1   Magnetostatics

In class demonstration   Two parrell wires connected to high voltage source:

- Parallel current: The wires move towards each

- Antiparallel current: The wires move away from each other

This implies some sort of force acting on each wire which cannot not be explained by the electrostatics we have covered thus far...

1.1   Lorentz Force Law

$$\mathbf{F}_{\mathrm{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

where Q and $\mathbf{v}$ describe the moving test charge. Compared to the force of the electric field

$$\mathbf{F}_{\mathrm{elec}} = Q\mathbf{E}$$

Thus the total force is give by the Lorentz Force Law

$$\boxed{\mathbf{F}_{\mathrm{tot}} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

1.1.1   Magnetic Fields

Skip discussion so we can move straight to

1.1.2   Magnetic Forces

Cyclotron motion   A $\mathbf{B}$ field is place into to page, and a test charge has velocity $\mathbf{v}$ moving up on the y-axis: So using the right-hand-rule (RHR) the force from the magnetic field points to the left with magnitude

$$F_{\mathrm{mag}} = qvB = \text{acceleration} = \frac{mv^2}{R}$$

where we have a circular motion similar to large masses rotating under gravity i.e. the "cylotron" radius

$$R = \frac{mv}{qB}$$

Cycloid motion   Add an $\mathbf{E} \perp \mathbf{B} \dots \mathbf{B} = B\hat{\mathbf{x}}$ $\mathbf{E} = E\hat{\mathbf{z}}$
The velocity of the test charge is given by

$$\mathbf{v} = \left(0, \frac{\mathrm{d}y(t)}{\mathrm{d}t}, \frac{\mathrm{d}z(t)}{\mathrm{d}t}\right) = (0, \dot{y}, \dot{z})$$

Where the charge is constrained in the yz plane. The math says

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\hat{\mathbf{y}} - B\dot{y}\hat{\mathbf{z}}$$

So the force is

$$\begin{aligned}
\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= q(E\hat{\mathbf{z}} + B\dot{z}\hat{\mathbf{y}}) = m\mathbf{a} = m(\ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}})
\end{aligned}$$

This gives us the equations of motion

$$qB\dot{z} = m\ddot{y}$$
$$qE - qB\dot{y} = m\ddot{z}$$

And using the "cyclotron frequency"

$$\omega \equiv \frac{qB}{m} \quad \sim \frac{\text{C.T}}{\text{kg}} \sim \frac{\text{rad}}{\text{s}}$$

We can rewrite the equations of motion as

$$\ddot{y} = \omega\dot{z}$$
$$\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

Which we can solve with the general solution

$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{E}{B}t + C_3$$
$$z(t) = C_2\cos(\omega t) - C_1\sin(\omega t) + C_4$$

with the intial conditions

$$\dot{y}(0) = \dot{z}(0) = 0$$
$$y(0) = z(0) = 0$$

The solution is

$$y(t) = \frac{E}{\omega B}(\omega t - \sin(\omega t))$$
$$z(t) = \frac{E}{\omega B}(1 - \cos(\omega t))$$

NOTE: $v = \omega R$ so the force $F = q(E + v \times B)$ we have a length scale

$$\frac{E}{B} \sim \omega R, \quad E \sim v \times B, \quad v \sim E/B$$

$$\boxed{R \equiv \frac{E}{\omega B}}$$

So the equations are simplified to

$$y(t) = R(\omega t - \sin(\omega t)) \quad z(t) \qquad\qquad = R(1 - \cos(\omega t))$$
$$(y - \omega tR)^2 = R^2\sin^2(\omega t) \quad (z - R)^2 = R^2\cos^2(\omega t)$$

thus

$$R^2(c^2 + s^2) = R^2 = (z - R)^2 + (y - \omega tR)^2$$

Since

$$\dot{y} = v = \omega R = \frac{E}{B}$$

and the equation is similar to the circle $x^2 + y^2 = R^2$ so the motion is a cycloid. So we can imagine the motion of the charge as rolling a circle along the y axis! [insert cycloid path]
    NOTE:

$$\boxed{\textbf{magnetic fields do no work}}$$

As a charge q moves $dl = \mathbf{v}\,dt$ the change in word is

$$d\omega_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,dt = 0$$

But why did the wires move in the demonstration?

Currents

We usually define the current direction I using a positive charge (conventional current) because the the negative charge moving in the opposite direction of the positive charge is equivalent i.e.

$$q_+ v_+ = (-q)(-v_+)$$

The current is charge/time

$$I \sim \mathrm{C/s} \sim \mathrm{A}$$

The the line charge has current

$$\mathbf{I} = \lambda \mathbf{v} \sim \mathrm{C/m\,m/s} = \mathrm{A/m}$$

In a neutral wire (or in general)

$$I = \lambda_+ v_+ + \lambda_- v_-$$

The force on a current carrying wire:

$$F_{\mathrm{mag}} = \int (\mathbf{v} \times \mathbf{B})\, \mathrm{d}q = \int (\mathbf{v} \times \mathbf{B}) \lambda\, \mathrm{d}l$$

$$= \int (\mathbf{I} \times \mathbf{B})\, \mathrm{d}l = \int I(\mathrm{d}\mathbf{l} \times \mathbf{B})$$

where we assume that the current is constant *usually* so

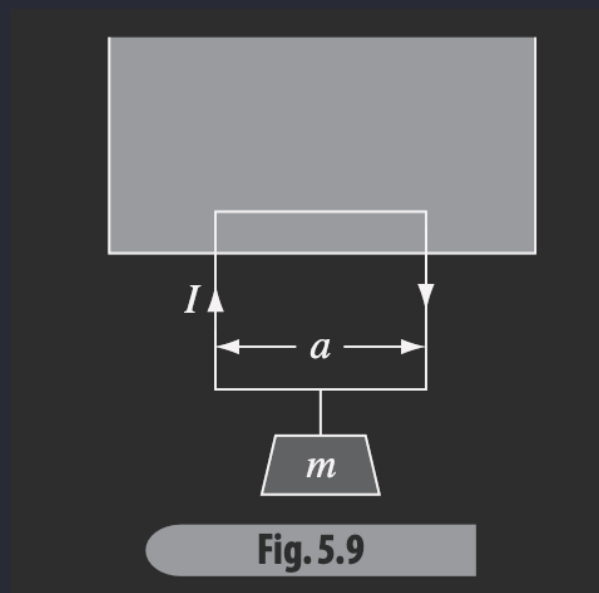$$\boxed{F_{\mathrm{mag}} = I \int (\mathrm{d}\mathbf{l} \times \mathbf{B})}$$

Figure 1.1: Magnetic field above point into page

Example: The current will move clockwise in the loop because on the top side in order to have the force point upwards to counteract the mass. The force on the sides of the loop will cancel out, so the magnitude is

$$|\mathbf{F}_{\mathrm{mag}}| = IaB = mg$$

Now increasing I will increase the force which means the whole wire move upwards with velocity $\mathbf{u}$ and a change in height $\delta h = u\delta t$. This now looks like work

$$\delta W_{\mathrm{mag}} = IaB\delta h$$

Who is doing the work to keep lifting the wire? The battery! This is analogous to normal force on applying force to a mass on an incline; the normal force does no work, but only redirects the force upwards. So magnetic force is like the normal force.

Surface Current Density   When charge flows on a surface, this density

$$\mathbf{K} \equiv \frac{\mathrm{d}\mathbf{I}}{\mathrm{d}l_\perp}$$

where we can image $l_\perp$ as a thin ribbon (1.2) and K is the current per unit width thus
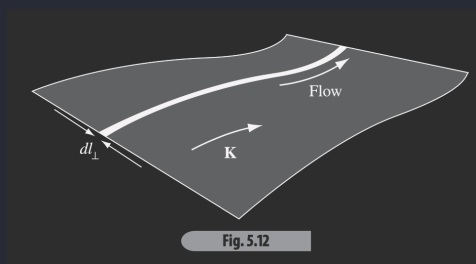
$$\mathbf{K} = \sigma\mathbf{v}$$

Figure 1.2: Surface current density

Magnetic force in 2D   From the 1D $\mathbf{F}_{\mathrm{mag}} = I\int \mathrm{d}\mathbf{l} \times \mathbf{B}$, so for 2D

$$\mathbf{F}_{\mathrm{mag}} = \int (\mathbf{v} \times \mathbf{B})\sigma \, \mathrm{d}a = \int (\mathbf{K} \times \mathbf{B}) \, \mathrm{d}a$$

For a charge dist. in a volume

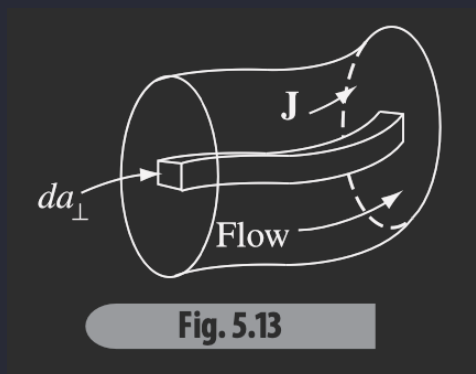$$\mathbf{J} \equiv \frac{\mathbf{I}}{a_\perp} \sim \text{current/unit area}$$

Figure 1.3: Volume current density

So we can find this (mobile) volume charge density with

$$\mathbf{J} = \rho\mathbf{v}$$

Figure 1.4: Volume current density

Think of this steady current as a column of water moving along this pipe. Rather than looking at each individual water molecule, we can look at the flow of the water as a whole.

In a **B** field there is a force

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\rho \, d\tau = \int (\mathbf{J} \times \mathbf{B}) \, d\tau$$

With current I & uniform cross-sectional area (a wire), the volume charge density is

$$\mathbf{J} = \frac{I}{\pi a^2}$$

With $J \propto$ distance from the center of the wire thus

$$J = ks, \quad k \sim \text{constant}$$

The current is

$$I = \int dI = \int \mathbf{J} \cdot d\mathbf{a}_\perp$$

$$= \int (ks)s \, ds \, d(\phi)$$

$$= 2\pi k \int_0^a s^2 \, ds = 2\pi k \frac{a^3}{3}$$

Total current crossing a surface

$$I = \int_S J \, d\mathbf{a}_\perp = \int_S \mathbf{J} \cdot d\mathbf{a}_\perp$$

If the surface is *closed*: if there is a source inside the surface, there is a net flow coming out of the surface and vice versa.

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = -\frac{d}{dt} \int_V \rho \, d\tau = -\int_V \left(\frac{d\rho}{dt} \right) d\tau$$

From the divergence theorem

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{J}) \, d\tau$$

or also known as the "continuity equation"

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$$

1.2   Biot-Savart Law

- Stationare charge $\rightarrow$ const. $\mathbf{E}$ $\rightarrow$ electrostatics (Coloumbs Law)

- Steady currents $\rightarrow$ const. $\mathbf{B}$ $\rightarrow$ magnetostatics (Biot-Savart Law)

1.2.1   Steady Currents

ASSUME

$$\frac{\partial \rho}{\partial t} = 0 \implies \frac{\partial \mathbf{J}}{\partial t} = 0$$

For most technologies, computers, etc.

- freq $\rightarrow \sim$ 10-100 MHz; which is fast and reaches a limit due to the speed of light. . . (lets not worry about this for now)
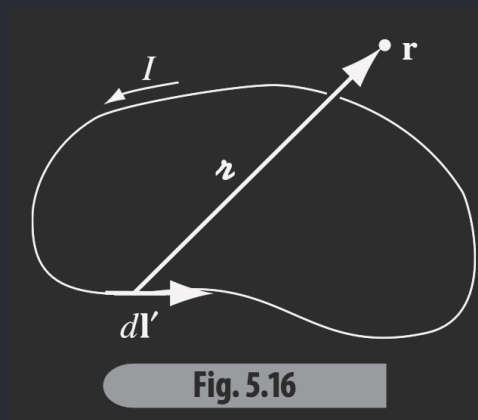

Figure 1.5: Biot-Savart Law along a wire

The $\mathbf{B}$ field of a steady current line currents is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\imath}}{\imath^2} \, \mathrm{d}l'$$
$$= \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\mathbf{l}' \times \hat{\imath}}{\imath^2}$$

where we integrate along the current path and μ_0 is the permeability of free space

$$\mu_0 \approx 4\pi \times 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}$$

Units of magnetic field:

$$B \sim \frac{\mathrm{N}}{\mathrm{A.m}} \sim \frac{\mathrm{V.s}}{\mathrm{m}^2} \sim T \rightarrow \text{Tesla}$$

Where the Earth has a magnetic field of a few μ T compared to a NMR machine which has a field of a few T.

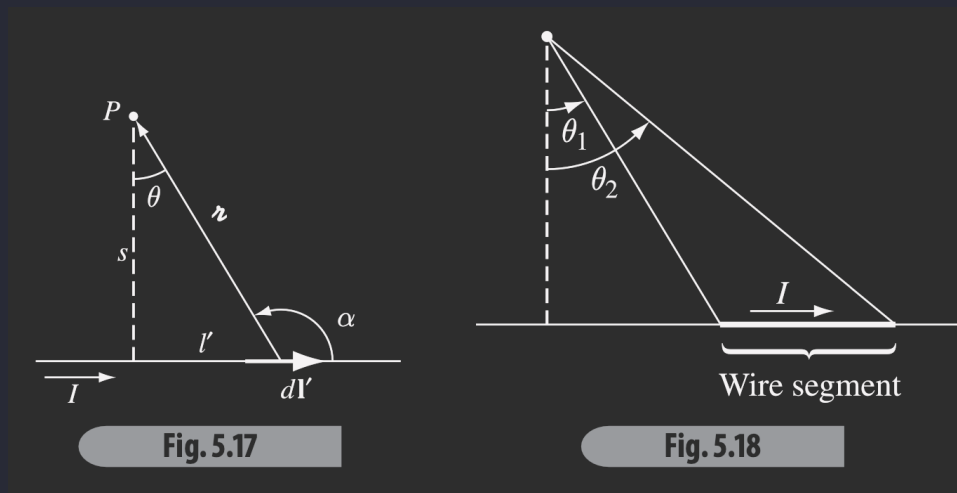**Example:**   Find **B** field a distance s from a long striaght current carrying wire I.

Figure 1.6: B-field a disance along a wire

From the Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\imath}}{\imath^2}$$

So looking at each part:

- $d\mathbf{l}' \times \hat{\imath}$ at P tells us $\mathbf{B} \sim (-\hat{\mathbf{y}})$ (RHR) with magnitude

$$dl' \sin \alpha = dl' \cos \theta$$

- $l' = s \tan \theta$ so

$$dl' = s \sec^2 \theta \, d\theta$$

- $s = \imath \cos \theta$ so

$$\frac{1}{\imath^2} = \frac{\cos^2 \theta}{s^2}$$

Thus

$$|\mathbf{B}| = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\cos\theta}{s}\, \mathrm{d}\theta$$

$$= \frac{\mu_0 I}{4\pi s} \int_{-\pi/2}^{\pi/2} \cos\theta\, \mathrm{d}\theta = \frac{\mu_0 I}{2\pi s}$$

We can imagine this as a circle around the wire capturing this B-field or wrapping your fingers around the wire where the thumb point in the direction of the current (RHR).
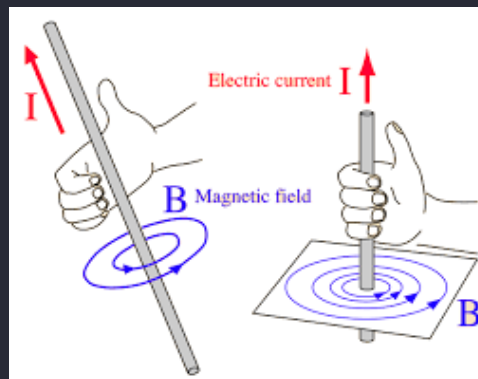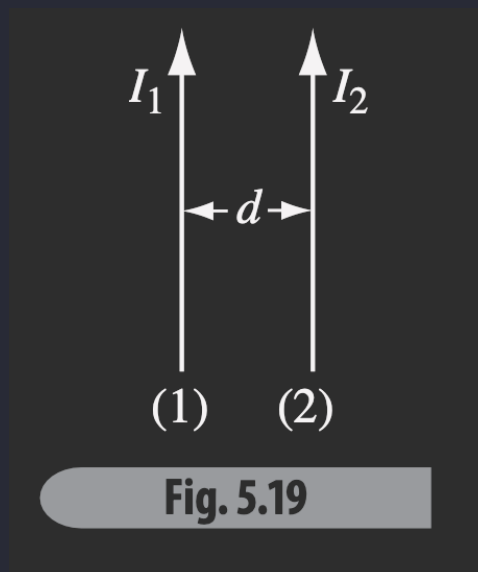

Figure 1.7: Right-hand rule

Figure 1.8: Demo

Back to demo   We can now see the forces on the wires due to each other's magnetic fields.

1.3　Divergence and Curl of B

1.3.1　Straight Line Currents

Integrating $\mathbf{B}$ around a circle of radius s for a long straight wire:

$$\oint \mathbf{B} \cdot d\mathbf{l} \quad \mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$

$$= \oint \frac{\mu_0 I}{2\pi s} s \, d\phi = \mu_0 I$$

which is independent of s!

　　Even for any loop around the wire $d\mathbf{l} = ds\,\hat{\boldsymbol{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$ so

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int \frac{\mu_0 I}{2\pi s} s \, d\phi = \mu_0 I$$

For a bundle of straight wires,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

From our intuitin we can use the divergence theorem to get

$$\oint \mathbf{B} \cdot d\mathbf{l} = \in (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 I = \mu \int \mathbf{J} \cdot d\mathbf{a}$$

or "Ampere's Law"

$$\boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}}$$

1.3.2　Divergence and Curl of B

How about div of B?

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} \, d\tau'$$

so

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla} \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) d\tau'$$

where the quantity in the integral is

$$\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \cdot (\boldsymbol{\nabla} \times \mathbf{J}) - \mathbf{J}\left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = 0$$

this is proved in Griffiths Prob 1.63... anyways this tells us

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B} = 0}$$

1.3.3　Ampere's Law

Again

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$

where

$$\in (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu \int \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{\text{enc}}$$

thus

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}}$$

for a current enclosed by an "Ampèrian oop"

Example: The magnetic field of a long straight wire with steady current I is

$$\oint B \cdot \mathrm{dl} = B \int s \, \mathrm{d}\phi = B2\pi s = \mu I_{\text{enc}}$$

$$\implies B = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

which is easier to get to compared to the last time we did this.

Example: Find $\mathbf{B}$ due to infinite uniform surface current $\mathbf{K} = K_0 = \hat{\mathbf{x}}$:

Drawing an amperian loop as a rectangle perpendicular to current above and below the current sheet: $\mathrm{dl} = 2Bl$ and

$$I_{\text{enc}} = K_0 l$$

so

$$B = \frac{\mu_0 K_0}{2}$$

Example: An infinitely long solenoid... using an amperian loop: a rectangle around the middle of the solenoid and outside it.

$$2Bl = \mu_0 I(nl)$$

but we can make the loop outside the solenoid go infinitely far away so

$$Bl = \mu_0 n I l$$

or the classic result

$$\boxed{B = \mu_0 n I}$$

1.4 Magnetic Vector Potential

The Vector Potential

From the divergence of $\mathbf{B}$ we introduce a vector potential of the magnetic field such that

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \implies \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{V}) = 0$$

so we define

$$\boxed{\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}}$$

and from Ampere's Law

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

In electrostatics we had

$$\mathbf{E} = -\boldsymbol{\nabla} V \quad V \to V + C \quad \text{"gauge freedom"}$$

where "gauge freedom" is a fancy way of saying we can add a constant to the potential and it won't change the physics.

Similarly for $\mathbf{B}$:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} F = 0$$

where we want to make

$$\boldsymbol{\nabla} \cdot A \to 0$$

Method: Say we have a potential $\mathbf{A}_0$ where $\mathbf{B}_0 = \boldsymbol{\nabla} \times \mathbf{A}_0$ i.e.

$$\mathbf{A}_0 \to \mathbf{A} + \boldsymbol{\nabla} \lambda \equiv \mathbf{A}$$

Thus

$$\mathbf{B} = \boldsymbol{\nabla} \times A = \boldsymbol{\nabla} \times (\mathbf{A}_0 + \boldsymbol{\nabla}\lambda) = \boldsymbol{\nabla} \times \mathbf{A}_0 + 0 = \mathbf{B}_0$$

but we know that

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A}_0 + \underbrace{\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\lambda)}_{\nabla^2 \lambda} = 0,$$

So we get $\boldsymbol{\nabla} \cdot \mathbf{A}\Gamma = \Gamma\nabla^2\lambda = -\boldsymbol{\nabla} \cdot \mathbf{A}_0$. To find this λ we can you the same Poisson's equation method like in electrostatics.

$\implies$ From electrostatics:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \to V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\imath} \, d\tau'$$

So for magnetostatics

$$\lambda = \frac{1}{4\pi} \int \frac{\boldsymbol{\nabla} \cdot \mathbf{A}_0}{\imath} \, d\tau'$$

Why did we do all this?

$\implies$ All for Ampèr's law:

$$\boldsymbol{\nabla} \times \mathbf{B} = -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

where we fnially get

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

Now we can just find the vector potential from a current distribution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r})}{\imath} \, \mathrm{d}\tau'$$

which is a much easier way to find the magnetic field rather than our previous Bio-Savart Law which had a curl inside the integral! for line and sheet currents

$$\rightarrow \int \frac{\mathbf{I}}{\imath} \, \mathrm{d}l' \quad \int \frac{\mathbf{K}}{\imath} \, \mathrm{d}a'$$

Before we tackle this example heavy in math, we recommend reading the Feynmane lectures Website.

Example   A spherical shell, radius, R, uniform fixed surface charge σ set to spin at ω. Find $\mathbf{A}(\mathbf{r})$
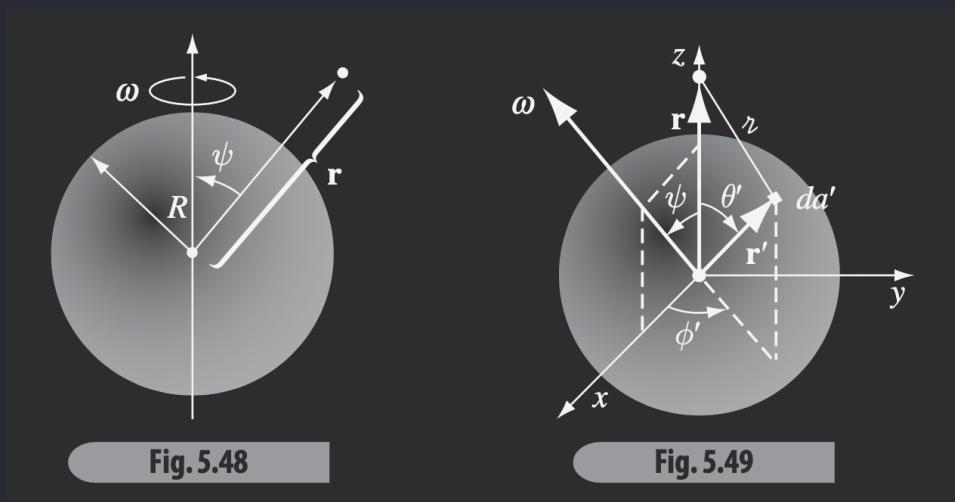

Figure 1.9: All the components

All the stuff we need:

- The vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{\imath} \, \mathrm{d}a'$$

- With current density $\mathbf{K} = \sigma \mathbf{v}$ and separation vector $\imath = \mathbf{r} - \mathbf{r}'$ with magnitude

$$\imath = \sqrt{R^2 + r'^2 - 2Rr' \cos\theta}$$

- The area element

$$\mathrm{d}a' = R^2 \sin\theta' \, \mathrm{d}r' \, \mathrm{d}\theta' \, \mathrm{d}\phi'$$

  and from position vector

$$\mathbf{r}' = R(\sin\theta' \cos\phi', \sin\theta' \sin\phi', \cos\theta')$$

- The velocity

$$\mathbf{v} = \omega \times \mathbf{r}' = R \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\theta' & 0 & \omega\cos\theta' \\ \sin\theta'\cos\phi' & \sin\theta'\sin\phi' & \cos\theta' \end{vmatrix}$$
$$= \omega R[-(\cos\psi \sin\theta' \sin\phi')\hat{\mathbf{x}} + (\cos\psi \sin\theta' \cos\phi' - \sin\psi \cos\theta')\hat{\mathbf{y}} + (\sin\psi \sin\theta' \sin\phi')\hat{\mathbf{z}}]$$

We do this so that any terms including $\sin \phi'$ or $\cos \phi'$ will integrate to zero:

$$\int_0^{2\pi} \sin \phi' = \int_0^{2\pi} \cos \phi' \, d\phi' = 0$$

So we can a much simpler

$$\mathbf{v} = -\omega R[\sin \psi \cos \theta' \hat{\mathbf{y}}]$$

Thus we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sigma(-\omega R \sin \psi) R^2 2\pi \hat{\mathbf{y}} \int_0^{\pi} \frac{\cos \theta' \sin \theta'}{\sqrt{R^2 + r^2 - 2Rr \cos \theta'}} \, d\theta'$$

after integration with substitution... we have two results

- If $|\mathbf{r}| < R$ the integral goes to

$$\int \to \frac{2r}{3R^2}$$

- If $|\mathbf{r}| > R$ the integral goes to

$$\int \to \frac{2R}{3r^2}$$

We can see that

$$\boldsymbol{\omega} \times \mathbf{r} = (-\omega R \sin \psi)\hat{\mathbf{y}}$$

which will transform our tilted form to the original form (Fig.).

Finally

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 R \sigma}{3}(\boldsymbol{\omega} \times \mathbf{r}) & \text{Inside sphere} \\ \frac{\mu_0 R^4 \sigma}{3r^2}(\boldsymbol{\omega} \times \mathbf{r}) & \text{Outside sphere} \end{cases}$$

And letting $\boldsymbol{\omega} \propto \hat{\mathbf{z}}$, and $\mathbf{r}$ at (r, θ, ϕ):

$$\mathbf{A}(r, \theta, \phi) = \begin{cases} \frac{\mu_0 R \sigma}{3}\omega r \sin \theta \hat{\phi} & \text{Inside sphere} \\ \frac{\mu_0 R^4 \sigma}{3r^2}\omega \sin \theta \hat{\phi} & \text{Outside sphere} \end{cases}$$

Thus the magnetic field in side the sphere is

$$\mathbf{B}(|\mathbf{r}| < R) = \boldsymbol{\nabla} \times \mathbf{A} = \frac{2}{3}\mu_0 R \sigma \omega \hat{\mathbf{z}}$$

Suprisingly a uniform magnetic field!

Or (from Feynman) a sphere with wire wrapped around it like a solenoid.

1.4.1 Multipole Expansion

Recalling what we did for the electric field

$$\frac{1}{\imath} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$
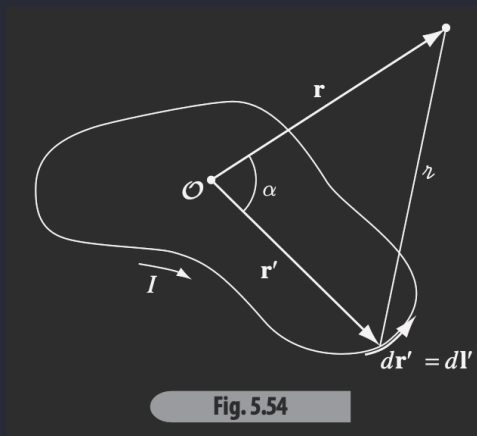

Figure 1.10: Field point P for loop of line current

For the line charge, the vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} I \int \frac{\mathrm{d}\mathbf{l}'}{\imath}$$

$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\,\mathrm{d}\mathbf{l}'$$

$$= \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint \mathrm{d}\mathbf{l}'}_{\text{mono}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\,\mathrm{d}\mathbf{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2 P_2(\cos\alpha)\,\mathrm{d}\mathbf{l}'}_{\text{quadrupole}} + \dots\right]$$

Where we can see that the monopole term is zero as magnetic monopoles do not exist. Looking at the dipole term

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha\,\mathrm{d}\mathbf{l}'$$

where the integral has $\hat{\mathbf{r}}\cdot\mathbf{r}'$ so

$$\oint(\hat{\mathbf{r}}\cdot\mathbf{r})\,\mathrm{d}\mathbf{l}' = -\hat{\mathbf{r}}\times\int \mathrm{d}\mathbf{a}' = \int \mathrm{d}\mathbf{a}'\times\hat{\mathbf{r}}$$

This gives us

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2}\mathbf{m}\times\hat{\mathbf{r}}$$

where

$$\mathbf{m} = I\int \mathrm{d}\mathbf{a}'$$

The field lines rather go through the loop and because we have no monopole term.