Problems for Taylor's Classical Mechanics

Contents

1 Newton's Laws of Motion

1.1 Given two vectors $\mathbf{b} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{z}}$ find $\mathbf{b} + \mathbf{c}, 5\mathbf{b} + 2\mathbf{c}, \mathbf{b} \cdot \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$.

$$\mathbf{b} + \mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{x}} + \hat{\mathbf{z}} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

$$5\mathbf{b} + 2\mathbf{c} = 5\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{x}} + 2\hat{\mathbf{z}} = 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$$

$$\mathbf{b} \cdot \mathbf{c} = \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} + \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 1 + 0 + 0 + 0 = 1$$

$$\mathbf{b} \times \mathbf{c} = \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$$

1.2 Given Vectors $\mathbf{b} = (1, 2, 3),\ \mathbf{c} = (3, 2, 1)$ compute 1.1

$$\mathbf{b} + \mathbf{c} = (4, 4, 4)$$

$$5\mathbf{b} + 2\mathbf{c} = (5, 10, 15) + (6, 4, 2) = (11, 14, 17)$$

$$\mathbf{b} \cdot \mathbf{c} = 1(3) + 2(2) + 3(1) = 10$$

$$\mathbf{b} \times \mathbf{c} = \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = -4\hat{\mathbf{x}} + 8\hat{\mathbf{y}} - 4\hat{\mathbf{z}}$$

1.3 Pythagorean Theorem for three dimensions

First find the magntitude of the vector $\mathbf{a} = x + y$ made up of the x and y components

$$|\mathbf{a}| = \sqrt{x^2 + y^2}$$

Then the magnitude of the vector $\mathbf{r} = a + z$ made up of the x, y, and z components

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$
$$r^2 = x^2 + y^2 + z^2$$

1.4 Find $\angle$ between vectors $\mathbf{b} = (1, 2, 4),\ \mathbf{c} = (4, 2, 1)$ using dot product

$$\mathbf{b} \cdot \mathbf{c} = 1(4) + 2(2) + 4(1) = 12$$

$$|\mathbf{b}| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$$

$$|\mathbf{c}| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = \frac{12}{21}$$

$$\theta = \cos^{-1}\frac{12}{21} = 55°$$

1.5 Angle between cube body diagonal and face diagonal

Face diagonal vector $\mathbf{P} = (1, 1, 0)$ and Body Diagonal $\mathbf{Q} = (1, 1, 1)$

$$\mathbf{P} \cdot \mathbf{Q} = 1 + 1 + 0 = 2$$

$$|\mathbf{P}| = \sqrt{2} \qquad |\mathbf{Q}| = \sqrt{3}$$

$$\cos\theta = \frac{2}{\sqrt{6}}$$

$$\theta = 35°$$

1.6 Find scalar s for orthogonal vectors $\mathbf{B} = \hat{\mathbf{x}} + s\hat{\mathbf{y}}, \; \mathbf{C} = \hat{\mathbf{x}} - s\hat{\mathbf{y}}$

The dot product of orthgonal vectors is zero:

$$\begin{gathered}
\mathbf{B} \cdot \mathbf{C} = 0 \\
(1, s) \cdot (1, -s) = 1 - s^2 = 0 \\
s^2 = 1 \\
s = \pm 1
\end{gathered}$$

1.7 Prove the 2 definitions of scalar product are equal

Treat vector $\mathbf{r}$ stricly in the x axis: $\mathbf{r} = (x, 0, 0)$ and $\mathbf{s} = (s_x, s_y, s_z)$:

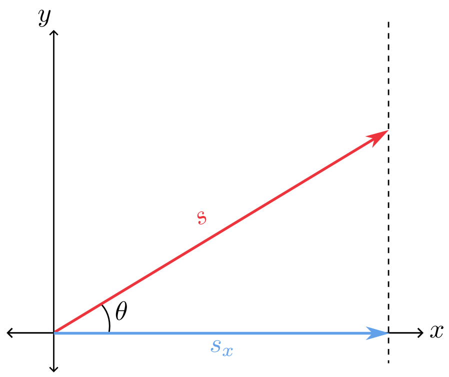

Figure 1.1

the x component of the vector s_x is equivalent to $s\cos\theta$...

$$\begin{aligned}
\mathbf{r} \cdot \mathbf{s} &= |\mathbf{r}||\mathbf{s}|\cos\theta & &= \sum_{i=1}^{3} r_i s_i \\
&= xs\cos\theta & &= xs_x \\
&= xs_x
\end{aligned}$$

1.8 Prove dot product is distributive and differentiable

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} &= \sum_{i=1}^{3} (a_i + b_i) c_i \\
&= \sum_{i=1}^{3} a_i c_i + \sum_{i=1}^{3} b_i c_i \\
&= \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) &= \frac{d}{dt} \sum_{i=1}^{3} a_i b_i \\
&= \sum_{i=1}^{3} \frac{d}{dt} a_i b_i \\
&= \sum_{i=1}^{3} \frac{da_i}{dt} b_i + \sum_{i=1}^{3} a_i \frac{db_i}{dt} \\
&= \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}
\end{aligned}$$

1.9 Show law of cosines from the identity $(\mathbf{a}+\mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b}$

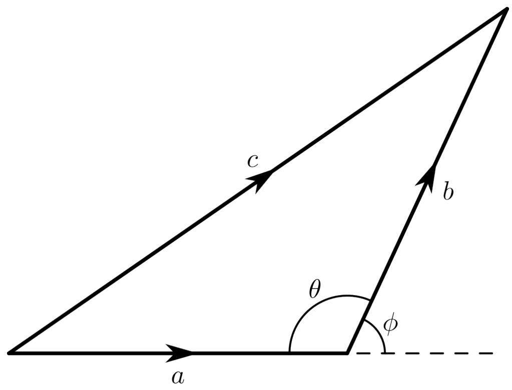

Figure 1.2: Law of Cosine: $c^2 = a^2 + b^2 - 2ab\cos\theta$

Using the identity $\cos\phi = \cos(\pi - \phi) = -\cos\theta$

$$\begin{aligned}
\mathbf{c}^2 &= (\mathbf{a}+\mathbf{b})^2 \\
&= a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} \\
&= a^2 + b^2 + 2|\mathbf{a}||\mathbf{b}|\cos\phi \\
&= a^2 + b^2 - 2ab\cos\theta \\
&= a^2 + b^2 - 2ab\cos\theta
\end{aligned}$$

1.11 Describe the orbit of a particle with the position function $r(t) = \hat{\mathbf{x}} b\cos\omega t + \hat{\mathbf{y}} c\sin\omega t$

This is a parametric representation of an ellipse using trigonometric functions $x = b\cos\omega t$, $y = c\sin\omega t$ equivalent to the standard ellipse equation:

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$$

The particle is moving counter-clockwise in the x-y plane with semi-major(longer) axis and semi-minor(short) axis c and b.

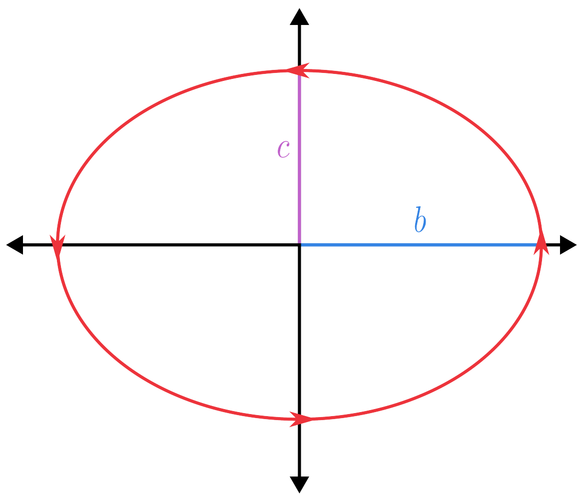

Figure 1.3: Ellipse with semi-major axis b and semi-minor axis c

1.13 For a fixed unit vector $\mathbf{u}$ show the any vector $\mathbf{b}$ satisfies $b^2 = (\mathbf{b} \cdot \mathbf{u})^2 + (\mathbf{b} \times \mathbf{u})^2$

The magnitude of a unit vector is 1

$$b^2 = (b \sin\theta)^2 + (b \cos\theta)^2$$

This is equivalent to the Pythagorean Theorem.

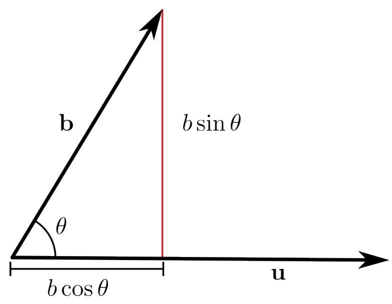

Figure 1.4

1.15 Show $\mathbf{r} \times \mathbf{s}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{s}$ with magnitude $rs\sin\theta$ given by the right-hand rule

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Choosing $\mathbf{r} = (r, 0, 0), \mathbf{s} = (s_x, s_y, 0)$ where $s_y = s\sin\theta$

$$\begin{aligned}\mathbf{r} \times \mathbf{s} &= \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & 0 & 0 \\ s_x & s_y & 0 \end{bmatrix} \\ &= -r_x s_y \hat{\mathbf{z}} \\ &= rs\sin\theta\hat{\mathbf{z}}\end{aligned}$$

The result is a vector stricly in the z direction, orthogonal to the x-y plane.

1.17 (a) Prove the vector product is distributive as in: $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = \mathbf{r} \times \mathbf{u} + \mathbf{r} \times \mathbf{v}$ (b) and differentiable by product rule

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \frac{d\mathbf{r}}{dt} \times \mathbf{s} + \mathbf{r} \times \frac{d\mathbf{s}}{dt}$$

(a) The components of the vector cross product $\mathbf{p} = \mathbf{r} \times \mathbf{s}$

$$\begin{aligned} p_x &= r_y s_z - r_z s_y \\ p_y &= r_z s_x - r_x s_z \\ p_z &= r_x s_y - r_y s_x \end{aligned} \tag{1.9}$$

Starting with the x componenet of the the resultant vector

$$\begin{aligned}\mathbf{r} \times (\mathbf{u} + \mathbf{v}) &= \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ u_x + v_x & u_y + v_y & u_z + v_z \end{bmatrix} \\ &= r_y(u_z + v_z) - r_z(u_y + v_y) \\ &= (r_y u_z - r_z u_y) + (r_y v_z - r_z v_y) \\ &= (r \times u)_x + (r \times v)_x\end{aligned}$$

The same can be done for the y and z components to show that the product is distributive.

(b) Using (1.9) starting with just the x component again

$$\begin{aligned}\frac{d}{dt}[(\mathbf{r}\times\mathbf{s})_x] &= \frac{d}{dt}[r_y s_z - r_z s_y]\\ &= \frac{dr_y}{dt}s_z + r_y\frac{ds_z}{dt} - \frac{dr_z}{dt}s_y - r_z\frac{ds_y}{dt} = \left(\frac{dr_y}{dt}s_z - \frac{dr_z}{dt}s_y\right) + \left(r_y\frac{ds_z}{dt} + r_z\frac{ds_y}{dt}\right)\\ &= \left[\frac{d\mathbf{r}}{dt}\times\mathbf{s} + \mathbf{r}\times\frac{d\mathbf{s}}{dt}\right]_x\end{aligned}$$

id. can the done in the y and z components to prove the product rule.

1.19 If $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ are the position, velocity, and acceleration vectors of a particle, prove that

$$\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] = \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r})$$

Using the product rule for the dot product

$$\begin{aligned}\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] &= \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{r})\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\frac{d\mathbf{r}}{dt}\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\mathbf{a}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\mathbf{v}\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r})\end{aligned}$$

The cross product of a vector with itself is zero. $\mathbf{a}\times\mathbf{a} = 0$ and the dot product of orthogonal vectors is zero.

1.21 Show the volume of a parallelepiped defined by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$

In geometry, the cross product refers to the positive area of a parallelgram(directed area product) which is the base area of the parallelepiped.

$$\mathbf{b}\times\mathbf{c} = bc\sin\theta = \text{base area}$$

The dot product equates to the volume of the parallelepiped with height $\mathbf{a}\cos\phi$... scalar triple product

1.23 The unknown vector $\mathbf{b}$ satisfies $\mathbf{b}\cdot\mathbf{v} = \lambda$ and $\mathbf{b}\times\mathbf{v} = \mathbf{c}$. Find $\mathbf{v}$ in terms of λ, $\mathbf{b}$, and $\mathbf{c}$.

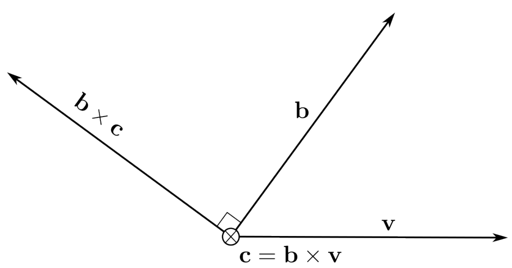

Figure 1.5: Visual example of vector $\mathbf{b}$ and $\mathbf{v}$ with vector $\mathbf{c}$ pointing into the page

$\mathbf{v}$ can be expressed as a linear combination of 2 orthogonal vectors

$$\mathbf{v} = \alpha\mathbf{b} + \beta\mathbf{b}\times\mathbf{c}$$

Taking the dot product to solve for α

$$\begin{aligned}
\mathbf{b} \cdot \mathbf{v} &= \mathbf{b} \cdot (\alpha\mathbf{b} + \beta\mathbf{b} \times \mathbf{c}) \\
&= \alpha\mathbf{b} \cdot \mathbf{b} + \beta\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) \\
\lambda &= \alpha b^2 \\
\alpha &= \frac{\lambda}{b^2}
\end{aligned}$$

1.5 shows that $\mathbf{b} \times \mathbf{c}$ is orthogonal to $\mathbf{b}$ so the dot product is zero. Solving for β

$$\begin{aligned}
\mathbf{b} \times \mathbf{v} &= \mathbf{b} \times (\alpha\mathbf{b} + \beta\mathbf{b} \times \mathbf{c}) \\
&= \alpha\mathbf{b} \times \mathbf{b} + \beta\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) \\
&= \beta\mathbf{b} \times (\mathbf{b} \times \mathbf{c}) \\
&= \beta\mathbf{b}(b^2 c) \\
&= \beta(-b^2 c)
\end{aligned}$$

The direction of the resultant triple product is in the negative direction of $\mathbf{c}$ so $\beta = -\frac{1}{b^2}$

$$\mathbf{v} = \frac{\lambda}{b^2}\mathbf{b} - \frac{1}{b^2}\mathbf{b} \times \mathbf{c}$$

1.25 Find the general solution for the first-order differential equation $df/dt = -3f$

$$\begin{aligned}
\frac{df}{dt} &= -3f \\
\int \frac{1}{f} df &= \int -3dt \\
\ln f &= -3t + C \\
f &= e^{-3t+C} \\
f &= Ae^{-3t}
\end{aligned}$$

1.29 Go over the steps from (1.25) to (1.29) for the conservation of momentum for $N = 4$ particles

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$$(\text{net force on particle}) = \mathbf{F}_\alpha = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} \tag{1.25}$$

Where $\mathbf{F}_{\alpha\beta}$ denotes the force on particle α due to particle β

$$\dot{\mathbf{p}}_\alpha = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} \tag{1.26}$$

This is in accordance to Newton's second law, same as the rate of change of momentum $\mathbf{p}_\alpha$. For the total momentum of the particle $\mathbf{P}$

$$\dot{\mathbf{P}} = \sum_\alpha^N \dot{\mathbf{p}}_\alpha = \sum_\alpha^N \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \sum_\alpha^N \mathbf{F}_\alpha^{\text{ext}} \tag{1.27}$$

Reorganizing double sum

$$\sum_\alpha^N \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} = \sum_\alpha^N \sum_{\beta > \alpha} (\mathbf{F}_{\alpha\beta} + \mathbf{F}_{\beta\alpha}) \tag{1.28}$$

Since the terms in double sum (1.28) is zero by Newton's third law

$$\dot{\mathbf{P}} = \sum_\alpha^N \mathbf{F}_\alpha^{\text{ext}} \equiv \mathbf{F}^{\text{ext}} \tag{1.29}$$

(1.25) and (1.26) for $N = 4$ particles

$$\dot{\mathbf{p}}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}}$$
$$\dot{\mathbf{p}}_2 = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}}$$
$$\dot{\mathbf{p}}_3 = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}}$$

Summation of momentum (1.27) $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3 + \dot{\mathbf{p}}_4$

$$\begin{aligned}\dot{\mathbf{P}} &= (\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}}) + (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}}) \\ &+(\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}}) + (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}})\end{aligned}$$

Reorganizing the double sum like (1.28)

$$\begin{aligned}\dot{\mathbf{P}} &= (\mathbf{F}_{12} + \mathbf{F}_{21}) + (\mathbf{F}_{13} + \mathbf{F}_{31}) + (\mathbf{F}_{14} + \mathbf{F}_{41}) + (\mathbf{F}_{23} + \mathbf{F}_{32}) \\ &+(\mathbf{F}_{24} + \mathbf{F}_{42}) + (\mathbf{F}_{34} + \mathbf{F}_{43}) + (\mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}})\end{aligned}$$

By Newton's third law, the opposing forces cancel out

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}} \equiv \mathbf{F}^{\text{ext}}$$

1.31 The law of conservation of momentum says that if there are no external forces on this pair of particles, then their total momentum must be constant. Use this to prove that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} \equiv \mathbf{F}^{\text{ext}}$$

For a two particle system

$$\dot{\mathbf{P}} = \mathbf{F}_{12} + \mathbf{F}_{21} + \mathbf{F}^{\text{ext}}$$

If there are no external forces, then $\mathbf{F}^{\text{ext}} = 0$ and for total momentum to be constant, $\dot{\mathbf{P}} = 0$. Therefore the interparticle forces obey the third law i.e. $\mathbf{F}_{12} = -\mathbf{F}_{21}$

1.33 Prove the magnetic forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, between two steady current loops obey Newton's 3rd law

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Hints: for currents I_1 and I_2, and points r_1 and r_2. According to Bio-Savart Law, the force on the segment $d\mathbf{r}_1$ due to $d\mathbf{r}_2$ of loop 2 is

$$\frac{\mu_0}{4\pi}\frac{I_1 I_2}{s^2} d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r_1} - \mathbf{r_2}$. The force $\mathbf{F_{12}}$ is found integrating over both loops. The unit vector is equivalent to

$$\hat{\mathbf{s}} = \frac{\mathbf{s}}{s} = \frac{\mathbf{r_1} - \mathbf{r_2}}{|\mathbf{r_1} - \mathbf{r_2}|}$$

and the *BAC* - *CAB* identity for the triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Integrating over both loops the force on loop 1 due to loop 2

$$\mathbf{F}_{12} = \oint \oint \frac{\mu_0}{4\pi}\frac{I_1 I_2}{s^2} d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

Using the *BAC* - *CAB* identity

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \left[\oint \oint \frac{d\mathbf{r}_2 (d\mathbf{r}_1 \cdot \hat{\mathbf{s}})}{s^2} - \oint \oint \frac{\hat{\mathbf{s}}(d\mathbf{r}_1 \cdot d\mathbf{r}_2)}{s^2} \right]$$

In the first term, the dot product $d\mathbf{r_1} \cdot \hat{\mathbf{s}}$ is projection of in the direction of $\mathbf{s}$ and the integral of the closed current loop is zero

$$\oint_{C_2} \oint_{C_1} \frac{d\mathbf{r}_2(d\mathbf{r}_1 \cdot \hat{\mathbf{s}})}{s^3} = \oint_{C_2} d\mathbf{r_2} \oint_{C_1} \frac{ds}{s^2} = 0$$

We end up with the force on loop 1 due to loop 2 as

$$\mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} \frac{\hat{\mathbf{s}}(d\mathbf{r}_1 \cdot d\mathbf{r}_2)}{s^2} = -\mathbf{F}_{21}$$

1.35 A golf ball is hit from ground level due east at a velocity v_0 at an angle θ above the horizontal. Neglecting air resistance use Newton's Second Law to find position as a function of time, the time for the ball to hit the ground, and the range of the ball. x measures east, y north, and z vertically up.

Newton's second law states $\mathbf{F} = m\ddot{\mathbf{r}}$ where $\ddot{\mathbf{r}} = \mathbf{g} = -g\hat{\mathbf{z}}$ is the gravitational force. We can find the position of the ball by integrating twice with respect to time

$$\begin{aligned} \ddot{x} &= 0 & \ddot{y} &= 0 & \ddot{z} &= -g \\ \dot{x} &= 0 & \dot{y} &= 0 & \dot{z} &= -gt + v_0 \sin\theta \\ x(t) &= 0 & y(t) &= 0 & z(t) &= -\frac{1}{2}gt^2 + v_0 t \sin\theta \end{aligned}$$

The time for the ball to hit the ground is when $z(t) = 0$

$$-\frac{1}{2}gt^2 + v_0 t \sin\theta = 0$$
$$t = \frac{2v_0 \sin\theta}{g}$$

To get the range of the ball we subsitute t from above into $x(t)$

$$x(t) = v_0 \cos\theta \frac{2v_0 \sin\theta}{g}$$
$$x(t) = \frac{2v_0^2 \sin\theta \cos\theta}{g}$$
$$x(t) = \frac{v_0^2 \sin 2\theta}{g}$$

1.37 A student kicks a frictionless puck with initial speed v_0, so that it slides straight up a plane that is inclined at an angle θ above the horizontal (a) Write down Newton's second law for the puck and solve to give its position as a function of time (b) How long will the puck take to return to its starting point?

(a) Having the x-axis on the plane parallel to the incline we get the force equation

$$F_x = m\ddot{x} = -mg\sin\theta$$

Solving for the position of the puck by integrating twice with respect to time and using the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$

$$\ddot{x} = -g\sin\theta$$
$$\dot{x} = g\cos\theta t + v_0$$
$$x(t) = -\frac{1}{2}g\sin\theta t^2 + v_0 t$$

(b) Solving for when the puck returns to its starting point $x(t) = 0$

$$-\frac{1}{2}g\sin\theta t^2 + v_0 t = 0$$
$$t = \frac{2v_0}{g\sin\theta}$$

1.39 Show the ball lands a distance $R = 2v_o^2 \sin\theta \cos(\theta+\phi) / \left(g \cos^2\phi\right)$ and $R_{max} = v_o^2/[g(1+\sin\phi)]$

Using θ as the angle above the incline and ϕ as the angle of the incline plane the components of the inital velocity are

$$v_{ox} = v_0 \cos\theta \quad v_{oy} = v_0 \sin\theta \quad v_{oz} = 0$$

Newton's second law

$$\begin{aligned}
F_x &= -mg\sin\phi & F_y &= -mg\cos\phi & F_z &= 0 \\
\ddot{x} &= -g\sin\phi & \ddot{y} &= -g\cos\phi & \ddot{z} &= 0 \\
\dot{x} &= -g\sin\phi t + v_{ox} & \dot{y} &= -g\cos\phi t + v_{oy} & \dot{z} &= 0 \\
x(t) &= -\frac{1}{2}g\sin\phi t^2 + v_{ox}t & y(t) &= -\frac{1}{2}g\cos\phi t^2 + v_{oy}t & z(t) &= 0
\end{aligned}$$

The range of the ball is when $y(t) = 0$ as it lands on the incline plane

$$\begin{aligned}
0 &= -\frac{1}{2}g\cos\phi t^2 + v_{oy}t \\
t &= \frac{2v_{oy}}{g\cos\phi}
\end{aligned}$$

Substituting t into $x(t)$ and using the identity $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$ simplifies the range...

$$\begin{aligned}
x(t) &= -\frac{1}{2}g\sin\phi\left(\frac{2v_{oy}}{g\cos\phi}\right)^2 + v_{ox}\left(\frac{2v_{oy}}{g\cos\phi}\right) \\
R &= \frac{-2v_o^2}{g\cos^2\phi}\sin^2\theta\sin\phi + \frac{2v_o^2}{g\cos\phi}\sin\theta\cos\theta \\
&= \frac{2v_o^2\sin\theta}{g\cos^2\phi}\left(-\sin\theta\sin\phi + \cos\theta\cos\phi\right) \\
R &= \frac{2v_o^2\sin\theta}{g\cos^2\phi}\cos(\theta+\phi)
\end{aligned}$$

Set identity $\sin\alpha\cos\beta = 1/2[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$$\begin{aligned}
\frac{dR}{d\theta} &= \frac{2v_o^2}{g\cos^2\phi}\left(\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)\right) \\
0 &= \cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi) \\
0 &= \cos(\theta + (\theta+\phi)) \\
\pi/2 &= 2\theta + \phi \\
\theta &= \frac{\pi - 2\phi}{4} \\
R_{max} &= \frac{2v_o^2\sin\left(\frac{\pi-2\phi}{4}\right)}{g\cos^2\phi}\cos\left(\frac{\pi-2\phi}{4} + \phi\right) \\
&= \frac{v_o^2}{g}\frac{\sin\left(\frac{\pi-2\phi}{2}+\phi\right) + \sin -\phi}{\cos^2\phi} \\
&= \frac{v_o^2}{g}\frac{\sin\left(\frac{\pi}{2}\right) - \sin\phi}{1-\sin^2\phi} \\
&= \frac{v_o^2}{g}\frac{1-\sin\phi}{(1+\sin\phi)(1-\sin\phi)} \\
R_{max} &= \frac{v_o^2}{g[1+\sin\phi]}
\end{aligned}$$

1.41 An astronaut in gravity-free space is twirling a mass m on the end of a string of length R in a circle, with constant angular velocity ω. Write down Newton's second law in polar coordinates and find the tension in the string.

Newton's second law in polar coordinates

$$\begin{aligned} F_r &= m\ddot{r} - mr\dot{\phi}^2 \\ F_\theta &= mr\ddot{\phi} + 2m\dot{r}\dot{\phi} \end{aligned} \tag{1.48}$$

The only force acting on the mass is the tension in the string. The tension is in the radial direction, so $F_r = -T$ and the mass is moving in a circle of radius $r = R$ so $\ddot{r} = \dot{r} = 0$. Since the angular velocity $\dot{\phi} = \omega$ is constant, $\ddot{\phi} = 0$. Newton's second law (1.48) then simplifies to $F_r = -mr\dot{\phi}^2$ and $F_\theta = 0$. Solving for tension we get

$$\begin{aligned} -T &= -mr\dot{\phi}^2 \\ T &= mr\omega^2 \end{aligned}$$

1.43 (a) Prove that the unit vector $\hat{\mathbf{r}}$ of two-dimensional polar coorddinates is equal to

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \tag{1.59}$$

and find a corresponding expression for $\hat{\boldsymbol{\phi}}$ (b) Assuming that ϕ depends on the time t, differentiate your answers in part (a) to give an alternative proof of the results (1.42) and (1.46) for the time derivatives $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$.

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\phi}\hat{\boldsymbol{\phi}} \tag{1.42}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = -\dot{\phi}\hat{\mathbf{r}} \tag{1.46}$$

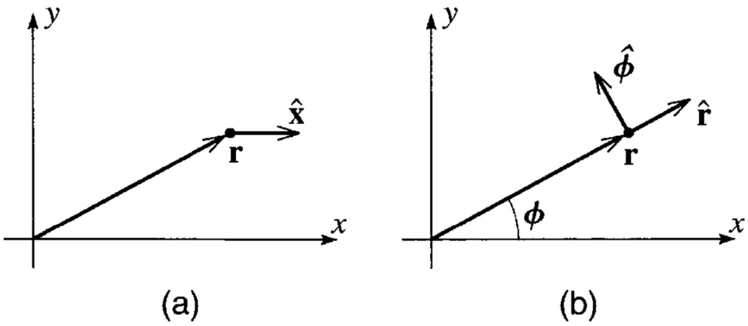


Figure 1.6: Unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ on the cartesian plane

(a) Figure 1.6 shows that the x and y component of the radial unit vector are $\hat{\mathbf{r}}_x = \cos\phi$ and $\hat{\mathbf{r}}_y = \sin\phi$. For the angular unit vector, the x and y components are $\hat{\boldsymbol{\phi}}_x = -\sin\phi$ and $\hat{\boldsymbol{\phi}}_y = \cos\phi$. The unit vector can be expressed as

$$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}_x + \hat{\boldsymbol{\phi}}_y = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \tag{1.60}$$

(b) Keep in mind that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constants. Differentiating (1.59) and (1.60) with respect to time

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} &= -\dot{\phi}\hat{\mathbf{x}}\sin\phi + \dot{\phi}\hat{\mathbf{y}}\cos\phi = \dot{\phi}\hat{\boldsymbol{\phi}} \\ \frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} &= -\dot{\phi}\hat{\mathbf{x}}\cos\phi - \dot{\phi}\hat{\mathbf{y}}\sin\phi = -\dot{\phi}\hat{\mathbf{r}} \end{aligned}$$

1.45 Prove that if $\mathbf{v}(t)$ is any vector that depends on time but which has constant magnitude, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. Prove the converse that $|\mathbf{v}(t)|$ is constant.

Hint: Consider the derivative of $\mathbf{v}^2$ Since the magnitude of $\mathbf{v}(t)$ is also $\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$, the derivative of $\mathbf{v}^2$ is tells us if the magnitude is constant.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\cdot\mathbf{v}(t))\\ &= 2\dot{\mathbf{v}}(t)\cdot\mathbf{v}(t)\end{aligned}$$

The magnitude of $\mathbf{v}(t)$ is constant if $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 = 0$. Since the dot product is zero, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$. The converse is also true because having $\dot{\mathbf{v}}(t)$ orthogonal to $\mathbf{v}(t)$ means that $|\mathbf{v}(t)|$ is always constant from the definition of the dot product.

1.47 (a) Make a sketch to illustrate the three cylindrical polar coordinates ρ, ϕ, z with a position of a point P. Let P' denote the projection of P onto the xy plane. (b) Describe the three unit vectors $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ and write the expansion of the position vector $\mathbf{r} = (x, y, z)$ in terms of these unit vectors. (c) Differentiate the last answers twice to find the cylindrical component of acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ of the particle.

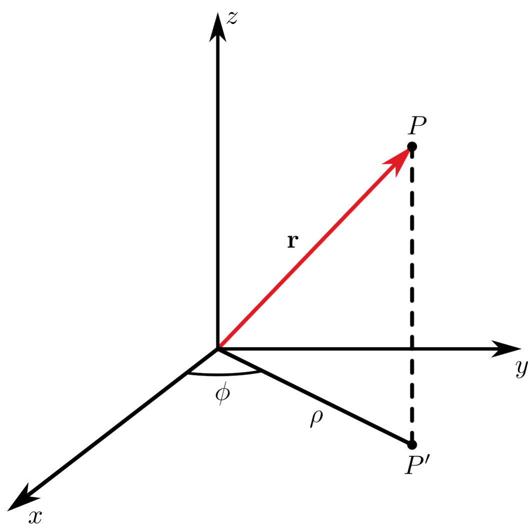


Figure 1.7: Cylindrical polar coordinates ρ, ϕ, z with a position of a point P

(a) Figure 1.7 shows the three cylindrical polar coordinates ρ, ϕ, z. $\rho = \sqrt{x^2 + y^2}$ is the distance of P from the projected point P' on the xy-plane. $\phi = \arctan y/x$ is the angle between the x-axis and the line from origin to P'. z is the height of P from the xy plane.
(b) $\hat{\boldsymbol{\rho}}$ is in the direction outward and orthogonal to the z axis. $\hat{\boldsymbol{\phi}}$ is perpendicular to $\hat{\boldsymbol{\rho}}$ and pointing counterclockwise along the tangent of a circle centered on the z axis. $\hat{\mathbf{z}}$ is in the direction of the z axis. The position vector $\mathbf{r} = (x, y, z)$ can be expressed as

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

(c) Differentiating twice with respect to time and substituting (1.42) and $\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = -\dot{\phi}\hat{\boldsymbol{\rho}}$ from (1.46)

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}\\ \ddot{\mathbf{r}} &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}\end{aligned}$$

1.49 Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. If we use cylindrical polar coordinates (ρ, ϕ, z) for its position (Problem 1.47), then ρ is fixed at $\rho = R$, while ϕ and z can vary at will. Write down and solve Newton's second law for the general motion of the puck, including the effects of gravity. Describe the puck's motion.

The forces on the puck consist of the normal force and the gravitational force. The normal force is in the radial direction and the gravitational force is in the negative z direction.

$$F = N\hat{\boldsymbol{\rho}} - mg\hat{\mathbf{z}} \tag{1.49}$$

Since ρ is fixed, $\dot{\rho} = \ddot{\rho} = 0$ Newton's second law in cylindrical polar coordinates:

$$\begin{aligned}
F_\rho &= m(\ddot{\rho} - \rho\dot{\phi}^2) = -mR\dot{\phi}^2 = N \\
F_\phi &= m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) = mR\ddot{\phi} = 0 \\
F_z &= m\ddot{z} = -mg
\end{aligned}$$

From the F_ϕ equation, $\ddot{\phi} = 0$ so $\dot{\phi}$ or the angular velocity of the ball is constant. From the F_z equation, $\ddot{z} = -g$ and integrating twice with respect to time gives us $z(t) = -\frac{1}{2}gt^2 + v_0 t + z_0$ where v_0 is the initial velocity and z_0 is the initial height. This shows us that the puck is in free fall along the z axis. With these equations of motion we can imagine the puck tracing a helical path with a downard increasing pitch.

1.51 Solve the differential equation for the skateboard given by

$$\ddot{\phi} = -\frac{g}{R}\sin\phi$$

and make a plot of ϕ against time for two or three periods. Make a plot of the approximate solution $\phi = \phi_0 \cos\omega t$ for the same time interval, where $\omega = \sqrt{g/R}$ and using the initial value $\phi_0 = \pi/2$

Python code for solving the differential equation

```python
import scipy as sp
import numpy as np
import matplotlib.pyplot as plt

# 1.51
# initial conditions
phi0 = np.pi / 2
dot_phi0 = 0

# constants
R = 5 # [m]
g = 9.8 # [m/s^2]

# differential equation
def ddot_phi(phi, t):
    return [phi[1], -g / R * np.sin(phi[0])]

# time
t = np.linspace(0, 10, 1000)

# solve the differential equation
pos, vel = sp.integrate.odeint(ddot_phi, [phi0, dot_phi0], t).T

# approximate solution
omega = np.sqrt(g / R)
pos_approx = phi0 * np.cos(omega * t)

# plot the solution
plt.plot(t, pos)

```

```
# plot the approximate solution
plt.plot(t, pos_approx, '--')

plt.xlabel('t [s]')
plt.ylabel('$\phi$ [rad]')

# create legend
plt.legend(['$\phi(t)$', '$\phi_{approx}(t)$'])
plt.show()
```

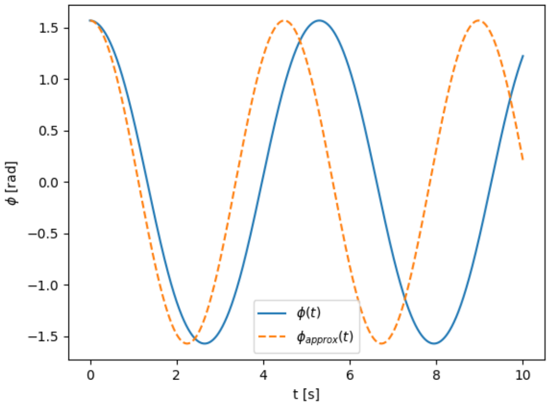


Figure 1.8: Plot of ϕ against time for two periods

Figure 1.8 shows the plot of ϕ and the approximate solution ϕ_{approx} against time for two periods. The approximate solution has a faster period than the actual solution, and the actual solution is not a perfect sinusoidal wave.

2 Projectiles and Charged Particles

2.1 The ratio of the quadratic and linear terms of drag is

$$\frac{f_q}{f_l} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = [1.6 \times 10^3 \,\mathrm{s/m^2}] Dv \tag{2.7}$$

where $\beta = 1.6 \times 10^{-4}\,\mathrm{N\,s/m^2}$ and $\gamma = 0.25\,\mathrm{N\,s^2/m^4}$ for spherical projectiles at STP. When $f_q/f_l = 1$ both drag forces are equally important.

For the baseball $D = 7\mathrm{cm}$

$$\begin{aligned}
1 &= 1.6 \times 10^3 \,\mathrm{s/m^2} Dv \\
v &= \frac{1}{1.6 \times 10^3 \,\mathrm{s/m^2} D} \\
v &= \frac{1}{1.6 \times 10^3 \,\mathrm{s/m^2} (0.07\,\mathrm{m})} \\
v &= 0.9\,\frac{\mathrm{cm}}{\mathrm{s}}
\end{aligned}$$

For the beach ball $D = 70\mathrm{cm}$

$$\begin{aligned}
v &= \frac{1}{1.6 \times 10^3 \,\mathrm{s/m^2} 0.7\,\mathrm{m}} \\
&= 0.9\,\frac{\mathrm{mm}}{\mathrm{s}}
\end{aligned}$$

The linear term is negligible when $f_q/f_l \gg 1$.

2.2 From Stokes's law, the viscous drag on a sphere is

$$f_{lin} = 3\pi\eta Dv \tag{2.82}$$

At STP $\eta = 1.7 \times 10^{-5}\,\mathrm{N\,s/m^2}$

From (2.3) and (2.4)

$$\begin{aligned}
f_{lin} &= 3\pi\eta Dv \\
bv &= 3\pi\eta Dv \\
\beta Dv &= 3\pi\eta Dv \\
\beta &= 3\pi\eta \\
\beta &= 1.6 \times 10^{-4}\,\mathrm{N\,s/m^2}
\end{aligned}$$

2.3 (a) Show ratio of drag can be written as $f_q/f_l = R/48$ for a sphere of radius R is Reynolds' number

$$R = \frac{DvQ}{\eta} \tag{2.83}$$

(b) Find R for a steel ball bearing $D = 2\mathrm{mm}$, $v = 5\mathrm{cm\,s^{-1}}$, $Q = 1.3\mathrm{g/cm^3}$, and $\eta = 12\,\mathrm{N\,s/m^2}$

(a) For a sphere $K = 1/4$ and the surface of the swept cross section of a sphere $A = \pi D^2/4$. Substituting (2.84) and (2.82)

$$\frac{f_q}{f_l} = \frac{KQAv^2}{3\pi\eta Dv} = \frac{1/4Q\frac{\pi D^2}{4}v}{3\pi\eta D} = \frac{1}{48}\frac{DvQ}{\eta} = \frac{R}{48}$$

(b) From (2.83)

$$R = \frac{0.002(0.05)(1300)}{12} = 0.011$$

2.4 (a) Show that the rate (mass/time) is QAv for quadratic drag (b) Show the net force of drag is $F_d = QAv^2$ (c) Given

$$f_q = KQAv^2 \tag{2.84}$$

Show (2.84) $\rightarrow$ (2.3) and verify γ from (2.4) given the density of air at STP is $Q = 1.29\text{kg/m}^3$ and $K = 1/4$ for a sphere.

(a) The volume of the fluid swept by the projectile is $v_{swept} = Avt$ where A is the cross sectional area of the projectile. The mass of the fluid swept is $m_f = Qv_{swept} = QAvt$. The rate at which the projectile encounters the fluid is the time derivative $\frac{\mathrm{d}m_f}{\mathrm{d}t} = QAv$.
(b) The net force of drag is equivalent to the time derivative of momentum of the fluid swept by the projectile. As the fluid is accelerated from rest to velocity v the change in momentum is

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}m_f}{\mathrm{d}t}v + m_f\frac{\mathrm{d}v}{\mathrm{d}t} = QAv^2$$

(c) (2.84) in the form of (2.3) is shown as

$$f_q = KQAv^2 = cv^2$$

Where the constant $c = KQA$. Substituting c into (2.4)

$$\gamma D^2 = KQA$$

The cross sectional area of a sphere is $A = \pi D^2/4$ so

$$\begin{aligned}
\gamma D^2 &= KQ\frac{\pi D^2}{4} \\
\gamma &= \frac{KQ\pi}{4} \\
\gamma &= \frac{1.29\,\mathrm{kg/m^3}\pi}{16} \\
\gamma &= 0.253\,\mathrm{N\,s^2/m^4}
\end{aligned}$$

2.5 Describe a projectile subject to linear drag is thrown vertically down where $v_{yo} > v_t$. Plot v_y vs t for $v_{yo} = 2v_t$

While $v_y > v_t$ the drag force is larger than the magnitude of weight and the projectile slows down at an exponential rate until it reaches terminal velocity. For when $v_{yo} = 2v_t$, using (2.30) the equation of motion is

$$v_y = v_t + (2v_t - v_t)e^{t/\tau} = v_t(1 + e^{-t/\tau})$$

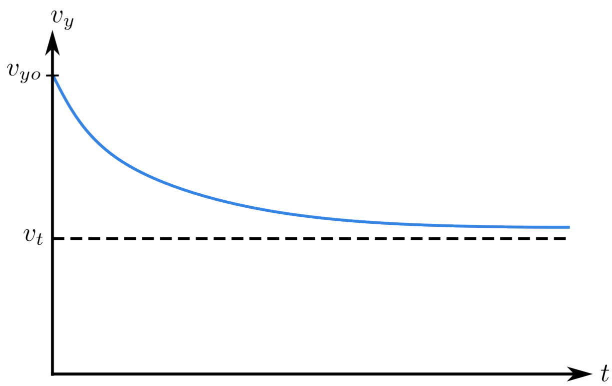


Figure 2.1: Plot of v_y vs t for $v_{yo} = 2v_t$

2.7 For the motion of a one-dimensional particle subject to a force that depends only on velocity, $F = F(v)$. Newton's second law $F = m\frac{\mathrm{d}v}{\mathrm{d}t}$ is rewritten as $m\,\mathrm{d}v/F(v) = \mathrm{d}t$. Integrating both sides gives

$$t = m\int_{v_0}^{v}\frac{dv'}{F(v')}$$

For the special case of a constant $F(v) = F_o$

$$\begin{aligned}
t &= m\int_{v_0}^{v}\frac{dv'}{F_o}\\
t &= \frac{m}{F_o}\int_{v_0}^{v}dv'\\
t &= \frac{m}{F_o}(v - v_o)\\
v &= v_o + \frac{F_o}{m}t
\end{aligned}$$

This is the first of the SUVAT equations where $a = F_o/m$.

2.9 Using seperation of variables (2.29) is rewritten as

$$\frac{m\,dv_y}{v_y - v_t} = -bdt$$

Integrating both sides from time 0 to t

$$\begin{aligned}
\int_0^t \frac{m\,dv_y}{v_y - v_t} &= -b\int_0^t dt\\
m\int_{v_{yo}}^{v_y}\frac{dv_y'}{v_y' - v_t} &= -bt\\
m\ln\left|\frac{v_y - v_t}{v_{yo} - v_t}\right| &= -bt\\
\ln\left|\frac{v_y - v_t}{v_{yo} - v_t}\right| &= -\frac{bt}{m}\\
\left|\frac{v_y - v_t}{v_{yo} - v_t}\right| &= e^{-t/\tau}\\
v_y &= v_t + (v_{yo} - v_t)e^{-t/\tau}
\end{aligned}$$

where $\tau = m/b$ and $v_y = v_{yo}$ when $t = 0$. This is the same as (2.30).

2.11 An object is thrown vertically upward with initial velocity v_o in a linear medium. (a) Measuring *y upward*, write $v_y(t)$ and $y(t)$. (b) Find the time at y_{max}. (c) Show $y_{max} = v_o^2/2g$ as the drag coefficient approaches zero. [*Hint:* Use the Taylor series approximation $\ln(1+\delta) \approx \delta - \frac{1}{2}\delta^2$ for large v_t]

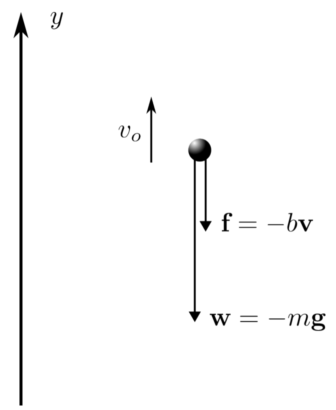


Figure 2.2: Free body diagram of an object thrown vertically upward with initial velocity v_o.

(a) The equation of motion is

$$\begin{aligned}
m\dot{v}_y &= -mg - bv_y \\
m\dot{v}_y &= -b(v_y + v_t) \\
m\frac{\mathrm{d}v_y}{v_y + v_t} &= -b\,\mathrm{d}t \\
m\int_{v_o}^{v_y} \frac{\mathrm{d}v_y'}{v_y' + v_t} &= -b\int_0^t \mathrm{d}t' \\
m\ln\left|\frac{v_y + v_t}{v_o + v_t}\right| &= -bt \\
\frac{v_y + v_t}{v_o + v_t} &= e^{-t/\tau} \\
v_y &= -v_t + (v_o + v_t)e^{-t/\tau}
\end{aligned}$$

This is the same as (2.30) but v_t has a reversed sign. The position is (2.35) with v_t replaced by $-v_t$

$$y(t) = -v_t t + (v_o + v_t)\tau(1 - e^{-t/\tau})$$

(b) The time at y_{max} is found by setting $v_y = 0$ and solving for t

$$\begin{aligned}
0 &= -v_t + (v_o + v_t)e^{-t/\tau} \\
\frac{v_t}{v_o + v_t} &= e^{-t/\tau} \\
\ln\left(\frac{v_t}{v_o + v_t}\right) &= -\frac{t}{\tau} \\
t &= -\tau\ln\left(\frac{v_t}{v_o + v_t}\right) \\
t_{max} &= \tau\ln\left(1 + \frac{v_o}{v_t}\right)
\end{aligned}$$

Substituting the time at the highest point into $y(t)$ gives the maximum height

$$\begin{aligned}
y(t_{max}) &= -v_t t_{max} + (v_o + v_t)\tau(1 - e^{-t_{max}/\tau}) \\
y_{max} &= -v_t\tau\ln\left(1 + \frac{v_o}{v_t}\right) + (v_o + v_t)\tau\left[1 - e^{-\ln\left(1 + \frac{v_o}{v_t}\right)}\right] \\
&= -v_t\tau\ln\left(1 + \frac{v_o}{v_t}\right) + (v_o + v_t)\tau\left[1 - \frac{v_t}{v_o + v_t}\right] \\
y_{max} &= \tau\left[-v_t\ln\left(1 + \frac{v_o}{v_t}\right) + v_o\right]
\end{aligned}$$

(c) When the drag force is small v_o/v_t is very small, so using the Taylor series approximation

$$\begin{aligned} y_{max} &= \tau\left[-v_t\left[\frac{v_o}{v_t}-\frac{1}{2}\left(\frac{v_o}{v_t}\right)^2\right]+v_o\right] \\ &= \tau\left[\frac{v_o^2}{2v_t}\right] \quad \text{given} \quad v_t = g\tau \\ y_{max} &= \frac{v_o^2}{2g} \end{aligned}$$

2.13 A mass m is constained to move on the x axis and subject to a net force $F(x) = -kx$ where k is a positive constant. The mass is released from rest at $x = x_o$ and $t = 0$. Find the mass's speed as a function of x given

$$v^2 = v_o^2 + \frac{2}{m}\int_{x_o}^{x} F(x')\,\mathrm{d}x' \tag{2.85}$$

Find x as a function of t through seperation of variables, integrating from time 0 to t.

With initial velocity $v_o = 0$ (2.85) is rewritten as

$$v^2 = -\frac{2k}{m}\int_{x_o}^{x} x'\,\mathrm{d}x' = \frac{k}{m}(x_o^2 - x^2) \quad \text{or} \quad v = -\omega\sqrt{x_o^2 - x^2}$$

where $\omega^2 = k/m$, the angular frequency. The sign is negative because the mass is moving in the negative x direction first due to the force $F(x) = -kx$. Separating variables to find $x(t)$

$$\begin{aligned} \frac{-\,\mathrm{d}x}{\sqrt{x_o^2 - x^2}} &= \omega dt \\ \int_{x_o}^{x} \frac{-\,\mathrm{d}x'}{\sqrt{x_o^2 - x'^2}} &= \omega\int_0^t dt' \\ \arccos\left(\frac{x}{x_o}\right) &= \omega t \\ x(t) &= x_o\cos(\omega t) \end{aligned}$$

where the integral comes from the identity $\frac{\mathrm{d}}{\mathrm{d}u}\arccos(u/a) = \frac{-1}{\sqrt{a^2-u^2}}$

2.15 A projectile launched with velocity (v_{xo}, v_{yo}) with no air resistance. Show the horizontal range is $2v_{xo}v_{yo}/g$.

The equation of motion for the projectile with no drag is $\ddot{\mathbf{r}} = \mathbf{g}$. Integrating the components

$$\begin{aligned} \ddot{x} &= 0 & \ddot{y} &= -g \\ \dot{x} &= v_{xo} & \dot{y} &= -gt \\ x &= v_{xo}t + x_o & y &= -\frac{1}{2}gt^2 + v_{yo}t + y_o \\ x &= v_{xo}t & y &= -\frac{1}{2}gt^2 + v_{yo}t \end{aligned}$$

where we set $x_o = y_o = 0$. The projectile hits the ground at $y = 0$ and solving for t

$$\begin{aligned} 0 &= -\frac{1}{2}gt^2 + v_{yo}t \\ t_{range} &= \frac{2v_{yo}}{g} \end{aligned}$$

Substituting t_{range} into x

$$x = \frac{2v_{xo}v_{yo}}{g}$$

which is the horizontal range of the projectile in a vacuum.

2.17 Eliminate t from (2.36) to give y as a function of x verifying (2.37)

Solving the first equation of (2.36) for t

$$\begin{aligned} x &= v_{xo}\tau\left(1 - e^{-t/\tau}\right) \\ e^{-t/\tau} &= 1 - \frac{x}{v_{xo}\tau} \\ -\frac{t}{\tau} &= \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \\ t &= -\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \end{aligned}$$

and substituting into the second equation

$$\begin{aligned} y &= (v_{yo} + v_t)\tau\left(1 - e^{-t/\tau}\right) - v_t t \\ &= (v_{yo} + v_t)\tau\left(1 - 1 + \frac{x}{v_{xo}\tau}\right) + v_t\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \\ y &= \frac{v_{yo} + v_t}{v_{xo}}x + v_t\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \end{aligned}$$

which is the same as (2.37).

2.19 (a) Find y as a function of x for a projectile with no air resistance. (b) Show that (2.37) reduces to part (a) when air resistance is switched off(τ and v_t approach infinity). [*Hint:* Use the Taylor series approximation for $\ln(1 - \epsilon)$]

(a) A projectile with no air resistance has position $\mathbf{r} = (x, y) = (v_{xo}t, v_{yo}t - \frac{1}{2}gt^2)$. To find y as a function of x, subsitute $t = x/v_{xo}$ into y

$$\begin{aligned} y &= v_{yo}\frac{x}{v_{xo}} - \frac{1}{2}g\frac{x^2}{v_{xo}^2} \\ y &= \frac{v_{yo}}{v_{xo}}x - \frac{1}{2}g\frac{x^2}{v_{xo}^2} \end{aligned}$$

(b) Using the Taylor series approximation for $\ln(1 - \epsilon)$

$$\ln\left(1 - \frac{x}{v_{xo}\tau}\right) = -\frac{x}{v_{xo}\tau} - \frac{1}{2}\frac{x^2}{v_{xo}^2\tau^2}$$

Substituting into (2.37)

$$\begin{aligned} y &= \frac{v_{yo} + v_t}{v_{xo}}x + v_t\tau\left(-\frac{x}{v_{xo}\tau} - \frac{1}{2}\frac{x^2}{v_{xo}^2\tau^2}\right) \\ &= \frac{v_{yo} + v_t}{v_{xo}}x - v_t\frac{x}{v_{xo}} - \frac{1}{2}\frac{v_t}{v_{xo}^2}\frac{x^2}{\tau} \\ &= \frac{v_{yo}}{v_{xo}}x - \frac{1}{2}\frac{v_t}{v_{xo}}\frac{x^2}{\tau} \quad \text{using} \quad v_t = g\tau \\ y &= \frac{v_{yo}}{v_{xo}}x - \frac{1}{2}g\frac{x^2}{v_{xo}^2} \end{aligned}$$

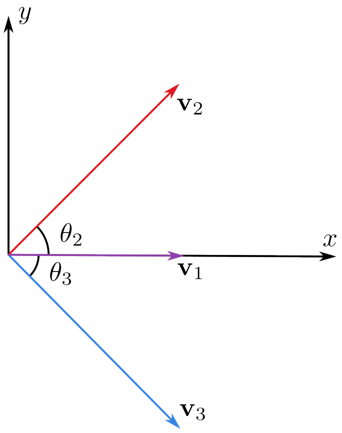


Figure 2.3: Projectile from gun at angle θ above the horizontal.

2.21 Ignoring air resistance, use cylindrical polar coordinates to show

$$z = \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2$$

Setting $\phi = 0$ and $z = 0$ at $t = 0$, the projectile is fired at an angle θ above the horizontal. From velocity is $v_{\rho o} = v_o \cos\theta, v_{zo} = v_o \sin\theta$. Solving the equations of motion

$$\begin{aligned} \ddot{\rho} &= 0 & \ddot{z} &= -g \\ \dot{\rho} &= v_{\rho o} & \dot{z} &= v_{zo} - gt \\ \rho &= v_o t \cos\theta & z &= v_o t \sin\theta - \frac{1}{2}gt^2 \end{aligned}$$

Substituting $t = \rho/v_{\rho o}$ into z

$$\begin{aligned} z &= v_o \frac{\rho}{v_{\rho o}} \sin\theta - \frac{1}{2} g \frac{\rho^2}{v_{\rho o}^2} \\ z &= \rho \tan\theta - \frac{1}{2} g \frac{\rho^2}{v_o^2} \sec^2\theta \end{aligned}$$

Taking the derivative of z with respect to θ and setting it equal to zero to find the maximum height.

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}\theta} &= \rho \sec^2\theta - \frac{1}{2} g \frac{\rho^2}{v_o^2} 2 \sec^2\theta \tan\theta \\ 0 &= \rho \sec^2\theta (1 - \frac{g\rho}{v_o^2} \tan\theta) \\ \tan\theta &= \frac{v_o^2}{g\rho} \end{aligned}$$

Substituting θ into z

$$\begin{aligned} z &= \rho \frac{v_o^2}{g\rho} - \frac{1}{2} g \frac{\rho^2}{v_o^2} (1 + \tan^2\theta) \\ z &= \frac{v_o^2}{2g} - \frac{g}{2v_o^2}\rho^2 \end{aligned}$$

where the maximum height is $z_{max} = v_o^2/2g$.

2.23 Find the terminal speeds in air of (a) steel ball bearing $D = 3\,\mathrm{mm}$, (b) 16 pound steel shot, and (c) 200 pound parachutist in free fall in the fetal position. Assume drag is purely quadratic. Density of steel is $8\,\mathrm{g/cm^3}$ and the parachutist is a sphere of density $1\,\mathrm{g/cm^3}$.

The terminal speed is given by

$$v_t = \sqrt{\frac{mg}{\gamma D^2}}$$

where mass m is given by the density $m = QV$ and $V = \pi D^3/6$ for a sphere. The new equation for terminal speed is

$$v_t = \sqrt{\frac{QVg}{\gamma D^2}} = \sqrt{\frac{Q\pi Dg}{6\gamma}}$$

for an unknown diameter, solve for D from the density $Q = m/V$

$$D = \left(\frac{\pi Q}{6m}\right)^{1/3}$$

and substitute into the equation for terminal speed

$$v_t = \sqrt{\frac{mg}{\gamma}\left(\frac{\pi Q}{6m}\right)^{1/3}}$$

(a) Using the diameter $D = 0.003\,\mathrm{m}$, density of steel $Q = 8\,\mathrm{g/cm^3}$, and $\gamma = 0.25\,\mathrm{N\,s^2/m^4}$, the terminal speed is

$$\begin{aligned}
v_t &= \sqrt{\frac{Q\pi Dg}{6\gamma}} \\
&= \sqrt{\frac{8000\,\mathrm{kg/m^3}\cdot\pi\cdot 0.003\,\mathrm{m}\cdot 9.8\,\mathrm{m/s^2}}{6\cdot 0.25\,\mathrm{N\,s^2/m^4}}} \\
&= 22\,\mathrm{m/s}
\end{aligned}$$

(b) For a steel shot $m = 7.26\,\mathrm{kg}$, $Q = 8000\,\mathrm{kg/m^3}$, the terminal speed is

$$v_t = \sqrt{\frac{7.26\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s^2}}{0.25\,\mathrm{N\,s^2/m^4}}\left(\frac{\pi 8000\,\mathrm{kg/m^3}}{6\cdot 7.26\,\mathrm{kg}}\right)^{1/3}} = 140\,\mathrm{m/s}$$

(c) For a parachutist $m = 91\,\mathrm{kg}$, $Q = 1000\,\mathrm{kg/m^3}$, the terminal speed is

$$v_t = \sqrt{\frac{91\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s^2}}{0.25\,\mathrm{N\,s^2/m^4}}\left(\frac{\pi 1000\,\mathrm{kg/m^3}}{6\cdot 91\,\mathrm{kg}}\right)^{1/3}} = 107\,\mathrm{m/s}$$

2.25 For horizontal motion under quadratic drag, derive the results of (2.49) and (2.51) and verify the constant $\tau = m/cv_o$ is time.

The equation of motion for horizontal motion under quadratic drag is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -cv^2$$

Separating variables and integrating

$$\begin{aligned}
\int_{v_o}^{v}\frac{m\,\mathrm{d}v'}{v'^2} &= -c\int_0^t \mathrm{d}t' \\
\frac{m}{v_o} - \frac{m}{v} &= -ct \\
\frac{1}{v} &= \frac{1}{v_o} + \frac{ct}{m} \\
v &= \frac{v_o}{1 + \frac{cv_ot}{m}} = \frac{v_o}{1 + t/\tau}
\end{aligned}$$

Integrating again to find x from time 0 to t

$$\int_0^x \mathrm{d}x' = \int_0^t \frac{v_o}{1+t'/\tau}\,\mathrm{d}t'$$
$$x = v_o\tau\ln\left(1+\frac{t}{\tau}\right)$$

Verifying the constant $\tau = m/cv_o$ using dimensional analysis: The unit of coefficient c is

$$[c] = \left[\frac{F}{v^2}\right] = \frac{\mathrm{M\,L/T^2}}{\mathrm{L^2/T^2}} = \mathrm{M\,L^{-1}}$$

and with $[m] = \mathrm{M}$ and $[v_o] = \mathrm{L/T}$, the unit of τ is

$$[\tau] = \frac{\mathrm{M}}{\mathrm{M\,L^{-1}\cdot L/T}} = \mathrm{T}$$

2.27 Write Newton's second law for a particle of mass m sliding up a frictionless incline of angle θ with the horizontal and subject to quadratic drag and solve for v as a function of t. How long does the upward journey last?

The equation of motion is

$$m\dot{v} = -mg\sin\theta - cv^2 = -c(v_t^2+v^2)$$

where $v_t^2 = mg\sin\theta/c$. Separating variables and integrating

$$\int_{v_o}^{v}\frac{m\,\mathrm{d}v'}{v_t^2+v'^2} = -c\int_0^t \mathrm{d}t'$$
$$\frac{m}{v_t}[\arctan(v/v_t) - \arctan(v_o/v_t)] = -ct$$
$$\arctan(v/v_t) - \arctan(v_o/v_t) = -\frac{cv_o t}{m}$$
$$v = v_t\tan\left(\arctan(v_o/v_t) - \frac{cv_o t}{m}\right)$$

The time it takes to reach the top of the incline is when $v=0$ and solving for t

$$0 = \arctan(v_o/v_t) - \frac{cv_o t}{m}$$
$$t = \frac{m}{cv_o}\arctan(v_o/v_t)$$

2.29 Compare the speeds of a skydiver subject to quadratic drag who has a terminal speed of 50 m/s for times $t = 1,\ 5,\ 10,\ 20$ and 30 seconds.

The equation of motion for a skydiver subject to quadratic drag is

$$m\dot{v} = -mg - cv^2$$

where $v_t^2 = mg/c$. Separating variables and integrating gives

$$v(t) = v_t\tanh(gt/v_t) \tag{2.57}$$

In a vacuum the equation for velocity is $v_c(t) = gt$.

t	$v(t)$	$v_c(t)$
1	9.7	9.8
5	38	49
10	48	98
20	49.96	196
30	49.999	294

2.31 (a) Find the terminal speed of a basketball of diameter $D = 0.24\,\mathrm{m}$ and mass $m = 0.6\,\mathrm{kg}$. (b) How long does it take to hit the ground from a height of $h = 30\,\mathrm{m}$ and what is its speed at impact?

(a) Assuming quadratic drag, the terminal speed is

$$\begin{aligned} v_t &= \sqrt{\frac{mg}{\gamma D^2}} \\ &= \sqrt{\frac{0.6\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2}}{0.25\,\mathrm{N\,s^2/m^4} \cdot 0.24^2\mathrm{m^2}}} \\ &= 20.2\,\mathrm{m/s} \end{aligned}$$

(b) Solving (2.58) for t when $y = 30\,\mathrm{m}$ and $v_t = 20.2\,\mathrm{m/s}$

$$t = \frac{v_t}{g} \operatorname{arccosh}\left(e^{gy/v_t^2}\right) = \frac{20.2\,\mathrm{m/s}}{9.8\,\mathrm{m/s^2}} \operatorname{arccosh}\left(e^{9.8\,\mathrm{m/s^2} \cdot 30\,\mathrm{m}/20.2^2\mathrm{m/s^2}}\right) = 2.78\,\mathrm{s}$$

The speed at impact is given by (2.57)

$$v(t) = v_t \tanh(gt/v_t) = 20.2\,\mathrm{m/s} \tanh\big(9.8\,\mathrm{m/s^2} \cdot 2.8\,\mathrm{s}/20.2\,\mathrm{m/s}\big) = 17.6\,\mathrm{m/s}$$

In a vacuum the time to hit the ground is $t = \sqrt{2y/g} = 2.47\,\mathrm{s}$ and the speed is $v = \sqrt{2gy} = 25.3\,\mathrm{m/s}$.

2.33 (a) Sketch the hyperbolic functions

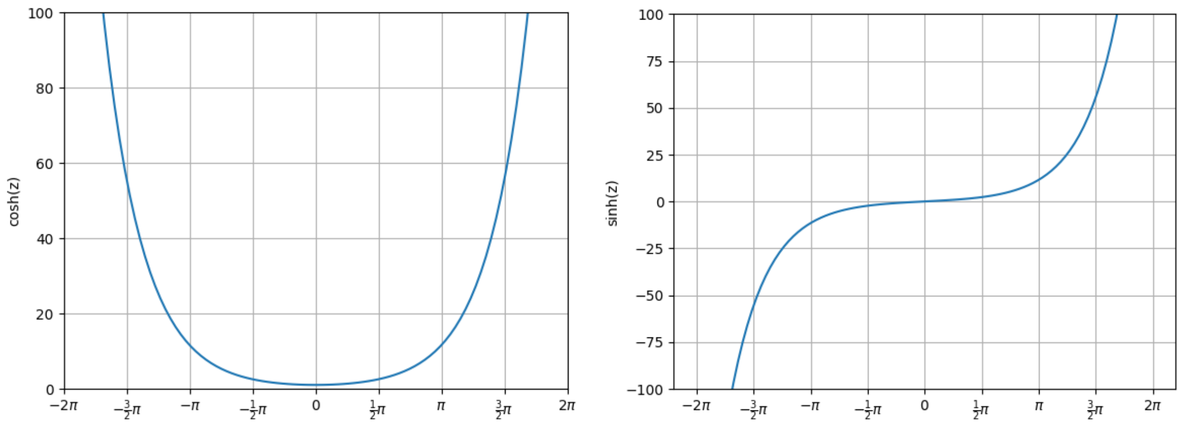
$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

for any z, real or complex. (b) Show $\cosh z = \cos(iz)$ and the same relation for $\sinh z$. (c) Find the derivative and integral of $\cosh z$ and $\sinh z$? (d) Show $\cosh^2 z - \sinh^2 z = 1$. (e) Show that

$$\int dx/\sqrt{1 + x^2} = \operatorname{arcsinh} x$$

[*Hint:* Use the substitution $x = \sinh u$ and the identity from (d)]

(a) The hyperbolic functions are plotted in Figure 2.4.



(a) $y = \cosh z$ (b) $y = \sinh z$

Figure 2.4: Graph of hyberbolic functions

(b) Using Euler's formula $e^{iz} = \cos z + i \sin z$ the exponents are rewritten as

$$e^{i(iz)} = \cos(iz) + i\sin(iz) = e^{-z} \quad \text{and} \quad e^{-i(iz)} = \cos(iz) - i\sin(iz) = e^{z}$$

and substituting into the hyperbolic function

$$
\begin{aligned}
\cosh z &= \frac{e^z + e^{-z}}{2} \\
&= \frac{e^{i(iz)} + e^{-i(iz)}}{2} \\
&= \cos(iz)
\end{aligned}
$$

and the relation for $\sinh z$ is

$$
\begin{aligned}
\sinh z &= \frac{e^z - e^{-z}}{2} \\
&= \frac{e^{i(iz)} - e^{-i(iz)}}{2} \\
&= -i\sin(iz)
\end{aligned}
$$

(c) The derivative of $\cosh z$ and $\sinh z$ are

$$\frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \frac{e^z - e^{-z}}{2} = \sinh z \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \frac{e^z + e^{-z}}{2} = \cosh z$$

similarly, the integration gives

$$\int \cosh z \,\mathrm{d}z = \sinh z + C \qquad \int \sinh z \,\mathrm{d}z = \cosh z + C$$

(d) Using the relation from (b)

$$\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1$$

(e) Using the substitution $x = \sinh u$ and the identity from (d)

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \int \frac{\cosh(u)\,\mathrm{d}u}{\sqrt{1+\sinh^2 u}} = \int \mathrm{d}u = u = \operatorname{arcsinh} x$$

2.35 (a) Show the steps from (2.52) to (2.57) and (2.58). (b) Using the parameter $\tau = v_t/g$, show when $t = \tau$, v is 76% of the terminal speed. Also show when $t = 2\tau$ and 3τ. (c) Show when $t \gg \tau$, $v \approx v_t t + C$ where C is a constant. (d) Show when t is small, position is $y \approx gt^2/2$.

(a) The equation of motion for a particle subject to quadratic drag is

$$m\dot{v} = mg - cv^2 \tag{2.52}$$

where $v_t^2 = mg/c$. Rewriting (2.52) with the sub $m = v_t^2 c/g$

$$\dot{v} = g - \frac{c}{m}v^2 = g\left(1 - \frac{v^2}{v_t^2}\right)$$

From seperation of variables and integrating

$$
\begin{aligned}
\frac{1}{1 - v^2/v_t^2}\,\mathrm{d}v &= g\,\mathrm{d}t \\
\int \frac{1}{1 - v'^2/v_t^2}\,\mathrm{d}v' &= g\int \mathrm{d}t'
\end{aligned}
$$

The integral on the left is solved using the u-sub $u = v'/v_t$, $\mathrm{d}u = \mathrm{d}v'/v_t$ and $\int 1/(1-u^2)\,\mathrm{d}u = \operatorname{arctanh} u$

$$\frac{v_t}{g}\operatorname{arctanh}(v/v_t) = t$$

Solving for v gives

$$v(t) = v_t \tanh(gt/v_t)$$

Integrating again using $\int \tanh u \, \mathrm{d}u = \ln(\cosh u)$

$$y(t) = \int v(t)\,\mathrm{d}t = v_t \int \tanh(gt/v_t)\,\mathrm{d}t = \frac{v_t^2}{g}\ln(\cosh(gt/v_t))$$

(b) Rewriting with the new parameter τ

$$v(t) = v_t \tanh(t/\tau) \quad \text{and} \quad y(t) = v_t\tau\ln(\cosh(t/\tau))$$

The values of $v(t)$ are

t	$v(t)$	percentage
τ	$\tanh(1) = 0.76v_t$	76%
2τ	$\tanh(2) = 0.96v_t$	96%
3τ	$\tanh(3) = 0.995v_t$	99.5%

(c) When $t \gg \tau$, $t/\tau \to \infty$

$$v(t) \approx \lim_{t/\tau\to\infty} v_t \tanh(t/\tau) = v_t$$

and the position is

$$y(t) \approx \int v(t)\,\mathrm{d}t = v_t t + C$$

(d) When t is small, Using Taylor series approximation for $\ln(1+\delta) \approx \delta$ and

$$\cosh x = \frac{e^x + e^{-x}}{2} \approx \frac{1}{2}\big(1 + x + x^2/2 + 1 - x + x^2/2\big) = 1 + \frac{x^2}{2}$$

The position is

$$\begin{aligned} y(t) &= v_t\tau\ln\left[1 + \frac{t^2}{2\tau^2}\right] \\ &= v_t\tau\left[\frac{t^2}{2\tau^2}\right] \\ &= \frac{v_t}{2\tau}t^2 \quad \text{using} \quad v_t = g\tau \\ y(t) &= \frac{1}{2}gt^2 \end{aligned}$$

2.37 Integrate (2.55) using the partial fraction

$$\frac{1}{1-u^2} = \frac{1}{2}\left[\frac{1}{1+u} + \frac{1}{1-u}\right]$$

The integral of the partial fraction is

$$\begin{aligned} \int \frac{1}{1-u^2}\,\mathrm{d}u &= \frac{1}{2}\int \frac{1}{1+u}\,\mathrm{d}u + \frac{1}{2}\int \frac{1}{1-u}\,\mathrm{d}u \\ &= \frac{1}{2}\ln(1+u) - \frac{1}{2}\ln(1-u) + C \\ &= \frac{1}{2}\ln\left[\frac{1+u}{1-u}\right] + C \end{aligned}$$

Integrating (2.55) using the partial fraction where $u = v/v_t$

$$
\begin{aligned}
\int \frac{1}{1 - v^2/v_t^2}\,\mathrm{d}v &= gt \\
\int \frac{1}{1-u^2}\,\mathrm{d}u &= gt/v_t \\
\ln\left[\frac{1+u}{1-u}\right] &= 2gt/v_t \\
\frac{1+u}{1-u} &= e^{2gt/v_t} \\
1+u &= e^{2gt/v_t} - ue^{2gt/v_t} \\
u &= \frac{e^{2gt/v_t} - 1}{e^{2gt/v_t} + 1} \\
u &= \frac{e^{2gt/v_t} - 1}{e^{2gt/v_t} + 1}\frac{e^{-gt/v_t}}{e^{-gt/v_t}} \\
u &= \frac{e^{gt/v_t} - e^{-gt/v_t}}{e^{gt/v_t} + e^{-gt/v_t}} \\
u &= \frac{2\cosh(gt/v_t)}{2\cosh(gt/v_t)} \\
u &= \tanh(gt/v_t) \\
v &= v_t\tanh(gt/v_t)
\end{aligned}
$$

2.39 (a) Write the equation of motion for a cyclist coasting to a stop subject to quadratic drag and constant frictional force f_{fr}. Solve for v as a function of t (b) With $f_{fr} = 3\,\mathrm{N}$, drag coefficient $c = 0.20\,\mathrm{N\,s^2/m^2}$, and mass $m = 80\,\mathrm{kg}$, find the time to slow from an initial speed of 20 m/s to 15 m/s, 10 m/s, 5 m/s, and time to come to a full stop.

(a) The equation of motion is

$$m\dot{v} = -cv^2 - f_{fr}$$

Separating variables and integrating

$$\frac{-m}{f_{fr}}\int_{v_o}^{v}\frac{\mathrm{d}v'}{cv'^2/f_{fr} + 1} = \int_0^t \mathrm{d}t'$$

With $u = \sqrt{c/f_{fr}}v'$ and $\mathrm{d}u = \sqrt{c/f_{fr}}\,\mathrm{d}v'$

$$
\begin{aligned}
t &= \frac{-m}{\sqrt{f_{fr}c}}\int_{v'=v_o}^{v}\frac{\mathrm{d}u}{u^2+1} \\
&= \frac{-m}{\sqrt{f_{fr}c}}\arctan u\Big|_{v'=v_o}^{v} \\
t &= \frac{-m}{\sqrt{f_{fr}c}}\left[\arctan\sqrt{\frac{c}{f_{fr}}}v - \arctan\sqrt{\frac{c}{f_{fr}}}v_o\right]
\end{aligned}
$$

(b) The time to slow from an initial speed of $v_o = 20$ m/s to $v = 15$ m/s, 10 m/s, 5 m/s, and 0 m/s are

v	t
15	6.34 s
10	18.4 s
5	48.3 s
0	142 s

2.41 For a baseball thrown vertically upward and subject to quadratic drag, find v as a function of y and the maximum height is

$$y_{max} = \frac{v_t^2}{2g}\ln\left(\frac{v_t^2+v_o^2}{v_t^2}\right)$$

Compute y_{max} for $v_o = 20\,\mathrm{m/s}$ and $v_t = 35\,\mathrm{m/s}$ and compare to the result in a vacuum.

Measuring y upwards The equation of motion is

$$m\dot{v} = -mg - cv^2$$

Where $v_t^2 = mg/c$. Rewriting with the sub $c = mg/v_t^2$

$$\dot{v} = -g - \frac{g}{v_t^2}v^2 = -g\left(1+\frac{v^2}{v_t^2}\right)$$

Using (2.86)

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}y} &= -g\left(1+\frac{v^2}{v_t^2}\right)\\
v_t^2\int_{v_o}^{v}\frac{1}{v_t^2+v'^2}\,\mathrm{d}v'^2 &= -2g\int_0^y \mathrm{d}y'\\
v_t^2\int_{v_o}^{v}\frac{1}{v_t^2+v'^2}\,\mathrm{d}v'^2 &= -2gy\\
v_t^2\ln\left(\frac{v_t^2+v^2}{v_t^2+v_o^2}\right) &= -2gy
\end{aligned}$$

The maximum height is when $v = 0$ and solving for y

$$\begin{aligned}
v_t^2\ln\left(\frac{v_t^2+0}{v_t^2+v_o^2}\right) &= -2gy_{max}\\
y_{max} &= \frac{v_t^2}{2g}\ln\left(\frac{v_t^2+v_o^2}{v_t^2}\right)
\end{aligned}$$

For $v_o = 20\,\mathrm{m/s}$ and $v_t = 35\,\mathrm{m/s}$, the maximum height is $y = 17.66$ m. In a vacuum the maximum height is $y_{max} = v_o^2/2g = 20.4$ m.

2.43 A basketball of mass $m = 600$ g and diameter $D = 24$ cm is thrown from a height of 2 m with an initial velocity $v_o = 20\,\mathrm{m/s}$ at 45° above the horizontal. (a) Numerically solve the equations of motion given by (2.61) for the ball's position and plot the trajectory as well as its trajectory in the abscence of air resistance. (b) Find how far the ball travels in the horizontal direction before hitting the ground as well as its corresponding range in a vacuum.

(a) The equation of motion is

$$\begin{aligned}
\dot{v}_x &= -\frac{c}{m}v_x\sqrt{v_x^2+v_y^2}\\
\dot{v}_y &= -g - \frac{c}{m}v_y\sqrt{v_x^2+v_y^2}
\end{aligned}$$

solving the equations of motion numerically using Scipy's RK4 method

```python
import scipy as sp
import numpy as np
import matplotlib.pyplot as plt

# initial conditions
v_xo = 15 * np.sin(np.pi/4)
v_yo = v_xo
x_o = 0
y_o = 2

# constants
gamma = 0.25
g = 9.8
m = 0.6
D = 0.24
c = gamma * D ** 2
v_t = np.sqrt(m*g/c)

# time
t = np.linspace(0, 10, 1000)

# differential equations
def soe(t, s):
    x = s[0]
    y = s[1]
    v_x = s[2]
    v_y = s[3]
    dxdt = v_x
    dydt = v_y
    dv_xdt = -c/m * v_x * np.sqrt(v_x**2 + v_y**2)
    dv_ydt = -g - c/m * v_y * np.sqrt(v_x**2 + v_y**2)
    return [dxdt, dydt, dv_xdt, dv_ydt]

# solveing the differential equations
sol = sp.integrate.solve_ivp(soe, [0, 10], [x_o, y_o, v_xo, v_yo], t_eval=t)

# equation of motion in a vacuum
x_vac = v_xo * t
y_vac = y_o + v_yo * t - 0.5 * g * t**2

# plot the results
plt.plot(sol.y[0], sol.y[1])
plt.plot(x_vac, y_vac, 'r--') # solution in a vacuum
plt.ylim(0, 8)
plt.xlim(0, 25)
plt.xlabel('x')
plt.ylabel('y')
plt.grid(True)
plt.legend(['quadratic drag', 'vacuum'])
plt.show()

# calculate the range when y = 0
# find the index of the first value of y that is less than zero
idx = np.where(sol.y[1] < 0)[0][0]

# calculate the range
range = sol.y[0][idx]
print("range = ", range)

# another way to calculate the range
it = np.nditer(sol.y[1], flags=['f_index'])
for i in it:
    if i < 0:
        # print("range = ", sol.y[0][it.index])
        break

# range in a vacuum
index = np.where(y_vac < 0)[0][0]
range_vac = x_vac[index]
print("range_vac = ", range_vac)
```

OUTPUT :

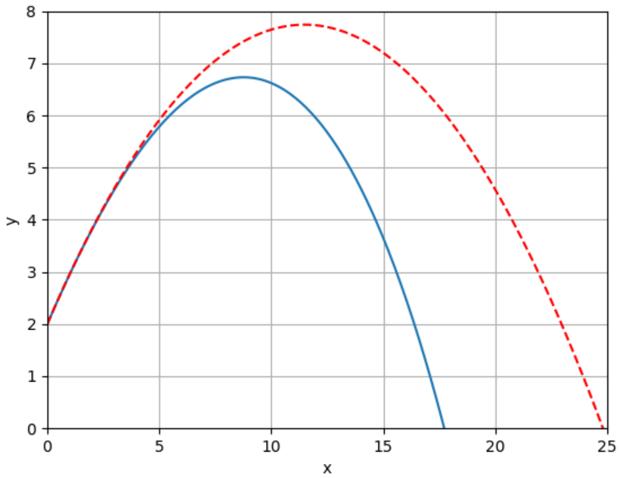


Figure 2.5: Trajectory of a basketball thrown from a height of 2 m with an initial velocity v_o and subject to quadratic drag (solid curve). In a vacuum, the trajectory is given by the dashed curve.

```
range =  17.741443936603282
range_vac =  24.844292311959776
```

in a vacuum, the equations of motion are

$$x = v_{xo}t$$
$$y = y_o + v_{yo}t - \frac{1}{2}gt^2$$

(b) The distance traveled in the horizontal direction before hitting the ground is $x = 17.7$ m. In a vacuum, the distance traveled is $x = 24.8$ m.

2.45 (a) Using Euler's relation (2.76), prove that any complex number $z = x + iy$ can be written in the form $z = re^{i\theta}$. (b) Write $z = 3 + 4i$ in the form $z = re^{i\theta}$. (c) Write $z = 2e^{-i\pi/3}$ in the form $x + iy$.

(a) Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

r and θ relate to the points on the unit circle in the complex plane
(b) Let $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$ with $x = 3$ and $y = 4$

$$r = \sqrt{3^2 + 4^2} = 5 \quad \text{and} \quad \theta = \arctan(4/3) = 0.93\,\text{rad}$$

hence $z = 5e^{0.93i}$ (c) Let $r = 2$ and $\theta = -\pi/3$

$$x = r\cos\theta = 2\cos(-\pi/3) = 1$$
$$y = r\sin\theta = 2\sin(-\pi/3) = -\sqrt{3}$$
$$z = 1 - i\sqrt{3}$$

2.47 For each of the following two pairs of numbers compute $z+w$, $z-w$, zw, and z/w. (a) $z=6+8i$ and $w=3-4i$ (b) $z=8e^{i\pi/3}$ and $w=4e^{i\pi/6}$

(a) With $z=10e^{0.93i}$ and $w=5e^{-0.93i}$ where $\theta=\arctan(4/3)=0.93\,\text{rad}$

$$\begin{aligned}
z+w &= 6+8i+3-4i = 9+4i\\
z-w &= 6+8i-(3-4i) = 3+12i\\
zw &= (10e^{i\theta})(5e^{-i\theta}) = 50\\
\frac{z}{w} &= \frac{6+8i}{3-4i} = \frac{6+8i}{3-4i}\frac{3+4i}{3+4i} = \frac{-14+48i}{25} = -0.56+1.92i
\end{aligned}$$

or

$$\frac{z}{w} = \frac{10e^{i\theta}}{5e^{-i\theta}} = 2e^{2i\theta} = 2e^{1.86i} = -0.56+1.92i$$

(b) With $z=4+4\sqrt{3}i$ and $w=2\sqrt{3}+2i$

$$\begin{aligned}
z+w &= (4+2\sqrt{3})+(4\sqrt{3}+2)i\\
z-w &= (4-2\sqrt{3})+(4\sqrt{3}-2)i\\
zw &= 8e^{i\pi/3}4e^{i\pi/6} = 32e^{i\pi/2} = 32i\\
\frac{z}{w} &= \frac{8e^{i\pi/3}}{4e^{i\pi/6}} = 2e^{i\pi/6} = \sqrt{3}+i
\end{aligned}$$

2.49 Consider the complex number $z=e^{i\theta}=\cos\theta+i\sin\theta$. (a) By evaluating z^2 two different ways, prove the trig identities $\cos 2\theta=\cos^2\theta-\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. (b) Use the same technique to find corresponding identities for $\cos 3\theta$ and $\sin 3\theta$.

(a)

$$\begin{aligned}
z^2 &= e^{i\theta}e^{i\theta} = e^{i2\theta} = \cos 2\theta + i\sin 2\theta\\
&= (\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)\\
&= (\cos^2\theta-\sin^2\theta)+(2\cos\theta\sin\theta)i
\end{aligned}$$

Equating the real and imaginary parts

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta-\sin^2\theta\\
\sin 2\theta &= 2\cos\theta\sin\theta
\end{aligned}$$

(b)

$$\begin{aligned}
z^3 &= e^{i\theta}e^{i\theta}e^{i\theta} = e^{i3\theta} = \cos 3\theta + i\sin 3\theta\\
&= (\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\\
&= (\cos\theta+i\sin\theta)(\cos^2\theta-\sin^2\theta+i2\cos\theta\sin\theta)\\
&= (\cos^3\theta-3\cos\theta\sin^2\theta)+i(3\cos^2\theta\sin\theta-\sin^3\theta)
\end{aligned}$$

Equating the real and imaginary parts

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta-3\cos\theta\sin^2\theta\\
\sin 3\theta &= 3\cos^2\theta\sin\theta-\sin^3\theta
\end{aligned}$$

2.51 Use the series definition (2.72) of e^z to prove that $e^ze^w=e^{z+w}$.

Grouping each term of z^nw^m where $n+m=p$.

$$\begin{aligned}
e^ze^w &= \left[1+z+z^2/2!+\cdots\right]\left[1+w+w^2/2!+\cdots\right]\\
&= \left[1+(z+w)+\frac{1}{2!}(z^2+2zw+w^2)+\frac{1}{3!}(z^3+3z^2w+3zw^2+w^3)\right]+\cdots
\end{aligned}$$

The coefficient $1/N!$ in each term are factored out by $1/(N-1)! = N/N!$ e.g. $z^2w/2! = 3z^2w/3!$. Each term is a binomial expansion of $(z+w)^p$ where the first term is $p = 0$, the second is $p = 1$ etc. Hence

$$\begin{aligned} e^z e^w &= \left[1 + (z+w) + \frac{1}{2!}(z+w)^2 + \frac{1}{3!}(z+w)^3 + \cdots\right] \\ &= e^{z+w} \end{aligned}$$

Q.E.D.

2.53 A charged particle of mass m and charge q moves in uniform electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, both pointing in the z direction. The net force on the particle is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Write down the equation of motion into its three components and solve the equations.

The components of each vector are

$$\mathbf{v} = (v_x, v_y, v_z) \qquad \mathbf{E} = (0,0,E) \qquad \mathbf{B} = (0,0,B) \qquad \mathbf{v} \times \mathbf{B} = (v_y B, -v_x B, 0)$$

The equation of motion is written as

$$\begin{aligned} \dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x \\ m\dot{v}_z &= qE \end{aligned}$$

where $\omega = qB/m$ is the cyclotron frequency. Using the complex number $\eta = v_x + iv_y$ the solution is in the general form

$$\eta = Ae^{-iwt}$$

where A is a constant. The general solution for x and y are the real and imaginary parts of $\xi = x + iy$ and constant $C = x_o + iy_o$ given by (2.80)

$$\begin{aligned} \xi &= Ce^{-iwt} \\ &= (x_o + iy_o)e^{-iwt} \\ &= (x_o + iy_o)(\cos(\omega t) - i\sin(\omega t)) \\ &= (x_o\cos(\omega t) + y_o\sin(\omega t)) + i(y_o\cos(\omega t) - x_o\sin(\omega t)) \end{aligned}$$

The solution for x and y are

$$\begin{aligned} x &= x_o\cos(\omega t) + y_o\sin(\omega t) \\ y &= y_o\cos(\omega t) - x_o\sin(\omega t) \end{aligned}$$

Solving for the motion in the z direction

$$\begin{aligned} \dot{v}_z &= \frac{qE}{m} \\ v_z &= \frac{qE}{m}t + v_{zo} \\ z &= \frac{qE}{2m}t^2 + v_{zo}t + z_o \end{aligned}$$

The particle moves in a circular path in the xy plane with an increaseing velocity in the z direction which combine to form a helix with increasing pitch in 3D space.

2.55 A charged particle of mass m and charge q moves in uniform electric and magnetic fields, $\mathbf{E}$ pointing in the y direction and $\mathbf{B}$ in the z direction (an arrangement called "crossed E and B fields"). Suppose the particle is initially at the origin and is given a kick at time $t = 0$ along the x axis with $v_x = v_{xo}$ (positive or negative). (a) Write down the equation of motion for the particle and resolve it into its three components. Show that the motion remains in the plane $z = 0$. (b) Prove that there is a unique value of v_{xo}, called drift speed v_{dr}, for which the particle moves undeflected through the fields. (This is the basis of velocity selectors, which select particles traveling at one chosen speed froma beam with many different speeds.) (c) Solve the equations of motion to give the particle's velocity as a function of t, for arbitrary values of v_{xo} (d) Integrate the velocity to find the position as a function of t and sketch the trajectory for various values of v_{xo}.

(a) The components of each vector are

$$\mathbf{v} = (v_x, v_y, v_z) \qquad \mathbf{E} = (0, E, 0) \qquad \mathbf{B} = (0, 0, B) \qquad \mathbf{v} \times \mathbf{B} = (v_y B, v_x B, 0)$$

The equation of motion is written as

$$\begin{aligned}
\dot{v}_x &= \omega v_y \\
\dot{v}_y &= -\omega v_x + \frac{E\omega}{B} \\
\dot{v}_z &= 0
\end{aligned}$$

Since $\dot{v}_z = 0$, the motion remains in the plane $z = 0$.
(b) An undeflected particle will have $\dot{v}_y = 0$ and $\dot{v}_x = 0$ which is solved by

$$\begin{aligned}
0 = \omega v_y \quad \text{and} \quad 0 &= -\omega v_x + \frac{E\omega}{B} \\
v_y = 0 \quad \text{and} \quad v_x &= \frac{E}{B}
\end{aligned}$$

The drift speed is $v_{dr} = E/B$ which is constant and equivalent to the initial velocity v_{xo}.
(c) Solving the first two equations of motion by substituting $v_u = v_x - v_{dr}$ and $\dot{v}_u = \dot{v}_x$

$$\begin{aligned}
\dot{v}_u &= \omega v_y \\
\dot{v}_y &= -\omega v_u
\end{aligned}$$

which is the same as the equations of motion in Problem (2.53). The general solution is a complex number $\eta = Ae^{-iwt}$. A is then given by the initial conditions at $t = 0$ which is

$$A = v_{uo} + iv_{yo} = v_{xo} - v_{dr}$$

This gives the solution on the complex plane which can be decomposed into its x and y components

$$\begin{aligned}
\eta &= (v_{xo} - v_{dr})e^{-iwt} \\
v_u + iv_y &= (v_{xo} - v_{dr})(\cos(\omega t) - i\sin(\omega t))
\end{aligned}$$

The solution for v_x, rewritten using $v_x = v_u + v_{dr}$, and v_y are

$$\begin{aligned}
v_x &= (v_{xo} - v_{dr})\cos(\omega t) + v_{dr} \\
v_y &= -(v_{xo} - v_{dr})\sin(\omega t)
\end{aligned}$$

Defining $R = (v_{xo} - v_{dr})/\omega$ the equations are simplified to

$$\begin{aligned}
v_x &= R\omega\cos(\omega t) + v_{dr} \\
v_y &= -R\omega\sin(\omega t)
\end{aligned}$$

(d) integrating from time 0 to t to solve for the x position

$$\int_0^x \mathrm{d}x = x_o + \int_0^t R\omega\cos(\omega t) + v_{dr}\,\mathrm{d}t$$
$$x = R\sin(\omega t) + +v_{dr}t\Big|_0^t + x_o$$
$$x = R\sin(\omega t) + v_{dr}t + x_o$$

same for the y position

$$y = y_o + \int_0^t -R\omega\sin(\omega t)\,\mathrm{d}t$$
$$= R\cos(\omega t)\Big|_0^t + y_o$$
$$y = R\cos(\omega t) + R + y_o$$

solving for the initial conditions x_o and y_o using $x(0) = 0$ and $y(0) = 0$

$$x_o = 0$$
$$y_o = -2R$$

The equation for the trajectory is

$$x = R\sin(\omega t) + v_{dr}t$$
$$y = R\cos(\omega t) - R$$

or

$$x = \frac{v_{xo} - v_{dr}}{\omega}\sin(\omega t) + v_{dr}t$$
$$y = \frac{v_{xo} - v_{dr}}{\omega}(\cos(\omega t) - 1)$$

with $\omega = 1$ and $v_{dr} = 1$ the trajectory for various values of v_{xo} is shown by Figure 2.6

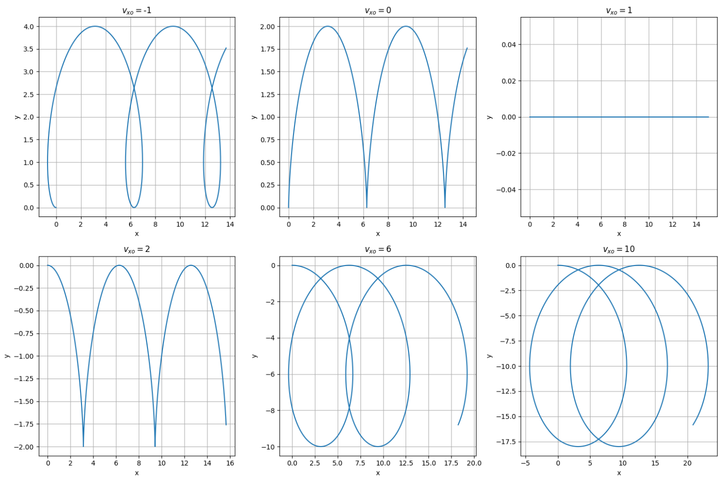


Figure 2.6: Trajectory of a charged particle in crossed electric and magnetic fields with intial velocities $v_{xo} = -1,\ 0,\ 1,\ 2,\ 6$ and 10 m/s. When the initial velocity is equivalent to the drift speed $v_{dr} = 1$, the particle moves undeflected.

3 Momentum and Angular Momentum

3.1 The speed of the shell relative to the ground is defined as $v_s = v + v_g$ or $v_g = v_s - v$ where v_g is the speed of the gun relative to the ground. Using conservation of momentum

$$\begin{aligned} P_i &= P_f \\ 0 &= mv_s + Mv_g \\ 0 &= mv_s + M(v_s - v) \\ v_s(m+M) &= Mv \\ v_s &= \frac{Mv}{m+M} \\ v_s &= v\frac{1}{1+m/M} \end{aligned}$$

3.3 Let the mass of each fragment be m and the mass of the shell be $3m$. The total momentum is

$$\begin{aligned} 3m\mathbf{v}_o &= m\mathbf{v}_1 + m\mathbf{v}_2 + m\mathbf{v}_3 \\ 2\mathbf{v}_o &= \mathbf{v}_2 + \mathbf{v}_3 \end{aligned}$$

since $\mathbf{v}_1 = \mathbf{v}_o$. Split into components

$$\begin{aligned} 2v_o &= v_2(\cos\theta_2 + \cos\theta_3) \\ 0 &= v_2(\sin\theta_2 + \sin\theta_3) = \sin\theta_2 + \sin\theta_3 \end{aligned}$$

where $v_2 = v_3$. Since $\theta_2 = \theta_3 + \pi/2$

$$\theta_3 = -(\pi/2 - \theta_2)$$

and

$$\begin{aligned} \cos(\theta_3) &= \cos(-(\pi/2 - \theta_2)) = \sin(\theta_2) \\ \sin(\theta_3) &= \sin(-(\pi/2 - \theta_2)) = -\cos(\theta_2) \end{aligned}$$

in the second equation

$$\begin{aligned} 0 &= \sin\theta_2 - \cos\theta_2 \\ \sin\theta_2 &= \cos\theta_2 \\ \tan\theta_2 &= 1 \\ \theta_2 &= \pi/4 \end{aligned}$$

and $\theta_3 = -(\pi/2 - \pi/4) = -\pi/4$. Substituting back into the first equation

$$\begin{aligned} 2v_o &= v_2(\cos(\pi/4) + \cos(-\pi/4)) \\ 2v_o &= v_2\frac{2}{\sqrt{2}} \\ v_2 &= v_o\sqrt{2} \end{aligned}$$

The three velocities are sketched in Figure 3.1.

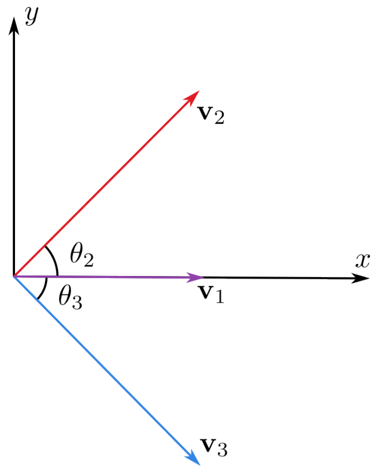


Figure 3.1: A shell exploding into three pieces. When $\mathbf{v}_1$ is solely in the positive x direction, $\theta_2 = \pi/4$ and $\theta_3 = -\pi/4$.

3.5 In an elastic collision the bodies stay seperated after the collision. The conservation of momentum:

$$P_i = P_f$$
$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$$

where $\mathbf{v}_2 = 0$ and $m_1 = m_2$ so the equation becomes

$$\mathbf{v}_1 = \mathbf{v}'_1 + \mathbf{v}'_2$$

From the conservation of energy:

$$\begin{aligned} E_i &= E_f \\ \frac{1}{2}m_1 v_1^2 &= \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 \\ v_1^2 &= v_1'^2 + v_2'^2 \end{aligned}$$

Squaring the first equation

$$\begin{aligned} v_1^2 &= v_1'^2 + v_2'^2 + 2\mathbf{v}'_1 \cdot \mathbf{v}'_2 \\ 0 &= 2\mathbf{v}'_1 \cdot \mathbf{v}'_2 \\ \mathbf{v}'_1 \cdot \mathbf{v}'_2 &= 0 \end{aligned}$$

The dot product is zero when the vectors are perpendicular, therefore the angle between the two vectors is $\pi/2$ or $90°$ Q.E.D.

3.7 The equation of the rocket's motion given by is

$$v - v_o = v_{ex} \ln \frac{m_o}{m} \tag{3.8}$$

Since $v_o = 0$ the velocity of the rocket is

$$\begin{aligned} v &= v_{ex} \ln \frac{m_o}{m} \\ &= 3000\,\mathrm{m/s} \ln \frac{2 \times 10^6\,\mathrm{kg}}{1 \times 10^6\,\mathrm{kg}} \\ &= 3000\,\mathrm{m/s} \ln 2 \\ v &= 2100\,\mathrm{m/s} \end{aligned}$$

solving for thrust

$$\begin{aligned}
\text{thrust} &= -\dot{m}v_{ex} \\
&= -\frac{\mathrm{d}m}{\mathrm{d}t}v_{ex} \\
&= -\frac{1\times10^6\,\mathrm{kg}}{120\,\mathrm{s}} * 3000\,\mathrm{m/s} \\
&= --2.5\times10^7\,\mathrm{kg\,m/s^2}
\end{aligned}$$

where thrust is in newtons. In comparison, the thrust is larger than the initial weight:

$$m_o g = 2\times10^6\,\mathrm{kg} * 9.81\,\mathrm{m/s^2} = 1.96\times10^7\,\mathrm{kg\,m/s^2}$$

3.9 The equation $m_o g = -\dot{m}v_{ex}$ describes when the magnitude of thrust equals the initial weight. Solving for the minimum exhaust speed

$$\begin{aligned}
v_{ex} &= \frac{m_o g}{-\dot{m}} \\
&= \frac{2\times10^6\,\mathrm{kg}\times 9.81\,\mathrm{m/s^2}\times 120\,\mathrm{s}}{-1\times10^6\,\mathrm{kg}} \\
&= -2350\,\mathrm{m/s}
\end{aligned}$$

3.11 (a) The change in total momentum of the system is given by

$$\mathrm{d}P = m\,\mathrm{d}v + \mathrm{d}m\,v_{ex} \tag{3.4}$$

Since there is a net external force, $\mathrm{d}P = F^{ext}\,\mathrm{d}t$. Dividing both sides by $\mathrm{d}t$

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= m\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}v_{ex} \\
F^{ext} &= m\dot{v} + \dot{m}v_{ex}
\end{aligned}$$

hence, the equation of motion is

$$m\dot{v} = -\dot{m}v_{ex} + F^{ext} \tag{3.29}$$

(b) In the Earth's gravitational field the external force is $F^{ext} = -mg$. Assuming a constant ejected mass $\dot{m} = -k$, the mass of the rocket is $m = m_o - kt$ where m_o is the initial mass of the rocket. Substituting into the equation of motion

$$m\dot{v} = -\dot{m}v_{ex} - mg \tag{3.30}$$

separating variables and integrating

$$\begin{aligned}
\dot{v} &= \frac{k}{m}v_{ex} - g \\
\mathrm{d}v &= \left(\frac{kv_{ex}}{m_o - kt} - g\right)\mathrm{d}t \\
\int_{v_o}^{v} &= \int_0^t \frac{k}{m_o - kt}\,\mathrm{d}t - \int_0^t g\,\mathrm{d}t
\end{aligned}$$

Using u-sub: $u = m_o - kt$ and $\mathrm{d}u = -k\,\mathrm{d}t$, where $u(0) = m_o$, $u(t) = m_o - kt = m$, and $v_o = 0$

$$\begin{aligned}
v - v_o &= -v_{ex}\int_{u(0)}^{u(t)} \frac{1}{u}\,\mathrm{d}u - gt \\
v &= -v_{ex}\ln u\Big|_{u(0)}^{u(t)} - gt \\
v &= -v_{ex}\ln\frac{m}{m_o} - gt \\
v &= v_{ex}\ln\frac{m_o}{m} - gt
\end{aligned}$$

(c) From Problem 3.7 at $t = 120$ s: $m_o/m = 2$, and $v_{ex} = 3000$m/s. The speed of the rocket at this time is $v = 900\,\mathrm{m/s}$. At $g = 0$ the speed is $2100\,\mathrm{m/s}$ from Problem 3.7.
(d) If $\dot{m}v_{ex} < mg$ then the magnitude of the thrust is less than the weight of the rocket. Therefore, the rocket will not be able to lift off the ground until enough mass has been ejected.

3.13 Integrating $v(t)$ from Problem 3.11(b)

$$\int_0^y y\,\mathrm{d}y = \int_0^t v_{ex}\ln\frac{m_o}{m} - gt\,\mathrm{d}t$$
$$y = v_{ex}\int_0^t \ln m_o - \ln(m_o - kt)\,\mathrm{d}t - \frac{1}{2}gt^2$$

Using u-sub:

$$u = m_o - kt \qquad\qquad u(0) = m_o$$
$$\mathrm{d}u = -k\,\mathrm{d}t \qquad\qquad u(t) = m_o - kt = m$$

which gives

$$y = v_{ex}t\ln m_o + \frac{v_{ex}}{k}\int_0^t \ln(u)\,\mathrm{d}u - \frac{1}{2}gt^2$$
$$= v_{ex}t\ln m_o + \frac{v_{ex}}{k}\,(u\ln u - u)\Big|_{m_o}^{m} - \frac{1}{2}gt^2$$

using $kt = m_o - m$ and $t = (m_o - m)/k$, the first term is

$$\frac{m_o v_{ex}}{k}\ln m_o - \frac{m v_{ex}}{k}\ln m_o$$

and the second term is

$$\frac{m v_{ex}}{k}\ln m - \frac{m_o v_{ex}}{k}\ln m_o + v_{ex}t$$

and combing the terms gives

$$v_{ex}t + \frac{mv_{ex}}{k}(\ln m_o - \ln m_o) = v_{ex}t - \frac{mv_{ex}}{k}\ln\Big(\frac{m_o}{m}\Big)$$

so, the height of the rocket is

$$y(t) = v_{ex}t - \frac{1}{2}gt^2 - \frac{mv_{ex}}{k}\ln\Big(\frac{m_o}{m}\Big)$$

Q.E.D.
After $t = 120\,\mathrm{s}$, $m/k = 120$ s. The height of the rocket is

$$y(120) = 3000\,\mathrm{m/s} * 120\,\mathrm{s} - \frac{1}{2}9.8\,\mathrm{m/s^2} * (120\,\mathrm{s})^2 - 120\,\mathrm{s} * 3000\,\mathrm{m/s} * \ln 2$$
$$= 40\,000\,\mathrm{m} \quad \text{or} \quad 40\,\mathrm{km}$$

3.15 Position of three particles with masses $m_1 = m_2$ and $m_3 = 10m_1$:

$$\mathbf{r}_1 = (1, 1, 0)$$
$$\mathbf{r}_2 = (1, -1, 0)$$
$$\mathbf{r}_3 = (0, 0, 0)$$

where $M = m_1 + m_2 + m_3 = 12m_1$, the total mass. The CM is defined to be

$$\mathbf{R} = \frac{1}{M}\sum m_a \mathbf{r}_a$$

where the three components are

$$X = \frac{1}{M}(m_1x_1 + m_2x_2 + m_3x_3) = \frac{1}{6}, \qquad Y = 0, \qquad Z = 0$$

The center of mass is drawn in Figure 3.2.

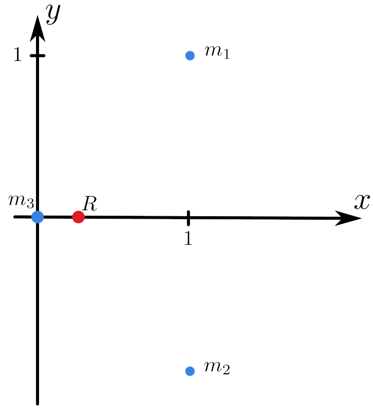


Figure 3.2: Three particles of mass m_1, m_2 and m_3 at positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ respectively. The center of mass is at R which is close to the larger mass m_3.

3.17 The masses of the earth and moon are approximately

$$M_e \approx 6.0 \times 10^{24}\,\text{kg} \quad \text{and} \quad M_m \approx 7.4 \times 10^{22}\,\text{kg}$$

where the distance between the center to center is $r = 3.8 \times 10^5$ km. Treating the center of the earth as the origin, The position of the CM is

$$\begin{aligned} R &= \frac{1}{M_e + M_m}(M_e\mathbf{r}_e + M_m\mathbf{r}_m) \\ &= \frac{M_m}{M_e + M_m} r \\ &= 4600\,\text{km} \end{aligned}$$

Compared to the radius of the earth, $R_e = 6400\,\text{km}$, the CM is located inside the earth.

3.19 (a) The trajectory is still a parabola if the projectile exploded in midair.
(b) Since the CM remains at the target position $R = 100$ m, if one piece landed at $r_1 = 200$ m. The second piece must be at $r_2 = 0$, or 100 m shy of the target position. Checking the CM

$$R = \frac{1}{2m}(200m + 0) = 100\,\text{m}$$

(c) If the pieces land at different times, shown by Figure 3.3, the CM changes; The first piece that lands on the ground (beyond the target) undergoes perfect inelastic collision with the ground, and stops immediately. The second piece still has momentum, so the CM will move in the direction of the second piece until it lands on the ground. Hence, the CM will have a position $R < 100$ m.

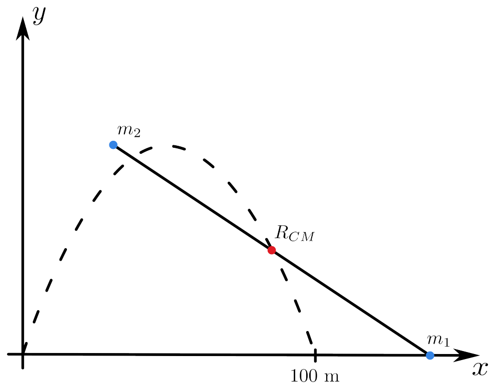


Figure 3.3: When the first piece lands on the ground, it loses all of its momentum while the second piece is still in projectile motion. The center of mass R_{CM} will always be at the midpoint of the line connecting the two pieces.

3.21 The axis of symmetry lies on the y-axis, so the x and z component of the CM is $Z = X = 0$. The y-component is given by

$$Y = \frac{1}{M}\int \sigma y \, \mathrm{d}A$$

where $\sigma = M/A$ is the area density. Using change of variables in polar coordinates: the area of a semicircle is $A = \pi R^2/2$ and position $y = r\sin\theta$. Given $\mathrm{d}A = r\,\mathrm{d}r\,,\mathrm{d}\theta$ The CM position is

$$\begin{aligned} Y &= \frac{2}{\pi R^2}\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \frac{2}{\pi R^2}\frac{R^3}{3}\int_0^{\pi}\sin\theta \\ Y &= \frac{4}{3\pi}R \end{aligned}$$

3.23 (a) The equation of motion in vector form is $\mathbf{r} = \mathbf{v}_o t - \mathbf{g}t^2/2$. Plotting grenade trajectory with parameters $\mathbf{v}_o = (4, 4)$, $g = 1$, from $0 \le t \le 4$:

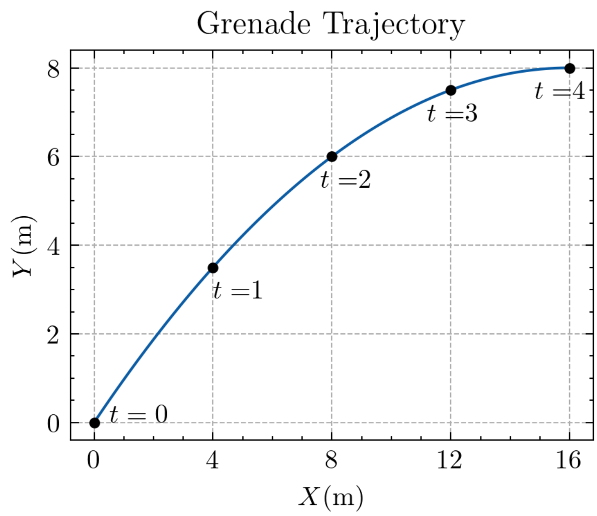


Figure 3.4: The trajectory of a grenade with initial velocity $\mathbf{v}_o = (4, 4)$ and $g = 1$.

(b) From the conservation of momentum

$$P_i = P_f$$
$$2m\mathbf{v} = m\mathbf{v}_1 + m\mathbf{v}_2$$
$$\mathbf{v}_2 = 2\mathbf{v} - \mathbf{v}_1$$

where $\mathbf{v}_1 = \mathbf{v} + \Delta\mathbf{v}$, hence $\mathbf{v}_2 = \mathbf{v} - \Delta\mathbf{v}$.
(c) Given $\Delta\mathbf{v} = (1, 3)$. Python Code:

```python
# 3.23
import scipy as sp
import numpy as np
import matplotlib.pyplot as plt
import scienceplots

plt.style.use('science')

# change dpi
plt.rcParams['figure.dpi'] = 300

# constants
g = 1; vx = 4; vy = 4
t = np.linspace(0, 4, 100)

# equations
x = vx * t
y = vy * t - 0.5 * g * t**2
def xf(t):
    return vx * t
def yf(t):
    return vy * t - 0.5 * g * t**2

# plot
plt.plot(x, y)
plt.plot([0, xf(1), xf(2), xf(3), xf(4)], [0, yf(1), yf(2), yf(3), yf(4)], 'ko',
    markersize=3)
for i, txt in enumerate(['1', '2', '3', '4']):
    plt.annotate('$t=$' + txt, (xf(i+1)-0.4*i, yf(i+1)-0.7))
plt.annotate('$t = 0$', (0.5, 0))
plt.xticks(np.arange(0, 20, 4))
plt.yticks(np.arange(0, 10, 2))
plt.xlabel('$X$(m)')
plt.ylabel('$Y$(m)')
plt.grid(True, linestyle='--')
plt.title('Grenade Trajectory')

# using vectors for equations of motion
# solving for initial velocity at t = 4 before explosion
vyo = vy - g * 4
vxo = vx
v_o = np.array([vxo, vyo])

# velocities after explosion
dv = np.array([1, 3])
v_1o = v_o + dv
v_2o = v_o - dv

# constants
t_vector = np.linspace(4, 9, 100)
g_vector = np.array([0, 1])

# equation of motion in vector form
r1 = np.zeros((len(t_vector)+1, 2))
r2 = np.zeros((len(t_vector)+1, 2))
r1[0] = np.array([xf(4), yf(4)])
r2[0] = np.array([xf(4), yf(4)])
for i, val in enumerate(t_vector):
    time = val - 4
    r1[i+1] = r1[0] + v_1o * time - 0.5 * g_vector * time**2
```

```python
    r2[i+1] = r2[0] + v_2o * time - 0.5 * g_vector * time**2
# as a function of t
def r1f(t):
    return r1[0] + v_1o * (t-4) - 0.5 * g_vector * (t-4)**2
def r2f(t):
    return r2[0] + v_2o * (t-4) - 0.5 * g_vector * (t-4)**2

# plotting the trajectories
plt.figure(2)
plt.plot(x, y)
plt.plot(r1[:, 0], r1[:, 1], 'tab:red')
plt.plot(r2[:, 0], r2[:, 1], 'tab:purple')

# plotting the points for t = [0,4]
plt.plot([0, xf(1), xf(2), xf(3), xf(4)], [0, yf(1), yf(2), yf(3), yf(4)], 'ko',
    markersize=3)
plt.annotate('$t = 0$', (0.5, -2)) # at origin t = 0
for i, txt in enumerate(np.arange(1, 5, 1)):
    plt.annotate(txt, (xf(i+1)-.3, yf(i+1)+1))

# plot line between points and its midpoint
# plt.plot([r1f(5)[0],r2f(5)[0]], [r1f(5)[1], r2f(5)[1]], 'k--')
# into a for loop
for i, val in enumerate(np.arange(5, 10, 1)):
    plt.plot([r1f(val)[0],r2f(val)[0]], [r1f(val)[1], r2f(val)[1]], 'k-.')
    plt.plot([(r1f(val)[0]+r2f(val)[0])/2], [(r1f(val)[1]+r2f(val)[1])/2], 'ko',
            markerfacecolor='white', markersize=3)
    plt.annotate(str(val), ((r1f(val)[0]+r2f(val)[0])/2+1, (r1f(val)[1]+r2f(val)[1])
    /2-0.4))

# plot points for t = [5,9]
for i, val in enumerate(np.arange(5, 10, 1)):
    plt.plot(r1f(val)[0], r1f(val)[1], 'ko', markersize=3)
    plt.plot(r2f(val)[0], r2f(val)[1], 'ko', markersize=3)

# labels and axes
plt.legend(['$R_o$', '$R_1$', '$R_2$'])
plt.title('Grenade Trajectory after Explosion')
plt.xlabel('$X$(m)')
plt.ylabel('$Y$(m)')
plt.xticks(np.arange(0, 45, 5))
plt.show()
```

OUTPUT:

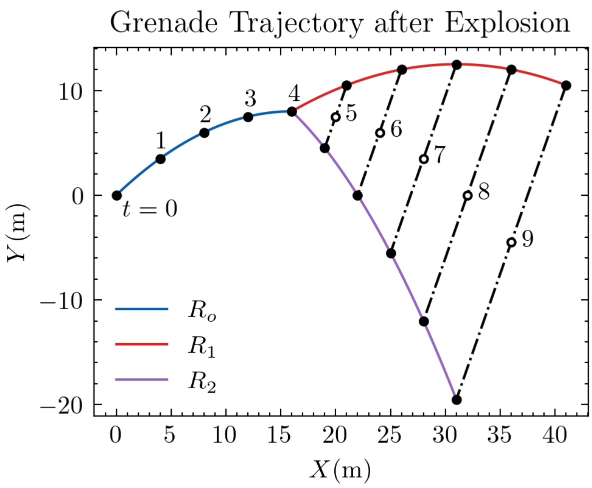


Figure 3.5: R_o is the trajectory of a grenade from time $t = [0, 4]$ before the explosion. After the grenade explodes, the two pieces follow the trajectories R_1 and R_2. The position is marked at $t = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$ and 9 s.

Since the two pieces have the same mass, the CM is at the midpoint of the line connecting the two pieces as shown in Figure 3.5. This follows the initial parabolic trajectory of the grenade before the explosion.

3.25 From the conservation of angular momentum

$$\begin{aligned} l_o &= l_f \\ mr_o^2\omega_o &= mr^2\omega \\ \omega &= \frac{r_o^2}{r^2}\omega_o \end{aligned}$$

3.27 The planets position in polar coordinates:

$$\mathbf{r} = r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}} = r\hat{\mathbf{r}}$$

(a) Given $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$, the angular momentum is

$$\begin{aligned} \boldsymbol{\ell} &= \mathbf{r} \times \mathbf{p} \\ &= m\mathbf{r} \times \dot{\mathbf{r}} \\ &= mr\hat{\mathbf{r}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}) \\ &= mr^2\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} \\ \boldsymbol{\ell} &= mr^2\dot{\phi}\hat{\mathbf{z}} \end{aligned}$$

where $\dot{\phi} = \omega$, the angular velocity. Hence, the magnitude is $\ell = mr^2\omega$.
(b) The change in area of the orbiting planet is given by the area of the triangle of base r and height $r\Delta\phi = r(\phi(t + \Delta t) - \phi(t))$ which is

$$\Delta A = \frac{1}{2}r^2\Delta\phi$$

dividing both sides by Δt and taking the limit $\Delta t \to 0$

$$\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{2}r^2\frac{\Delta\phi}{\Delta t}$$

which gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\omega = \frac{\ell}{2m}$$

The rate of change of area is constant, hence the swept areas are equal for equal changes in times.

3.29 Given the initial angular momentum of the spherical asteroid to be

$$\ell_o = \frac{2}{5}M_oR_o^2\omega_o$$

where $M = \rho V$ and the volume of the sphere $V = 4/3\pi R^3$. When $R = 2R_o$, the angular momentum is conserved:

$$\begin{aligned} \frac{2}{5}MR^2\omega &= \frac{2}{5}M_oR_o^2\omega_o \\ \omega &= \frac{V_o}{4V}\omega_o \\ \omega &= \frac{1}{32}\omega_o \end{aligned}$$

3.31 Moment of inertia as an integral

$$I = \int r^2 \,\mathrm{d}m$$

where $\mathrm{d}m = \sigma\,\mathrm{d}A$. The area density of a uniform disc is $\sigma = M/A$ where $A = \pi R^2$, the area of a circle. Using polar coordinates, change of variables gives

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$$

The moment of inertia is

$$\begin{aligned}
I &= \sigma \int_0^R \int_0^{2\pi} (r^2) r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \sigma \int_0^R r^3\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\
&= \sigma \frac{R^4}{4} 2\pi \\
I &= \frac{1}{2} M R^2
\end{aligned}$$

3.33 A uniform thin square of side $2b$ lies on the xy plane and rotates about an axis through its center and perpendicular to the square itself. The distance of the point mass from the axis is

$$r = \sqrt{x^2 + y^2}$$

where $x = y$ for a square with its center at the origin. With $dm = \sigma\,\mathrm{d}A$, and area of the square $A = (2b)^2$, the moment of inertia is

$$\begin{aligned}
I &= \sigma \int r^2\,\mathrm{d}A \\
&= \sigma \int_{-b}^{b} \int_{-b}^{b} (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{8}{3} \sigma b^4 \\
&= \frac{2}{3} M b^2
\end{aligned}$$

3.35 (a) The free-body diagram of the disk is shown in Figure 3.6.
(b) Given the moment of inertia about point P is $I_P = \frac{3}{2} M R^2$ and the external torque $\Gamma^{ext} = RMg \sin\gamma$. From conservation of anuglar momentum

$$\begin{aligned}
\dot{L} &= \Gamma^{ext} \\
I_p \dot{\omega} &= RMg \sin\gamma \\
\frac{3}{2} M R^2 \dot{\omega} &= RMg \sin\gamma \\
R\dot{\omega} &= \frac{2}{3} g \sin\gamma
\end{aligned}$$

where $R\dot{\omega} = \dot{v}$ is the angular acceleration.
(c) Applying $\dot{L} = \Gamma^{ext}$ to the rotation about the CM: Finding the frictional force from Newton's Second law only requires the component parallel to the incline

$$f = Mg \sin\gamma - M\dot{v}$$

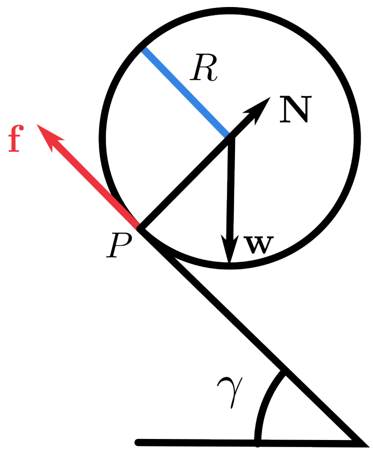


Figure 3.6: A uniform solid disk of mass M and radius R is rolling without slipping down an incline at an angle γ to the horizontal. The point of contact between the disk and incline is at P. The forces acting on the point of contact are the normal force $\mathbf{N}$, the weight $\mathbf{w}$ and the frictional force $\mathbf{f}$.

The torque about the CM is $\Gamma^{ext} = fR$. The angular acceleration is

$$\begin{aligned} fR &= I_{CM}\dot{\omega} \\ MgR\sin\gamma - MR\dot{v} &= \frac{1}{2}MR^2\dot{\omega} \\ gR\sin\gamma &= \frac{1}{2}R\dot{v} + \dot{v} \\ \dot{v} &= \frac{2}{3}g\sin\gamma \end{aligned}$$

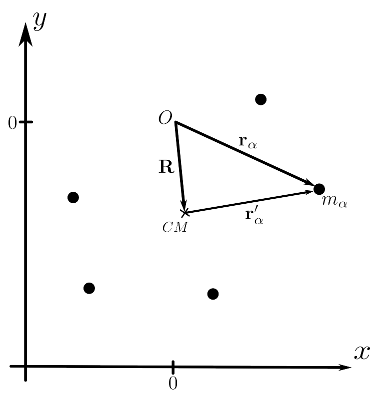


Figure 3.7: A system of N particles of masses m_α at positions $\mathbf{r}_\alpha$ relative to origin O. The center of mass is at $\mathbf{R}$ and the position of m_α relative to the CM is $\mathbf{r}'_\alpha$.

3.37 (a) From Figure 3.7, the position of the center of mass is

$$\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$$

(b) Given $\sum \mathbf{r}_\alpha = \mathbf{R}$ and the mass of the system $M = \sum m_\alpha$:

$$\begin{aligned}\sum m_\alpha \mathbf{r}'_\alpha &= \sum m_\alpha (\mathbf{r}_\alpha - \mathbf{R}) \\ &= \sum m_\alpha \mathbf{r}_\alpha - \sum m_\alpha \mathbf{R} \\ &= M\mathbf{R} - M\mathbf{R} \\ &= 0\end{aligned}$$

Obviously, the CM is at the origin if the *frame of reference* is at the CM.
(c) The angular momentum about the CM is

$$\mathbf{L} = \sum \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha = \sum m_\alpha \mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha$$

taking the time derivative

$$\begin{aligned}\dot{\mathbf{L}} &= \sum m_\alpha \dot{\mathbf{r}}'_\alpha \times \dot{\mathbf{r}}'_\alpha + \sum m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{r}}'_\alpha \\ &= \sum m_\alpha \mathbf{r}'_\alpha \times \ddot{\mathbf{r}}'_\alpha \\ &= \sum m_\alpha \mathbf{r}'_\alpha \times (\ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}}) \\ &= \sum \mathbf{r}'_\alpha \times m_\alpha \ddot{\mathbf{r}}_\alpha - \ddot{\mathbf{R}} \sum m_\alpha \mathbf{r}'_\alpha \\ &= \sum \mathbf{r}'_\alpha \times \mathbf{F}^{ext}_\alpha \\ &= \mathbf{\Gamma}^{ext}\end{aligned}$$

where the internal forces cancel out from Newton's third law. Hence, the angular momentum about the CM is equal to the external torque.

4 Energy

4.2 From the origin O to point $P = (1, 1)$ a two dimensional force $\mathbf{F} = (x^2, 2xy)$ moves a point along three paths where the work done by the force is

$$W = \int_O^P \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_O^P F_x \,\mathrm{d}x + F_y \,\mathrm{d}y$$

(a) Splitting the path into two parts $O \to Q = (1, 0)$ and $Q \to P$, we have two integrals

$$W = \int_O^Q F_x \,\mathrm{d}x + \int_Q^P F_y \,\mathrm{d}y$$

where the first integral accounts for just the x component of force $F_x = x^2$ and the second integral accounts for just the y component of force when $x = 1$; $F_y = 2(1)y$. Thus

$$W = \int_0^1 x^2 \,\mathrm{d}x + \int_0^1 2y \,\mathrm{d}y = \frac{4}{3}$$

(b) The path follows the parabola $y = x^2$ from $O \to P$. From $\mathrm{d}y = 2x \,\mathrm{d}x$ the integral can be rewritten in terms of just x

$$W = \int_0^1 x^2 \,\mathrm{d}x + \int_0^1 2x(x^2) \,\mathrm{d}y = \frac{1}{3} + \int_0^1 4x^4 \,\mathrm{d}x = \frac{17}{15}$$

(c) Path follows the parametric curve $x = t^3$ and $y = t^2$ where the differentials are: $dx = 3t^2 \,\mathrm{d}t$ and $dy = 2t \,\mathrm{d}t$. Thus the work done on the path is

$$W = \int_0^1 (t^6)(3t^2 \,\mathrm{d}t) + \int_0^1 (2t^3)(2t \,\mathrm{d}t) = \frac{1}{3} + \frac{4}{5} = \frac{19}{15}$$

4.3 Same as Problem 4.2 but with a force $\mathbf{F} = (-y, x)$ and three different paths from $P = (1, 0) \to Q = (0, 1)$.
(a) This path follows a straight line $y = 0$ from $P \to O$ and then $x = 0$ from $O \to Q$. Thus the work done is

$$W = \int_P^O F_x \,\mathrm{d}x + \int_O^Q F_y \,\mathrm{d}y = 0$$

(b) A straight line from $P \to Q$ is given by $y = -x + 1$ and the differential $dy = -dx$. Thus the work done is

$$W = \int_P^Q F_x \,\mathrm{d}x + F_y \,\mathrm{d}y = \int_1^0 (-(-x + 1)) \,\mathrm{d}x + (x)(-\,\mathrm{d}x) = \int_1^0 -1 \,\mathrm{d}x = 1$$

(c) The path of a quarter circle centered on the origin in polar coordinates is given by

$$x = r \cos\phi \qquad y = r \sin\phi$$

where $r = 1$, $\phi = 0 \to \pi/2$ and the differentials are

$$dx = \cos\phi dr - r \sin\phi \,\mathrm{d}\phi = -\sin\phi \,\mathrm{d}\phi \qquad dy = \sin\phi dr + r\cos\phi \,\mathrm{d}\phi = \cos\phi \,\mathrm{d}\phi$$

Thus the work done is

$$W = \int_P^Q F_x \,\mathrm{d}x + F_y \,\mathrm{d}y = \int_0^{\pi/2} (-\sin\phi)(-\sin\phi \,\mathrm{d}\phi) + (\cos\phi)(\cos\phi \,\mathrm{d}\phi) = \int_0^{\pi/2} \mathrm{d}\phi = \frac{\pi}{2}$$

4.5 (a) Given the force of gravity $\mathbf{F} = -mg\hat{\mathbf{y}}$ and vertical height from 1 to 2 $h = y_2 - y_1$, the work done by gravity is

$$W_g(1 \to 2) = \int_1^2 \mathbf{F} \cdot \mathrm{d}r = \int_0^h -mg\,\mathrm{d}y = -mgh$$

Since the force $\mathbf{F}$ depends only on position and the work done by is independent of the path taken, the force is conservative.
(b) The gravitational potential energy of the particle is

$$U_g(\mathbf{r}) = -W_g(0 \to \mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_0^{\mathbf{r}} -mg\,\mathrm{d}y = mgy$$

where $\mathbf{r} = y\hat{\mathbf{y}}$ is the position vector of the particle.

4.7 (a) Given the gravitational force has magnitude $F_y = -m\gamma y^2$, the work done by gravity is

$$W = \int_1^2 F_y\,\mathrm{d}y = \int_1^2 m\gamma y^2\,\mathrm{d}y = \frac{1}{3}m\gamma(y_2^3 - y_1^3)$$

The gravity is still conservative since the work done by gravity is independent of the path taken and the force depends only on position. Hence, the corresponding potential energy is

$$U_g(\mathbf{r}) = -W(0 \to \mathbf{r}) = -\int_0^y F_y \cdot \mathrm{d}y' = \frac{1}{3}m\gamma y^3$$

(b)

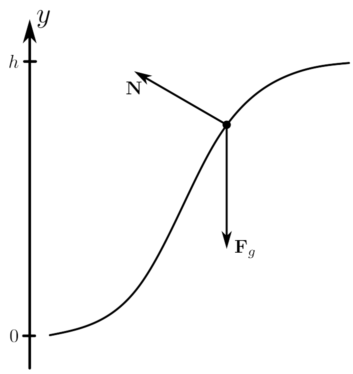


Figure 4.1: A threaded bead on a wire with two forces acting on it; The force of gravity $\mathbf{F}_g$ is conservative and the normal force $\mathbf{N}$ is non-conservative.

(c) The bead is initially released from rest at a height h. From conservation of energy:

$$E_i = E_f \tag{4.1}$$

$$\frac{1}{3}m\gamma h^3 = \frac{1}{2}mv^2 \tag{4.2}$$

$$v = \sqrt{\frac{2}{3}\gamma h^3} \tag{4.3}$$

where v is the speed of the bead at the bottom of the wire.

4.9 (a) Assuming the force of a one-dimensional spring $F = -kx$ is conservative, potential energy is

$$U(x) = -\int_0^x F\,\mathrm{d}x' = \frac{1}{2}kx^2$$

where x is the displacement of the spring from its equilibrium position.
(b) From Newton's second law, the new equilibrium position x_o is found when the spring force and gravity are equal.

$$0 = F + F_g = -kx_o + mg \implies x_o = \frac{mg}{k}$$

When $y = 0$, $U = 0$. Thus the potential energy is zero at position $x = x_o$:

$$U(x_o) = \frac{1}{2}k(x_o)^2 - mg(x_o) = 0$$

The total potential energy of the system at position $x = y + x_o$ is

$$\begin{aligned} U(x) = U_{sp} + U_g &= \frac{1}{2}k(y + x_o)^2 - mg(y + x_o) \\ &= \frac{1}{2}ky^2 + kyx_o - mgy + \frac{1}{2}kx_o^2 - mgx_o \end{aligned}$$

Since $kyx_o - mgy = 0$ and the last two terms are the potential energy at the new equilibrium $U(x_o) = 0$, the total potential energy is $U(x) = \frac{1}{2}ky^2$.

4.11 Finding the partial derivatives of the functions with constants a, b, c:
(a) $f(x, y, z) = ax^2 + bxy + cy^2$:

$$\frac{\partial f}{\partial x} = 2ax + by \qquad \frac{\partial f}{\partial y} = bx + 2cy \qquad \frac{\partial f}{\partial z} = 0$$

(b) $g(x, y, z) = \sin\big(axyz^2\big)$:

$$\frac{\partial g}{\partial x} = ayz^2 \cos\big(axyz^2\big) \qquad \frac{\partial g}{\partial y} = axz^2 \cos\big(axyz^2\big) \qquad \frac{\partial g}{\partial z} = 2axyz \cos\big(axyz^2\big)$$

(c) $h(x, y, z) = ar$ where $r = \sqrt{x^2 + y^2 + z^2}$: Since

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

The partial derivatives of h are

$$\frac{\partial h}{\partial x} = \frac{ax}{r} \qquad \frac{\partial h}{\partial y} = \frac{ay}{r} \qquad \frac{\partial h}{\partial z} = \frac{az}{r}$$

4.13 Calculating the gradient $\boldsymbol{\nabla} f$ of
(a) $f(x, y, z) = \ln(r) = \ln\Big(\sqrt{x^2 + y^2 + z^2}\Big)$:

$$\frac{\partial f}{\partial x} = \frac{x}{r^2} \qquad \frac{\partial f}{\partial y} = \frac{y}{r^2} \qquad \frac{\partial f}{\partial z} = \frac{z}{r^2}$$
$$\boldsymbol{\nabla} f = \frac{x}{r^2}\hat{\mathbf{x}} + \frac{y}{r^2}\hat{\mathbf{y}} + \frac{z}{r^2}\hat{\mathbf{z}} = \frac{\hat{\mathbf{r}}}{r}$$

(b) $f = r^n = (x^2 + y^2 + z^2)^{n/2}$ where n is a constant:

$$\frac{\partial f}{\partial x} = nr^{n-1}\frac{x}{r} = nr^{n-2}x \qquad \frac{\partial f}{\partial y} = nr^{n-2}y \qquad \frac{\partial f}{\partial z} = nr^{n-2}z$$
$$\boldsymbol{\nabla} f = nr^{n-2}x\hat{\mathbf{x}} + nr^{n-2}y\hat{\mathbf{y}} + nr^{n-2}z\hat{\mathbf{z}} = nr^{n-1}\hat{\mathbf{r}}$$

(c) $f = g(r)$ where $g(r)$ is some unspecified function of r:

$$\frac{\partial f}{\partial x} = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r} \qquad \frac{\partial f}{\partial y} = g'(r)\frac{y}{r} \qquad \frac{\partial f}{\partial z} = g'(r)\frac{z}{r}$$
$$\boldsymbol{\nabla} f = g'(r)\frac{x}{r}\hat{\mathbf{x}} + g'(r)\frac{y}{r}\hat{\mathbf{y}} + g'(r)\frac{z}{r}\hat{\mathbf{z}} = g'(r)\hat{\mathbf{r}}$$

4.15 Using the approximate formula for the change in f:

$$\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} \tag{4.35}$$

For $f(\mathbf{r}) = x^2 + 2y^2 + 3z^2$, The approximation of moving from $\mathbf{r} = (1, 1, 1)$ to $(1.01, 1.03, 1.05)$:

$$\begin{aligned}\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} &= (2x\hat{\mathbf{x}} + 4y\hat{\mathbf{y}} + 6z\hat{\mathbf{z}}) \cdot (0.01\hat{\mathbf{x}} + 0.03\hat{\mathbf{y}} + 0.05\hat{\mathbf{z}})\\ &= 0.02 + 0.12 + 0.30 = 0.44\end{aligned}$$

The exact change in f is

$$\Delta f = f(1.01, 1.03, 1.05) - f(1, 1, 1) = 0.4494$$

4.17 A charge q experiences a constant force $\mathbf{F} = q\mathbf{E}_o$ where $\mathbf{E}_o$ is a uniform electric field.
(a) The work done by the force from point 1 to 2

$$W = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = q\mathbf{E}_o \cdot (\mathbf{r}_1 - \mathbf{r}_2)$$

which is independent of the path hence it is a conservative force. Thus the potential energy is

$$U(\mathbf{r}) = -W(0 \to \mathbf{r}) = -q\mathbf{E}_o \cdot \mathbf{r}$$

(b) Checking that $\mathbf{F}$ is derivable from potential energy U:

$$\begin{aligned}\mathbf{F} = -\boldsymbol{\nabla} U &= -\frac{\partial U}{\partial x}\hat{\mathbf{x}} - \frac{\partial U}{\partial y}\hat{\mathbf{y}} - \frac{\partial U}{\partial z}\hat{\mathbf{z}}\\ &= -\frac{\partial}{\partial x}(-q\mathbf{E}_o \cdot \mathbf{x})\hat{\mathbf{x}} - \frac{\partial}{\partial y}(-q\mathbf{E}_o \cdot \mathbf{y})\hat{\mathbf{y}} - \frac{\partial}{\partial z}(-q\mathbf{E}_o \cdot \mathbf{z})\hat{\mathbf{z}}\\ &= q\mathbf{E}_o(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) = q\mathbf{E}_o\end{aligned}$$

4.18 (a) If the vector $\boldsymbol{\nabla} f$ is perpendicular to the surface through r, then (4.35) becomes

$$\mathrm{d}f = \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = 0 \tag{4.4}$$

since the dot product of perpendicular vectors is zero. Thus f is constant on the surface.
(b) Choosing a small displacement $\mathrm{d}\mathbf{r} = \epsilon\mathbf{u}$:

$$\mathrm{d}f = \boldsymbol{\nabla} f \cdot (\epsilon\mathbf{u}) = \epsilon\boldsymbol{\nabla} f \cdot \mathbf{u} = \epsilon|\boldsymbol{\nabla} f||\mathbf{u}|\cos\theta \tag{4.5}$$

the corresponding maximum value of $\mathrm{d}f$ is when $\theta = 0$ where $\mathbf{u}$ is in the same direction as $\boldsymbol{\nabla} f$.

4.19 (a) For a surface of constant f, $f = x^2 + 4y^2$ is an ellipse in the xy plane cenetered at the origin with semi-major axis $a = \sqrt{f}$ and semi-minor axis $b = \sqrt{f}/2$. Since z is unspecified, the surface is an infinitely long elliptical cylinder.
(b) The gradient of f is

$$\boldsymbol{\nabla} f = 2x\hat{\mathbf{x}} + 8y\hat{\mathbf{y}}$$

For a surface $f = 5$ at the point $(1, 1, 1)$, the gradient is $\boldsymbol{\nabla} f = 2\hat{\mathbf{x}} + 8\hat{\mathbf{y}}$. From Problem 4.18, $0 = \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r}$ describes that $\boldsymbol{\nabla} f$ is normal to this surface. Thus the unit normal vector is

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{2\hat{\mathbf{x}} + 8\hat{\mathbf{y}}}{\sqrt{68}} = \frac{1\hat{\mathbf{x}} + 4\hat{\mathbf{y}}}{\sqrt{17}}$$

or $-\hat{\mathbf{n}}$ corresponding to the opposite direction. Moving along the direction of $\mathbf{n}$ maximizes the rate of change of f.

4.20 Finding the curl, $\boldsymbol{\nabla} \times \mathbf{F}$, for the forces:
(a) $\mathbf{F} = k\mathbf{r}$

$$\begin{aligned}\boldsymbol{\nabla} \times k\mathbf{r} &= \boldsymbol{\nabla} \times (kx\hat{\mathbf{x}} + ky\hat{\mathbf{y}} + kz\hat{\mathbf{z}}) \\ &= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} = 0\end{aligned}$$

(b) $\mathbf{F} = (Ax, By^2, Cz^3)$ where A, B, C are constants:

$$\boldsymbol{\nabla} \times (Ax, By^2, Cz^3) = 0$$

(c) $\mathbf{F} = (Ay^2, Bx, Cz)$:

$$\begin{aligned}\boldsymbol{\nabla} \times (Ay^2, Bx, Cz) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ay^2 & Bx & Cz \end{vmatrix} \\ &= (0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (B - 2Ay)\hat{\mathbf{z}} = (B - 2Ay)\hat{\mathbf{z}}\end{aligned}$$

4.21 Given the gravitational force

$$\mathbf{F} = -\frac{GmM}{r^2}\hat{\mathbf{r}} = -\frac{GmM}{r^3}\mathbf{r}$$

The curl of $\mathbf{F}$ is

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \boldsymbol{\nabla} \times \frac{GmM}{r^3}\mathbf{r} \\ &= \frac{GmM}{r^3}\boldsymbol{\nabla} \times \mathbf{r} \\ &= \frac{GmM}{r^3}\boldsymbol{\nabla} \times (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \\ &= \frac{GmM}{r^3}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \frac{GmM}{r^3}(0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) = 0\end{aligned}$$

Thus the gravitational force is conservative. The potential energy is

$$U(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \frac{GmM}{r}$$

4.23 Find which of the following forces are conservative, and for those that are, find the potential and verify that $\mathbf{F} = -\boldsymbol{\nabla} U$:
(a) $\mathbf{F} = k(x, 2y, 3z)$ where k is a constant. The force is conservative when the curl is zero

$$\boldsymbol{\nabla} \times \mathbf{F} = k\boldsymbol{\nabla} \times (x, 2y, 3z) = 0$$

Thus the force is conservative and the potential energy is

$$U(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = -\frac{1}{2}k(x^2 + 2y^2 + 3z^2)$$

Verification of $\mathbf{F} = -\boldsymbol{\nabla} U$:

$$\begin{aligned}-\boldsymbol{\nabla} U &= -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \\ &= k(x, 2y, 3z) = \mathbf{F}\end{aligned}$$

(b) $\mathbf{F} = k(y, x, 0)$: The curl of $\mathbf{F}$ is

$$\boldsymbol{\nabla} \times \mathbf{F} = k\boldsymbol{\nabla} \times (y, x, 0) = 0$$

Thus the force is conservative and the potential energy is

$$U(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_0^{\mathbf{r}} F_x(x, 0)\,\mathrm{d}x' + F_y(x, y)\,\mathrm{d}y' = -k(xy)$$

where the first integral is zero since $F_x = 0$. Differentiating the potential gives

$$-\boldsymbol{\nabla}U = -\left(\frac{\partial}{\partial x}(-kxy), \frac{\partial}{\partial y}(-kxy)\right) = k(y, x) = \mathbf{F}$$

(c) $\mathbf{F} = k(-y, x, 0)$: The curl of $\mathbf{F}$ is

$$\boldsymbol{\nabla} \times \mathbf{F} = k\boldsymbol{\nabla} \times (-y, x, 0) = 2k\hat{\mathbf{z}}$$

Thus the force is not conservative.

4.25 (a)

$$\begin{aligned}\oint_\Gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_2^1 \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_1^1 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0\end{aligned}$$

(b) If $\boldsymbol{\nabla} \times \mathbf{F} = 0$,

$$\oint_\Gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}\mathbf{A} = 0$$

(c) Integrating $\mathbf{F}$ around the closed path Γ:

$$\begin{aligned}\oint_\Gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = &\int_B^{B+b} F_x(x, C, z)\,\mathrm{d}x - \int_B^{B+b} F_x(x, C+c, z)\,\mathrm{d}x \\ &+ \int_C^{C+c} F_y(B+b, y, z)\,\mathrm{d}y - \int_C^{C+c} F_y(B, y, z)\,\mathrm{d}y\end{aligned}$$

where the first two terms are the path along the top and bottom of the rectangle and the last two terms are the path along the left and right sides of the rectangle. Looking at the first two terms,

$$-\int_B^{B+b} F_x(x, C+c, z) - F_x(x, C, z)\,\mathrm{d}x = -\int_B^{B+b}\int_C^{C+c} \frac{\partial F_x(x, y, z)}{\partial y}\,\mathrm{d}y\,\mathrm{d}x = -\int_S \frac{\partial F_x}{\partial y}\,\mathrm{d}A$$

The last two terms are

$$\int_C^{C+c} F_y(B+b, y, z) - F_y(B, y, z)\,\mathrm{d}y = \int_C^{C+c}\int_B^{B+b} \frac{\partial F_y(x, y, z)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_S \frac{\partial F_y}{\partial x}\,\mathrm{d}A$$

Hence the integral is

$$\oint_\Gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\,\mathrm{d}A$$

where $\boldsymbol{\nabla} \times \mathbf{F} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$. Thus

$$\oint_\Gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}A$$

4.27 From the time-dependent PE for any fixed time t

$$U(\mathbf{r}, t) = -\int_{\mathbf{r}_o}^{\mathbf{r}} \mathbf{F}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}' \tag{4.48}$$

and the force as a gradient of PE:

$$-\boldsymbol{\nabla} U(\mathbf{r}, t) = \frac{\partial U}{\partial r} \int \mathbf{F}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}' + \frac{\partial U}{\partial t} \int \mathbf{F}(\mathbf{r}', t) \cdot \mathrm{d}\mathbf{r}'$$

Since t is fixed, and by the fundamental theorem of calculus

$$-\boldsymbol{\nabla} U(\mathbf{r}, t) = \mathbf{F}(\mathbf{r}, t)$$

For closer inspection, the change in U from a small displacement $\mathrm{d}\mathbf{r}$ and

$$\mathrm{d}U = U(\mathbf{r} + \mathrm{d}\mathbf{r}\,, t + \mathrm{d}t) - U(\mathbf{r}, t)$$

rewritten as partial derivatives

$$\mathrm{d}U = \boldsymbol{\nabla} U \cdot \mathrm{d}\mathbf{r} + \frac{\partial U}{\partial t}\,\mathrm{d}t$$

Given the force is conservative $\mathbf{F} = -\boldsymbol{\nabla} U$ and the change in T is defined as $\mathrm{d}T = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ the work done by the force in the displacement $\mathrm{d}\mathbf{r}$:

$$\begin{aligned} \mathrm{d}U &= -\,\mathrm{d}T + \frac{\partial U}{\partial t}\,\mathrm{d}t \\ \mathrm{d}(U + T) &= \frac{\partial U}{\partial t}\,\mathrm{d}t \end{aligned}$$

Hence, the total mechanical energy is conserved only when $\partial U / \partial t = 0$ and false otherwise.

4.28 (a) From conservation of energy, the total mechanical energy is

$$E = T + U = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

Solving for velocity $\dot{x}$:

$$\dot{x} = \pm\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2\right)}$$

(b) At the spring's max displacement $x_{max} = A$, the kinetic energy is zero hence a total energy

$$E = U = \frac{1}{2} kA^2$$

subbing into the equation for velocity

$$\dot{x} = \pm\sqrt{\frac{2}{m}\left(\frac{1}{2}kA^2 - \frac{1}{2}kx^2\right)} = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

Solving for the time to go from the origin $x = 0$ at time $t = 0$ to a position x with (4.58):

$$t = \int_0^x \frac{\mathrm{d}x'}{\dot{x}(x')} = \sqrt{\frac{m}{k}} \int_0^x \frac{\mathrm{d}x'}{\sqrt{(A^2 - x'^2)}} = \sqrt{\frac{m}{k}}\, \arcsin\!\left(\frac{x}{A}\right)$$

Solving x as a function of t:

$$x = A \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

where the function $\sin(\omega t)$ has a period $T = 2\pi/\omega = 2\pi\sqrt{m/k}$.

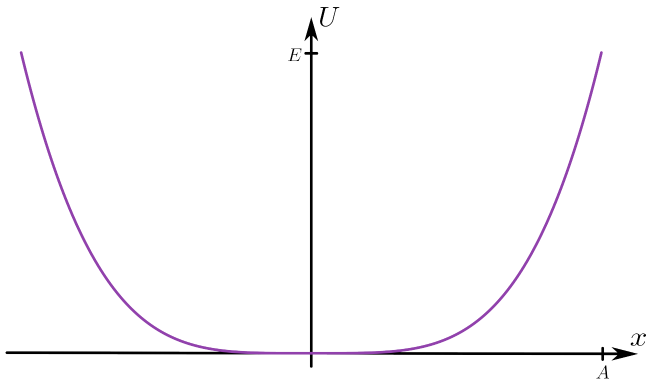


Figure 4.2: A linear system of a mass m with potential energy $U = kx^4$ force F_f.

4.29 (a) Figure 4.2 shows the mass moves initially at $t = 0$ from $x = 0$ to the max displacement $x = A$ where $E = U = kA^4$. Then it oscillates to and fro from $x = A$ to $x = -A$.
(b) Using (4.58) to find the time it takes to go from $x = 0$ to $x = A$:

$$t = \sqrt{\frac{m}{2}} \int_{x_o}^{x} \frac{\mathrm{d}x'}{E - U(x')} = \sqrt{\frac{m}{2k}} \int_0^A \frac{\mathrm{d}x'}{\sqrt{A^4 - x'^4}} \tag{4.6}$$

where the period of oscillation $\tau = 4t$ describes the mass moving from $x = 0$ to $x = A$ and back to $x = -A$ and finally back to the initial posistion $x = 0$. Hence the period of oscillation

$$\tau = 4\sqrt{\frac{m}{2k}} \int_0^A \frac{\mathrm{d}x'}{\sqrt{A^4 - x'^4}} = \sqrt{\frac{8m}{k}} \int_0^A \frac{\mathrm{d}x'}{\sqrt{A^4 - x'^4}}$$

(c) Changing the variable of integration to $u = x'/A$, $\mathrm{d}u = \mathrm{d}x'/A$: the limits of integration are $u = 0 \to 1$ and the integrand is $1/\sqrt{1/A^4(1 - x'^4/A^4)} = 1/(A^2\sqrt{1 - u^4})$. Substituting into the integral:

$$\tau = \frac{1}{A}\sqrt{\frac{8m}{k}} \int_0^1 \frac{\mathrm{d}u}{\sqrt{1 - u^4}} = \frac{1}{A}\gamma$$

where γ is a constant independent of A. This clearly shows that that period τ is inversely proportional to the amplitude A, or simply $\tau \propto 1/A$.
(d) For the case $m = k = A = 1$, the period of oscillation is

$$\tau = \sqrt{8} \int_0^1 \frac{\mathrm{d}u}{\sqrt{1 - u^4}}$$

where the integral is numerically evaluated as 1.31 thus $\tau = 3.71$.

4.31 (a) The total energy E of the two masses in the Atwood machine is

$$E = T_1 + T_2 + U_1 + U_2 = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - m_1gx_1 - m_2gx_2$$

where $x_1 = -x_2 = x$. The total energy is then

$$E = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 - m_1gx + m_2gx = \frac{1}{2}\dot{x}^2(m_1 + m_2) + (m_2 - m_1)gx$$

(b) Differentiating E with respect to time:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (m_1 + m_2)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x} = 0$$

or rewritten as

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

Finding the equation of motion through Newton's second law: the equation of motion for the two masses are

$$m_1\ddot{x}_1 = m_1 g - T_1$$
$$m_2\ddot{x}_2 = T_2 - m_2 g$$

since the tensions $T_1 = T_2 = T$ are equal. Adding the two equations together

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

Hence the equation of motions are the same.

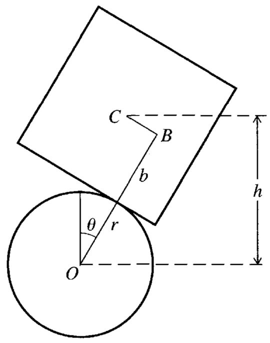


Figure 4.3: A cube, of side $2b$ and center C, placed on a fixed cylinder of radius r and center O. The cube is constrained to roll from side to side without slipping on the cylinder.

4.33 (a) The forces that constrains the cube to roll without slipping is the frictional force and the normal force which do no work. With the origin centered at O, the potential energy of the cube is $U = mgh$ where the heigh h is the vertical components of the lines $OB = (r + b)$ and $BC = r\theta$, the distance the cube rolls from the top of the cylinder:

$$U(\theta) = mgh = mg[(r + b)\cos\theta + r\theta\sin\theta] \tag{4.59}$$

(b) Choosing $r = m = g = 1$, the plot of $U(\theta)$ is shown in Figure 4.4.

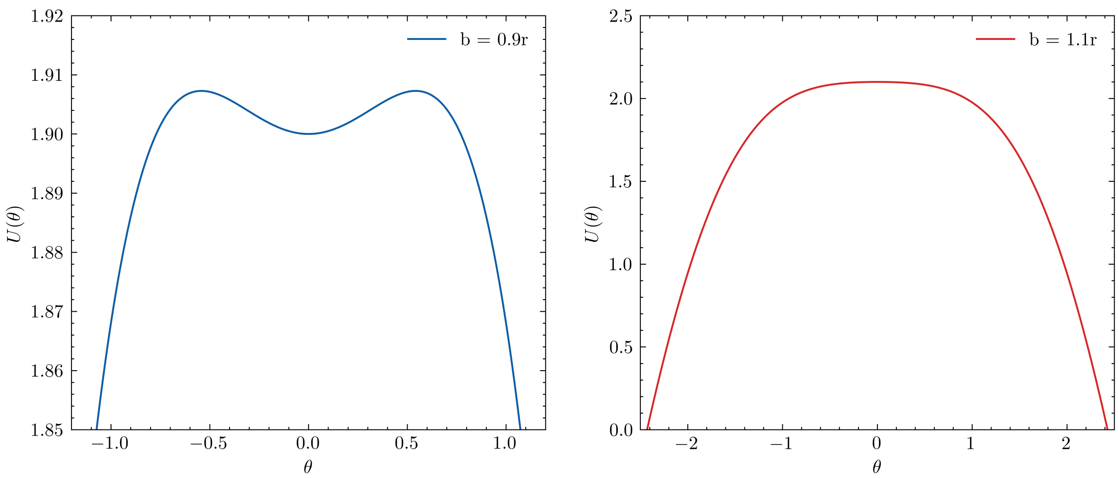


Figure 4.4: The potential energy $U(\theta)$ of the cube as a function of θ.

(c) The equilibrium position $\theta = 0$ is stable for $b = 0.9r$, or when $b < r$, and unstable for $b = 1.1r$ when $b > r$. For the case $b = 0.9r$, there are two unstable equilibrium points further away at the two maximum points in Figure 4.4.

4.35 The Atwood machine of Figure 4.3 now has a pulley of radius R and moment of inertia I
(a) Given the kinetic energy of the pulley is $T_p = \frac{1}{2}I\omega^2$, the total energy is

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\omega^2 - m_1gx_1 - m_2gx_2$$

subbing $x_1 = -x_2 = x$ and $\omega = \dot{x}/R$:

$$E = \frac{1}{2}\dot{x}^2(m_1 + m_2 + I/R^2) + (m_2 - m_1)gx$$

Differentiating E with respect to time:

$$0 = (m_1 + m_2 + I/R^2)\dot{x}\ddot{x} + (m_2 - m_1)g\dot{x}$$

or rewritten as

$$(m_1 + m_2 + I/R^2)\ddot{x} = (m_1 - m_2)g$$

Finding the equation of motion through Newton's second law:

$$\begin{aligned} m_1\ddot{x}_1 &= m_1g - T_1 \\ m_2\ddot{x}_2 &= T_2 - m_2g \\ I\dot{\omega} &= T_1R - T_2R \end{aligned}$$

where $\dot{\omega} = \ddot{x}/R$ and thus $I\ddot{x}/R^2 = T_1 - T_2$. Adding the first two equations together

$$\begin{aligned} (m_1 + m_2)\ddot{x} &= (m_1 - m_2)g - (T_1 - T_2) \\ (m_1 + m_2)\ddot{x} + (T_1 - T_2) &= (m_1 - m_2)g \\ (m_1 + m_2 + I/R^2)\ddot{x} &= (m_1 - m_2)g \end{aligned}$$

Hence the equation of motions are the same.

4.37 (a) Letting the potential energy $U = mgh$ be zero when $\phi = 0$, the height of mass M is $h = R(1 - \cos\phi)$, and the height of mass m is lowered by the arc length $h = -R\phi$. Therefore, the total potential energy is

$$U(\phi) = MgR(1 - \cos\phi) - mgR\phi$$

(b) Differentiating U with respect to ϕ to find a possible equilibrium position:

$$\frac{\mathrm{d}U}{\mathrm{d}\phi} = MgR\sin\phi - mgR = 0$$

Thus the equilibrium position is when $\sin\phi = m/M$. Since the range of $\sin\phi$ is $[-1, 1]$, the equilibrium position is only possible when $m/M \in [-1, 1]$. Therefore, when $m > M$ there is no equilibrium position. For the case $m = M$, the equilibrium position is $\phi = \pi/2$. When $m < M$, there are two equilibrium positions where mass M deviates symmetrically from $\pi/2$ by an angle ϕ_1 from both directions. To find stability, we differentiate again:

$$\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2} = MgR\cos\phi$$

where the second derivative is positive when $\phi < \pi/2$ and is negative when $\phi > \pi/2$ hence the equilibrium position for the case of $m < M$ the lower equilibrium position is stable and the upper equilibrium position is unstable. For $m = M$, the equilibrium position is unstable since the second derivative is zero. In terms of torque, the clockwise torque on mass M must be equal and opposite to the counterclockwise torque on mass m for equilibrium.
(c) Plotting $U(\phi)$ with $g = R = M = 1$ is shown in Figure 4.5. For the case $m = 0.7M$, when the

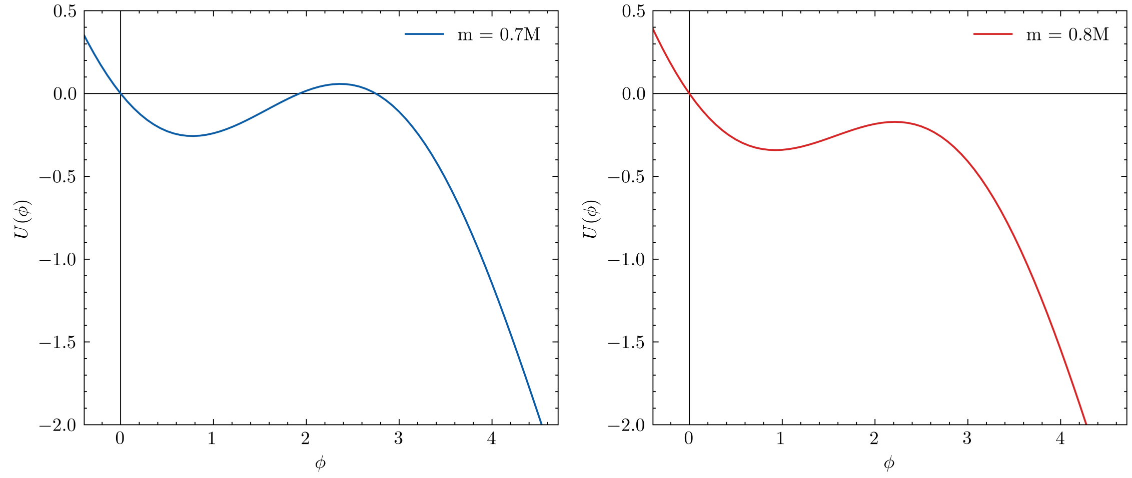


Figure 4.5: The potential energy $U(\phi)$ of the system as a function of ϕ for the cases $m = 0.7M$ and $m = 0.8M$.

system is released from rest at $\phi = 0$, the masses will oscillate about the lower equilibrium position $\phi = \arcsin(0.7) = 0.78$ rad. When $m = 0.8M$, there is no equilibrium position and the masses will keep rotating ccw.
(d) The critical value of m/M is when the potential energy at the second equilibrium position is zero, or when the x axis is tangent to the relative maximum of the graph.

$$0 = MgR(1 - \cos\phi) - mgR\phi$$
$$\cos\phi = 1 - \frac{m}{M}\phi$$

combining with $\sin\phi = m/M$:

$$\cos\phi = 1 - \phi\sin\phi$$

The numerical solution is $\phi = 2.33$ rad, so the critical value is $m/M = \sin\phi = 0.72$.

4.39 (a) Problem 4.28a: The potential energy of a simple pendulum is given by

$$U(\phi) = mgl(1 - \cos\phi)$$

The total energy is

$$E = T + U = \frac{1}{2}ml^2\dot{\phi}^2 + mgl(1 - \cos\phi)$$

where linear velocity $v = l\dot{\phi}$. At the max amplitude Φ, $T = 0$ so the total energy is $E = mgl(1 - \cos\Phi)$. $\dot{\phi}$ as a function of ϕ is

$$\dot{\phi} = \sqrt{\frac{2}{ml^2}}\sqrt{(E - mgl(1 - \cos\phi))} = \sqrt{\frac{2g}{l}}\sqrt{(1 - \cos\Phi) - (1 - \cos\phi)}$$

Using the form $t = \int \mathrm{d}\phi \,/\dot{\phi}$

$$t = \sqrt{\frac{l}{2g}}\int_0^{\Phi}\frac{\mathrm{d}\phi}{\sqrt{(1 - \cos\Phi) - (1 - \cos\phi)}}$$

since $\tau = 4t$ the period is

$$\tau = 4\sqrt{\frac{l}{2g}}\int_0^\Phi \frac{\mathrm{d}\phi}{\sqrt{(1-\cos\Phi)-(1-\cos\phi)}}$$

subbing the period for small oscillations $\tau_o = 2\pi\sqrt{l/g}$:

$$\tau = \tau_o\frac{\sqrt{2}}{\pi}\int_0^\Phi \frac{\mathrm{d}\phi}{\sqrt{(1-\cos\Phi)-(1-\cos\phi)}}$$

from the half angle identity $1-\cos(\phi) = 2\sin^2(\phi/2)$ the square root in the integrand is

$$\sqrt{(1-\cos\Phi)-(1-\cos\phi)} = \sqrt{2}\sqrt{\sin^2(\Phi/2)-\sin^2(\phi/2)}$$

therefore,

$$\tau = \tau_o\frac{1}{\pi}\int_0^\Phi \frac{\mathrm{d}\phi}{\sqrt{\sin^2(\Phi/2)-\sin^2(\phi/2)}}$$

with the substitution $\sin(\phi/2) = Au$ where $A = \sin(\Phi/2)$ and $\mathrm{d}\phi = \mathrm{d}u\,2A/\cos(\phi/2) = 2A/\sqrt{1-A^2u^2}$ from the relation

$$\sin^2(\phi/2) = A^2u^2 = 1-\cos^2(\phi/2) \to \cos(\phi/2) = \sqrt{1-A^2u^2}$$

The limits of integration are from $u = 0 \to u = 1$. The square root in the integrand simplifies to $\sqrt{A^2-A^2u^2} = A\sqrt{1-u^2}$, and thus the period is

$$\tau = \tau_o\frac{2}{\pi}\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^2}\sqrt{1-A^2u^2}}$$

and ignoring the last square root for small amplitudes and using the integral $\int \mathrm{d}u\,/\sqrt{1-u^2} = \arcsin(u)$:

$$\tau = \tau_o\frac{2}{\pi}\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \tau_o = 2\pi\sqrt{\frac{l}{g}}$$

(c) For a better approximation, using the binomial expansion $1/\sqrt{1-A^2u^2} \approx 1+\frac{1}{2}A^2u^2$:

$$\tau = \tau_o\frac{2}{\pi}\int_0^1 \frac{1+\frac{1}{2}A^2u^2}{\sqrt{1-u^2}}\,\mathrm{d}u = \tau_o + \tau_o\frac{A^2}{\pi}\int_0^1 \frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u$$

the integral is evaluated using the substitution $u = \sin(\theta)$ and $\mathrm{d}u = \cos(\theta)\,\mathrm{d}\theta$:

$$\int \frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u = \int \frac{\sin^2(\theta)}{\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,\mathrm{d}\theta = \int \sin^2(\theta)\,\mathrm{d}\theta$$

using the half angle identity $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$:

$$\int \sin^2(\theta)\,\mathrm{d}\theta = \frac{1}{2}\int (1-\cos(2\theta))\,\mathrm{d}\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)$$

subbing back $u = \sin(\theta)$:

$$\int_0^1 \frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u = \frac{1}{2}\arcsin(u) - \frac{1}{4}u\sqrt{1-u^2}\Big|_0^1 = \frac{\pi}{4}$$

subbing back to the original equation and using $A = \sin(\Phi/2)$ once more gives

$$\tau = \tau_o + \tau_o\frac{A^2}{\pi}\frac{\pi}{4} = \tau_o\left(1+\frac{1}{4}\sin^2(\Phi/2)\right)$$

Q.E.D
For $\Phi = 45°$, the second term approximation gives $\tau = 1.037\tau_o$ or a 3.7% correction compared to the small angle approx $\tau = \tau_o$ and a 0.3% error compared to the exact solution $\tau = 1.040\tau_o$.

4.41 Given $U = kr^n$, taking the derivative with respect to r:

$$F = -\frac{\partial U}{\partial r} = -knr^{n-1} = -\frac{n}{r}U$$

where the magnitude of force is also equivalent to the centripetal force $F = -mv^2/r$ (the sign specifies the inward direction). Therefore,

$$-\frac{mv^2}{r} = -\frac{n}{r}U$$
$$mv^2 = nU$$

Subbing into the KE equation

$$T = \frac{1}{2}mv^2 = \frac{n}{2}U$$

4.43 (a) For a central and spherically symmetric force

$$\mathbf{F} = f(r)\hat{\mathbf{r}} = \frac{f(r)}{r}\mathbf{r}$$

Taking the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{f(r)x}{r} & \frac{f(r)y}{r} & \frac{f(r)z}{r} \end{vmatrix}$$

looking at the x component of the curl

$$(\nabla \times \mathbf{F})_x = \frac{\partial}{\partial y}\left(\frac{f(r)z}{r}\right) - \frac{\partial}{\partial z}\left(\frac{f(r)y}{r}\right)$$

Since $\frac{\partial z}{\partial y} = \frac{\partial y}{\partial z} = 0$ the equation above is rewritten as

$$z\frac{\partial}{\partial y}\left(\frac{f(r)}{r}\right) - y\frac{\partial}{\partial z}\left(\frac{f(r)}{r}\right)$$

Next, from the relation

$$r = \sqrt{x^2 + y^2 + z^2}$$

and differentiating gives

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

and using the chain rule

$$\frac{\partial f(r)}{\partial y} = f'(r)\frac{y}{r}$$

and finally the quotient rule

$$\frac{\partial}{\partial y}\left(\frac{f(r)}{r}\right) = \frac{1}{r^2}\left(rf'(r)\frac{y}{r} - f(r)\frac{y}{r}\right) = \frac{y}{r^3}(rf'(r) - f(r))$$

therefore, the x component of the curl is

$$(\nabla \times \mathbf{F})_x = \frac{zy}{r^3}(rf'(r) - f(r)) - \frac{yz}{r^3}(rf'(r) - f(r)) = 0$$

Similarly, the y and z components of the curl are also zero. Hence, a central and spherically symmetric force is irrotational by $\nabla \times \mathbf{F} = 0$ and thus conservative.

(b) The curl in spherical polar is

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ (F_r) & r(F_\theta) & r\sin\theta(F_\phi) \end{vmatrix}$$

where in this case the central force $\mathbf{F} = f(r)\hat{\mathbf{r}}$ implies that the polar and azimuthal components are $F_\theta = F_\phi = 0$:

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f(r) & 0 & 0 \end{vmatrix} = \frac{1}{r^2 \sin\theta}\left(0 + r\hat{\boldsymbol{\theta}}\frac{\partial f(r)}{\partial \phi} - r\sin\theta\hat{\boldsymbol{\phi}}\frac{\partial f(r)}{\partial \theta}\right)$$

where the partial derivatives are

$$\frac{\partial r}{\partial \phi} = \frac{\partial \theta}{\partial \phi} = 0$$

therefore

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta}\left(0 + r\hat{\boldsymbol{\theta}}f'(r)\frac{\partial r}{\partial \phi} - r\sin\theta\hat{\boldsymbol{\phi}}f'(r)\frac{\partial r}{\partial \theta}\right) = 0$$

4.45 Work done by $\mathbf{F}(r) = f(\mathbf{r})\hat{\mathbf{r}}$ from point A to B seperated by an infinitesimal distance $\mathrm{d}r$ where $r_B = r_A + \mathrm{d}r$:

$$W_{AB} = f(\mathbf{r})\hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{r}$$

The work along both paths are equivalent due to the conservative nature of the force:

$$W_{ACB} = W_{ADB} \quad \text{or} \quad W_{AC} + W_{CB} = W_{AD} + W_{DB}$$

Since the magnitude of force is perpendicular on the paths AD and CB, the work done is zero:

$$W_{AD} = W_{CB} = 0$$

and the work done along the paths AC and DB are equivalent:

$$W_{AC} = W_{DB}$$
$$f(\mathbf{r}_A)\,\mathrm{d}r = f(\mathbf{r}_D)\,\mathrm{d}r$$

Since the two position vectors $\mathbf{r}_A$ and $\mathbf{r}_D$ have the same magnitude

$$|\mathbf{r}_A| = |\mathbf{r}_D| = r$$

the magnitude of the force is the same at both points:

$$f(\mathbf{r}_A) = f(\mathbf{r}_D) = f(r)$$

and the force is spherically symmetric.

4.47 From the conservation of kinetic energy

$$T_i = T_f$$
$$m_1 v_1^2 + m_2 v_2^2 = m_1 v_1'^2 + m_2 v_2'^2$$
$$m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)$$

and the conservation of momentum

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$
$$m_1(v_1 - v_1') = m_2(v_2' - v_2)$$

dividing the two equations gives

$$v_1 + v_1' = v_2 + v_2'$$
$$v_1 - v_2 = -(v_1' - v_2')$$

4.49 Given $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the potential energy of the two particle system is

$$U = \frac{\gamma}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\gamma}{|\mathbf{r}|} = \frac{\gamma}{r}$$

and the force on each particle is

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = -\frac{\gamma}{r^3}\mathbf{r}$$

Taking the gradient with respect to $\mathbf{r}_1$

$$-\boldsymbol{\nabla}_1 U = -\frac{\partial U}{\partial x_1}\hat{\mathbf{x}} - \frac{\partial U}{\partial y_1}\hat{\mathbf{y}} - \frac{\partial U}{\partial z_1}\hat{\mathbf{z}}$$

looking at just the x component

$$-\frac{\partial U}{\partial x_1} = -\frac{\partial}{\partial x_1}\frac{\gamma}{r} = \frac{\gamma}{r^2}\frac{\partial}{\partial x_1}r = \frac{\gamma}{r^2}\frac{\partial}{\partial x_1}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

and using the chain rule

$$\frac{\partial}{\partial x_1}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \frac{x_1 - x_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}$$

therefore

$$-\frac{\partial U}{\partial x_1} = \frac{\gamma}{r^3}(x_1 - x_2)$$

and similarly for the y and z components

$$-\frac{\partial U}{\partial y_1} = \frac{\gamma}{r^3}(y_1 - y_2), \qquad -\frac{\partial U}{\partial z_1} = \frac{\gamma}{r^3}(z_1 - z_2)$$

combining the three components

$$-\boldsymbol{\nabla}_1 U = \frac{\gamma}{r^3}\mathbf{r}$$

which is the same as the force on particle 1 $\mathbf{F}_{12}$. Similarly, the force on particle 2 is

$$-\boldsymbol{\nabla}_2 U = -\frac{\gamma}{r^3}\mathbf{r} = \mathbf{F}_{21}$$

where the negative sign arises from the partial derivative step.

4.51 With potential energy defined as $U_{34} = U_{34}(\mathbf{r}_3 - \mathbf{r}_4)$, the four particle system has a potential energy

$$\begin{aligned} U = {} & U_{12}(\mathbf{r}_1 - \mathbf{r}_2) + U_{13}(\mathbf{r}_1 - \mathbf{r}_3) + U_{14}(\mathbf{r}_1 - \mathbf{r}_4) \\ & + U_{23}(\mathbf{r}_2 - \mathbf{r}_3) + U_{24}(\mathbf{r}_2 - \mathbf{r}_4) + U_{34}(\mathbf{r}_3 - \mathbf{r}_4) \\ & + U_1^{ext}(\mathbf{r}_1) + U_2^{ext}(\mathbf{r}_2) + U_3^{ext}(\mathbf{r}_3) + U_4^{ext}(\mathbf{r}_4) \end{aligned}$$

The internal forces between a pair of particles are defined as

$$\mathbf{F}_{ij} = -\boldsymbol{\nabla}_i U_{ij} \quad \text{and} \quad \mathbf{F}_{ji} = -\boldsymbol{\nabla}_j U_{ij}$$

and the external force on particle α depends only on $\mathbf{r}_\alpha$:

$$\mathbf{F}_\alpha^{ext} = -\boldsymbol{\nabla}_\alpha U_\alpha^{ext}(\mathbf{r}_\alpha)$$

when $\boldsymbol{\nabla}_3$ acts on U the only the terms with $\mathbf{r}_3$ are affected:

$$-\boldsymbol{\nabla}_3 U = -\boldsymbol{\nabla}_3(U_{13} + U_{23} + U_{34} + U_3^{ext}) = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{ext} = \mathbf{F}_3^{net}$$

where $\mathbf{F}_3^{net}$ is the net force on particle 3.

4.53 An electron of mass m and charge $-e$ in circular orbit of radius r around a fixed proton of charge $+e$ where the Coulombic force has magnitude $F = ke^2/r^2$.
(a) The magnitude of force in terms of centripetal acceleration is

$$F = -\frac{mv^2}{r} = \frac{ke^2}{r^2} \quad \text{or} \quad mv^2 = \frac{ke^2}{r}$$

and the potential energy is

$$U = -\int F \cdot \mathrm{d}r = -\int \frac{ke^2}{r^2}\,\mathrm{d}r = -\frac{ke^2}{r}$$

the kinetic energy is then

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\frac{ke^2}{r} = -\frac{1}{2}U$$

and the total energy is

$$E = T + U = -\frac{1}{2}U + U = \frac{1}{2}U$$

(b) After electron 1 is knocked out of orbit by electron 2, electron 2 is in orbit of radius r'. The total energy of the system is

$$E = T_1 + T_2 + U_{12} + U_1 + U_2$$

(c) Long before the collision, electron 2 is an infinite distance away from electron 1 and thus

$$E_i = T_1 + T_2 + U_1$$

Long after the collision

$$E' = T_1' + T_2' + U_2'$$

Since the total energy is conserved

$$\begin{aligned}
E_i &= E' \\
T_1 + T_2 + U_1 &= T_1' + T_2' + U_2' \\
T_1' &= T_1 + T_2 + U_1 - T_2' - U_2' \\
&= -\frac{1}{2}U_1 + T_2 + U_1 + \frac{1}{2}U_2' - U_2' \\
&= T_2 + \frac{1}{2}(U_1 - U_2') \\
&= T_2 + \frac{1}{2}(-ke^2/r + ke^2/r') \\
T_1' &= T_2 + \frac{1}{2}ke^2\left(\frac{1}{r'} - \frac{1}{r}\right)
\end{aligned}$$