1. From the Lagrangian

$$\mathcal{L} = \frac{1}{m}(\mathbf{L}\times\mathbf{p}) + \frac{\kappa\mathbf{r}}{r} = \frac{1}{2m}(\mathbf{L}\times\mathbf{p} - \mathbf{p}\times\mathbf{L}) + \frac{\kappa\mathbf{r}}{r}$$

taking the derivative of a cross product is given by the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L}\times\mathbf{p}) = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\times\mathbf{p} + \mathbf{L}\times\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

so

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \frac{1}{m}\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\times\mathbf{p} + \mathbf{L}\times\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa\mathbf{r}}{r}\right)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\kappa\mathbf{r}}{r}\right) = \kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right)$$

and since the angular momentum is $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L}\times\mathbf{p}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\mathbf{p})\times\mathbf{p} + (\mathbf{r}\times\mathbf{p})\times\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

and since

$$\begin{aligned}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} = -\boldsymbol{\nabla}V = -\frac{\kappa}{r^2} &= -\frac{\kappa\mathbf{r}}{r^3}\\ \frac{\mathrm{d}L}{\mathrm{d}t} &= 0\end{aligned}$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L}\times\mathbf{p}) &= -m\frac{\kappa}{r^3}(r\times\dot{\mathbf{r}})\times\mathbf{r}\\ &= -m\frac{\kappa}{r^3}[\dot{\mathbf{r}}r^2 - \mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}})]\end{aligned}$$

where

$$\mathbf{r}\cdot\dot{\mathbf{r}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{r}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = r\dot{r}$$

so

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \frac{1}{m}\left[-m\frac{\kappa}{r^3}\left(\dot{\mathbf{r}}r^2 - \mathbf{r}(r\dot{r})\right)\right] + \kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right)\\ &= -\kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right) + \kappa\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right)\\ &= 0\end{aligned}$$

2. (a) Given

$$M_{\text{meson}} = m_1 + m_2 + A\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{m_1 m_2}$$

and

$$A = \left(\frac{2m_u}{\hbar}\right)^2 159\,\text{MeV/c}^2, \quad m_u = m_d = 308\,\text{MeV/c}^2, \quad m_s = 483\,\text{MeV/c}^2$$

Finding $S_1 \cdot S_2$:

$$\begin{aligned}
\mathbf{S}_T &= \mathbf{S}_1 + \mathbf{S}_2 \\
\mathbf{S}^2 &= (\mathbf{S}_1 + \mathbf{S}_2)^2 \\
&= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2
\end{aligned}$$

Where from the operator

$$\begin{aligned}
[J^2, J_z] = 0 \qquad & |j,m\rangle \\
J_z\,|j,m\rangle + \hbar m\,|j,m\rangle \\
J^2\,|j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle
\end{aligned}$$

so the eigenvalues of $\mathbf{S}$ are $\frac{1}{2}(\frac{1}{2}+1)\hbar^2$:

$$\begin{aligned}
\mathbf{S}^2 &= \frac{3}{2}\hbar^2 + \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \\
&= \frac{3}{2}\hbar^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2
\end{aligned}$$

so for the scalar case $s = 0$, $\mathbf{S}^2 = 0$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = -\frac{3}{4}\hbar^2$$

and for the vector case $s = 1$, $\mathbf{S}^2 = 2\hbar^2$:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(2 - \frac{3}{2}\hbar^2\right) = \frac{1}{4}\hbar^2$$

So

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \begin{cases} -\frac{3}{4}\hbar^2 & \text{spin-0} \\ \frac{1}{4}\hbar^2 & \text{spin-1} \end{cases}$$

So for pseudoscalar cases $\mathbf{S}_1 \cdot \mathbf{S}_2 = -\frac{3}{4}\hbar^2$:

- π (ud)

$$\begin{aligned}
M_\pi = 2m_u + A\frac{-3}{4m_u m_d} &= 2(308) + 4(308)^2 159\frac{-3}{4(308)(308)} \\
= 139\,\text{MeV/c}^2
\end{aligned}$$

- $K+$ (us)

$$\begin{aligned}
M_{K+} &= (308) + 483 - (308)^2 159\frac{3}{308(483)} \\
&= 487\,\text{MeV/c}^2
\end{aligned}$$

- K^0 (ds)

$$\begin{aligned} M_{K^0} &= (308) + 483 - (308)^2 159 \frac{3}{308(483)} \\ &= 487\,\mathrm{MeV/c^2} \end{aligned}$$

- η The masses of constituent parts:
  - $u\bar{u}$ and $d\bar{d}$:

$$M_{u\bar{u}} = M_{d\bar{d}} = 139\,\mathrm{MeV/c^2}$$

  - $s\bar{s}$:

$$\begin{aligned} M_{s\bar{s}} &= 2(483) - (308)^2 159 \frac{3}{483^2} \\ &= 772\,\mathrm{MeV/c^2} \end{aligned}$$

  so

$$\begin{aligned} M_\eta &= \frac{1}{6}(139) + \frac{1}{6}(139) + \frac{4}{6}(772) \\ &= 561\,\mathrm{MeV/c^2} \end{aligned}$$

- η'

$$\begin{aligned} M_{\eta'} &= \frac{1}{3}(139) + \frac{1}{3}(139) + \frac{1}{3}(772) \\ &= 350\,\mathrm{MeV/c^2} \end{aligned}$$

And for vector cases $\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{4}\hbar^2$:

- ρ (ud):

$$\begin{aligned} M_\rho &= 2(308) + 4(308)^2 159 \frac{1}{4(308)(308)} \\ &= 775\,\mathrm{MeV/c^2} \end{aligned}$$

- K^{*+} (us):

$$\begin{aligned} M_{K^{*+}} &= (308) + 483 + (308)^2 159 \frac{1}{308(483)} \\ &= 892\,\mathrm{MeV/c^2} \end{aligned}$$

- K^{*0} (ds):

$$M_{K^{*0}} = 892\,\mathrm{MeV/c^2}$$

- $\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$:

$$\begin{aligned} M_\omega &= \frac{1}{2}(775) + \frac{1}{2}(775) \\ &= 775\,\mathrm{MeV/c^2} \end{aligned}$$

- $\phi = s\bar{s}$:

$$\begin{aligned} M_\phi &= 2(483) + 308^2 159 \frac{1}{483^2} \\ &= 1031\,\mathrm{MeV/c^2} \end{aligned}$$

(b) With $m_c = 1250\,\mathrm{MeV/C^2}$ and the same stuff from part (a) the pseudoscalars are:

- $\eta_c(c\bar{c})$:

$$\begin{aligned} M_{\eta_c} &= 2(1250) - (308)^2 159 \frac{3}{1250^2} \\ &= 2471\,\mathrm{MeV/c^2} \end{aligned}$$

- $D^0(c\bar{u})$:

$$\begin{aligned} M_{D^0} &= 1250 + 308 - (308)^2 159 \frac{3}{308(1250)} \\ &= 1440\,\mathrm{MeV/c^2} \end{aligned}$$

- $D_s^+(c\bar{s})$:

$$\begin{aligned} M_{D_s^+} &= 1250 + 483 - (308)^2 159 \frac{3}{483(1250)} \\ &= 1658\,\mathrm{MeV/c^2} \end{aligned}$$

and the vector mesons are:

- $J/\psi(c\bar{c})$:

$$\begin{aligned} M_{J/\psi} &= 2(1250) + (308)^2 159 \frac{1}{1250^2} \\ &= 2510\,\mathrm{MeV/c^2} \end{aligned}$$

- $D^{*0}(c\bar{u})$:

$$\begin{aligned} M_{D^{*0}} &= 1250 + 308 + (308)^2 159 \frac{1}{308(1250)} \\ &= 1597\,\mathrm{MeV/c^2} \end{aligned}$$

- $D_s^{*+}(c\bar{s})$:

$$\begin{aligned} M_{D_s^{*+}} &= 1250 + 483 + (308)^2 159 \frac{1}{483(1250)} \\ &= 1758\,\mathrm{MeV/c^2} \end{aligned}$$

(c) Now the beauty mesons with $m_b = 4.5\,\mathrm{GeV/c^2} = 4500\,\mathrm{MeV/c^2}$: Pseudoscalars

- $\eta_b(b\bar{b})$:

$$\begin{aligned} M_{\eta_b} &= 2(4500) - (308)^2 159 \frac{3}{4500^2} \\ &= 8998\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^+(u\bar{b})$:

$$\begin{aligned} M_{B^+} &= 308 + 4500 - (308)^2 159 \frac{3}{308(4500)} \\ &= 4775\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^0(d\bar{d})$:

$$M_{B^0} = 4775\,\mathrm{MeV/c^2}$$

- $B_c^+(c\bar{b})$:

$$\begin{aligned} M_{B_c^+} &= 1250 + 4500 - (308)^2 159 \frac{3}{1250(4500)} \\ &= 5742\,\mathrm{MeV/c^2} \end{aligned}$$

and vector mesons are:

- $\Upsilon(b\bar{b})$:

$$\begin{aligned} M_\Upsilon &= 2(4500) + (308)^2 159 \frac{1}{4500^2} \\ &= 9001\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^{*+}(u\bar{b})$:

$$\begin{aligned} M_{B^{*+}} &= 308 + 4500 + (308)^2 159 \frac{1}{308(4500)} \\ &= 4819\,\mathrm{MeV/c^2} \end{aligned}$$

- $B^{*0}(d\bar{b})$:

$$M_{B^{*0}} = 4819\,\mathrm{MeV/c^2}$$

- $B_c^{*+}(c\bar{b})$:

$$\begin{aligned} M_{B_c^{*+}} &= 1250 + 4500 + (308)^2 159 \frac{1}{1250(4500)} \\ &= 5752\,\mathrm{MeV/c^2} \end{aligned}$$

(d) Comparing all these masses compared to the PDB

Meson	Calculated Mass	PDB Mass
π	139 MeV/c^2	139.570 61 MeV/c^2
K^+	487 MeV/c^2	493.677 MeV/c^2
K^0	487 MeV/c^2	497.614 MeV/c^2
η	561 MeV/c^2	547.862 MeV/c^2
η'	350 MeV/c^2	957.78 MeV/c^2
ρ	775 MeV/c^2	775.26 MeV/c^2
K^{*+}	892 MeV/c^2	891.66 MeV/c^2
K^{*0}	892 MeV/c^2	895.55 MeV/c^2
ω	775 MeV/c^2	782.65 MeV/c^2
ϕ	1031 MeV/c^2	1019 MeV/c^2
η_c	2471 MeV/c^2	2980 MeV/c^2
D^0	1440 MeV/c^2	1864 MeV/c^2
D_s^+	1658 MeV/c^2	1968 MeV/c^2
J/ψ	2510 MeV/c^2	3096 MeV/c^2
D^{*0}	1597 MeV/c^2	2006 MeV/c^2
D_s^{*+}	1758 MeV/c^2	2112 MeV/c^2
η_b	8998 MeV/c^2	9398 MeV/c^2
B^+	4775 MeV/c^2	5279 MeV/c^2
B^0	4775 MeV/c^2	5279 MeV/c^2
B_c^+	5742 MeV/c^2	6274 MeV/c^2
Υ	9001 MeV/c^2	9460 MeV/c^2
B^{*+}	4819 MeV/c^2	5325 MeV/c^2
B^{*0}	4819 MeV/c^2	5324 MeV/c^2
B_c^{*+}	5752 MeV/c^2	-

The light mesons are all pretty good estimates except for η'... For heavier mesons the estimates are not as good, but they are within the ballpark of the actual masses. Why is the η' so far off?