Physics 411: Mechanics

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Contents

1 Newtons Laws

The Four Horsemen of the Apocalypse (In Physics)

- Classical Mechanics
- Electromagnetism
- Statistical Mechanics
- Quantum Mechanics

Before 1900, there was no relativity or QM and the world was a simple place ...

Newton's 1st Law: The Law of Inertia
And object keeps going unless acted on by a force.
This only applies to and 'inertial frame'.

Newton's 2nd Law: $\mathbf{F} = m\mathbf{a}$
Sum notation: The position vector is

$$\mathbf{r} = (x, y, z) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$$

in the Cartesian coordinate system. The time derivative gives the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

and acceleration is the time derivative of velocity

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

Thus in vector notation, Newton's 2nd law is

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$$

where $\mathbf{r}(t)$ is and ordinary differential equation (ODE).
The basic idea of solving mechanics problems is writing down the ODEs and solving them.

What is mass? m is an 'inertial mass'.
In Newton's law of gravity

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

m is the 'gravitational mass' and $g \approx 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.
A larger mass has a larger inertia or 'resistance to being accelerated' (Taylor). Key fact: When acceleration is zero ($\mathbf{a} = 0$), the velocity is constant ($\mathbf{v} =$ constant).

Momentum: $\mathbf{p} = m\mathbf{v}$
The third law of motion in terms of momentum is

$$\mathbf{F} = \dot{\mathbf{p}} = m\dot{\mathbf{v}}$$

Newton's Third Law: $\mathbf{F}_{12} = -\mathbf{F}_{21}$
In a two body system, the total force of the system is

$$\mathbf{F}_t = \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

From the second law,

$$\dot{\mathbf{p}}_1 = \mathbf{F}_{21} \qquad \dot{\mathbf{p}}_2 = \mathbf{F}_{12}$$

adding these two equations gives

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$$

thus the total momentum of the system is conserved.
For a system of N particles, the total momentum is

$$\mathrm{d}t \sum_i \mathbf{p}_i = \frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = \mathbf{F}_{ext}$$

sometimes $\mathbf{p}_{tot} = \mathbf{P}$ where the capital P denotes the total momentum of the system.

2 A pendulum

How to solve a problem:

1. Write down the eq
2. Solve it
3. Understand the solution

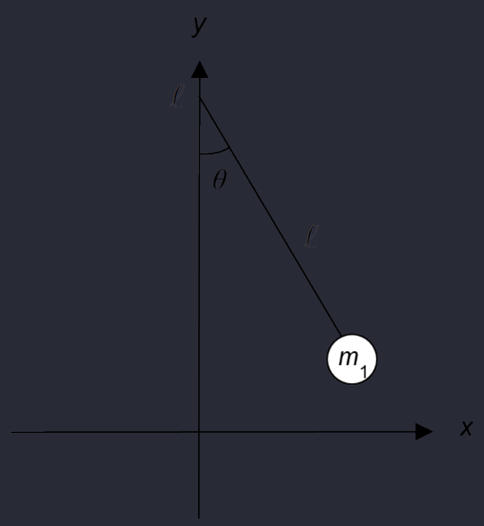

Figure 2.1: A pendulum with mass m and length l.

From Figure 2.1, we can write down Newton's 2nd law:

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} = m\ddot{\mathbf{r}} \\ F_x &= -mg\sin\theta = m\ddot{x} \\ F_y &= -mg\cos\theta + T\cos\theta = m\ddot{x} \end{aligned}$$

Using a right triangle we can find the angle using $\tan\theta = x/y$. Furthermore, we can use the constrain that the length of the pendulum is constant thus $x^2 + y^2 = l^2$. But solving this system of equations is difficult. Instead we now use a new coordinate system.

Quick Hack Using the arc length $l = L\theta$ and choosing a coordinate in the direction of the pendulums path, we can write the force equation as

$$F_l = -mg\sin\theta = m\ddot{l} = mL\ddot{\theta}$$

Thus the equation of motion is

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

which is a second order ODE. This can only be solved with two conditions. We can use the initial conditions (at $t = 0$) of the position $\theta(t = 0) = \theta_0$ and velocity $\dot{\theta}(0) = 0$.

Polar Coordinates From Taylor:

$$x = r\cos\phi$$
$$y = r\sin\phi$$

For an arbitrary vector $\mathbf{v}$ it has the Cartesian vector components

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

Where the magnitude of the unit vectors are equivalent:

$$|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = 1$$

and the magnitude of the vector is

$$\begin{aligned}|\mathbf{v}| &= \sqrt{\mathbf{v}\cdot\mathbf{v}} \\ &= \sqrt{v_x^2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + 2v_xv_y\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}v_y^2\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}} \\ &= \sqrt{v_x^2 + v_y^2}\end{aligned}$$

The vector $\mathbf{v}$ can be written in polar coordinates as

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\phi}$$

where radial vector is

$$\mathbf{r} = r\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

taking the time derivative of $\mathbf{r}$ gives the velocity

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}}$$

but how do we find $\dot{\hat{r}}$? We can look at the change in the direction of the radial unit vector for a small change in time Δt. Thus,

$$\Delta\hat{\mathbf{r}} \approx r\Delta\phi\hat{\boldsymbol{\phi}}$$

dividing both sides by Δt gives

$$\frac{\Delta\hat{\mathbf{r}}}{\Delta t} \approx r\frac{\Delta\phi}{\Delta t}\hat{\boldsymbol{\phi}} = r\dot{\phi}\hat{\boldsymbol{\phi}}$$

Therefore, the vector in polar coordinates is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}}$$

where the polar components v_r and v_ϕ are related to the radial and angular velocity respectively. Taking the time derivative of $\dot{\mathbf{r}}$ gives the acceleration

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}} \\ &= \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\hat{r}} + \dot{v}_\phi\hat{\boldsymbol{\phi}} + v_\phi\dot{\hat{\phi}}\end{aligned}$$

3 Polar Coordinates

using the geometric relation $\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$, we can write the acceleration as

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}}\end{aligned}$$

where $r\dot{\phi}^2 = r\omega^2$ is the centripetal acceleration and $r\ddot{\phi} = r\dot{\omega}$ is the tangential acceleration.
From the Pendulum problem we know that the string is taut $r = L$ thus the radial velocity is zero $\dot{r} = 0$.
Thus the force equation in the $\hat{\boldsymbol{\phi}}$ direction is

$$\begin{aligned}F_\phi &= mL\ddot{\phi} = -mg\sin\theta \\ \ddot{\phi} &= -\frac{g}{L}\sin\theta\end{aligned}$$

which is the same equation of motion.

Projectile in 2D The initial conditions of a general projectile is usually

$$\begin{aligned}F_x &= 0 = m\ddot{x} \\ F_y &= -mg = m\ddot{y}\end{aligned}$$

thus the equations of motion are

$$\begin{aligned}\ddot{x} &= 0 \\ \ddot{y} &= -g\end{aligned}$$

And solving these equations gives the position of the projectile

$$\begin{aligned}x(t) &= v_{ox}t \\ y(t) &= y_o v_{oy}t - \frac{1}{2}gt^2\end{aligned}$$

This can be expanded on with the addition of air resistance $\mathbf{f}$. This drag force is proportional to the velocity:

$$\mathbf{f} \propto -\hat{\mathbf{v}}$$

and there are two types of air resistance: linear

$$\mathbf{f}_l = -bv\hat{\mathbf{v}} = -b\mathbf{v}$$

and quadratic

$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

where we compare the terms with

$$\frac{f_l}{f_q} = \frac{cv}{b}$$

Linear $\mathbf{f}_l = -b\mathbf{v}$
From Newton's 2nd law

$$F_x = -bv_x = m\ddot{x} = m\dot{v}_x$$
$$F_y = -mg - bv_y = m\ddot{y} = m\dot{v}_y$$

For the case of uncoupled differential equations (such as pure horizontal motion), we can solve the horizontal equation

$$\dot{v}_x = -\frac{b}{m}v_x$$

which has a general solution

$$v_x = Ae^{-kt}$$

where

$$k = \frac{b}{m}, \quad A = v_{ox}$$

to find the position we have to integrate $\dot{x} = v_x$:

$$\begin{aligned} x &= x_o + \int_0^t v_x(t')\,\mathrm{d}t' \\ &= x_o + \left[-\frac{v_{xo}}{k}e^{-kt}\right]\Big|_0^t \end{aligned}$$

where we have a limit of $t \to \infty$
For pure vertical motion, we solve the equation

$$\dot{v}_y = -g - \frac{b}{m}v_y \tag{3.1}$$

and with the initial condition $\dot{v}_y = 0$ we can solve for the velocity

$$v_y = -\frac{mg}{b} = v_{ter}$$

where v_{ter} is the terminal velocity. To get position, we use a trick by rewriting the equation as

$$m\dot{v}_y = -mg - bv_y = -mg - b(v_y - v_{ter})$$

and we can solve similar to the horizontal case using the general solution

$$v_y - v_{ter} = Ae^{-kt} = (v_{oy} - v_{ter})e^{-kt}bye$$

4 Air Resistance

Last time:

$$\mathbf{f}_l = -b\mathbf{v} \quad \dot{\mathbf{r}} = \mathbf{v}$$
$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

In the case of linear, x motino has a range, y velocity has a terminal velocity v_t.

Horizontal Quadratic Drag

$$F_y = -mg - c|v_y|v_y$$
$$m\ddot{y} = F_y$$
$$m\dot{v}_y = -mg - c|v_y|v_y$$

when $v_y = 0$ we have the terminal velocity

$$v_{ter} = \sqrt{\frac{mg}{c}} \quad \text{or} \quad c = \frac{mg}{v_{ter}^2}$$

thus the equation of motion is

$$\dot{v}_y = -g - \frac{c}{m}v_y^2 = -g(1 - \frac{v_y^2}{v_{ter}^2}) = \frac{\mathrm{d}v_y}{\mathrm{d}t}$$

using separation of variables

$$\frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\,\mathrm{d}t$$

integrating both sides

$$\int_{v_{oy}}^{v_y} \frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\int_0^t \mathrm{d}t$$

where we get the integral using the hyperbolic tangent

$$v_t \operatorname{arctanh}\frac{v_y}{v_t} = -gt$$
$$v_y = -v_t\tanh(gt)$$

2D Motion For Quadratic

$$F_x = -cvv_x = -c\sqrt{v_x^2 + v_y^2}v_x = m\dot{v}_x$$
$$F_y = -mg - cvv_y = -mg - c\sqrt{v_x^2 + v_y^2}v_y = m\dot{v}_y$$

where $v = \sqrt{v_x^2 + v_y^2}$. For linear, it is simply

$$F_x = -bv_x = m\dot{v}_x$$
$$F_y = -mg - bv_y = m\dot{v}_y$$

5 Energy

Review: There are two requirements for conservation of angular momentum

1. Force is central
2. External torque is zero

Kinetici Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$. Taking the time derivative

$$\begin{aligned}\dot{T} &= \frac{1}{2}(\dot{\mathbf{v}}\cdot\mathbf{v} + \mathbf{v}\cdot\dot{\mathbf{v}}) \\ &= m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}\end{aligned}$$

and integrating over time t_1 to t_2

$$\int_{t_1}^{t_2} \dot{T}\,\mathrm{d}t = \Delta T = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = W(1\to 2)$$

since $\mathbf{v}\cdot\mathrm{d}t = \mathrm{d}\mathbf{r}$ and $\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ hints that this is a line integral.

Example:

$$\begin{aligned}\mathbf{F}(x,y) &= \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ \mathrm{d}vbr &= \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}}\end{aligned}$$

(a) $y = x$ from $a = (0,0)$ to $b = (1,1)$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + y\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 x\,\mathrm{d}x = 1\end{aligned}$$

(b) $y = x^2$ and $\mathrm{d}y = 2x\,\mathrm{d}x$

$$\begin{aligned}\int_a^b \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \int_0^1 (x\,\mathrm{d}x + x^2\,\mathrm{d}y) \\ &= \int_0^1 x\,\mathrm{d}x + \int_0^1 2x^2\,\mathrm{d}x = 1\end{aligned}$$

thus the line integral is independent of the path.

Conservative force

1. Given $\mathbf{F}(\mathbf{r})$, there is no dependence on $\mathbf{v}$, t.
2. $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is independent of the path.

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}'$$

For gravity

$$U_g(\mathbf{r}) = -\int_a^b \mathbf{F}_g\cdot\mathrm{d}\mathbf{r} = -\int_a^b -mg\,\mathrm{d}y' = mg(y_a - y_b)$$

Work-Kinetic Energy Theorem:

$$W(1 \to 2) = W(1 \to \mathcal{O}) + W(\mathcal{O} \to 2) = U(1) - U(2) = \Delta T$$

From this we have

$$\Delta T = T_2 - T_1 = U_1 - U_2$$

rearranging terms give the mechanical energy

$$T_2 + U_2 = T_1 + U_1 = E$$

and for N conservative forces in a system

$$E = T + U_1 + U_2 + \cdots + U_N$$

Energy: Part 2

Conservative Force: Potential Energy The mechanical energy

$$E = T + U$$

is made up of the sum of the kinetic and potential energy. This is useful for

- obtaining equations of motion (EOM)

e.g. finding the EOM for a simple pendulum of mass m, length L and initial angle θ. The kinetic energy is

$$T = \frac{1}{2}mL^2\dot{\theta}^2$$

where the magnitude of velocity is the tangential component $v = L\omega = L\dot{\theta}$. The potential energy is

$$U = -mgy = -mgL\cos\theta$$

and the conservation of energy tells us that the mechanical energy is constant

$$\begin{aligned} T + U &= \text{constant} = E \\ \frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta &= E \end{aligned}$$

and in the intial condition we know that the velocity is zero $\dot{\theta} = 0$ and thus

$$-mgL\cos\theta_{max} = E$$

taking the time derivative of the energy equation gives

$$\begin{aligned} mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\dot{\theta} &= 0 \\ \ddot{\theta} + \frac{g}{L}\sin\theta &= 0 \\ \ddot{\theta} &= -\frac{g}{L}\sin\theta \end{aligned}$$

taking the time derivative of the energy is a useful trick for finding the EOM when we are trying to solve for $\dot{v}^2$.

Last time we found the potential energy for a position $\mathbf{r}$ in a conservative force field $\mathbf{F}(\mathbf{r})$ is

$$U(\mathbf{r}) = -W(\mathbf{r}_0 \to \mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}'$$

from the fundamental theorem of calculus (derivatives and intergrals are inverses) we want to find a function where the derivative equals the conservative force: First we take an infinitesimal change in the position $\mathbf{r} \to \mathbf{r} + \mathrm{d}\mathbf{r}$ and the change in potential energy is

$$\begin{aligned} U(\mathbf{r} + \mathrm{d}\mathbf{r}) &= -\int_{\mathbf{r}_0}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' - \int_{\mathbf{r}}^{\mathbf{r}+\mathrm{d}\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}' \\ &= U(\mathbf{r}) - F(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \end{aligned}$$

where is know that the force is constant over a small distance. Moving the terms gives

$$\begin{aligned} U(\mathbf{r}+\mathrm{d}\mathbf{r}) - U(\mathbf{r}) &= -\mathbf{F}\cdot \mathrm{d}\mathbf{r} \\ &= -(F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z) \end{aligned}$$

where we use Cartesian Coordinates, and we know that the gradient of potential is

$$\begin{aligned} \boldsymbol{\nabla} U &= \hat{\mathbf{x}}\frac{\partial U}{\partial x} + \hat{\mathbf{y}}\frac{\partial U}{\partial y} + \hat{\mathbf{z}}\frac{\partial U}{\partial z} \\ &= -(F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}) = -\mathbf{F} \end{aligned}$$

Example 3: 1D motion If we know what U is as a function of x, we can find the force! At points where $E = U$ we call these classical turning points. At a region of a relative minimum, a particle below the threshold of the turning point will oscillate between the two turning points. And at $E > U_max$ the particle is unbound and will escape the forces that attracted it.

Example 4:

$$\begin{aligned} E &= T + U(x) \text{ is constant} \\ T &= \frac{1}{2}m\dot{x}^2 = E - U(x) \\ \dot{x}^2 &= \frac{2}{m}(E - U(x)) \\ \dot{x} &= \pm\sqrt{\frac{2}{m}(E - U(x))} \end{aligned}$$

using seperation of variables

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{\frac{2}{m}(E - U(x))} \\ \sqrt{\frac{m}{2}}\,\mathrm{d}t &= \frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\ \int_{t_1}^{t_2}\sqrt{\frac{m}{2}}\,\mathrm{d}t &= \int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E - U(x)}} \\ (t_2 - t_1) &= \sqrt{\frac{2}{m}}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{E - U(x)}} \end{aligned}$$

where the sign of the velocity is positive within the oscillating bounds of the turning points and changes sign as the particle moves past the turning point.

Energy: Part 3

Last time: Conservative force as a negative gradient of potential:

$$\mathbf{F} = -\boldsymbol{\nabla} U$$

with classical turning points at $E = U$.

Conditions of a conservative force

- Only depends on position $\mathbf{r}$ (or just constant)
- Work done is path independent (this is sometimes hard to check) $\Leftrightarrow \boldsymbol{\nabla} \times \mathbf{F} = 0$

What is curl? In 3D Cartesian coordinates

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \hat{\mathbf{x}} + \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \hat{\mathbf{y}} + \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \hat{\mathbf{z}} () \right)\end{aligned}$$

Mathematically we know that if the force vector is a gradient of a scalar potential

$$\mathbf{F} = \boldsymbol{\nabla} \phi = -\boldsymbol{\nabla} U \quad \Leftrightarrow \quad \boldsymbol{\nabla} \times \mathbf{F} = 0$$

The curl of a gradient is always zero! Short 'proof':

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

and the curl

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial U}{\partial x} & -\frac{\partial U}{\partial y} & -\frac{\partial U}{\partial z} \end{vmatrix} = 0$$

Proving that the line integral is path independent is a little difficult, but in general we know that the two different paths a and b from points 1 to 2 we can write the work as

$$\int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_2 - \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}_1 = \oint_{a-b} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_A (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A} = 0$$

where we invoke Stokes' Theorem to find the integral of the curl over the surface A is zero.

Conservative Force: $\mathbf{F} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is conservative, but is

$$\mathbf{F}(\mathbf{r}) = F(\mathbf{r})\hat{\mathbf{r}}$$

a central force always conservative? We will need to check the curl of the force to find out. But first we define spherical coordinates (r, θ, ϕ)

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

and the central force in spherical coordinates is

$$\mathbf{F} = F(\mathbf{r})\hat{\mathbf{r}} + 0\hat{\theta} + 0\hat{\phi}$$

the curl in spherical coordinates is

$$\begin{aligned}\nabla \times \mathbf{F} = &\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right]\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\hat{\theta} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]\hat{\phi}\end{aligned}$$

And since

$$\frac{\partial F}{\partial \theta} = \frac{\partial F}{\partial \phi} = 0$$

the curl is zero $\nabla \times \mathbf{F} = 0$ and thus $\mathbf{F}$ is a conservative central force.

Gravity Conservative? The force due to gravity

$$\mathbf{F}_g = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}$$

is a central force as it only depends on $\mathbf{r}$. e.g. for a a two mass system:

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Using Translational Invariance, we can shift the origin to the center of m_2

$$\mathbf{r}_2 = 0 \quad \mathbf{F}_{12} = -\frac{Gm_1m_2}{\mathbf{r}_1^3}\mathbf{r}_1$$

or

$$F_{12} = -\nabla_1 U = -(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial z_1})$$

where the potential energy due to the interaction between 1 and 2 is

$$U_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

from newtons 3rd law, the force on 2 due to 1 is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

so

$$\begin{aligned}-\nabla_1 U_{12} &\rightarrow \mathbf{F}_{21} = \nabla_1 U_{12} \\ \nabla_1 U_{12}(\mathbf{r}_1 0 \mathbf{r}_2) &= -\nabla_2 U_{12}(\mathbf{r}_2 0 \mathbf{r}_1) \\ u_{12}(\mathbf{x}) &\quad \mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2 \\ \nabla_1 U_{12}(\mathbf{x}) &= \nabla_x U_{12}(\mathbf{x}) = -\nabla_2 U_{12}(\mathbf{x})\end{aligned}$$

so

$$\mathbf{F}_{12} = -\nabla_1 U_{12} \qquad \mathbf{F}_{21} = -\nabla_2 U_{12}$$

and for N particles

$$\mathbf{F}_i = -\nabla_i U \qquad U = \sum_{i,j} U_{ij} + \sum_i U_i^{\text{ext}}$$

6 Oscillations

& Simple Harmonic Oscillators For the simple case of a mas on a spring, the spring force is $F_s = -k(x - x_o)$ where the force is conservative and the (elastic) potential energy is $U_s = \frac{1}{2}k(x - x_o)^2$.

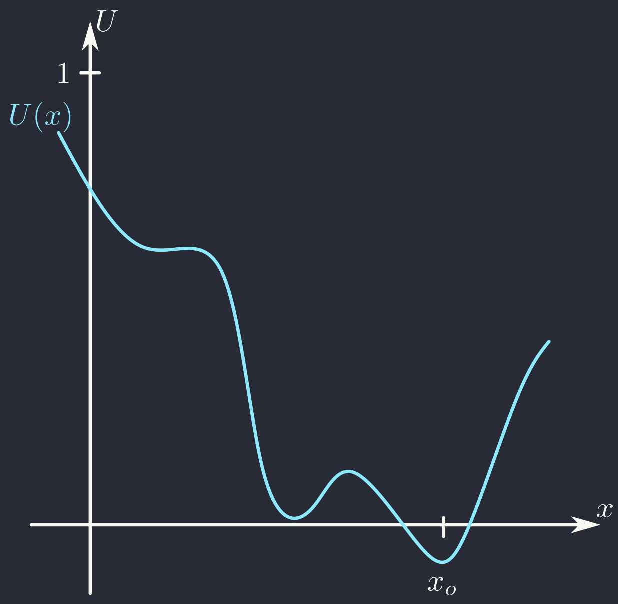

Figure 6.1: Arbitrary Potential Energy: $\mathbf{F} = -\boldsymbol{\nabla} U$

Aribtrary Potential energy For a equilibrium positon x_o we can take the taylor expansion of the potential energy

$$U(x) = U(x_o) + U'(x_o)\Delta x + \frac{1}{2}U''(x_o)\Delta x^2 + \dots$$

where $\Delta x = x - x_o$. Setting x_o to the reference point of U cancels the first term and the conservative nature tells us that the second term is also zero thus we are left with the third term where the spring constant is

$$k = U''(x_o)$$

To find the equations of motion, using N2L

$$m\ddot{x} = F = -k(x - x_o)$$
$$\ddot{x} = -\frac{k}{m}(x - x_o)$$

where we have a constant of angular frequency

$$\omega_o = \sqrt{\frac{k}{m}}$$

the solution could be a sinusoidal function

$$x(t) \approx \sin \omega_o t$$

but we are missing the initial value, so

$$x(t) \approx \sin \omega_o t + x_o$$

the general solution is linear combinations of the sine and cosine functions

$$x(t) = A\sin\omega_o t + B\cos\omega_o t + x_o$$
$$\dot{x}(t) = \omega_o A\cos\omega_o t - \omega_o B\sin\omega_o t$$

where we need 2 initial conditions to solve for A and B. e.g. $x(0)$ and $\dot{x}(0)$.

$$B = x(0) - x_o = \Delta x(0), \qquad A = \frac{\dot{x}(0)}{\omega_o}$$

Euler's Solution We can also use a general solution of the form

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where

$$\left|e^{i\theta}\right| = \cos^2\theta + \sin^2\theta = 1$$

taking the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{i\omega_o t} = i\omega_o e^{i\omega_o t}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{i\omega_o t} = -\omega_o^2 e^{i\omega_o t}$$

and the general solution is

$$x(t) = Ae^{i\omega_o t} + Be^{-i\omega_o t} + x_o$$

this does not mean that we have an imaginary solution, but rather we are using the geomtric nature of the solution.

Third Way We can also use a method where we introduce the phase

$$x(t) = A\cos(\omega_o t - \delta) + x_o$$
$$\dot{x}(t) = -A\omega_o\sin(\omega_o t - \delta)$$

for $t = 0$ we have

$$x(0) = A\cos(-\delta) + x_o = A\cos\delta + x_o$$
$$\dot{x}(0) = -A\omega_o\sin(-\delta) = A\omega_o\sin\delta$$

and the constants are found by squaring and adding the two equations

$$A^2 = (x(0) - x_o)^2 + \frac{\dot{x}(0)^2}{\omega_o^2} = \Delta x(0)^2 + \frac{\dot{x}(0)^2}{\omega_o^2}$$
$$\delta = \arctan\frac{\dot{x}(0)}{\omega_o(x(0) - x_o)} = \arctan\frac{\dot{x}(0)}{\omega_o\Delta x(0)}$$

Energy of the Oscillator The mechanical energy is $E = T + U$

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_o^2 A^2\sin^2(\omega_o t - \delta)$$
$$U = \frac{1}{2}k(x - x_o)^2 = \frac{1}{2}kA^2\cos^2(\omega_o t - \delta)$$

setting $x_o = 0$ we can work with a much simple case

$$U = \frac{1}{2}kx^2$$
$$T = \frac{1}{2}kx^2$$

using the third way where

$$
\begin{aligned}
x(t) &= A\cos(\omega_o t - \delta) \\
\dot{x}(t) &= -A\omega_o \sin(\omega_o t - \delta)
\end{aligned}
$$

we have

$$
\begin{aligned}
U &= \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_o t - \delta) \\
T &= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_o^2\sin^2(\omega_o t - \delta) &&= \frac{1}{2}kA^2\sin^2(\omega_o t - \delta)
\end{aligned}
$$

thus the total mechanical energy is

$$
E = T + U = \frac{1}{2}kA^2
$$

where this is the maximum potential energy of the system, or the potential energy at the maximum amplitude. This is also the classical turning point $E = U$. As time goes on, we can see that the energy oscillates betweeen being completely kinetic (T) and completely potential (U).

2D Oscillator We can have two cases of oscillation:

$$
\mathbf{F} = -k(\mathbf{r} - \mathbf{r}_o) \quad \text{isotropic oscillator}
$$

this is where each component share the same frequency, but different amplitudes and/or initial conditions

$$
\begin{aligned}
x(t) &= A_x\cos(\omega_o t - \delta_x) + x_o \\
y(t) &= A_y\cos(\omega_o t - \delta_y) + y_o
\end{aligned}
$$

for the anisotropic oscillator

$$
F_x = -k_x(x - x_o) \quad F_y = -k_y(y - y_o)
$$

the frequency is decoupled thus

$$
\begin{aligned}
x(t) &= A_x\cos(\omega_{ox} t - \delta_x) + x_o \\
y(t) &= A_y\cos(\omega_{oy} t - \delta_y) + y_o
\end{aligned}
$$

and if the ratio between the angular frequencies ω_{ox}/ω_{oy} are rational, the motion is periodic and the figure will be closed. But for irrational ratios, the motion is *quasiperiodic* and the figure is not closed (chaotic).

Oscillations: Damping

Damped Oscillator From last time the simple EOM for a spring is

$$m\ddot{x} = -k(x - x_o)$$

where the equilibrium position is x_o and the spring constant is k. When we add air resistance e.g. linear drag:

$$\mathbf{f} = -b\mathbf{v} \qquad m\ddot{x} = -kx - b\dot{x}$$

or

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

where we have two constants

$$\omega_o = \sqrt{\frac{k}{m}} \quad \beta = \frac{b}{2m}$$

where ω_o is the natural frequency and β is the damping coefficient. Rewriting in terms of the constants we get

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

where a general solution is

$$x = e^{rt}; \quad \dot{x} = re^{rt}; \quad \dot{x} = r^2 e^{rt}$$

plugging into the EOM gives

$$r^2 e^{rt} + 2\beta r e^{rt} + \omega_o^2 e^{rt} = 0$$
$$r^2 + 2\beta r + \omega_o^2 = 0$$

which is the characteristic (or auxiliary) equation, and the solution is in the form of the quadratic formula

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$$

thus the position is a linear combination of the two solutions

$$x(t) = C_1 e^{(-\beta + \sqrt{\beta^2 - \omega_o^2})t} + C_2 e^{(-\beta - \sqrt{\beta^2 - \omega_o^2})t}$$

At $\beta = 0$ (no damping)

$$\sqrt{\beta^2 - \omega_o^2} = \sqrt{-\omega_o^2} = i\omega$$

thus the solution of a SHO

$$x(t) = C_1 \exp(i\omega t) + C_2 \exp(-i\omega t)$$

Weak Damping For the case $\beta < \omega_o$ (underdamping)

$$\sqrt{\beta^2 - \omega_o^2} = i\sqrt{\omega_o^2 - \beta^2}$$

thus the solution is

$$x(t) = C_1 e^{(-\beta + i\sqrt{\omega_o^2 - \beta^2})t} + C_2 e^{(-\beta - i\sqrt{\omega_o^2 - \beta^2})t}$$
$$= e^{-\beta t}(C_1 \cos\left(\sqrt{\omega_o^2 - \beta^2}t\right) + C_2 \sin\left(\sqrt{\omega_o^2 - \beta^2}t\right))$$

we can simplify this with a new frequency term $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$ and therefore

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$

this is called underdamping because the amplitude oscillates and decays slowly.

Strong damping For the case $\beta > \omega_o$ we have the solution

$$x(t) = e^{-(\beta - \sqrt{\beta^2 - \omega_o^2})t}\left(C_1 + C_2 e^{-2\sqrt{\beta^2 - \omega_o^2}t}\right)$$

this called overdamping because the system cannot cannot complete a full oscillation, and decays exponentially to the equilibrium position. Thus we call the term

$$\text{decay parameter} = \beta - \sqrt{\beta^2 - \omega_o^2}$$

where the decaying tail is described by the decay parameter whereas the second term $-2\sqrt{\beta^2 - \omega_o^2}$ describes the fast initial damping of the system.

Large β For the case of $\beta \to \infty$ the decay parameter goes to zero:

$$\gamma = \beta - \sqrt{\beta^2 - \omega_o^2} \to 0$$

which is counter intuitive as the high damping coefficient results in a very slow exponential decay where it looks like a constant almost zero.

Critical Damping For the case $\beta = \omega_o$ we get

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t} \to x = e^{-\beta t}(C_1 + C_2 t)$$

where the extra factor of t comes from solving for a function $f(t)e^{-\beta t}$ to get the Constants. Pluggin this back into the initial EOM: $\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$

$$\dot{x} = e^{-\beta t} - te^{-\beta t} \qquad \ddot{x} = -2\beta e^{-\beta t} + te^{-\beta t}$$

so

$$-2\beta e^{-\beta t} + \beta^2 te^{-\beta t} + 2\beta e^{-\beta t} - 2\beta^2 te^{-\beta t} + \beta^2 te^{-\beta t} = 0$$

condition	γ
$\beta < \omega_o$	β
$\beta = \omega_o$	β
$\beta > \omega_o$	$\beta - \sqrt{\beta^2 - \omega_o^2}$

the critical damping will have the fastest decay of the system. The quickest way to stop an oscillating system is to apply a damping force at the natural frequency of the system.

NOTE: This all goes away when the magnitude of the damping force is not linear (e.g. quadratic drag). The linear EOM gives us something that can be easily analyzed, but for terms with higher powers (e.g. $\dot{x}^2$) the EOM becomes non-linear and the solutions are chaotic.

Driven Damped Oscillations

From last time: Note that the two parameters ω_o and β have the same units (rad/s) where we treat radians as a unitless quantity.

Time dependent force For the SHO we have a new EOM

$$m\ddot{x} + b\dot{x} + kx = F\cos(\omega t)$$

or in terms of the constants ω_o and β

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \frac{F(t)}{m} = f(t)$$

where $f(t)$ has the same units as acceleration/ force per unit mass. This is a inhomogeneous differential equation, but we can consider this as a combination of a homogeneous solution x_h and a particular solution x_p:

$$x_p(t) + x_h(t) = x(t)$$

denoting a differential operator

$$D = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_o^2$$

we know that

$$Dx_h(t) = 0 \qquad Dx_p(t) = f(t)$$

where from last time we know that the homogeneous solution is

$$x_h(t) = e^{-\beta t}(C_1\exp\left(t\sqrt{\beta^2 - \omega_o^2}\right) + C_2\exp\left(-t\sqrt{\beta^2 - \omega_o^2}\right))$$

and for the particular solution we can define the driving force as a sinusoidal function

$$f(t) = f_o\cos(\omega t) \quad \text{driving force}$$

where

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o\cos\omega t$$

or using Euler's formula we can define the EOM as the real part of the complex function

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f_o e^{i\omega t}$$

the particular solution is then

$$\begin{aligned} x &= Ce^{i\omega t} \\ \dot{x} = i\omega Ce^{i\omega t}, \qquad \ddot{x} &= -\omega^2 Ce^{i\omega t} \end{aligned}$$

subbing this back in to the EOM

$$-\omega^2 Ce^{i\omega t} + 2\beta i\omega Ce^{i\omega t} + \omega_o^2 Ce^{i\omega t} = f_o e^{i\omega t}$$

or

$$C = \frac{f_o}{\omega_o^2 - \omega^2 + 2i\beta\omega}$$

which is a complex number. Before taking the real part of the solution,

$$x(t) = \mathrm{Re}(Ce^{i\omega t})$$

we have to consider the complex part as well, so we rewrite C as

$$C = Ae^{-i\delta} = A\cos\delta - iA\sin\delta$$

so the real part of the solution is

$$x(t) = \mathrm{Re}(Ae^{i(\omega t-\delta)}) = A\cos(\omega t - \delta)$$

to find the amplitude A we have to multiply C by its complex conjugate

$$C^*C = |C|^2 = A^2$$

where we get the modulus squared. Thus we get

$$A^2 = \frac{f_o^2}{(\omega_o^2 - \omega^2)^2 + (2\beta\omega)^2}$$

to find δ we move some terms around

$$\begin{aligned} \frac{f_o}{\omega_o^2 - \omega^2 + 2i\beta\omega} &= Ae^{-i\delta} \\ f_o e^{i\delta} &= A(\omega_o^2 - \omega^2 + 2i\beta\omega) \\ f_o\cos\delta + if_o\sin\delta &= A(\omega_o^2 - \omega^2 + 2i\beta\omega) \end{aligned}$$

thus we have

$$\sin\delta = \frac{2A\beta\omega}{f_o} \qquad \cos\delta = \frac{A(\omega_o^2 - \omega^2)}{f_o}$$

dividing the two equations gives us the phase

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right)$$

The full solution is now

$$\begin{aligned}x(t) &= x_p(t) + x_h(t) \\ &= A\cos(\omega t - \delta) + C_1 e^{\left(-\beta+\sqrt{\beta^2-\omega_o^2}\right)t} + C_2 e^{\left(-\beta-\sqrt{\beta^2-\omega_o^2}\right)t}\end{aligned}$$

where ω is the driving frequency, and ω_o is the natural frequency. The last two exponential terms are known as the transient solution which decays very quick (exponentially) as shown in Figure 6.2 Finding

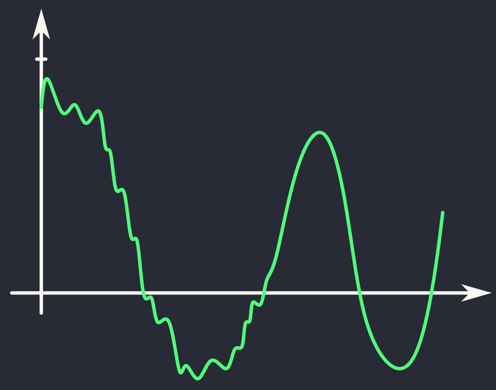

Figure 6.2: Driven Damped Oscillations

the maximum we look for where the derivative of A^2 is zero, or roughly

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left((\omega_o^2 - \omega^2)^2 + (2\beta\omega)^2\right) = 0$$

which gives us

$$\omega = \omega_2 = \sqrt{\omega_o^2 - 2\beta^2}$$

where at $B \ll \omega_o \to \omega \approx \omega_o$. Figure 6.3 shows that ω_2 is a resonant frequency where the amplitude is maximized.

$$A_{max} = \frac{f_o}{\sqrt{4\beta^2(\omega_o^2 - \omega^2)}} \approx \frac{f_o}{2\beta\omega} \quad \text{for} \quad \beta \ll \omega_o$$

What is δ? From the general solution, we can see that δ is a shift with respect to the driving force. This lag we can graph as a function of ω as shown in Figure 6.4

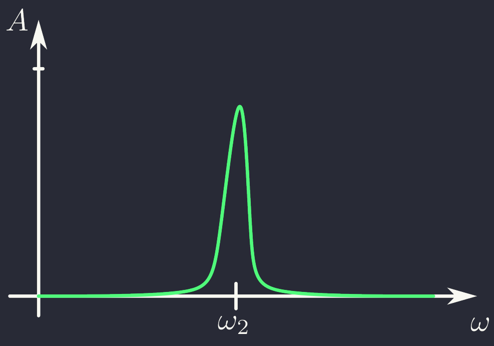

Figure 6.3: Resonance at ω_2

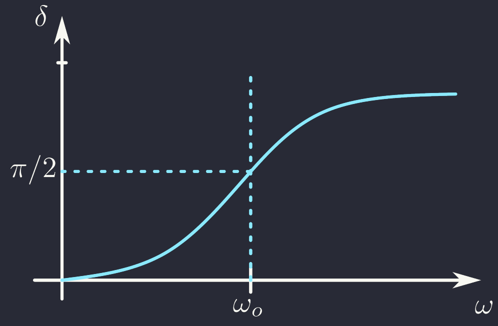

Figure 6.4: Phase shift δ