Problems for Griffiths' Electrodynamics

Contents

1 Vector Analysis

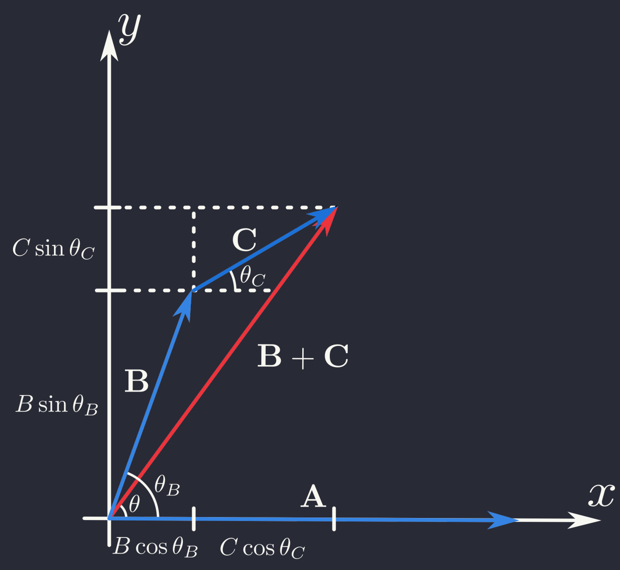

Figure 1.1: Three Coplanar Vectors

1.1 (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$
$$A(B+C)\cos\theta = AB\cos\theta_B + AC\cos\theta_C$$

Since $B\cos\theta_B + B\cos\theta_C = (B+C)\cos\theta$ from Figure 1.1, the distriubtive property holds true. The cross product also holds true since $B\sin\theta_B + B\sin\theta_C = (B+C)\sin\theta$, and multiplying by A on both sides gives the same result as the distriubtive property:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$A(B+C)\sin\theta = AB\sin\theta_B + AC\sin\theta_C$$

(b) In the general case in three-dimensional space, each vector has three components: $\mathbf{A} = (A_x, A_y, A_z)$. Therefore,

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot(B_x+C_x, B_y+C_y, B_z+C_z)\\ &= A_x(B_x+C_x)+A_y(B_y+C_y)+A_z(B_z+C_z)\\ &= A_xB_x+A_yB_y+A_zB_z+A_xC_x+A_yC_y+A_zC_z\\ &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\end{aligned}$$

1.2 Setting $\mathbf{A}=\mathbf{B}=(1,1,1)$ and $\mathbf{C}=(1,1,-1)$:

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times\mathbf{C} &\stackrel{?}{=} \mathbf{A}\times(\mathbf{B}\times\mathbf{C})\\ 0 &\stackrel{?}{=} (1,1,1)\times[(1,1,1)\times(1,1,-1)]\\ 0 &\stackrel{?}{=} (1,1,1)\times(-2,2,0)\\ 0 &\neq (-2,-2,4)\end{aligned}$$

where the cross product of parallel vectors $\mathbf{A}\times\mathbf{B}=0$. Therefore, the cross product is not associative.

1.3 Taking the dot product of a unit cube's body diagonals $\mathbf{A} = (1, 1, 1)$, $\mathbf{B} = (1, 1, -1)$:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= AB \cos\theta \\ 1 &= 3\cos\theta \\ \theta &= \arccos 1/3 \approx 70.53^\circ\end{aligned}$$

1.4 The cross product of two vectors coplanar to the shaded plane—$\mathbf{A} = (-1, 2, 0)$, $\mathbf{B} = (-1, 0, 3)$—is parallel to the normal unit vector $\hat{\mathbf{n}}$ of the plane:

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6, 3, 2)$$

where $\hat{\mathbf{n}} = \mathbf{C}/C$, and $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$. Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

1.5 Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}\end{aligned}$$

where the x component is

$$A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

From the "BAC-CAB" rule,

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)$$

where x component simplifes to

$$B_x(\cancel{A_xC_x} + A_yC_y + A_zC_z) - C_x(\cancel{A_xB_x} + A_yB_y + A_zB_z) = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

So,

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_x = [\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})]_x$$

and similary for the y and z components. Therefore, the "BAC–CAB" rule holds true.

1.6

$$\begin{aligned}[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) = 0 \\ &\quad - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C} \cdot \mathbf{A})}\end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \\ \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \times \mathbf{B}\end{aligned}$$

For the relation to hold true, either the vectors $\mathbf{A}$ and $\mathbf{C}$ are parallel ($\mathbf{A} \times \mathbf{A} = 0$) or $\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{C}$ ($\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = 0$).

1.7 Finding the seperation vector $\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = (4, 6, 8) - (2, 8, 7) = (2, -2, 1)$$
$$\mathscr{r} = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$
$$\hat{\boldsymbol{\mathscr{r}}} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$$

1.8 (a)

$$\begin{aligned}\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z &= (A_y \cos\phi + A_z \sin\phi)(B_y \cos\phi + B_z \sin\phi) \\ &\quad + (-A_y \sin\phi + A_z \cos\phi)(-B_y \sin\phi + B_z \cos\phi) \\ &= A_yB_y \cos^2\phi + A_zB_z \sin^2\phi + \cancel{A_yB_z \sin\phi\cos\phi} + \cancel{A_zB_y \sin\phi\cos\phi} \\ &\quad + A_yB_y \sin^2\phi - \cancel{A_yB_z \sin\phi\cos\phi} - \cancel{A_zB_y \sin\phi\cos\phi} + A_zB_z \cos^2\phi \\ &= A_yB_y(\sin^2\phi + \cos^2\phi) + A_zB_z(\sin^2\phi + \cos^2\phi) \\ \bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z &= A_yB_y + A_zB_z\end{aligned}$$

(b) To preserve length $|\bar{A}| = |A|$. Squaring both sides and expanding gives

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_x^2 + A_y^2 + A_z^2$$

in summation form,

$$\sum_{i=1}^{3} \bar{A}_i\bar{A}_i = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} R_{ij}A_j\right)\left(\sum_{k=1}^{3} R_{ik}A_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} R_{ij}R_{ik}A_jA_k$$

For the length to be preserved, the indices j and k must be equal. Therefore,

$$R_{ij}R_{ik} = \delta_{jk}$$

where δ_{ij} is the Kronecker delta. Thus the length is preserved if the rotation matrix is orthogonal, i.e.

$$R_{ij}R_{ik} = (R^T)_{ji}R_{ik} = \delta_{jk} \quad \text{or} \quad R^TR = I$$

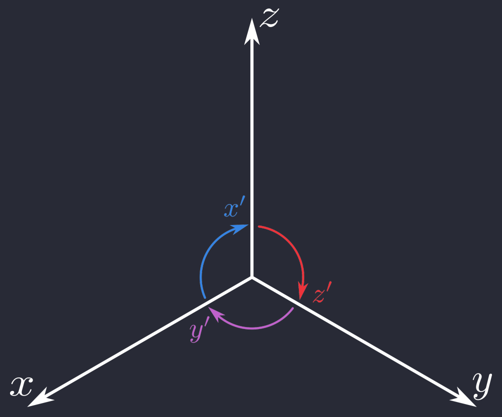

Figure 1.2: Rotation of 120° about an axis through the origin and point $(1, 1, 1)$

1.9 From Figure 1.2, the rotation is equivalent to changing the position of the basis vectors $\hat{\mathbf{x}} \to \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \to \hat{\mathbf{y}}$. Therefore, the rotation matrix is a permutation matrix:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1.10 (a) Under a **translation** of coordinates $\bar{y} = y - a$, the origin O and terminus $A = (x, y, z)$ of some vector are translated to

$$\begin{aligned} O &\to O' = (0, -a, 0) \\ A &\to A' = (x, y - a, z) \end{aligned}$$

therefore, the translated vector is

$$\overline{O'A'} = (x, y - a, z) - (0, -a, 0) = (x, y, z) = \overline{OA} = \mathbf{A}$$

which is the same as the original vector, so

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) Under **inversion** of coordinates, only the terminus changes

$$\begin{aligned} O &\to O' = (0, 0, 0) \\ A &\to A' = (-x, -y, -z) \end{aligned}$$

therefore, the inverted vector is

$$\overline{O'A'} = (-x, -y, -z) \quad \text{or} \quad \begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

(c) The components of cross product under inversion

$$\bar{\mathbf{A}} \times \bar{\mathbf{B}} = \begin{pmatrix} \bar{A}_y \bar{B}_z - \bar{A}_z \bar{B}_y \\ \bar{A}_z \bar{B}_x - \bar{A}_x \bar{B}_z \\ \bar{A}_x \bar{B}_y - \bar{A}_y \bar{B}_x \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}$$

which is the same as the original cross product $\mathbf{A} \times \mathbf{B}$. The cross product of two pseudovectors is also a pseudovector. Torque $\tau = \mathbf{r} \times \mathbf{F}$ and magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ are examples of pseudovectors.

(d) Scalar triple product under inversion

$$\begin{aligned} \bar{A} \cdot (\bar{B} \times \bar{C}) &= -\mathbf{A} \cdot (-\mathbf{B} \times -\mathbf{C}) \\ &= -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \end{aligned}$$

the scalar triple product changes sign under inversion.

1.11 (a) Finding gradient of $f(x, y, z) = x^2 + y^3 + z^4$:

$$\begin{aligned} \boldsymbol{\nabla} f &= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\ &= 2x\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + 4z^3\hat{\mathbf{z}} \end{aligned}$$

(b) Gradient of $f(x, y, z) = x^2 y^3 z^4$:

$$\boldsymbol{\nabla} f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$$

(c) Gradient of $f(x, y, z) = e^x \sin(y) \ln(z)$:

$$\boldsymbol{\nabla} f = e^x\hat{\mathbf{x}} + e^x \cos(y) \ln(z)\hat{\mathbf{y}} + \frac{e^x \sin(y)}{z}\hat{\mathbf{z}}$$

1.12 The height of the hill (in feet) is given by the function

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is north and x is east in miles. The gradient of h is

$$\nabla h = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10(2x - 8y + 28)\hat{\mathbf{y}}$$

(a) The top of the hill is a stationary point, so the summit is found by setting the gradient to zero which gives the system of equations

$$0 = 2y - 6x - 18 \qquad\qquad 0 = 2x - 8y + 28$$

adding the first equation to 3 times the second equation gives

$$\begin{aligned} &0 = 2y - 6x - 18 + 3(2x - 8y + 28) \\ &0 = -22y + 66 \\ &y = 3 \end{aligned}$$

substituting $y = 3$ into the first equation

$$0 = 2(3) - 6x - 18 \to x = -2$$

Therefore, the top of the hill is at $(-2, 3)$ or 2 miles west and 3 miles north of the origin.

(b) The height of the hill is simply $h(-2, 3) = 10(12) = 720$ feet.

(c) The steepness of the hill at $h(1, 1)$ is given by the magnitude of the gradient

$$\begin{aligned} |\nabla h| &= 10\sqrt{(2y - 6x - 18)^2 + (2x - 8y + 28)^2} \\ &= 10\sqrt{(2 - 6 - 18)^2 + (2 - 8 + 28)^2} \\ &= 10\sqrt{(-22)^2 + (22)^2} = 220\sqrt{2} \approx 311\,\mathrm{ft/mi} \end{aligned}$$

The direction of the steepest slope is given by the vector in the direction of the gradient at the point $\nabla h(1, 1) = 220(-\mathbf{x} + \mathbf{y})$, or simply northwest.

1.13 Given the seperation vector

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(a) Show that $\nabla\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$:

$$\mathscr{r}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

and each component of the gradient is

$$\begin{aligned} \frac{\partial}{\partial x}\left(\mathscr{r}^2\right) &= 2(x - x') \\ \frac{\partial}{\partial y}\left(\mathscr{r}^2\right) &= 2(y - y') \\ \frac{\partial}{\partial z}\left(\mathscr{r}^2\right) &= 2(z - z') \end{aligned}$$

so

$$\nabla\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$$

(b) Show $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$:

$$\frac{1}{\mathscr{r}} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2} = [(\ )]^{-1/2}$$

looking at the x component of the gradient (using chain rule),

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right) &= -\frac{1}{2}[(\ )]^{-3/2}\cdot 2(x-x') \\
&= -\frac{(x-x')}{\mathscr{r}^3}
\end{aligned}$$

and similarly for the y and z components:

$$\begin{aligned}
\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{z}} \\
&= -\frac{1}{\mathscr{r}^3}[(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}] = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}
\end{aligned}$$

Finally, substituting the unit vector $\hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}/\mathscr{r}$ gives us

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(c) The general formula for $\boldsymbol{\nabla}(\mathscr{r}^n)$:

$$\boldsymbol{\nabla}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\cdot\boldsymbol{\nabla}(\mathscr{r})$$

where

$$\begin{aligned}
\boldsymbol{\nabla}(\mathscr{r}) &= \frac{\partial}{\partial x}(\mathscr{r})\hat{\mathbf{x}} + \frac{\partial}{\partial y}(\mathscr{r})\hat{\mathbf{y}} + \frac{\partial}{\partial z}(\mathscr{r})\hat{\mathbf{z}} \\
&= \frac{1}{2}[(\ )]^{-1/2}\cdot 2(x-x')\hat{\mathbf{x}} + \dots \qquad \text{[similar to part (b)]} \\
&= \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \hat{\boldsymbol{\mathscr{r}}}
\end{aligned}$$

So the general formula is

$$\boldsymbol{\nabla}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\hat{\boldsymbol{\mathscr{r}}}$$

1.14 Given the rotation matrix

$$\begin{pmatrix}\bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix}$$

or the two equations

$$\begin{aligned}
\bar{y} &= y\cos\phi + z\sin\phi \\
\bar{z} &= -y\sin\phi + z\cos\phi
\end{aligned}$$

differentiating with respect to $\bar{y}$ and $\bar{z}$ respectively gives

$$\begin{aligned}
1 &= \frac{\partial y}{\partial\bar{y}}\cos\phi + \frac{\partial z}{\partial\bar{y}}\sin\phi \\
1 &= -\frac{\partial y}{\partial\bar{z}}\sin\phi + \frac{\partial z}{\partial\bar{z}}\cos\phi
\end{aligned}$$

this can only be true if

$$\frac{\partial y}{\partial\bar{y}} = \cos\phi, \quad \frac{\partial z}{\partial\bar{y}} = \sin\phi \quad \text{and} \quad \frac{\partial y}{\partial\bar{z}} = -\sin\phi, \frac{\partial z}{\partial\bar{z}} = \cos\phi$$

which satisfies the trig identity $\sin^2\phi + \cos^2\phi = 1$. Differentiating f with respect to the rotated coordinates $\bar{y}$ and $\bar{z}$ is given by

$$\begin{aligned}
\frac{\partial f}{\partial\bar{y}} &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial\bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial\bar{y}} = \frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi \\
\frac{\partial f}{\partial\bar{z}} &= \frac{\partial f}{\partial y}\frac{\partial y}{\partial\bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial\bar{z}} = -\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi
\end{aligned}$$

therefore, the gradient of f transforms as a vector under rotations given by

$$\overline{\nabla f} = \frac{\partial f}{\partial \bar{y}}\hat{\bar{\mathbf{y}}} + \frac{\partial f}{\partial \bar{z}}\hat{\bar{\mathbf{z}}} = \left(\frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi\right)\hat{\bar{\mathbf{y}}} + \left(-\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi\right)\hat{\bar{\mathbf{z}}}$$

or in matrix form

$$\overline{\nabla f} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \nabla f$$

where the gradient is a column vector

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

1.15 (a) Calculating divergence of $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned} \nabla \cdot \mathbf{v}_a &= \frac{\partial v_{ax}}{\partial x} + \frac{\partial v_{ay}}{\partial y} + \frac{\partial v_{az}}{\partial z} \\ &= 2x + 0 - 2x = 0 \end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\nabla \cdot \mathbf{v}_b = y + 2z + 3x$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\nabla \cdot \mathbf{v}_c = 0 + 2x + 2y = 2(x + y)$$

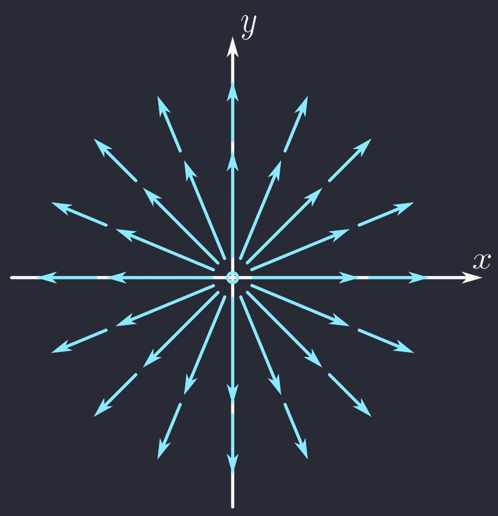

Figure 1.3: Sketch of the vector field $\mathbf{v} = \hat{\mathbf{r}}/r^2$

1.16 Given

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector. The vector functions is

$$v = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

where the components are

$$v_x = \frac{x}{r^3} \quad \text{and} \quad v_y = \frac{y}{r^3} \quad \text{and} \quad v_z = \frac{z}{r^3}$$

Looking at the x component of the divergence,

$$\begin{aligned}
[\boldsymbol{\nabla} \cdot \mathbf{v}]_x &= \frac{\partial v_x}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) \quad \text{using product rule...} \\
&= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} \\
&= \frac{1}{r^3} - \frac{3x^2}{r^5}
\end{aligned}$$

therefore, the divergence of $\mathbf{v}$ is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v} &= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \\
&= \frac{3}{r^3} - 3\frac{x^2+y^2+z^2}{r^5} \\
&= \frac{3}{r^3} - 3\frac{r^2}{r^5} = 0
\end{aligned}$$

This is consistent with the sketch in Figure 1.3 because the vector field is not ‘sourcing’ or ‘sinking’.

1.17 Given

$$\bar{v}_y = v_y \cos\phi + v_z \sin\phi \quad \text{and} \quad \bar{v}_z = -v_y \sin\phi + v_z \cos\phi$$

Calculating the derivatives

$$\begin{aligned}
\frac{\partial \bar{v}_y}{\partial \bar{y}} &= \frac{\partial v_y}{\partial \bar{y}} \cos\phi + \frac{\partial v_z}{\partial \bar{y}} \sin\phi \\
\frac{\partial \bar{v}_z}{\partial \bar{z}} &= -\frac{\partial v_y}{\partial \bar{z}} \sin\phi + \frac{\partial v_z}{\partial \bar{z}} \cos\phi
\end{aligned}$$

from Problem 1.14,

$$\begin{aligned}
\frac{\partial f}{\partial \bar{y}} &= \frac{\partial f}{\partial y} \cos\phi + \frac{\partial f}{\partial z} \sin\phi \\
\frac{\partial f}{\partial \bar{z}} &= -\frac{\partial f}{\partial y} \sin\phi + \frac{\partial f}{\partial z} \cos\phi
\end{aligned}$$

therefore, the derivatives are rewritten as

$$\begin{aligned}
\frac{\partial \bar{v}_y}{\partial \bar{y}} &= \left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \bar{y}}\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial v_z}{\partial y}\frac{\partial z}{\partial \bar{z}}\right)\sin\phi \\
&= \left(\frac{\partial v_y}{\partial y}\cos\phi + \frac{\partial v_y}{\partial z}\sin\phi\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi\right)\sin\phi
\end{aligned}$$

and likewise,

$$\frac{\partial \bar{v}_z}{\partial \bar{z}} = -\left(-\frac{\partial v_y}{\partial y}\sin\phi + \frac{\partial v_y}{\partial z}\cos\phi\right)\sin\phi + \left(-\frac{\partial v_z}{\partial y}\sin\phi + \frac{\partial v_z}{\partial z}\cos\phi\right)\cos\phi$$

Finally adding the two equations together gives

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\bar{\mathbf{v}} &= \frac{\partial\bar{v}_y}{\partial\bar{y}} + \frac{\partial\bar{v}_z}{\partial\bar{z}} \\
&= \frac{\partial v_y}{\partial y}\cos^2\phi + \frac{\partial v_y}{\partial z}\sin\phi\cos\phi + \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi^2 \\
&+ \frac{\partial v_y}{\partial y}\sin^2\phi - \frac{\partial v_y}{\partial z}\sin\phi\cos\phi - \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\cos^2\phi \\
&= (\sin\phi^2 + \cos\phi^2)\left[\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] \\
&= \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}
\end{aligned}$$

which shows that the divergence transforms as a scalar under rotations.

1.18 Curl of vector functions from Problem 1.15: (a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v}_a &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0) \\
&= -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}
\end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v}_b &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x) \\
&= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}
\end{aligned}$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v}_c &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\
&= \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) \\
&= 0
\end{aligned}$$

1.19 From Figure 1.4, the sign of $\partial v_x/\partial y$ is positive, and the sign of $\partial v_y/\partial x$ is negative. Therefore, the curl

$$\boldsymbol{\nabla}\times\mathbf{v} = \hat{\mathbf{z}}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial z}\right)$$

is in the negative z direction, or into the page. This is consistent with the right-hand rule as the thumb points into the page.

1.20 Proof that the vector function

$$\mathbf{g} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3}$$

has zero divergence and curl given

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{and} \quad \frac{\partial x}{\partial y} = \frac{y}{x} = 0$$

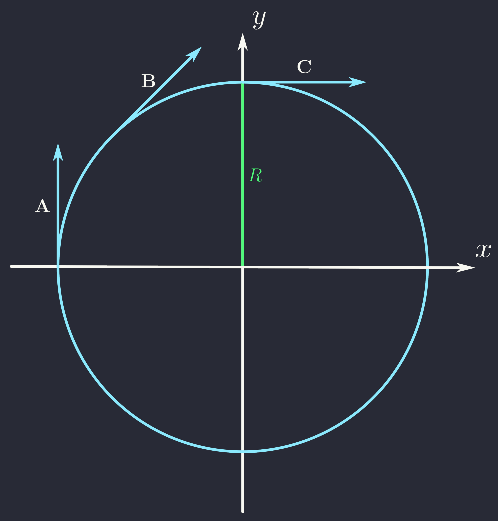

Figure 1.4: Sketch of the vector field pointing clockwise around a circle of radius R

From Problem 1.16, the divergence of $\mathbf{g}$ is

$$\begin{aligned}\nabla\cdot\mathbf{g} &= \frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y}+\frac{\partial g_z}{\partial z}\\ &= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\\ &= \frac{1}{r^3}-\frac{3x^2}{r^5}+\frac{1}{r^3}-\frac{3y^2}{r^5}+\frac{1}{r^3}-\frac{3z^2}{r^5}\\ &= \frac{3}{r^3}-\frac{3}{r^3}=0\end{aligned}$$

and the curl is

$$\begin{aligned}\nabla\times\mathbf{g} &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3}\end{vmatrix}\\ &= \hat{\mathbf{x}}(0-0)-\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}(0-0)=0\end{aligned}$$

1.21 Proving product rule for (i)

$$\begin{aligned}\nabla(fg) &= \frac{\partial(fg)}{\partial x}\hat{\mathbf{x}}+\frac{\partial(fg)}{\partial y}\hat{\mathbf{y}}+\frac{\partial(fg)}{\partial z}\hat{\mathbf{z}}\\ &= \left(\frac{\partial f}{\partial x}g+f\frac{\partial g}{\partial x}\right)\hat{\mathbf{x}}+\left(\frac{\partial f}{\partial y}g+f\frac{\partial g}{\partial y}\right)\hat{\mathbf{y}}+\left(\frac{\partial f}{\partial z}g+f\frac{\partial g}{\partial z}\right)\hat{\mathbf{z}}\\ &= f\left(\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial g}{\partial z}\right)+g\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\right)\\ &= f\nabla g+g\nabla f\end{aligned}$$

(iv)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) &= \boldsymbol{\nabla}\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\\
&= \boldsymbol{\nabla}\cdot[(A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}]\\
&= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x)\\
&= \left(\frac{\partial A_y}{\partial x}B_z + A_y\frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x}B_y - A_z\frac{\partial B_y}{\partial x}\right) + \left(\frac{\partial A_z}{\partial y}B_x + A_z\frac{\partial B_x}{\partial y} - \frac{\partial A_x}{\partial y}B_z - A_x\frac{\partial B_z}{\partial y}\right)\\
&+ \left(\frac{\partial A_x}{\partial z}B_y + A_x\frac{\partial B_y}{\partial z} - \frac{\partial A_y}{\partial z}B_x - A_y\frac{\partial B_x}{\partial z}\right)\\
&= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\\
&+ A_x\left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right) + A_y\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + A_z\left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right)\\
&= \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})
\end{aligned}$$

(v)

$$\begin{aligned}
\boldsymbol{\nabla}\times(f\mathbf{A}) &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ fA_x & fA_y & fA_z\end{vmatrix}\\
&= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y)\right) - \hat{\mathbf{y}}\left(\frac{\partial}{\partial x}(fA_z) - \frac{\partial}{\partial z}(fA_x)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x)\right)\\
&= \hat{\mathbf{x}}\left(f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right) - \hat{\mathbf{y}}\left(f\frac{\partial A_z}{\partial x} + A_z\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial z} - A_x\frac{\partial f}{\partial z}\right)\\
&+ \hat{\mathbf{z}}\left(f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right)\\
&= f\left[\hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]\\
&- \hat{\mathbf{x}}\left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial y}\right) + \hat{\mathbf{y}}\left(A_z\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial z}\right) - \hat{\mathbf{z}}\left(A_y\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial y}\right)\\
&= f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}f)
\end{aligned}$$

1.22 (a) If **A** and **B** are two vector functions, then

$$\begin{aligned}
(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} &= \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})\\
&= \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) + \hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)\\
&+ \hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)
\end{aligned}$$

This means that the direction of **A** points in the direction of where **B** moves fastest.

(b)

$$(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} = \frac{1}{r}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{r}$$

looking at the x component,

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\mathbf{r}}{r}\right) &= \frac{1}{r}\frac{\partial}{\partial x}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) + \mathbf{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \\ &= \frac{\hat{\mathbf{x}}}{r} - \mathbf{r}\frac{x^2}{r^3}\end{aligned}$$

therefore,

$$\begin{aligned}(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} &= \frac{1}{r}\left[x\left(\frac{\hat{\mathbf{x}}}{r} - \mathbf{r}\frac{x}{r^3}\right) + y\left(\frac{\hat{\mathbf{y}}}{r} - \mathbf{r}\frac{y}{r^3}\right) + z\left(\frac{\hat{\mathbf{z}}}{r} - \mathbf{r}\frac{z}{r^3}\right)\right] \\ &= \frac{1}{r}\left[\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r} - \mathbf{r}\frac{x^2+y^2+z^2}{r^3}\right] \\ &= \frac{1}{r}\left[\frac{\mathbf{r}}{r} - \frac{\mathbf{r}}{r}\right] = 0\end{aligned}$$

(c)

$$\begin{aligned}(v_a\cdot\boldsymbol{\nabla})v_b &= \left(x^2\frac{\partial}{\partial x} + 3xz^2\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}\right)(xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}) \\ &= x^2(y\hat{\mathbf{x}} + 0 + 3z\hat{\mathbf{z}}) + 3xz^2(x\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 0) - 2xz(0 + 2y\hat{\mathbf{y}} + 3x\hat{\mathbf{z}}) \\ &= (x^2y + 3x^2z^2)\hat{\mathbf{x}} + (6xz^3 - 4xyz)\hat{\mathbf{y}} + (3x^2z - 6x^2z)\hat{\mathbf{z}} \\ &= x^2(y + 3z^2)\hat{\mathbf{x}} + 2xz(3z^2 - 2y)\hat{\mathbf{y}} - 3x^2z\hat{\mathbf{z}}\end{aligned}$$

1.23 Proving the product rule for (ii) given the x component of the left hand side is

$$\begin{aligned}[\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x &= \frac{\partial(\mathbf{A}\cdot\mathbf{B})}{\partial x}\hat{\mathbf{x}} \\ &= \frac{\partial}{\partial x}(A_xB_x + A_yB_y + A_zB_z)\hat{\mathbf{x}} \\ &= A_x\frac{\partial B_x}{\partial x} + B_x\frac{\partial A_x}{\partial x} + A_y\frac{\partial B_y}{\partial x} + B_y\frac{\partial A_y}{\partial x} + A_z\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_z}{\partial x}\end{aligned}$$

Finding the x component of the right hand side of (ii)

$$\begin{aligned}[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})]_x &= \left[\mathbf{A}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z\end{vmatrix}\right]_x \\ &= \left[\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ () & -\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\end{vmatrix}\right]_x \\ &= A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\end{aligned}$$

and

$$[\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})]_x = B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - B_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

From Problem 1.22:

$$[(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}] = A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}$$

and likewise,

$$[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}] = B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}$$

Adding all four equations gives

$$
\begin{aligned}
&[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})+(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}+(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}]_x = \\
&\qquad A_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-A_z\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+B_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-B_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right) \\
&\qquad +A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}+B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z} \\
&\qquad = A_x\frac{\partial B_x}{\partial x}+A_y\left(\frac{\partial B_y}{\partial x}-\cancel{\frac{\partial B_x}{\partial y}}+\cancel{\frac{\partial B_x}{\partial y}}\right)+A_z\left(\frac{\partial B_z}{\partial x}-\cancel{\frac{\partial B_x}{\partial z}}+\cancel{\frac{\partial B_x}{\partial z}}\right) \\
&\qquad + B_x\frac{\partial A_x}{\partial x}+B_y\left(\frac{\partial A_y}{\partial x}-\cancel{\frac{\partial A_x}{\partial y}}+\cancel{\frac{\partial A_x}{\partial y}}\right)+B_z\left(\frac{\partial A_z}{\partial x}-\cancel{\frac{\partial A_x}{\partial z}}+\cancel{\frac{\partial A_x}{\partial z}}\right) \\
&\qquad = A_x\frac{\partial B_x}{\partial x}+B_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+B_y\frac{\partial A_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}+B_z\frac{\partial A_x}{\partial z} \\
&\qquad = [\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x
\end{aligned}
$$

and likewise for the y and z components.

For (vi), the x on the left hand side is

$$
\begin{aligned}
[\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x &= \left[\boldsymbol{\nabla}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{vmatrix}\right]_x \\
&= \left[\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ () & -(A_xB_z-A_zB_x) & A_xB_y-A_yB_x\end{vmatrix}\right]_x \\
&= \frac{\partial}{\partial y}(A_xB_y-A_yB_x)-\frac{\partial}{\partial z}(A_zB_x-A_xB_z) \\
&= A_x\frac{\partial B_y}{\partial y}+B_y\frac{\partial A_x}{\partial y}-A_y\frac{\partial B_x}{\partial y}-B_x\frac{\partial A_y}{\partial y} \\
&- A_z\frac{\partial B_x}{\partial z}-B_x\frac{\partial A_z}{\partial z}+A_x\frac{\partial B_z}{\partial z}+B_z\frac{\partial A_x}{\partial z}+ \\
&= A_x\left(\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z}-B_x\left(\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}
\end{aligned}
$$

On the right hand side, first we find the x component of the two new operations:

$$
\begin{aligned}
[A(\boldsymbol{\nabla}\cdot\mathbf{B})]_x &= \left[A\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)\right]_x \\
&= A_x\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)
\end{aligned}
$$

and likewise,

$$
[B(\boldsymbol{\nabla}\cdot\mathbf{A})]_x = B_x\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)
$$

Therefore, $[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + A(\boldsymbol{\nabla}\cdot\mathbf{B}) - B(\boldsymbol{\nabla}\cdot\mathbf{A})]_x =$

$$\begin{aligned}
& B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z} - \left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right) \\
& + A_x\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) - \left(B_x\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right) \\
& = A_x\left(\cancel{\frac{\partial B_x}{\partial x}} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} - \cancel{\frac{\partial B_x}{\partial x}}\right) - A_y\frac{\partial B_x}{\partial y} - A_z\frac{\partial B_x}{\partial z} \\
& - B_x\left(\cancel{\frac{\partial A_x}{\partial x}} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} - \cancel{\frac{\partial A_x}{\partial x}}\right) + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z} \\
& = A_x\left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) - A_y\frac{\partial B_x}{\partial y} - A_z\frac{\partial B_x}{\partial z} - B_x\left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z} \\
& = [\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x
\end{aligned}$$

and likewise for the y and z components.

1.24 Deriving the three quotient rules from the product rule: The gradient is

$$\begin{aligned}
\boldsymbol{\nabla}\left(\frac{f}{g}\right) &= \boldsymbol{\nabla}\big(fg^{-1}\big) = f\boldsymbol{\nabla}\big(g^{-1}\big) + g^{-1}\boldsymbol{\nabla}(f) \\
&= f(-g^{-2}\boldsymbol{\nabla}(g)) + g^{-1}\boldsymbol{\nabla}(f) \\
&= -\frac{f}{g^2}\boldsymbol{\nabla}(g) + \frac{g}{g}\frac{1}{g}\boldsymbol{\nabla}(f) \\
&= \frac{g\boldsymbol{\nabla}f - f\boldsymbol{\nabla}g}{g^2}
\end{aligned}$$

the divergence

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left(\frac{A}{g}\right) &= \boldsymbol{\nabla}\cdot\big(Ag^{-1}\big) = A(\boldsymbol{\nabla}\cdot g^{-1}) + g^{-1}(\boldsymbol{\nabla}\cdot A) \\
&= A(-g^{-2}(\boldsymbol{\nabla}\cdot g)) + \frac{g}{g}g^{-1}(\boldsymbol{\nabla}\cdot A) \\
&= \frac{g(\boldsymbol{\nabla}\cdot A) - A\boldsymbol{\nabla}\cdot g}{g^2}
\end{aligned}$$

and the curl

$$\begin{aligned}
\left[\boldsymbol{\nabla}\times\left(\frac{\mathbf{A}}{g}\right)\right]_x &= \frac{\partial}{\partial y}\left(\frac{A_z}{g}\right) - \frac{\partial}{\partial z}\left(\frac{A_y}{g}\right) \\
&= \frac{g\frac{\partial A_z}{\partial y} - A_z\frac{\partial g}{\partial y}}{g^2} - \frac{g\frac{\partial A_y}{\partial z} - A_y\frac{\partial g}{\partial z}}{g^2} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \left(A_y\frac{\partial g}{\partial z} - A_z\frac{\partial g}{\partial y}\right)\right] \\
&= \frac{g[\boldsymbol{\nabla}\times\mathbf{A}]_x - \mathbf{A}\times[\boldsymbol{\nabla}g]_x}{g^2}
\end{aligned}$$

and likewise for the y and z components. Therefore,

$$\boldsymbol{\nabla}\times\left(\frac{\mathbf{A}}{g}\right) = \frac{g(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}g)}{g^2}$$

1.25 (a) Calculating (iv) for the functions

$$\mathbf{A} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}; \qquad \mathbf{B} = 3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}$$

LHS:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\
&= \boldsymbol{\nabla} \cdot \left[(0 + 6xz)\hat{\mathbf{x}} - (0 - 9yz)\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}\right] \\
&= \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9yz) + \frac{\partial}{\partial z}\left(-2x^2 - 6y^2\right) \\
&= 6z + 9z + 0 = 15z
\end{aligned}$$

RHS:

$$\begin{aligned}
\mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) &= \mathbf{B} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} \\
&= \mathbf{B} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) &= \mathbf{A} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= \mathbf{A} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-2 - 3)\hat{\mathbf{z}}] \\
&= 3z(-5) = -15z
\end{aligned}$$

therefore,

$$\mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = 0 - (-15z) = 15z$$

(b) Checking (ii): LHS

$$\begin{aligned}
\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) &= \boldsymbol{\nabla}(x(3y) + 2y(-2x) + 3z(0)) \\
&= \boldsymbol{\nabla}(3xy - 4xy) = \boldsymbol{\nabla}(-xy) \\
&= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}
\end{aligned}$$

RHS:

$$\begin{aligned}
\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= \mathbf{A} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-5)\hat{\mathbf{z}}] \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = -10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}
\end{aligned}$$

and

$$\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mathbf{B} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = \mathbf{B} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0$$

and

$$\begin{aligned}
(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} &= \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right)(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) \\
&= 6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}
\end{aligned}$$

and

$$
\begin{aligned}
(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} &= \left(3y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}\right)(x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \\
&= 3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} &= (-10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}) + (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) \\
&= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}
\end{aligned}
$$

(c) For rule (vi), the left hand side is

$$
\begin{aligned}
\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= \boldsymbol{\nabla}\times\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\
&= \boldsymbol{\nabla}\times\left[6xz\hat{\mathbf{x}} + 9yz\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}\right] \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & -2x^2 - 6y^2 \end{vmatrix} \\
&= \hat{\mathbf{x}}(-12y - 9y) - \hat{\mathbf{y}}(-4x - 6x) + \hat{\mathbf{z}}(0) \\
&= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}
\end{aligned}
$$

and on the right hand side, the new terms are

$$
\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) = \mathbf{A}[0+0] = 0
$$

and

$$
\begin{aligned}
\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) &= \mathbf{B}[1+2+3] = 6\mathbf{B} \\
&= 6(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) = 18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}
\end{aligned}
$$

combining these with the terms from (iv) gives

$$
\begin{aligned}
(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} - (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} &= (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) - (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + 0 - (18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}) \\
&= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}
\end{aligned}
$$

1.26 Given the Laplacian of a scalar function T is

$$
\nabla^2 T = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}
$$

(a) $T_a = x^2 + 2xy + 3z + 4$:

$$
\nabla^2 T_a = 2 + 0 + 0 = 2
$$

(b) $T_b = \sin x \sin y \sin z$:

$$
\frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -\sin x \sin y \sin z = -T_b
$$

$$
\nabla^2 T_b = -3T_b
$$

(c) $T_c = e^{-5x}\sin 4y \cos 3z$: The components are

$$
\begin{aligned}
\frac{\partial^2 T_c}{\partial x^2} &= 25e^{-5x}\sin 4y\cos 3z = 25T_c \\
\frac{\partial^2 T_c}{\partial y^2} &= -16e^{-5x}\sin 4y\cos 3z = -16T_c \\
\frac{\partial^2 T_c}{\partial z^2} &= -9e^{-5x}\sin 4y\cos 3z = -9T_c
\end{aligned}
$$

therefore

$$\nabla^2 T_c = T_c(25 - 16 - 9) = 0$$

(d) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$: The laplacian of a vector function is

$$\nabla^2 \mathbf{v} = (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$$

and the components are

$$\begin{aligned}
\nabla^2 v_x &= \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = 2 + 0 + 0 = 2 \\
\nabla^2 v_y &= 0 + 0 + 6x = 6x \\
\nabla^2 v_z &= 0
\end{aligned}$$

therefore

$$\nabla^2 \mathbf{v} = 2\hat{\mathbf{x}} + 6x\hat{\mathbf{y}}$$

1.27 The divergence of curl is always zero:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\
&= \boldsymbol{\nabla} \cdot \left(\hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) - \hat{\mathbf{y}}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\
&= \left[\frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} \right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} \right) \right] + \left[\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} \right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial y} \right) \right] + \left[\frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} \right) - \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial z} \right) \right] \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) &= 0
\end{aligned}$$

where the last step hinges on the equality of cross derivatives:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial v}{\partial x_j} \right) = \frac{\partial}{\partial x_j}\left(\frac{\partial v}{\partial x_i} \right)$$

Checking for $v_a = x^2\hat{\mathbf{x}} + 2xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times v_a) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xz^2 & -2xz \end{vmatrix} \\
&= \boldsymbol{\nabla} \cdot \left[\hat{\mathbf{x}}(0 - 4xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}\big(2z^2 - 0\big) \right] \\
&= \boldsymbol{\nabla} \cdot \left[\frac{\partial}{\partial x}(-4xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}\big(2z^2\big) \right] \\
&= -4z + 0 + 4z = 0
\end{aligned}$$

1.28 The curl of gradient is always zero:

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) &= \boldsymbol{\nabla} \times \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \right) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} \\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial z} \right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial y} \right) \right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial z} \right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial x} \right) \right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y} \right) - \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x} \right) \right] \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) &= 0
\end{aligned}$$

where the last step uses the equality of cross derivatives again. Checking for $T = x^2y^3z^4$:

$$\frac{\partial T}{\partial x} = 2xy^3z^4, \quad \frac{\partial T}{\partial y} = 3x^2y^2z^4, \quad \text{and} \quad \frac{\partial T}{\partial z} = 4x^2y^3z^3$$

and

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) &= \boldsymbol{\nabla} \times \left(2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}\right) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix} \\
&= \hat{\mathbf{x}}\left(12x^2y^2z^4 - 12x^2y^2z^4\right) - \hat{\mathbf{y}}\left(8x^2y^3z^3 - 8x^2y^3z^3\right) + \hat{\mathbf{z}}\left(6x^2y^3z^3 - 6x^2y^3z^3\right) \\
&= 0
\end{aligned}$$

1.29 Calculating the line integral of the function $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$: from the origin to point $(1, 1, 1)$ along three different paths:

(a) $a = (0, 0, 0) \to b = (1, 0, 0) \to c = (1, 1, 0) \to d = (1, 1, 1)$ split to three paths:
(i) From $a \to b$: $\mathrm{d}l = \mathrm{d}x\,\hat{\mathbf{x}}$ and $\mathbf{v} = x^2\hat{\mathbf{x}}$.
(ii) From $b \to c$: $\mathrm{d}l = \mathrm{d}y\,\hat{\mathbf{y}}$ and $\mathbf{v} = 2yz\hat{\mathbf{y}} = 0$ since $z = 0$.
(iii) From $c \to d$: $\mathrm{d}l = \mathrm{d}z\,\hat{\mathbf{z}}$ and $\mathbf{v} = y^2\hat{\mathbf{z}} = 1\hat{\mathbf{z}}$ since $y = 1$.

$$\begin{aligned}
\int_a^b \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3} \\
\int_b^c \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 0\,\mathrm{d}y = 0 \\
\int_c^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 1\,\mathrm{d}z = 1 \\
\int_a^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \frac{1}{3} + 0 + 1 = \frac{4}{3}
\end{aligned}$$

(b) $a = (0, 0, 0) \to b = (0, 0, 1) \to c = (0, 1, 1) \to d = (1, 1, 1)$ split to three paths:
(i) From $a \to b$: $\mathrm{d}l = \mathrm{d}z\,\hat{\mathbf{z}}$ and $\mathbf{v} = y^2\hat{\mathbf{z}} = 0$ since $y = 0$.
(ii) From $b \to c$: $\mathrm{d}l = \mathrm{d}y\,\hat{\mathbf{y}}$ and $\mathbf{v} = 2yz\hat{\mathbf{y}} = 2y\hat{\mathbf{z}}$ since $y = 1$.
(iii) From $c \to d$: $\mathrm{d}l = \mathrm{d}x\,\hat{\mathbf{x}}$ and $\mathbf{v} = x^2\hat{\mathbf{x}}$.

$$\begin{aligned}
\int_a^b \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 0\,\mathrm{d}z = 0 \\
\int_b^c \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 2y\,\mathrm{d}y = 1 \\
\int_c^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3} \\
\int_a^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= 0 + 1 + \frac{1}{3} = \frac{4}{3}
\end{aligned}$$

(c) A straight line: Since $x = y = z$ and $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z$,
$\mathrm{d}l = \mathrm{d}x\,(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$ and $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}} = 4x^2\hat{\mathbf{x}}$.

$$\int_a^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^1 4x^2\,\mathrm{d}x = \frac{4}{3}$$

(d) The line integral around the closed loop:

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_{(a)} \mathbf{v} \cdot \mathrm{d}\mathbf{l} - \int_{(b)} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \frac{4}{3} - \frac{4}{3} = 0$$

1.30 Surface integral of $\mathbf{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ over the bottom of the box:
$z = 0$, $\mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}$ $\mathbf{v}\cdot\mathrm{d}\mathbf{A} = y(z^2-3)\,\mathrm{d}x\,\mathrm{d}y = -3y\,\mathrm{d}x\,\mathrm{d}y$, so

$$\int \mathbf{v}\cdot\mathrm{d}\mathbf{A} = \int_0^2 \mathrm{d}x \int_0^2 -3y\,\mathrm{d}y = -12$$

From Example 1.7 (v) has the same boundary line but a different surface integral, so the surface integral does not depend on just the boundary line. The surface over the closed surface of the box is

$$\oint \mathbf{v}\cdot\mathrm{d}\mathbf{A} = 20 + 12 = 32$$

since the direction of $\mathrm{d}\mathbf{A}$ on the bottom side is in the negative z direction for it to point 'outward'.

1.31 Calculating the volume integral of $T = z^2$ over the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$:

The equation of the plane containing the three vertices $A = (1,0,0)$, $B = (0,1,0)$, and $C(0,0,1)$: The vector normal to this plane $\mathbf{n} = (a,b,c)$ is the cross product of two vectors in the plane given by $\mathbf{AB} = (-1,1,0)$ and $\mathbf{AC} = (-1,0,1)$:

$$\mathbf{n} = \begin{vmatrix} x & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1,1,1)$$

the equation of the plane is therefore

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_o) &= 0 \\ (1,1,1)\cdot[(x,y,z)-A] &= 0 \\ x+y+z &= 1 \end{aligned}$$

therefore, the boundary for x is $x = 0$ and $x = 1-y-z$; for y is $y = 0$ and $y = 1-z$; and for z is $z = 0$ and $z = 1$. The volume integral is therefore

$$\begin{aligned} \int T\,\mathrm{d}V &= \int_0^1 z^2\,\mathrm{d}z \int_0^{1-z} \mathrm{d}y \int_0^{1-y-z} \mathrm{d}x \\ &= \int_0^1 z^2\,\mathrm{d}z \int_0^{1-z} (1-y-z)\,\mathrm{d}y \\ &= \int_0^1 z^2\,\mathrm{d}z \left(y - y^2/2 - yz \Big|_0^{1-z} \right) \\ &= \int_0^1 z^2[(1-z) - (1-z)^2/2 - z(1-z)]\,\mathrm{d}z \\ &= \int_0^1 z^2(1 - z - 1/2 - z^2/2 + z - z + z^2)\,\mathrm{d}z \\ &= \int_0^1 z^2(1/2 - z + z^2/2)\,\mathrm{d}z \\ &= \int_0^1 (z^2/2 - z^3 + z^4/2)\,\mathrm{d}z \\ &= z^3/6 - z^4/4 + z^5/10 \Big|_0^1 \\ &= 1/6 - 1/4 + 1/10 = 1/60 \end{aligned}$$

1.32 Given $T = x^2 + 4xy + 2yz^3$,

$$\frac{\partial T}{\partial x} = 2x + 4y, \quad \frac{\partial T}{\partial y} = 4x + 2z^3, \quad \text{and} \quad \frac{\partial T}{\partial z} = 6yz^2$$

therefore

$$\boldsymbol{\nabla} T = \hat{\mathbf{x}}(2x + 4y) + \hat{\mathbf{y}}(4x + 2z^3) + \hat{\mathbf{z}}(6yz^2)$$

Checking the fundamental theorem for gradients using the points $a = (0,0,0) \to b = (1,1,1)$:

$$\int_a^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = T(b) - T(a) = 1^2 + 4(1)(1) + 2(1)(1)^3 - 0 = 7$$

For the three paths:

(a) $a \to (1,0,0) \to (1,1,0) \to b$;

(i) $a \to (1,0,0)$:

$$x : 0 \to 1; \quad y = z = \mathrm{d}y = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 2x\,\mathrm{d}x$$

and

$$\int_a^c \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2x\,\mathrm{d}x = 1$$

(ii) $(1,0,0) \to (1,1,0)$:

$$y : 0 \to 1; \quad x = 1, \quad z = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 4\,\mathrm{d}y$$

and

$$\int_c^d \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 4\,\mathrm{d}y = 4$$

(iii) $(1,1,0) \to b$:

$$z : 0 \to 1; \quad x = y = 1, \quad \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 6z^2\,\mathrm{d}z$$

and

$$\int_d^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 6z^2\,\mathrm{d}z = 2$$

therefore

$$\int_a^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 1 + 4 + 2 = 7$$

(b) $a \to (0,0,1) \to (0,1,1) \to b$;

(i) $a \to (0,0,1)$:

$$z : 0 \to 1; \quad x = y = \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 0$$

and

$$\int_a^c \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 0 = 0$$

(ii) $(0,0,1) \to (0,1,1)$:

$$z = 1, \quad x = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 2\,\mathrm{d}y$$

and

$$\int_c^d \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2\,\mathrm{d}y = 2$$

(iii) $(0,1,1) \to b$:

$$z: 0 \to 1; \quad y = z = 1, \quad \mathrm{d}y = \mathrm{d}z = 0; \quad \mathbf{dl} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \boldsymbol{\nabla} T \cdot \mathbf{dl} = (2x+4)\,\mathrm{d}x$$

and

$$\int_d^b \boldsymbol{\nabla} T \cdot \mathbf{dl} = \int_0^1 (2x+4)\,\mathrm{d}x = 5$$

therefore

$$\int_a^b \boldsymbol{\nabla} T \cdot \mathbf{dl} = 0 + 2 + 5 = 7$$

(c) the parabolic path $z = x^2$; $y = x$:

$$\mathrm{d}x = \mathrm{d}y\,, \quad \text{and} \quad \mathrm{d}z = 2x\,\mathrm{d}x\,; \quad \mathbf{dl} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}x\,\hat{\mathbf{y}} + 2x\,\mathrm{d}x\,\hat{\mathbf{z}}$$

and

$$\begin{aligned}\boldsymbol{\nabla} T \cdot \mathbf{dl} &= (2x+4y)\,\mathrm{d}x + (4x+2z^3)\,\mathrm{d}x + (6yz^2)2x\,\mathrm{d}x \\ &= 6x\,\mathrm{d}x + (4x+2x^6)\,\mathrm{d}x + (12x^6)\,\mathrm{d}x \\ &= 10x\,\mathrm{d}x + 14x^6\,\mathrm{d}x\end{aligned}$$

therefore

$$\begin{aligned}\int_a^b \boldsymbol{\nabla} T \cdot \mathbf{dl} &= \int_0^1 (10x + 14x^6)\,\mathrm{d}x \\ &= 5x^2 + 2x^7\Big|_0^1 = 7\end{aligned}$$

1.33 Checking the divergence theorem: For the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

the divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{v} = y + 2z + 3x$$

so the volume integral is

$$\begin{aligned}\int_V \boldsymbol{\nabla} \cdot \mathbf{v}\,\mathrm{d}\tau &= \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^2 \int_0^2 (2y + 4z + 6)\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^2 (4 + 8z + 12)\,\mathrm{d}z \\ &= 8 + 16 + 24 \\ \int_V \boldsymbol{\nabla} \cdot \mathbf{v}\,\mathrm{d}\tau &= 48\end{aligned}$$

The surface integral is evaluated over the six faces of the cube noted by Figure 1.29:
(i) $x = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 2y\,\mathrm{d}y\,\mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 2y\,\mathrm{d}y\,\mathrm{d}z = 8$$

(ii) $x = 0$, $\mathrm{d}\mathbf{A} = -\,\mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 0\,\mathrm{d}y\,\mathrm{d}z = 0$$

(iii) $y = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}z\,\hat{\mathbf{y}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 4z\,\mathrm{d}x\,\mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 4z\,\mathrm{d}x\,\mathrm{d}z = 16$$

(iv) $y = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

(v) $z = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 6x\,\mathrm{d}x\,\mathrm{d}y$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 6x\,\mathrm{d}x\,\mathrm{d}y = 24$$

(vi) $z = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

So the total flux is

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 8 + 0 + 16 + 0 + 24 + 0 = 48$$

therefore, the divergence theorem is verified.

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{v})\,\mathrm{d}V = \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A}$$

1.34 Testing Stokes' theorem for the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

using the triangular shaded area bounded by the vertices $O = (0,0,0)$, $A = (0,2,0)$, and $B = (0,0,2)$:

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{v} &= (0 - 2y)\hat{\mathbf{x}} - (3z - 0)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}} \quad \text{and} \quad \mathrm{d}\mathbf{A} = \mathrm{d}z\,\mathrm{d}y\,\hat{\mathbf{x}} \\ &= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}\end{aligned}$$

$x = 0$ on this surface, and the limits of integration are $y : 0 \to 2$ and $z = 0 \to z = 2 - y$:

$$(\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = -2y\,\mathrm{d}z\,\mathrm{d}y$$

Thus, the flux of the curl through the surface is

$$\begin{aligned}\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} &= \int_0^2 \int_0^{2-y} -2y\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_0^2 -2y(2-y)\,\mathrm{d}y \\ &= -2y^2 + \frac{2}{3}y^3 \Big|_0^2 = -8/3\end{aligned}$$

The line integral is evaluated over the three sides of the triangle:

(i) On the path OA:
$x = z = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 2yz\,\mathrm{d}y = 0$;

$$\int_{OA} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

(ii) On the path AB:
$x = 0$, $y = 2 - z$; $\mathrm{d}y = -\,\mathrm{d}z$; $\mathrm{d}\mathbf{l} = -\,\mathrm{d}z\,(\hat{\mathbf{y}} - \hat{\mathbf{z}})$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = -2yz\,\mathrm{d}z = -2(2-z)z\,\mathrm{d}z = (2z^2 - 4z)\,\mathrm{d}z$;

$$\int_{AB} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^2 (2z^2 - 4z)\,\mathrm{d}z = -8/3$$

(iii) On the path BO:
$x = y = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$;

$$\int_{BO} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

So, the circulation of $\mathbf{v}$ around the triangle is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0 + -8/3 + 0 = -8/3$$

thus,

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

1.35 By Corollary 1, the function

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$$

with curl

$$\boldsymbol{\nabla} \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}$$

the flux of the curl through the flat surface from Ex. 1.11

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \frac{4}{3}$$

is the same as a different surface shown by Figure 1.35 which has the same boundary. Integrating over the five faces:
(i) $x = 1, \quad \mathrm{d}\mathbf{A} = \mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{x}}; \quad (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = (4z^2 - 2)\,\mathrm{d}y\,\mathrm{d}z\,;$

$$\int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \int_0^1 \int_0^1 (4z^2 - 2)\,\mathrm{d}y\,\mathrm{d}z = -\frac{2}{3}$$

(ii) $z = 0, \quad \mathrm{d}\mathbf{A} = -\,\mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}; \quad (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = 0;$

$$\int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = 0$$

(iii) $y = 1, \quad \mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}z\,\hat{\mathbf{y}}; \quad (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = 0;$

$$\int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = 0$$

(iv) Similar to (iii), $(\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = 0;$

$$\int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = 0$$

(v) $z = 1, \quad \mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}; \quad (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = 2\,\mathrm{d}x\,\mathrm{d}y\,;$

$$\int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \int_0^1 \int_0^1 2\,\mathrm{d}x\,\mathrm{d}y = 2$$

So,

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = -\frac{2}{3} + 0 + 0 + 0 + 2 = \frac{4}{3}$$

Thus the flux of the curl through a surface depends only on the boundary line.

1.36 (a) From the product rule

$$\boldsymbol{\nabla} \times (f\mathbf{A}) = f(\boldsymbol{\nabla} \times \mathbf{A}) - A \times (\boldsymbol{\nabla} f)$$

and integrating over a surface

$$\int_S \boldsymbol{\nabla} \times (f\mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \int_S f(\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} - \int_S [\mathbf{A} \times (\boldsymbol{\nabla} f)] \cdot \mathrm{d}\mathbf{a}$$

or rewritten as

$$\int_S f(\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \int_S \boldsymbol{\nabla} \times (f\mathbf{A}) \cdot \mathrm{d}\mathbf{a} + \int_S [\mathbf{A} \times (\boldsymbol{\nabla} f)] \cdot \mathrm{d}\mathbf{a}$$

invoking Stokes' theorem $\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint_P \mathbf{v} \cdot \mathrm{d}\mathbf{l}$:

$$\int_S f(\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \oint_P f(\mathbf{A}) \cdot \mathrm{d}\mathbf{l} + \int_S [\mathbf{A} \times (\boldsymbol{\nabla} f)] \cdot \mathrm{d}\mathbf{a}$$

(b) From the product rule for divergence:

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$$

or rewritten as

$$\mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) + \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$$

integrating both sides over a volume:

$$\int_V \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) \, \mathrm{d}V = \int_V \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) \, \mathrm{d}V + \int_V \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \, \mathrm{d}V$$

Using the divergence theorem $\int_V (\boldsymbol{\nabla} \cdot \mathbf{v}) \, \mathrm{d}V = \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a}$:

$$\int_V \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) \, \mathrm{d}V = \oint_S (\mathbf{A} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{a} + \int_V \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) \, \mathrm{d}V$$

1.37 Given the relation of Cartesian to spherical coordinates:

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad \text{and} \quad z = r\cos\theta$$

To find the formula for r, take the sum of the squares of the three equations; Solve for θ using the third equation; and solve for ϕ by dividing the second equation by the first:

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos\frac{z}{r}, \quad \text{and} \quad \phi = \arctan\frac{y}{x}$$

1.38 From the position vector

$$\begin{aligned}
\mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\
\mathbf{r} &= r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}} \\
\mathbf{r} &= r\hat{\mathbf{r}}(\theta, \phi)
\end{aligned}$$

where the unit vector $\hat{\mathbf{r}}(\theta, \phi)$ is dependent on θ and ϕ. The new basis vectors are in the same direction as the partial derivatives with respect to r, θ, and ϕ, so

$$\hat{\mathbf{r}} = \frac{e_r}{|e_r|}, \quad \hat{\boldsymbol{\theta}} = \frac{e_\theta}{|e_\theta|}, \quad \text{and} \quad \hat{\boldsymbol{\phi}} = \frac{e_\phi}{|e_\phi|}$$

The partial derivatives are

$$\begin{aligned}
e_r &= \frac{\partial \mathbf{r}}{\partial r} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
e_\theta &= \frac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta\cos\phi\hat{\mathbf{x}} + r\cos\theta\sin\phi\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}} \\
e_\phi &= \frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}}
\end{aligned}$$

and the magnitude

$$\begin{aligned}
|e_r| &= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} \\
&= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} \\
&= \sqrt{\sin^2\theta + \cos^2\theta} = 1 \\
|e_\theta| &= \sqrt{r^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)} \\
&= \sqrt{r^2(\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta)} \\
&= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r \\
|e_\phi| &= \sqrt{r^2(\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi)} \\
&= \sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} \\
&= \sqrt{r^2\sin^2\theta} = r\sin\theta
\end{aligned}$$

thus, the unit vectors are:

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}
\end{aligned}$$

or in matrix form:

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

where $a = Qx$ is an orthogonal matrix, so $Q^T = Q^{-1}$ and $Q^T Q = I$. Multiplying both sides by Q^T:

$$Q^T a = Q^T Q x \to x = Q^T a$$

thus, the inverse formula is

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}$$

or in vector form:

$$\begin{aligned}
\hat{\mathbf{x}} &= \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}} \\
\hat{\mathbf{y}} &= \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}} \\
\hat{\mathbf{z}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}
\end{aligned}$$

1.39 (a) Divergence theorem for $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$ using a volume of a sphere of radius R centered at the origin: The divergence is

$$\boldsymbol{\nabla}\cdot\mathbf{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(r^2)\right) = 4r$$

and the volume and surface elements are

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \quad \text{and} \quad \mathrm{d}\mathbf{a}_1 = (r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}$$

So

$$\begin{aligned}
\int_V (\boldsymbol{\nabla}\cdot\mathbf{v}_1)\cdot\mathrm{d}V &= \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 4r(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\
&= \int_0^R 4r^3\,\mathrm{d}r\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\
&= (R^4)(2)(2\pi) = 4\pi R^4
\end{aligned}$$

and

$$\begin{aligned}
\oint_S \mathbf{v}_1\cdot\mathrm{d}\mathbf{a}_1 &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= r^4\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\
&= (r^4)(2)(2\pi) = 4\pi r^4
\end{aligned}$$

where $r = R$ on the surface of the sphere. Therefore,

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{v}_1)\cdot\mathrm{d}V = \oint_S \mathbf{v}_1\cdot\mathrm{d}\mathbf{a}_1$$

(b) For $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2(1/r^2)\big) = 0$$

So

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{v}_2)\cdot \mathrm{d}V = 0$$

and

$$\begin{aligned}\oint_S \mathbf{v}_2\cdot \mathrm{d}\mathbf{a}_1 &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{1}{r^2}(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi = 4\pi\end{aligned}$$

1.40 Given the function

$$\mathbf{v} = (r\cos\theta)\hat{\mathbf{r}} + (r\sin\theta)\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\hat{\boldsymbol{\phi}}$$

the divergence in spherical coordinates is

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 v_r\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(v_\phi) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^3\cos\theta\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\big(r\sin^2\theta\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) \\ &= 3\cos\theta + 2\cos\theta - \sin\phi \\ &= 5\cos\theta - \sin\phi\end{aligned}$$

Checking the divergence theorem using a volume of a inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin:
The volume and surface elements are

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \quad \text{and} \quad \mathrm{d}\mathbf{a}_1 = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}$$

The volume integral for the hemisphere is

$$\begin{aligned}\int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\cdot\mathrm{d}V &= \int_{r=0}^{R}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(5\cos\theta - \sin\phi)(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\ &= \int_0^R r^2\,\mathrm{d}r\int_0^{\pi/2}\sin\theta\int_0^{2\pi}(5\cos\theta - \sin\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \frac{R^3}{3}\int_0^{\pi/2} 5\phi\cos\theta + \cos\phi\Big|_0^{2\pi}\,\mathrm{d}\theta \\ &= \frac{R^3}{3}(10\pi)\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta \\ \text{using} \quad u = \sin\theta, &\quad \mathrm{d}u = \cos\theta\,\mathrm{d}\theta\,; \quad \int u\,\mathrm{d}u = \frac{u^2}{2} \\ &= \frac{5\pi R^3}{3}\sin^2\theta\Big|_0^{\pi/2} = \frac{5\pi R^3}{3}\end{aligned}$$

The surface integral is split into two parts: the top surface of the hemisphere and the circular base.
(i) The top surface of the hemisphere where $r = R$:

$$\begin{aligned}\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a}_1 &= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= r^3\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\ &= \pi r^3 = \pi R^3\end{aligned}$$

(ii) The circular base of the hemisphere where $\theta = \pi/2$ and $\mathbf{a}_2 = r\,\mathrm{d}r\,\mathrm{d}\phi\,\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}\oint_S \mathbf{v}\cdot \mathrm{d}\mathbf{a}_2 &= \int_{r=0}^{R}\int_{\phi=0}^{2\pi}(r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\phi \\ &= \sin(\pi/2)\int_0^R r^2\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi \\ &= (1)\frac{R^3}{3}(2\pi) = \frac{2\pi R^3}{3}\end{aligned}$$

So, the total surface integral is

$$\oint_S \mathbf{v}\cdot \mathrm{d}\mathbf{a}_1 + \oint_S \mathbf{v}\cdot \mathrm{d}\mathbf{a}_2 = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

1.41

$$T = r(\cos\theta + \sin\theta\cos\phi)$$

The partial derivatives are:

$$\begin{aligned}\frac{\partial T}{\partial r} &= \cos\theta + \sin\theta\cos\phi \\ \frac{\partial T}{\partial \theta} &= r(-\sin\theta + \cos\theta\cos\phi) \\ \frac{\partial T}{\partial \phi} &= -r\sin\theta\sin\phi\end{aligned}$$

thus, the gradient of T in spherical is

$$\begin{aligned}\boldsymbol{\nabla} T &= \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \\ &= (\cos\theta + \sin\theta\cos\phi)\hat{\mathbf{r}} + (-\sin\theta + \cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - (\sin\phi)\hat{\boldsymbol{\phi}}\end{aligned}$$

and partial derivative in the laplacian are:

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) &= \frac{\partial}{\partial r}\big(r^2(\cos\theta + \sin\theta\cos\phi)\big) = 2r(\cos\theta + \sin\theta\cos\phi) \\ \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) &= r\frac{\partial}{\partial \theta}(\sin\theta(-\sin\theta + \cos\theta\cos\phi)) = r\frac{\partial}{\partial \theta}\left(-\sin^2\theta + \sin\theta\cos\theta\cos\phi\right) \\ &= -2r\sin\theta\cos\theta + r\cos^2\theta\cos\phi - r\sin^2\theta\cos\phi \\ \frac{\partial^2 T}{\partial \phi^2} &= -r\sin\theta\cos\phi\end{aligned}$$

The laplacian of T in spherical is

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

Simplifying each term: (i) The first term:

$$\frac{2}{r}(\cos\theta + \sin\theta\cos\phi)$$

(ii) The second term:

$$-\frac{2}{r}\cos\theta + \frac{\cos^2\theta\cos\phi}{r\sin\theta} - \frac{1}{r}\sin\theta\cos\phi$$

(iii) The third term:

$$-\frac{\cos\phi}{r\sin\theta}$$

adding all three terms:

$$\begin{aligned}
\nabla^2 T &= \frac{2}{r}(\cos\theta + \sin\theta\cos\phi) - \frac{2}{r}\cos\theta + \frac{\cos^2\theta\cos\phi}{r\sin\theta} - \frac{1}{r}\sin\theta\cos\phi - \frac{\cos\phi}{r\sin\theta} \\
&= \frac{2}{r}(\sin\theta\cos\phi) + \frac{\cos^2\theta\cos\phi - \cos\phi}{r\sin\theta} - \frac{1}{r}\sin\theta\cos\phi \\
&= \frac{2}{r\sin\theta}(\sin^2\theta\cos\phi) + \frac{\cos^2\theta\cos\phi - \cos\phi}{r\sin\theta} - \frac{1}{r\sin\theta}\sin^2\theta\cos\phi \\
&= \frac{1}{r\sin\theta}(\sin^2\theta\cos\phi + \cos^2\theta\cos\phi - \cos\phi) \\
&= \frac{1}{r\sin\theta}(\cos\phi)(\sin^2\theta + \cos^2\theta - 1) = 0
\end{aligned}$$

Converting T to Cartesian coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad \text{and} \quad z = r\cos\theta$$

So

$$T = z + x$$

The laplacian of T in Cartesian coordinates is

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Testing the gradient theorem using the path $O \to A = (2,0,0) \to B = (0,2,0) \to C = (0,0,2)$: Given the general infinitesimal displacement

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}$$

and the gradient of T in spherical coordinates

$$\boldsymbol{\nabla} T = (\cos\theta + \sin\theta\cos\phi)\hat{\mathbf{r}} + (-\sin\theta + \cos\theta\cos\phi)\hat{\boldsymbol{\theta}} + (-\sin\phi)\hat{\boldsymbol{\phi}}$$

(i) On the path OA:

$$\theta = \pi/2, \quad \phi = 0, \quad \mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}}; \quad (\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{l} = 1\,\mathrm{d}r$$

So

$$\int_{OA} (\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{l} = \int_0^2 1\,\mathrm{d}r = 2$$

(ii) On the path AB:

$$r = 2, \quad \theta = \pi/2, \quad \mathrm{d}\mathbf{l} = 2\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}; \quad (\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{l} = -2\sin\phi\,\mathrm{d}\phi$$

So

$$\int_{AB} (\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{l} = \int_0^{\pi/2} -2\sin\phi\,\mathrm{d}\phi = -2$$

(iii) On the path BC:

$$r = 2, \quad \phi = \pi/2, \quad \mathrm{d}\mathbf{l} = 2\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}}; \quad (\boldsymbol{\nabla} T)\cdot\mathrm{d}\mathbf{l} = -2\sin\theta\,\mathrm{d}\theta$$

So

$$\int_{BC} (\boldsymbol{\nabla} T) \cdot \mathrm{d}\mathbf{l} = \int_{\pi/2}^{0} -2 \sin\theta \, \mathrm{d}\theta = 2$$

therefore the total line integral is

$$\int_{OC} (\boldsymbol{\nabla} T) \cdot \mathrm{d}\mathbf{l} = 2 + -2 + 2 = 2$$

For the left hand side of the gradient theorem:
At C:

$$r = 2, \quad \theta = 0, \quad \phi = 0; \quad T = 2(\cos 0 + \sin 0 \cos 0) = 2$$

At O:

$$r = 0; \quad T = 0$$

So

$$T(C) - T(O) = 2 + 0 = 2$$

which is the same as the total line integral, so the gradient theorem holds.

1.42 Cylindrical coordinates are related to Cartesian coordinates by

$$x = s\cos\phi, \quad y = s\sin\phi, \quad \text{and} \quad z = z$$

and the position vector is

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$
$$\mathbf{r} = s\cos\phi\hat{\mathbf{x}} + s\sin\phi\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

The unit vectors are in the same direction as the partial derivatives with respect to s, ϕ, and z, so

$$\hat{\mathbf{s}} = \frac{e_s}{|e_s|}, \quad \hat{\boldsymbol{\phi}} = \frac{e_\phi}{|e_\phi|}, \quad \text{and} \quad \hat{\mathbf{z}} = \frac{e_z}{|e_z|}$$

where e_u is the new basis vector given by

$$e_u = \frac{\partial \mathbf{r}}{\partial u}$$

The partial derivatives are

$$e_s = \frac{\partial \mathbf{r}}{\partial s} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$
$$e_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = -s\sin\phi\hat{\mathbf{x}} + s\cos\phi\hat{\mathbf{y}}$$
$$e_z = \frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{z}}$$

and the magnitude

$$|e_s| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$
$$|e_\phi| = \sqrt{s^2\sin^2\phi + s^2\cos^2\phi} = s$$
$$|e_z| = 1$$

thus, the unit vectors are:

$$\begin{aligned}
\hat{\mathbf{s}} &= \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\
\hat{\mathbf{z}} &= \hat{\mathbf{z}}
\end{aligned}$$

The cylindrical unit vectors in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in matrix form:

$$\begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

Which is an orthogonal matrix $a = Qx$, so the Cartesian unit vectors is found by multiplying a by the transpose of Q:

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

or in vector form:

$$\begin{aligned}
\hat{\mathbf{x}} &= \cos\phi\hat{\mathbf{s}} - \sin\phi\hat{\boldsymbol{\phi}} \\
\hat{\mathbf{y}} &= \sin\phi\hat{\mathbf{s}} + \cos\phi\hat{\boldsymbol{\phi}} \\
\hat{\mathbf{z}} &= \hat{\mathbf{z}}
\end{aligned}$$

1.43 (a) Finding the divergence of the function

$$\mathbf{v} = s(2 + \sin^2\phi)\hat{\mathbf{s}} + s\sin\phi\cos\phi\hat{\boldsymbol{\phi}} + 3z\hat{\mathbf{z}}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \\
&= \frac{1}{s}\frac{\partial}{\partial s}\big(s(s(2+\sin^2\phi))\big) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z) \\
&= 2(2+\sin^2\phi) + \cos^2\phi - \sin^2\phi + 3 \\
&= 7 + \sin^2\phi + \cos^2\phi = 8
\end{aligned}$$

(b) Testing divergence theorem using a quarter cylinder of radius 2 and height 5 in quadrant I:
LHS: The volume elements is

$$\mathrm{d}V = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z\,,$$

so the volume integral is

$$\begin{aligned}
\int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\cdot\mathrm{d}V &= \int_{s=0}^{2}\int_{\phi=0}^{\pi/2}\int_{z=0}^{5} 8(s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z) \\
&= 8\int_0^2 s\,\mathrm{d}s\int_0^{\pi/2}\mathrm{d}\phi\int_0^5\mathrm{d}z \\
&= 8(2)(\pi/2)(5) = 40\pi
\end{aligned}$$

RHS: There are 5 surfaces: the top, bottom, and 3 sides.
(i) The top surface:

$$z = 5, \quad \mathrm{d}a = s\,\mathrm{d}s\,\mathrm{d}\phi\,\hat{\mathbf{z}}; \quad \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 15s\,\mathrm{d}s\,\mathrm{d}\phi$$

So

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int_{\phi=0}^{\pi/2} \int_{s=0}^{2} 15s \,\mathrm{d}s \,\mathrm{d}\phi = 15 \int_0^{\pi/2} \mathrm{d}\phi \int_0^2 s \,\mathrm{d}s = 15\pi$$

(ii) The bottom surface:

$$z = 0, \quad \mathrm{d}a = s \,\mathrm{d}s \,\mathrm{d}\phi \,\hat{\mathbf{z}}; \quad \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 0$$

(iii) The surface on the xy plane:

$$\phi = \pi/2 \quad \mathrm{d}a = \mathrm{d}s \,\mathrm{d}z \,\hat{\boldsymbol{\phi}}; \quad \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 0$$

(iv) The surface on the xz plane:

$$\phi = 0, \quad \mathrm{d}a = -\,\mathrm{d}s \,\mathrm{d}z \,\hat{\boldsymbol{\phi}}; \quad \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 0$$

(v) The curved surface:

$$s = 2, \quad \mathrm{d}a = 2 \,\mathrm{d}\phi \,\mathrm{d}z \,\hat{\mathbf{s}}; \quad \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 4(2 + \sin^2\phi) \,\mathrm{d}\phi \,\mathrm{d}z$$

So

$$\begin{aligned}
\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} &= \int_{z=0}^{5} \int_{\phi=0}^{\pi/2} 4(2 + \sin^2\phi) \,\mathrm{d}\phi \,\mathrm{d}z \\
&= 4 \int_0^5 \mathrm{d}z \int_0^{\pi/2} (4(1/2) + \sin^2\phi) \,\mathrm{d}\phi \\
\text{using} \quad \sin^2\phi &= \frac{1 - \cos(2\phi)}{2} \\
&= 4 \int_0^5 \mathrm{d}z \int_0^{\pi/2} (2 + (1 - \cos(2\phi))/2) \,\mathrm{d}\phi \\
&= 10 \int_0^{\pi/2} (5 - \cos(2\phi)) \,\mathrm{d}\phi \\
&= 10 \left(5\phi - \frac{\sin(2\phi)}{2} \Big|_0^{\pi/2} \right) = 25\pi
\end{aligned}$$

So the total surface integral is

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 15\pi + 0 + 0 + 0 + 25\pi = 40\pi$$

which is the same as the volume integral, so the divergence theorem holds.
(c) The curl of $\mathbf{v}$ in cylindrical coordinates is

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v} &= \frac{1}{s} \begin{vmatrix} \hat{\mathbf{s}} & s\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ v_s & (s)v_\phi & v_z \end{vmatrix} \\
&= \frac{1}{s} \begin{vmatrix} \hat{\mathbf{s}} & s\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ s(2 + \sin^2\phi) & (s)s\sin\phi\cos\phi & 3z \end{vmatrix} \\
&= 0\hat{\mathbf{s}} - \frac{1}{s}(0 - 0) + \frac{1}{s}(2s\sin\phi\cos\phi - 2s\sin\phi\cos\phi) \\
&= 0
\end{aligned}$$

1.44 (a)

$$\int_2^6 (3x^2 - 2x - 1)\delta(x-3)\,\mathrm{d}x = (3(3)^2 - 2(3) - 1) = 20$$

(b)

$$\int_0^5 \cos x\delta(x-\pi)\,\mathrm{d}x = \cos\pi = -1$$

(c)

$$\int_0^3 x^3\delta(x+1)\,\mathrm{d}x = 0$$

since $\delta(x+1) = 0$ when $x = -1$ from the bounds of the integral $x = 0 \to 3$.
(d)

$$\int_{\infty}^{-\infty} \ln(x+3)\delta(x+2)\,\mathrm{d}x = \ln(-2+3) = 0$$

1.45 (a)

$$\int_{-2}^{2} (2x+3)\delta(3x)\,\mathrm{d}x = \int_{-2}^{2} (2x+3)\frac{1}{3}\delta(x)\,\mathrm{d}x = \frac{1}{3}(2(0)+3) = 1$$

(b) Since $\delta(1-x) = \delta(-(x-1)) = \delta(x-1)$:

$$\int_0^2 (x^3 + 3x + 2)\delta(1-x)\,\mathrm{d}x = \int_0^2 (x^3 + 3x + 2)\delta(x-1)\,\mathrm{d}x = (1)^3 + 3(1) + 2 = 6$$

(c) Using $\delta(3x-1) = \frac{1}{3}\delta(x-1/3)$:

$$\int_{-1}^{1} 9x^2\delta(3x-1)\,\mathrm{d}x = \int_{-1}^{1} 3x^2\delta(x-1/3)\,\mathrm{d}x = 3(1/3)^2 = 1/3$$

(d)

$$\int_{-\infty}^{a} \delta(x-b)\,\mathrm{d}x = \begin{cases} 1 & \text{if } a > b \\ 0 & \text{if } a < b \end{cases}$$

1.46 (a) Using integration by parts:

$$\begin{aligned}
\int_{-a}^{a} x\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x))\,\mathrm{d}x &= x\delta(x)\Big|_a^a - \int_a^a x\delta(x)\,\mathrm{d}x \\
&= x\delta(x)\Big|_a^a - (0) = 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}(x\delta(x)) &= x\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x)) + \delta(x) = 0 \\
\text{thus}\quad x\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x)) &= -\delta(x)
\end{aligned}$$

where $\delta(a) = \delta(-a) = 0$
(b) Using integration by parts again:

$$\begin{aligned}
\int_{-a}^{a} f(x)\frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x &= f(x)\theta(x)\Big|_{-a}^{a} - \int_{-a}^{a}\frac{\mathrm{d}f}{\mathrm{d}x}\theta(x)\,\mathrm{d}x \\
&= f(a) - \int_{0}^{a}\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \\
&= f(a) - (f(a) - f(0)) = f(0) \\
\int_{-a}^{a} f(x)\frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x &= f(0)
\end{aligned}$$

which is the same as the definition of the delta function, so

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \delta(x)$$

1.47 (a) The expression for volume charge density of a point charge q at $\mathbf{r}'$ where

$$\int_V \rho(\mathbf{r})\,\mathrm{d}V = q \quad \text{at} \quad \mathbf{r} = \mathbf{r}'$$

is

$$\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}')$$

(b) The volume charge density of an electric dipole:

$$\begin{aligned}
\rho(\mathbf{r}_d) &= \rho(\mathbf{r})\Big|_{\mathbf{r}'=\mathbf{r}} - \rho(\mathbf{r})\Big|_{\mathbf{r}'=0} \\
&= q\delta^3(\mathbf{r} - \mathbf{a}) - q\delta^3(\mathbf{r})
\end{aligned}$$

where $\mathbf{r}_d$ is the position vector of the dipole. (c) The volume charge density of a spherical shell of radius R and total charge Q:

$$\int_V f\delta^3(r - R)\,\mathrm{d}V = f(R) = Q$$

where V is all space, so

$$Q = \int_0^{\infty}\int_0^{\pi} f\delta(r - R)(r^2\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and since the charge density is uniform in θ and ϕ, f only depends on r; $f = f(r)$.

$$\begin{aligned}
Q &= \int_0^{\infty} f(r)r^2\delta(r - R)\,\mathrm{d}r\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\
&= R^2[f(R)](2)(2\pi) = 4\pi R^2 f(R)
\end{aligned}$$

and since f is constant,

$$f = \frac{Q}{4\pi R^2}$$

thus, the volume charge density is

$$\rho(r) = \frac{Q}{4\pi R^2}\delta^3(r - R)$$

1.48 (a)

$$\int (r^2 + \mathbf{r}\cdot\mathbf{a} + a^2)\delta^3(\mathbf{r}-\mathbf{a})\,\mathrm{d}V = a^2 + \mathbf{a}\cdot\mathbf{a} + a^2 = 3a^2$$

(b) Given V is a cube of side 2 centered at the origin and $\mathbf{b} = 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$:

$$\begin{aligned}\int_V |\mathbf{r}-\mathbf{b}|^2\delta^3(5\mathbf{r})\,\mathrm{d}V &= \int_0^1\int_0^1\int_0^1 |\mathbf{r}-\mathbf{b}|^2\delta(5x)\delta(5y)\delta(5z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{1}{5^3}|-\mathbf{b}|^2 = \frac{1}{5^3}(4^2+3^2) = \frac{1}{5}\end{aligned}$$

(c) Given V is a sphere of radius 6 centered at the origin and $\mathbf{c} = 5\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$:
The magnitude $c = \sqrt{5^2+3^2+2^2} = \sqrt{38}$, is outside the sphere (magnitude $r = \sqrt{36}$). Therefore,

$$\int_V [r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4]\delta^3(\mathbf{r}-\mathbf{c})\,\mathrm{d}V = 0$$

(d) Given V is a sphere of radius 1.5 centered at $(2,2,2)$ and

$$\mathbf{d} = (1,2,3), \quad \mathbf{e} = (3,2,1)$$

Checking if the delta function is inside the sphere:

$$|\mathbf{e} - (2,2,2)| = \sqrt{1+0+1} = \sqrt{2} < 1.5$$

so the delta function is inside the sphere and the integral is

$$\begin{aligned}\int_V \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})\,\mathrm{d}V &= \int_V \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{r}-\mathbf{e})\,\mathrm{d}V \\ &= \mathbf{e}\cdot(\mathbf{d}-\mathbf{e}) \\ &= (3,2,1)\cdot(-2,0,2) = -4\end{aligned}$$

1.49 Evaluating the integral

$$J = \int_V e^{-r}\left(\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2}\right)\mathrm{d}V$$

where V is a sphere of radius R centered at the origin.
(i)

$$J = \int_V e^{-r}4\pi\delta^3(\mathbf{r})\,\mathrm{d}V = 4\pi e^0 = 4\pi$$

(ii) Using the product rule of divergence:

$$\boldsymbol{\nabla}\cdot(f\mathbf{A}) = f(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla}f)$$

integrating over a volume and using the divergence theorem:

$$\int \boldsymbol{\nabla}\cdot(f\mathbf{A})\,\mathrm{d}V = \int f(\boldsymbol{\nabla}\cdot\mathbf{A})\,\mathrm{d}V + \int \mathbf{A}\cdot(\boldsymbol{\nabla}f)\,\mathrm{d}V = \oint f\mathbf{A}\cdot\mathrm{d}\mathbf{a}$$

or

$$\int_V f(\boldsymbol{\nabla}\cdot\mathbf{A})\,\mathrm{d}V = -\int_V \mathbf{A}\cdot(\boldsymbol{\nabla}f)\,\mathrm{d}V + \oint f\mathbf{A}\cdot\mathrm{d}\mathbf{a}$$

where

$$f = e^{-r} \quad \text{and} \quad \mathbf{A} = \frac{\hat{\mathbf{r}}}{r^2}$$

(a) Computing the first term:

$$\boldsymbol{\nabla} f = -e^{-r}\hat{\mathbf{r}}; \quad \mathbf{A}\cdot(\boldsymbol{\nabla} f) = -\frac{e^{-r}}{r^2}$$

So

$$\begin{aligned}\int_V \frac{e^{-r}}{r^2}\,\mathrm{d}V &= \int_0^R\int_0^\pi\int_0^{2\pi}\frac{e^{-r}}{r^2}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \int_0^R\int_0^\pi\int_0^{2\pi} e^{-r}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= 4\pi(1-e^{-R})\end{aligned}$$

(b) For the second term: the surface element is on the boundary of the sphere $r = R$.

$$\mathrm{d}\mathbf{a} = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}; \quad f\mathbf{A}\cdot\mathrm{d}\mathbf{a} = e^{-R}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

So

$$e^{-R}\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi = 4\pi e^{-R}$$

adding (a) and (b) gives the total volume integral:

$$J = 4\pi(1-e^{-R}) + 4\pi e^{-R} = 4\pi$$

1.50 (a)

$$\mathbf{F}_1 = x^2\hat{\mathbf{z}} \quad \text{and} \quad \mathbf{F}_2 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

The divergence of the two vector fields:

$$\boldsymbol{\nabla}\cdot\mathbf{F}_1 = 0 \quad \text{and} \quad \boldsymbol{\nabla}\cdot\mathbf{F}_2 = 3$$

and curl:

$$\boldsymbol{\nabla}\times\mathbf{F}_1 = -2x\hat{\mathbf{y}} \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{F}_2 = 0$$

Thus $\mathbf{F}_2$ can be written as a gradient of some scalar potential

$$\mathbf{F}_2 = -\boldsymbol{\nabla} V \quad \text{where} \quad V = -\frac{1}{2}(x^2+y^2+z^2)$$

and $\mathbf{F}_1$ can be written as a curl of some vector potential

$$\mathbf{F}_1 = \boldsymbol{\nabla}\times\mathbf{A}$$

where

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, \quad \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = 0, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$$

so one solution can be found setting

$$A_y = x^3/3, \quad A_y = 0, \quad A_z = 0$$

thus $A = x^3/3\hat{\mathbf{x}}$.

(b) Finding the vector and scalar potentials of the vector field

$$\mathbf{F}_3 = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$$

The gradient of a scalar:

$$\mathbf{F}_3 = -\boldsymbol{\nabla} V_3$$

so

$$\frac{\partial V_3}{\partial x} = -yz, \quad \frac{\partial V_3}{\partial y} = -zx, \quad \frac{\partial V_3}{\partial z} = -xy$$

A particular solution is

$$V_3 = -xyz$$

The curl of a vector (a little harder):

$$\mathbf{F}_3 = \boldsymbol{\nabla} \times \mathbf{V}_3$$

so

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, \quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = zx, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy$$

integrating the first equation with respect to y:

$$\begin{aligned}\frac{\partial A_z}{\partial y} &= yz + \frac{\partial A_y}{\partial z} \\ \int \frac{\partial A_z}{\partial y}\,\mathrm{d}y &= \int yz\,\mathrm{d}y + \int \frac{\partial A_y}{\partial z}\,\mathrm{d}y \\ A_z &= \frac{1}{2}y^2 z + f\end{aligned}$$

where the function f is an arbitrary function. Integrating the first equation again but with respect to z gives:

$$A_y = -\frac{1}{2}yz^2 + g$$

Repeating for all three equations:

$$\begin{aligned}A_z &= \frac{1}{2}y^2 z + f \quad \text{and} \quad A_y = -\frac{1}{2}yz^2 + g \\ A_x &= \frac{1}{2}z^2 x + h \quad \text{and} \quad A_z = -\frac{1}{2}zx^2 + i \\ A_y &= \frac{1}{2}x^2 y + j \quad \text{and} \quad A_x = -\frac{1}{2}xy^2 + k\end{aligned}$$

a particular solution can be found by setting $f = g = h = i = j = k = 0$, and each component is a linear combination of the two possible solutions:

$$\begin{aligned}2A_x &= \frac{1}{2}z^2 x + f - \frac{1}{2}xy^2 + k \\ 2A_x &= \frac{1}{2}z^2 x - \frac{1}{2}xy^2 \\ A_x &= \frac{1}{4}(z^2 x - xy^2)\end{aligned}$$

and similarly for A_y and A_z gives the vector potential:

$$\mathbf{A} = \frac{1}{4}[x(z^2 - y^2)\hat{\mathbf{x}} + y(x^2 - z^2)\hat{\mathbf{y}} + z(y^2 - x^2)\hat{\mathbf{z}}]$$

1.51 $(d) \to (a)$: Given $\mathbf{F} = -\boldsymbol{\nabla} V$, the curl of the gradient is always zero;

$$\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times (-\boldsymbol{\nabla} V) = 0$$

$(a) \to (c)$: From Stokes' theorem;

$$\int_S (\boldsymbol{\nabla} \times F)\, \mathrm{d}\mathbf{a} = \oint F \cdot \mathrm{d}\mathbf{l} = 0$$

$(c) \to (b)$:

$$\oint F \cdot \mathrm{d}\mathbf{l} = \int_a^b F \cdot \mathrm{d}\mathbf{l} - \int_a^b F \cdot \mathrm{d}\mathbf{l} = 0$$

where the integrals are two different paths but equal, and thus independent of path. $(b) \to (c)$ and $(c) \to (a)$ are just the same steps in reverse.

1.52 $(d) \to (a)$: Given $\mathbf{F} = \boldsymbol{\nabla} \times A$, the divergence of curl is always zero;

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times A) = 0$$

$(a) \to (c)$: From the divergence theorem;

$$\int_V (\boldsymbol{\nabla} \cdot F)\, \mathrm{d}V = \oint F \cdot \mathrm{d}\mathbf{a} = 0$$

$(c) \to (b)$:

$$\oint F \cdot \mathrm{d}\mathbf{a} = \int F \cdot \mathrm{d}\mathbf{a}_1 + \int F \cdot \mathrm{d}\mathbf{a}_2 = 0$$

where $\mathbf{a}_2 = -\mathbf{a}_1$ and the integrals are two different surfaces but equal, and thus depend only on the boundary. $(b) \to (c)$ and $(c) \to (a)$ are just the same steps in reverse.

1.53 (a) From Problem 1.15,

$$\begin{aligned}
\mathbf{v}_a &= x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}} \\
\mathbf{v}_b &= xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}} \\
\mathbf{v}_c &= y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}
\end{aligned}$$

and the curl of each vector field:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_a &\neq 0 \\
\boldsymbol{\nabla} \times \mathbf{v}_b &\neq 0 \\
\boldsymbol{\nabla} \times \mathbf{v}_c &= 0
\end{aligned}$$

Thus $\mathbf{v}_c$ can be written as a gradient of some scalar potential $\mathbf{v}_c = -\boldsymbol{\nabla} V$:

$$\begin{aligned}
\frac{\partial V}{\partial x} &= -y^2; \quad V = -y^2 x + f \\
\frac{\partial V}{\partial y} &= -(2xy + z^2); \quad V = -y^2 x - yz^2 + g \\
\frac{\partial V}{\partial z} &= -2yz; \quad V = -yz^2 + h
\end{aligned}$$

where f, g, and h are arbitrary functions, thus one solution can be found setting $g = 0$ and solving for f or h;

$$V = -y^2 x - yz^2$$

(b) Given $\boldsymbol{\nabla} \cdot \mathbf{v}_a = 0$ thus $\mathbf{v}_a$ can be written as a curl of some vector potential $\mathbf{v}_a = \boldsymbol{\nabla} \times \mathbf{A}$:

$$\begin{aligned} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} &= x^2; \quad & A_z &= x^2 y + f, \quad & A_y &= -x^2 z + g \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} &= 3xz^2; \quad & A_x &= xz^3 + h, \quad & A_z &= -\frac{3}{2}x^2 z^2 + i \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= -2xz; \quad & A_y &= -x^2 z + j, \quad & A_x &= 2xyz + k \end{aligned}$$

where f, g, h, i, j, and k are arbitrary functions where a solution can be found by setting one function to zero and solving for the others: e.g., $h = 0$;

$$A_x = xz^3; \quad A_z = 0; \quad A_y = -x^2 z$$

or

$$\mathbf{A} = xz^3 \hat{\mathbf{x}} - x^2 z \hat{\mathbf{y}}$$

Checking the solution

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -x^2 z & 0 \end{vmatrix} \\ &= x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}} \end{aligned}$$

Another solution can be found by setting $f = 0$:

$$A_z = x^2 y; \quad A_y = 0; \quad A_x = xz^3 + 2xyz$$

or

$$\mathbf{A} = 2xyz \hat{\mathbf{x}} + \frac{1}{3} x^3 y \hat{\mathbf{z}}$$

Checking the solution

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 + 2xyz & 0 & x^2 y \end{vmatrix} \\ &= x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}} \end{aligned}$$

Another solution can be found by setting all arbitrary functions to zero and taking the linear combination of the two solutions:

$$2A_x = xz^3 + 2xyz A_x = \frac{1}{2}(xz^3 + 2xyz)$$

and thus

$$A_y = \frac{1}{2} - x^2 z; \quad A_z = \frac{1}{2} x^2 y - \frac{3}{4} x^2 z^2$$

or

$$\mathbf{A} = \frac{1}{2}(xz^3 + 2xyz)\hat{\mathbf{x}} - \frac{1}{2} x^2 z \hat{\mathbf{y}} + \frac{1}{2} x^2 y - \frac{3}{4} x^2 z^2 \hat{\mathbf{z}}$$

Checking the solution

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}(xz^3 + 2xyz) & -\frac{1}{2} x^2 z & \frac{1}{2} x^2 y - \frac{3}{4} x^2 z^2 \end{vmatrix} \\ &= x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}} \end{aligned}$$

1.61 (a) Show that

$$\int_V (\boldsymbol{\nabla} T)\,\mathrm{d}V = \oint_S T\,\mathrm{d}\mathbf{a}$$

Letting $\mathbf{v} = \mathbf{c}T$ where $\mathbf{c}$ is a constant: using the product rule of divergence:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(f\mathbf{A}) &= f(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla} f)\\ \boldsymbol{\nabla}\cdot(T\mathbf{c}) &= T(\boldsymbol{\nabla}\cdot\mathbf{c}) + \mathbf{c}\cdot(\boldsymbol{\nabla} T)\\ \boldsymbol{\nabla}\cdot(T\mathbf{c}) &= \mathbf{c}\cdot(\boldsymbol{\nabla} T)\end{aligned}$$

where $\boldsymbol{\nabla}\cdot\mathbf{c} = 0$ since $\mathbf{c}$ is a constant. Integrating over a volume and using the divergence theorem:

$$\int_V \boldsymbol{\nabla}\cdot(T\mathbf{c})\,\mathrm{d}V = \int_V \mathbf{c}\cdot(\boldsymbol{\nabla} T)\,\mathrm{d}V = \oint_S T\mathbf{c}\cdot\mathrm{d}\mathbf{a}$$

or moving the constant $\mathbf{c}$ outside the integral:

$$\begin{aligned}\mathbf{c}\int_V \boldsymbol{\nabla} T\,\mathrm{d}V &= \mathbf{c}\oint_S T\cdot\mathrm{d}\mathbf{a}\\ \int_V \boldsymbol{\nabla} T\,\mathrm{d}V &= \oint_S T\cdot\mathrm{d}\mathbf{a}\end{aligned}$$

(b) Show that

$$\int_V (\boldsymbol{\nabla}\times\mathbf{v})\,\mathrm{d}V = -\oint_S \mathbf{v}\times\mathrm{d}\mathbf{a}$$

replacing $\mathbf{v}$ with $\mathbf{v}\times\mathbf{c}$ in the divergence theorem:

$$\int_V \boldsymbol{\nabla}\cdot(\mathbf{v}\times\mathbf{c})\,\mathrm{d}V = \oint_S (\mathbf{v}\times\mathbf{c})\cdot\mathrm{d}\mathbf{a} \tag{1.1}$$

and using the product rule of divergence:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{v}\times\mathbf{c}) &= \mathbf{c}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) - \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{c})\\ \boldsymbol{\nabla}\cdot(\mathbf{v}\times\mathbf{c}) &= \mathbf{c}\cdot(\boldsymbol{\nabla}\times\mathbf{v})\end{aligned}$$

where $\boldsymbol{\nabla}\times\mathbf{c} = 0$ since $\mathbf{c}$ is a constant. So subbing into the divergence theorem and moving the constant $\mathbf{c}$ outside the integral:

$$\mathbf{c}\int_V (\boldsymbol{\nabla}\times\mathbf{v})\,\mathrm{d}V = \oint_S (\mathbf{v}\times\mathbf{c})\cdot\mathrm{d}\mathbf{a}$$

and using the scalar triple product identity:

$$(\mathbf{v}\times\mathbf{c})\cdot\mathrm{d}\mathbf{a} = \mathbf{v}\cdot(\mathbf{c}\times\mathrm{d}\mathbf{a}) = \mathbf{c}\cdot(\mathrm{d}\mathbf{a}\times\mathbf{v}) = -\mathbf{c}\cdot(\mathbf{v}\times\mathrm{d}\mathbf{a})$$

so

$$\begin{aligned}\mathbf{c}\int_V (\boldsymbol{\nabla}\times\mathbf{v})\,\mathrm{d}V &= -\mathbf{c}\oint_S (\mathbf{v}\times\mathrm{d}\mathbf{a})\\ \int_V (\boldsymbol{\nabla}\times\mathbf{v})\,\mathrm{d}V &= -\oint_S (\mathbf{v}\times\mathrm{d}\mathbf{a})\end{aligned}$$

(c) [Green's identity] Show that

$$\int_V [T\nabla^2 U\,\mathrm{d}V + (\boldsymbol{\nabla} T)\cdot(\boldsymbol{\nabla} U)]\,\mathrm{d}V = \oint_S (T\boldsymbol{\nabla} U)\cdot\mathrm{d}a$$

letting $\mathbf{v} = T\boldsymbol{\nabla}U$ in the divergence theorem:

$$\int_V \boldsymbol{\nabla} \cdot (T\boldsymbol{\nabla}U)\,\mathrm{d}V = \oint_S (T\boldsymbol{\nabla}U) \cdot \mathrm{d}\mathbf{a}$$

using the product rule of divergence:

$$\boldsymbol{\nabla} \cdot (T\boldsymbol{\nabla}U) = T(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}U) + (\boldsymbol{\nabla}U) \cdot (\boldsymbol{\nabla}T)$$
$$\boldsymbol{\nabla} \cdot (T\boldsymbol{\nabla}U) = T(\nabla^2 U) + (\boldsymbol{\nabla}T) \cdot (\boldsymbol{\nabla}U)$$

where $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}U = \nabla^2 U$ and the dot product is commutative. Subbing into the divergence theorem:

$$\int_V [T\nabla^2 U\,\mathrm{d}V + (\boldsymbol{\nabla}T) \cdot (\boldsymbol{\nabla}U)]\,\mathrm{d}V = \oint_S (T\boldsymbol{\nabla}U) \cdot \mathrm{d}a$$

(d) [Green's second identity] Show that

$$\int_V (T\nabla^2 U - U\nabla^2 T)\,\mathrm{d}V = \oint_S (T\boldsymbol{\nabla}U - U\boldsymbol{\nabla}T) \cdot \mathrm{d}a$$

using the product rule of divergence:

$$\boldsymbol{\nabla} \cdot (U\boldsymbol{\nabla}T) = U\nabla^2 T + (\boldsymbol{\nabla}T) \cdot (\boldsymbol{\nabla}U)$$

and letting $\mathbf{v} = U\boldsymbol{\nabla}T$ in the divergence theorem:

$$\int_V \boldsymbol{\nabla} \cdot (U\boldsymbol{\nabla}T)\,\mathrm{d}V = \oint_S (U\boldsymbol{\nabla}T) \cdot \mathrm{d}\mathbf{a}$$
$$\int_V [U\nabla^2 T + (\boldsymbol{\nabla}T) \cdot (\boldsymbol{\nabla}U)]\,\mathrm{d}V = \oint_S (U\boldsymbol{\nabla}T) \cdot \mathrm{d}\mathbf{a}$$

and taking the difference with the result from (c):

$$\oint_S (T\boldsymbol{\nabla}U) \cdot \mathrm{d}\mathbf{a} - \oint_S (U\boldsymbol{\nabla}T) \cdot \mathrm{d}\mathbf{a} = \oint_S (T\boldsymbol{\nabla}U - U\boldsymbol{\nabla}T) \cdot \mathrm{d}\mathbf{a}$$
$$\int_V [T\nabla^2 U\,\mathrm{d}V - U\nabla^2 T]\,\mathrm{d}V = \oint_S (T\boldsymbol{\nabla}U - U\boldsymbol{\nabla}T) \cdot \mathrm{d}\mathbf{a}$$

where $(\boldsymbol{\nabla}T) \cdot (\boldsymbol{\nabla}U)$ cancels out on the left hand side.
(e) Show that

$$\int_S \boldsymbol{\nabla}T \times \mathrm{d}\mathbf{a} = -\oint_P T\,\mathrm{d}\mathbf{l}$$

letting $\mathbf{v} = \mathbf{c}T$ in Stokes' theorem:

$$\int_S \boldsymbol{\nabla} \times (\mathbf{c}T) \cdot \mathrm{d}\mathbf{a} = \oint_P (\mathbf{c}T) \cdot \mathrm{d}\mathbf{l}$$

using the product rule of curl:

$$\boldsymbol{\nabla} \times (\mathbf{c}T) = T(\boldsymbol{\nabla} \times \mathbf{c}) - \mathbf{c} \times (\boldsymbol{\nabla}T)$$
$$\boldsymbol{\nabla} \times (\mathbf{c}T) = -\mathbf{c} \times (\boldsymbol{\nabla}T)$$

where $\boldsymbol{\nabla} \times \mathbf{c} = 0$ since $\mathbf{c}$ is a constant. And subbing into Stokes' theorem:

$$\int_S (-\mathbf{c} \times (\boldsymbol{\nabla}T)) \cdot \mathrm{d}\mathbf{a} = \oint_P (\mathbf{c}T) \cdot \mathrm{d}\mathbf{l}$$

and using the scalar triple product identity

$$(-\mathbf{c} \times (\boldsymbol{\nabla}T)) \cdot \mathrm{d}\mathbf{a} = -\mathbf{c} \cdot (\boldsymbol{\nabla}T \times \mathrm{d}\mathbf{a})$$

thus the constant $\mathbf{c}$ can move out of the integral and cancel out giving the result

$$\int_S \boldsymbol{\nabla}T \times \mathrm{d}\mathbf{a} = -\oint_P T\,\mathrm{d}\mathbf{l}$$

1.62 The vector area of the surface S is

$$\mathbf{a} \equiv \int_S \mathrm{d}\mathbf{a}$$

(a) The vector area of a hemispherical bowl of radius R:

$$\mathbf{a} = \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{r}}$$

where unit vector

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

changes with θ and ϕ, so the vector area is computes for each component

$$\begin{aligned}
\mathbf{a} &= \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, (\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}) \\
&= \int_0^{2\pi} \int_0^{\pi/2} () \cos\phi \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{x}} + \int_0^{2\pi} \int_0^{\pi/2} () \sin\phi \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{y}} \\
&+ \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin\theta\cos\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{z}}
\end{aligned}$$

for the first two components

$$\int_0^{2\pi} \cos\phi \, \mathrm{d}\phi = \int_0^{2\pi} \sin\phi \, \mathrm{d}\phi = 0$$

and for the third component

$$\int_0^{2\pi} \int_0^{\pi/2} R^2 \sin\theta\cos\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{z}} = R^2\hat{\mathbf{z}}(2\pi) \int_0^{\pi/2} \sin\theta\cos\theta \, \mathrm{d}\theta$$

using the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\int_0^{\pi/2} \sin\theta\cos\theta \, \mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/2} \sin(2\theta) \, \mathrm{d}\theta = \frac{1}{2}\left(-\frac{1}{2}\cos(2\theta)\Big|_0^{\pi/2}\right) = \frac{1}{2}$$

thus the vector area is

$$\mathbf{a} = \pi R^2 \hat{\mathbf{z}}$$

(b) Using 1.61(a) with $T = 1$:

$$\begin{aligned}
\int_V (\boldsymbol{\nabla} T) \, \mathrm{d}V &= \oint_S T \, \mathrm{d}\mathbf{a} \\
\int_V (\boldsymbol{\nabla} 1) \, \mathrm{d}V &= \oint_S \mathrm{d}\mathbf{a} \\
0 &= \oint_S \mathrm{d}\mathbf{a} = \mathbf{a}
\end{aligned}$$

since the gradient of a constant is always zero.
(c) The vector area of a closed surface can be split into two surfaces with the same boundary but opposite normals:

$$\oint_S \mathrm{d}\mathbf{a} = \int_{S_1} \mathrm{d}\mathbf{a} - \int_{S_2} \mathrm{d}\mathbf{a} = 0$$

Thus the vector area of the two surfaces are equal as long as the boundary is the same.
(d) Show that the integral around the boundary line

$$\mathbf{a} = \frac{1}{2}\oint_S \mathbf{r} \times \mathrm{d}\mathbf{l}$$

A cone subtended by the loop at the origin (vertex at the origin) can be split into infinitesimal triangular wedges with base d$\mathbf{l}$ and height $\mathbf{r}$, and the area of each wedge is

$$\mathrm{d}A = \frac{1}{2}\mathbf{r} \times \mathrm{d}\mathbf{l}$$

since the cross product is the area of the parallelogram formed by the two vectors or in this case the area of the triangle. Summing over all the wedges along the closed loop gives the total area

$$\frac{1}{2}\oint_S \mathbf{r} \times \mathrm{d}\mathbf{l} = \mathbf{a} = \oint_S \mathrm{d}A$$

(e) Show that

$$\oint (\mathbf{c} \cdot \mathbf{r})\,\mathrm{d}\mathbf{l} = \mathbf{a} \cdot \mathbf{c}$$

for any constant vector $\mathbf{c}$. Letting $T = \mathbf{c} \cdot \mathbf{r}$ from the result in 1.61(e):

$$\int \boldsymbol{\nabla} T \times \mathrm{d}\mathbf{a} = -\oint T\,\mathrm{d}\mathbf{l}$$
$$\int \boldsymbol{\nabla}(\mathbf{c} \cdot \mathbf{r}) \times \mathrm{d}\mathbf{a} = -\oint (\mathbf{c} \cdot \mathbf{r})\,\mathrm{d}\mathbf{l}$$

using the product rule of gradients:

$$\boldsymbol{\nabla}(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c} \times (\boldsymbol{\nabla} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\nabla} \times \mathbf{c}) + (\mathbf{c} \cdot \boldsymbol{\nabla})\mathbf{r} + (\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{c}$$

where $\mathbf{r} = (x, y, z)$; $\boldsymbol{\nabla} \times \mathbf{r} = 0$, $\boldsymbol{\nabla} \times c = 0$, $\boldsymbol{\nabla}\mathbf{c} = 0$, and

$$(\mathbf{c} \cdot \boldsymbol{\nabla})\mathbf{r} = \mathbf{c}$$

So

$$\boldsymbol{\nabla}(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c}$$

therefore

$$\begin{aligned}
-\oint (\mathbf{c} \cdot \mathbf{r})\,\mathrm{d}\mathbf{l} &= \int \mathbf{c} \times \mathrm{d}\mathbf{a} \\
&= \mathbf{c} \times \int \mathrm{d}\mathbf{a} \\
&= \mathbf{c} \times \mathbf{a} \\
&= -\mathbf{a} \times \mathbf{c} \\
\oint (\mathbf{c} \cdot \mathbf{r})\,\mathrm{d}\mathbf{l} &= \mathbf{a} \times \mathbf{c}
\end{aligned}$$

1.63 (a) Finding the divergence of

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}\right) = \frac{1}{r^2}$$

Using divergence theorem: the surface integral is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \left(\frac{\hat{\mathbf{r}}}{R}\right) \cdot (R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{r}})$$
$$= R \int_0^\pi \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = 4\pi R$$

and the volume integral

$$\int (\boldsymbol{\nabla} \cdot \mathbf{v}) \, \mathrm{d}V = \int \frac{1}{r^2} (r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi)$$
$$= \int_0^R \mathrm{d}r \int_0^\pi \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = 4\pi R$$

There is no delta function because the divergence theorem holds true on both sides. The general formula for divergence is

$$\boldsymbol{\nabla} \cdot (r^n \hat{\mathbf{r}}) = \frac{n}{r} r^n = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 r^n\right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^{n+2}\right) = \frac{n+2}{r^2} r^{n+1} = (n+2) r^{n-1}$$

except for $n = -2$ the divergence is $4\pi\delta^3(\mathbf{r})$.

(b) Finding the curl of $\mathbf{v} = r^n \hat{\mathbf{r}}$:

$$\boldsymbol{\nabla} \times (r^n \hat{\mathbf{r}}) = 0$$

since we have no θ or ϕ components. From 1.61(b):

$$\int (\boldsymbol{\nabla} \times \mathbf{v}) \, \mathrm{d}V = 0$$

and since the cross product

$$\mathbf{v} \times \mathrm{d}\mathbf{a} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r^n & 0 & 0 \\ R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi & 0 & 0 \end{vmatrix} = 0$$

the surface integral is also zero:

$$-\oint \mathbf{v} \times \mathrm{d}\mathbf{a} = 0$$

2 Electrostatics

2.1 (a) Given twelve equal charges, q situated on corners of a regular 12-sided polygon, the net force is

$$\vec{F}_a = \sum_{i=1}^{12} \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_i^2} \hat{\mathscr{r}} = 0$$

since the forces on each pair of charges (e.g., 12 and 6 o' clock) opposite to each other cancel out.
(b) If one of the charges is removed at 6 o' clock, the net force is strictly due to the the source charge at 12 o' clock:

$$\mathbf{F}_b = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_{12}^2} \hat{\mathscr{r}}_{12}$$

where $\mathbf{F}_b$ points from 12 to 6 o' clock.
(c) For 13 equal charges, the net force is still $\mathbf{F}_c = 0$ because the symmetry of the arrangement is preserved.
(d) Removing one of the charges $\mathbf{r}_i'$ is equivalent to the superposition of a source charge, $-q$, at $\mathbf{r}_i'$ and the original configuration. The net force is then

$$\mathbf{F}_d = \mathbf{F}_c - \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i = -\frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$

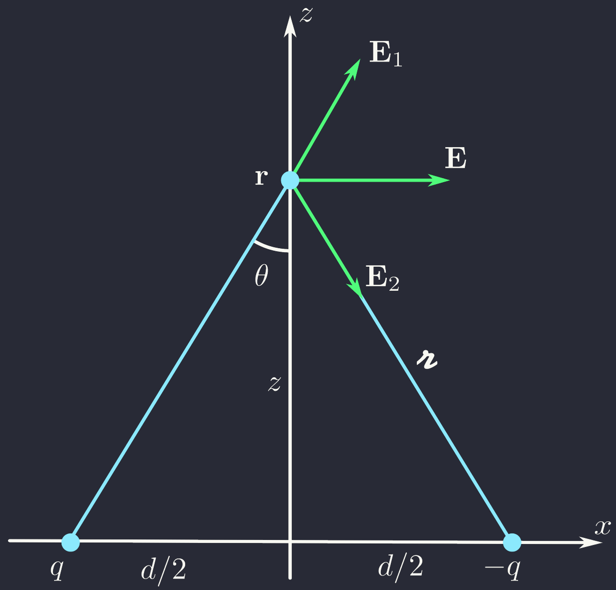

Figure 2.1: An electric field at a distance z from the midpoint between equal and opposite charges ($\pm q$) separated by a distance d. The charge at $x = d/2$ is $-q$.

2.2 The vertical componets of the electric field cancel out and the horizontal components add up:

$$E_x = 2\frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}^2} \sin\theta$$

where $E_x = E\cos\theta$, $\mathscr{r} = \sqrt{z^2 + (d/2)^2}$, and $\sin\theta = d/(2\mathscr{r})$, so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{qd}{[z^2 + (d/2)^2]^{3/2}} \hat{\mathbf{x}}$$

2.3

$$\mathbf{r} = z\hat{\mathbf{z}}, \quad \mathbf{r}' = x\hat{\mathbf{x}}, \quad \mathrm{d}l = \mathrm{d}x\,;$$
$$\boldsymbol{\mathscr{r}} = z\hat{\mathbf{z}} - x\hat{\mathbf{x}}, \quad \mathscr{r} = \sqrt{z^2 + x^2}, \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}$$

With uniform line charge λ and the limts of integration $[0, L]$,

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda\,\mathrm{d}l}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \\
&= \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{[z^2 + x^2]^{3/2}}\,\mathrm{d}x \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\hat{\mathbf{z}} \int_0^L \frac{1}{(z^2 + x^2)^{3/2}} - \hat{\mathbf{x}} \int_0^L \frac{x}{(z^2 + x^2)^{3/2}} \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\hat{\mathbf{z}} \left(\frac{x}{z^2\sqrt{z^2 + x^2}} \right) \Big|_0^L + \hat{\mathbf{x}} \left(\frac{1}{\sqrt{z^2 + x^2}} \right) \Big|_0^L \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\hat{\mathbf{z}} \left(\frac{L}{z^2\sqrt{z^2 + L^2}} - \frac{0}{z^2\sqrt{z^2 + 0^2}} \right) + \hat{\mathbf{x}} \left(\frac{1}{\sqrt{z^2 + L^2}} - \frac{1}{\sqrt{z^2 + 0^2}} \right) \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[\hat{\mathbf{z}} \left(\frac{L}{z\sqrt{z^2 + L^2}} \right) + \hat{\mathbf{x}} \left(\frac{1}{\sqrt{z^2 + L^2}} - \frac{1}{z} \right) \right] \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\hat{\mathbf{x}} \left(\frac{z}{\sqrt{z^2 + L^2}} - 1 \right) + \hat{\mathbf{z}} \left(\frac{L}{\sqrt{z^2 + L^2}} \right) \right]
\end{aligned}$$

For $z \gg L$,

$$\mathbf{E} \approx \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z^2}\hat{\mathbf{z}}$$

From far away, the line looks like a point charge $q = \lambda L$.

2.4 One segment of the square loop is equivalent to Ex. 2.2, but with line segment length $2L \to a$ and electric field distance $z_o \to \sqrt{z_o^2 + a^2/4}$. So, the magnitude of the electric field from one segment is

$$\begin{aligned}
E &= \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z_o\sqrt{z_o^2 + L^2}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{z^2 + a^2/4}\sqrt{z^2 + a^2/2}}
\end{aligned}$$

Due to the symmetry of the loop, the electric field components in the x-direction cancel out, and the electric field components in the z-direction add up:

$$\begin{aligned}
\mathbf{E} &= 4E\cos\theta\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{z^2 + a^2/4}\sqrt{z^2 + a^2/2}} \frac{z}{\sqrt{z^2 + a^2/4}}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda a z}{(z^2 + a^2/4)\sqrt{z^2 + a^2/2}}\hat{\mathbf{z}}
\end{aligned}$$

2.5 The horizontal components of the electric field cancel out, and the vertical components conspire:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda}{\mathscr{r}^2} \cos\theta\hat{\mathbf{z}}\,\mathrm{d}l$$

where geometrically $\mathscr{r} = \sqrt{z^2 + r^2}$ and $\cos\theta = z/\mathscr{r}$. So,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \, \mathrm{d}l$$

and the line integral is over the circumference of the circle, so $\mathrm{d}\mathbf{l} = r\,\mathrm{d}\theta$ and the limits of integration are $[0, 2\pi]$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{\lambda z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \int_0^{2\pi} r \, \mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda z (2\pi r)}{(z^2 + r^2)^{3/2}}
\end{aligned}$$

2.6 The electric field is only in the z-direction where $\cos\theta = z/\mathscr{r}$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{\mathscr{r}^2} \cos\theta \hat{\mathbf{z}} \, \mathrm{d}\mathbf{a} \\
&= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \, \mathrm{d}\mathbf{a}
\end{aligned}$$

Using polar coordinates: since $\mathrm{d}\mathbf{a} = r\,\mathrm{d}r\,\mathrm{d}\theta$

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0} \sigma z (2\pi) \hat{\mathbf{z}} \int_0^R \frac{r}{(z^2 + r^2)^{3/2}} \, \mathrm{d}r \\
&= \frac{\sigma}{2\epsilon_0} z \hat{\mathbf{z}} \left[-\frac{1}{\sqrt{z^2 + r^2}} \right]_0^R \\
&= \frac{\sigma}{2\epsilon_0} z \left[\frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}} \right] \hat{\mathbf{z}} \\
\mathbf{E} &= \frac{\sigma}{2\epsilon_0} \left[1 - \frac{1}{\sqrt{z^2 + R^2}} \right] \hat{\mathbf{z}}
\end{aligned}$$

when $R \to \infty$,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma \hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}}$$

for $z \gg R$,

$$-\frac{1}{\sqrt{z^2 + R^2}} = -\frac{1}{z} \left(1 + \frac{R^2}{z^2} \right)^{-1/2} \approx -\frac{1}{z} \left(1 - \frac{1}{2} \frac{R^2}{z^2} \right) = -\frac{1}{z} + \frac{1}{2} \frac{R^2}{z^3}$$

where the binomial theorem approximation $(1 + x)^n \approx 1 + nx$ is used. So,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left[\frac{1}{z} - \frac{1}{z} + \frac{1}{2} \frac{R^2}{z^3} \right] \hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0} \frac{\pi R^2 \sigma}{z^2} \hat{\mathbf{z}}$$

or a point charge $q = \pi R^2 \sigma$ from far away.

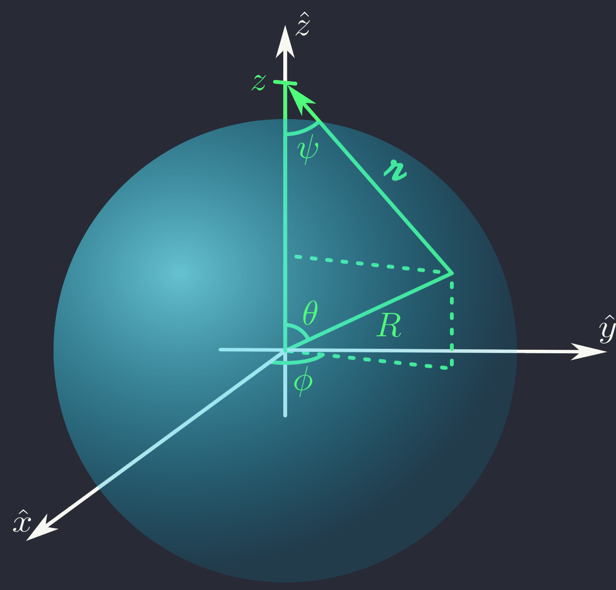

Figure 2.2: An electric field a distance z from the center of a spherical surface of radius R that carries a charge density σ.

2.7 Once again, the electric field is in the z-direction:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}^2}\cos\psi\,\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a} \tag{2.1}$$

From the law of cosines, $\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$; Geometrically, $\cos\psi = \dfrac{z - R\cos\theta}{\mathscr{r}}$; the surface area element is $\mathrm{d}\mathbf{a} = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\theta$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{\sigma R^2(z - R\cos\theta)}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}(2\pi\sigma R^2)\int_0^{\pi}\frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\,\mathrm{d}\theta\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}(2\pi\sigma R^2)f(\theta)\hat{\mathbf{z}}
\end{aligned}$$

using the substitution $u = \cos\theta$: $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$, and the limits of integration are $[\cos 0, \cos\pi]$. So,

$$f(\theta) = \int_{-1}^{1}\frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}}\,\mathrm{d}u = f(u)$$

substituting again with $v = \sqrt{z^2 + R^2 - 2zRu}$; $\mathrm{d}v = -\dfrac{zR}{v}\,\mathrm{d}u$; and $u = \dfrac{1}{2zR}(z^2 + R^2 - v^2)$:

$$\begin{aligned}
f(v) &= -\frac{1}{zR}\int\frac{z - \frac{1}{2z}(z^2 + R^2 - v^2)}{v^3}v\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{2z^2 - (z^2 + R^2 - v^2)}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{v^2 + z^2 - R^2}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\left(1 + \frac{z^2 - R^2}{v^2}\right)\mathrm{d}v \\
&= -\frac{1}{2z^2R}\left(v - \frac{z^2 - R^2}{v}\right)
\end{aligned}$$

back substituting $v = \sqrt{z^2 + R^2 - 2zRu}$,

$$
\begin{aligned}
f(u) &= -\frac{1}{2z^2R}\left(\frac{z^2 + R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}} - \frac{z^2 - R^2}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= -\frac{1}{2z^2R}\left(\frac{2R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{zu - R}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} - \frac{-z - R}{\sqrt{z^2 + R^2 + 2zR}}\right)
\end{aligned}
$$

Taking the positive square root: $\sqrt{z^2 + R^2 - 2zR} = (R - z)$ if $R > z$, but $(z - R)$ if $R < z$. So, for the case $z < R$ (inside the sphere) the electric field is

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} - \frac{-z - R}{R + z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + \frac{z + R}{R + z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + 1\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + \frac{R - z}{R - z}\right)\hat{\mathbf{z}} \\
&= 0
\end{aligned}
$$

For the case $z > R$ (outside the sphere) the electric field is

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{z - R} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{4\pi\sigma R^2}{z^2}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}
\end{aligned}
$$

This makes sense: From outside the sphere, the point charge q is the charge-per-area σ times the surface area of the sphere $4\pi R^2$, or simply $q = 4\pi R^2\sigma$.

2.8 Finding the field inside and outside a solid sphere of radius R with a uniform volume charge density ρ is similar to Prob. 2.7. Outside the solid sphere the total charge q contributes to the electric field as if it were a point charge:

$$
\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}
$$

Inside the solid sphere, only the volume of the solid sphere less than r contributes to the electric field. The volume of the total sphere is $V = \frac{4}{3}\pi R^3$, and the volume of the sphere less than r is $V' = \frac{4}{3}\pi r^3$. So, electric field inside the solid sphere is

$$
\begin{aligned}
\mathbf{E}_{in} &= \frac{V'}{V}\mathbf{E}_{out} \\
&= \frac{r^3}{R^3}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}
\end{aligned}
$$

or

$$
\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\mathbf{r}
$$

2.9 (a) The electric field in some region is $\mathbf{E} = kr^3\hat{\mathbf{r}}$ in spherical coordinates, where k is a constant. The differential form of Gauss's law is

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

and the radial component of divergence in spherical coordinates is

$$\begin{aligned}
[\boldsymbol{\nabla} \cdot \mathbf{E}]_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r^3\right) \\
&= 5kr^2
\end{aligned}$$

So, the charge density is

$$\rho = 5\epsilon_0 k r^2$$

(b) The total charge inside a sphere of radius R is found using Gauss's law:

$$\begin{aligned}
\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{Q}{\epsilon_o} \\
Q &= \epsilon_o \oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} \\
&= \epsilon_o \int (kR^3\hat{\mathbf{r}}) \cdot (R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{r}}) \\
&= \epsilon_o \int_0^{2\pi} \int_0^{\pi} (kR^5 \sin\theta) \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= 4\pi\epsilon_o k R^5
\end{aligned}$$

or using Gauss's theorem:

$$\begin{aligned}
\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \int (\boldsymbol{\nabla} \cdot \mathbf{E}) \, \mathrm{d}\tau \\
Q &= \epsilon_o \int_0^{2\pi} \int_0^{\pi} \int_0^{R} 5kr^2 (r^2 \sin\theta) \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= 4\pi\epsilon_o k R^5
\end{aligned}$$

2.10 For simplicity, using a cube of length 1:

$$y = 1, \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{r^3}, \quad \mathrm{d}a = \mathrm{d}x \, \mathrm{d}z \, \hat{\mathbf{y}}; \quad \mathbf{E} \cdot \mathrm{d}a = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}$$

the limits of integration are $x = [0, 1]$ and $z = [0, 1]$:

$$\begin{aligned}
\oint \mathbf{E} \cdot \mathrm{d}a &= \frac{1}{4\pi\epsilon_0} q \int \frac{1}{r^3} \, \mathrm{d}x \, \mathrm{d}z \\
&= \frac{1}{4\pi\epsilon_0} q \int_0^1 \int_0^1 \frac{1}{(x^2 + 1 + z^2)^{3/2}} \, \mathrm{d}x \, \mathrm{d}z \\
&= \frac{1}{4\pi\epsilon_0} q \int_0^1 \left[\frac{x}{(1 + z^2)\sqrt{x^2 + 1 + z^2}} \Bigg|_0^1 \right] \mathrm{d}z \\
&= \frac{1}{4\pi\epsilon_0} q \int_0^1 \frac{1}{(1 + z^2)\sqrt{2 + z^2}} \, \mathrm{d}z \\
&= \frac{1}{4\pi\epsilon_0} q \arctan\left(\frac{z}{\sqrt{z^2 + 2}} \right) \Bigg|_0^1 \\
&= \frac{1}{4\pi\epsilon_0} q \left(\frac{\pi}{6} \right) = \frac{q}{24\epsilon_o}
\end{aligned}$$

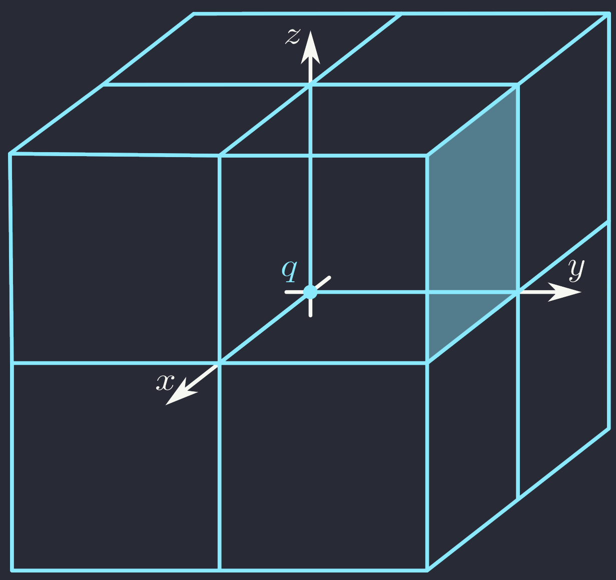

Figure 2.3: 8 cubes with a charge q at the center.

Where the first integral is solved using the trig identity $x = \tan(u)\sqrt{z^2+1}$, and similarly, the second integral uses $z = \tan(u)\sqrt{2}$.

The simpler solution is though the superposition of 8 cubes with the charge in the center of the larger cube, and the surface that encloses the larger cube is made of 24 squares equivalent to the shaded region as shown in Figure 2.3. Therefore, the flux through the shaded region is $\frac{1}{24}$ of the total flux $\frac{q}{\epsilon_o}$.

2.11 For a spherical shell of radius R with a uniform surface charge density σ, the enclosed charge in side the sphere is $Q_{enc} = 0$, thus the electric field inside the sphere is

$$\mathbf{E}_i = 0$$

and using the sphericla symmetry of a Gaussian surface, the electric field outside the sphere is

$$\begin{aligned}
\oint \mathbf{E}_o \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\
|\mathbf{E}_0| \int \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o}(4\pi\sigma R^2) \\
\mathbf{E}_o(4\pi r^2) &= \frac{1}{\epsilon_o}(4\pi\sigma R^2)\hat{\mathbf{r}} \\
\mathbf{E}_o &= \frac{\sigma R^2}{\epsilon_o r^2}\hat{\mathbf{r}}
\end{aligned}$$

2.12 Inside a solid sphere, the total charge enclosed in the Gaussian surface is

$$Q_{enc} = V'\rho = \frac{4}{3}\pi r^3 \rho$$

where V' is the volume of the sphere enclosed by the Gaussian surface. Using Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc} = \frac{1}{\epsilon_o}\frac{4}{3}\pi r^3 \rho$$

using the spherical symmetry of the Gaussian surface, the electric field is

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = |\mathbf{E}| \int \mathrm{d}a = |\mathbf{E}|(4\pi r^2)$$

Thus

$$|\mathbf{E}|(4\pi r^2) = \frac{1}{\epsilon_o}\frac{4}{3}\pi r^3 \rho$$

or

$$\mathbf{E} = \frac{1}{3\epsilon_o} r\rho\hat{\mathbf{r}} = \frac{1}{3\epsilon_o}\rho\mathbf{r}$$

Since the total charge of the solid sphere is $q = \frac{4}{3}\pi R^3 \rho$, the electric field can be rewritten as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\mathbf{r}$$

which is the same as Prob. 2.8.

2.13 Finding the electric field a distance s from an infinitely long straight wire that carries a uniform line charge λ. Using a Gaussian cylinder of radius s and length L, enclosed charge is $Q_{enc} = \lambda L$. Using Gauss's law and the symmetry of the cylinder,

$$\begin{aligned}\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int \mathrm{d}s' \, \mathrm{d}\phi' \, \mathrm{d}z' &= \frac{1}{\epsilon_o}\lambda L \\ E(2\pi s L) &= \frac{1}{\epsilon_o}\lambda L\end{aligned}$$

or

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o s}\hat{\mathbf{s}} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{\mathbf{s}}$$

which is similar to Eq. 2.9.

2.14 Find the electric field inside a sphere that carries a charge density proportional to the distance from the origin, $\rho = kr$, where k is a constant: The enclosed charge is

$$Q_{enc} = \int \rho \, \mathrm{d}\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} kr(r^2 \sin\theta) \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = \pi k r^4$$

Using Gauss's law and the spherical symmetry of the Gaussian surface,

$$\begin{aligned}\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int \mathrm{d}a &= \frac{1}{\epsilon_o}\pi k r^4 \\ E(4\pi r^2) &= \frac{1}{\epsilon_o}\pi k r^4\end{aligned}$$

or

$$\mathbf{E} = \frac{1}{4\pi\epsilon_o}\pi k r^2 \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} k r \mathbf{r}$$

2.15 A thick spherical shell with charge density

$$\rho = \frac{k}{r^2} \quad (a \le r \le b)$$

The electric field in the three regions:
(i) $r < a$

$$Q_{enc} = 0; \mathbf{E} = 0$$

(ii) $a \leq r \leq b$

$$Q_{enc} = \int_0^{2\pi} \int_0^{\pi} \int_a^{r} \rho(r^2 \sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi \int_a^{r} \frac{k}{r^2}(r^2)\,\mathrm{d}r = 4\pi k(r-a)$$

And from Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc}$$
$$|\mathbf{E}| \int \mathrm{d}a = \frac{1}{\epsilon_o} 4\pi k(r-a)$$
$$E(4\pi r^2) = \frac{1}{\epsilon_o} 4\pi k(r-a)$$

or

$$\mathbf{E} = \frac{k(r-a)}{\epsilon_o r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{4\pi k(r-a)}{r^3}\mathbf{r}$$

(iii) $r > b$

$$Q_{enc} = \int_0^{2\pi} \int_0^{\pi} \int_a^{b} \rho(r^2 \sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi k(b-a)$$

And from Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc}$$
$$|\mathbf{E}| \int \mathrm{d}a = \frac{1}{\epsilon_o} 4\pi k(b-a)$$
$$E(4\pi r^2) = \frac{1}{\epsilon_o} 4\pi k(b-a)$$

or

$$\mathbf{E} = \frac{k(b-a)}{\epsilon_o r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{4\pi k(b-a)}{r^3}\mathbf{r}$$

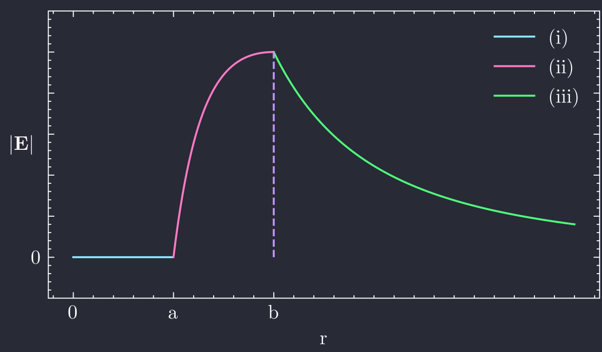

Figure 2.4: Plot of $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

2.16 A long coaxial cable carries a uniform *volume* charge charge density ρ on the inner cylinder (radius a), and a uniform *surface* charge density σ on the outer cylindrical shell (radius b). This surface charge is negative, and the cable as a whole is electrically neutral. Find the electric field in the three regions:

(i) Inside the inner cylinder $r < a$: The enclosed charge is

$$Q_{enc} = \rho\pi s^2 l$$

where l is the length of the Gaussian cylinder. Using Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc}$$
$$|\mathbf{E}| \int \mathrm{d}a = \frac{1}{\epsilon_o} \rho \pi s^2 l$$
$$E(2\pi s l) = \frac{1}{\epsilon_o} \rho \pi s^2 l$$
$$\mathbf{E} = \frac{\rho s}{2\epsilon_o} \hat{\mathbf{s}}$$

(ii) Between the cylinders $a \leq r \leq b$: The enclosed charge is

$$Q_{enc} = \rho \pi a^2 l$$

thus the electric field is

$$E(2\pi s l) = \frac{1}{\epsilon_o} \rho \pi a^2 l$$
$$\mathbf{E} = \frac{\rho a^2}{2\epsilon_o s} \hat{\mathbf{s}}$$

(iii) Outside the cable $r > b$: The enclosed charge is

$$Q_{enc} = \rho \pi a^2 l - \sigma \pi b l = 0$$

thus the electric field is $\mathbf{E} = 0$

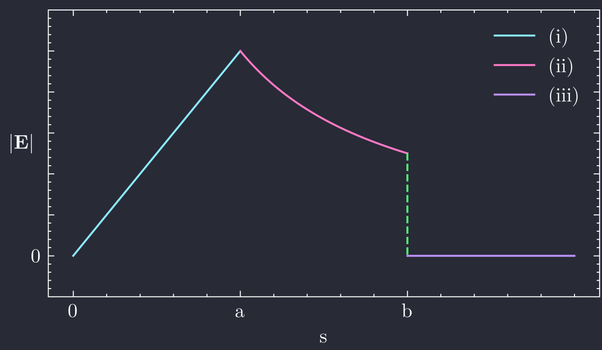

Figure 2.5: Plot of $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

2.17 Finding the electric field, as a function of y, where $y = 0$ is the center of an infinite plane slab, of thickness $2d$, carrying a uniform volume charge density ρ. For the case $y > 2d$ The enclosed charge is

$$Q_{enc} = \rho(2d)A = 2\rho A d$$

where A is the area of the Gaussian pillbox. Using Gauss's law,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc}$$
$$|\mathbf{E}| \int \mathrm{d}a = \frac{1}{\epsilon_o} 2\rho A d$$
$$E(2A) = \frac{1}{\epsilon_o} 2\rho A d$$
$$\mathbf{E} = \frac{\rho d}{\epsilon_o} \hat{\mathbf{y}}$$

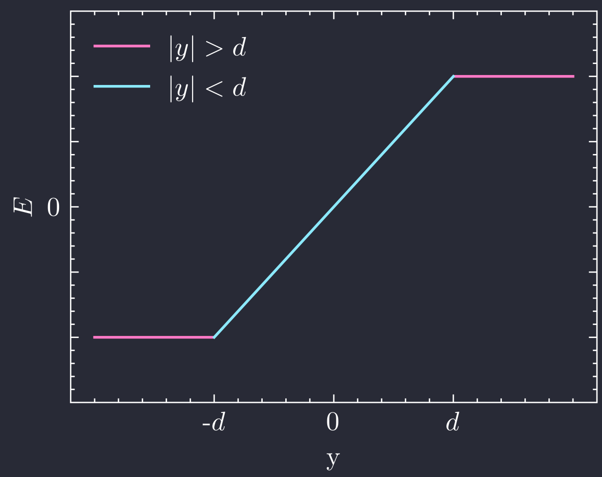

Figure 2.6: Plot of $|\mathbf{E}|$ as a function of y

For the case $0 < y < 2d$, the enclosed charge is

$$Q_{enc} = 2\rho y A$$

and the electric field is

$$E(2A) = \frac{1}{\epsilon_o}\rho y A$$
$$\mathbf{E} = \frac{\rho y}{\epsilon_o}\hat{\mathbf{y}}$$

In the $-y$ direction, E is negative as shown in Figure 2.6.

2.18 For two spheres of radius R and charge density $+\rho$ and $-\rho$, respectively, are partially overlapping. From Prob 2.12, the electric field inside a sphere of radius R with a uniform volume charge density ρ is

$$\mathbf{E} = \frac{1}{3\epsilon_o}\rho\mathbf{r}$$

where $\mathbf{r}$ is the position vector from the center of the sphere. The electric field for each sphere is

$$\mathbf{E}_1 = \frac{1}{3\epsilon_o}\rho\mathbf{r}_1$$
$$\mathbf{E}_2 = -\frac{1}{3\epsilon_o}\rho\mathbf{r}_2$$

Thus the total electric field is

$$\begin{aligned}\mathbf{E} &= \frac{1}{3\epsilon_o}\rho(\mathbf{r}_1 - \mathbf{r}_2) \\ &= \frac{1}{3\epsilon_o}\rho\mathbf{d}\end{aligned}$$

where $\mathbf{d}$ is the vector from the positive center to the negative center. Thus the electric field is constant inside the overlapping region.

2.19 The electric field inside a sphere of radius R with a uniform volume charge density ρ is

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,\mathrm{d}\tau' \tag{2.8}$$

The curl of (2.8) is

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\mathbf{r}')\, \mathrm{d}\tau'$$

From Prob 1.63, $\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$, thus

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

2.20 From the conservative nature of the electric field, the curl of the electric field is zero: (a) $\mathbf{E} = k[xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}]$;

$$\boldsymbol{\nabla} \times \mathbf{E}_a = k \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = k[-2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}] \neq 0$$

Thus (a) is not a possible electric field.
(b) $\mathbf{E}_b = k[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}]$;

$$\boldsymbol{\nabla} \times \mathbf{E}_b = k \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} = 0$$

So (b) is a possible electric field. Finding a potential using the origin as the reference point:

$$V = -\int_O^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{2.21}$$

Setting the path of integration into three parts:
(I) From $O \to A = (x, 0, 0)$;

$$\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \mathbf{E} \cdot \mathrm{d}\mathbf{l} = ky^2\,\mathrm{d}x = 0; \quad \int_O^A \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0$$

(II) From $A \to B = (x, y, 0)$;

$$\mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \mathbf{E} \cdot \mathrm{d}\mathbf{l} = k(2xy + z^2)\,\mathrm{d}y = 2kxy\,\mathrm{d}y\,; \quad \int_A^B \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 2kx \int_0^y y'\,\mathrm{d}y' = kxy^2$$

(III) From $B \to C = (x, y, z)$;

$$\mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 2kyz; \quad \int_B^C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 2ky \int_0^z z'\,\mathrm{d}z' = kyz^2$$

So the potential is

$$V = -\left(\int_O^A \mathbf{E} \cdot \mathrm{d}\mathbf{l} + \int_A^B \mathbf{E} \cdot \mathrm{d}\mathbf{l} + \int_B^C \mathbf{E} \cdot \mathrm{d}\mathbf{l} \right) = -k(xy^2 + yz^2)$$

Checking the potential function using the gradient:

$$-\boldsymbol{\nabla} V = k(y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}) = \mathbf{E}_b$$

2.21 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

The electric field outside the sphere is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

and from Problem 2.8, the electric field inside the sphere is

$$\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$$

For points outside the sphere ($r > R$),

$$V(r) = -\int_{\infty}^{r} \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

For points inside the sphere ($r < R$),

$$\begin{aligned} V(r) &= -\int_{\infty}^{R} \mathbf{E}\cdot \mathrm{d}\mathbf{l} - \int_{R}^{r} \mathbf{E}\cdot \mathrm{d}\mathbf{l} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\int_{R}^{r} r'\,\mathrm{d}r' \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\left(\frac{r'^2}{2}\right)\bigg|_{R}^{r} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right) \end{aligned}$$

The gradient of V for $r > R$:

$$-\boldsymbol{\nabla} V = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} = \mathbf{E}_{out}$$

and for $r < R$:

$$-\boldsymbol{\nabla} V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}} = \mathbf{E}_{in}$$

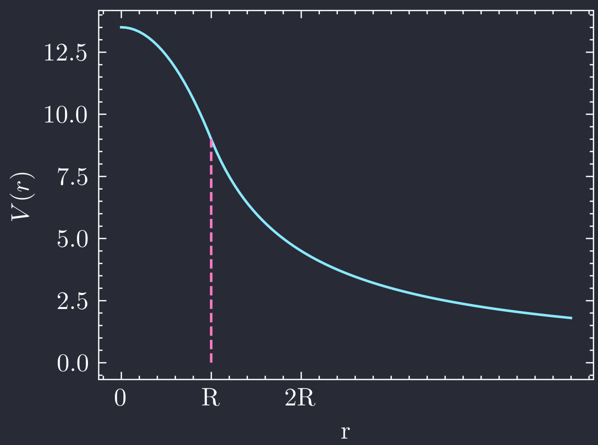

Figure 2.7: Plot of $V(r)$ as a function of r where $q = 1\,\mathrm{nC}$ and $R = 1\,\mathrm{m}$.

2.22 From Problem 2.13, the electric field a distance s from an infinitely long straight wire is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{\mathbf{s}}$$

where λ is the linear charge density of the wire. Setting the reference point at an arbitrary point s_0,

$$\begin{aligned} V(s) &= -\int_{s_0}^{s} \mathbf{E}\cdot \mathrm{d}\mathbf{l} \\ &= -\frac{1}{4\pi\epsilon_0}\int_{s_0}^{s}\frac{2\lambda}{s'}\,\mathrm{d}s' \\ &= -\frac{1}{4\pi\epsilon_0}2\lambda\ln\left(\frac{s}{s_0}\right) \end{aligned}$$

And the gradient of V is

$$-\boldsymbol{\nabla} V = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{\mathbf{s}} = \mathbf{E}$$

2.23 From Problem 2.15, the electric field in the three regions are:

(i) Inside the inner sphere, $r < a$:

$$\mathbf{E}_1 = 0$$

(ii) Between the spheres, $a \leq r \leq b$:

$$\mathbf{E}_2 = \frac{k(r-a)}{\epsilon_o r^2}\hat{\mathbf{r}}$$

(iii) Outside the outer sphere, $r > b$:

$$\mathbf{E}_3 = \frac{k(b-a)}{\epsilon_o r^2}\hat{\mathbf{r}}$$

To find the potential at the center, the reference point is set to ∞, and the line element is $\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}}$:

$$\begin{aligned}
V(r) &= -\int_{\infty}^{O} \mathbf{E}\cdot\mathrm{d}\mathbf{l} \\
&= -\int_{\infty}^{b} \mathbf{E}_3\cdot\mathrm{d}\mathbf{l} - \int_{b}^{a} \mathbf{E}_2\cdot\mathrm{d}\mathbf{l} - \int_{a}^{0} \mathbf{E}_1\cdot\mathrm{d}\mathbf{l} \\
&= -\int_{\infty}^{b} \frac{k(b-a)}{\epsilon_o r^2}\,\mathrm{d}r - \int_{b}^{a} \frac{k(r-a)}{\epsilon_o r^2}\,\mathrm{d}r - \int_{a}^{0} 0\,\mathrm{d}r \\
&= \frac{k}{\epsilon_o}\frac{b-a}{b} - \frac{k}{\epsilon_o}\left(\ln\left(\frac{a}{b}\right) + 1 - \frac{a}{b}\right) \\
&= -\frac{k}{\epsilon_o}\ln\left(\frac{a}{b}\right)
\end{aligned}$$

2.24 From Problem 2.16, the potential difference betwen a point on the axis ($s = 0$) and a point on the outside cylinder ($s = b$) goes through two distinct electric fields:

(i) Inside the inner cylinder, $s < a$:

$$\mathbf{E}_1 = \frac{\rho s}{2\epsilon_o}\hat{\mathbf{s}}$$

(ii) Between the cylinders, $a \leq s \leq b$:

$$\mathbf{E}_2 = \frac{\rho a^2}{2\epsilon_o s}\hat{\mathbf{s}}$$

So the potential difference is (using the line element $\mathrm{d}\mathbf{l} = \mathrm{d}s\,\hat{\mathbf{s}}$):

$$\begin{aligned}
V(b) - V(0) &= -\int_{0}^{b} \mathbf{E}\cdot\mathrm{d}\mathbf{l} \\
&= -\int_{0}^{a} \mathbf{E}_1\cdot\mathrm{d}\mathbf{l} - \int_{a}^{b} \mathbf{E}_2\cdot\mathrm{d}\mathbf{l} \\
&= -\int_{0}^{a} \frac{\rho s}{2\epsilon_o}\,\mathrm{d}s - \int_{a}^{b} \frac{\rho a^2}{2\epsilon_o s}\,\mathrm{d}s \\
&= -\frac{\rho}{2\epsilon_0}\left(\int_{0}^{a} s\,\mathrm{d}s + a^2\int_{a}^{b}\frac{1}{s}\,\mathrm{d}s\right) \\
&= -\frac{\rho}{2\epsilon_o}\left(\frac{a^2}{2} + a^2\ln\left(\frac{b}{a}\right)\right) \\
&= -\frac{\rho a^2}{4\epsilon_o}\left(1 + 2\ln\left(\frac{b}{a}\right)\right)
\end{aligned}$$

2.25 From Griffiths

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i} \tag{2.27}$$

and

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}\ell' \quad \text{and} \quad V = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,\mathrm{d}a' \tag{2.30}$$

(a.1) Two point charges $+q$ a distance d apart: Find the potential a distance z above the center of the charges: Using Eq. (2.27), the potential is

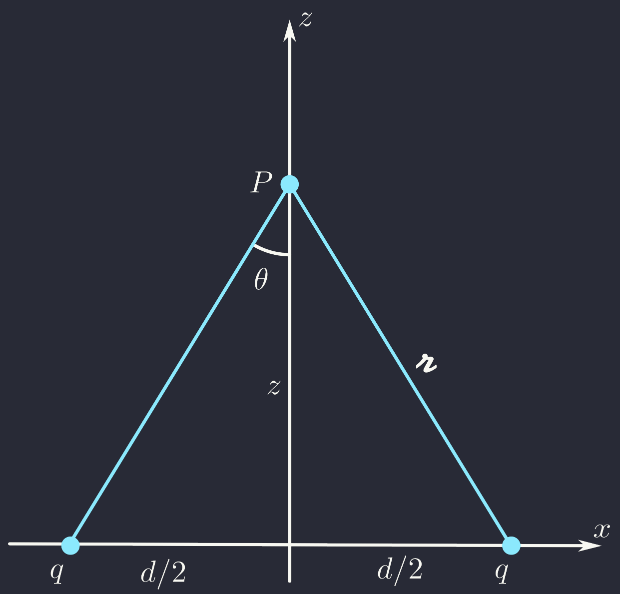

Figure 2.8: Two point charges $+q$ a distance d apart.

$$V_a = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{z^2+\frac{d^2}{4}}} + \frac{q}{\sqrt{z^2+\frac{d^2}{4}}}\right)$$

$$\boxed{V_a = \frac{1}{4\pi\epsilon_0}\frac{2q}{\sqrt{z^2+\frac{d^2}{4}}}}$$

(a.2) Computing the electric field $\mathbf{E} = -\boldsymbol{\nabla}V$:

$$\begin{aligned}\mathbf{E}_a &= -\frac{\partial V_a}{\partial z}\hat{\mathbf{z}} \\ &= -\frac{1}{4\pi\epsilon_0}\frac{-1}{2}\frac{2q(2z)}{\left(z^2+\frac{d^2}{4}\right)^{3/2}}\hat{\mathbf{z}}\end{aligned}$$

simplifying to

$$\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2+\frac{d^2}{4}\right)^{3/2}}\hat{\mathbf{z}}}$$

which is the same as Ex. 2.1

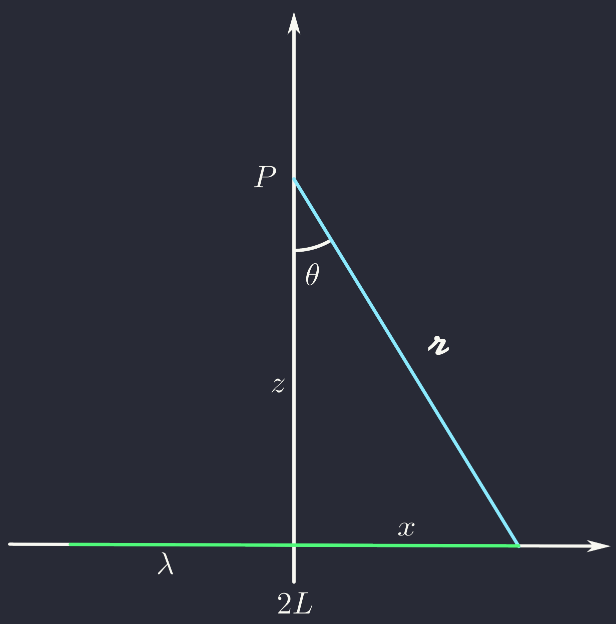

Figure 2.9: A line charge of density λ.

(b.1) Using Eq. (2.30), the potential is

$$V_b = \frac{1}{4\pi\epsilon_0}\lambda \int_{-L}^{L} \frac{1}{\sqrt{z^2+x^2}}\,\mathrm{d}x$$

To solve the integral, we can use the substitution from the trig identity

$$\begin{aligned} \cosh^2 u - \sinh^2 u = 1 \\ \implies z^2\cosh^2 u &= z^2 + z^2\sinh^2 u \\ &= z^2 + x^2 \end{aligned}$$

where

$$\begin{aligned} x = z\sinh u &\implies u = \operatorname{arcsinh}\frac{x}{z} \\ \mathrm{d}x = z\cosh u\,\mathrm{d}u \end{aligned}$$

Thus the integral becomes

$$\begin{aligned} V_b &= \frac{1}{4\pi\epsilon_0}\lambda \int \frac{\cancel{z\cosh u}}{\cancel{z\cosh u}}\,\mathrm{d}u \\ &= \frac{1}{4\pi\epsilon_0}\lambda u\Big|_{-L}^{L} \\ &= \frac{1}{4\pi\epsilon_0}\lambda\left[\operatorname{arcsinh}\frac{L}{z} - \operatorname{arcsinh}\frac{-L}{z}\right] \end{aligned}$$

Using $\operatorname{arcsinh}(a) = \ln\left|a + \sqrt{a^2+1}\right|$:

$$\begin{aligned} \implies \operatorname{arcsinh}(\frac{L}{z}) &= \ln\left|\frac{L}{z} + \sqrt{\left(\frac{L}{z}\right)^2 + 1}\right| \\ &= \ln\left|\frac{1}{z}(L + \sqrt{L^2+z^2})\right| \end{aligned}$$

so the potential is

$$\boxed{V_b = \frac{1}{4\pi\epsilon_0}\lambda \ln\left|\frac{L+\sqrt{L^2+z^2}}{-L+\sqrt{L^2+z^2}}\right|}$$

(b.2) The electric field is

$$\begin{aligned}
\mathbf{E}_b &= -\frac{\partial V_b}{\partial z}\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\left[\frac{1}{L+\sqrt{L^2+z^2}}\left(\frac{1}{2}\frac{2z}{\sqrt{L^2+z^2}}\right) - \frac{1}{-L+\sqrt{L^2+z^2}}\left(\frac{1}{2}\frac{2z}{\sqrt{L^2+z^2}}\right)\right]\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\frac{z}{\sqrt{L^2+z^2}}\left[\frac{-L+\sqrt{L^2+z^2}}{-L^2+(L^2+z^2)} - \frac{L+\sqrt{L^2+z^2}}{-L^2+(L^2+z^2)}\right]\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\lambda\frac{-2Lz}{z^2\sqrt{L^2+z^2}}\hat{\mathbf{z}}
\end{aligned}$$

simplifying to

$$\boxed{\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{L^2+z^2}}\hat{\mathbf{z}}}$$

which is the same as Ex. 2.2

(c.1) Using Eq. (2.30) and polar coordinates, the potential is

$$\begin{aligned}
V_c &= \frac{1}{4\pi\epsilon_0}\sigma\int_0^{2\pi}\int_0^R \frac{1}{\sqrt{z^2+r^2}}r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0}2\pi\sigma\int_0^R \frac{r}{\sqrt{z^2+r^2}}\,\mathrm{d}r
\end{aligned}$$

substituting $u = z^2+r^2$; $\mathrm{d}u = 2r\,\mathrm{d}r$:

$$\begin{aligned}
V_c &= \frac{1}{4\pi\epsilon_0}\pi\sigma\int\frac{1}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{4\pi\epsilon_0}\pi\sigma 2\sqrt{z^2+r^2}\Big|_0^R
\end{aligned}$$

thus

$$\boxed{V_c = \frac{\sigma}{2\epsilon_0}\left[\sqrt{z^2+R^2}-z\right]}$$

(c.2) The electric field is

$$\begin{aligned}
\mathbf{E}_c &= -\frac{\partial V_c}{\partial z}\hat{\mathbf{z}} \\
&= -\frac{\sigma}{2\epsilon}\left[\frac{z}{\sqrt{z^2+R^2}}-1\right]\hat{\mathbf{z}}
\end{aligned}$$

thus

$$\boxed{\mathbf{E}_c = \frac{\sigma}{2\epsilon_0}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}}$$

which is the same as Problem 2.6:

2.6 The electric field is only in the z-direction where $\cos\theta = z/\mathscr{r}$:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}^2}\cos\theta\,\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a}$$
$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a}$$

Using polar coordinates: since $\mathrm{d}\mathbf{a} = r\,\mathrm{d}r\,\mathrm{d}\theta$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{\sigma z}{(z^2+r^2)^{3/2}}\hat{\mathbf{z}}r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \frac{1}{4\pi\epsilon_0}\sigma z(2\pi)\hat{\mathbf{z}}\int_0^R \frac{r}{(z^2+r^2)^{3/2}}\,\mathrm{d}r$$
$$= \frac{\sigma}{2\epsilon_0}z\hat{\mathbf{z}}\left[-\frac{1}{\sqrt{z^2+r^2}}\right]_0^R$$
$$= \frac{\sigma}{2\epsilon_0}z\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}$$
$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}$$

(d) if the right-hand charge of Fig. 2.8 is replaced by a charge $-q$, the potential at P using Eq. (2.27) is

$$V_d = 0 \implies \mathbf{E}_d = 0$$

which contradicts the result from Prob 2.2. This is because point P does not give us any information about the electric field which points in the x-direction. In fact any reference point on the z-axis will give us the same result.

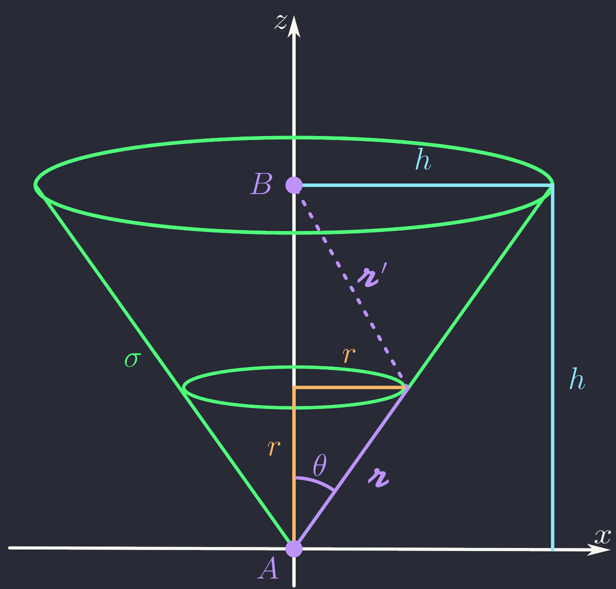

Figure 2.10: Empty ice cream cone with surface charge density σ.

2.26

(i) Potential at A: Geometrically, we can see from the large right triangle that

$$\mathscr{r}^2 = h^2 + h^2$$
$$\implies \mathscr{r} = h\sqrt{2}, \quad h = \frac{\mathscr{r}}{\sqrt{2}}$$

and from the smaller right triangle

$$\mathscr{r}^2 = 2r^2 \implies r = \frac{\mathscr{r}}{\sqrt{2}}$$

We can find the potential at A using Eq. (2.30) and integrate the rings of the cone along the slant length $0 \to h\sqrt{2}$ which gives us the area element $\mathrm{d}a = 2\pi r\,\mathrm{d}\mathscr{r}$:

$$\begin{aligned}
V(A) &= \frac{1}{4\pi\epsilon_0}\int_0^{h\sqrt{2}} \frac{\sigma}{\mathscr{r}} 2\pi r\,\mathrm{d}\mathscr{r} \\
&= \frac{\sigma}{2\epsilon_0\sqrt{2}}\int_0^{h\sqrt{2}} \mathrm{d}\mathscr{r} \\
&= \frac{\sigma}{2\epsilon_0\sqrt{2}}\,\mathscr{r}\Big|_0^{h\sqrt{2}} \\
V(A) &= \frac{\sigma h}{2\epsilon_0}
\end{aligned}$$

(ii) Potential at B: Using the law of cosines,

$$\mathscr{r}'^2 = h^2 + \mathscr{r}^2 - 2h\,\mathscr{r}\cos\theta$$

where

$$\begin{aligned}
\cos\theta &= \frac{h}{\mathscr{r}} \\
&= \frac{h}{h\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\implies \mathscr{r}' &= \sqrt{h^2 + \mathscr{r}^2 - h\,\mathscr{r}\sqrt{2}}
\end{aligned}$$

so the potential at B is

$$\begin{aligned}
V(B) &= \frac{1}{4\pi\epsilon_0}\int_0^{h\sqrt{2}} \frac{\sigma}{\mathscr{r}'} 2\pi r\,\mathrm{d}\mathscr{r} \\
&= \frac{\sigma}{2\epsilon_0\sqrt{2}}\int_0^{h\sqrt{2}} \frac{\mathscr{r}}{\sqrt{h^2 + \mathscr{r}^2 - h\,\mathscr{r}\sqrt{2}}}\,\mathrm{d}\mathscr{r}
\end{aligned}$$

I just used integral-calculator for this one...

$$\begin{aligned}
V(B) &= \frac{\sigma}{2\epsilon_0\sqrt{2}}\left[h\sqrt{2}\ln\left(1+\sqrt{2}\right)\right] \\
V(B) &= \frac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right)
\end{aligned}$$

Finally the potential difference between A and B is

$$V(B) - V(A) = \frac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right) - \frac{\sigma h}{2\epsilon_0}$$

$$\boxed{V(B) - V(A) = \frac{\sigma h}{2\epsilon_0}\left[\ln\left(1+\sqrt{2}\right) - 1\right]}$$

2.34 For a solid sphere radius R and charge q

(a) From Problem 2.21

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

and

$$W = \frac{1}{2}\int \rho V \, \mathrm{d}\tau \tag{2.43}$$

So the energy is

$$\begin{aligned}
W &= \frac{\rho}{2}\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(3-\frac{r^2}{R^2}\right)r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{\rho}{2}\frac{1}{4\pi\epsilon_0}\frac{q}{2R}4\pi\int_0^{R}\left(3r^2-\frac{r^4}{R^2}\right)\mathrm{d}r \\
&= \frac{\rho q}{4R\epsilon_0}\left[r^3-\frac{r^5}{5R^2}\right]_0^R \\
&= \frac{\rho q}{4R\epsilon_0}\left[R^3-\frac{R^3}{5}\right] \\
&= \frac{\rho q}{5\epsilon_0}R^2
\end{aligned}$$

where the charge over the volume of the sphere is $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, thus

$$W = \frac{q}{5\epsilon_0}R^2\frac{q}{\frac{4}{3}\pi R^3}$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}}$$

(b) Integrating over all space using

$$W = \frac{\epsilon_0}{2}\int E^2\,\mathrm{d}\tau \tag{2.45}$$

Where the electric field is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \quad \mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$$

so the energy is

$$\begin{aligned}
W &= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}4\pi q^2\left[\int_0^R\frac{r^2}{R^6}r^2\mathrm{d}r + \int_R^\infty\frac{1}{r^4}r^2\mathrm{d}r\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\int_0^R\frac{r^4}{R^6}\mathrm{d}r + \int_R^\infty\frac{1}{r^2}\mathrm{d}r\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{r^5}{5R^6}\bigg|_0^R - \frac{1}{R}\bigg|_R^\infty\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{R^5}{5R^6}+\frac{1}{R}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\frac{6}{5R} \\
W &= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}
\end{aligned}$$

checkmark.

(c) For a spherical volume of radius a and

$$W = \frac{\epsilon_0}{2}\left(\int_V E^2\mathrm{d}\tau + \oint_S V\mathbf{E}\cdot\mathrm{d}\mathbf{a}\right) \tag{2.44}$$

we can assume the volume is outside the charged sphere so

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

From part (b), the first term is

$$\begin{aligned}\frac{\epsilon_0}{2}\int_V E^2 \mathrm{d}\tau &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{1}{5R} - \frac{1}{a} + \frac{1}{R}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{6}{5R} - \frac{1}{a}\right]\end{aligned}$$

the second term is at $r = a$

$$\begin{aligned}\frac{\epsilon_0}{2}\oint_V V\mathbf{E}\cdot \mathrm{d}\mathbf{a} &= \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}\int \frac{q}{r}\frac{q}{r^2}r^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{4\pi\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}\frac{1}{r}\bigg|_{r=a} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\end{aligned}$$

so the total energy is

$$\begin{aligned}W &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left[\frac{6}{5R} - \frac{1}{a}\right] + \frac{1}{4\pi\epsilon_0}\frac{q^2}{2a} \\ &= \frac{1}{4\pi\epsilon_0}\frac{3q^2}{5R}\end{aligned}$$

As $a \to \infty$ the $\int V\mathbf{E}\cdot \mathrm{d}\mathbf{a}$ term goes to zero.

2.39 Two cavities radii a and b in a conducting sphere of radius R with a point charge q_a and q_b respectively in each cavity.

(a) Surface charge densities:

On the surface of cavity a the charge density is simply

$$\sigma_a = \frac{-q_a}{4\pi a^2}$$

and

$$\sigma_b = \frac{-q_b}{4\pi b^2}$$

respectively. For the surface of the conducting sphere, the charge density is positive and equal to the superposition of the two charges:

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$$

(b) The field outside the conductor is equivalent to a point charge at the center of the sphere with the sum of the charges:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$$

(c) The field in cavity a with respect to the center of the cavity is

$$\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{a^2}\hat{\mathbf{a}}$$

and in cavity b is

$$\mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{b^2}\hat{\mathbf{b}}$$

(d) The field due to to the cavity charge is zero in the exterior of the cavity, so there is no Force on q_a or q_b.

(e) If a charge q_c was brought near the conductor from outside, there would be a change in (a) σ_R and (b) $\mathbf{E}_{out}$.

2.47 Net force of of the southern hemisphere extering on the northern hemisphere (solid sphere) with an inside Electric field (Problem 2.8)

$$E_{in} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$$

where the total force is

$$\mathbf{F} = Q\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{R^3}\mathbf{r}$$

Finding the net force exerted by the southern hemisphere: integrate $\mathrm{d}F = \mathbf{F}/V$ over the southern hemisphere:

$$\begin{aligned}
\mathrm{d}F &= \frac{1}{\frac{4}{3}\pi R^3}\frac{1}{4\pi\epsilon_0}\frac{Q^2}{R^3}\mathbf{r}\mathrm{d}\tau \\
&= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}\mathbf{r}\mathrm{d}\tau
\end{aligned}$$

The symmetry of the sphere implies that the Force is only in the z-direction i.e. $F_z = F\cos\theta$, so integrating over the southern hemisphere:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/2}\int_0^R F_z\mathrm{d}\tau &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R r\cos\theta r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{3Q^2}{16\pi^2\epsilon_0 R^6}(2\pi)\left(\frac{r^4}{4}\right)\Bigg|_0^R\int_0^{\pi/2}\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{3Q^2}{32\pi\epsilon_0 R^2}\frac{\sin^2 x}{2}\Bigg|_0^{\pi/2} \\
&= \boxed{\frac{3Q^2}{64\pi\epsilon_0 R^2}}
\end{aligned}$$

USE GRIFFITHS 5th EDITION FROM NOW ON

3 Potentials

3.4

(a) Average field over a spherical surface due to charges outside the sphere is the same at the center:

For a charge q a distance z above the center of the sphere, we can use the same geometrical argument from HW 2 Problem 2.7: The average field at over the surface will be in the negative z direction

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\psi(-\hat{\mathbf{z}})$$

where (using law of cosines)

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta \qquad \cos\phi = \frac{z - R\cos\theta}{\mathscr{r}}$$

The surface element is $\mathrm{d}a = R^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$, so the average field is

$$\begin{aligned}
\mathbf{E}_{\text{avg}} &= \frac{1}{4\pi R^2}\frac{1}{4\pi\epsilon_0}(-qR^2)\hat{\mathbf{z}}\int_0^{2\pi}\int_0^{\pi}\frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{4\pi}\frac{1}{4\pi\epsilon_0}(-q)\hat{\mathbf{z}}(2\pi)\int_0^{\pi}\frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{-q}{2}\hat{\mathbf{z}}f
\end{aligned}$$

The integral evaluates to (from Problem 2.7): Using the substitution $u = \cos\theta$: $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$, and the limits of integration are $[\cos 0, \cos\pi]$. So,

$$\begin{aligned}
f(\theta) &= -\int_1^{-1}\frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}}\mathrm{d}u \\
&= \int_{-1}^{1}\frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}}\mathrm{d}u
\end{aligned}$$

substituting again with $v = \sqrt{z^2 + R^2 - 2zRu}$; $\mathrm{d}v = \frac{zR}{v}\mathrm{d}u$; and $u = \frac{1}{2zR}(z^2 + R^2 - v^2)$:

$$\begin{aligned}
f(v) &= -\frac{1}{zR}\int\frac{z - \frac{1}{2z}(z^2 + R^2 - v^2)}{v^3}v\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{2z^2 - (z^2 + R^2 - v^2)}{v^2}\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{v^2 + z^2 - R^2}{v^2}\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\left(1 + \frac{z^2 - R^2}{v^2}\right)\mathrm{d}v \\
&= -\frac{1}{2z^2R}\left(v - \frac{z^2 - R^2}{v}\right)
\end{aligned}$$

substituting back in $v = \sqrt{z^2 + R^2 - 2zRu}$,

$$
\begin{aligned}
f(u) &= -\frac{1}{2z^2 R}\left(\frac{z^2 + R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}} - \frac{z^2 - R^2}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= -\frac{1}{2z^2 R}\left(\frac{2R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{zu - R}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} - \frac{-z - R}{\sqrt{z^2 + R^2 + 2zR}}\right) \\
&= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} + \frac{z + R}{z + R}\right) \\
&= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} + 1\right)
\end{aligned}
$$

where the positive root $\sqrt{z^2 + R^2 - 2zR} = (z - R)$ for $z > R$, so

$$
\mathbf{E}_{\text{avg}} = \frac{1}{4\pi\epsilon_0}\left(-\frac{q}{2z^2}\right)\left(\frac{z - R}{z - R} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}}
$$

Simplfying to

$$
\boxed{\mathbf{E}_{\text{avg}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}}
$$

which is the same as the field at the center of the sphere. For a collection of particles, we can use superposition and find the net field as the sum of the fields at the center from each charge.

(b) For charges inside the sphere we can use the result from before: for one charge

$$
\mathbf{E}_{\text{avg}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}}
$$

but now the positive root is $\sqrt{z^2 + R^2 - 2zR} = (R - z)$ for $z < R$, so

$$
\begin{aligned}
\mathbf{E}_{\text{avg}} &= -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left(\frac{z - R}{R - z} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}(-1 + 1)\hat{\mathbf{z}} \\
\mathbf{E}_{\text{avg}} &= 0
\end{aligned}
$$

And we can superimpose the fields from a collection of charges

$$
\boxed{\mathbf{E}_{\text{avg}} = 0 + 0 + \cdots = 0}
$$

3.7 Charges $+q$ & $-2q$ are respectively $z = 3d$ & $z = d$ above the xy plane (grounded conductor). Find the force of the charge $+q$:

We can use the method of images and replace the grounded conductor with two charges $-q$ at $z = -3d$ and $+2q$ at $z = -d$. Thus the force on $+q$ is the superposition of the forces from the three charges: The seperation vectors are

$$\begin{aligned} \mathscr{r}_{-2q} &= 2d\hat{\mathbf{z}} \\ \mathscr{r}_{+2q} &= 4d\hat{\mathbf{z}} \\ \mathscr{r}_{-q} &= 6d\hat{\mathbf{z}} \end{aligned}$$

Finally, the force on charge $+q$ is

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_{-2q} + \mathbf{F}_{+2q} + \mathbf{F}_{-q} \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{-2q^2}{(2d)^2} + \frac{2q^2}{(4d)^2} + \frac{-q^2}{(6d)^2}\right)\hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{36}\right)\hat{\mathbf{z}} \end{aligned}$$

which simplifies to

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{29q^2}{72d^2}\hat{\mathbf{z}}}$$

3.8 From Griffiths, where the configuration has another point charge

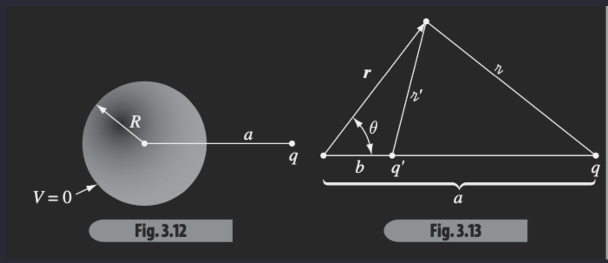

Figure 3.1: From Griffiths Example 3.2

$$q' = -\frac{R}{a} q \tag{3.15}$$

placed a distance

$$b = \frac{R^2}{a} \tag{3.16}$$

thus the potential of the config

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) \tag{3.17}$$

(a) Using law of cosines, show that Eq. (3.17) can be written as

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right]$$

From Fig. 3.1, we can see that

$$\mathscr{r} = \sqrt{r^2 + a^2 - 2ra\cos\theta} \qquad \mathscr{r}' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$$

so using Eq. (3.15) and Eq. (3.16) we can rewrite

$$\begin{aligned}
\frac{q'}{\mathscr{r}'} &= \frac{-R}{a}\frac{q}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}} \\
&= -\frac{1}{\sqrt{\frac{a^2}{R^2}}}\frac{q}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}} \\
&= -\frac{q}{\sqrt{r^2 a/R^2 + (R^2/a)^2 a^2/R^2 - 2r(R^2/a)\cos\theta a^2/R^2}} \\
\frac{q'}{\mathscr{r}'} &= -\frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}
\end{aligned}$$

Now we can rewrite Eq. (3.17) as

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right]$$

(b) Finding the induced charge on the sphere & integrating to get total induced charge: The normal component of the potential is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so

$$
\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} \\
&= -\epsilon_0 \frac{1}{4\pi\epsilon_0} q \left(-\frac{1}{2}\right) \left[\frac{2r - 2a\cos\theta}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}} - \frac{2ra^2/R^2 - 2a\cos\theta}{(R^2 + (ra/R)^2 - 2ra\cos\theta)^{3/2}} \right] \bigg|_{r=R} \\
&= \frac{q}{4\pi} \left[\frac{R - a\cos\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} - \frac{a^2/R - a\cos\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} \right] \\
&= \frac{q}{4\pi} \left[\frac{R - a^2/R}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} \right]
\end{aligned}
$$

which simplifies to

$$
\boxed{\sigma(\theta) = \frac{q}{4\pi R} \frac{R^2 - a^2}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}}}
$$

Integrating to get the total induced charge using the surface element $\mathrm{d}a = R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$:

$$
\begin{aligned}
Q &= \int \sigma \mathrm{d}a \\
Q &= \frac{q}{4\pi R}(R^2 - a^2)(2\pi R^2) \int_0^\pi \frac{\sin\theta}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} \mathrm{d}\theta \\
\text{using} \quad & u = R^2 + a^2 - 2Ra\cos\theta; \quad \mathrm{d}u = 2Ra\sin\theta\, \mathrm{d}\theta \\
&= \frac{qR}{2}(R^2 - a^2) \frac{1}{2Ra} \int \frac{1}{u^{3/2}} \mathrm{d}u \\
&= \frac{qR}{2}(R^2 - a^2) \frac{1}{2Ra} \frac{-2}{\sqrt{u}} \\
&= -\frac{q}{2a}(R^2 - a^2) \frac{1}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} \bigg|_0^\pi \\
&= -\frac{q}{2a}(R^2 - a^2) \left[\frac{1}{\sqrt{R^2 + a^2 + 2Ra}} - \frac{1}{\sqrt{R^2 + a^2 - 2Ra}} \right]
\end{aligned}
$$

From Fig. 3.1, we can see that $R < a$ so the positive root is $\sqrt{R^2 + a^2 - 2Ra} = (a - R)$. Now the total induced charge is

$$
\begin{aligned}
Q &= -\frac{q}{2a}(R^2 - a^2) \left[\frac{1}{a + R} - \frac{1}{a - R} \right] \\
&= -\frac{q}{2a}(R^2 - a^2) \left[\frac{a - R}{(a + R)(a - R)} - \frac{a + R}{(a - R)(a + R)} \right] \\
&= -\frac{q}{2a}(R^2 - a^2) \left[\frac{a - R - (a + R)}{a^2 - R^2} \right] \\
&= -\frac{q}{2a}(R^2 - a^2) \left[\frac{-2R}{-(R^2 - a^2)} \right]
\end{aligned}
$$

Thus the total induced charge is

$$
\boxed{Q = -\frac{R}{a} q = q'}
$$

(c) The energy of the config:

First we find the force on q due the induced charge q' which are separated by a distance $a - b$:

$$
\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qq'}{(a - b)^2} \hat{\mathbf{n}}
$$

Using Eq. (3.15) and Eq. (3.16)

$$\begin{aligned}
\mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{R}{a}\frac{q^2}{(a-R^2/a)^2}\hat{\mathbf{a}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{R}{a}\frac{q^2}{\frac{1}{a^2}(a^2-R^2)^2}\hat{\mathbf{a}} \\
\mathbf{F} &= -\frac{1}{4\pi\epsilon_0}\frac{Raq^2}{(a^2-R^2)^2}\hat{\mathbf{a}}
\end{aligned}$$

Now we can determine the energy by calculating the work it takes to bring q from infinity: The line element is $\mathrm{d}\ell = \mathrm{d}a\hat{\mathbf{a}}$ since the force is in the negative a direction; so the work required to *oppose* the force is

$$\begin{aligned}
W &= -\int_\infty^a \mathbf{F}\cdot \mathrm{d}\ell \\
&= -\frac{1}{4\pi\epsilon_0}Rq^2\int_\infty^a \frac{a'}{(a'^2-R^2)^2}(-\,\mathrm{d}a') \\
\text{using}\quad & u = a'^2 - R^2; \quad \mathrm{d}u = 2a'\,\mathrm{d}a' \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\int\frac{1}{2u^2}\,\mathrm{d}u \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2u}\right] \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2(a^2-R^2)}\right]\Bigg|_\infty^a \\
&= \frac{1}{4\pi\epsilon_0}Rq^2\left[-\frac{1}{2(a^2-R^2)}-0\right]
\end{aligned}$$

which simplifies to

$$\boxed{W = -\frac{1}{4\pi\epsilon_0}\frac{Rq^2}{2(a^2-R^2)}}$$

3.10 For a second image charge q'' inside the center of the sphere (it must not be outside the sphere) with potential

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) + V_0$$

where

$$V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{\mathscr{r}''} = \frac{1}{4\pi\epsilon_0}\frac{q''}{R} \implies q'' = 4\pi\epsilon_0 R V_0$$

So for a neutral conducting sphere the potential should be zero at the surface, i.e. the magnitude of the image charges q' and q'' are equal and opposite:

$$q' = -q''$$

The distance from the second image charge and q is a, so

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{qq''}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0} q\left[\frac{q'a^2}{(a^2-R^2)^2} - \frac{q'}{a^2}\frac{(a^2-R^2)^2}{(a^2-R^2)^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a^2-R^2)^2}\left[a^2 - a^2 + 2R^2 - R^4/a^2\right] \end{aligned}$$

Using Eq. (3.15) $q' = -\frac{R}{a}q$

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{(a^2-R^2)^2}\left(\frac{-R}{a}\right)\left[2R^2 - R^4/a^2\right] \\ &= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(a^2-R^2)^2}\left(\frac{R^3}{a^3}\right)\left[2a^2 - R^2\right] \end{aligned}$$

So the force of attraction has magnitude

$$\boxed{F_{\text{att}} = \frac{1}{4\pi\epsilon_0}\frac{q^2R^3}{a^3(a^2-R^2)^2}\left[2a^2 - R^2\right]}$$

3.11 Force between point charge q and spherical conductor of total charge q:
We can use a second image charge (at the center of the sphere) where

$$q'' + q' = q$$

So the force between q and the conductor is

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(a-b)^2} + \frac{qq''}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}q\left[\frac{q'a^2}{(a^2-R^2)^2} + \frac{q-q'}{a^2}\right] \end{aligned}$$

Using Eq. (3.15) $q' = -\frac{R}{a}q$:

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}q\left[\frac{-Rqa}{(a^2-R^2)^2} + \frac{q+Rq/a}{a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}q^2\left[-\frac{Ra}{(a^2-R^2)^2} + \frac{a+R}{a^3}\right] \end{aligned}$$

So the force is attractive when $[\,] < 0$, or a critical value of a_c at

$$\begin{aligned} \frac{Ra_c}{(a_c^2-R^2)^2} &= \frac{a_c+R}{a_c^3} \\ Ra_c^4 &= (a_c^2-R^2)^2(a_c+R) \\ Ra_c^4 &= (a_c^4-2a_c^2R^2+R^4)(a_c+R) \\ Ra_c^4 &= a_c^5 - 2a_c^3R^2 + R^4a_c + Ra_c^4 - 2a_c^2R^3 + R^5 \\ 0 &= a_c^5 - 2a_c^3R^2 - 2a_c^2R^3 + R^4a_c + R^5 \end{aligned}$$

From the hint, the solution must be in the form

$$a_c = R\frac{1+\sqrt{5}}{2}$$

which is the golden ratio i.e. in the quadratic form

$$\phi^2 - \phi - 1 = 0$$

So

$$a_c = R\phi \implies \phi = \frac{a_c}{R}$$

With this intuition, we can divide our quintic equation by R^5:

$$\begin{aligned} 0 &= \frac{a_c^5}{R^5} - 2\frac{a_c^3}{R^3} - 2\frac{a_c^2}{R^2} + \frac{a_c^4}{R^4} + 1 \\ &= \phi^5 - 2\phi^3 - 2\phi^2 + \phi + 1 \end{aligned}$$

then we can factor it by dividing by the golden ratio equation $\phi^2 - \phi - 1 = 0$ (using polynomial long division):

$$\begin{array}{r|l} & x^3 + x^2 - 1 \\ \hline x^2 - x - 1 & x^5 - 2x^3 - 2x^2 + x + 1 \\ & -x^5 + x^4 + x^3 \\ \hline & x^4 - x^3 - 2x^2 \\ & -x^4 + x^3 + x^2 \\ \hline & -x^2 + x + 1 \\ & x^2 - x - 1 \\ \hline & 0 \end{array}$$

Thus

$$0 = (\phi^2 - \phi - 1)(\phi^3 + \phi^2 - 1)$$

Where the quadratic equation factors to the golden ratio (positive root),

$$\begin{aligned} \phi^2 - \phi - 1 = 0 \\ \implies \phi = \frac{1+\sqrt{5}}{2} = \frac{a_c}{R} \\ \implies a_c = R\frac{1+\sqrt{5}}{2} \end{aligned}$$

So the force is attractive when the separation distance is less than the critical value a_c.

3.13 Two semi-infinite grounded conducting planes meeting as shown in Fig. 3.2 To set up the problem

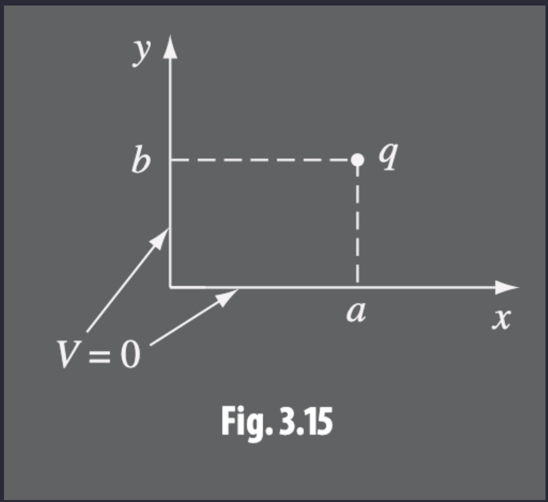

Figure 3.2: From Griffiths

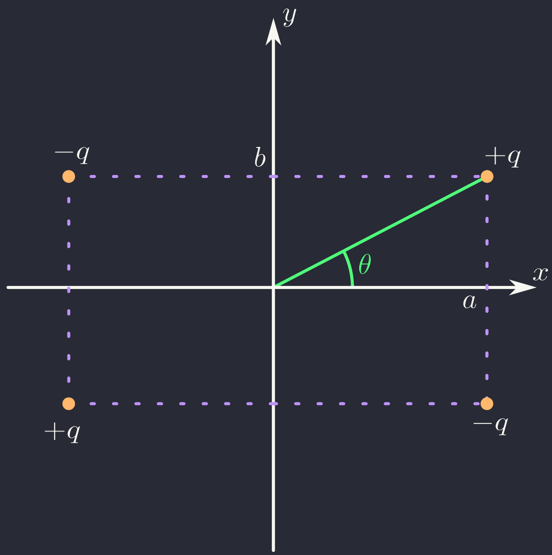

Figure 3.3: Three image charges

so we can have a potential of zero at the planes, we can place image charges $-q$ at $(a, -b)$ and $(-a, b)$, and place an image charge $+q$ at $(-a, -b)$ to balance the potentials at the axes.

The potential in the region $x > 0, y > 0$ is:

$$V = \frac{1}{4\pi\epsilon_0} q \left[\frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2 + z^2}} \right.$$
$$\left. - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} \right]$$

The force on q is (using Fig. 3.3):

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} q^2 \left[-\frac{1}{(2a)^2}\hat{\mathbf{x}} - \frac{1}{(2b)^2}\hat{\mathbf{y}} \right.$$
$$\left. + \frac{1}{(2\sqrt{a^2+b^2})^2}(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}) \right]$$

where

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

So

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left[-\frac{1}{a^2}\hat{\mathbf{x}} - \frac{1}{b^2}\hat{\mathbf{y}} + \left(\frac{a}{(a^2+b^2)^{3/2}}\hat{\mathbf{x}} + \frac{b}{(a^2+b^2)^{3/2}}\hat{\mathbf{y}}\right)\right]$$

$$\boxed{\mathbf{F} = \frac{q^2}{16\pi\epsilon_0}\left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\hat{\mathbf{y}}\right]}$$

The work done to bring q from infinity to the origin:
From $\infty \to (a,b)$ the line element is $\mathrm{d}\ell = \mathrm{d}a\,\hat{\mathbf{x}} + \mathrm{d}b\,\hat{\mathbf{y}}$ so

$$\begin{aligned} W &= -\int_{\infty}^{(a,b)} \mathbf{F}\cdot \mathrm{d}\ell \\ &= -\frac{q^2}{16\pi\epsilon_0}\int_{\infty}^{(a,b)} \left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\hat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\hat{\mathbf{y}}\right]\cdot \mathrm{d}\ell \\ &= -\frac{q^2}{16\pi\epsilon_0}\int_{\infty}^{(a,b)} \left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\mathrm{d}a + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\mathrm{d}b\right] \end{aligned}$$

Evaluating the two integrals using

$$\int -\frac{1}{a^2}\mathrm{d}a = \frac{1}{a}$$
$$\int \frac{a}{(a^2+b^2)^{3/2}}\mathrm{d}a = -\frac{1}{\sqrt{a^2+b^2}}$$

gives the work done is

$$W = -\frac{q^2}{16\pi\epsilon_0}\left[-\frac{1}{\sqrt{a^2+b^2}} + \frac{1}{a} - \frac{1}{\sqrt{a^2+b^2}} + \frac{1}{b}\right]$$

which simplifies to

$$\boxed{W = \frac{q^2}{16\pi\epsilon_0}\left[\frac{2}{\sqrt{a^2+b^2}} - \frac{1}{a} - \frac{1}{b}\right]}$$

We can solve the problem with the method of images, as long as the angle ϕ divides $180°$ into an integer, e.g., $\phi = 180, 90, 60, 45, 36, 30, 20, 18, 15, 12, 10, 9, 6, 5, 4, 3, 2, 1, 0.5, \ldots$

We would place a 'mirror' at each ϕ division and place an image charge that mirrors the point charge q (then the next image charge q mirrors the previous image charge $-q$ with opposite charge) and repeat the process until we have a symmetric configuration of charges as shown in Fig. 3.4.

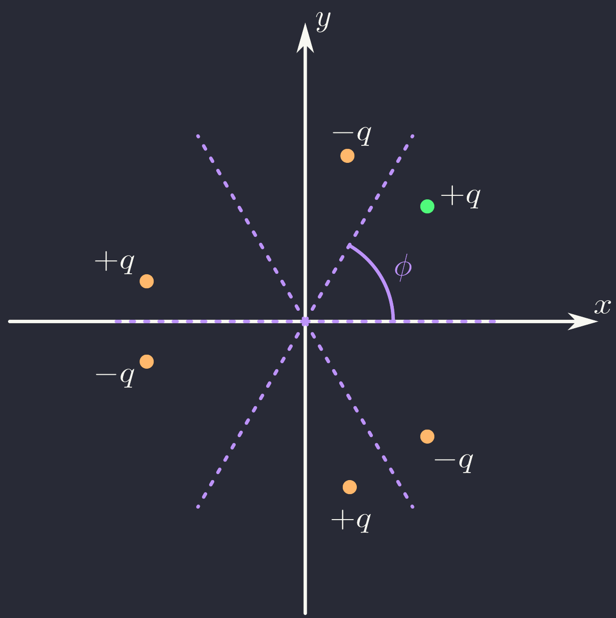

Figure 3.4: Method of images for $\phi = 60°$