Homework 1

Due 1/24 9pm

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1. Given: the 2D Cartesian relation to polar coordinates

$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi, \quad \hat{\phi} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi \tag{1}$$

We can write v as a linear combination of $\hat{\mathbf{r}}$ and $\hat{\phi}$

$$\begin{aligned} \mathbf{v} &= v_r \hat{\mathbf{r}} + v_\phi \hat{\boldsymbol{\phi}} \\ &= v_r(\hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi) + v_\phi(-\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi) \\ &= (v_r \cos\phi - v_\phi \sin\phi)\hat{\mathbf{x}} + (v_r \sin\phi + v_\phi \cos\phi)\hat{\mathbf{y}} \end{aligned}$$

and since we know the vector in Cartesian coordinates is

$$\mathbf{v} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$$

we can equate the components to get

$$\begin{aligned} v_x &= v_r \cos\phi - v_\phi \sin\phi \\ v_y &= v_r \sin\phi + v_\phi \cos\phi \end{aligned}$$

multiplying the first equation by $\cos\phi$ and the second by $\sin\phi$ and adding them together

$$\begin{aligned} v_x \cos\phi &= v_r \cos^2\phi - v_\phi \sin\phi \cos\phi \\ v_y \sin\phi &= v_r \sin^2\phi + v_\phi \sin\phi \cos\phi \\ v_x \cos\phi + v_y \sin\phi &= v_r(\cos^2\phi + \sin^2\phi) \end{aligned}$$

or simply

$$v_r = v_x \cos\phi + v_y \sin\phi$$

Likewise,

$$\begin{aligned} v_y \cos\phi &= v_r \sin\phi \cos\phi + v_\phi \sin^2\phi \\ v_x \sin\phi &= v_r \sin\phi \cos\phi - v_\phi \cos^2\phi \end{aligned}$$

and subtracting the second equation from the first

$$v_y \cos\phi - v_x \sin\phi = v_\phi(\sin^2\phi + \cos^2\phi)$$

Therefore we get the components of $\mathbf{v}$ in polar coordinates

$$\boxed{v_r = v_x \cos\phi + v_y \sin\phi}$$

$$\boxed{v_\phi = -v_x \sin\phi + v_y \cos\phi}$$

Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are *constant*, the time derivatives of (1) are

$$\begin{aligned} \dot{\hat{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} &= \hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\phi) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\phi) \\ &= (-\dot{\phi}\sin\phi)\hat{\mathbf{x}} + (\dot{\phi}\cos\phi)\hat{\mathbf{y}} = \dot{\phi}\hat{\phi} \end{aligned}$$

and

$$\begin{aligned} \dot{\hat{\phi}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi} &= -\hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}t}(\sin\phi) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\phi) \\ &= (-\dot{\phi}\cos\phi)\hat{\mathbf{x}} + (-\dot{\phi}\sin\phi)\hat{\mathbf{y}} = -\dot{\phi}\hat{\mathbf{r}} \end{aligned}$$

2. From Taylor Problem 1.45: [*Hint:* Consider the derivative of $\mathbf{v}^2$].
Since the magnitude of $\mathbf{v}(t)$ is also $\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$, the derivative of $\mathbf{v}^2$ is tells us if the magnitude is constant.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\cdot\mathbf{v}(t))\\ &= 2\dot{\mathbf{v}}(t)\cdot\mathbf{v}(t)\end{aligned}$$

The magnitude of $\mathbf{v}(t)$ is constant if $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 = 0$. Since the dot product is zero, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$.

3. Using the product rule for the dot product $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}\cdot\mathbf{y}) = \dot{\mathbf{x}}\cdot\mathbf{y} + \mathbf{x}\cdot\dot{\mathbf{y}}$

$$\begin{aligned}\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] &= \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{r})\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\frac{d\mathbf{r}}{dt}\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\mathbf{a}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\mathbf{v}\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r})\end{aligned}$$

The cross product of a vector with itself is zero ($\mathbf{v}\times\mathbf{v} = 0$), and the dot product of orthogonal vectors are zero (acceleration and position are perpendicular based on the result from Problem 2) QED

4. Given

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \tag{3}$$

(a) Moving everything to one side:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

multiplying by $\dot{\theta}$ and grouping terms:

$$\begin{aligned}\dot{\theta}\ddot{\theta} + \frac{g}{l}\dot{\theta}\sin\theta &= 0\\ \dot{\theta}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}\right) + \frac{g}{l}\sin\theta\frac{\mathrm{d}}{\mathrm{d}t}(\theta) &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\theta}^2\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g}{l}\cos\theta\right) &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta\right) &= 0\end{aligned}$$

so the integral constant is

$$\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = C$$

Initially the pendulum starts at rest, $\dot{\theta}(t=0) = 0$ and $\theta(0) = \theta_o$ thus

$$\boxed{C = -\frac{g}{l}\cos\theta_o}$$

(b) Rewriting $X = C$

$$\begin{aligned}\frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta &= -\frac{g}{l}\cos\theta_o\\ \dot{\theta} &= \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_o)}\end{aligned}$$

using separation of variables

$$\mathrm{d}\theta = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_o)}\,\mathrm{d}t$$

Integrating gives the analytic solution

$$\boxed{\theta(t) = \sqrt{\frac{2g}{l}}\int_0^T \sqrt{\cos\theta - \cos\theta_o}\,\mathrm{d}t}$$

where T is the period of the pendulum.
(c) The period of the pendulum is the time it takes to complete one cycle. Since

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

using separation of variables

$$\mathrm{d}t = \frac{1}{\dot{\theta}}\,\mathrm{d}\theta$$

Integrating both sides gives the period

$$\int \mathrm{d}t = \int \frac{1}{\dot{\theta}}\,\mathrm{d}\theta$$

$$\boxed{T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_o} \frac{1}{\sqrt{\cos\theta - \cos\theta_o}}\,\mathrm{d}\theta}$$

where the constant '4' comes from the fact that the total cycle is 4 times the period it takes to go from path from θ_o to 0.