

# 1 Electric Fields in Matter

## 1.1 Polarization

### 1.1.1 Dielectrics

In dielectrics, “All charges are attached to specific atoms or molecules” (Griffith, pg.166)

### 1.1.2 Induced Dipoles

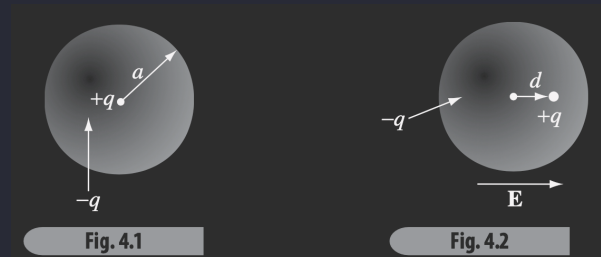


Figure 1.1: Left: Simple Nucleus  $+q$  surrounded by spherical cloud  $-q$  of radius  $a$ . Right: In an external electric field  $E$  the nucleus shifts right  $d$  and the cloud shifts left.

For a simple model of the atom (Fig. 1.1), the electric field at  $d$  is

$$E_d = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}$$

where the dipole moment is  $p = qd$ . We have equilibrium when

$$F_{\text{ext}} = qE_{\text{ext}} = qE_d$$

So using the dipole moment

$$|\mathbf{p}| = qd = 4\pi\epsilon_0 a^3 E_{\text{ext}} = \alpha E_{\text{ext}}$$

Here we have this “atomic polarizability”

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon v$$

where  $v$  is the volume of the atom. Comment: this crude approximation is still accurate by a factor of 4.

In general we have a vector

$$\mathbf{p} = \hat{\alpha} \mathbf{E}$$

where  $\hat{\alpha}$  is the polarizability tensor. For a linear dielectric relation between  $E$  and  $p$  we get

$$\hat{\alpha} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}$$

### 1.1.3 Alignment of Polar Molecules

The dipole will experience a torque in an  $E$ -field

$$\begin{aligned} \mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= \left( \frac{\mathbf{d}}{2} \times q\mathbf{E} \right) + \left( -\frac{\mathbf{d}}{2} \times q\mathbf{E} \right) \\ &= q\mathbf{d} \times \mathbf{E} \end{aligned}$$

thus

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

which implies that there is a force that acts to align  $\mathbf{p} \parallel \mathbf{E}$ .

**What if  $\mathbf{E}$  is not uniform?**

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q\mathbf{E} + q\mathbf{E} = q\Delta E$$

assuming small  $d$  in  $E_x$ , then

$$\Delta E_x = (\nabla E_x) \cdot \mathbf{d}$$

So the total change in the field is

$$\Delta \mathbf{E} = (\mathbf{d} \cdot \nabla) \mathbf{E}$$

thus

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}$$

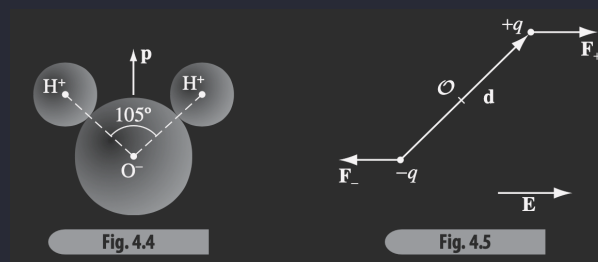


Figure 1.2: Dipole moment in oxygen molecule, and in an electric field.

**Example:** Problem 4.5 Using the method of images

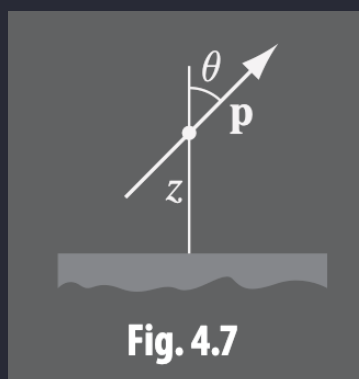


Figure 1.3: Infinitely grounded conductor with dipole at an angle  $\theta$  from the normal plane and nailed in place.

Where look at  $\mathbf{p}_i$  coordinate (pointing up) thus  $2z$  away we have the dipole pointing perpendicular to the image dipole i.e.

$$\mathbf{E}_i = \frac{1}{4\pi\epsilon_0} \frac{p_i}{r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

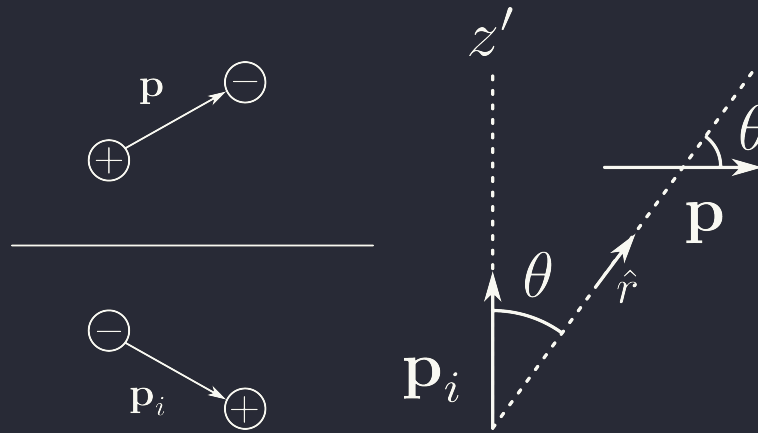


Figure 1.4: Left: Method of images using image dipole. Right: Coordinate using image dipole up as  $z'$ .

where

$$\mathbf{p} = p \cos \theta \hat{\mathbf{r}} + p \sin \theta \hat{\theta}$$

So the torque  $\mathbf{N} = \mathbf{p} \times \mathbf{E}_i$  is

$$\begin{aligned} \mathbf{N} &= \frac{1}{4\pi\epsilon_0} \frac{p^2}{(2z)^3} (\cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \times (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \\ &= \frac{1}{4\pi\epsilon_0} \frac{p^2}{(2z)^3} (\cos \theta \sin \theta \hat{\phi} + 2 \cos \theta \sin \theta (-\hat{\phi})) \\ &= \frac{1}{4\pi\epsilon_0} \frac{p^2 \cos \theta \sin \theta}{(2z)^3} (-\hat{\phi}) \\ &= \frac{1}{4\pi\epsilon_0} \frac{p^2 \sin(2\theta)}{8\pi\epsilon_0 (8z^3)} (-\hat{\phi}) \end{aligned}$$

So

$$\begin{cases} 0 < \theta < \frac{\pi}{2} & N \sim -\hat{\phi} \\ \frac{\pi}{2} < \theta < \pi & N \sim \hat{\phi} \end{cases}$$

Which means the dipole can either align perpendicularly up or down depending on the angle  $\theta$ .

### 1.1.4 Polarization

$\mathbf{P} \equiv$  dipole moment per unit volume

i.e. the little  $\mathbf{p}$  is

$$\mathbf{p} = \mathbf{P} d\tau$$

## 1.2 The Field of a Polarized Object

### 1.2.1 Bound Charges

For a single dipole

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{z}}}{r^2}$$

and using the dipole moment per unit volume def:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{z}}}{z^2} d\tau'$$

recalling the math fact

$$\nabla' \left( \frac{1}{z} \right) = \frac{\hat{\mathbf{z}}}{z^2}$$

thus

$$V = \frac{1}{4\pi\epsilon_0} \int \mathbf{P}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{z} \right) d\tau'$$

using another math fact

$$\nabla \cdot (F\mathbf{A}) = F(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla F)$$

So we can rewrite the integral

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \left[ \int_V \nabla \cdot \left( \frac{\mathbf{P}}{z} \right) d\tau' - \int_V \frac{1}{z} \nabla \cdot \mathbf{P} d\tau' \right] \\ &= \frac{1}{4\pi\epsilon_0} \left[ \oint_S \frac{1}{z} \mathbf{P} \cdot d\mathbf{a}' - \int_V \frac{1}{z} \nabla \cdot \mathbf{P} d\tau' \right] \end{aligned}$$

where we used the divergence theorem for the first term. For charge densities

$$\begin{cases} \text{surface charge} & \sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} \\ \text{volume charge} & \rho_b \equiv -\nabla \cdot \mathbf{P} \end{cases}$$

then

$$V = \frac{1}{4\pi\epsilon_0} \left[ \oint_S \frac{\sigma_b}{z} da' + \int_V \frac{\rho_b}{z} d\tau' \right]$$

### 1.2.2 Physical Interpretation of Bound Charges

So for a charge neutral sphere with an applied  $E$ -field, we can imagine this sphere as two oppositely charge spheres superimposed on each other but slightly shifted (Fig. 1.5). Thus we can imagine a collection of



Figure 1.5: Charge neutral sphere with applied  $E$ -field.

dipoles for each atom in a material with alternating charges. This is actually wrong (read Berry Phases in Electronic Structure Theory by David Vanderbilt).

**Example:** Find  $\mathbf{E}$  of a uniformly polarized sphere of radius  $R$ .

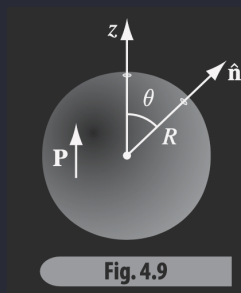


Figure 1.6: Uniformly polarized sphere of radius  $R$ .

We choose  $\mathbf{P} \propto \hat{\mathbf{z}}$  as shown in Fig. 1.6. The bound volume and surface charges are

$$\rho_b = -\nabla \cdot \mathbf{P} = 0 \quad \text{but} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos \theta$$

Thus using the result from before

$$V = \frac{1}{4\pi\epsilon_0} \left[ \oint_S \frac{\sigma_b}{z} da' + \int_V \frac{\rho_b}{z} d\tau' \right]$$

or

$$V(r, \theta) = \begin{cases} \frac{P}{3\epsilon_0} r \cos \theta & r \leq R \\ \frac{P}{3\epsilon_0} \frac{R^3}{r^2} \cos \theta & r > R \end{cases}$$

Recalling  $z = r \cos \theta$   $\mathbf{E} = -\nabla V$  we get

$$\mathbf{E}_{\text{in}} = -\frac{P}{3\epsilon_0} \hat{\mathbf{z}} = -\frac{1}{3\epsilon_0} \mathbf{P}$$

and for outside the potential is

$$V_{\text{out}} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

with the total dipole moment

$$\mathbf{p} = \frac{4\pi}{3} R^3 \mathbf{P}$$

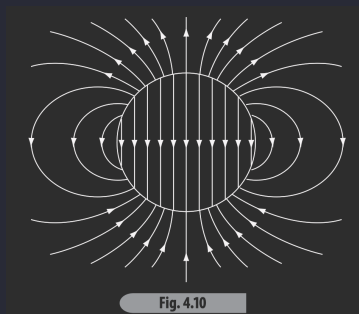


Figure 1.7: Polarized sphere in an external  $E$ -field.

So the polarized sphere is similar to the spherical conductor but with  $E$ -fields inside it (Fig. 1.7).

### 1.3 The Displacement Field

$$\rho_b = -\nabla \cdot \mathbf{P} \quad \text{and} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

The  $\rho_{\text{free}} \rightarrow$  anything *not* due to Polarization

$$\rho = \rho_b + \rho_f$$

and from Gauss' Law

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \implies \epsilon_0 \nabla \cdot \mathbf{E} &= \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f \end{aligned}$$

where  $\mathbf{E}$  is the total electric field. Moving some terms around we get

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

where we now define the electric displacement field

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

which has the same laws as  $\mathbf{E}$ :

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f} \quad \boxed{\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f, \text{enc}}}$$

**Example:** Long straight wire, with uniform  $\lambda$  (charge per length), surrounded by rubber insulation out to radius  $a$ ; Find  $\mathbf{D}$

Using the Gaussian surface (cylinder) of length  $l$  and radius  $s$  so the enclosed charge is

$$\begin{aligned} \oint \mathbf{D} \cdot d\mathbf{a} &= 2\pi s l \mathbf{D} = \lambda l \\ \mathbf{D} &= \frac{\lambda}{2\pi s} \hat{\mathbf{s}} \end{aligned}$$

$\implies$  holds for both  $s \leq a$  and  $s > a$ .

For  $s > a$

$$\mathbf{P}_{\text{out}} = 0 \implies \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s} \hat{\mathbf{s}}$$

But for inside  $s \leq a$  we can't determine  $\mathbf{P}_{\text{in}}$  yet!

**Comments** While the two equations

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

are similar, the  $E$ -field has a Coulomb law

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}}$$

but there is no equivalent for  $\mathbf{D}$

~~$$\mathbf{D} = \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}}$$~~

since there is this tensor relation  $\mathbf{p} = \hat{\alpha} \mathbf{E}$ . Furthermore, the curl is also different:

$$\nabla \times \mathbf{D} = \nabla \times (\epsilon_0 \mathbf{E} + \mathbf{P}) = \epsilon_0 \cancel{\nabla \times \mathbf{E}} + \nabla \times \mathbf{P}$$

since  $\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0$ .

**What is  $\mathbf{D}$ ?** Units  $\frac{\text{C}}{\text{m}^2}$  which is the same as  $\sigma$  (surface charge density).

## 1.4 Linear Dielectrics

### 1.4.1 Susceptibility, Permittivity, and Dielectric constant

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

where  $\chi_e$  is the electric susceptibility (dimensionless). For here, we will assume linear (isotropic & homogeneous) dielectrics.

- vacuum:  $\chi_e = 0$
- air: 1.00054
- salt:  $\sim 4.9$
- Si:  $\sim 11$
- water:  $\sim 80$  (water is already polarized!)
- $\text{SrTiO}_3$ :  $\sim 50,000$  at low temperatures

$\mathbf{E}$  is the *total* electric field.

Starting with

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ &= \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} \\ &= \epsilon_0 (1 + \chi_e) \mathbf{E} \quad \epsilon = \epsilon_0 (1 + \chi_e) \\ &= \epsilon \mathbf{E} \end{aligned}$$

where  $\epsilon$  is the *permittivity* of the material and the relative permittivity or *dielectric constant* is

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

**Example:** Metal sphere with charge  $Q$ , radius  $a$ , surrounded by a linear dielectric,  $\epsilon$ , out to radius  $b$ . Find potential at the center (relative to  $\infty$ ).

Inside the metal sphere  $\mathbf{E} = \mathbf{P} = \mathbf{D} = 0$ . Drawing the Gaussian surface between the sphere and the dielectric we get

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{\text{f, enc}}$$

for  $r > a$

$$\mathbf{D} = \frac{Q}{4\pi r^2} \hat{\mathbf{r}}$$

Now finding  $\mathbf{E}$  from  $\mathbf{D} = \epsilon \mathbf{E}$  i.e. in the vacuum  $\epsilon = \epsilon_0$  and in the dielectric  $\epsilon = \epsilon$ :

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi \epsilon r^2} & a < r < b \\ \frac{Q}{4\pi \epsilon_0 r^2} & r > b \end{cases}$$

The potential at the origin is therefore

$$\begin{aligned} V &= - \int_{\infty}^0 \mathbf{E} \cdot d\boldsymbol{\ell} = - \int_{\infty}^b \frac{Q}{4\pi \epsilon_0 r^2} dr - \int_b^a \frac{Q}{4\pi \epsilon r^2} dr - \int_a^0 0 dr \\ &= \frac{Q}{4\pi} \left( \frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right) \end{aligned}$$

Now we can find  $\mathbf{P}$  since  $\mathbf{E}$  is fixed by  $\mathbf{D}$ :

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon r^2} \hat{\mathbf{r}} \quad (\text{in the dielectric})$$

Thus we can get the volume bound charge

$$\rho_b = -\nabla \cdot \mathbf{P} = 0 \quad \nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = 0 \quad \text{except at } r = 0$$

and

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon b^2} & \text{outer} \\ -\frac{\epsilon_0 \chi_e Q}{4\pi \epsilon a^2} & \text{inner} \end{cases}$$

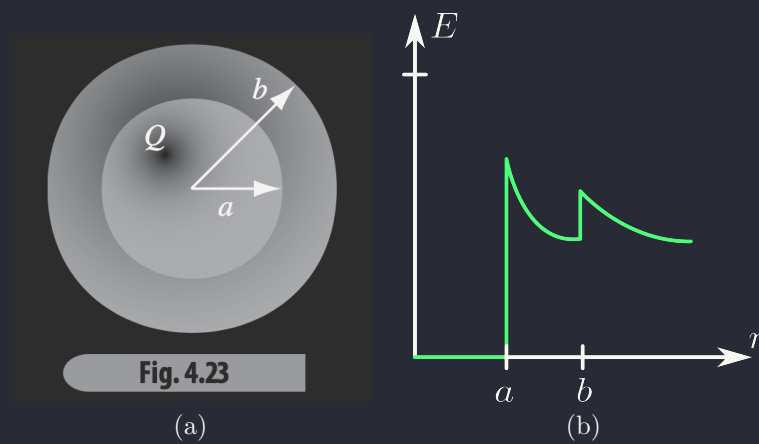


Figure 1.8: (a) Dielectric sphere surrounding metal sphere. (b) Electric field of resulting system.



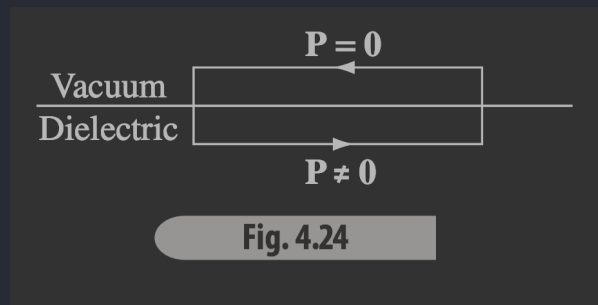


Figure 1.9: Interface between polarized dielectric and vacuum.

Since  $\mathbf{P}$  is zero on the vacuum side but not on the dielectric side

$$\oint \mathbf{P} \cdot d\boldsymbol{\ell} \neq 0$$

In addition, the displacement field  $\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P}$  implies

$$\oint \mathbf{D} \cdot d\mathbf{a} \neq 0 \neq \nabla \times \mathbf{D}$$

Furthermore, the proportionality factor  $\epsilon_0 \chi_e$  is different on both sides. For the homogeneous linear dielectric

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = 0$$

where

$$\begin{cases} \chi_f \mathbf{E}_{\text{vac}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \\ \chi_e \mathbf{D} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \end{cases} \implies \mathbf{E} = \epsilon \mathbf{E}$$

or

$$\mathbf{E} = \frac{1}{\epsilon} \mathbf{D}$$

**Example:** Parallel plate capacitor with insulating material of dielectric constant  $\epsilon_r$

$$\mathbf{E} \rightarrow \frac{\mathbf{E}}{\epsilon_r}$$

and the potential difference between the plates is

$$V = - \int \mathbf{E} \cdot d\boldsymbol{\ell} = -Ed \rightarrow \frac{Ed}{\epsilon_r}$$

And the capacitance is

$$C = \frac{Q}{V} \rightarrow \frac{\epsilon_r Q}{V} = \epsilon_r C$$

### Boundary Value Problems with Linear Dielectrics

From the given properties of the displacement vector

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ &= \epsilon_0 (1 + \chi_e) \mathbf{E} \end{aligned}$$

where  $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$  so

$$\mathbf{D} = \epsilon \mathbf{E}$$

We can get the bound charge densities

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left( \frac{\epsilon_0 \chi_e}{\epsilon} \mathbf{D} \right), \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

which implies

$$\rho_b = -\left( \frac{\chi_e}{1 + \chi_e} \right) \rho_f$$

Since the net charge resides in the surface of the dielectric, i.e.  $\rho = 0$  so from Laplace's equation

$$\epsilon_{\text{above}} \mathbf{E}_{\text{above}}^\perp - \epsilon_{\text{below}} \mathbf{E}_{\text{below}}^\perp = \sigma_f$$

or in terms of voltage (from  $\mathbf{E} = -\nabla V$ )

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f$$

And since the potential is still continuous

$$V_{\text{above}} = V_{\text{below}}$$

**Example:** Sphere of homogeneous linear dielectric material in a uniform  $\mathbf{E}_0$ . Find  $\mathbf{E}$  in sphere.

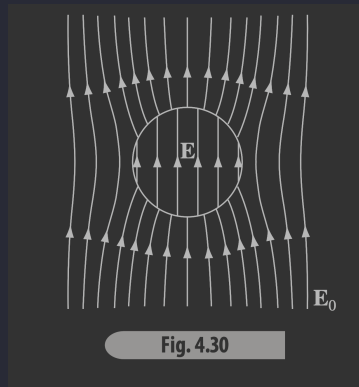


Figure 1.10: Homogeneous linear dielectric sphere in uniform E-field  $\mathbf{E}_0$ .

Using B.C. to solve Laplace's equation

- (i)  $V_{\text{in}} = V_{\text{out}}$  at  $r = R$
- (ii)  $\epsilon \frac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial r}$  at  $r = R$  where  $\epsilon = \epsilon_0 \epsilon_r$
- (iii)  $V_{\text{out}} = -E_0 r \cos \theta$  at  $r \gg R$

From the Legendre polynomials

$$V_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

and outside the sphere

$$V_{\text{out}} = \underbrace{-E_0 r \cos \theta}_{\text{external field}} + \underbrace{\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)}_{\text{due to sphere}}$$

and at  $r = R \rightarrow V_{\text{in}} = V_{\text{out}}$  so

$$\sum_l A_l R^l P_l(\cos \theta) = -E_0 R \cos \theta + \sum_l \frac{B_l}{R^{l+1}} P_l(\cos \theta)$$

Remembering that  $P_0 = 1$  and  $P_1 = \cos \theta$  we get from (i)

$$\begin{cases} A_1 R = -\epsilon_0 R + \frac{B_1}{R^2} & l = 1 \\ A_l R^l = \frac{B_l}{R^{l+1}} & l \neq 1 \end{cases}$$

and from (ii)

$$\epsilon_r \sum_l A_l R^{l+1} P_l(\cos \theta) = -\epsilon_0 \cos \theta - \sum_l \frac{(l+1)B_l}{R^{l+2}} P_l(\cos \theta)$$

so the orthogonality of the Legendre polynomials gives

$$\begin{cases} \epsilon_r A_1 = -\epsilon_0 - \frac{2B_1}{R^3} & l = 1 \\ \epsilon_r l A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} & l \neq 1 \end{cases}$$

After staring at it for some time and looking through the math

$$\begin{cases} A_l = B_l = 0 & l \neq 1 \\ A_1 = -\frac{3}{\epsilon_r + 2} E_0 \quad \text{and} \quad B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0 & l = 1 \end{cases}$$

Thus the potentials are

$$V_{\text{in}}(r, \theta) = -\frac{3\epsilon_0}{\epsilon_r + 2} r \cos \theta = -\frac{3\epsilon_0}{\epsilon_r + 2} z$$

so the electric field is

$$\mathbf{E}_{\text{in}} = -\nabla V = \frac{3}{\epsilon_r + 2} \mathbf{E}_0$$

The field inside the sphere will go to zero if the dielectric constant is infinite.

**Another example:** Everything below plane  $z = 0$  is linear dielectric with susceptibility  $\chi_e$ . Calculating the force on a charge  $q$  a distance  $d$  above the origin.

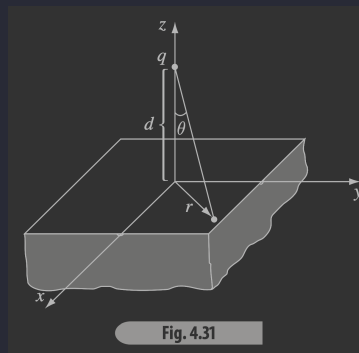


Figure 1.11: Everything below  $z = 0$  is a dielectric

Earlier we found a relationship in the volume of a dielectric sphere being  $\rho_b \propto \rho_f = 0$ , so we only need to worry about the bound surface charge density

$$\begin{aligned} \sigma_b &= \mathbf{P} \cdot \hat{\mathbf{n}} = P_z, \quad \mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \\ &= P_z = \epsilon_0 \chi_e E_z \end{aligned}$$

where  $E_z$  is the  $z$ -component of the TOTAL field. The total field due to  $q$  and  $\sigma_b$ :

