1 Newtons Laws

The Four Horsemen of the Apocalypse (In Physics)

- Classical Mechanics
- Electromagnetism
- Statistical Mechanics
- Quantum Mechanics

Before 1900, there was no relativity or QM and the world was a simple place ...

Newton's 1st Law: The Law of Inertia
And object keeps going unless acted on by a force.
This only applies to and 'inertial frame'.

Newton's 2nd Law: $\mathbf{F} = m\mathbf{a}$
Sum notation: The position vector is

$$\mathbf{r} = (x, y, z) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$$

in the Cartesian coordinate system. The time derivative gives the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

and acceleration is the time derivative of velocity

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

Thus in vector notation, Newton's 2nd law is

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$$

where $\mathbf{r}(t)$ is and ordinary differential equation (ODE).
The basic idea of solving mechanics problems is writing down the ODEs and solving them.

What is mass? m is an 'inertial mass'.
In Newton's law of gravity

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

m is the 'gravitational mass' and $g \approx 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.
A larger mass has a larger inertia or 'resistance to being accelerated' (Taylor). Key fact: When acceleration is zero ($\mathbf{a} = 0$), the velocity is constant ($\mathbf{v} = \text{constant}$).

Momentum: $\mathbf{p} = m\mathbf{v}$
The third law of motion in terms of momentum is

$$\mathbf{F} = \dot{\mathbf{p}} = m\dot{\mathbf{v}}$$

Newton's Third Law: $\mathbf{F}_{12} = -\mathbf{F}_{21}$
In a two body system, the total force of the system is

$$\mathbf{F}_t = \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

From the second law,

$$\dot{\mathbf{p}}_1 = \mathbf{F}_{21} \qquad \dot{\mathbf{p}}_2 = \mathbf{F}_{12}$$

adding these two equations gives

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0$$

thus the total momentum of the system is conserved.
For a system of N particles, the total momentum is

$$\mathrm{d}t \sum_i \mathbf{p}_i = \frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = \mathbf{F}_{ext}$$

sometimes $\mathbf{p}_{tot} = \mathbf{P}$ where the capital P denotes the total momentum of the system.

2 A pendulum

How to solve a problem:

1. Write down the eq
2. Solve it
3. Understand the solution

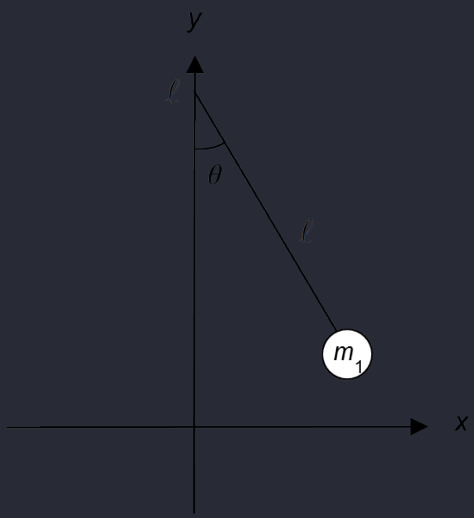

Figure 2.1: A pendulum with mass m and length l.

From Figure 2.1, we can write down Newton's 2nd law:

$$\begin{aligned}
\mathbf{F} &= m\mathbf{a} = m\ddot{\mathbf{r}} \\
F_x &= -mg\sin\theta = m\ddot{x} \\
F_y &= -mg\cos\theta + T\cos\theta = m\ddot{x}
\end{aligned}$$

Using a right triangle we can find the angle using $\tan\theta = x/y$. Furthermore, we can use the constrain that the length of the pendulum is constant thus $x^2 + y^2 = l^2$. But solving this system of equations is difficult. Instead we now use a new coordinate system.

Quick Hack Using the arc length $l = L\theta$ and choosing a coordinate in the direction of the pendulums path, we can write the force equation as

$$F_l = -mg\sin\theta = m\ddot{l} = mL\ddot{\theta}$$

Thus the equation of motion is

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

which is a second order ODE. This can only be solved with two conditions. We can use the initial conditions (at $t = 0$) of the position $\theta(t = 0) = \theta_0$ and velocity $\dot{\theta}(0) = 0$.

Polar Coordinates From Taylor:

$$x = r\cos\phi$$
$$y = r\sin\phi$$

For an arbitrary vector $\mathbf{v}$ it has the Cartesian vector components

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

Where the magnitude of the unit vectors are equivalent:

$$|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = 1$$

and the magnitude of the vector is

$$\begin{aligned}
|\mathbf{v}| &= \sqrt{\mathbf{v}\cdot\mathbf{v}} \\
&= \sqrt{v_x^2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} + 2v_xv_y\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}v_y^2\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}} \\
&= \sqrt{v_x^2 + v_y^2}
\end{aligned}$$

The vector $\mathbf{v}$ can be written in polar coordinates as

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\phi}$$

where radial vector is

$$\mathbf{r} = r\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

taking the time derivative of $\mathbf{r}$ gives the velocity

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}}$$

but how do we find $\dot{\hat{r}}$? We can look at the change in the direction of the radial unit vector for a small change in time Δt. Thus,

$$\Delta\hat{\mathbf{r}} \approx r\Delta\phi\hat{\boldsymbol{\phi}}$$

dividing both sides by Δt gives

$$\frac{\Delta\hat{\mathbf{r}}}{\Delta t} \approx r\frac{\Delta\phi}{\Delta t}\hat{\boldsymbol{\phi}} = r\dot{\phi}\hat{\boldsymbol{\phi}}$$

Therefore, the vector in polar coordinates is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} = v_r\hat{\mathbf{r}} + v_\phi\hat{\boldsymbol{\phi}}$$

where the polar components v_r and v_ϕ are related to the radial and angular velocity respectively. Taking the time derivative of $\dot{\mathbf{r}}$ gives the acceleration

$$\begin{aligned}
\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}} \\
&= \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\hat{r}} + \dot{v}_\phi\hat{\boldsymbol{\phi}} + v_\phi\dot{\hat{\phi}}
\end{aligned}$$

3 Polar Coordinates

using the geometric relation $\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$, we can write the acceleration as

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (r\ddot{\phi} + 2r\ddot{\phi})\hat{\boldsymbol{\phi}} \\ &= a_r\hat{\mathbf{r}} + a_\phi\hat{\boldsymbol{\phi}}\end{aligned}$$

where $r\dot{\phi}^2 = r\omega^2$ is the centripetal acceleration and $r\ddot{\phi} = r\dot{\omega}$ is the tangential acceleration.
From the Pendulum problem we know that the string is taut $r = L$ thus the radial velocity is zero $\dot{r} = 0$. Thus the force equation in the $\hat{\boldsymbol{\phi}}$ direction is

$$\begin{aligned}F_\phi &= mL\ddot{\phi} = -mg\sin\theta \\ \ddot{\phi} &= -\frac{g}{L}\sin\theta\end{aligned}$$

which is the same equation of motion.

Projectile in 2D The initial conditions of a general projectile is usually

$$\begin{aligned}F_x &= 0 = m\ddot{x} \\ F_y &= -mg = m\ddot{y}\end{aligned}$$

thus the equations of motion are

$$\begin{aligned}\ddot{x} &= 0 \\ \ddot{y} &= -g\end{aligned}$$

And solving these equations gives the position of the projectile

$$\begin{aligned}x(t) &= v_{ox}t \\ y(t) &= y_o v_{oy}t - \frac{1}{2}gt^2\end{aligned}$$

This can be expanded on with the addition of air resistance $\mathbf{f}$. This drag force is proportional to the velocity:

$$\mathbf{f} \propto -\hat{\mathbf{v}}$$

and there are two types of air resistance: linear

$$\mathbf{f}_l = -bv\hat{\mathbf{v}} = -b\mathbf{v}$$

and quadratic

$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

where we compare the terms with

$$\frac{f_l}{f_q} = \frac{cv}{b}$$

Linear $\mathbf{f}_l = -b\mathbf{v}$
From Newton's 2nd law

$$\begin{aligned} F_x &= -bv_x = m\ddot{x} = m\dot{v}_x \\ F_y &= -mg - bv_y = m\ddot{y} = m\dot{v}_y \end{aligned}$$

For the case of uncoupled differential equations (such as pure horizontal motion), we can solve the horizontal equation

$$\dot{v}_x = -\frac{b}{m} v_x$$

which has a general solution

$$v_x = Ae^{-kt}$$

where

$$k = \frac{b}{m}, \quad A = v_{ox}$$

to find the position we have to integrate $\dot{x} = v_x$:

$$\begin{aligned} x &= x_o + \int_0^t v_x(t')\,\mathrm{d}t' \\ &= x_o + \left[-\frac{v_{xo}}{k} e^{-kt}\right]\Bigg|_0^t \end{aligned}$$

where we have a limit of $t \to \infty$
For pure vertical motion, we solve the equation

$$\dot{v}_y = -g - \frac{b}{m} v_y \tag{3.1}$$

and with the initial condition $\dot{v}_y = 0$ we can solve for the velocity

$$v_y = -\frac{mg}{b} = v_{ter}$$

where v_{ter} is the terminal velocity. To get position, we use a trick by rewriting the equation as

$$m\dot{v}_y = -mg - bv_y = -mg - b(v_y - v_{ter})$$

and we can solve similar to the horizontal case using the general solution

$$v_y - v_{ter} = Ae^{-kt} = (v_{oy} - v_{ter})e^{-kt} bye$$

4 Air Resistance

Last time:

$$\mathbf{f}_l = -b\mathbf{v} \quad \dot{\mathbf{r}} = \mathbf{v}$$
$$\mathbf{f}_q = -cv^2\hat{\mathbf{v}}$$

In the case of linear, x motino has a range, y velocity has a terminal velocity v_t.

Horizontal Quadratic Drag

$$F_y = -mg - c|v_y|v_y$$
$$m\ddot{y} = F_y$$
$$m\dot{v}_y = -mg - c|v_y|v_y$$

when $v_y = 0$ we have the terminal velocity

$$v_{ter} = \sqrt{\frac{mg}{c}} \quad \text{or} \quad c = \frac{mg}{v_{ter}^2}$$

thus the equation of motion is

$$\dot{v}_y = -g - \frac{c}{m}v_y^2 = -g(1 - \frac{v_y^2}{v_{ter}^2}) = \frac{\mathrm{d}v_y}{\mathrm{d}t}$$

using separation of variables

$$\frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\,\mathrm{d}t$$

integrating both sides

$$\int_{v_{oy}}^{v_y} \frac{1}{1 - \frac{v_y^2}{v_t^2}}\,\mathrm{d}v_y = -g\int_0^t \mathrm{d}t$$

where we get the integral using the hyperbolic tangent

$$v_t \operatorname{arctanh}\frac{v_y}{v_t} = -gt$$
$$v_y = -v_t \tanh(gt)$$

2D Motion For Quadratic

$$F_x = -cvv_x = -c\sqrt{v_x^2 + v_y^2}v_x = m\dot{v}_x$$
$$F_y = -mg - cvv_y = -mg - c\sqrt{v_x^2 + v_y^2}v_y = m\dot{v}_y$$

where $v = \sqrt{v_x^2 + v_y^2}$. For linear, it is simply

$$F_x = -bv_x = m\dot{v}_x$$
$$F_y = -mg - bv_y = m\dot{v}_y$$