

1 Vector Analysis

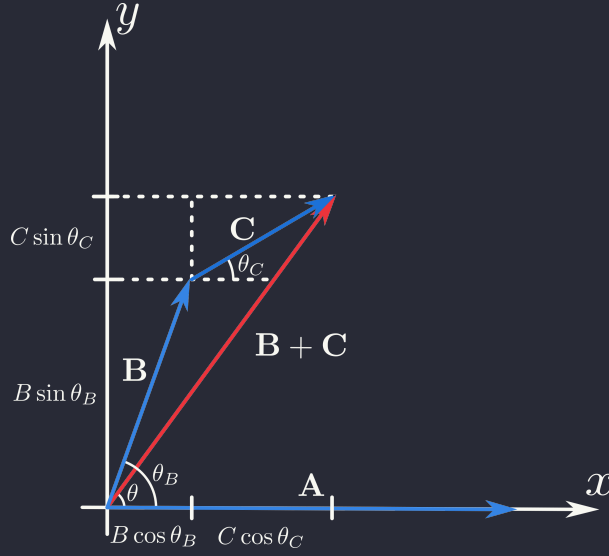


Figure 1.1: Three Coplanar Vectors

1.1 (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\ A(B + C) \cos \theta &= AB \cos \theta_B + AC \cos \theta_C\end{aligned}$$

Since $B \cos \theta_B + C \cos \theta_C = (B + C) \cos \theta$ from Figure 1.1, the distributive property holds true. The cross product also holds true since $B \sin \theta_B + C \sin \theta_C = (B + C) \sin \theta$, and multiplying by A on both sides gives the same result as the distributive property:

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ A(B + C) \sin \theta &= AB \sin \theta_B + AC \sin \theta_C\end{aligned}$$

(b) In the general case in three-dimensional space, each vector has three components: $\mathbf{A} = (A_x, A_y, A_z)$. Therefore,

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot (B_x + C_x, B_y + C_y, B_z + C_z) \\ &= A_x(B_x + C_x) + A_y(B_y + C_y) + A_z(B_z + C_z) \\ &= A_x B_x + A_y B_y + A_z B_z + A_x C_x + A_y C_y + A_z C_z \\ &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}\end{aligned}$$

1.2 Setting $\mathbf{A} = \mathbf{B} = (1, 1, 1)$ and $\mathbf{C} = (1, 1, -1)$:

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &\stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \\ 0 &\stackrel{?}{=} (1, 1, 1) \times [(1, 1, 1) \times (1, 1, -1)] \\ 0 &\stackrel{?}{=} (1, 1, 1) \times (-2, 2, 0) \\ 0 &\neq (-2, -2, 4)\end{aligned}$$

where the cross product of parallel vectors $\mathbf{A} \times \mathbf{B} = 0$. Therefore, the cross product is not associative.

1.3 Taking the dot product of a unit cube's body diagonals $\mathbf{A} = (1, 1, 1)$, $\mathbf{B} = (1, 1, -1)$:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= AB \cos \theta \\ 1 &= 3 \cos \theta \\ \theta &= \arccos 1/3 \approx 70.53^\circ\end{aligned}$$

1.4 The cross product of two vectors coplanar to the shaded plane— $\mathbf{A} = (-1, 2, 0)$, $\mathbf{B} = (-1, 0, 3)$ —is parallel to the normal unit vector $\hat{\mathbf{n}}$ of the plane:

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \mathbf{C} \\ \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} &= (6, 3, 2)\end{aligned}$$

where $\hat{\mathbf{n}} = \mathbf{C}/C$, and $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$. Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

1.5 Proving the “BAC–CAB” rule for three-dimensional vectors:

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix}\end{aligned}$$

where the x component is $A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$. Similarly,

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}(A_x C_x + A_y C_y + A_z C_z) - \mathbf{C}(A_x B_x + A_y B_y + A_z B_z)$$

where x component simplifies to

$$B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z B_z) = A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

the same is done for the y and z components. Therefore, the “BAC–CAB” rule holds true.

1.6

$$\begin{aligned}[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) = 0 \\ &\quad - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C} \cdot \mathbf{A})}\end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \\ \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \times \mathbf{B}\end{aligned}$$

For the relation to hold true, either the vectors \mathbf{A} and \mathbf{C} are parallel ($\mathbf{A} \times \mathbf{C} = 0$) or \mathbf{B} is perpendicular to both \mathbf{A} and \mathbf{C} ($\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = 0$).

1.7 Finding the separation vector \mathbf{z} :

$$\begin{aligned}\mathbf{z} &= \mathbf{r} - \mathbf{r}' = (4, 6, 8) - (2, 8, 7) = (2, -2, 1) \\ z &= \sqrt{2^2 + (-2)^2 + 1^2} = 3 \\ \hat{\mathbf{z}} &= \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\end{aligned}$$

1.8 (a)

$$\begin{aligned}\bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z &= (A_y \cos \phi + A_z \sin \phi)(B_y \cos \phi + B_z \sin \phi) \\ &\quad + (-A_y \sin \phi + A_z \cos \phi)(-B_y \sin \phi + B_z \cos \phi) \\ &= A_y B_y \cos^2 \phi + A_z B_z \sin^2 \phi + \cancel{A_y B_z \sin \phi \cos \phi} + \cancel{A_z B_y \sin \phi \cos \phi} \\ &\quad + A_y B_y \sin^2 \phi - \cancel{A_y B_z \sin \phi \cos \phi} - \cancel{A_z B_y \sin \phi \cos \phi} + A_z B_z \cos^2 \phi \\ &= A_y B_y (\sin^2 \phi + \cos^2 \phi) + A_z B_z (\sin^2 \phi + \cos^2 \phi) \\ \bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z &= A_y B_y + A_z B_z\end{aligned}$$

(b) To preserve length $|\bar{\mathbf{A}}| = |\mathbf{A}|$. Squaring both sides and expanding gives

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_x^2 + A_y^2 + A_z^2$$

in summation form,

$$\sum_{i=1}^3 \bar{A}_i \bar{A}_i = \sum_{i=1}^3 \left(\sum_{j=1}^3 R_{ij} A_j \right) \left(\sum_{k=1}^3 R_{ik} A_k \right) = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 R_{ij} R_{ik} A_j A_k$$

For the length to be preserved, the indices j and k must be equal. Therefore,

$$R_{ij} R_{ik} = \delta_{jk}$$

where δ_{ij} is the Kronecker delta. Thus the length is preserved if the rotation matrix is orthogonal, i.e.

$$R_{ij} R_{ik} = (R^T)_{ji} R_{ik} = \delta_{jk} \quad \text{or} \quad R^T R = I$$

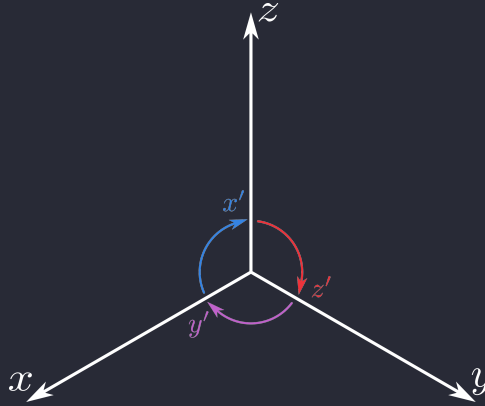


Figure 1.2: Rotation of 120° about an axis through the origin and point $(1, 1, 1)$

1.9 From Figure 1.2, the rotation is equivalent to changing the position of the basis vectors $\hat{\mathbf{x}} \rightarrow \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{y}}$. Therefore, the rotation matrix is a permutation matrix:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1.10 (a) Under a **translation** of coordinates $\bar{y} = y - a$, the origin O and terminus $A = (x, y, z)$ of some vector are translated to

$$\begin{aligned} O &\rightarrow O' = (0, -a, 0) \\ A &\rightarrow A' = (x, y - a, z) \end{aligned}$$

therefore, the translated vector is

$$\overline{O'A'} = (x, y - a, z) - (0, -a, 0) = (x, y, z) = \overline{OA} = \mathbf{A}$$

which is the same as the original vector, so

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) Under **inversion** of coordinates, only the terminus changes

$$\begin{aligned} O &\rightarrow O' = (0, 0, 0) \\ A &\rightarrow A' = (-x, -y, -z) \end{aligned}$$

therefore, the inverted vector is

$$\overline{O'A'} = (-x, -y, -z) \quad \text{or} \quad \begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

(c) The components of cross product under inversion

$$\bar{\mathbf{A}} \times \bar{\mathbf{B}} = \begin{pmatrix} \bar{A}_y \bar{B}_z - \bar{A}_z \bar{B}_y \\ \bar{A}_z \bar{B}_x - \bar{A}_x \bar{B}_z \\ \bar{A}_x \bar{B}_y - \bar{A}_y \bar{B}_x \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}$$

which is the same as the original cross product $\mathbf{A} \times \mathbf{B}$. The cross product of two pseudovectors is also a pseudovector. Torque $\tau = \mathbf{r} \times \mathbf{F}$ and magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ are examples of pseudovectors.

(d) Scalar triple product under inversion

$$\begin{aligned} \bar{A} \cdot (\bar{B} \times \bar{C}) &= -\mathbf{A} \cdot (-\mathbf{B} \times -\mathbf{C}) \\ &= -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \end{aligned}$$

the scalar triple product changes sign under inversion.

1.11 (a) Finding gradient of $f(x, y, z) = x^2 + y^3 + z^4$:

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \\ &= 2x \hat{\mathbf{x}} + 3y^2 \hat{\mathbf{y}} + 4z^3 \hat{\mathbf{z}} \end{aligned}$$

(b) Gradient of $f(x, y, z) = x^2 y^3 z^4$:

$$\nabla f = 2xy^3z^4 \hat{\mathbf{x}} + 3x^2y^2z^4 \hat{\mathbf{y}} + 4x^2y^3z^3 \hat{\mathbf{z}}$$

(c) Gradient of $f(x, y, z) = e^x \sin(y) \ln(z)$:

$$\nabla f = e^x \hat{\mathbf{x}} + e^x \cos(y) \ln(z) \hat{\mathbf{y}} + \frac{e^x \sin(y)}{z} \hat{\mathbf{z}}$$

1.12 The height of the hill (in feet) is given by the function

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is north and x is east in miles. The gradient of h is

$$\nabla h = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10(2x - 8y + 28)\hat{\mathbf{y}}$$

(a) The top of the hill is a stationary point, so the summit is found by setting the gradient to zero which gives the system of equations

$$0 = 2y - 6x - 18$$

$$0 = 2x - 8y + 28$$

adding the first equation to 3 times the second equation gives

$$0 = 2y - 6x - 18 + 3(2x - 8y + 28)$$

$$0 = -22y + 66$$

$$y = 3$$

substituting $y = 3$ into the first equation

$$0 = 2(3) - 6x - 18 \rightarrow x = -2$$

Therefore, the top of the hill is at $(-2, 3)$ or 2 miles west and 3 miles north of the origin.

(b) The height of the hill is simply $h(-2, 3) = 10(12) = 720$ feet.

(c) The steepness of the hill at $h(1, 1)$ is given by the magnitude of the gradient

$$\begin{aligned} |\nabla h| &= 10\sqrt{(2y - 6x - 18)^2 + (2x - 8y + 28)^2} \\ &= 10\sqrt{(2 - 6 - 18)^2 + (2 - 8 + 28)^2} \\ &= 10\sqrt{(-22)^2 + (22)^2} = 220\sqrt{2} \approx 311 \text{ ft/mi} \end{aligned}$$

The direction of the steepest slope is given by the vector in the direction of the gradient at the point $\nabla h(1, 1) = 220(-\mathbf{x} + \mathbf{y})$, or simply northwest.

1.13 Given the separation vector

$$\mathbf{z} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad z = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(a) Show that $\nabla(z^2) = 2\mathbf{z}$:

$$\nabla(z^2) = 2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}} = 2\mathbf{z}$$

(b)

$$\nabla\left(\frac{1}{z}\right) = \frac{\partial}{\partial x}\left(\frac{1}{z}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{z}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{z}\right)\hat{\mathbf{z}}$$

looking at the x component,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{z}\right) &= -\frac{1}{z^2}\frac{\partial}{\partial x}(z) \\ &= -\frac{1}{z^2}\frac{\partial}{\partial x}\left(\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\right) \\ &= -\frac{1}{z^2}\frac{1}{2}\frac{2(x - x')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ &= -\frac{x - x'}{z^3} \end{aligned}$$

therefore,

$$\nabla\left(\frac{1}{z}\right) = -\frac{1}{z^3}[(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}] = -\frac{\mathbf{z}}{z^3} = -\frac{\hat{\mathbf{z}}}{z^2}$$

(c) The general formula is

$$\nabla(z^n) = n z^{n-1} \hat{\mathbf{z}}$$

1.14 Given the rotation matrix

$$\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

or the two equations

$$\begin{aligned} \bar{y} &= y \cos \phi + z \sin \phi \\ \bar{z} &= -y \sin \phi + z \cos \phi \end{aligned}$$

differentiating with respect to \bar{y} and \bar{z} respectively gives

$$\begin{aligned} 1 &= \frac{\partial y}{\partial \bar{y}} \cos \phi + \frac{\partial z}{\partial \bar{y}} \sin \phi \\ 1 &= -\frac{\partial y}{\partial \bar{z}} \sin \phi + \frac{\partial z}{\partial \bar{z}} \cos \phi \end{aligned}$$

this can only be true if

$$\frac{\partial y}{\partial \bar{y}} = \cos \phi, \quad \frac{\partial z}{\partial \bar{y}} = \sin \phi \quad \text{and} \quad \frac{\partial y}{\partial \bar{z}} = -\sin \phi, \quad \frac{\partial z}{\partial \bar{z}} = \cos \phi$$

which satisfies the trig identity $\sin^2 \phi + \cos^2 \phi = 1$. Differentiating f with respect to the rotated coordinates \bar{y} and \bar{z} is given by

$$\begin{aligned} \frac{\partial f}{\partial \bar{y}} &= \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y} \cos \phi + \frac{\partial f}{\partial z} \sin \phi \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \bar{z}} = -\frac{\partial f}{\partial y} \sin \phi + \frac{\partial f}{\partial z} \cos \phi \end{aligned}$$

therefore, the gradient of f transforms as a vector under rotations given by

$$\overline{\nabla f} = \frac{\partial f}{\partial \bar{y}} \hat{\mathbf{y}} + \frac{\partial f}{\partial \bar{z}} \hat{\mathbf{z}} = \left(\frac{\partial f}{\partial y} \cos \phi + \frac{\partial f}{\partial z} \sin \phi \right) \hat{\mathbf{y}} + \left(-\frac{\partial f}{\partial y} \sin \phi + \frac{\partial f}{\partial z} \cos \phi \right) \hat{\mathbf{z}}$$

or in matrix form

$$\overline{\nabla f} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \nabla f$$

where the gradient is a column vector

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

1.15 (a) Calculating divergence of $v_a = x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}}$:

$$\begin{aligned} \nabla \cdot v_a &= \frac{\partial v_{ax}}{\partial x} + \frac{\partial v_{ay}}{\partial y} + \frac{\partial v_{az}}{\partial z} \\ &= 2x + 0 - 2x = 0 \end{aligned}$$

(b) $v_b = xy \hat{\mathbf{x}} + 2yz \hat{\mathbf{y}} + 3zx \hat{\mathbf{z}}$:

$$\nabla \cdot v_b = y + 2z + 3x$$

(c) $v_c = y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}}$:

$$\nabla \cdot v_c = 0 + 2x + 2y = 2(x + y)$$

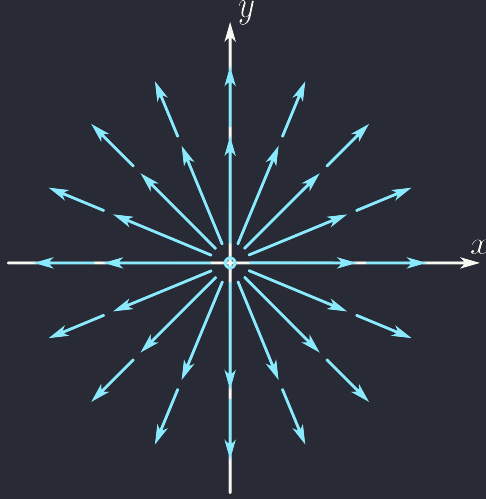


Figure 1.3: Sketch of the vector field $\mathbf{v} = \hat{\mathbf{r}}/r^2$

1.16 Given

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector. The vector functions is

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

where the components are

$$v_x = \frac{x}{r^3} \quad \text{and} \quad v_y = \frac{y}{r^3} \quad \text{and} \quad v_z = \frac{z}{r^3}$$

Looking at the x component of the divergence,

$$\begin{aligned} [\nabla \cdot \mathbf{v}]_x &= \frac{\partial v_x}{\partial x} \\ &= \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) \quad \text{using chain rule...} \\ &= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} \\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} \end{aligned}$$

therefore, the divergence of \mathbf{v} is

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \left(\frac{1}{r^3} - \frac{3x^2}{r^5} \right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5} \right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right) \\ &= \frac{3}{r^3} - 3 \frac{x^2 + y^2 + z^2}{r^5} \\ &= \frac{3}{r^3} - 3 \frac{r^2}{r^5} = 0 \end{aligned}$$

This is consistent with the sketch in Figure 1.3 because the vector field is not ‘sourcing’ or ‘sinking’.

1.17 Given

$$\bar{v}_y = v_y \cos \phi + v_z \sin \phi \quad \text{and} \quad \bar{v}_z = -v_y \sin \phi + v_z \cos \phi$$

Calculating the derivatives

$$\begin{aligned} \frac{\partial \bar{v}_y}{\partial \bar{y}} &= \frac{\partial v_y}{\partial y} \cos \phi + \frac{\partial v_z}{\partial y} \sin \phi \\ \frac{\partial \bar{v}_z}{\partial \bar{z}} &= -\frac{\partial v_y}{\partial z} \sin \phi + \frac{\partial v_z}{\partial z} \cos \phi \end{aligned}$$

from Problem 1.14,

$$\begin{aligned} \frac{\partial f}{\partial \bar{y}} &= \frac{\partial f}{\partial y} \cos \phi + \frac{\partial f}{\partial z} \sin \phi \\ \frac{\partial f}{\partial \bar{z}} &= -\frac{\partial f}{\partial y} \sin \phi + \frac{\partial f}{\partial z} \cos \phi \end{aligned}$$

therefore, the derivatives are rewritten as

$$\begin{aligned} \frac{\partial \bar{v}_y}{\partial \bar{y}} &= \left(\frac{\partial v_y}{\partial y} \frac{\partial y}{\partial \bar{y}} + \frac{\partial v_y}{\partial z} \frac{\partial z}{\partial \bar{y}} \right) \cos \phi + \left(\frac{\partial v_z}{\partial y} \frac{\partial y}{\partial \bar{y}} + \frac{\partial v_z}{\partial z} \frac{\partial z}{\partial \bar{y}} \right) \sin \phi \\ &= \left(\frac{\partial v_y}{\partial y} \cos \phi + \frac{\partial v_y}{\partial z} \sin \phi \right) \cos \phi + \left(\frac{\partial v_z}{\partial y} \cos \phi + \frac{\partial v_z}{\partial z} \sin \phi \right) \sin \phi \end{aligned}$$

and likewise,

$$\frac{\partial \bar{v}_z}{\partial \bar{z}} = -\left(-\frac{\partial v_y}{\partial y} \sin \phi + \frac{\partial v_y}{\partial z} \cos \phi \right) \sin \phi + \left(-\frac{\partial v_z}{\partial y} \sin \phi + \frac{\partial v_z}{\partial z} \cos \phi \right) \cos \phi$$

Finally adding the two equations together gives

$$\begin{aligned} \nabla \cdot \bar{\mathbf{v}} &= \frac{\partial \bar{v}_y}{\partial \bar{y}} + \frac{\partial \bar{v}_z}{\partial \bar{z}} \\ &= \frac{\partial v_y}{\partial y} \cos^2 \phi + \frac{\partial v_y}{\partial z} \sin \phi \cos \phi + \frac{\partial v_z}{\partial y} \sin \phi \cos \phi + \frac{\partial v_z}{\partial z} \sin^2 \phi \\ &\quad + \frac{\partial v_y}{\partial y} \sin^2 \phi - \frac{\partial v_y}{\partial z} \sin \phi \cos \phi - \frac{\partial v_z}{\partial y} \sin \phi \cos \phi + \frac{\partial v_z}{\partial z} \cos^2 \phi \\ &= (\sin^2 \phi + \cos^2 \phi) \left[\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] \\ &= \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \end{aligned}$$

which shows that the divergence transforms as a scalar under rotations.

1.18 Curl of vector functions from Problem 1.15: (a) $\mathbf{v}_a = x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}}$:

$$\begin{aligned} \nabla \times \mathbf{v}_a &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\ &= \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0) \\ &= -6xz \hat{\mathbf{x}} + 2z \hat{\mathbf{y}} + 3z^2 \hat{\mathbf{z}} \end{aligned}$$

(b) $\mathbf{v}_b = xy \hat{\mathbf{x}} + 2yz \hat{\mathbf{y}} + 3zx \hat{\mathbf{z}}$:

$$\begin{aligned} \nabla \times \mathbf{v}_b &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \\ &= \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x) \\ &= -2y \hat{\mathbf{x}} - 3z \hat{\mathbf{y}} - x \hat{\mathbf{z}} \end{aligned}$$

(c) $\mathbf{v}_c = y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}}$:

$$\begin{aligned}\nabla \times \mathbf{v}_c &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\ &= \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) \\ &= 0\end{aligned}$$

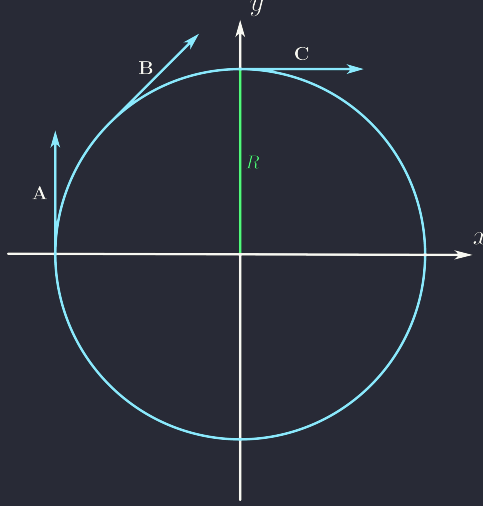


Figure 1.4: Sketch of the vector field pointing clockwise around a circle of radius R

1.19 From Figure 1.4, the sign of $\partial v_x / \partial y$ is positive, and the sign of $\partial v_y / \partial x$ is negative. Therefore, the curl

$$\nabla \times \mathbf{v} = \hat{\mathbf{z}} \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right)$$

is in the negative z direction, or into the page. This is consistent with the right-hand rule as the thumb points into the page.

1.20 Proof that the vector function

$$\mathbf{g} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3}$$

has zero divergence and curl given

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{and} \quad \frac{\partial x}{\partial y} = \frac{y}{x} = 0$$

From Problem 1.16, the divergence of \mathbf{g} is

$$\begin{aligned}\nabla \cdot \mathbf{g} &= \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} \\ &= \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5} \\ &= \frac{3}{r^3} - \frac{3}{r^3} = 0\end{aligned}$$

and the curl is

$$\begin{aligned}\nabla \times \mathbf{g} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} \\ &= \hat{\mathbf{x}}(0-0) - \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-0) = 0\end{aligned}$$

1.21 Proving product rule for (i)

$$\begin{aligned}\nabla(fg) &= \frac{\partial(fg)}{\partial x} \hat{\mathbf{x}} + \frac{\partial(fg)}{\partial y} \hat{\mathbf{y}} + \frac{\partial(fg)}{\partial z} \hat{\mathbf{z}} \\ &= \left(\frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x} \right) \hat{\mathbf{x}} + \left(\frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y} \right) \hat{\mathbf{y}} + \left(\frac{\partial f}{\partial z} g + f \frac{\partial g}{\partial z} \right) \hat{\mathbf{z}} \\ &= f \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} \right) + g \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right) \\ &= f \nabla g + g \nabla f\end{aligned}$$

(iv)

$$\begin{aligned}\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \nabla \cdot [(A_y B_z - A_z B_y) \hat{\mathbf{x}} + (A_z B_x - A_x B_z) \hat{\mathbf{y}} + (A_x B_y - A_y B_x) \hat{\mathbf{z}}] \\ &= \frac{\partial}{\partial x} (A_y B_z - A_z B_y) + \frac{\partial}{\partial y} (A_z B_x - A_x B_z) + \frac{\partial}{\partial z} (A_x B_y - A_y B_x) \\ &= \left(\frac{\partial A_y}{\partial x} B_z + A_y \frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x} B_y - A_z \frac{\partial B_y}{\partial x} \right) + \left(\frac{\partial A_z}{\partial y} B_x + A_z \frac{\partial B_x}{\partial y} - \frac{\partial A_x}{\partial y} B_z - A_x \frac{\partial B_z}{\partial y} \right) \\ &\quad + \left(\frac{\partial A_x}{\partial z} B_y + A_x \frac{\partial B_y}{\partial z} - \frac{\partial A_y}{\partial z} B_x - A_y \frac{\partial B_x}{\partial z} \right) \\ &= B_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + B_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + B_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ &\quad + A_x \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \right) + A_y \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \right) + A_z \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) \\ &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot (\nabla \times \mathbf{B})\end{aligned}$$

(v)

$$\begin{aligned}\nabla \times (f\mathbf{A}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix} \\ &= \hat{\mathbf{x}} \left(\frac{\partial}{\partial y} (fA_z) - \frac{\partial}{\partial z} (fA_y) \right) - \hat{\mathbf{y}} \left(\frac{\partial}{\partial x} (fA_z) - \frac{\partial}{\partial z} (fA_x) \right) + \hat{\mathbf{z}} \left(\frac{\partial}{\partial x} (fA_y) - \frac{\partial}{\partial y} (fA_x) \right) \\ &= \hat{\mathbf{x}} \left(f \frac{\partial A_z}{\partial y} + A_z \frac{\partial f}{\partial y} - f \frac{\partial A_y}{\partial z} - A_y \frac{\partial f}{\partial z} \right) - \hat{\mathbf{y}} \left(f \frac{\partial A_z}{\partial x} + A_z \frac{\partial f}{\partial x} - f \frac{\partial A_x}{\partial z} - A_x \frac{\partial f}{\partial z} \right) \\ &\quad + \hat{\mathbf{z}} \left(f \frac{\partial A_y}{\partial x} + A_y \frac{\partial f}{\partial x} - f \frac{\partial A_x}{\partial y} - A_x \frac{\partial f}{\partial y} \right) \\ &= f \left[\hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \\ &\quad - \hat{\mathbf{x}} \left(A_y \frac{\partial f}{\partial z} - A_z \frac{\partial f}{\partial y} \right) + \hat{\mathbf{y}} \left(A_z \frac{\partial f}{\partial x} - A_x \frac{\partial f}{\partial z} \right) - \hat{\mathbf{z}} \left(A_x \frac{\partial f}{\partial y} - A_y \frac{\partial f}{\partial x} \right) \\ &= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)\end{aligned}$$

1.22 (a) If \mathbf{A} and \mathbf{B} are two vector functions, then

$$\begin{aligned} (\mathbf{A} \cdot \nabla) \mathbf{B} &= \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\ &= \hat{\mathbf{x}} \left(A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right) + \hat{\mathbf{y}} \left(A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z} \right) \\ &\quad + \hat{\mathbf{z}} \left(A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z} \right) \end{aligned}$$

This means that the direction of \mathbf{A} points in the direction of where \mathbf{B} moves fastest.

(b)

$$(\hat{\mathbf{r}} \cdot \nabla) \hat{\mathbf{r}} = \frac{1}{r} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \frac{(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{r}$$

looking at the x component,

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\mathbf{r}}{r} \right) &= \frac{1}{r} \frac{\partial}{\partial x} (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) + \mathbf{r} \frac{\partial}{\partial x} \left(\frac{1}{r} \right) \\ &= \frac{\hat{\mathbf{x}}}{r} - \mathbf{r} \frac{x^2}{r^3} \end{aligned}$$

therefore,

$$\begin{aligned} (\hat{\mathbf{r}} \cdot \nabla) \hat{\mathbf{r}} &= \frac{1}{r} \left[x \left(\frac{\hat{\mathbf{x}}}{r} - \mathbf{r} \frac{x^2}{r^3} \right) + y \left(\frac{\hat{\mathbf{y}}}{r} - \mathbf{r} \frac{y^2}{r^3} \right) + z \left(\frac{\hat{\mathbf{z}}}{r} - \mathbf{r} \frac{z^2}{r^3} \right) \right] \\ &= \frac{1}{r} \left[\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r} - \mathbf{r} \frac{x^2 + y^2 + z^2}{r^3} \right] \\ &= \frac{1}{r} \left[\frac{\mathbf{r}}{r} - \frac{\mathbf{r}}{r} \right] = 0 \end{aligned}$$

(c)

$$\begin{aligned} (v_a \cdot \nabla) v_b &= \left(x^2 \frac{\partial}{\partial x} + 3xz^2 \frac{\partial}{\partial y} - 2xz \frac{\partial}{\partial z} \right) (xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}) \\ &= x^2(y\hat{\mathbf{x}} + 0 + 3z\hat{\mathbf{z}}) + 3xz^2(x\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 0) - 2xz(0 + 2y\hat{\mathbf{y}} + 3x\hat{\mathbf{z}}) \\ &= (x^2y + 3x^2z^2)\hat{\mathbf{x}} + (6xz^3 - 4xyz)\hat{\mathbf{y}} + (3x^2z - 6x^2z)\hat{\mathbf{z}} \\ &= x^2(y + 3z^2)\hat{\mathbf{x}} + 2xz(3z^2 - 2y)\hat{\mathbf{y}} - 3x^2z\hat{\mathbf{z}} \end{aligned}$$

1.23 Proving the product rule for (ii) given the x component of the left hand side is

$$\begin{aligned} [\nabla(\mathbf{A} \cdot \mathbf{B})]_x &= \frac{\partial(\mathbf{A} \cdot \mathbf{B})}{\partial x} \hat{\mathbf{x}} \\ &= \frac{\partial}{\partial x} (A_x B_x + A_y B_y + A_z B_z) \hat{\mathbf{x}} \\ &= A_x \frac{\partial B_x}{\partial x} + B_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial B_y}{\partial x} + B_y \frac{\partial A_y}{\partial x} + A_z \frac{\partial B_z}{\partial x} + B_z \frac{\partial A_z}{\partial x} \end{aligned}$$

Finding the x component of the right hand side of (ii)

$$\begin{aligned}
[\mathbf{A} \times (\nabla \times \mathbf{B})]_x &= \left[\mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \right]_x \\
&= \left[\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ () & -(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}) & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{vmatrix} \right]_x \\
&= A_y \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) - A_z \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right)
\end{aligned}$$

and

$$[\mathbf{B} \times (\nabla \times \mathbf{A})]_x = B_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - B_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)$$

From Problem 1.22:

$$[(\mathbf{A} \cdot \nabla) \mathbf{B}] = A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}$$

and likewise,

$$[(\mathbf{B} \cdot \nabla) \mathbf{A}] = B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z}$$

Adding all four equations gives

$$\begin{aligned}
&[\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}]_x = \\
&A_y \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) - A_z \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + B_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - B_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\
&+ A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\
&= A_x \frac{\partial B_x}{\partial x} + A_y \left(\frac{\partial B_y}{\partial x} - \cancel{\frac{\partial B_x}{\partial y}} + \cancel{\frac{\partial B_x}{\partial y}} \right) + A_z \left(\frac{\partial B_z}{\partial x} - \cancel{\frac{\partial B_x}{\partial z}} + \cancel{\frac{\partial B_x}{\partial z}} \right) \\
&+ B_x \frac{\partial A_x}{\partial x} + B_y \left(\frac{\partial A_y}{\partial x} - \cancel{\frac{\partial A_x}{\partial y}} + \cancel{\frac{\partial A_x}{\partial y}} \right) + B_z \left(\frac{\partial A_z}{\partial x} - \cancel{\frac{\partial A_x}{\partial z}} + \cancel{\frac{\partial A_x}{\partial z}} \right) \\
&= A_x \frac{\partial B_x}{\partial x} + B_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + B_y \frac{\partial A_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + B_z \frac{\partial A_x}{\partial z} \\
&= [\nabla(\mathbf{A} \cdot \mathbf{B})]_x
\end{aligned}$$

and likewise for the y and z components.

For (vi), the x on the left hand side is

$$\begin{aligned}
[\nabla \times (\mathbf{A} \times \mathbf{B})]_x &= \left[\nabla \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \right]_x \\
&= \left[\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ () & -(A_x B_z - A_z B_x) & A_x B_y - A_y B_x \end{vmatrix} \right]_x \\
&= \frac{\partial}{\partial y} (A_x B_y - A_y B_x) - \frac{\partial}{\partial z} (A_z B_x - A_x B_z) \\
&= A_x \frac{\partial B_y}{\partial y} + B_y \frac{\partial A_x}{\partial y} - A_y \frac{\partial B_x}{\partial y} - B_x \frac{\partial A_y}{\partial y} \\
&- A_z \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_z}{\partial z} + A_x \frac{\partial B_z}{\partial z} + B_z \frac{\partial A_x}{\partial z} + \\
&= A_x \left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} - B_x \left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z}
\end{aligned}$$

On the right hand side, first we find the x component of the two new operations:

$$\begin{aligned} [A(\nabla \cdot \mathbf{B})]_x &= \left[A \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) \right]_x \\ &= A_x \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) \end{aligned}$$

and likewise,

$$[B(\nabla \cdot \mathbf{A})]_x = B_x \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

Therefore, $[(\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + A(\nabla \cdot \mathbf{B}) - B(\nabla \cdot \mathbf{A})]_x =$

$$\begin{aligned} & B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} - \left(A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right) \\ & + A_x \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - \left(B_x \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \right) \\ & = A_x \left(\cancel{\frac{\partial B_x}{\partial x}} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} - \cancel{\frac{\partial B_x}{\partial x}} \right) - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} \\ & - B_x \left(\cancel{\frac{\partial A_x}{\partial x}} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} - \cancel{\frac{\partial A_x}{\partial x}} \right) + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\ & = A_x \left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} - B_x \left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\ & = [\nabla \times (\mathbf{A} \times \mathbf{B})]_x \end{aligned}$$

and likewise for the y and z components.

1.24 Deriving the three quotient rules from the product rule: The gradient is

$$\begin{aligned} \nabla \left(\frac{f}{g} \right) &= \nabla (fg^{-1}) = f\nabla(g^{-1}) + g^{-1}\nabla(f) \\ &= f(-g^{-2}\nabla(g)) + g^{-1}\nabla(f) \\ &= -\frac{f}{g^2}\nabla(g) + \frac{g}{g} \frac{1}{g}\nabla(f) \\ &= \frac{g\nabla f - f\nabla g}{g^2} \end{aligned}$$

the divergence

$$\begin{aligned} \nabla \cdot \left(\frac{\mathbf{A}}{g} \right) &= \nabla \cdot (Ag^{-1}) = A(\nabla \cdot g^{-1}) + g^{-1}(\nabla \cdot \mathbf{A}) \\ &= A(-g^{-2}(\nabla \cdot g)) + \frac{g}{g} g^{-1}(\nabla \cdot \mathbf{A}) \\ &= \frac{g(\nabla \cdot \mathbf{A}) - A\nabla \cdot g}{g^2} \end{aligned}$$

and the curl

$$\begin{aligned} \left[\nabla \times \left(\frac{\mathbf{A}}{g} \right) \right]_x &= \frac{\partial}{\partial y} \left(\frac{A_z}{g} \right) - \frac{\partial}{\partial z} \left(\frac{A_y}{g} \right) \\ &= \frac{g \frac{\partial A_z}{\partial y} - A_z \frac{\partial g}{\partial y}}{g^2} - \frac{g \frac{\partial A_y}{\partial z} - A_y \frac{\partial g}{\partial z}}{g^2} \\ &= \frac{1}{g^2} \left[g \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \left(A_y \frac{\partial g}{\partial z} - A_z \frac{\partial g}{\partial y} \right) \right] \\ &= \frac{g[\nabla \times \mathbf{A}]_x - \mathbf{A} \times [\nabla g]_x}{g^2} \end{aligned}$$

and likewise for the y and z components. Therefore,

$$\nabla \times \left(\frac{\mathbf{A}}{g} \right) = \frac{g(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla g)}{g^2}$$

1.25 (a) Calculating (iv) for the functions

$$\mathbf{A} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}; \quad \mathbf{B} = 3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}$$

LHS:

$$\begin{aligned} \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\ &= \nabla \cdot [(0 + 6xz)\hat{\mathbf{x}} - (0 - 9yz)\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}] \\ &= \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9yz) + \frac{\partial}{\partial z}(-2x^2 - 6y^2) \\ &= 6z + 9z + 0 = 15z \end{aligned}$$

RHS:

$$\begin{aligned} \mathbf{B} \cdot (\nabla \times \mathbf{A}) &= \mathbf{B} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} \\ &= \mathbf{B} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0 \end{aligned}$$

and

$$\begin{aligned} \mathbf{A} \cdot (\nabla \times \mathbf{B}) &= \mathbf{A} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\ &= \mathbf{A} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-2 - 3)\hat{\mathbf{z}}] \\ &= 3z(-5) = -15z \end{aligned}$$

therefore,

$$\mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = 0 - (-15z) = 15z$$

(b) Checking (ii): LHS

$$\begin{aligned} \nabla(\mathbf{A} \cdot \mathbf{B}) &= \nabla(x(3y) + 2y(-2x) + 3z(0)) \\ &= \nabla(3xy - 4xy) = \nabla(-xy) \\ &= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}} \end{aligned}$$

RHS:

$$\begin{aligned} \mathbf{A} \times (\nabla \times \mathbf{B}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\ &= \mathbf{A} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-5)\hat{\mathbf{z}}] \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = -10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}} \end{aligned}$$

and

$$\mathbf{B} \times (\nabla \times \mathbf{A}) = \mathbf{B} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = \mathbf{B} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0$$

and

$$\begin{aligned} (\mathbf{A} \cdot \nabla) \mathbf{B} &= \left(x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z} \right) (3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) \\ &= 6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} \end{aligned}$$

and

$$\begin{aligned} (\mathbf{B} \cdot \nabla) \mathbf{A} &= \left(3y \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y} \right) (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \\ &= 3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}} \end{aligned}$$

therefore,

$$\begin{aligned} \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} &= (-10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}) + (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) \\ &= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}} \end{aligned}$$

(c) For rule (vi), the left hand side is

$$\begin{aligned} \nabla \times (\mathbf{A} \times \mathbf{B}) &= \nabla \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\ &= \nabla \times [6xz\hat{\mathbf{x}} + 9yz\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}] \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & -2x^2 - 6y^2 \end{vmatrix} \\ &= \hat{\mathbf{x}}(-12y - 9y) - \hat{\mathbf{y}}(-4x - 6x) + \hat{\mathbf{z}}(0) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}} \end{aligned}$$

and on the right hand side, the new terms are

$$\mathbf{A}(\nabla \cdot \mathbf{B}) = \mathbf{A}[0 + 0] = 0$$

and

$$\begin{aligned} \mathbf{B}(\nabla \cdot \mathbf{A}) &= \mathbf{B}[1 + 2 + 3] = 6\mathbf{B} \\ &= 6(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) = 18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}} \end{aligned}$$

combining these with the terms from (iv) gives

$$\begin{aligned} (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{A} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{A} &= (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) - (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + 0 - (18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}} \end{aligned}$$

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