3.16 From Griffiths:

$$V(x,y) = \frac{2V_0}{\pi} \arctan\left(\frac{\sin \pi y/a}{\sinh \pi x/a}\right)$$

where the surface charge density is given by a Gaussian pillbox

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

since normal of the surface is $\hat{\mathbf{n}} = \hat{\mathbf{x}}$,

$$\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial}{\partial x}\left(\frac{2V_0}{\pi} \arctan(b)\right) \\
&= -\frac{2V_0\epsilon_0}{\pi} \frac{1}{1+b^2} \left(\frac{-\sin(\pi y/a)}{\sinh^2(\pi x/a)}\right) \cosh(\pi x/a) \frac{\pi}{a} \\
&= \frac{2V_0\epsilon_0}{a} \frac{\sin(\pi y/a)}{\sinh^2(\pi x/a) + \sin^2(\pi y/a)} \cosh(\pi x/a)
\end{aligned}$$

where at the boundary $x = 0$,

$$\sinh(0) = 0, \cosh(0) = 1$$

so

$$\sigma = \frac{2V_0\epsilon_0}{a} \frac{\sin(\pi y/a)}{\sin^2(\pi y/a)} = \boxed{\frac{2V_0\epsilon_0}{a} \sin(\pi y/a)}$$

3.18 Since only the top plate has a potential V_0 the boundary conditions are

(i) $V = V_0$ at $z = a$

(ii) $V = 0$ at $x = 0$, $y = 0$, $z = 0$, $x = a$, and $y = a$

We look for solutions

$$V(x, y, z) = X(x)Y(y)Z(z)$$

and solving for Laplace's equation

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0$$

where

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = C_1, \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = C_2, \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = C_3 \quad \text{with} \quad C_1 + C_2 + C_3 = 0$$

The textbook suggests that $C_1 = -k^2$, $C_2 = -l^2$, and $C_3 = k^2 + l^2$ where k, l are constants. We can solve for X, Y, Z separately:

$$\begin{aligned} X(x) &= A\sin(kx) + B\cos(kx) \\ Y(y) &= C\sin(ly) + D\cos(ly) \\ Z(z) &= Ee^{\sqrt{k^2+l^2}z} + Fe^{-\sqrt{k^2+l^2}z} \end{aligned}$$

So for the boundary conditions (ii)

$$\begin{aligned} \left.\frac{\mathrm{d}^2X}{\mathrm{d}x^2}\right|_{x=0} &= -A^2k^2\sin^2(0) - B^2k^2\cos^2(0) = 0 \implies B = 0 \\ \left.\frac{\mathrm{d}^2X}{\mathrm{d}x^2}\right|_{x=a} &= -A^2k^2\sin^2(ka) = 0 \implies k = \frac{n\pi}{a} \quad \text{for} \quad n = 1, 2, 3, \cdots \\ \left.\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}\right|_{y=0} &= -C^2l^2\sin^2(0) - D^2l^2\cos^2(0) = 0 \implies D = 0 \\ \left.\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}\right|_{y=a} &= -C^2l^2\sin^2(la) = 0 \implies l = \frac{m\pi}{a} \quad \text{for} \quad m = 1, 2, 3, \cdots \\ \left.\frac{\mathrm{d}^2Z}{\mathrm{d}z^2}\right|_{z=0} &= E(k^2 + l^2) + F(k^2 + l^2) = 0 \implies F = -E \end{aligned}$$

thus

$$\begin{aligned} Z(z) &= Ee^{\sqrt{k^2+l^2}z} - Ee^{-\sqrt{k^2+l^2}z} \\ &= 2E\sinh\left(\sqrt{k^2 + l^2}z\right) \\ &= 2E\sinh\left(\sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{a})^2 z}\right) \\ &= 2E\sinh\left(\frac{\pi}{a}\sqrt{n^2 + m^2 z}\right) \end{aligned}$$

The potential is now

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi}{a}\sqrt{n^2 + m^2 z}\right)$$

where we can solve for the Constants C_{nm} using the boundary condition (i) $V = V_0$ at $z = a$:

$$V_0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{\pi}{a}\sqrt{n^2 + m^2 a}\right)$$

Doing the Fourier series by multiplying by $\sin\left(\frac{n'\pi x}{a}\right)\sin\left(\frac{m'\pi y}{a}\right)$ and integrating over $x, y = [0, a]$:

$$\begin{aligned}&\int_0^a\int_0^a V_0\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dxdy\\ &\quad=\sum\sum C_{nm}\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}a\right)\int_0^a\int_0^a \sin^2(\frac{n\pi x}{a})\sin^2(\frac{m\pi y}{a})dxdy\end{aligned}$$

where the integral

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)dx = \begin{cases}\frac{a}{2} & \text{if } n = n'\\ 0 & \text{if } n \neq n'\end{cases}$$

So

$$V_0\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dxdy = C_{nm}\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}a\right)\frac{a^2}{4}$$

where

$$C_{nm}\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}a\right) = \begin{cases}0 & \text{if } n \text{ or } m \text{ is even}\\ \frac{16V_0}{\pi^2 nm} & \text{if } n \text{ and } m \text{ are odd}\end{cases}$$

Thus the potential is

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n\text{ odd}}\sum_{m\text{ odd}}\frac{1}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\frac{\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}z\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}}$$

The potential at the center is

$$V(a/2,a/2,a/2) = \frac{16V_0}{\pi^2}\sum_{n\text{ odd}}\sum_{m\text{ odd}}\frac{1}{nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\frac{\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}a/2\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}$$

where a calculator gives us

$$V(a/2,a/2,a/2) \approx 0.167V_0$$

Which roughly $V_0/6$ or a sixth of the potential in the center of a cube with V_0 on each face.

3.25 The potential inside and outside the sphere is given by (Griffiths 3.78 & 3.79)

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & (r \le R) \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) & (r \ge R) \end{cases}$$

where (3.81 & 3.84)

$$B_l = A_l R^{2l+1} \tag{3.81}$$

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta) \sin\theta d\theta \tag{3.84}$$

Since the charge density is constant $\sigma_0(\theta) = \sigma_0$ and using $u = \cos\theta; \cos(0) = 1, \cos(\pi) = -1$ and $du = -\sin\theta d\theta$ we get

$$A_l = -\frac{\sigma_0}{2\epsilon_0 R^{l-1}} \int_1^{-1} P_l(u) du$$

where $P_l(u)$ is odd for odd l and even for even l so the integral is zero for odd l and non-zero for even l:

$$\int_1^{-1} P_l(u) du = \begin{cases} 0 & \text{if } l \text{ is even} \\ -2\int_0^1 P_l(u) du & \text{if } l \text{ is odd} \end{cases}$$

Thus

$$A_l = -\frac{\sigma_0}{2\epsilon_0 R^{l-1}} (-2) \int_0^1 P_l(u) du = \begin{cases} 0 & \text{if } l \text{ is odd} \\ \frac{\sigma_0}{\epsilon_0 R^{l-1}} \int_0^1 P_l(u) du & \text{if } l \text{ is even} \end{cases}$$

So for the odd P_l

$$\begin{aligned} \int_0^1 P_1(u) du &= \int_0^1 u du = \frac{1}{2} \\ \int_0^1 P_3(u) du &= \frac{1}{2} \int_0^1 (5u^3 - 3u) du = -\frac{1}{8} \\ \int_0^1 P_5(u) du &= \frac{1}{8} \int_0^1 (63u^5 - 70u^3 + 15u) du = \frac{1}{16} \end{aligned}$$

and for the even P_l the integral is zero; therfore

$$A_0 = A_2 = A_4 = A_6 = 0$$

$$A_1 = \frac{\sigma_0}{\epsilon_0} \frac{1}{2}, \quad A_3 = \frac{\sigma_0}{\epsilon_0 R^2} \left(\frac{-1}{8}\right), \quad A_5 = \frac{\sigma_0}{\epsilon_0 R^4} \frac{1}{16}$$

and

$$B_0 = B_2 = B_4 = B_6 = 0$$

$$B_1 = \frac{\sigma_0}{\epsilon_0} R^3 \frac{1}{2}, \quad B_3 = \frac{\sigma_0}{\epsilon_0} R^5 \left(\frac{-1}{8}\right), \quad B_5 = \frac{\sigma_0}{\epsilon_0} R^7 \frac{1}{16}$$

So finally the potential is

$$\boxed{V(r,\theta) = \frac{\sigma_0}{\epsilon_0} \left[\frac{r}{2} P_1(\cos\theta) - \frac{r^3}{8R^2} P_3(\cos\theta) + \frac{r^5}{16R^4} P_5(\cos\theta)\right] \quad (r \le R)}$$

and

$$\boxed{V(r,\theta) = \frac{\sigma_0}{\epsilon_0} \left[\frac{R^3}{2r^2} P_1(\cos\theta) - \frac{R^5}{8r^4} P_3(\cos\theta) + \frac{R^7}{16r^6} P_5(\cos\theta)\right] \quad (r \ge R)}$$

3.29 Given charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta$$

for a sphere of radius R and constant k. The monopole term is

$$\int \rho \mathrm{d}\tau = kR \int \left(\frac{1}{r^2}R - 2r\sin\theta\right) r^2 \sin\theta dr d\theta d\phi$$

Where the radial integral is

$$\int_0^R R - 2r dr = 0$$

Moving on to the dipole term

$$\int r\cos\theta \rho \mathrm{d}\tau = kR \int (r\cos\theta)(R-2r)\sin\theta \sin\theta dr d\theta d\phi$$

where the polar integral is

$$\int_0^\pi \sin\theta\cos\theta d\theta = 0$$

Maybe the quadrupole term will be non-zero:

$$\begin{aligned}
\int r^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho \mathrm{d}\tau &= \frac{1}{2}kR \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \big((3\cos^2\theta - 1)(R-2r)\sin\theta\big) dr d\theta d\phi \\
&= \frac{1}{2}kR(2\pi)\Big(-\frac{\pi}{8}\Big)\Big(-\frac{R^4}{6}\Big) \\
&= \frac{\pi^2 kR^5}{48}
\end{aligned}$$

So far from the sphere for points on the z-axis the potential is

$$\boxed{V(z) = \frac{1}{4\pi\epsilon_0}\frac{\pi^2 KR^5}{48z^3}}$$

3.38 Given

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) \tag{3.103}$$

and

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \tag{3.104}$$

The dipole moment has vector components

$$\mathbf{p} = p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\theta}$$

so now we can sub this into (3.104):

$$\begin{aligned}
\mathbf{E}_{\text{dip}}(r,\theta) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}3(p\cos\theta)\hat{\mathbf{r}} - (p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\theta}) \\
\text{Eq. } 3.103 &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta})
\end{aligned}$$

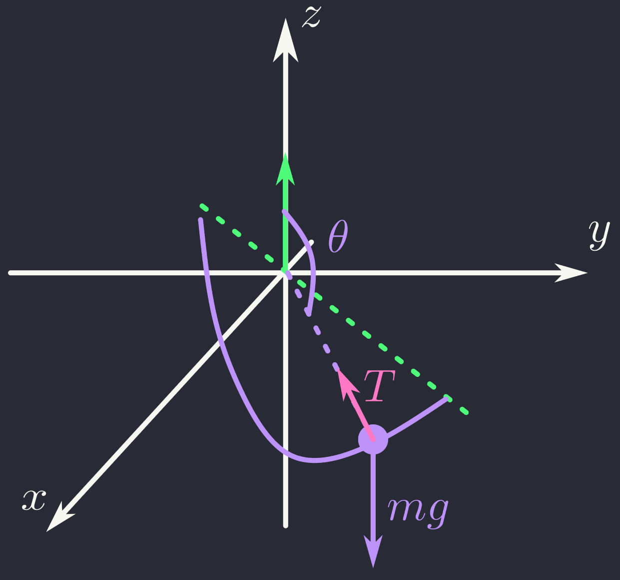

Figure 4.1: The electric field of a dipole at a distance r from the origin.

3.61 Given the potential from (3.103) and the force

$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$$

Pretending this is a pendulum as shown in Fig. 4.1 the force is

$$\mathbf{F} = -T\hat{\mathbf{r}} - mg\hat{\mathbf{z}} = -T\hat{\mathbf{r}} - mg(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta})$$

T is the tension given by the centripetal acceleration

$$\begin{aligned} ma_c &= T - mg\cos(\pi - \theta) \\ m\frac{v^2}{r} &= T + mg\cos\theta \end{aligned}$$

where v is the velocity given by energy conservation

$$\begin{aligned} KE &= \Delta PE \\ \frac{1}{2}mv^2 &= -mgr\cos\theta \\ \implies v^2 &= -2gr\cos\theta \end{aligned}$$

So

$$T = m\frac{v^2}{r} - mg\cos\theta = -2mg\cos\theta - mg\cos\theta = -3mg\cos\theta$$

Thus we get the force

$$\begin{aligned} \mathbf{F} &= 3mg\cos\theta\hat{\mathbf{r}} - mg(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}) \\ &= mg(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) \end{aligned}$$

Which is telling us that the electric charge will swing in an arc like a pendulum i.e.

$$mg \equiv \frac{qp}{4\pi\epsilon_0 r^3}$$

or we can think of the charge q as an inertial mass and group the other terms as the gravitational constant.