1 Energy

4.2 From the origin O to point $P = (1, 1)$ a two dimensional force $\mathbf{F} = (x^2, 2xy)$ moves a point along three paths where the work done by the force is

$$W = \int_O^P \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_O^P F_x \, \mathrm{d}x + F_y \, \mathrm{d}y$$

(a) Splitting the path into two parts $O \to Q = (1, 0)$ and $Q \to P$, we have two integrals

$$W = \int_O^Q F_x \, \mathrm{d}x + \int_Q^P F_y \, \mathrm{d}y$$

where the first integral accounts for just the x component of force $F_x = x^2$ and the second integral accounts for just the y component of force when $x = 1$; $F_y = 2(1)y$. Thus

$$W = \int_0^1 x^2 \, \mathrm{d}x + \int_0^1 2y \, \mathrm{d}y = \frac{4}{3}$$

(b) The path follows the parabola $y = x^2$ from $O \to P$. From $\mathrm{d}y = 2x \, \mathrm{d}x$ the integral can be rewritten in terms of just x

$$W = \int_0^1 x^2 \, \mathrm{d}x + \int_0^1 2x(x^2) \, \mathrm{d}y = \frac{1}{3} + \int_0^1 4x^4 \, \mathrm{d}x = \frac{17}{15}$$

(c) Path follows the parametric curve $x = t^3$ and $y = t^2$ where the differentials are: $dx = 3t^2 \, \mathrm{d}t$ and $dy = 2t \, \mathrm{d}t$. Thus the work done on the path is

$$W = \int_0^1 (t^6)(3t^2 \, \mathrm{d}t) + \int_0^1 (2t^3)(2t \, \mathrm{d}t) = \frac{1}{3} + \frac{4}{5} = \frac{19}{15}$$

4.3 Same as Problem 4.2 but with a force $\mathbf{F} = (-y, x)$ and three different paths from $P = (1, 0) \to Q = (0, 1)$.
(a) This path follows a straight line $y = 0$ from $P \to O$ and then $x = 0$ from $O \to Q$. Thus the work done is

$$W = \int_P^O F_x \, \mathrm{d}x + \int_O^Q F_y \, \mathrm{d}y = 0$$

(b) A straight line from $P \to Q$ is given by $y = -x + 1$ and the differential $dy = -dx$. Thus the work done is

$$W = \int_P^Q F_x \, \mathrm{d}x + F_y \, \mathrm{d}y = \int_1^0 (-(-x+1)) \, \mathrm{d}x + (x)(-\,\mathrm{d}x) = \int_1^0 -1 \, \mathrm{d}x = 1$$

(c) The path of a quarter circle centered on the origin in polar coordinates is given by

$$x = r\cos\phi \qquad y = r\sin\phi$$

where $r = 1$, $\phi = 0 \to \pi/2$ and the differentials are

$$dx = \cos\phi dr - r\sin\phi \, \mathrm{d}\phi = -\sin\phi \, \mathrm{d}\phi \qquad dy = \sin\phi dr + r\cos\phi \, \mathrm{d}\phi = \cos\phi \, \mathrm{d}\phi$$

Thus the work done is

$$W = \int_P^Q F_x \, \mathrm{d}x + F_y \, \mathrm{d}y = \int_0^{\pi/2} (-\sin\phi)(-\sin\phi \, \mathrm{d}\phi) + (\cos\phi)(\cos\phi \, \mathrm{d}\phi) = \int_0^{\pi/2} \mathrm{d}\phi = \frac{\pi}{2}$$

4.5 (a) Given the force of gravity $\mathbf{F} = -mg\hat{\mathbf{y}}$ and vertical height from 1 to 2 $h = y_2 - y_1$, the work done by gravity is

$$W_g(1 \to 2) = \int_1^2 \mathbf{F} \cdot \mathrm{d}r = \int_0^h -mg\,\mathrm{d}y = -mgh$$

Since the force $\mathbf{F}$ depends only on position and the work done by is independent of the path taken, the force is conservative.
(b) The gravitational potential energy of the particle is

$$U_g(\mathbf{r}) = -W_g(0 \to \mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_0^{\mathbf{r}} -mg\,\mathrm{d}y = mgy$$

where $\mathbf{r} = y\hat{\mathbf{y}}$ is the position vector of the particle.

4.7 (a) Given the gravitational force has magnitude $F_y = -m\gamma y^2$, the work done by gravity is

$$W = \int_1^2 F_y\,\mathrm{d}y = \int_1^2 m\gamma y^2\,\mathrm{d}y = \frac{1}{3}m\gamma(y_2^3 - y_1^3)$$

The gravity is still conservative since the work done by gravity is independent of the path taken and the force depends only on position. Hence, the corresponding potential energy is

$$U_g(\mathbf{r}) = -W(0 \to \mathbf{r}) = -\int_0^y F_y \cdot \mathrm{d}y' = \frac{1}{3}m\gamma y^3$$

(b)

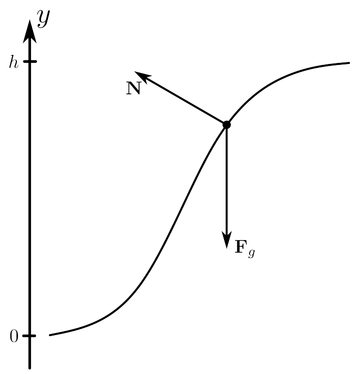

Figure 1.1: A threaded bead on a wire with two forces acting on it; The force of gravity $\mathbf{F}_g$ is conservative and the normal force $\mathbf{N}$ is non-conservative.

(c) The bead is initially released from rest at a height h. From conservation of energy:

$$E_i = E_f \tag{1.1}$$
$$\frac{1}{3}m\gamma h^3 = \frac{1}{2}mv^2 \tag{1.2}$$
$$v = \sqrt{\frac{2}{3}\gamma h^3} \tag{1.3}$$

where v is the speed of the bead at the bottom of the wire.

4.9 (a) Assuming the force of a one-dimensional spring $F = -kx$ is conservative, potential energy is

$$U(x) = -\int_0^x F\,\mathrm{d}x' = \frac{1}{2}kx^2$$

where x is the displacement of the spring from its equilibrium position.
(b) From Newton's second law, the new equilibrium position x_o is found when the spring force and gravity are equal.

$$0 = F + F_g = -kx_o + mg \implies x_o = \frac{mg}{k}$$

When $y = 0$, $U = 0$. Thus the potential energy is zero at position $x = x_o$:

$$U(x_o) = \frac{1}{2}k(x_o)^2 - mg(x_o) = 0$$

The total potential energy of the system at position $x = y + x_o$ is

$$\begin{aligned} U(x) = U_{sp} + U_g &= \frac{1}{2}k(y + x_o)^2 - mg(y + x_o) \\ &= \frac{1}{2}ky^2 + kyx_o - mgy + \frac{1}{2}kx_o^2 - mgx_o \end{aligned}$$

Since $kyx_o - mgy = 0$ and the last two terms are the potential energy at the new equilibrium $U(x_o) = 0$, the total potential energy is $U(x) = \frac{1}{2}ky^2$.

4.11 Finding the partial derivatives of the functions with constants a, b, c:
(a) $f(x, y, z) = ax^2 + bxy + cy^2$:

$$\frac{\partial f}{\partial x} = 2ax + by \qquad \frac{\partial f}{\partial y} = bx + 2cy \qquad \frac{\partial f}{\partial z} = 0$$

(b) $g(x, y, z) = \sin\big(axyz^2\big)$:

$$\frac{\partial g}{\partial x} = ayz^2 \cos\big(axyz^2\big) \qquad \frac{\partial g}{\partial y} = axz^2 \cos\big(axyz^2\big) \qquad \frac{\partial g}{\partial z} = 2axyz \cos\big(axyz^2\big)$$

(c) $h(x, y, z) = ar$ where $r = \sqrt{x^2 + y^2 + z^2}$: Since

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

The partial derivatives of h are

$$\frac{\partial h}{\partial x} = \frac{ax}{r} \qquad \frac{\partial h}{\partial y} = \frac{ay}{r} \qquad \frac{\partial h}{\partial z} = \frac{az}{r}$$

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