1 Magnetic Fields in Matter

1.1 Magnetization

1.1.1 Paramagnets, Diamagnet, Ferromagnets

1.1.2 Forces and Torques on Magnetic Dipoles

Using $\mathbf{F} = I\int \mathrm{dl} \times \mathbf{B}$ we can find the directions of the forces on each section of the current loop

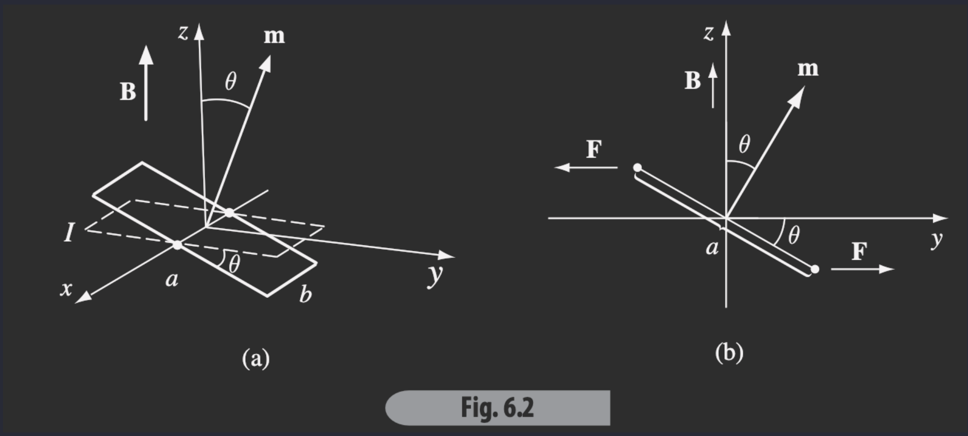

Figure 1.1: Magnetic field due to a current loop

Since the sum of the forces $\sum \mathbf{F} = 0$ so the magnitude of force on each segment is

$$F = IbB$$

and the torque on the loop is

$$\mathbf{N} = aF\sin\theta\hat{\mathbf{x}}$$

Or

$$\mathbf{N} = IabB\sin\theta\hat{\mathbf{x}} \quad \text{using} \quad m = Iab$$
$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

which encapsulates the defintion of "paramagnetism" i.e. the magnetic moments align with the magnetic field. This is similar to the electric form

$$\mathbf{N}_E = \mathbf{p} \times \mathbf{E}$$

. . . in a uniform field The net force $\to 0$

$$\mathbf{F} = I\oint \mathrm{dl} \times \mathbf{B} = 0$$

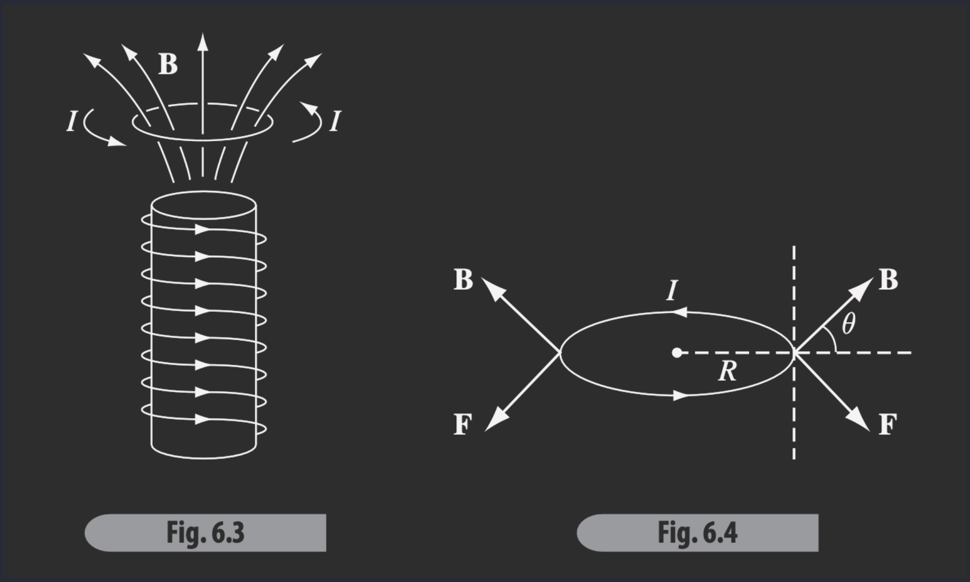

Figure 1.2: Solenoid

non-uniform field Using RHR we can see that the horizontal components cancel out from symmetry and we are left with a downward force on the dipole with magnitude

$$|\mathbf{F}| = 2\pi RIB\sin\theta$$

At the other end of the dipole, the force will be upwards; so for the pure dipole

$$\mathbf{F}_B = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})$$
$$\mathbf{F}_E = \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E})$$

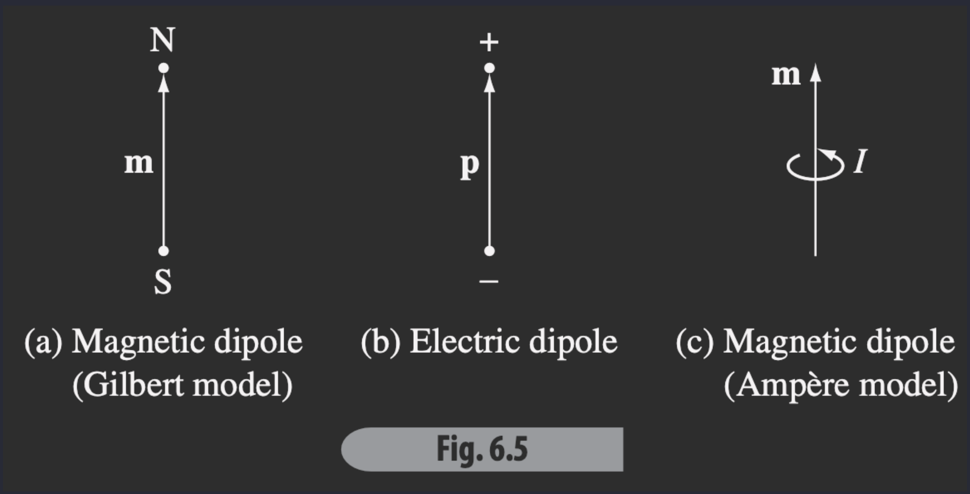

Figure 1.3: Early models of magnetism

From the electric dipole, we could easiliy infer a magnetic dipole or "Gilbert" model.

1.1.3 Effect of B on atomic orbitals

For an electron circulating a nucleus,

- Period $T = 2\pi R/v$
- $\sim$ stead current $I = -e/T = -ev/(2\pi R)$
- Thus a magnetic moment $\mathbf{m} = I\mathbf{a} = I\pi R^2(-\hat{\mathbf{z}}) = -\frac{1}{2}evR\hat{\mathbf{z}}$

The centripetal orbital force keeping the electron in orbit is entirely due to the coulomb force

$$\mathbf{F}_{\text{orb}} = \mathbf{F}_{\text{coul}}$$
$$m_e\frac{v^2}{R} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2}$$

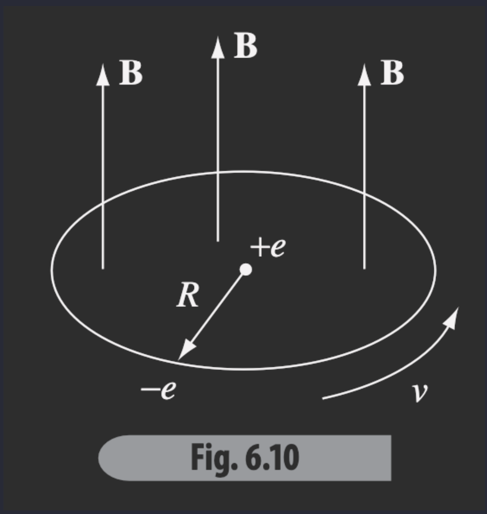

Figure 1.4: Magnetic field due to an electron

From the LHR (for electron) the force points inwards, so

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + ev'B = m_e\frac{v^2}{R}$$
$$\implies ev'B = \frac{mv'^2}{R} - \frac{mv^2}{R} = \frac{m}{R}(v'^2 - v^2)$$
$$= \frac{m}{R}(v + v')(v' - v) \quad \text{using} \quad v' - v = \delta v$$

where

$$\delta v = \frac{ev'B}{m/R}\frac{1}{v' + v} \approx \frac{eRB}{2} \quad v' \approx v$$

Using

$$\mathbf{m} = -\frac{1}{2}evR\hat{\mathbf{z}} \implies \delta m = -\frac{e}{2}R\delta v\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}$$

This opposite alignment of the magnetic moment is called “diamagnetism”. Andre Geim shared the Ig Nobel prize for levitating a frog in a magnetic field Wiki.

1.1.4 Magnetization

The magnetization is defined by the vector quantity

$$\mathbf{M} = \text{magnetic dipole moment/unit volume}$$

which is analagous to the electric polarization $\mathbf{P}$:

1.2 The field of a magnetized object

1.2.1 Bound Currents

The field produced by $\mathbf{M}$? For each tiny dipole moment

$$\mathbf{m} = \mathbf{M}\,\mathrm{d}\tau$$

we can use a vector potential

$$\mathbf{A}_m(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

So the total vector potential (using product rule 7 [insert ref]) and $\boldsymbol{\nabla}'\frac{1}{\mathscr{r}} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

$$\begin{aligned}\mathbf{A}_{\mathrm{tot}}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\mathrm{d}\tau' \\ &= \frac{\mu_0}{4\pi}\int\mathbf{M}(\mathbf{r}')\times\boldsymbol{\nabla}'\frac{1}{\mathscr{r}}\,\mathrm{d}\tau' \\ &= \frac{\mu_0}{4\pi}\left(\int\frac{1}{\mathscr{r}}[\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')]\,\mathrm{d}\tau' - \int\boldsymbol{\nabla}'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]\mathrm{d}\tau'\right)\end{aligned}$$

where the second term is

$$-\int\boldsymbol{\nabla}'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]\mathrm{d}\tau' = \oint\frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times\mathrm{d}\mathbf{a}']$$

from Stokes' theorem of sorts. The two integrals represent the volume and surface current

$$\begin{aligned}\mathbf{J}_b &\equiv \boldsymbol{\nabla}\times\mathbf{M} \qquad \mathbf{A}\sim\int\frac{\mathbf{J}_b}{\mathscr{r}}\,\mathrm{d}\tau \\ \mathbf{K}_b &\equiv \mathbf{M}\times\hat{\mathbf{n}} \qquad \mathbf{A}\sim\oint\frac{\mathbf{K}_b}{\mathscr{r}}\,\mathrm{d}a\end{aligned}$$

so

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_b}{\mathscr{r}}\,\mathrm{d}\tau' + \frac{\mu_0}{4\pi}\oint\frac{\mathbf{K}_b}{\mathscr{r}}\,\mathrm{d}a'$$

Example Find $\mathbf{B}$ of a uniformly magnetized sphere

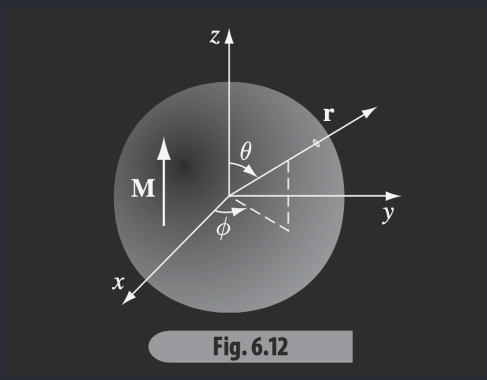

Figure 1.5: Magnetized sphere

Thre curl of a constant vector is zero so

$$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = 0$$

and for the surface current, we can see that the cross product is always in the azimuthal direction:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta \hat{\boldsymbol{\phi}}$$

Previously we found the field of a charged spinning sphere

$$\mathbf{K}_e = \sigma \mathbf{v} = \sigma\omega R \sin\theta \hat{\boldsymbol{\phi}}$$

so we can think of the magnetic counterpart as $\sigma\omega R = M$ where the solution inside the sphere is

$$\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M} \quad \text{inside}$$

Outside the sphere, we can think of this as a perfect dipole

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$$

1.2.2 Physical Interpretation of Bound Currents

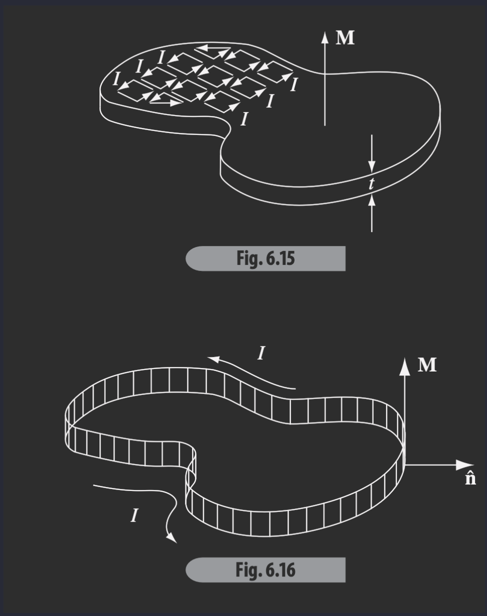

Figure 1.6: Physical interpretation of bound currents

For a magnetization $\mathbf{M} = M\hat{\mathbf{z}}$ the little loops of current (area a) cancel out in the bulk of the material (of thickness t) but at the edges, they line up and create a train of current. The magnitude

$$|\mathbf{m}| = Mat = Ia \implies I_b = Mt$$

Then the current density is

$$\mathbf{K} = \mathbf{M}(\hat{\mathbf{z}} \times \hat{\mathbf{n}}) = \frac{I_b}{t}(\hat{\mathbf{z}} \times \hat{\mathbf{n}})$$

So the current in the x direction of two adjacent loops is

$$I_x = M_z(y + \mathrm{d}y) - M_z(y) = \frac{\partial M_z}{\partial y}\,\mathrm{d}y$$

or

$$J_b\Big|_x = \frac{\partial M_z}{\partial y}$$

also adding the contribution from the y direction

$$J_b\Big|_x = \frac{\partial M_z}{\partial y}\hat{\mathbf{x}} - \frac{\partial M_y}{\partial x}\hat{\mathbf{z}} \implies \mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M}$$

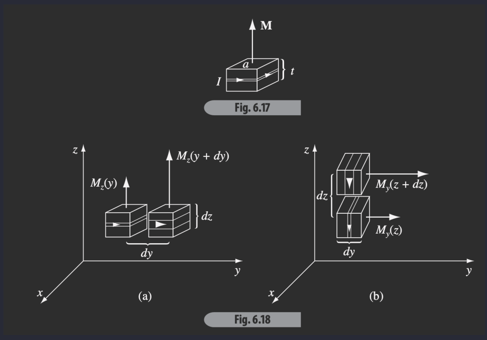

Figure 1.7: Physical interpretation of bound currents

Example A cylinder along the z axis with a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$: Find the magnetic field $\mathbf{B}$

THe volume current is

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} = 0$$

and the surface current is

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}$$

so

$$\mathbf{B} = \mu_0 nI\hat{\mathbf{z}} = \mu_0 M\hat{\mathbf{z}}$$

since the surface current density is $K \sim \frac{\mathrm{d}I}{\mathrm{d}l_\perp}$

Example Finding the magnetici field for a very long cylinder along z with a mamgetization

$$\mathbf{M} = ks^2\hat{\boldsymbol{\phi}}$$

The surface current density is

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = -kR^2\hat{\mathbf{z}}$$

and the volume current density is

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} = 3ks\hat{\mathbf{z}}$$

To find the current we can first find the total current due to the volume current density for each cross sectional area

$$\begin{aligned} I_J &= \int \mathbf{J}_b \cdot \mathrm{d}\mathbf{a} \\ &= \int 3ks(s\,\mathrm{d}s)\,\mathrm{d}\phi = 2\pi kR^3 \end{aligned}$$

then for the surface current density we integrate over the surface along the z axis

$$I_K = \int -kR^2 R\,\mathrm{d}\phi = -2\pi kR^3$$

Then using Ampere's law; an amperian circular loop far away

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0(I_J + I_K) = 0 \implies \mathbf{B} = 0$$

but inside the cyclinder it starts from zero and increases until we get to the surface where it drops to zero again.

1.3 The H Field

1.3.1 Ampere's Law in Magnetized Materials

In bound currents,

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} \quad \text{or} \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

where similar to the electric case $\rho_{\text{tot}} = \rho_{\text{free}} + \rho_{\text{bound}}$, we have a total current density

$$\mathbf{J}_{\text{tot}} = \mathbf{J}_{\text{free}} + \mathbf{J}_b$$

and from Ampere's law

$$\begin{aligned}\frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) = \mathbf{J}_{\text{tot}} &= \mathbf{J}_{\text{free}} + \mathbf{J}_b \\ &= \mathbf{J}_{\text{free}} + \boldsymbol{\nabla} \times \mathbf{M}\end{aligned}$$

and moving all the curl terms to one side

$$\boldsymbol{\nabla} \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_{\text{free}}$$

or defining the auxiliary field

$$\boxed{\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{free}}}$$

In integral form we can use the Stoke's theorem to get

$$\boxed{\oint \mathbf{H} \cdot \mathrm{dl} = I_{\text{free}}}$$

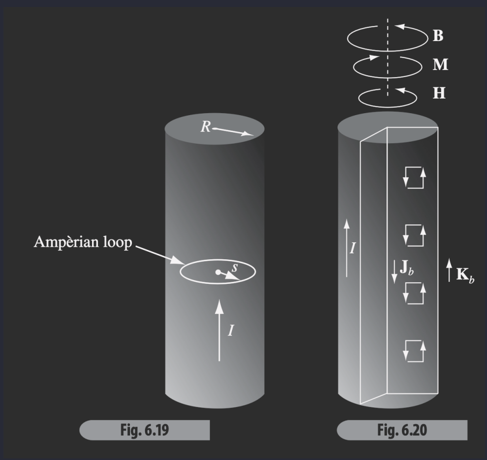

Figure 1.8: At the cross section, the currents from the dipoles point downward while there is a net current $\mathbf{K}_b$ upwards at the boundary.

Example A long copper rod of radius R, carrying a uniform free current I... find $\mathbf{H}$:
Copper is a weakly diamagnetic material so the magnetization responding to the field opposes it

- Fields $\sim \hat{\boldsymbol{\phi}}$
- Magnetizations $\sim \hat{\boldsymbol{\phi}}$

So for an Amperian loop of radius s inside the rod we have

$$\oint \mathbf{H} \cdot d\mathbf{l} = H(2\pi s) = I_{\text{f, enc}} = I\frac{\pi s^2}{\pi R^2}$$
$$\implies \mathbf{H}_{\text{in}} = I\frac{s}{2\pi R^2}\hat{\boldsymbol{\phi}} \quad s < R$$

For outside $s > R$,

$$\mathbf{H}_{\text{out}} = I\frac{1}{2\pi s}\hat{\boldsymbol{\phi}} \quad s > R$$

So the H-field increases linearly with s inside the rod and drops off by $\sim 1/s$ outside the rod Outside the rod, we can find the magnetic field since the magnetization is zero $\mathbf{M} = 0$ so

$$\mathbf{B}_{\text{in}} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$

but we can't readily find the B-field inside since we are not sure what $\mathbf{M}$ is inside.

1.3.2 Deceptive Parallels

From electrostatics

$$\nabla \cdot \mathbf{D} = \frac{\rho_{\text{free}}}{\epsilon_0} \cancel{\to} \text{Coulomb's Law} \to \nabla \times \mathbf{D} \neq 0$$
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \to \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \to \nabla \times \mathbf{E} = 0$$

but in magnetostatics

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \cdot \mathbf{H} = \cancel{\frac{1}{\mu_0}\nabla \cdot \mathbf{B}} - \nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{M}$$

where the source—e.g., a bar magnet $\nabla \cdot \mathbf{M} \neq 0$—will elucidate a B-field even if $\mathbf{H} = 0$.

1.3.3 Boundary Conditions

From $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$,

$$H_{\perp,\text{above}} - H_{\perp,\text{below}} = -(M_{\perp,\text{above}} - M_{\perp,\text{below}})$$

where the curl $\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}}$ so

$$\mathbf{H}_{\parallel,\text{above}} - \mathbf{H}_{\parallel,\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

which points in opposite directions i.e. a discontinuity of the H-field above and below the sheet current.

For the B-field

$$B_{\perp,\text{above}} - B_{\perp,\text{below}} = 0$$

for the tangential components and

$$\mathbf{B}_{\parallel,\text{above}} - \mathbf{B}_{\parallel,\text{below}} = \mu_0\mathbf{K}_f \times \hat{\mathbf{n}}$$

for a continuous B-field across the boundary.

1.4 Linear (and Nonlinear) Media

1.4.1 Magnetic Susceptibility + Permeability

From electrostatics:

$$\mathbf{P} = \epsilon \chi_e \mathbf{E} + O(\mathbf{E}^2)$$

where we can assume a linear media and omit the higher order $O(\mathbf{E}^2)$ terms.

In magnetostatics:

$$\mathbf{M} = \chi_m \mathbf{H} + O(\mathbf{H}^2)$$

where χ_m is the magnetic susceptibility and we omit the higher order terms for a linear media. This is roughly $\chi_m \sim 10^{-5}$ for typical values. In relation to the B-field

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} \qquad \mu = \mu_0(1 + \chi_m)$$

In free space $\chi_m = 0$ and $\mu = \mu_0$.

Example An infinite solenoid, n turnes/length, current I, filled with linear material $\chi_m \ldots$ find field in solenoid.

$$\oint \mathbf{H} \cdot \mathrm{dl} = I_{\text{f, enc}} \implies \mathbf{H} = nI\hat{\mathbf{z}} \quad \text{inside}$$

So the B-field is

$$\mathbf{B} = \mu\mathbf{H} = \mu_0(1 + \chi_m)\mathbf{H}$$

- Paramagnets $\chi_m > 0$
- Diamagnets $\chi_m < 0$