1 Calculus of Variations

Why do we care?

- What is the shortest distance between two points in a 2D plane?
- What is the shortest path between two points on a sphere?
- What is the fastest path for a ball to roll down a hill?
- For a car driving on a flat path $A \to B$, what shape of a pot hole will minimize the time it takes to get from $A \to B$?

For some path $a \to b$, we have a path defined as an integral

$$S = \int_a^b f(x, y, y')\mathrm{d}x$$

with a *Goal*: find $y(x)$ that minimizes S (path).

Path Length:

$$l = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{1 + y'^2}\mathrm{d}x$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. To minimize $y = f(x)$ it is equivalent to finding where

$$f'(x) = 0$$

where we note that this could be a maximium point, but it is usually a minimum in these cases. Another look at this function:

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

we can define a small change in the path $y(x)$ as

$$y(x) + \delta y(x)$$

where

$$\delta y(x_2) = 0 \quad \delta y(x_2) = 0$$

so the change in the path is

$$\delta S = \int_a^b \delta f \mathrm{d}x$$

and from the change of variables

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \qquad \delta y' = \frac{\mathrm{d}}{\mathrm{d}x}\delta y$$

thus we have

$$\delta S = \int_a^b \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x$$

this is the line integral of the change in the new path

$$\delta S = S_{new} - S_{old}$$

looking at the second term: using integration by parts

$$\int_a^b \left(\frac{\partial f}{\partial y}\frac{\mathrm{d}}{\mathrm{d}x}\delta y\right)\mathrm{d}x = \left[\frac{\partial f}{\partial y'}\delta y\right]\Bigg|_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y \mathrm{d}x$$

the first term is zero because $\delta y(a) = \delta y(b) = 0$. Thus we have

$$\delta S = \int_a^b \left[\frac{\partial f}{\partial y} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\right]\delta y \mathrm{d}x$$

Near a minimum, $\delta S = 0$ for any small δy. So the the terms in the brackets must be zero as well! This gives us the **Euler-Lagrange Equation**:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0$$

NOTE: δS is the variation of S(some number) under δy(a function).

Example: Shortest path between two points $a \to b$ in a 2D cartesian plane.

Goal: find $y(x)$ tha minimzes the path length $l = \int_a^b \sqrt{1+y'^2}\mathrm{d}x$ where $f(x, y, y') = \sqrt{1+y'^2}$.

$$\frac{\partial f}{\partial y} = 0$$
$$\frac{\partial f}{\partial y'} = \frac{2y'}{2\sqrt{1+y'^2}} = \frac{y'}{\sqrt{1+y'^2}}$$

From the EL:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{\partial f}{\partial y} = 0$$

and

$$\begin{aligned}
\frac{y'}{\sqrt{1+y'^2}} &= Const = C \\
y'^2 &= C(1+y'^2) \\
y'^2 &= \frac{C}{1-C} \\
y' &= \pm\sqrt{\frac{C}{1-C}} = \pm k \\
y &= \pm kx + b
\end{aligned}$$

which is just a straight line as we expected.

Example: The Brachistochrone.

Goal: Find $y(x)$ that minimizes $t = \int_a^b \mathrm{d}t$ where

$$t = \frac{s}{v} \to \mathrm{d}t = \frac{\mathrm{d}s}{v}$$

and v is the velocity which can be found using the conservation of energy:

$$\frac{1}{2}mv^2 = mg(y_o - y) \to v = \sqrt{2g(y_o - y)}$$

thus we have

$$\mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o - y)}}\mathrm{d}x$$

where $f(x, y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2g(y_o-y)}}$. Using EL:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}\sqrt{2g(y_o - y)}}$$
$$\frac{\partial f}{\partial y} = \frac{\sqrt{1+y^2}\sqrt{2g}}{(2g(y_o - y))^{3/2}}$$

From the initial conditions: $x = y = 0, \quad \dot{x} = \dot{y} = 0$ (assuming no friction). So

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} \to \frac{\partial f}{\partial y} = \frac{\sqrt{1+y'^2}}{\sqrt{2gy^3}}$$

and if $y(x)$ satisfies EL, then we have a conserved quantity (prove in HW):

$$f - y'\frac{\partial f}{\partial y'} = const$$

we will soon find out that this is the energy of the system...we can solve for $y(x)$ using the conserved quantity:

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gy}}\frac{y'}{\sqrt{1+y'^2}}$$

so

$$f - y'\frac{\partial f}{\partial y'} = C$$
$$\frac{1}{\sqrt{2gy}}\left(\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right) = C$$
$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}}(1 + y'^2 - y'^2) = \sqrt{\frac{1}{4ga}}$$

using

$$\frac{1}{y(1+y'^2)} = \frac{1}{2a} \implies 1 + y'^2 = \frac{2a}{y} \implies y'^2 = \frac{2a}{y} - 1$$

this can be solved using separation of variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{2a}{y} - 1} \implies \mathrm{d}x = \mathrm{d}y\sqrt{\frac{y}{2a-y}}$$

and integration both sides:

$$\int_a^b \mathrm{d}y\sqrt{\frac{y}{2a-y}} = x - x_a = x$$

using a change of variables $y = a(1-\cos\theta)$; $\quad \mathrm{d}y = a\sin\theta\mathrm{d}\theta$ and a substitution $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{(1-\cos\theta)(1+\cos\theta)}$:

$$\int_a^b a\sin\theta\mathrm{d}\theta\sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}} = \int_a^b a(1-\cos\theta)\mathrm{d}\theta = a\theta - a\sin\theta$$

this is a parametric equation:

$$x = a(\theta - \sin\theta) = x(\theta)$$
$$y = a(1-\cos\theta) = y(\theta)$$

where $\theta = \omega t$. This is a curve traced by a point on a wheel AKA cycloid. When we choose a variable time we get

$$x(t) = a(\omega t - \sin\omega t)$$
$$y(t) = a(1-\cos\omega t)$$

and thus we get $\omega = \sqrt{\frac{g}{a}}$. To find a we use the coordiate of the lower second point to find the curve that goes through the two points.

Example: Find two functions $x(u)$, $y(u)$ where the path

$$S = \int_a^b f(x, x', y, y', u)\mathrm{d}u$$

is minimized/stationary. We will get two EL equations:

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial x'} = 0$$
$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial y'} = 0$$

e.g. for a distance between two points:

$$L = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_a^b \sqrt{x'^2 + y'^2}\mathrm{d}u \quad \text{using} \quad \mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}u}\mathrm{d}u = y'\mathrm{d}u$$

and from the EL equations:

$$\frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial x'} = 0 = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{x'}{\sqrt{x'^2+y'^2}}\right)$$
$$\implies C_1 = \frac{x'}{\sqrt{x'^2+y'^2}} \quad C_2 = \frac{y'}{\sqrt{x'^2+y'^2}}$$

this also tells us that

$$\frac{y'}{x'} = const = \frac{\mathrm{d}y}{\mathrm{d}x}$$

For N unknown functions in time t:

$$S = \int_a^b f(x_1, x_1', \ldots, x_N, x_N', u)\mathrm{d}u$$

where f has $2N+1$ variables.

Generalized Coordinates: $q_1, q_2, \ldots, q_N$ we would define the Lagrangian

$$\mathcal{L}(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t)$$

and minimize the action

$$S = \int \mathcal{L}\mathrm{d}t$$

and N EL equations gives the trajectory for the path of minimal action.