1 Center of Mass & Conservation of Momentum

For N particles, the center of mass is

$$\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i$$

where $M = \sum_i m_i$ is the total mass of the system. This is similar to the 'weighted' average! Taking the time derivative of $\mathbf{R}$ gives the total momentum

$$\dot{\mathbf{R}} = \frac{1}{M}\sum_i m_i \dot{\mathbf{r}}_i = \frac{1}{M}\sum_i \mathbf{p}_i = \mathbf{P}$$

From Newton's 3rd Law

$$\sum \dot{\mathbf{p}}_i = \mathbf{F}_{ext}$$

and from the second Law

$$M\ddot{\mathbf{R}} = \mathbf{F}_{ext}$$

in integral form

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\,\mathrm{d}m$$

and using the mass density $\mathrm{d}m = \rho\,\mathrm{d}V$ we can write

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\rho\,\mathrm{d}V = \frac{1}{M}\int \mathbf{r}\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

For a uniform solid semisphere lying on the xy plane with radius $R = 1$ and mass $M = 2\pi/3$, the CM is

$$\begin{aligned} z &= \frac{1}{M}\int \rho z\,\mathrm{d}m \\ &= \frac{1}{M}\int z\pi r^2\,\mathrm{d}z \\ &= \frac{1}{M}\int_0^1 \pi z(1-z^2)\,\mathrm{d}z \\ &= \frac{\pi}{M}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{3}{8} \end{aligned}$$

Angular Momentum

For the singular particle, the angular momentum is

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}$$

and the total angular momentum of an multi particle system is

$$\mathbf{L} = \sum_i \boldsymbol{\ell}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

and the time derivative of $\mathbf{L}$ is

$$\dot{\mathbf{L}} = \sum_i \dot{\mathbf{r}}_i \times \mathbf{p}_i + \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \boldsymbol{\Gamma}_i$$

where $\dot{\mathbf{r}}_i \times \mathbf{p}_i = 0$ since $\dot{\mathbf{r}}_i$ is parallel to $\mathbf{p}_i$. Since $\mathbf{F}_i$ is the force on the ith particle,

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{ext}$$

Plugging into the time derivative of angular momentum

$$\dot{\mathbf{L}} = \sum_i \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{ij} + \sum_i \mathbf{F}_i^{ext}$$

In terms of a matrix, the double sum skips the diagonal elements and thus we can pair the indices that are reflected on the diagonal

$$\sum_i \sum_{j>i} (\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}) = \sum_i \sum_{j>i} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}$$

where we use the associativity of the cross product and N3L $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. In addition the force must be central along the line connecting the two particles. Thus we get

$$\dot{\mathbf{L}} = \sum_i \Gamma_i^{ext}$$

The direction of the angular momentum is along the axis of rotation.

A car To move a car forward, you exert a torque clockwise on the wheels, and from the conservation of angular momentum the car will typically want to rotate counter clockwise which feels like the weight is being pushed back. The torque on the car will increase the friction on the rear wheel (increasing traction) and thus RWD are better at high accelerations.