Homework 4

Due 2/14

1. (a) Imagining the electron as solid sphere, the moment of inertia is $I = \frac{2}{5}m_e r^2$. The speed at a point on its 'equator' is given by the tangential velocity $v = \omega r \implies \omega = \frac{v}{r}$. And if this electron is spinning with angular momentum $\ell = \hbar/2$, the speed of this point is

$$\begin{aligned} \ell &= I\omega \\ \frac{\hbar}{2} &= \frac{2}{5}m_e r^2 \frac{v}{r} \\ v &= \frac{5\hbar}{4m_e r} \end{aligned}$$

(b) If we have probed down to 10^{-18} and still haven't found any structure, then we would think that the radius of this electron solid sphere is $r < 10^{-18}$. Then the speed of the point on the equator is roughly

$$v > \frac{5 \times 10^{-34}}{4 \times 9 \times 10^{-31} \times 10^{-18}} \approx 10^{14}\,\text{m/s}$$

which is much faster than the speed of light $c = 3 \times 10^8$ m/s. So an electron is probably not spinning.

2. Given that the neutron, proton, and electron are all spin $S = 1/2$ particles, the total spin of the beta decay

$$n \to p + e^-$$

on the left side is $S_L = 1/2$ and on the right side we have $S_R = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. So the angular momentum is not conserved in this process. For the correct conservation of angular momentum, we need to include the neutrino in the decay

$$n \to p + e^- + \bar{\nu}_e$$

if we were to suppose the electron antineutrino had spin $S = 1/2$, then the total spin states could be $1/2$ or $3/2$ and the angular momentum would be conserved. We could also suppose it has spin $S = 3/2$ and the possible total spin states could be $1/2, 3/2$ or, $5/2$ which also conserves angular momentum. This means that any half integer spin would conserve angular momentum in the beta decay.

3. If the J_i's are Hermitian, then

$$J_i^\dagger = J_i, \quad J_j^\dagger = J_i, \quad J_k^\dagger = J_k$$

and the commutator is defined as

$$[J_i, J_j] = J_i J_j - J_j J_i$$

taking the Hermitian conjugate of the commutator

$$\begin{aligned} [J_i, J_j]^\dagger &= (J_i J_j - J_j J_i)^\dagger \\ &= (J_i J_j)^\dagger - (J_j J_i)^\dagger \\ &= J_j^\dagger J_i^\dagger - J_i^\dagger J_j^\dagger \\ &= J_j J_i - J_i J_j \\ &= -(J_i J_j - J_j J_i) = -[J_i, J_j] \end{aligned}$$

where on the 3rd step we know that the Hermitian adjoint (conjugate *transpose*) of a product of matrices is the product of the Hermitian adjoints in reverse order i.e. $(AB)^\dagger = B^\dagger A^\dagger$ because transposes do this. taking the Hermitian adjoint of the right hand side:

$$(if_{ijk}J_k)^\dagger = -if_{ijk}^\dagger J_k$$

so

$$-[J_i, J_j] = -if^\dagger_{ijk} J_k \rightarrow [J_i, J_j] = if^\dagger_{ijk} J_k$$

and in order for the commutator relation to be true, the structure constants must be real i.e. $f_{ijk} = f^\dagger_{ijk}$ (real numbers are Hermitian).

4. f as a sum:

$$f(x, y, z) = f_+(x, y, z) + f_-(x, y, z)$$

Parities are:

$$P(f_+) = +f_+, \quad P(f_-) = -f_- \quad P(f(x, y, z)) = f(-x, -y, -z)$$

where the parity of f is just the inversion(reflection + 180 degree rotation). Thus the parity of the RHS:

$$\begin{aligned} P(f(x, y, z)) &= P(f_+(x, y, z)) + P(f_-(x, y, z)) \\ f(-x, -y, z) &= (+f_+(x, y, z)) + (-f_-(x, y, z)) \end{aligned}$$

solving for f_+ and substituting $f_- = f - f_+$:

$$\begin{aligned} f_+(x, y, z) &= f(-x, -y, z) + f_-(x, y, z) \\ &= f(-x, -y, z) + f(x, y, z) - f_+(x, y, z) \\ 2f_+(x, y, z) &= f(-x, -y, z) + f(x, y, z) \\ f_+(x, y, z) &= \frac{1}{2}(f(x, y, z) + f(-x, -y, z)) \end{aligned}$$

and similarly by substituting the result back into $f_- = f - f_+$:

$$f_-(x, y, z) = \frac{1}{2}(f(x, y, z) - f(-x, -y, z))$$

we can see that

$$f_+(x, y, z) + f_-(x, y, z) = f(x, y, z)$$

where f_+ and f_- is an eigenfunction of the parity operator with eigenvalues $+1$ and -1 respectively.

5. (a) Given that for EM & Strong interactions, the parity must be conserved. For the decay

$$\eta \rightarrow 2\pi$$

the parity of η is $P(\eta) = -1$ and the parity of 2π is $P_{tot} = (P(\pi))^2 = (-1)^2 = 1$. So the parity is not conserved in this decay and thus forbidden for EM & Strong interactions.
(b) For the decay

$$\eta \rightarrow 3\pi$$

we can see that the parity is conserved: $P(\eta) = P_{tot} = (P(\pi))^3 = -1$. Since a G-parity violation forbids decay under Strong interactions, the G-parity of the two sides are:

$$G(\eta) = (-1)^0 C = 1(+1) = +1, \quad G(3\pi) = (-1)^3 = -1$$

so G-parity conservation is violated and the decay is forbidden under Strong interactions, but allowed for EM interactions due to Parity conservation.