4.15 Thick spherical shell of inner radius a, outer radius b, with polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}}$$

(a) E-field in all regions using bound charges: The volume bound charge is

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$$

and the surface bound charges are ($\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ at $r = a$)

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} -\dfrac{k}{a} & r = a \\ \dfrac{k}{b} & r = b \end{cases}$$

Using Gauss's Law

$$\oint \mathbf{E}\cdot \mathrm{d}\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

(i) $r < a$: $Q_{\text{enc}} = 0$, so $\mathbf{E} = 0$

(ii) $a < r < b$: The enclosed charge is the inner surface charge plus the volume charge:

$$\begin{aligned} Q_{\text{enc}} &= \oint_S \sigma_b\,\mathrm{d}\mathbf{a} + \int_V \rho_b\,\mathrm{d}\tau \\ &= \int -\frac{k}{a}a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi + \int -\frac{k}{r^2}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= -4\pi ka - 4\pi k(r-a) = -4\pi kr \end{aligned}$$

So using Gauss's Law

$$\begin{aligned} \oint \mathbf{E}\cdot\mathrm{d}\mathbf{a} &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ |\mathbf{E}|\oint \mathrm{d}a &= -\frac{4\pi kr}{\epsilon_0} \\ |\mathbf{E}|4\pi r^2 &= -\frac{4\pi kr}{\epsilon_0} \\ |\mathbf{E}| &= -\frac{k}{\epsilon_0 r} \end{aligned}$$

or

$$\mathbf{E} = -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}}$$

(iii) $r > b$: The total enclosed charge of a dielectric is zero (from last HW 4.14), so $\mathbf{E} = 0$

(b) Using

$$\oint D\cdot\mathrm{d}\mathbf{a} = Q_{\text{free}} \tag{4.23}$$

and

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$$

the total free enclosed charge is zero, so

$$\mathbf{D} = 0$$

Thus

$$\epsilon_0 \mathbf{E} + \mathbf{P} = 0$$

or

$$\mathbf{E} = -\frac{1}{\epsilon_0}\mathbf{P} = \begin{cases} 0 & r < a \quad \text{and} \quad r > b \\ -\dfrac{k}{\epsilon_0 r}\hat{\mathbf{r}} & a < r < b \end{cases}$$

4.17 Bar electret from Prob. 4.11 has $\rho_b = 0$: From divergence theorem

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{D})\, \mathrm{d}\tau = \oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{\text{free}} = 0 \implies \boldsymbol{\nabla} \cdot \mathbf{D} = 0 \tag{4.22}$$

So, the field lines for $\mathbf{D}$ are closed loops as shown in Fig.

4.19