1 Bound States

Two Types:

- Binding Energy < Rest mass energy: Nonrelativistic bound state e.g. Hydrogen atom (-13.6 eV < 1GeV rest mass of proton).
- Binding Energy > Rest mass energy: Relativistic bound state e.g. light meson.

Hydrogen Atom: The potential energy is given by

$$V(r) = -\frac{e^2}{r}$$

or the coulomb potential. The Hamiltonian is given by the Schrödinger equation

$$H\psi = -\frac{\hbar}{2m}\nabla^2\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

where $V(r)$ is the central potential with spherical symmetry SO(3). But there also is an enhanced symmetry.

Noether's Theorem: Symmetry $\leftrightarrow$ Conservation Law. e.g.

- SO(3) $\leftrightarrow$ Conservation of Angular momentum.
- SO(1,3) $\leftrightarrow$ linear momentum (Poincare symmetry)
- T-reversal $\leftrightarrow$ energy
- $\mathrm{U}(1)_{em}$ $\leftrightarrow$ electric charge

so from the central potential, we know that angular momentum $\mathbf{L}$ is conserved. But for $1/r$ there is a SO(4) symmetry from the LRL (Laplace-Runge-Lenz) vector

$$\mathcal{L} = \frac{1}{m}\mathbf{L}\times\mathbf{p} + \frac{\kappa\mathbf{r}}{r}$$

where

$$V(r) = -\frac{\kappa}{r}$$

the energy eigenvalues of the hydrogen atom are given by

$$E_n = -\frac{13.6\,\text{eV}}{n^2} = -\frac{me^4}{2\hbar^2 n^2} = -\frac{1}{2}\frac{\alpha m_e c^2}{n^2}$$

where $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine structure constant.

Degeneracy n^2 e.g. For SO(3), $(2l+1)$ degeneracy

$$\sum_{l=0}^{n-1}(2l+1) = 2\sum_{l=0}^{n-1} l + n = 2\frac{(n-1)(n)}{2} + n = n^2$$

n	l	m	degeneracy
1	0	0	1
2	0	0	1
	1	-1,0,1	3
3	0	0	1
	1	-1,0,1	3
	2	-2,-1,0,1,2	5

Positronium (e^+e^- bound state) has the same energy levels as the hydrogen atom the energy eigen value is given by first looking at the reduced mass

$$\mu = \frac{1}{\frac{1}{m_e} + \frac{1}{m_e}} = \frac{m_1 m_2}{m_1 + m_2} \approx m_1 \quad \text{if} \quad m_1 \ll m_2$$

but here $m_1 = m_2 = m_e$ so $\mu = \frac{m_e}{2}$. The energy eigenvalues are given by

$$E_n = \frac{1}{2} - \frac{13.6\,\mathrm{eV}}{n^2} = -\frac{6.8\,\mathrm{eV}}{n^2}$$

We can do this for Muonium (μ^+e^- bound state) and Pionic Hydrogen(π^+e^- bound state).

Fine Structure

1. Relativistic Correction

$$T = E - m_e c^2$$

2. Spin-Orbit Coupling
3. Lambd Shift (QED)
4. Hyperfine Splitting aka zeeman effect

Quiz Review

- For the Positronium:

$$C : (-1)^{l+s} = (-1)^n$$

where $l + s = n$ (the selection rule for Positronium decay). *for n photons, $C = (-1)^n$. FOr the ground states $l = 0$ so the spin is

$$S : \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

where we have a triplet state $S = 1$ and a singlet state $S = 0$. For this singlet:

$$S = 0 \implies (-1)^0 = 1 = (-1)^2$$

or two photons can be emitted (para-positronium). For the triplet state:

$$S = 1 \implies (-1)^1 = -1 = (-1)^3$$

or three photons can be emitted (ortho-positronium). The mass of each photon for two photons is roughtly a half of the mass of the positronium $E_\gamma = 511\,\text{keV}$. For three photons $E_\gamma < 511$ keV.

- Binding Energy vs. Rest Mass Energy: Quarkonium ($q\bar{q}$): *uds* light quarks, *cbt* heavy quarks.
  - Heavy Quarkonium: $c\bar{c}$: Charmonium (J/ψ), $b\bar{b}$: Bottomonium (Υ), $t\bar{t}$: Toponium *does not exist* (very heavy so it decays really fast $\sim 10^{-25}$s vs $\tau_{\text{bound state}} \sim 10^{-23}$ sec).

For Charmoniun, the reduced mass is

$$\mu = \frac{m_c m_c}{m_c + m_c} \approx \frac{m_c}{2}$$

and the energy of the Hydrogen atom is

$$E_n = -\frac{me^4}{2\hbar^2 n^2} = -\frac{1}{2}\frac{\alpha mc^2}{n^2}$$

and for the Charmonium:

$$E_n = -\frac{4}{9}\frac{1}{2}\frac{\alpha m_c c^2}{n^2} \quad \text{incorrect}$$

where we have to adjust for the charge of the quark $e \to \frac{2}{3}e$ and the potential: For electron coulomb potential we know that

$$V = -\frac{e^2}{r} = -\frac{e^2}{\hbar r}\frac{\hbar c}{r} = -\frac{\alpha \hbar c}{r}$$

but for quarks there is a different potential from the strong interaction (gluon)

$$V(r) = -\frac{\alpha_s \hbar c}{r} - \frac{4}{9}\frac{\alpha \hbar c}{r} \qquad \alpha_s = \frac{g_s^2}{\hbar c} \gg \alpha$$

which is much larger than the coulomb potential (suppressed second term), but there is a transition to a linear potential as the distance get very large.

$$V(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} + F_o r \quad \text{QCD Potential}$$

there also is a color factor $\frac{4}{3}$ based on the three colors of the quarks. So the energy is given by

$$E_n = -\frac{4}{3}\frac{1}{2}\frac{\alpha_s m_c c^2}{n^2}$$

- Decay of Charmonium:

$$J/\psi \to \pi^+\pi^-\pi^0 \quad \text{or} \quad D^+D^-$$

For the ground state, $m_{J/\psi} = 3.1$ GeV. And the total rest mass of D^+D^- is kinematically forbidden $m_{D^+} + m_{D^-} = 3.7$ GeV. We have a decay to 3 pions due to the G-parity conservation $(-1)^I C$ or $(-1)^n$.

OZI rule (Okubo, Zweig, Iizuka) Cutting a hard gluon line in the Feynman diagram separates the quarks and the decay is suppressed. For soft gluon lines, cutting a line does not separate the quarks and the decay is not suppressed.

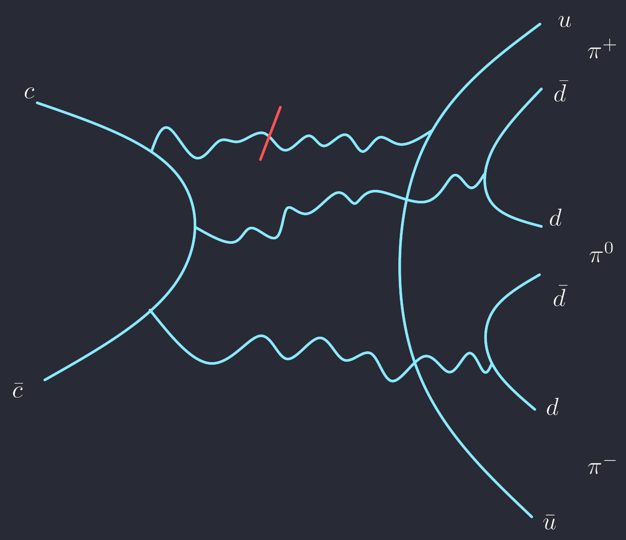

Figure 1.1: OZI Rule

- Light Mesons: $q\bar{q}$ where $q = u, d, s$. There are nine spin-0 (pseudo scalar) mesons and nine spin-1 (vector) mesons. (insert figure 5.11 from Griffiths). From the lie algebra of the spin-0 nonet

$$3 \otimes \bar{3} = 8 \oplus 1$$

where the 1 is the η' meson. and we break down the 8 into

$$8 \to 2 \oplus 3 \oplus 2 \oplus 1$$

where they refer to the top row, middle row pions, bottom row and η meson. For the vector mesons. For the isospin doublet:

$$u = \left|\frac{1}{2}\frac{1}{2}\right\rangle \qquad d = \left|\frac{1}{2} - \frac{1}{2}\right\rangle$$

and for the antiquarks:

$$\bar{u} = \left|\frac{1}{2} - \frac{1}{2}\right\rangle \qquad \bar{d} = -\left|\frac{1}{2}\frac{1}{2}\right\rangle$$

and the pions are given by

$$\begin{aligned}
\pi^+ &= \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle = -u\bar{d} \\
\pi^- &= \left|\frac{1}{2} - \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} - \frac{1}{2}\right\rangle = d\bar{u} \\
\pi^0 &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2} - \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle + \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} - \frac{1}{2}\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})
\end{aligned}$$

for the corner mesons:

$$K^0 = d\bar{s} \qquad \bar{K}^0 = \bar{d}s \qquad K^+ = u\bar{s} \qquad K^- = \bar{u}s$$

and the η mesons are

$$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$
$$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

For spin-1 mesons, in terms of the flavor

$$\rho^+, \rho^0, \rho^- = \pi^+, \pi^0, \pi^-$$

and the same for the K^* mesons. The difference is in the center

$$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \qquad \phi = s\bar{s}$$