Problems for Griffiths' Electrodynamics

Contents

1 Vector Analysis

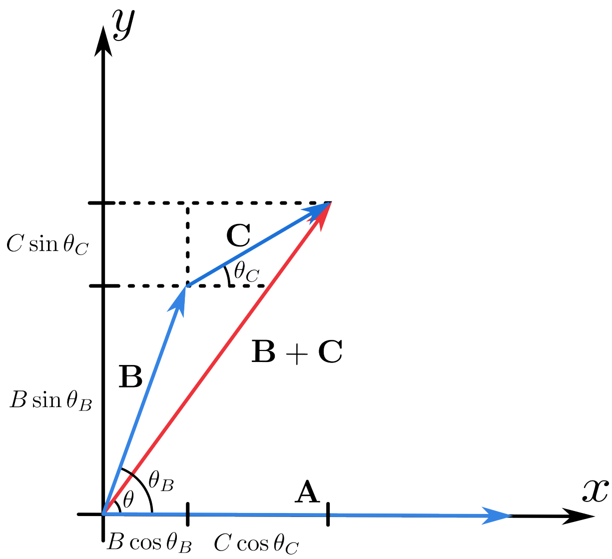

Figure 1.1: Three Coplanar Vectors

1.1 (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$
$$A(B+C)\cos\theta = AB\cos\theta_B + AC\cos\theta_C$$

Since $B\cos\theta_B + B\cos\theta_C = (B+C)\cos\theta$ from Figure 1.1, the distriubtive property holds true. The cross product also holds true since $B\sin\theta_B + B\sin\theta_C = (B+C)\sin\theta$, and multiplying by A on both sides gives the same result as the distriubtive property:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$A(B+C)\sin\theta = AB\sin\theta_B + AC\sin\theta_C$$

(b) In the general case in three-dimensional space, each vector has three components:
$\mathbf{A} = (A_x, A_y, A_z)$. Therefore,

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot(B_x+C_x, B_y+C_y, B_z+C_z)\\ &= A_x(B_x+C_x)+A_y(B_y+C_y)+A_z(B_z+C_z)\\ &= A_xB_x+A_yB_y+A_zB_z+A_xC_x+A_yC_y+A_zC_z\\ &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\end{aligned}$$

1.2 Setting $\mathbf{A}=\mathbf{B}=(1,1,1)$ and $\mathbf{C}=(1,1,-1)$:

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times\mathbf{C} &\stackrel{?}{=} \mathbf{A}\times(\mathbf{B}\times\mathbf{C})\\ 0 &\stackrel{?}{=} (1,1,1)\times[(1,1,1)\times(1,1,-1)]\\ 0 &\stackrel{?}{=} (1,1,1)\times(-2,2,0)\\ 0 &\neq (-2,-2,4)\end{aligned}$$

where the cross product of parallel vectors $\mathbf{A}\times\mathbf{B}=0$. Therefore, the cross product is not associative.

1.3 Taking the dot product of a unit cube's body diagonals $\mathbf{A} = (1, 1, 1)$, $\mathbf{B} = (1, 1, -1)$:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= AB \cos\theta \\ 1 &= 3 \cos\theta \\ \theta &= \arccos 1/3 \approx 70.53^\circ \end{aligned}$$

1.4 The cross product of two vectors coplanar to the shaded plane—$\mathbf{A} = (-1, 2, 0)$, $\mathbf{B} = (-1, 0, 3)$—is parallel to the normal unit vector $\hat{\mathbf{n}}$ of the plane:

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$
$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6, 3, 2)$$

where $\hat{\mathbf{n}} = \mathbf{C}/C$, and $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$. Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

1.5 Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix} \end{aligned}$$

where the x component is $A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$. Similarly,

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)$$

where x component simplifes to

$$B_x(\cancel{A_xC_x} + A_yC_y + A_zC_z) - C_x(\cancel{A_xB_x} + A_yB_y + A_zB_z) = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

the same is done for the y and z components. Therefore, the "BAC–CAB" rule holds true.

1.6

$$\begin{aligned} [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) = 0 \\ - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C} \cdot \mathbf{A})} \end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \\ \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \times \mathbf{B} \end{aligned}$$

For the relation to hold true, either the vectors $\mathbf{A}$ and $\mathbf{C}$ are parallel ($\mathbf{A} \times \mathbf{A} = 0$) or $\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{C}$ ($\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = 0$).