1 Vector Analysis

1.1 What is a Vector?

In type we use boldface $\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}$, where we we can do some simple operations as such:

- Adding and Subtraction: $\mathbf{A} \pm \mathbf{B} = \mathbf{C}$ or aligning the head to the tail
- Multiplication:
  - Scalar: $\mathbf{A} \to 2\mathbf{A}$
  - Dot Product: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = AB\cos\theta$
  - Cross Product: $\mathbf{A} \times \mathbf{B} = AB\sin\theta$, and $\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$

Components of a Vector In 3D space, we often use the familiar Cartesian coordinates, e.g.

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

and we can add components by adding the components:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}$$

and likewise for subtraction. For the dot product:

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

or more shortly

$$= \sum_{i,j} A_iB_j\delta_{ij}$$

where δ_{ij} is the Kronecker delta. The cross product is a bit trickier...

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{\mathbf{x}} - (A_xB_z - A_zB_x)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}$$

This can also be written in short form using the Levi-Civita symbol (look it up)

Scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

This is the also the volume of the parallelepiped formed by the three vectors. NOTE that

$$\cancel{(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}}$$

since you can't cross a scalar with a vector.

Vector triple product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Using the BAC-CAB rule.

Some important vectors We define a position vector

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = r\hat{\mathbf{r}}$$

where the unit vector is

$$\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}$$

and an infinitesimal displacement vector

$$\mathrm{d}l = \mathrm{d}x\hat{\mathbf{x}} + \mathrm{d}y\hat{\mathbf{y}} + \mathrm{d}z\hat{\mathbf{z}}$$

In EM we define a source point $\mathbf{r}'$ (e.g. a charge) and a field point $\mathbf{r}$ that give us the seperation vector

$$\boldsymbol{\mathscr{r}} = \mathbf{r} = \mathbf{r}'$$

with magnitude

$$|\boldsymbol{\mathscr{r}}| = |\mathbf{r} - \mathbf{r}'|$$

and unit vector

$$\hat{\mathbf{r}} = \frac{\mathbf{r} - \mathbf{r}'}{\mathbf{r} - \mathbf{r}'}$$

1.2 Differential Calculus

And ordinary derivative $\frac{\mathrm{d}F}{\mathrm{d}x}$ is a change in $F(x)$ in $\mathrm{d}x$

$$\mathrm{d}F = \left(\frac{\partial F}{\partial x}\right)\mathrm{d}x$$

...geometrically, it's the slope

Gradient for functions of 2 or more variables, generalize for $h(x, y)$

$$\mathrm{d}h = \left(\frac{\partial h}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial h}{\partial y}\right)\mathrm{d}y$$

it's a scalar so $\mathrm{d}h = (\boldsymbol{\nabla} h) \cdot (\mathrm{d}l)$ where

$$\boldsymbol{\nabla} h = \frac{\partial h}{\partial x}\hat{\mathbf{x}} + \frac{\partial h}{\partial y}\hat{\mathbf{y}}$$

In 3D

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

If $\boldsymbol{\nabla} u = 0$, we are at an extremum (max, min, or shoulder/saddle point) Rewriting:

$$\boldsymbol{\nabla} T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)T(x, y, z)$$

wehere we can assume the ∇ as an "operator" acting on T:

1. Scalars like T: $\boldsymbol{\nabla} T$, "grad T", generalized slope
2. Dot product on $\mathbf{V}$: $\boldsymbol{\nabla} \cdot \mathbf{V}$, "divergence" or "div"
3. Cross product : $\boldsymbol{\nabla} \times \mathbf{V}$, "curl" or "rotatation"

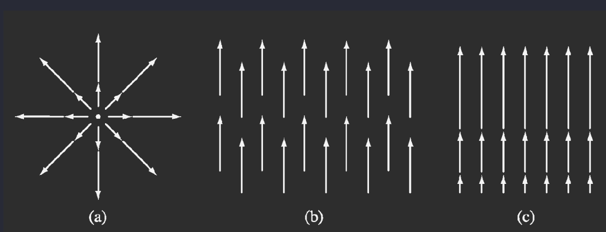

Figure 1.1: Divergence of field lines

Divergence From the Figure, we can see that (a) & (c) diverges, and (b) does not.

Geometrical Interpretation: Sources of positive divergence is a source or "faucet", and negative divergence is a sink or "drain".

Curl

$$\boldsymbol{\nabla} \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

E.g. for $\mathbf{V} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, $\boldsymbol{\nabla} \times \mathbf{V} = 2\hat{\mathbf{z}}$.

Combining Multiple Operations Two ways to get scalar from two functions:

$$fg \quad \text{or} \quad \mathbf{A} \cdot \mathbf{B}$$

Two ways to get vector from two functions:

$$f\mathbf{A} \quad \text{or} \quad \mathbf{A} \times \mathbf{B}$$

And we have 3 'derivatives': div, grad, and curl.

Product rule:

$$\partial_x(fg) = f\partial_x g + g\partial_x f$$

i $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f$

ii $\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + B \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}$

…(see Griffiths for more)

Second Derivatives Combining $\boldsymbol{\nabla}, \boldsymbol{\nabla}\cdot, \boldsymbol{\nabla}\times$

$\boldsymbol{\nabla} T$ is a vector

i

$$\begin{aligned} \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} T) &= (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}} + \partial_y + \hat{\mathbf{z}}\partial_z)\cdot(\hat{\mathbf{x}}\partial_x T + \hat{\mathbf{y}}\partial_y T + \hat{\mathbf{z}}\partial_z T) \\ &= \partial_x^2 T + \partial_y^2 T + \partial_z^2 T \\ &= \nabla^2 T \end{aligned}$$

ii $\boldsymbol{\nabla}\times(\boldsymbol{\nabla} T) = 0$

iii $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \dots$ ignored

iv $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = 0$

v $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

Integral Calculus: line, surface and volume integrals

"Fundamental theorem for gradients" Start with a scalar $T(x, y, z)$: from $a \to b$, in small steps $\mathrm{d}T = \boldsymbol{\nabla}\cdot T\,\mathrm{d}\boldsymbol{\ell}$

Total change in T:

$$\boxed{\int_a^b \mathrm{d}T = \int_a^b \boldsymbol{\nabla} T\cdot\mathrm{d}\boldsymbol{\ell} = T(b) - T(a)}$$

This line integral is path independent but $\int_a^b \mathbf{F}\cdot\mathrm{d}\ell$ is *not*!

Divergence Theorem, "Gauss' Theorem", or "Green's Theorem"

$$\boxed{\int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\mathrm{d}\tau = \oint_S v\cdot\mathrm{d}\mathbf{a}}$$

where V is the volume enclosed by the surface S. The divergence of a field in a volume is equivalent to the flux of the field at the boundary or surface.

Geometrical Interpretation: The "source"(or faucet) should present a flux (or flow) out through the surface.

Fundamental Theorem of Curls: Stokes' Theorem

$$\boxed{\oint_S (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{a} = \oint_P \mathbf{v}\cdot\mathrm{d}\boldsymbol{\ell}}$$

We have a 2D surfaces S bounded by a closed 1D perimeter P.

In 3D, imagine a soap bubble in a wire frame. We can change the surface of the bubble by moving the wire frame and making almost making a bubble, but the perimeter that bounds the bubble is still the wireframe.

Example:

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + 4yz^2\hat{\mathbf{z}}$$

On a surface S square on the $y - z$ plane:

$$\oint \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} =$$

RHS: We can split this into 4 integrals going counterclockwise around the square: First $x = 0, z = 0, y : 0 \to 1$: $dx = dz = 0$ $\int_0^1 3y^2 \mathrm{d}y = 1$

Second $\int_0^1 4z^2 \mathrm{d}z = 4/3$
Third: -1
Fourth: 0
Summing them all gives: $\oint \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell} = 4/3$
LHS: The curl gives: $4z^2 - 2x, -(0-0), 2z$ so

$$\oint (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint$$

...

1.3 Dirac Delta Function

Considering a vector function

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}$$

[insert picture of field lines]

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = 0$$

The div theorem doesn't obviously tell what volume and surface to choose, but we can choose a sphere of radius R and its corresponding surface:

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \frac{1}{R^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = 4\pi$$

where the polar angle is $\theta : 0 \to \pi$ and the azimuthal angle is $\phi : 0 \to 2\pi$.

We think that the divergence is zero (thus the theorem does not hold), but we make a tiny mistake:

$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ everywhere except at the origin $r \to 0$
and we stumbled on the Dirac delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1$$

where

$$\int f(x)\delta(x)\mathrm{d}x = f(0)$$

Shifting the delta function:

$$\delta(x-a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$

so

$$\int f(x)\delta(x-a)\mathrm{d}x = f(a)$$

Multiplying by a constant:

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Generalizing to 3D:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

Examples:

$$\int_V (\boldsymbol{\nabla} \cdot (\mathbf{v}))\mathrm{d}\tau = \int 4\pi\delta^3(\mathbf{r}) = 4\pi$$

and

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$