}_2 = -(m_1+m_2)gL_1\phi_1 \end{aligned}$$

and for ϕ_2:

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial\dot{\phi}_2} = m_2L_2^2\dot{\phi}_2 + m_2L_1L_2\dot{\phi}_1 \\
&\frac{\partial\mathcal{L}}{\partial\phi_2} = -m_2gL_2\phi_2 \\
&\implies m_2L_1L_2\ddot{\phi}_1 + m_2L_2^2\ddot{\phi}_2 = -m_2gL_2\phi_2
\end{aligned}$$

This is a matrix in the form

$$M = \begin{pmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix} \quad K = \begin{pmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{pmatrix}$$

here the K matrix is diagonal (opposite from the previous example). But we solve this the same way

$$\det\big(K - \omega^2 M\big) = 0$$

where ω^2 is the eigenvalue of $M^{-1}K$.

Equal Mass and Length Case Assume $m_1 = m_2 = m, L_1 = L_2 = L$, then

$$M = mL^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad K = mgL\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

From the simple pendulum we know that

$$\omega_0 = \sqrt{\frac{g}{L}}, \quad g = \omega_0^2 L$$

so we can rewrite

$$K = m\omega_0^2L^2\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant of the matrix is

$$\det\big(K - \omega^2 M\big) = m^2L^4\begin{pmatrix} 2\omega_0^2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{pmatrix} = 0$$

which gives us the equation

$$\begin{aligned}
2(\omega_0^2 - \omega^2)^2 - \omega^4 = 0 \\
\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0
\end{aligned}$$

so the two normal frequencies are

$$\omega_1^2 = (2-\sqrt{2})\omega_0^2, \quad \omega_2^2 = (2+\sqrt{2})\omega_0^2$$

Normal Modes To find the normal modes, we substitute ω_1, ω_2 into the equation again:

$$\begin{aligned}
K - \omega_1^2 M &= mL^2\omega_0^2\begin{pmatrix} 2-(4-2\sqrt{2}) & -(2-\sqrt{2}) \\ -(2-\sqrt{2}) & 1-(2-\sqrt{2}) \end{pmatrix} \\
&= mL^2\omega_0^2\begin{pmatrix} 2\sqrt{2}-2 & 2-\sqrt{2} \\ 2-\sqrt{2} & \sqrt{2}-1 \end{pmatrix} \\
&= mL^2\omega_0^2(\sqrt{2}-1)\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}
\end{aligned}$$

So

$$(k - \omega_1^2 M)\mathbf{a}_1 = 0, \quad \mathbf{a}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

and the first normal mode is

$$
\begin{aligned}
\phi(t) &= A_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_1 t - \delta_1) \\
\phi_1(t) &= A_1 \cos(\omega_1 t - \delta_1) \\
\phi_2(t) &= \sqrt{2} A_1 \cos(\omega_1 t - \delta_1)
\end{aligned}
$$

where the two pendulums are moving exactly in phase (or the 2nd pendulum angle is always $\sqrt{2}$ times the first pendulum). The second normal mode is

$$
\phi(t) = A_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_2 t - \delta_2)
$$

Lecture 32: **4/15/24**

Review For the General Case of Coupled Oscillators:

$$M\ddot{q} = -K\mathbf{q}$$

where $\mathbf{q} = \mathbf{a}e^{i\omega t}$, and ω^2 is an eigenvalue of $M^{-1}K$.

$$(K - \omega^2 M)\mathbf{a} = 0 \implies \det\big(K - \omega^2 M\big) = 0$$

which gives the normal frequency, and to determine $\mathbf{a}$:

$$\mathbf{q} = \sum_i A_i \mathbf{a}_i \cos(\omega_i t - \delta_i)$$

where we have $2n$ unknowns and $2n$ initial conditions.

Nodes Of a String

For a string of mass M under tension T, we separate the string into small nodes of length ℓ, and the nodes deviate y_i to form a segmented wave. Assuming y_i is small: N2L gives

$$m\ddot{y}_2 = F_y = -T\sin\theta_1 - T\sin\theta_2$$
$$\sin\theta_1 = \frac{y_i - y_{i-1}}{\ell}, \quad \sin\theta_2 = \frac{y_i - y_{i+1}}{\ell}$$

and

$$\begin{aligned} m\ddot{y}_1 &= -T\frac{y_i - y_{i-1}}{l} - T\frac{y_i - y_{i+1}}{\ell} \\ &= \frac{T}{\ell}(y_{i-1} - 2y_i + y_{i+1}) \\ \implies M\ddot{\mathbf{y}} &= -K\mathbf{y} \end{aligned}$$

e.g. For $n = 2$ we have the two equations

$$\begin{aligned} i = 1: &\quad \ddot{y}_1 = \frac{T}{m\ell}(y_2 - 2y_1) \\ i = 2: &\quad \ddot{y}_2 = \frac{T}{m\ell}(y_1 - 2y_2) \end{aligned}$$

which can be written in matrix form

$$M = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad K = \frac{T}{m\ell}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and for n nodes we have a tri-diagonal matrix for K:

$$K = \frac{T}{m\ell}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

Solving for the normal modes where $n = 2$:

$$\det\big(K - \omega^2 M\big) = 0$$
$$\omega_1^2 = \omega_0^2, \quad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\omega_2^2 = 3\omega_0^2, \quad \mathbf{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$n \to \infty$**?** We take the limit of the continuous string...

$$m = \frac{M}{n} \to 0, \quad \ell = \frac{L}{n+1} \to 0$$

since these quanities go to zero, we have to define a nonzero quantity

$$\mu = \frac{M}{L} \approx \frac{m}{\ell}$$

The equation of motion is

$$\ddot{y}_i = \frac{T}{m\ell}(y_{i-1} - 2y_i + y_{i+1})$$

and since $y \to y(x), \qquad x \in [0, L]$, we can Taylor expand

$$y_{i+1} = y_i + y_i'\ell + \frac{1}{2}y_i''\ell^2 y_{i-1} = y_i - y_i'(-\ell) + \frac{1}{2}y_i''\ell^2$$

where the first two terms cancel out, so we have

$$\ddot{y} = \frac{T}{m\ell}y''\ell^2 = \frac{T}{\mu}y'' = c^2 y''$$

A solution to y is exponential in the form

$$\begin{aligned} y(x) &= a(x)e^{i\omega t} \\ -\omega^2 a(x) &= c^2 a''(x) \\ a'' &= -\frac{\omega^2}{c^2}a = -k^2 a \end{aligned}$$

where k is the wave vector. The general solution is

$$\begin{aligned} &a(x) = C_1 \sin kx + C_2 \cos kx \\ &a(0) = 0 \implies C_2 = 0 \\ &a(L) = 0 \implies \sin(kL) = 0 = \sin(n\pi) \implies k = \frac{n\pi}{L} \end{aligned}$$

so

$$k_n = \frac{n\pi}{L}, \quad \omega_n = \frac{n\pi c}{L}$$

Which gives us

$$\begin{aligned} a_n(x) &= A_n \sin\left(\frac{n\pi}{L}x\right) \\ y(x,t) &= \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)e^{i\omega_n t} \end{aligned}$$

The initial conditions tell us

$$\begin{aligned} y(x,0) = f(x) &= \sum_n A_n \sin\left(\frac{m\pi}{L}x\right) \\ A_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x \end{aligned}$$

this is from the Fourier coefficient:

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = \delta_{nm}$$

13 Hamiltonian Mechanics

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$

Is a Legendre transform where we change variables for $(q, \dot{q}) \to (q, p)$. This results in the Hamitlon's equations:

$$2n : \quad \frac{\partial \mathcal{H}}{\partial p_i} = -\frac{\partial \mathcal{H}}{\partial q_i}$$

compared to the E-L equations:

$$n : \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

Any function $f(q,p)$ which are dependent on time, i.e., $q(t), p(t)$, we can differentiate with respect to time:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \cancel{\frac{\partial f}{\partial t}} + \frac{\partial f}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} \\ &= \frac{\partial f}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial \mathcal{H}}{\partial q} \\ &= \{f, \mathcal{H}\}\end{aligned}$$

This is the Poisson Bracket

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = [f, g]_p$$

this is similar to the commutator in QM:

$$i\hbar \frac{\mathrm{d}O}{\mathrm{d}t} = [O, H]$$

Examples:

$$[q,p]_p = 1 \quad [x,p] = i\hbar$$
$$[L_x, L_y]_p = L_z \quad [L_x L_y] = i\hbar L_z$$

Why use Lagrangian when we have the Hamiltonian?

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

We must first define the canonical momentum before we can use the Hamiltonian which also requires a transformation involving the Lagrangian.

$\mathcal{H} = T + U$**. . .** This formulation for the Hamiltonian is only useful in natural coordinates where $\{q_i\} \leftrightarrow \{p_i\}$ does not depend on time. Otherwise, $\mathcal{H} \neq T + U$. But this also doesn't mean that energy is not conserved (it can still be conserved).

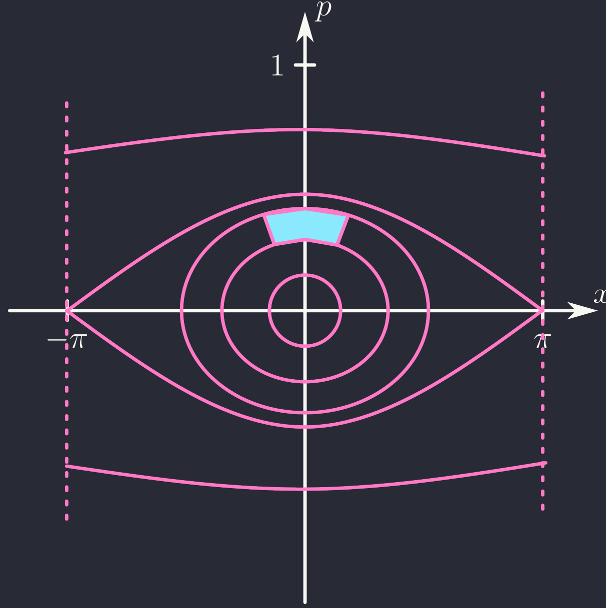

Figure 13.1: Phase Space for a Pendulum

Phase Space & Liouville's Theorem The phase space is a 2n-dimensional space

$$\bar{z} = (q_1, \dots, q_n, p_1, \dots, p_n)$$

For a Pendulum we can see the phase space for different initial conditions.
The volume of a region in the phase space which we can represent using the phase velocity

$$\mathbf{v}_z = \dot{\mathbf{z}} = (\dot{q}, \dot{p})$$

The the volumet element is the flux of the phase velocity through the surface:

$$\delta V = \oint_S \mathbf{v}_z \cdot \mathrm{d}\mathbf{A}$$

and from the divergence theorem:

$$\oint_S \mathbf{v}_z \cdot \mathrm{d}\mathbf{A} = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{v}_z \, \mathrm{d}V$$

And if the volume doesn't change with time then the divergence of the phase velocity is zero, i.e.,

$$\boldsymbol{\nabla} \cdot \mathbf{v}_z = 0$$

Or

$$\boldsymbol{\nabla} \cdot v_z = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

where

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

so we find that

$$\boldsymbol{\nabla} \cdot v_z = \frac{\partial \mathcal{H}}{\partial q_i \partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i \partial q_i} = 0$$

This tells us that the volume enclosed by a surface is conserved as it moves around in phase space.

Example: Particles in a Volume We can consider a volume in phase space with density or distribution function f where

$$N = \int f(x,p,t)\,\mathrm{d}V$$
$$\implies \frac{\mathrm{d}f}{\mathrm{d}t} = 0 \quad \text{Vlasov Equation}$$

The total derivative is zero, and in phase space the density of the volume would change. In otherwords,

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial f}{\partial x} + \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\partial f}{\partial p} = 0$$
$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + F\frac{\partial f}{\partial p} = 0$$

where v is the velocity and F is the force. In branch of mathmatics, this describes Symplectic Geometry or Symplectic Manifolds.

Nonlinear Dynamics

Last time: Hamiltonian Mechanics; Poisson Brackets; Liouville's Theorem:

$$[f, g] = 0$$

Then f, g are independent. In otherwords, $[f, \mathcal{H}] = 0$ means that f is conserved.

Phaase space Portrait:

$$\begin{cases} \dot{x} = \frac{p}{m} \\ \dot{p} = -\frac{\partial u}{\partial x} = -u' \end{cases}$$

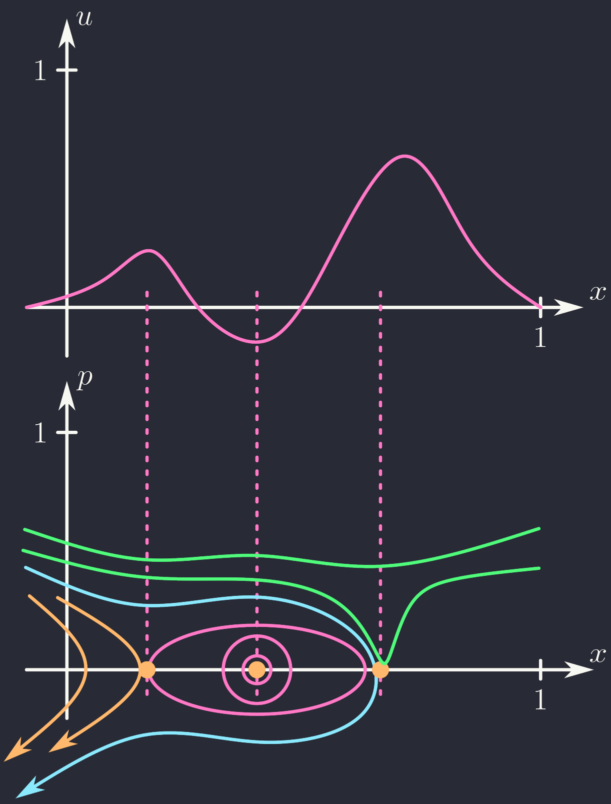

Figure 13.2: Phase Space Portrait

We can see at the center point, or equilibrium point, of the phase space portrait, we have a closed path around it (pink orbit). The orbits from further away that do not go past the first critical point circle back around as shown by the orange orbit, and if we start with higher potential energy than the first critical point(and not the third), it maintains a blue orbit. And potentials larger than the third critical point will have a green orbit.
The critical points are given by

$$\dot{x} = F(x, y)$$
$$\dot{y} = G(x, y)$$

where (x_0, y_0) is a critical point if $F(x_0, y_0) = G(x_0, y_0) = 0$.

1D Motion Under $u(x)$

$$F = \frac{y}{m}$$
$$G = -u'(x)$$

and from the Jacobian

$$J = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -u''(x) & 0 \end{pmatrix}$$

Which has two eigenvalues λ_1, λ_2:

$$\det J = \lambda_1 \lambda_2 = \frac{u''(x)}{m}$$
$$\operatorname{tr} J = \lambda_1 + \lambda_2 = 0$$

Since $u''(x) < 0$, both λ_1, λ_2 are real and opposite signs $\implies$ saddle point.
If $u''(x) > 0$, then $\lambda_1, \lambda_2 = \pm iv$ are purely imaginary $\implies$ center point.

Brusselator

$$\dot{x} = a - (1+b)x + x^2 y$$
$$\dot{y} = bx - x^2 y$$

Setting the two equations to zero, we find the critical points:

$$a - (1+b)x + x^2 y = 0 \implies a - x = 0 \to x = a$$
$$bx - x^2 y = 0 \implies y = \frac{b}{a}$$

So we have only one critical point at $(a, b/a)$. The Jacobian is

$$J_{(a,b/a)} = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}$$
$$\operatorname{tr} J = b - 1 - a^2 = b - (1 + a^2) = \lambda_1 + \lambda_2$$
$$\det J = a^2 = \lambda_1 \lambda_2$$

If a, b are positive real numbers,

$$b < 1 + a^2$$

So the eigenvalues are both negative $\lambda_1, \lambda_2 < 0$, and a same sign implies that every orbit spirals inward to the critical point (attracter).
If $\lambda_1, \lambda_2 > 0$ then the critical point is a repeller. Interestingly, $b > 1 + a^2$ creates a limit cycle, where the critical poin is outside are repellers. The two eigenvalues tell us the behavior along a particular direction, e.g., how the system behaves near a critical point.

Rayleigh's equation

$$\ddot{x} - \epsilon \dot{x}(1 - \dot{x}^2) + x = 0$$
$$\to \dot{x} = y$$
$$\dot{y} = -x + \epsilon y(1 - y^2)$$

where an obvious critical point is $(0, 0)$. The path of the system for smaller ϵ will be approaching a limit cycle around the critical point. Furthermore, for different ϵ values, the shape of the limit cycle will change unintuitively.

Lorenz System

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

For the $\sigma = 10, \rho = 28, \beta = 8/3$ values, the system has a chaotic behavior, aka the lorentz attractor.

Order And Chaos

What is chaos? Two points very close together in phase space $\delta \sim e^{\lambda t}$ will diverge exponentially so. The Lyapunov exponent λ describes this rate of diveregence.

$$\text{Chaos} \Leftrightarrow \text{Sensitivity to initial condition} \Leftrightarrow \text{Loss of predictability} \Leftrightarrow \text{Butterfly Effect}$$

The Butterfly effect is not a causal relationship (like the traditional metaphor).

WHen is a system chaotic?

- Nonlinear
- # of degrees of freedom $\geq 2 \geq$ 2D
- # of conserved quanties $<$ # of degrees of freedom.

If not 3, then the system is "integrable" (in the sense of Liouville). The conserved quantities, $[f, H]_p = 0$ and $[f_1, f_2] = 0$.

Examples:

- A 1D system is always integrable, or not chaotic.
- A 2 body central force problem: We have 6 degrees of freedom, and 6 conserved quantities
  - Center of Mass $R = (X, Y, Z)$
  - Momentum $\mathbf{P}_R$
  - Hamiltonian H
  - L_z
  - L^2

  If we add a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ that interacts with the two bodies, the symmetry conserves L_z but not L^2, so the system is chaotic.
- Rigid Body Motion: 3 dof (euler angles) and 3 conserved quantities
  - H, L_z, L^2

  means the system is not chaotic.
- The 3 body problem is chaotic.

Hierarchical Problem The Solar System is a predictable system because of the hierarchy of the masses relative to the sun:

$$m_{\text{moon}} \ll m_{\text{earth}} \ll m_{\text{sun}}$$

For example, the Hamiltonian of the Sun-Earth-Moon system would be approximately

$$H = H_{E+S} + \delta H_M$$

which makes the system "nearly integrable".

Driven, Damped Pendulum From N2L

$$mL\ddot{\phi} = -bL\dot{\phi} - mg\sin\phi + F(t)$$
$$\rightarrow \ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{g}{L}\sin\phi = \frac{F(t)}{mL}$$

which is similar to the driven damped oscillator, but now we have a non-linear sine term. Rewriting the equation with the familar damping coefficient ant natural frequency:

$$2\beta = \frac{b}{m} \quad \omega_0^2 = \frac{g}{L}$$

we get

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \frac{F}{mL} = \frac{F_0}{mL}\cos(\omega t)$$

For small $\phi \ll 1$ we can approximate $\sin\phi \approx \phi$ and we have a solution which is a linear combination of the homogenous(transient) and particular solutions:

$$\phi(t) = \phi_h(t) + \phi_p(t)$$
$$= e^{-\beta t}A_1\cos(\omega_1 t) + A\cos(\omega t - \delta)$$

and for $t \rightarrow \infty$

$$\phi(t) = A\cos(\omega t - \delta)$$

which is dependent on the driving force and the resonant frequency of the system. Using the next order term

$$\sin\phi \approx \phi - \frac{\phi^3}{6}$$

we get an equation of the form

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0\left(\phi - \frac{\phi^3}{6}\right) = \frac{F_0}{mL}\cos(\omega t)$$
$$\rightarrow \ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi = \frac{F_0}{mL}\cos(\omega t) + \frac{\omega_0^2}{6}A^3\cos^3(\omega t - \delta)$$

and using the trigonometric identity

$$\cos^3 x = \frac{1}{4}(\cos(3x) + 3\cos x)$$

so we have a form

$$\phi(t) = A\cos(\omega t - \delta) + B\cos(3\omega t - \delta)$$

Final Lecture: DDP and Chaos The DDP is a nonlinear system but in one dimension... but in fact we have a dynamic variable from the driving force:

$$\dot{\theta} = \omega \implies \theta = \omega t$$
$$\to \ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \frac{F_0}{mL}\cos(\theta)$$

So there are two degrees of freedom θ, ϕ. In addition, there are no conserved quantities in the system. If the driving force depends on the force of gravity, i.e.,

$$\gamma = \frac{F_0}{mg} \to \frac{F_0}{mL}\cos\theta = \gamma\omega_0^2\cos\theta$$

At small γ we have a predictable motion where the period is the same as the driving force frequency $P = 2\pi/\omega$. For larger $\gamma = 1.073$ we have a doubling of the period; the position at each expected period drops slightly before going back to the original position. At $\gamma = 1.081$ we have a period of 4.

Period Doubling Cascade At very large γ, the period will increase exponentially to infinity, and the system will fall into "chaos". These bifucating points (the spliiting of the period), will bifurcate very quickly at a rate known as the Feigenbaum number δ.