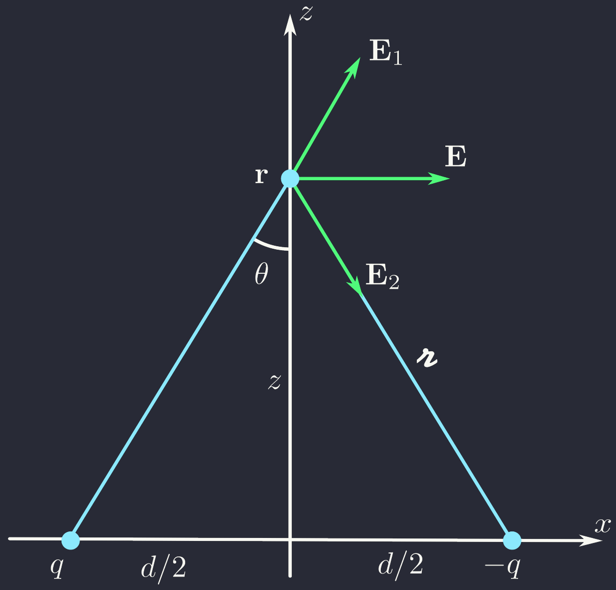

Figure 2.1: An electric field at a distance z from the midpoint between equal and opposite charges ($\pm q$) separated by a distance d. The charge at $x = d/2$ is $-q$.

2.2 The vertical componets of the electric field cancel out and the horizontal components add up:

$$E_x = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\sin\theta$$

where $E_x = E\cos\theta$, $\mathscr{r} = \sqrt{z^2 + (d/2)^2}$, and $\sin\theta = d/(2\mathscr{r})$, so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{[z^2 + (d/2)^2]^{3/2}}\hat{\mathbf{x}}$$

2.5 The horizontal components of the electric field cancel out, and the vertical components conspire:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda}{\mathscr{r}^2}\cos\theta\hat{\mathbf{z}}\,\mathrm{d}\mathrm{l}$$

where geometrically $\mathscr{r} = \sqrt{z^2+r^2}$ and $\cos\theta = z/\mathscr{r}$. So,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}\,\mathrm{d}\mathrm{l}$$

and the line integral is over the circumference of the circle, so $\mathrm{d}\mathbf{l} = r\,\mathrm{d}\theta$ and the limits of integration are $[0, 2\pi]$:

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{\lambda z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}\int_0^{2\pi} r\,\mathrm{d}\theta \\ &= \frac{1}{4\pi\epsilon_0}\frac{\lambda z(2\pi r)}{(z^2+r^2)^{3/2}}\end{aligned}$$

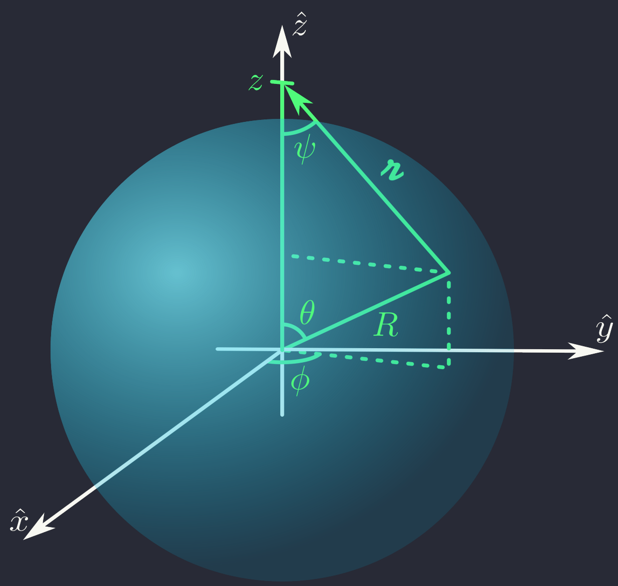

Figure 2.2: An electric field a distance z from the center of a spherical surface of radius R that carries a charge density σ.

2.7 Once again, the electric field is in the z-direction:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{\mathscr{r}^2} \cos\psi \hat{\mathbf{z}} \, \mathrm{d}\mathbf{a} \tag{2.1}$$

From the law of cosines, $\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$; Geometrically, $\cos\psi = \dfrac{z - R\cos\theta}{\mathscr{r}}$; the surface area element is $\mathrm{d}\mathbf{a} = R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\theta$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \frac{\sigma R^2 (z - R\cos\theta)}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} (2\pi\sigma R^2) \int_0^{\pi} \frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} \sin\theta \, \mathrm{d}\theta \, \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} (2\pi\sigma R^2) f(\theta) \hat{\mathbf{z}}
\end{aligned}$$

using the substitution $u = \cos\theta$: $\mathrm{d}u = -\sin\theta \, \mathrm{d}\theta$, and the limits of integration are $[\cos 0, \cos\pi]$. So,

$$f(\theta) = \int_{-1}^{1} \frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}} \, \mathrm{d}u = f(u)$$

substituting again with $v = \sqrt{z^2 + R^2 - 2zRu}$; $\mathrm{d}v = -\dfrac{zR}{v} \, \mathrm{d}u$; and $u = \dfrac{1}{2zR}(z^2 + R^2 - v^2)$:

$$\begin{aligned}
f(v) &= -\frac{1}{zR} \int \frac{z - \frac{1}{2z}(z^2 + R^2 - v^2)}{v^3} v \, \mathrm{d}v \\
&= -\frac{1}{2z^2 R} \int \frac{2z^2 - (z^2 + R^2 - v^2)}{v^2} \, \mathrm{d}v \\
&= -\frac{1}{2z^2 R} \int \frac{v^2 + z^2 - R^2}{v^2} \, \mathrm{d}v \\
&= -\frac{1}{2z^2 R} \int \left(1 + \frac{z^2 - R^2}{v^2}\right) \mathrm{d}v \\
&= -\frac{1}{2z^2 R} \left(v - \frac{z^2 - R^2}{v}\right)
\end{aligned}$$

back substituting $v = \sqrt{z^2 + R^2 - 2zRu}$,

$$\begin{aligned}
f(u) &= -\frac{1}{2z^2R}\left(\frac{z^2+R^2-2zRu}{\sqrt{z^2+R^2-2zRu}} - \frac{z^2-R^2}{\sqrt{z^2+R^2-2zRu}}\right)\Bigg|_{-1}^{1} \\
&= -\frac{1}{2z^2R}\left(\frac{2R^2-2zRu}{\sqrt{z^2+R^2-2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{zu-R}{\sqrt{z^2+R^2-2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{z-R}{\sqrt{z^2+R^2-2zR}} - \frac{-z-R}{\sqrt{z^2+R^2+2zR}}\right)
\end{aligned}$$

Taking the positive square root: $\sqrt{z^2+R^2-2zR} = (R-z)$ if $R > z$, but $(z-R)$ if $R < z$. So, for the case $z < R$ (inside the sphere) the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z-R}{R-z} - \frac{-z-R}{R+z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z-R}{R-z} + \frac{z+R}{R+z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z-R}{R-z} + 1\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z-R}{R-z} + \frac{R-z}{R-z}\right)\hat{\mathbf{z}} \\
&= 0
\end{aligned}$$

For the case $z > R$ (outside the sphere) the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z-R}{z-R} + \frac{z+R}{z+R}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{4\pi\sigma R^2}{z^2}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}
\end{aligned}$$

This makes sense: From outside the sphere, the point charge q is the charge-per-area σ times the surface area of the sphere $4\pi R^2$, or simply $q = 4\pi R^2\sigma$.

2.8 Finding the field inside and outside a solid sphere of radius R with a uniform volume charge density ρ is similar to Prob. 2.7. Outside the solid sphere the total charge q contributes to the electric field as if it were a point charge:

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Inside the solid sphere, only the volume of the solid sphere less than r contributes to the electric field. The volume of the total sphere is $V = \frac{4}{3}\pi R^3$, and the volume of the sphere less than r is $V' = \frac{4}{3}\pi r^3$. So, electric field inside the solid sphere is

$$\begin{aligned}\mathbf{E}_{in} &= \frac{V'}{V}\mathbf{E}_{out} \\ &= \frac{r^3}{R^3}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}\end{aligned}$$

or

$$\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\mathbf{r}$$

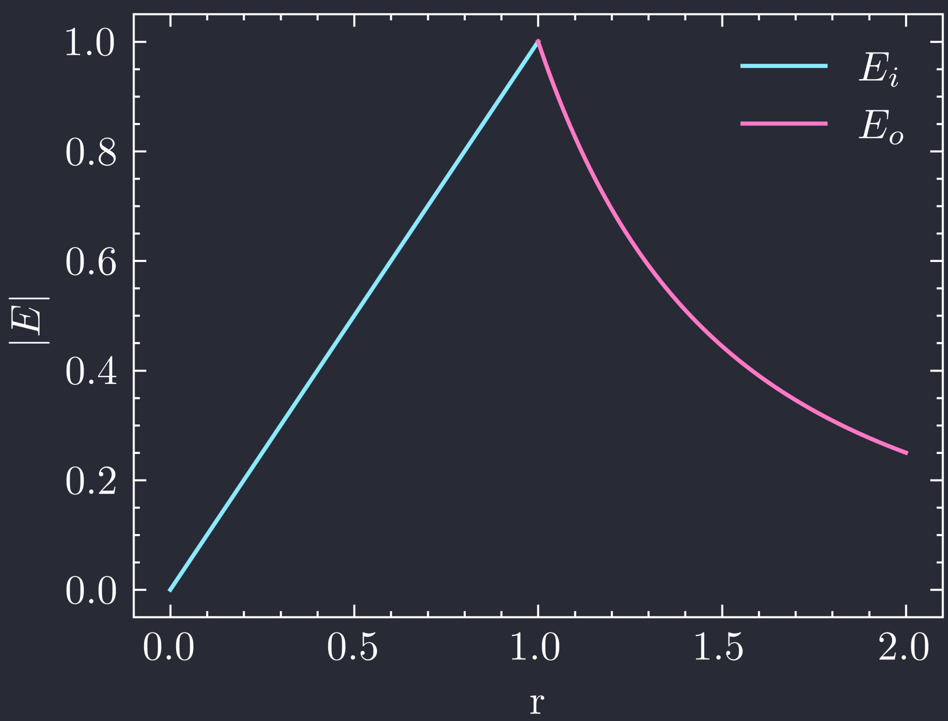

Figure 2.3: Magnitude of Electric field $|E|$ as a function of r inside and outside a solid. Where $q = 9\text{nC}$ and $R = 1\text{m}$.

2.12 For a spherical shell of radius R with a uniform surface charge density σ, the enclosed charge in side the sphere is $Q_{enc} = 0$, thus the electric field inside the sphere is

$$\mathbf{E}_i = 0$$

and using the sphericla symmetry of a Gaussian surface, the electric field outside the sphere is

$$\begin{aligned}
\oint \mathbf{E}_o \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\
|\mathbf{E}_0| \int \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o}(4\pi\sigma R^2) \\
\mathbf{E}_o(4\pi r^2) &= \frac{1}{\epsilon_o}(4\pi\sigma R^2)\hat{\mathbf{r}} \\
\mathbf{E}_o &= \frac{\sigma R^2}{\epsilon_o r^2}\hat{\mathbf{r}}
\end{aligned}$$

2.16 A thick spherical shell with charge density

$$\rho = \frac{k}{r^2} \quad (a \leq r \leq b)$$

The electric field in the three regions:
(i) $r < a$

$$Q_{enc} = 0; \mathbf{E} = 0$$

(ii) $a \leq r \leq b$

$$Q_{enc} = \int_0^{2\pi} \int_0^{\pi} \int_a^r \rho(r^2 \sin\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi = 4\pi \int_a^r \frac{k}{r^2}(r^2)\, \mathrm{d}r = 4\pi k(r-a)$$

And from Gauss's law,

$$\begin{aligned} \oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int \mathrm{d}a &= \frac{1}{\epsilon_o} 4\pi k(r-a) \\ E(4\pi r^2) &= \frac{1}{\epsilon_o} 4\pi k(r-a) \end{aligned}$$

or

$$\mathbf{E} = \frac{k(r-a)}{\epsilon_o r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{4\pi k(r-a)}{r^3}\mathbf{r}$$

(iii) $r > b$

$$Q_{enc} = \int_0^{2\pi} \int_0^{\pi} \int_a^b \rho(r^2 \sin\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi = 4\pi k(b-a)$$

And from Gauss's law,

$$\begin{aligned} \oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int \mathrm{d}a &= \frac{1}{\epsilon_o} 4\pi k(b-a) \\ E(4\pi r^2) &= \frac{1}{\epsilon_o} 4\pi k(b-a) \end{aligned}$$

or

$$\mathbf{E} = \frac{k(b-a)}{\epsilon_o r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{4\pi k(b-a)}{r^3}\mathbf{r}$$

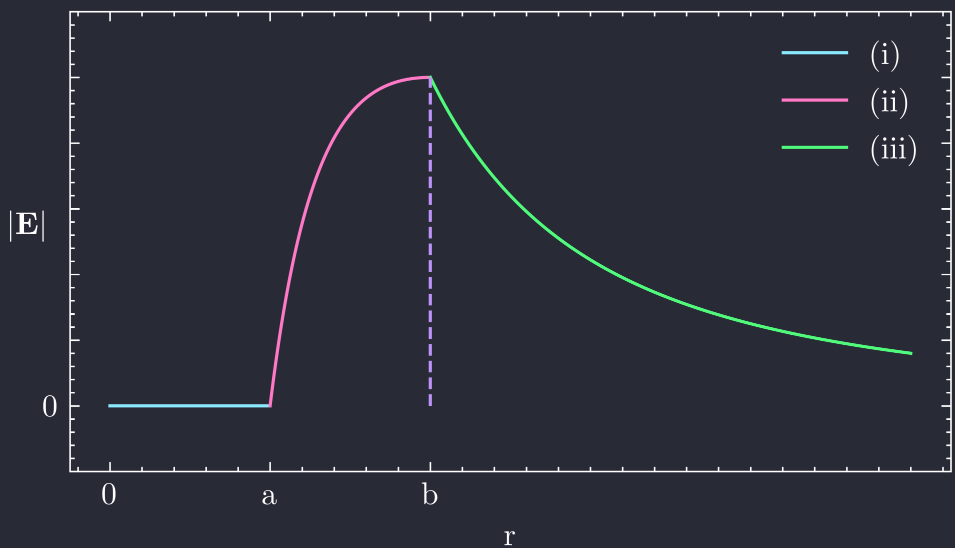

Figure 2.4: Plot of $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

2.18 Finding the electric field, as a function of y, where $y = 0$ is the center of an infinite plane slab, of thickness $2d$, carrying a uniform volume charge density ρ. For the case $y > 2d$ The enclosed charge is

$$Q_{enc} = \rho(2d)A = 2\rho Ad$$

where A is the area of the Gaussian pillbox. Using Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc}$$
$$|\mathbf{E}| \int \mathrm{d}a = \frac{1}{\epsilon_o} 2\rho Ad$$
$$E(2A) = \frac{1}{\epsilon_o} 2\rho Ad$$
$$\mathbf{E} = \frac{\rho d}{\epsilon_o} \hat{\mathbf{y}}$$

For the case $0 < y < 2d$, the enclosed charge is

$$Q_{enc} = 2\rho y A$$

and the electric field is

$$E(2A) = \frac{1}{\epsilon_o} \rho y A$$
$$\mathbf{E} = \frac{\rho y}{\epsilon_o} \hat{\mathbf{y}}$$

In the $-y$ direction, E is negative as shown in Figure 2.5.

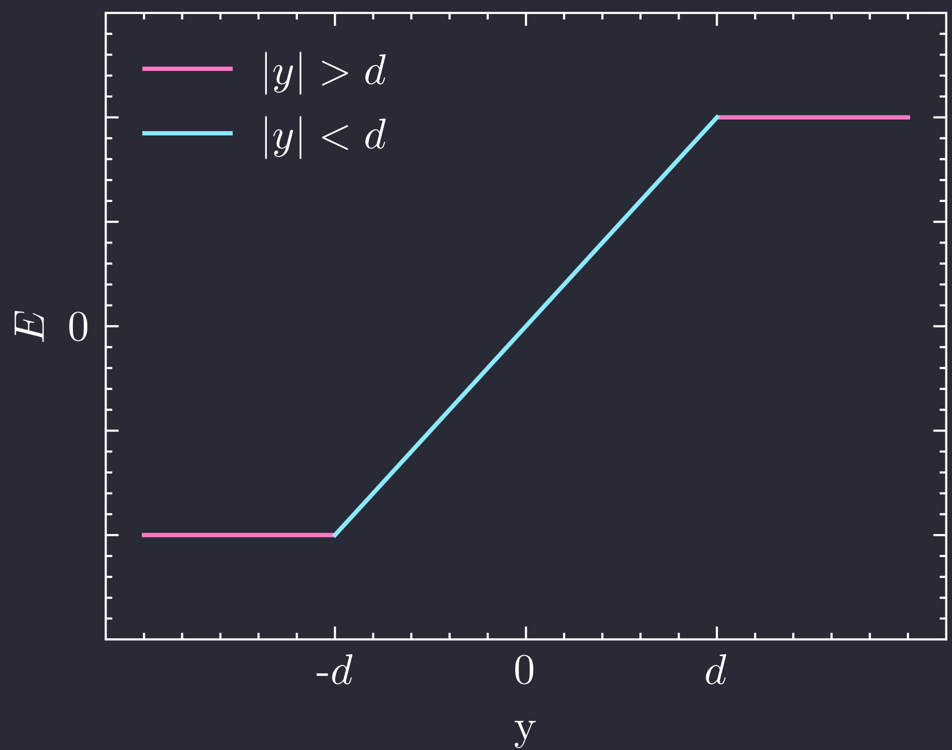

Figure 2.5: Plot of $|\mathbf{E}|$ as a function of y