1 Energy

4.2 From the origin O to point $P = (1, 1)$ a two dimensional force $\mathbf{F} = (x^2, 2xy)$ moves a point along three paths where the work done by the force is

$$W = \int_O^P \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_O^P F_x \,\mathrm{d}x + F_y \,\mathrm{d}y$$

(a) Splitting the path into two parts $O \to Q = (1, 0)$ and $Q \to P$, we have two integrals

$$W = \int_O^Q F_x \,\mathrm{d}x + \int_Q^P F_y \,\mathrm{d}y$$

where the first integral accounts for just the x component of force $F_x = x^2$ and the second integral accounts for just the y component of force when $x = 1$; $F_y = 2(1)y$. Thus

$$W = \int_0^1 x^2 \,\mathrm{d}x + \int_0^1 2y \,\mathrm{d}y = \frac{4}{3}$$

(b) The path follows the parabola $y = x^2$ from $O \to P$. From $\mathrm{d}y = 2x \,\mathrm{d}x$ the integral can be rewritten in terms of just x

$$W = \int_0^1 x^2 \,\mathrm{d}x + \int_0^1 2x(x^2) \,\mathrm{d}y = \frac{1}{3} + \int_0^1 4x^4 \,\mathrm{d}x = \frac{17}{15}$$

(c) Path follows the parametric curve $x = t^3$ and $y = t^2$ where the differentials are: $dx = 3t^2 \,\mathrm{d}t$ and $dy = 2t \,\mathrm{d}t$. Thus the work done on the path is

$$W = \int_0^1 (t^6)(3t^2 \,\mathrm{d}t) + \int_0^1 (2t^3)(2t \,\mathrm{d}t) = \frac{1}{3} + \frac{4}{5} = \frac{19}{15}$$

4.3 Same as Problem 4.2 but with a force $\mathbf{F} = (-y, x)$ and three different paths from $P = (1, 0) \to Q = (0, 1)$.
(a) This path follows a straight line $y = 0$ from $P \to O$ and then $x = 0$ from $O \to Q$. Thus the work done is

$$W = \int_P^O F_x \,\mathrm{d}x + \int_O^Q F_y \,\mathrm{d}y = 0$$

(b) A straight line from $P \to Q$ is given by $y = -x + 1$ and the differential $dy = -dx$. Thus the work done is

$$W = \int_P^Q F_x \,\mathrm{d}x + F_y \,\mathrm{d}y = \int_1^0 (-(-x + 1)) \,\mathrm{d}x + (x)(-\,\mathrm{d}x) = \int_1^0 -1 \,\mathrm{d}x = 1$$

(c) The path of a quarter circle centered on the origin in polar coordinates is given by

$$x = r \cos\phi \qquad y = r \sin\phi$$

where $r = 1$, $\phi = 0 \to \pi/2$ and the differentials are

$$dx = \cos\phi dr - r \sin\phi \,\mathrm{d}\phi = -\sin\phi \,\mathrm{d}\phi \qquad dy = \sin\phi dr + r\cos\phi \,\mathrm{d}\phi = \cos\phi \,\mathrm{d}\phi$$

Thus the work done is

$$W = \int_P^Q F_x \,\mathrm{d}x + F_y \,\mathrm{d}y = \int_0^{\pi/2} (-\sin\phi)(-\sin\phi \,\mathrm{d}\phi) + (\cos\phi)(\cos\phi \,\mathrm{d}\phi) = \int_0^{\pi/2} \mathrm{d}\phi = \frac{\pi}{2}$$

4.5 (a) Given the force of gravity $\mathbf{F} = -mg\hat{\mathbf{y}}$ and vertical height from 1 to 2 $h = y_2 - y_1$, the work done by gravity is

$$W_g(1 \to 2) = \int_1^2 \mathbf{F} \cdot \mathrm{d}r = \int_0^h -mg \, \mathrm{d}y = -mgh$$

Since the force $\mathbf{F}$ depends only on position and the work done by is independent of the path taken, the force is conservative.
(b) The gravitational potential energy of the particle is

$$U_g(\mathbf{r}) = -W_g(0 \to \mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_0^{\mathbf{r}} -mg \, \mathrm{d}y = mgy$$

where $\mathbf{r} = y\hat{\mathbf{y}}$ is the position vector of the particle. The potential energy is a function