Homework 5

Due 2/21

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1. (a) If f is independent of y, then

$$\frac{\partial f}{\partial y} = 0$$

and using the Euler-Lagrange (EQ) equation, we have

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

so

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

and for the derivative to be zero,

$$\frac{\partial f}{\partial y'} = \text{constant}$$

(b) Since f is independent of x,

$$\frac{\partial f}{\partial x} = 0$$

Using the Euler-Lagrange equation, we have

$$\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)$$

Using chain rule to differentiate f with respect to x for a general function $f(x, y, y')$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x, y, y') &= \frac{\partial f}{\partial x}\left(\frac{\mathrm{d}}{\mathrm{d}x} x\right) + \frac{\partial f}{\partial y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{\partial f}{\partial y'}\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
\frac{\mathrm{d}}{\mathrm{d}x} f(y, y') &= 0 + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''
\end{aligned}$$

and substituting what we got from the EL equation,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(y, y') = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] y' + \frac{\partial f}{\partial y'} y'' = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] y' + \frac{\partial f}{\partial y'}\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x} f = \frac{\mathrm{d}}{\mathrm{d}x}\left(y' \frac{\partial f}{\partial y'}\right)$$

from chain rule. Moving everything to one side:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f - \frac{\mathrm{d}}{\mathrm{d}x}\left(y' \frac{\partial f}{\partial y'}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(f - y' \frac{\partial f}{\partial y'}\right) &= 0
\end{aligned}$$

which is only true if

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}$$

2. (a) In 2D, we know the length of a short segment is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

and in 3D

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$$

transforming to spherical coordinates:

$$\begin{aligned}
&x = r\cos\phi\sin\theta \quad \mathrm{d}x = \mathrm{d}r\cos\phi\sin\theta - r\sin\phi\sin\theta\mathrm{d}\phi + r\cos\phi\cos\theta\mathrm{d}\theta \\
&y = r\sin\phi\sin\theta \quad \mathrm{d}y = \mathrm{d}r\sin\phi\sin\theta + r\cos\phi\sin\theta\mathrm{d}\phi + r\sin\phi\cos\theta\mathrm{d}\theta \\
&z = r\cos\theta \quad \mathrm{d}z = \mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta
\end{aligned}$$

for the sphere of radius $r \to R$ and $\mathrm{d}r = 0$ since radius is constant, so

$$\begin{aligned}
\mathrm{d}x &= -R\sin\phi\sin\theta\mathrm{d}\phi + R\cos\phi\cos\theta\mathrm{d}\theta \\
\mathrm{d}y &= R\cos\phi\sin\theta\mathrm{d}\phi + R\sin\phi\cos\theta\mathrm{d}\theta \\
\mathrm{d}z &= -R\sin\theta\mathrm{d}\theta
\end{aligned}$$

and squaring each term:

$$\begin{aligned}
\mathrm{d}x^2 &= R^2\sin^2\phi\sin^2\theta\mathrm{d}\phi^2 + R^2\cos^2\phi\cos^2\theta\mathrm{d}\theta^2 - 2R^2\sin\phi\sin\theta\cos\phi\cos\theta\mathrm{d}\phi\mathrm{d}\theta \\
\mathrm{d}y^2 &= R^2\cos^2\phi\sin^2\theta\mathrm{d}\phi^2 + R^2\sin^2\phi\cos^2\theta\mathrm{d}\theta^2 + 2R^2\sin\phi\sin\theta\cos\phi\cos\theta\mathrm{d}\phi\mathrm{d}\theta \\
\mathrm{d}z^2 &= R^2\sin^2\theta\mathrm{d}\theta^2
\end{aligned}$$

when we add all three equations we can see that the last term in $\mathrm{d}x^2$ and $\mathrm{d}y^2$ cancel out, and grouping the like terms we get

$$\begin{aligned}
R^2\sin^2\phi\sin^2\theta\mathrm{d}\phi^2 + R^2\cos^2\phi\sin^2\theta\mathrm{d}\phi^2 &= R^2\sin^2\theta\mathrm{d}\phi^2(\sin^2\phi + \cos^2\phi) \\
&= R^2\sin^2\theta\mathrm{d}\phi^2
\end{aligned}$$

and

$$\begin{aligned}
&R^2\cos^2\phi\cos^2\theta\mathrm{d}\theta^2 + R^2\sin^2\phi\cos^2\theta\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\theta^2 \\
&\qquad = R^2\cos^2\theta\mathrm{d}\theta^2(\cos^2\phi + \sin^2\phi) + R^2\sin^2\theta\mathrm{d}\theta^2 \\
&\qquad = R^2\mathrm{d}\theta^2(\cos^2\theta + \sin^2\theta) \\
&\qquad = R^2\mathrm{d}\theta^2
\end{aligned}$$

so the length of a short segment in spherical coordinates is

$$\begin{aligned}
\mathrm{d}s &= \sqrt{R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2} \\
&= \sqrt{R^2\mathrm{d}\theta^2\left(\frac{\mathrm{d}\theta^2}{\mathrm{d}\theta^2} + \sin^2\theta\frac{\mathrm{d}\phi^2}{\mathrm{d}\theta^2}\right)} \\
\text{using}\quad \frac{\mathrm{d}\phi}{\mathrm{d}\theta} &= \phi'(\theta) \\
&= R\sqrt{1 + \sin^2\theta\phi'(\theta)^2}\mathrm{d}\theta
\end{aligned}$$

and the total path length L is found by integrating $\mathrm{d}s$ from θ_a to θ_b:

$$L = R\int_{\theta_a}^{\theta_b}\sqrt{1 + \sin^2\theta\phi'(\theta)^2}\mathrm{d}\theta$$

(b) We find that the integral function is independent of ϕ or

$$f = f(\theta, \phi') = \sqrt{1 + \sin^2\theta\phi'(\theta)^2}$$

so from Problem 1a, we know that

$$\frac{\partial f}{\partial \phi'} = \text{constant} = C$$
$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta \phi'(\theta)}{\sqrt{1+\sin^2\theta \phi'(\theta)^2}}$$

setting one of the points to be the north pole, $\theta_a = 0$, so the constant is

$$\frac{\sin^2(0)\phi'(0)}{\sqrt{1+\sin^2(0)\phi'(0)^2}} = 0 = C$$

solving for $\phi'(\theta)$

$$\frac{\sin^2\theta \phi'(\theta)}{\sqrt{1+\sin^2\theta \phi'(\theta)^2}} = 0$$
$$\phi'(\theta) = 0$$

using separation of variables,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = 0$$
$$\int \mathrm{d}\phi = \int 0 \mathrm{d}\theta$$
$$\phi(\theta) = C_2$$

Since ϕ is a constant, this is equivalent to to a slice of the sphere through the north pole, which has a cross section of a circle with radius R. The path follows the circumference of the circle, from the north pole to the point at θ_b.

3. (a) Finding the 2D length element in polar coordinates:

$$x = r\cos\phi \quad \mathrm{d}x = \mathrm{d}r\cos\phi - r\sin\phi\mathrm{d}\phi$$
$$y = r\sin\phi \quad \mathrm{d}y = \mathrm{d}r\sin\phi + r\cos\phi\mathrm{d}\phi$$

so the length element is

$$\begin{aligned}
\mathrm{d}x^2 + \mathrm{d}y^2 &= (\mathrm{d}r\cos\phi - r\sin\phi\mathrm{d}\phi)^2 + (\mathrm{d}r\sin\phi + r\cos\phi\mathrm{d}\phi)^2 \\
&= \mathrm{d}r^2\cos^2\phi + r^2\sin^2\phi\mathrm{d}\phi^2 - 2r\sin\phi\mathrm{d}r\mathrm{d}\phi \\
&+ \mathrm{d}r^2\sin^2\phi + r^2\cos^2\phi\mathrm{d}\phi^2 + 2r\cos\phi\mathrm{d}r\mathrm{d}\phi \\
&= \mathrm{d}r^2(\cos^2\phi + \sin^2\phi) + r^2\mathrm{d}\phi^2(\sin^2\phi + \cos^2\phi) \\
&= \mathrm{d}r^2 + r^2\mathrm{d}\phi^2
\end{aligned}$$

so the length element in polar coordinates is

$$\mathrm{d}l = \sqrt{\mathrm{d}r^2 + r^2\mathrm{d}\phi^2}$$

and solving for $\mathrm{d}\phi$,

$$\begin{aligned}
\mathrm{d}l^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 \\
\mathrm{d}\phi^2 &= \frac{\mathrm{d}l^2 - \mathrm{d}r^2}{r^2} \\
\mathrm{d}\phi &= \frac{1}{r}\sqrt{\mathrm{d}l^2 - \mathrm{d}r^2} \\
\mathrm{d}\phi &= \frac{1}{r}\sqrt{1 - \left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)^2}\mathrm{d}l
\end{aligned}$$

plugging into the area integral:

$$\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2}r^2\mathrm{d}\phi \\
&= \int_0^L \frac{1}{2}r^2\left[\frac{1}{r}\sqrt{1 - \left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)^2}\mathrm{d}l\right] \\
&= \frac{1}{2}\int_0^L r\sqrt{1 - \left(\frac{\mathrm{d}r}{\mathrm{d}l}\right)^2}\mathrm{d}l
\end{aligned}$$

so the function in the integrand is

$$f = f(r, r') = r\sqrt{1 - r'^2}$$

(b) From Problem 1b we know that the conserved quantity we know that f is independent of l, so

$$f - r'\frac{\partial f}{\partial r'} = \text{constant} = K$$

using the partial derivative

$$\frac{\partial f}{\partial r'} = \frac{r}{2\sqrt{1 - r'^2}}(-2r') = -\frac{rr'}{\sqrt{1 - r'^2}}$$

So the conserved quantity is

$$\begin{aligned}
K &= r\sqrt{1 - r'^2} + \frac{rr'^2}{\sqrt{1 - r'^2}} \\
&= \frac{r(1 - r'^2) + rr'^2}{\sqrt{1 - r'^2}} \\
K &= \frac{r}{\sqrt{1 - r'^2}}
\end{aligned}$$

rearranging for r';

$$\begin{aligned}
K\sqrt{1-r'^2} &= r \\
K^2(1-r'^2) &= r^2 \\
K^2 - K^2 r'^2 &= r^2 \\
K^2 r'^2 &= r^2 - K^2 \\
r'^2 &= \frac{r^2}{K^2} - 1 \\
r' &= \sqrt{\frac{r^2}{K^2} - 1} = \frac{\mathrm{d}r}{\mathrm{d}l}
\end{aligned}$$

using separation of variables:

$$\mathrm{d}l = \frac{\mathrm{d}r}{\sqrt{r^2/K^2 - 1}}$$

using the substitution $u = r/K$ and $\mathrm{d}u = \mathrm{d}r/K$:

$$\begin{aligned}
\mathrm{d}l &= \frac{K\mathrm{d}u}{\sqrt{u^2-1}} \\
\int \mathrm{d}l &= K \int \frac{\mathrm{d}u}{\sqrt{u^2-1}} \\
l &= K \operatorname{arccosh}(r/K) + C \\
\operatorname{arccosh}(r/K) &= \frac{l-C}{K} \\
r &= K \cosh\left(\frac{l-C}{K}\right)
\end{aligned}$$

Since the constraint l is constant as the total length of the curve (K & C are also constants), $r =$ constant is a solution to the equation or the radius of the curve is constant. This is only true for circles which have a constant radial distance from the origin, so circles leads to the maximum area integral.

4. (a) The kinetic energy of the particle is

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$$

so the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - U$$

where $U = U(r,\theta,\phi)$ and the Euler-Lagrange equations are

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right)\\
\frac{\partial\mathcal{L}}{\partial\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)\\
\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right)
\end{aligned}$$

For the first equation:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial r} &= mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - \frac{\partial}{\partial r}U(r)\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = m\ddot{r}
\end{aligned}$$

so

$$\begin{aligned}
m\ddot{r} &= mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - \frac{\partial U}{\partial r}\\
-\frac{\partial U}{\partial r} &= m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)
\end{aligned}$$

For the second equation:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= mr^2\dot{\phi}^2\sin\theta\cos\theta - \frac{\partial U}{\partial\theta}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}
\end{aligned}$$

so

$$\begin{aligned}
mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= mr^2\dot{\phi}^2\sin\theta\cos\theta - \frac{\partial U}{\partial\theta}\\
-\frac{\partial U}{\partial\theta} &= m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\dot{\phi}^2\sin\theta\cos\theta)\\
-\frac{1}{r}\frac{\partial U}{\partial\theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)
\end{aligned}$$

and for the third equation:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= -\frac{\partial U}{\partial\phi}\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\sin^2\theta\right)\\
&= 2mr\dot{r}\dot{\phi}\sin^2\theta + mr^2\ddot{\phi}\sin^2\theta + 2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta
\end{aligned}$$

so

$$\begin{aligned}
mr^2\ddot{\phi}\sin^2\theta &= -2mr\dot{r}\dot{\phi}\sin^2\theta - 2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta - \frac{\partial U}{\partial\phi}\\
-\frac{\partial U}{\partial\phi} &= m(r^2\ddot{\phi}\sin^2\theta + 2r\dot{r}\dot{\phi}\sin^2\theta + 2r^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta)\\
-\frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi} &= m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)
\end{aligned}$$

where we have the 3 equations of motion:

$$\begin{aligned}
-\frac{\partial U}{\partial r} &= m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta) \\
-\frac{1}{r}\frac{\partial U}{\partial \theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) \\
-\frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} &= m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)
\end{aligned}$$

where from N2L in spherical coordinates, the components of acceleration are

$$\begin{aligned}
a_r &= \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta \\
a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta \\
a_\phi &= r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta
\end{aligned}$$

and the conservative force is

$$\mathbf{F} = -\boldsymbol{\nabla} U = -\frac{\partial U}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

To compare with N2L we start with the unit vectors in spherical coordinates:

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
\hat{\theta} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\
\hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}
\end{aligned}$$

and from the velocity equation we know that the derivative of the radial unit vector is

$$\dot{\hat{r}} = \dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}$$

for the colatitude unit vector in the θ direction

$$\begin{aligned}
\dot{\hat{\theta}} &= (-\dot{\theta}\sin\theta\cos\phi - \dot{\phi}\cos\theta\sin\phi)\hat{\mathbf{x}} + (-\dot{\theta}\sin\theta\sin\phi + \dot{\phi}\cos\theta\cos\phi)\hat{\mathbf{y}} - \dot{\theta}\cos\theta\hat{\mathbf{z}} \\
&= \dot{\phi}\cos\theta\hat{\boldsymbol{\phi}} - \dot{\theta}\hat{\mathbf{r}}
\end{aligned}$$

(linear combination of unit vectors) and for the azimuthal unit vector in the ϕ direction

$$\begin{aligned}
\dot{\hat{\phi}} &= -\dot{\phi}\sin\phi\hat{\mathbf{x}} - \dot{\phi}\cos\phi\hat{\mathbf{y}} \\
&= -\dot{\phi}\sin\theta\hat{\mathbf{r}} - \dot{\phi}\cos\theta\hat{\boldsymbol{\theta}}
\end{aligned}$$

with this in hand taking the time derivative of velocity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}\right)$$

the first term is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{\mathbf{r}}) &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}
\end{aligned}$$

the second term is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\theta}\hat{\theta}\right) &= (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}} + r\dot{\theta}(\dot{\phi}\cos\theta\hat{\boldsymbol{\phi}} - \dot{\theta}\hat{\mathbf{r}}) \\
&= (-r\dot{\theta}^2)\hat{\mathbf{r}} + (\dot{r}\ddot{\theta})\hat{\boldsymbol{\theta}} + (r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}}
\end{aligned}$$

and the third term is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}\right) &= (\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}} + r\dot{\phi}\sin\theta(-\dot{\phi}\sin\theta\hat{\mathbf{r}} - \dot{\phi}\cos\theta\hat{\boldsymbol{\theta}}) \\ &= (-r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + (-r\dot{\phi}^2\sin\theta\cos\theta)\hat{\boldsymbol{\theta}} + (\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}}\end{aligned}$$

so combing all the terms:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\boldsymbol{\theta}} \\ &\quad + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}}\end{aligned}$$

5. (a) In the frame where the cart is at rest at $x' = x - v_o t$ this is simply the Brachistochrone where the time of travel is

$$T = \int_A^B \frac{\mathrm{d}s}{v}$$

and the short segment length is

$$\mathrm{d}s = \sqrt{\mathrm{d}x'^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x'}\right)^2}\,\mathrm{d}x' \quad \text{or} \quad \sqrt{1+y'^2}\mathrm{d}x$$

and from the conservation of energy

$$\frac{1}{2}mv^2 = mgy \quad \text{or} \quad v = \sqrt{2gy}$$

so

$$T = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\mathrm{d}x'$$

where the integral function is independent of x;

$$f = f(y, y') = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

so using the second form of the EL equation from Problem 1b, we have

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

using using the partial derivative

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gy}}\frac{y'}{\sqrt{1+y'^2}}$$

so the conserved quantity is

$$\begin{aligned}
C &= \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - \frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} \\
&= \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}}\left[\frac{\cancel{\sqrt{2gy}}\sqrt{1+y'^2}\sqrt{1+y'^2}}{\cancel{\sqrt{2gy}}} - y'^2\right] \\
&= \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} \\
\text{or} \quad 2C^2 g &= \frac{1}{y(1+y'^2)}
\end{aligned}$$

setting a new constant

$$2C^2 g = \frac{1}{2a} \quad \text{where} \quad C = \sqrt{\frac{1}{4ga}}$$

we can now solve for y':

$$\begin{aligned}
\frac{1}{2a} &= \frac{1}{y(1+y'^2)} \\
1 + y'^2 &= \frac{2a}{y} \\
y'^2 &= \frac{2a}{y} - 1 \\
y' &= \sqrt{\frac{2a}{y} - 1} \quad \text{or} \quad \sqrt{\frac{2a-y}{y}}
\end{aligned}$$

using separation of variables:

$$\frac{\mathrm{d}y}{\mathrm{d}x'} = \sqrt{\frac{2a-y}{y}}$$
$$\int \mathrm{d}y \sqrt{\frac{y}{2a-y}} = \int \mathrm{d}x'$$

and using the substitution $y = a(1-\cos\theta)$; $\quad \mathrm{d}y = \mathrm{d}\theta a \sin\theta$ and

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\cos\theta}\sqrt{1+\cos\theta}$$

so

$$\begin{aligned} x' &= \int \mathrm{d}\theta a \sin\theta \sqrt{\frac{a(1-\cos\theta)}{2a - a(1-\cos\theta)}} \\ &= \int \mathrm{d}\theta a \sqrt{1-\cos\theta}\cancel{\sqrt{1+\cos\theta}}\frac{\sqrt{1-\cos\theta}}{\cancel{\sqrt{1+\cos\theta}}} \\ &= a\int \mathrm{d}\theta(1-\cos\theta) = a(\theta - \sin\theta) \end{aligned}$$

since the $\theta = \omega t$ we have the parametric equation for the path in the reference frame

$$x'(t) = a(\omega t - \sin(\omega t))$$
$$y(t) = a(1-\cos(\omega t))$$

since $x = x' + v_o t$ the x position in the original frame is

$$x(t) = a(\omega t - \sin(\omega t)) + v_o t$$

(b) Using the initial conditions

$$x = y = 0,\ \dot{x} = v_o,\ \dot{y} = 0$$

we first solve for ω:

$$\dot{x} = a\omega(1-\cos(\omega t)) + v_o$$
$$\dot{y} = a\omega\sin(\omega t)$$

we know at $t=0$ that $\dot{x} = v_o$ and $\dot{y} = 0$. Since $\ddot{y} = g$ from the gravitational force we can solve for ω:

$$\ddot{y}(t) = a\omega^2\cos(\omega t)$$
$$\ddot{y}(0) = a\omega^2 = g \implies \omega = \sqrt{\frac{g}{a}}$$

At the boundary point B we know that the cycloid completes one cycle so $\omega t = 2\pi$:

$$x'(t_B) = a(2\pi - \sin(2\pi)) = L$$
$$L = 2\pi a \implies a = \frac{L}{2\pi} \implies \omega = \sqrt{\frac{2\pi g}{L}}$$

so

$$x(t) = \frac{L}{2\pi}\left[\sqrt{\frac{2\pi g}{L}}t - \sin\left(\sqrt{\frac{2\pi g}{L}}t\right)\right] + v_o t$$
$$y(t) = \frac{L}{2\pi}\left[1 - \cos\left(\sqrt{\frac{2\pi g}{L}}t\right)\right]$$

(c) Sketching the path

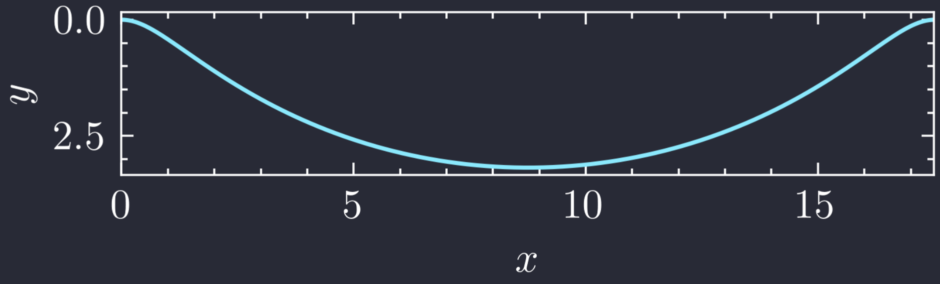

Figure 5.1: Numerically computed track shape