1 Electrostatics

1.1 The Electric Field

given charge q: find force on Q , where $\mathbf{F}$ depends on $\boldsymbol{\mathscr{r}}, \mathbf{v}_i, \mathbf{a}_i$

1.1.1 Coulombs law

Coulomb's Law empirically,

$$\mathbf{F}_Q = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

where $k = \frac{1}{4\pi\epsilon_0}$ and the permittivity of free space is $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{Nm}^2$

The force is attractive if $sgn(qQ) = -1$ and repulsive if $= +1$.

Principal of superposition:

$$\begin{aligned}\mathbf{F}_T &= \mathbf{F}_{Q1} + \mathbf{F}_{Q2} + \dots \\ &= \frac{1}{4\pi\epsilon_0} Q\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \dots\right) \\ &= Q\mathbf{E}_T\end{aligned}$$

where $\mathbf{E}_T$ is the total electric field due to all of the source (point) charges.

$\mathbf{E}$ doesn't depend on Q

- $\mathbf{E} \sim F/Q$

Example: $\mathbf{E}$ field midway above two charges q: The electric fields are zero in the x and y direction:

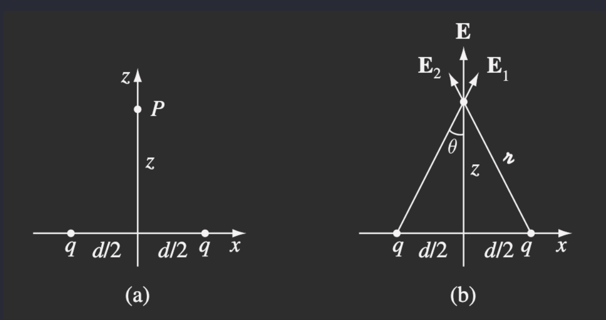

Figure 1.1: Griffiths Example 2.1

$$E_x = E_y = 0$$

But we can sum the fields in the z direction:

$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta$$

where

$$\mathscr{r} = \left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2} \qquad \cos\theta = \frac{z}{\mathscr{r}}$$

so

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2+\left(\frac{d}{2}\right)^2\right]^{3/2}}$$

Far away: $z \gg d$, so $d \to 0$ thus

$$E_z \approx \frac{1}{4\pi\epsilon_0}\frac{2qz}{z^3} = \frac{1}{4\pi\epsilon_0}\frac{2}{z^2}$$

Continuous Charge Distributions

- line: charge per unit length λ; $\mathrm{d}q = \lambda \mathrm{d}\ell$
- surface: charge per unit area σ; $\mathrm{d}q = \sigma \mathrm{d}a$
- volume: charge per unit volume ρ; $\mathrm{d}q = \rho \mathrm{d}\tau$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\mathrm{d}q$$

e.g. for a volume charge:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\mathrm{d}\tau'$$

where $'$ denotes the source charge in (no $'$ is a field point)

Example: Find $\mathbf{E}$ at z above a straight line segment of length $2L$ with uniform line charge λ. If we

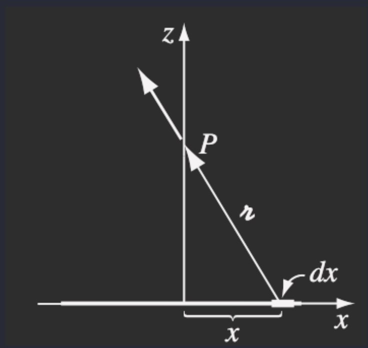

Figure 1.2: Griffiths Example 2.2

treat dq as a point particle, then we can use Ex 2.1 likewise but integrate over the line segment.

First we catalog what we know:

- Field point P is at $\mathbf{r} = z\hat{\mathbf{z}}$
- Sources at $\mathbf{r}' = x\hat{\mathbf{x}}$; $\mathrm{d}\ell' = \mathrm{d}x$
- $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = z\hat{\mathbf{z}} - x\hat{\mathbf{x}}$
- $\mathscr{r} = \sqrt{x^2+z^2}$
- $\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\hat{\mathbf{z}}-x\hat{\mathbf{x}}}{\sqrt{x^2+z^2}}$

The electric field is then (line charge)

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{+L}\frac{\lambda}{\mathscr{r}^2}\hat{\mathscr{r}}\mathrm{d}x = \frac{1}{4\pi\epsilon_0}\lambda\int_{-L}^{+L}\frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{(z^2+x^2)^{3/2}}\mathrm{d}x \\
&= \frac{1}{4\pi\epsilon_0}\lambda\left[z\hat{\mathbf{z}}\int_{-L}^{L}\frac{\mathrm{d}x}{(z^2+x^2)^{3/2}} - \hat{\mathbf{x}}\int_{-L}^{L}\frac{x\mathrm{d}x}{(z^2+x^2)^{3/2}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\lambda\left[z\hat{\mathbf{z}}\frac{x}{z^2\sqrt{z^2+x^2}}\Bigg|_{-L}^{L} - \cancel{\hat{\mathbf{x}}\frac{1}{\sqrt{z^2+x^2}}\Bigg|_{-L}^{L}}\right]
\end{aligned}$$

we can easily see that the x component is zero through the geometrical symmetry of the line centered at the origin (like Ex 2.1). Simplifying gives us

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{\mathbf{z}}$$

Checks and balances:

- $\hat{\mathbf{z}}$ is expected!
- $$z \gg L \quad \sqrt{z^2+L^2} \approx z \quad E(P, z \gg L) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2}$$
  where we can treat this as a point charge $q = 2\lambda L$ when we are far away.

1.2 Divergence and curl of E: Gauss' Law

'flux' of field lines

$$\Phi = \oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{a}$$

What is Φ for point charge at origin surrounded by a spherical surface?

$$\begin{aligned}
\Phi &= \int \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{q_{enc}}{\epsilon_0}
\end{aligned}$$

A bunch of charges surrounded by a surface: $\mathbf{E}_T = \sum \mathbf{E}_i$

$$\Phi = \oint \mathbf{E}_T\cdot\mathrm{d}\mathbf{a} = \sum_i \oint \mathbf{E}_i\cdot\mathrm{d}\mathbf{a} = \sum_i \frac{q_i}{\epsilon_0}$$

Thus we have an integral form of Gauss's law:

$$\boxed{\oint \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

where $Q = \sum q_i$.

From the theorem of divergence:

$$\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a} = \int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\mathrm{d}\tau \quad \text{and} \quad Q = \int_V \rho\mathrm{d}\tau$$

so

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{E})\mathrm{d}\tau = \int_V \rho\mathrm{d}\tau \rightarrow \text{good for all volume}$$

therefore we have the differential form of Gauss' Law:

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}}$$

Three ways Gauss's law makes life nice: Gaussian surfaces

- spherical: gaussian sphere
- cylindrical: gaussian cylinder
- planar: gaussian pillbox

1.2.1 Applications of Gauss's Law

$$\int \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{q_{enc}}{\epsilon_0} \to \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

1. (Simple spherical) What is $\mathbf{E}$ outside a uniformly charged solid sphere of radius R and total charge Q? The spherical Gaussian surface implies a symmetry where we should *only have a radial component* $\mathbf{E} = E_r$.

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q}{\epsilon_0}$$
$$E \oint \mathrm{d}\mathbf{a} = E \cdot 4\pi r^2 = \frac{Q}{\epsilon_0}$$
$$\implies \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

where the integral is equivalent to the surface area of the sphere. This is also $\implies$ a field of a *point*.

2. (Simple cylindrical) A long cylinder (radius a) of charge density $\rho = ks$ ($\propto$ distance from axis) where k is a constant and s is the radial distance from the axis. What is $\mathbf{E}$ inside the cylinder? The cylindrical Gaussian surface has radius s and length ℓ:

$$\oint \mathbf{E}.\mathrm{d}\mathbf{a} = \frac{Q_enc}{\epsilon_0}; \quad Q_{enc} = \int \rho \mathrm{d}\tau = \int (ks')\mathrm{d}s'\mathrm{d}\phi\mathrm{d}z = \frac{2}{3}\pi k \ell s^3$$

When using the divergence theorem, note that only the curved part of the cylinder contributes to the flux. Thus,

$$\int \mathbf{E}\mathrm{d}\mathbf{a} \to E \int \mathrm{d}a = E(2\pi s \ell)$$
$$\implies \mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}$$

If we were to find the field outside the cylinder we would find that the enclosed charge is constant Q_{enc} thus the field is proportional to $1/s$.

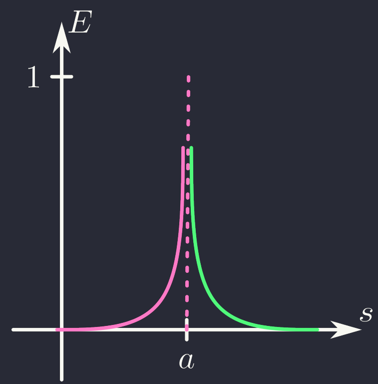

Figure 1.3: Electric field as a function of s

3. (Simple infinite plane) with uniform surface charge σ. Symmetry implies that $\mathbf{E}$ is perpendicular to the plane. The Gaussian pillbox (either box or cylinder) will have a field of

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

1.2.2 The curl of E

$$\nabla \times \mathbf{E} = 0, \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

calculating

$$\int_a^b \mathbf{E} \cdot \mathrm{d}\ell, \quad \mathrm{d}\ell = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\theta\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\phi\hat{\boldsymbol{\phi}}$$

So the integral is

$$\frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a} - \frac{q}{b}\right)$$

This means:

- path independent!
- if $a = b$ then $\oint \mathbf{E} \cdot \mathrm{d}\ell = 0$ (ℓ is a vector but I don't know how to bold it)

We can now use Stokes' theorem: $\oint \mathbf{v} \cdot \mathrm{d}\ell = \int_S (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a}$ or

$$\oint \mathbf{E} \cdot \mathrm{d}\ell = \int_S (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{a} = 0 \implies \nabla \times \mathbf{E} = 0$$

1.3 Electric potential

Any function f with zero curl is the gradient of a scalar function: $\nabla \times (\nabla f) = 0$ (curl of gradient is always 0!)

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\ell$$

implies all paths give some value.

$V \sim$ "electric potential"

$$\begin{aligned} V(\mathbf{b}) - V(\mathbf{a}) &= -\left(\int_{\mathcal{O}}^{b} \mathbf{E} \cdot \mathrm{d}\ell\right) - \left(-\int_{\mathcal{O}}^{a} \mathbf{E} \cdot \mathrm{d}\ell\right) \\ &= -\int_{\mathcal{O}}^{b} - \int_{\dashv} O\mathbf{E} \cdot \mathrm{d}\ell \\ &= -\int_a^b \mathbf{E} \cdot \mathrm{d}\ell \end{aligned}$$

And from the fundamental theorem for gradients: $T(\mathbf{b}) - T(\mathbf{a}) = \int_a^b (\nabla T) \cdot \mathrm{d}\ell$

$$\implies \mathbf{E} = -\nabla V$$

i "potential" is a terrible name

ii $\mathbf{E} = (E_x, E_y, E_z)$ vs V with only *one* value at every point in space! Otherwise we would have to deal with

$$(\nabla \times \mathbf{E})_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

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$$V'(\mathbf{r}) = -\int_{O'}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\ell = -\int_{O'}^{O} \mathbf{E} \cdot \mathrm{d}\ell - \int_{O}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\ell = C + V(\mathbf{r})$$

$\implies \mathbf{E} = -\nabla V$