1 Vector Analysis

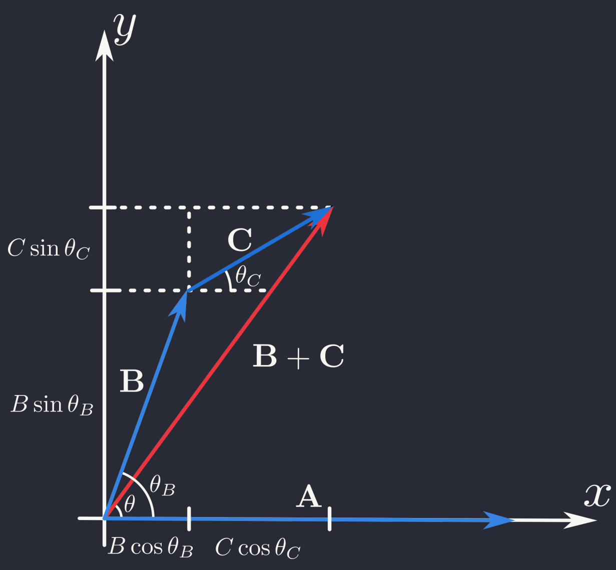

Figure 1.1: Three Coplanar Vectors

1.1 (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$
$$A(B+C)\cos\theta = AB\cos\theta_B + AC\cos\theta_C$$

Since $B\cos\theta_B + B\cos\theta_C = (B+C)\cos\theta$ from Figure 1.1, the distriubtive property holds true. The cross product also holds true since $B\sin\theta_B + B\sin\theta_C = (B+C)\sin\theta$, and multiplying by A on both sides gives the same result as the distriubtive property:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$A(B+C)\sin\theta = AB\sin\theta_B + AC\sin\theta_C$$

(b) In the general case in three-dimensional space, each vector has three components: $\mathbf{A} = (A_x, A_y, A_z)$. Therefore,

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot(B_x+C_x, B_y+C_y, B_z+C_z) \\
&= A_x(B_x+C_x) + A_y(B_y+C_y) + A_z(B_z+C_z) \\
&= A_xB_x + A_yB_y + A_zB_z + A_xC_x + A_yC_y + A_zC_z \\
&= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}
\end{aligned}$$

1.2 Setting $\mathbf{A}=\mathbf{B}=(1,1,1)$ and $\mathbf{C}=(1,1,-1)$:

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\times\mathbf{C} &\stackrel{?}{=} \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \\
0 &\stackrel{?}{=} (1,1,1)\times[(1,1,1)\times(1,1,-1)] \\
0 &\stackrel{?}{=} (1,1,1)\times(-2,2,0) \\
0 &\neq (-2,-2,4)
\end{aligned}$$

where the cross product of parallel vectors $\mathbf{A}\times\mathbf{B}=0$. Therefore, the cross product is not associative.

1.3 Taking the dot product of a unit cube's body diagonals $\mathbf{A} = (1, 1, 1)$, $\mathbf{B} = (1, 1, -1)$:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= AB\cos\theta \\ 1 &= 3\cos\theta \\ \theta &= \arccos 1/3 \approx 70.53^\circ\end{aligned}$$

1.4 The cross product of two vectors coplanar to the shaded plane—$\mathbf{A} = (-1, 2, 0)$, $\mathbf{B} = (-1, 0, 3)$—is parallel to the normal unit vector $\hat{\mathbf{n}}$ of the plane:

$$\mathbf{A}\times\mathbf{B} = \mathbf{C}$$

$$\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3\end{vmatrix} = (6, 3, 2)$$

where $\hat{\mathbf{n}} = \mathbf{C}/C$, and $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$. Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

1.5 Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z\end{vmatrix} \\ &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x\end{vmatrix}\end{aligned}$$

where the x component is $A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$. Similarly,

$$\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)$$

where x component simplifes to

$$B_x(\cancel{A_xC_x} + A_yC_y + A_zC_z) - C_x(\cancel{A_xB_x} + A_yB_y + A_zB_z) = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

the same is done for the y and z components. Therefore, the "BAC–CAB" rule holds true.

1.6

$$\begin{aligned}[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})] + [\mathbf{B}\times(\mathbf{C}\times\mathbf{A})] + [\mathbf{C}\times(\mathbf{A}\times\mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A}\cdot\mathbf{C})} - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) + \mathbf{C}(\mathbf{B}\cdot\mathbf{A}) = 0 \\ &\quad - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{A}(\mathbf{C}\cdot\mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C}\cdot\mathbf{A})}\end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (\mathbf{A}\times\mathbf{B})\times\mathbf{C} \\ \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A}\cdot\mathbf{B}) + \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C}\times\mathbf{A}) = (\mathbf{C}\times\mathbf{A})\times\mathbf{B}\end{aligned}$$

For the relation to hold true, either the vectors $\mathbf{A}$ and $\mathbf{C}$ are parallel ($\mathbf{A}\times\mathbf{A} = 0$) or $\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{C}$ ($\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{C} = 0$).

1.7 Finding the seperation vector $\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' = (4, 6, 8) - (2, 8, 7) = (2, -2, 1)$$
$$\mathscr{r} = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$
$$\hat{\boldsymbol{\mathscr{r}}} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$$

1.8 (a)

$$\begin{aligned}\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z &= (A_y\cos\phi + A_z\sin\phi)(B_y\cos\phi + B_z\sin\phi)\\ &\quad + (-A_y\sin\phi + A_z\cos\phi)(-B_y\sin\phi + B_z\cos\phi)\end{aligned}$$
$$= A_yB_y\cos^2\phi + A_zB_z\sin^2\phi + \cancel{A_yB_z\sin\phi\cos\phi} + \cancel{A_zB_y\sin\phi\cos\phi}$$
$$+A_yB_y\sin^2\phi - \cancel{A_yB_z\sin\phi\cos\phi} - \cancel{A_zB_y\sin\phi\cos\phi} + A_zB_z\cos^2\phi$$
$$= A_yB_y(\sin^2\phi + \cos^2\phi) + A_zB_z(\sin^2\phi + \cos^2\phi)$$
$$\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z = A_yB_y + A_zB_z$$

(b) To preserve length $|\bar{A}| = |A|$. Squaring both sides and expanding gives

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_x^2 + A_y^2 + A_z^2$$

in summation form,

$$\sum_{i=1}^{3}\bar{A}_i\bar{A}_i = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}R_{ij}A_j\right)\left(\sum_{k=1}^{3}R_{ik}A_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}R_{ij}R_{ik}A_jA_k$$

For the length to be preserved, the indices j and k must be equal. Therefore,

$$R_{ij}R_{ik} = \delta_{jk}$$

where δ_{ij} is the Kronecker delta. Thus the length is preserved if the rotation matrix is orthogonal, i.e.

$$R_{ij}R_{ik} = (R^T)_{ji}R_{ik} = \delta_{jk} \quad \text{or} \quad R^TR = I$$

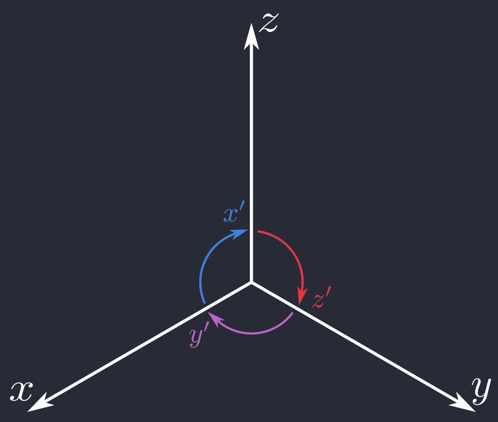

Figure 1.2: Rotation of 120° about an axis through the origin and point $(1, 1, 1)$

1.9 From Figure 1.2, the rotation is equivalent to changing the position of the basis vectors $\hat{\mathbf{x}} \to \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \to \hat{\mathbf{y}}$. Therefore, the rotation matrix is a permutation matrix:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1.10 (a) Under a **translation** of coordinates $\bar{y} = y - a$, the origin O and terminus $A = (x, y, z)$ of some vector are translated to

$$\begin{aligned} O &\to O' = (0, -a, 0) \\ A &\to A' = (x, y - a, z) \end{aligned}$$

therefore, the translated vector is

$$\overline{O'A'} = (x, y - a, z) - (0, -a, 0) = (x, y, z) = \overline{OA} = \mathbf{A}$$

which is the same as the original vector, so

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) Under **inversion** of coordinates, only the terminus changes

$$\begin{aligned} O &\to O' = (0, 0, 0) \\ A &\to A' = (-x, -y, -z) \end{aligned}$$

therefore, the inverted vector is

$$\overline{O'A'} = (-x, -y, -z) \quad \text{or} \quad \begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

(c) The components of cross product under inversion

$$\bar{\mathbf{A}} \times \bar{\mathbf{B}} = \begin{pmatrix} \bar{A}_y \bar{B}_z - \bar{A}_z \bar{B}_y \\ \bar{A}_z \bar{B}_x - \bar{A}_x \bar{B}_z \\ \bar{A}_x \bar{B}_y - \bar{A}_y \bar{B}_x \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}$$

which is the same as the original cross product $\mathbf{A} \times \mathbf{B}$. The cross product of two pseudovectors is also a pseudovector. Torque $\tau = \mathbf{r} \times \mathbf{F}$ and magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ are examples of pseudovectors.
(d) Scalar triple product under inversion

$$\begin{aligned} \bar{A} \cdot (\bar{B} \times \bar{C}) &= -\mathbf{A} \cdot (-\mathbf{B} \times -\mathbf{C}) \\ &= -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \end{aligned}$$

the scalar triple product changes sign under inversion.

1.11 (a) Finding gradient of $f(x, y, z) = x^2 + y^3 + z^4$:

$$\begin{aligned} \boldsymbol{\nabla} f &= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\ &= 2x\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + 4z^3\hat{\mathbf{z}} \end{aligned}$$

(b) Gradient of $f(x, y, z) = x^2 y^3 z^4$:

$$\boldsymbol{\nabla} f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$$

(c) Gradient of $f(x, y, z) = e^x \sin(y) \ln(z)$:

$$\boldsymbol{\nabla} f = e^x\hat{\mathbf{x}} + e^x \cos(y) \ln(z)\hat{\mathbf{y}} + \frac{e^x \sin(y)}{z}\hat{\mathbf{z}}$$

1.12 The height of the hill (in feet) is given by the function

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is north and x is east in miles. The gradient of h is

$$\boldsymbol{\nabla} h = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10(2x - 8y + 28)\hat{\mathbf{y}}$$

(a) The top of the hill is a stationary point, so the summit is found by setting the gradient to zero which gives the system of equations

$$0 = 2y - 6x - 18 \qquad\qquad 0 = 2x - 8y + 28$$

adding the first equation to 3 times the second equation gives

$$\begin{aligned} 0 &= 2y - 6x - 18 + 3(2x - 8y + 28) \\ 0 &= -22y + 66 \\ y &= 3 \end{aligned}$$

substituting $y = 3$ into the first equation

$$0 = 2(3) - 6x - 18 \to x = -2$$

Therefore, the top of the hill is at $(-2, 3)$ or 2 miles west and 3 miles north of the origin.
(b) The height of the hill is simply $h(-2, 3) = 10(12) = 720$ feet.
(c) The steepness of the hill at $h(1, 1)$ is given by the magnitude of the gradient

$$\begin{aligned} |\boldsymbol{\nabla} h| &= 10\sqrt{(2y - 6x - 18)^2 + (2x - 8y + 28)^2} \\ &= 10\sqrt{(2 - 6 - 18)^2 + (2 - 8 + 28)^2} \\ &= 10\sqrt{(-22)^2 + (22)^2} = 220\sqrt{2} \approx 311\,\mathrm{ft/mi} \end{aligned}$$

The direction of the steepest slope is given by the vector in the direction of the gradient at the point $\boldsymbol{\nabla} h(1, 1) = 220(-\mathbf{x} + \mathbf{y})$, or simply northwest.

1.13 Given the seperation vector

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(a) Show that $\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}} = 2\boldsymbol{\mathscr{r}}$$

(b)

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{z}}$$

looking at the x component,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right) &= -\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial x}(\mathscr{r}) \\ &= -\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial x}\left(\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\right) \\ &= -\frac{1}{\mathscr{r}^2}\frac{1}{2}\frac{2(x - x')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ &= -\frac{x - x'}{\mathscr{r}^3} \end{aligned}$$

therefore,

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}^3}[(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}] = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(c) The general formula is

$$\nabla(\mathscr{r}^n) = n\mathscr{r}^{n-1}\hat{\boldsymbol{\mathscr{r}}}$$

1.14 Given the rotation matrix

$$\begin{pmatrix}\bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix}$$

or the two equations

$$\bar{y} = y\cos\phi + z\sin\phi$$
$$\bar{z} = -y\sin\phi + z\cos\phi$$

differentiating with respect to $\bar{y}$ and $\bar{z}$ respectively gives

$$1 = \frac{\partial y}{\partial\bar{y}}\cos\phi + \frac{\partial z}{\partial\bar{y}}\sin\phi$$
$$1 = -\frac{\partial y}{\partial\bar{z}}\sin\phi + \frac{\partial z}{\partial\bar{z}}\cos\phi$$

this can only be true if

$$\frac{\partial y}{\partial\bar{y}} = \cos\phi, \quad \frac{\partial z}{\partial\bar{y}} = \sin\phi \quad \text{and} \quad \frac{\partial y}{\partial\bar{z}} = -\sin\phi, \frac{\partial z}{\partial\bar{z}} = \cos\phi$$

which satisfies the trig identity $\sin^2\phi + \cos^2\phi = 1$. Differentiating f with respect to the rotated coordinates $\bar{y}$ and $\bar{z}$ is given by

$$\frac{\partial f}{\partial\bar{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial\bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial\bar{y}} = \frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi$$
$$\frac{\partial f}{\partial\bar{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial\bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial\bar{z}} = -\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi$$

therefore, the gradient of f transforms as a vector under rotations given by

$$\overline{\nabla f} = \frac{\partial f}{\partial\bar{y}}\hat{\bar{\mathbf{y}}} + \frac{\partial f}{\partial\bar{z}}\hat{\bar{\mathbf{z}}} = \left(\frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi\right)\hat{\bar{\mathbf{y}}} + \left(-\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi\right)\hat{\bar{\mathbf{z}}}$$

or in matrix form

$$\overline{\nabla f} = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\nabla f$$

where the gradient is a column vector

$$\nabla f = \begin{pmatrix}\frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{pmatrix}$$

1.15 (a) Calculating divergence of $v_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\nabla\cdot v_a = \frac{\partial v_{ax}}{\partial x} + \frac{\partial v_{ay}}{\partial y} + \frac{\partial v_{az}}{\partial z}$$
$$= 2x + 0 - 2x = 0$$

(b) $v_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\nabla\cdot v_b = y + 2z + 3x$$

(c) $v_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\nabla\cdot v_c = 0 + 2x + 2y = 2(x+y)$$

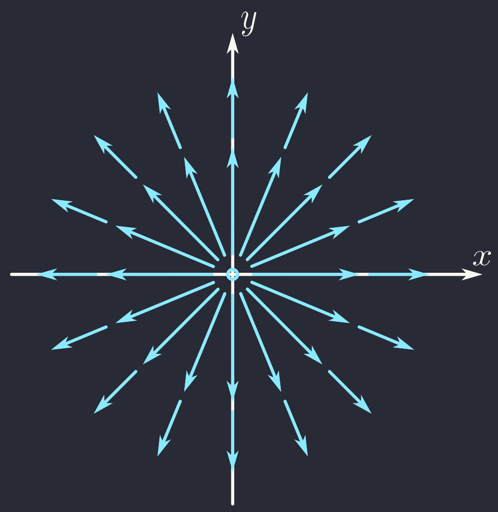

Figure 1.3: Sketch of the vector field $\mathbf{v} = \hat{\mathbf{r}}/r^2$

1.16 Given

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector. The vector functions is

$$v = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

where the components are

$$v_x = \frac{x}{r^3} \quad \text{and} \quad v_y = \frac{y}{r^3} \quad \text{and} \quad v_z = \frac{z}{r^3}$$

Looking at the x component of the divergence,

$$\begin{aligned}
[\nabla \cdot \mathbf{v}]_x &= \frac{\partial v_x}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) \quad \text{using chain rule...} \\
&= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{1}{r^3} - \frac{3x^2}{r^5}
\end{aligned}$$

therefore, the divergence of $\mathbf{v}$ is

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \\
&= \frac{3}{r^3} - 3\frac{x^2 + y^2 + z^2}{r^5} \\
&= \frac{3}{r^3} - 3\frac{r^2}{r^5} = 0
\end{aligned}$$

This is consistent with the sketch in Figure 1.3 because the vector field is not ‘sourcing’ or ‘sinking’.

1.17 Given

$$\bar{v}_y = v_y \cos\phi + v_z \sin\phi \quad \text{and} \quad \bar{v}_z = -v_y \sin\phi + v_z \cos\phi$$

Calculating the derivatives

$$\begin{aligned}\frac{\partial \bar{v}_y}{\partial \bar{y}} &= \frac{\partial v_y}{\partial \bar{y}} \cos\phi + \frac{\partial v_z}{\partial \bar{y}} \sin\phi \\ \frac{\partial \bar{v}_z}{\partial \bar{z}} &= -\frac{\partial v_y}{\partial \bar{z}} \sin\phi + \frac{\partial v_z}{\partial \bar{z}} \cos\phi\end{aligned}$$

from Problem 1.14,

$$\begin{aligned}\frac{\partial f}{\partial \bar{y}} &= \frac{\partial f}{\partial y} \cos\phi + \frac{\partial f}{\partial z} \sin\phi \\ \frac{\partial f}{\partial \bar{z}} &= -\frac{\partial f}{\partial y} \sin\phi + \frac{\partial f}{\partial z} \cos\phi\end{aligned}$$

therefore, the derivatives are rewritten as

$$\begin{aligned}\frac{\partial \bar{v}_y}{\partial \bar{y}} &= \left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \bar{y}}\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial v_z}{\partial y}\frac{\partial z}{\partial \bar{z}}\right)\sin\phi \\ &= \left(\frac{\partial v_y}{\partial y}\cos\phi + \frac{\partial v_y}{\partial z}\sin\phi\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi\right)\sin\phi\end{aligned}$$

and likewise,

$$\frac{\partial \bar{v}_z}{\partial \bar{z}} = -\left(-\frac{\partial v_y}{\partial y}\sin\phi + \frac{\partial v_y}{\partial z}\cos\phi\right)\sin\phi + \left(-\frac{\partial v_z}{\partial y}\sin\phi + \frac{\partial v_z}{\partial z}\cos\phi\right)\cos\phi$$

Finally adding the two equations together gives

$$\begin{aligned}\boldsymbol{\nabla}\cdot\bar{\mathbf{v}} &= \frac{\partial \bar{v}_y}{\partial \bar{y}} + \frac{\partial \bar{v}_z}{\partial \bar{z}} \\ &= \frac{\partial v_y}{\partial y}\cos^2\phi + \frac{\partial v_y}{\partial z}\sin\phi\cos\phi + \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi^2 \\ &+ \frac{\partial v_y}{\partial y}\sin^2\phi - \frac{\partial v_y}{\partial z}\sin\phi\cos\phi - \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\cos^2\phi \\ &= (\sin\phi^2 + \cos\phi^2)\left[\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] \\ &= \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\end{aligned}$$

which shows that the divergence transforms as a scalar under rotations.

1.18 Curl of vector functions from Problem 1.15: (a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{v}_a &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz\end{vmatrix} \\ &= \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0) \\ &= -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}\end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{v}_b &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx\end{vmatrix} \\ &= \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x) \\ &= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}\end{aligned}$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_c &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\
&= \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) \\
&= 0
\end{aligned}$$

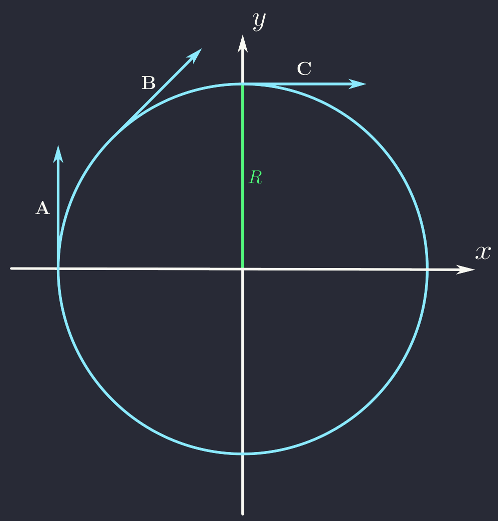

Figure 1.4: Sketch of the vector field pointing clockwise around a circle of radius R

1.19 From Figure 1.4, the sign of $\partial v_x/\partial y$ is positive, and the sign of $\partial v_y/\partial x$ is negative. Therefore, the curl

$$\boldsymbol{\nabla} \times \mathbf{v} = \hat{\mathbf{z}}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial z}\right)$$

is in the negative z direction, or into the page. This is consistent with the right-hand rule as the thumb points into the page.

1.20 Proof that the vector function

$$\mathbf{g} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3}$$

has zero divergence and curl given

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{and} \quad \frac{\partial x}{\partial y} = \frac{y}{x} = 0$$

From Problem 1.16, the divergence of $\mathbf{g}$ is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{g} &= \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) \\
&= \frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5} \\
&= \frac{3}{r^3} - \frac{3}{r^3} = 0
\end{aligned}$$

and the curl is

$$\begin{aligned}\nabla \times \mathbf{g} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} \\ &= \hat{\mathbf{x}}(0-0) - \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-0) = 0\end{aligned}$$

1.21 Proving product rule for (i)

$$\begin{aligned}\nabla(fg) &= \frac{\partial(fg)}{\partial x}\hat{\mathbf{x}} + \frac{\partial(fg)}{\partial y}\hat{\mathbf{y}} + \frac{\partial(fg)}{\partial z}\hat{\mathbf{z}} \\ &= \left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right)\hat{\mathbf{x}} + \left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right)\hat{\mathbf{y}} + \left(\frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\right)\hat{\mathbf{z}} \\ &= f\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\right) + g\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) \\ &= f\nabla g + g\nabla f\end{aligned}$$

(iv)

$$\begin{aligned}\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \nabla\cdot\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \nabla\cdot[(A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}] \\ &= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x) \\ &= \left(\frac{\partial A_y}{\partial x}B_z + A_y\frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x}B_y - A_z\frac{\partial B_y}{\partial x}\right) + \left(\frac{\partial A_z}{\partial y}B_x + A_z\frac{\partial B_x}{\partial y} - \frac{\partial A_x}{\partial y}B_z - A_x\frac{\partial B_z}{\partial y}\right) \\ &+ \left(\frac{\partial A_x}{\partial z}B_y + A_x\frac{\partial B_y}{\partial z} - \frac{\partial A_y}{\partial z}B_x - A_y\frac{\partial B_x}{\partial z}\right) \\ &= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &+ A_x\left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right) + A_y\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + A_z\left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) \\ &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) + \mathbf{A}\cdot(\nabla\times\mathbf{B})\end{aligned}$$

(v)

$$\begin{aligned}\nabla\times(f\mathbf{A}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y)\right) - \hat{\mathbf{y}}\left(\frac{\partial}{\partial x}(fA_z) - \frac{\partial}{\partial z}(fA_x)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x)\right) \\ &= \hat{\mathbf{x}}\left(f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right) - \hat{\mathbf{y}}\left(f\frac{\partial A_z}{\partial x} + A_z\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial z} - A_x\frac{\partial f}{\partial z}\right) \\ &+ \hat{\mathbf{z}}\left(f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right) \\ &= f\left[\hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \\ &- \hat{\mathbf{x}}\left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial y}\right) + \hat{\mathbf{y}}\left(A_z\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial z}\right) - \hat{\mathbf{z}}\left(A_y\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial y}\right) \\ &= f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)\end{aligned}$$

1.22 (a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, then

$$\begin{aligned}
(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} &= \left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}) \\
&= \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right)+\hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_y}{\partial z}\right) \\
&+\hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x}+A_y\frac{\partial B_z}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)
\end{aligned}$$

This means that the direction of $\mathbf{A}$ points in the direction of where $\mathbf{B}$ moves fastest.
(b)

$$(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} = \frac{1}{r}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\frac{(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})}{r}$$

looking at the x component,

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\mathbf{r}}{r}\right) &= \frac{1}{r}\frac{\partial}{\partial x}(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})+\mathbf{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \\
&= \frac{\hat{\mathbf{x}}}{r}-\mathbf{r}\frac{x^2}{r^3}
\end{aligned}$$

therefore,

$$\begin{aligned}
(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} &= \frac{1}{r}\left[x\left(\frac{\hat{\mathbf{x}}}{r}-\mathbf{r}\frac{x}{r^3}\right)+y\left(\frac{\hat{\mathbf{y}}}{r}-\mathbf{r}\frac{y}{r^3}\right)+z\left(\frac{\hat{\mathbf{z}}}{r}-\mathbf{r}\frac{z}{r^3}\right)\right] \\
&= \frac{1}{r}\left[\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{r}-\mathbf{r}\frac{x^2+y^2+z^2}{r^3}\right] \\
&= \frac{1}{r}\left[\frac{\mathbf{r}}{r}-\frac{\mathbf{r}}{r}\right]=0
\end{aligned}$$

(c)

$$\begin{aligned}
(v_a\cdot\boldsymbol{\nabla})v_b &= \left(x^2\frac{\partial}{\partial x}+3xz^2\frac{\partial}{\partial y}-2xz\frac{\partial}{\partial z}\right)(xy\hat{\mathbf{x}}+2yz\hat{\mathbf{y}}+3zx\hat{\mathbf{z}}) \\
&= x^2(y\hat{\mathbf{x}}+0+3z\hat{\mathbf{z}})+3xz^2(x\hat{\mathbf{x}}+2z\hat{\mathbf{y}}+0)-2xz(0+2y\hat{\mathbf{y}}+3x\hat{\mathbf{z}}) \\
&= (x^2y+3x^2z^2)\hat{\mathbf{x}}+(6xz^3-4xyz)\hat{\mathbf{y}}+(3x^2z-6x^2z)\hat{\mathbf{z}} \\
&= x^2(y+3z^2)\hat{\mathbf{x}}+2xz(3z^2-2y)\hat{\mathbf{y}}-3x^2z\hat{\mathbf{z}}
\end{aligned}$$

1.23 Proving the product rule for (ii) given the x component of the left hand side is

$$\begin{aligned}
[\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x &= \frac{\partial(\mathbf{A}\cdot\mathbf{B})}{\partial x}\hat{\mathbf{x}} \\
&= \frac{\partial}{\partial x}(A_xB_x+A_yB_y+A_zB_z)\hat{\mathbf{x}} \\
&= A_x\frac{\partial B_x}{\partial x}+B_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial B_y}{\partial x}+B_y\frac{\partial A_y}{\partial x}+A_z\frac{\partial B_z}{\partial x}+B_z\frac{\partial A_z}{\partial x}
\end{aligned}$$

Finding the x component of the right hand side of (ii)

$$
\begin{aligned}
[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})]_x &= \left[\mathbf{A}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_x & B_y & B_z\end{vmatrix}\right]_x \\
&= \left[\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ () & -\left(\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}\right) & \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\end{vmatrix}\right]_x \\
&= A_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-A_z\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)
\end{aligned}
$$

and

$$
[\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})]_x = B_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-B_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)
$$

From Problem 1.22:

$$
[(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}] = A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}
$$

and likewise,

$$
[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}] = B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}
$$

Adding all four equations gives

$$
\begin{aligned}
&[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})+(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}+(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}]_x = \\
&\quad A_y\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)-A_z\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+B_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-B_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right) \\
&\quad +A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}+B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z} \\
&= A_x\frac{\partial B_x}{\partial x}+A_y\left(\frac{\partial B_y}{\partial x}-\cancel{\frac{\partial B_x}{\partial y}}+\cancel{\frac{\partial B_x}{\partial y}}\right)+A_z\left(\frac{\partial B_z}{\partial x}-\cancel{\frac{\partial B_x}{\partial z}}+\cancel{\frac{\partial B_x}{\partial z}}\right) \\
&\quad +B_x\frac{\partial A_x}{\partial x}+B_y\left(\frac{\partial A_y}{\partial x}-\cancel{\frac{\partial A_x}{\partial y}}+\cancel{\frac{\partial A_x}{\partial y}}\right)+B_z\left(\frac{\partial A_z}{\partial x}-\cancel{\frac{\partial A_x}{\partial z}}+\cancel{\frac{\partial A_x}{\partial z}}\right) \\
&= A_x\frac{\partial B_x}{\partial x}+B_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial B_y}{\partial y}+B_y\frac{\partial A_x}{\partial y}+A_z\frac{\partial B_z}{\partial z}+B_z\frac{\partial A_x}{\partial z} \\
&= [\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x
\end{aligned}
$$

and likewise for the y and z components.
For (vi), the x on the left hand side is

$$
\begin{aligned}
[\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x &= \left[\boldsymbol{\nabla}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\right]_x \\
&= \left[\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ () & -(A_xB_z-A_zB_x) & A_xB_y-A_yB_x\end{vmatrix}\right]_x \\
&= \frac{\partial}{\partial y}(A_xB_y-A_yB_x)-\frac{\partial}{\partial z}(A_zB_x-A_xB_z) \\
&= A_x\frac{\partial B_y}{\partial y}+B_y\frac{\partial A_x}{\partial y}-A_y\frac{\partial B_x}{\partial y}-B_x\frac{\partial A_y}{\partial y} \\
&- A_z\frac{\partial B_x}{\partial z}-B_x\frac{\partial A_z}{\partial z}+A_x\frac{\partial B_z}{\partial z}+B_z\frac{\partial A_x}{\partial z}+ \\
&= A_x\left(\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z}-B_x\left(\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}
\end{aligned}
$$

On the right hand side, first we find the x component of the two new operations:

$$
\begin{aligned}
[A(\boldsymbol{\nabla}\cdot\mathbf{B})]_x &= \left[A\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)\right]_x \\
&= A_x\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)
\end{aligned}
$$

and likewise,

$$
[B(\boldsymbol{\nabla}\cdot\mathbf{A})]_x = B_x\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)
$$

Therefore, $[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}-(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}+A(\boldsymbol{\nabla}\cdot\mathbf{B})-B(\boldsymbol{\nabla}\cdot\mathbf{A})]_x =$

$$
\begin{aligned}
&B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}-\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right)\\
&+A_x\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)-\left(B_x\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\right)\\
&=A_x\left(\cancel{\frac{\partial B_x}{\partial x}}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}-\cancel{\frac{\partial B_x}{\partial x}}\right)-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z}\\
&-B_x\left(\cancel{\frac{\partial A_x}{\partial x}}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}-\cancel{\frac{\partial A_x}{\partial x}}\right)+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}\\
&=A_x\left(\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z}-B_x\left(\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}\\
&=[\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x
\end{aligned}
$$

and likewise for the y and z components.

1.24 Deriving the three quotient rules from the product rule: The gradient is

$$
\begin{aligned}
\boldsymbol{\nabla}\left(\frac{f}{g}\right) &= \boldsymbol{\nabla}\left(fg^{-1}\right) = f\boldsymbol{\nabla}\left(g^{-1}\right)+g^{-1}\boldsymbol{\nabla}(f)\\
&= f(-g^{-2}\boldsymbol{\nabla}(g))+g^{-1}\boldsymbol{\nabla}(f)\\
&= -\frac{f}{g^2}\boldsymbol{\nabla}(g)+\frac{g}{g}\frac{1}{g}\boldsymbol{\nabla}(f)\\
&= \frac{g\boldsymbol{\nabla}f-f\boldsymbol{\nabla}g}{g^2}
\end{aligned}
$$

the divergence

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\left(\frac{A}{g}\right) &= \boldsymbol{\nabla}\cdot\left(Ag^{-1}\right) = A(\boldsymbol{\nabla}\cdot g^{-1})+g^{-1}(\boldsymbol{\nabla}\cdot A)\\
&= A(-g^{-2}(\boldsymbol{\nabla}\cdot g))+\frac{g}{g}g^{-1}(\boldsymbol{\nabla}\cdot A)\\
&= \frac{g(\boldsymbol{\nabla}\cdot A)-A\boldsymbol{\nabla}\cdot g}{g^2}
\end{aligned}
$$

and the curl

$$
\begin{aligned}
\left[\boldsymbol{\nabla}\times\left(\frac{\mathbf{A}}{g}\right)\right]_x &= \frac{\partial}{\partial y}\left(\frac{A_z}{g}\right)-\frac{\partial}{\partial z}\left(\frac{A_y}{g}\right)\\
&= \frac{g\frac{\partial A_z}{\partial y}-A_z\frac{\partial g}{\partial y}}{g^2}-\frac{g\frac{\partial A_y}{\partial z}-A_y\frac{\partial g}{\partial z}}{g^2}\\
&= \frac{1}{g^2}\left[g\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\left(A_y\frac{\partial g}{\partial z}-A_z\frac{\partial g}{\partial y}\right)\right]\\
&= \frac{g[\boldsymbol{\nabla}\times\mathbf{A}]_x-\mathbf{A}\times[\boldsymbol{\nabla}g]_x}{g^2}
\end{aligned}
$$

and likewise for the y and z components. Therefore,

$$\nabla \times \left(\frac{\mathbf{A}}{g} \right) = \frac{g(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla g)}{g^2}$$

1.25 (a) Calculating (iv) for the functions

$$\mathbf{A} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}; \qquad \mathbf{B} = 3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}$$

LHS:

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\
&= \nabla \cdot \left[(0 + 6xz)\hat{\mathbf{x}} - (0 - 9yz)\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}} \right] \\
&= \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9yz) + \frac{\partial}{\partial z}\left(-2x^2 - 6y^2\right) \\
&= 6z + 9z + 0 = 15z
\end{aligned}$$

RHS:

$$\begin{aligned}
\mathbf{B} \cdot (\nabla \times \mathbf{A}) &= \mathbf{B} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} \\
&= \mathbf{B} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A} \cdot (\nabla \times \mathbf{B}) &= \mathbf{A} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= \mathbf{A} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-2 - 3)\hat{\mathbf{z}}] \\
&= 3z(-5) = -15z
\end{aligned}$$

therefore,

$$\mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = 0 - (-15z) = 15z$$

(b) Checking (ii): LHS

$$\begin{aligned}
\nabla(\mathbf{A} \cdot \mathbf{B}) &= \nabla(x(3y) + 2y(-2x) + 3z(0)) \\
&= \nabla(3xy - 4xy) = \nabla(-xy) \\
&= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}
\end{aligned}$$

RHS:

$$\begin{aligned}
\mathbf{A} \times (\nabla \times \mathbf{B}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= \mathbf{A} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-5)\hat{\mathbf{z}}] \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = -10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}
\end{aligned}$$

and

$$\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mathbf{B} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = \mathbf{B} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0$$

and

$$\begin{aligned}(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} &= \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z} \right)(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) \\ &= 6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}\end{aligned}$$

and

$$\begin{aligned}(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} &= \left(3y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y} \right)(x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \\ &= 3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}\end{aligned}$$

therefore,

$$\begin{aligned}\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} &= (-10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}) + (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) \\ &= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}\end{aligned}$$

(c) For rule (vi), the left hand side is

$$\begin{aligned}\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) &= \boldsymbol{\nabla} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\ &= \boldsymbol{\nabla} \times \left[6xz\hat{\mathbf{x}} + 9yz\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}\right] \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & -2x^2 - 6y^2 \end{vmatrix} \\ &= \hat{\mathbf{x}}(-12y - 9y) - \hat{\mathbf{y}}(-4x - 6x) + \hat{\mathbf{z}}(0) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}\end{aligned}$$

and on the right hand side, the new terms are

$$\mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) = \mathbf{A}[0 + 0] = 0$$

and

$$\begin{aligned}\mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) &= \mathbf{B}[1 + 2 + 3] = 6\mathbf{B} \\ &= 6(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) = 18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}\end{aligned}$$

combining these with the terms from (iv) gives

$$\begin{aligned}(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} - (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} &= (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) - (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + 0 - (18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}\end{aligned}$$

1.26 Given the Laplacian of a scalar function T is

$$\nabla^2 T = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

(a) $T_a = x^2 + 2xy + 3z + 4$:

$$\nabla^2 T_a = 2 + 0 + 0 = 2$$

(b) $T_b = \sin x \sin y \sin z$:

$$\frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -\sin x \sin y \sin z = -T_b$$

$$\nabla^2 T_b = -3T_b$$

(c) $T_c = e^{-5x} \sin 4y \cos 3z$: The components are

$$\begin{aligned}\frac{\partial^2 T_c}{\partial x^2} &= 25e^{-5x} \sin 4y \cos 3z = 25T_c \\ \frac{\partial^2 T_c}{\partial y^2} &= -16e^{-5x} \sin 4y \cos 3z = -16T_c \\ \frac{\partial^2 T_c}{\partial z^2} &= -9e^{-5x} \sin 4y \cos 3z = -9T_c\end{aligned}$$

therefore

$$\nabla^2 T_c = T_c(25 - 16 - 9) = 0$$

(d) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$: The laplacian of a vector function is

$$\nabla^2 \mathbf{v} = (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$$

and the components are

$$\begin{aligned}\nabla^2 v_x &= \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = 2 + 0 + 0 = 2 \\ \nabla^2 v_y &= 0 + 0 + 6x = 6x \\ \nabla^2 v_z &= 0\end{aligned}$$

therefore

$$\nabla^2 \mathbf{v} = 2\hat{\mathbf{x}} + 6x\hat{\mathbf{y}}$$

1.27 The divergence of curl is always zero:

$$\begin{aligned}\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \boldsymbol{\nabla} \cdot \left(\hat{\mathbf{x}} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) - \hat{\mathbf{y}} \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + \hat{\mathbf{z}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\ &= \left[\frac{\partial}{\partial x} \left(\frac{\partial v_z}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial v_z}{\partial x} \right) \right] + \left[\frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial v_x}{\partial y} \right) \right] + \left[\frac{\partial}{\partial z} \left(\frac{\partial v_y}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial v_y}{\partial z} \right) \right] \\ \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) &= 0\end{aligned}$$

where the last step hinges on the equality of cross derivatives:

$$\frac{\partial}{\partial x_i} \left(\frac{\partial v}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left(\frac{\partial v}{\partial x_i} \right)$$

Checking for $v_a = x^2\hat{\mathbf{x}} + 2xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times v_a) &= \boldsymbol{\nabla}\cdot\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & 2xz^2 & -2xz\end{vmatrix}\\
&= \boldsymbol{\nabla}\cdot\left[\hat{\mathbf{x}}(0-4xz) - \hat{\mathbf{y}}(-2z-0) + \hat{\mathbf{z}}\big(2z^2-0\big)\right]\\
&= \boldsymbol{\nabla}\cdot\left[\frac{\partial}{\partial x}(-4xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}\big(2z^2\big)\right]\\
&= -4z + 0 + 4z = 0
\end{aligned}$$

1.28 The curl of gradient is always zero:

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}T) &= \boldsymbol{\nabla}\times\left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)\\
&= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z}\end{vmatrix}\\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial y}\right)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial x}\right)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)\right]\\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}T) &= 0
\end{aligned}$$

where the last step uses the equality of cross derivatives again. Checking for $T = x^2y^3z^4$:

$$\frac{\partial T}{\partial x} = 2xy^3z^4, \quad \frac{\partial T}{\partial y} = 3x^2y^2z^4, \quad \text{and} \quad \frac{\partial T}{\partial z} = 4x^2y^3z^3$$

and

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}T) &= \boldsymbol{\nabla}\times\big(2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}\big)\\
&= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3\end{vmatrix}\\
&= \hat{\mathbf{x}}\big(12x^2y^2z^4 - 12x^2y^2z^4\big) - \hat{\mathbf{y}}\big(8x^2y^3z^3 - 8x^2y^3z^3\big) + \hat{\mathbf{z}}\big(6x^2y^3z^3 - 6x^2y^3z^3\big)\\
&= 0
\end{aligned}$$

1.29 Calculating the line integral of the function $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$: from the origin to point $(1,1,1)$ along three different paths:
(a) $a = (0,0,0) \to b = (1,0,0) \to c = (1,1,0) \to d = (1,1,1)$ split to three paths:
(i) From $a \to b$: $\mathrm{d}l = \mathrm{d}x\,\hat{\mathbf{x}}$ and $\mathbf{v} = x^2\hat{\mathbf{x}}$.
(ii) From $b \to c$: $\mathrm{d}l = \mathrm{d}y\,\hat{\mathbf{y}}$ and $\mathbf{v} = 2yz\hat{\mathbf{y}} = 0$ since $z = 0$.
(iii) From $c \to d$: $\mathrm{d}l = \mathrm{d}z\,\hat{\mathbf{z}}$ and $\mathbf{v} = y^2\hat{\mathbf{z}} = 1\hat{\mathbf{z}}$ since $y = 1$.

$$\begin{aligned}
\int_a^b \mathbf{v}\cdot\mathrm{d}\mathbf{l} &= \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}\\
\int_b^c \mathbf{v}\cdot\mathrm{d}\mathbf{l} &= \int_0^1 0\,\mathrm{d}y = 0\\
\int_c^d \mathbf{v}\cdot\mathrm{d}\mathbf{l} &= \int_0^1 1\,\mathrm{d}z = 1\\
\int_a^d \mathbf{v}\cdot\mathrm{d}\mathbf{l} &= \frac{1}{3} + 0 + 1 = \frac{4}{3}
\end{aligned}$$

(b) $a = (0,0,0) \to b = (0,0,1) \to c = (0,1,1) \to d = (1,1,1)$ split to three paths:

(i) From $a \to b$: $\mathrm{d}l = \mathrm{d}z\,\hat{\mathbf{z}}$ and $\mathbf{v} = y^2\hat{\mathbf{z}} = 0$ since $y = 0$.
(ii) From $b \to c$: $\mathrm{d}l = \mathrm{d}y\,\hat{\mathbf{y}}$ and $\mathbf{v} = 2yz\hat{\mathbf{y}} = 2y\hat{\mathbf{z}}$ since $y = 1$.
(iii) From $c \to d$: $\mathrm{d}l = \mathrm{d}x\,\hat{\mathbf{x}}$ and $\mathbf{v} = x^2\hat{\mathbf{x}}$.

$$\int_a^b \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int_0^1 0\,\mathrm{d}z = 0$$
$$\int_b^c \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int_0^1 2y\,\mathrm{d}y = 1$$
$$\int_c^d \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$$
$$\int_a^d \mathbf{v}\cdot\mathrm{d}\mathbf{l} = 0 + 1 + \frac{1}{3} = \frac{4}{3}$$

(c) A straight line: Since $x = y = z$ and $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z$,
$\mathrm{d}l = \mathrm{d}x\,(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$ and $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}} = 4x^2\hat{\mathbf{x}}$.

$$\int_a^d \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int_0^1 4x^2\,\mathrm{d}x = \frac{4}{3}$$

(d) The line integral around the closed loop:

$$\oint \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int_{(a)} \mathbf{v}\cdot\mathrm{d}\mathbf{l} - \int_{(b)} \mathbf{v}\cdot\mathrm{d}\mathbf{l} = \frac{4}{3} - \frac{4}{3} = 0$$

1.30 Surface integral of $\mathbf{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ over the bottom of the box:
$z = 0$, $\mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}$ $\mathbf{v}\cdot\mathrm{d}\mathbf{A} = y(z^2-3)\,\mathrm{d}x\,\mathrm{d}y = -3y\,\mathrm{d}x\,\mathrm{d}y$, so

$$\int \mathbf{v}\cdot\mathrm{d}\mathbf{A} = \int_0^2 \mathrm{d}x \int_0^2 -3y\,\mathrm{d}y = -12$$

From Example 1.7 (v) has the same boundary line but a different surface integral, so the surface integral does not depend on just the boundary line. The surface over the closed surface of the box is

$$\oint \mathbf{v}\cdot\mathrm{d}\mathbf{A} = 20 + 12 = 32$$

since the direction of $\mathrm{d}\mathbf{A}$ on the bottom side is in the negative z direction for it to point 'outward'.

1.31 Calculating the volume integral of $T = z^2$ over the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$:
The equation of the plane containing the three vertices $A = (1,0,0)$, $B = (0,1,0)$, and $C(0,0,1)$: The vector normal to this plane $\mathbf{n} = (a,b,c)$ is the cross product of two vectors in the plane given by $\mathbf{AB} = (-1,1,0)$ and $\mathbf{AC} = (-1,0,1)$:

$$\mathbf{n} = \begin{vmatrix} x & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1,1,1)$$

the equation of the plane is therefore

$$\begin{aligned} \mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_o) &= 0 \\ (1,1,1)\cdot[(x,y,z) - A] &= 0 \\ x + y + z &= 1 \end{aligned}$$

therefore, the boundary for x is $x = 0$ and $x = 1 - y - z$; for y is $y = 0$ and $y = 1 - z$; and for z is $z = 0$ and $z = 1$. The volume integral is therefore

$$\begin{aligned}
\int T\,\mathrm{d}V &= \int_0^1 z^2\,\mathrm{d}z \int_0^{1-z} \mathrm{d}y \int_0^{1-y-z} \mathrm{d}x \\
&= \int_0^1 z^2\,\mathrm{d}z \int_0^{1-z} (1-y-z)\,\mathrm{d}y \\
&= \int_0^1 z^2\,\mathrm{d}z \left(y - y^2/2 - yz \Big|_0^{1-z} \right) \\
&= \int_0^1 z^2[(1-z) - (1-z)^2/2 - z(1-z)]\,\mathrm{d}z \\
&= \int_0^1 z^2(1 - z - 1/2 - z^2/2 + z - z + z^2)\,\mathrm{d}z \\
&= \int_0^1 z^2(1/2 - z + z^2/2)\,\mathrm{d}z \\
&= \int_0^1 (z^2/2 - z^3 + z^4/2)\,\mathrm{d}z \\
&= z^3/6 - z^4/4 + z^5/10\Big|_0^1 \\
&= 1/6 - 1/4 + 1/10 = 1/60
\end{aligned}$$

1.32 Given $T = x^2 + 4xy + 2yz^3$,

$$\frac{\partial T}{\partial x} = 2x + 4y, \quad \frac{\partial T}{\partial y} = 4x + 2z^3, \quad \text{and} \quad \frac{\partial T}{\partial z} = 6yz^2$$

therefore,

$$\boldsymbol{\nabla} T = \hat{\mathbf{x}}(2x + 4y) + \hat{\mathbf{y}}(4x + 2z^3) + \hat{\mathbf{z}}(6yz^2)$$

Checking the fundamental theorem for gradients using the points $a = (0,0,0) \to b = (1,1,1)$:

$$\int_a^b \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = T(b) - T(a) = 1^2 + 4(1)(1) + 2(1)(1)^3 - 0 = 7$$

For the three paths: (a) $a \to c = (1,0,0) \to d = (1,1,0) \to d$;
(i) $a \to c$:

$$y = z = \mathrm{d}y = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 2x\,\mathrm{d}x$$

and

$$\int_a^c \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2x\,\mathrm{d}x = 1$$

(ii) $c \to d$:

$$x = 1, \quad z = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 4\,\mathrm{d}y$$

and

$$\int_c^d \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = \int_0^1 4\,\mathrm{d}y = 4$$

(iii) $d \to b$:

$$x = y = 1, \quad \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \boldsymbol{\nabla} T \cdot \mathrm{d}\mathbf{l} = 6z^2\,\mathrm{d}z$$

and

$$\int_d^b \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 6z^2 \, \mathrm{d}z = 2$$

therefore,

$$\int_a^b \nabla T \cdot \mathrm{d}\mathbf{l} = 1 + 4 + 2 = 7$$

(b) $a \to c = (0, 0, 1) \to d = (0, 1, 1) \to b$;
(i) $a \to c$:

$$x = y = \mathrm{d}x = \mathrm{d}y = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}; \quad \nabla T \cdot \mathrm{d}\mathbf{l} = 0$$

and

$$\int_a^c \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 0 = 0$$

(ii) $c \to d$:

$$z = 1, \quad x = \mathrm{d}x = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}; \quad \nabla T \cdot \mathrm{d}\mathbf{l} = 2\,\mathrm{d}y$$

and

$$\int_c^d \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 2\,\mathrm{d}y = 2$$

(iii) $d \to b$:

$$y = z = 1, \quad \mathrm{d}y = \mathrm{d}z = 0; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}; \quad \nabla T \cdot \mathrm{d}\mathbf{l} = (2x + 4)\,\mathrm{d}x$$

and

$$\int_d^b \nabla T \cdot \mathrm{d}\mathbf{l} = \int_0^1 (2x + 4)\,\mathrm{d}x = 5$$

therefore,

$$\int_a^b \nabla T \cdot \mathrm{d}\mathbf{l} = 0 + 2 + 5 = 7$$

(c) the parabolic path $z = x^2$; $y = x$:

$$\mathrm{d}x = \mathrm{d}y\,, \quad \text{and} \quad \mathrm{d}z = 2x\,\mathrm{d}x\,; \quad \mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}x\,\hat{\mathbf{y}} + 2x\,\mathrm{d}x\,\hat{\mathbf{z}}$$

and

$$\begin{aligned}\nabla T \cdot \mathrm{d}\mathbf{l} &= (2x + 4y)\,\mathrm{d}x + (4x + 2z^3)\,\mathrm{d}x + (6yz^2)2x\,\mathrm{d}x \\ &= 6x\,\mathrm{d}x + (4x + 2x^6)\,\mathrm{d}x + (12x^6)\,\mathrm{d}x \\ &= 10x\,\mathrm{d}x + 14x^6\,\mathrm{d}x\end{aligned}$$

therefore,

$$\begin{aligned}\int_a^b \nabla T \cdot \mathrm{d}\mathbf{l} &= \int_0^1 (10x + 14x^6)\,\mathrm{d}x \\ &= 5x^2 + 2x^7 \Big|_0^1 = 7\end{aligned}$$

1.33 Checking the divergence theorem for the function:

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

and the volume as the cube of sides of lenght 2 with a corner at the origin:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = y + 2z + 3x$$

Thus the volume integral

$$\begin{aligned}
\int_V \boldsymbol{\nabla} \cdot \mathbf{v} \, \mathrm{d}V &= \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_0^2 \int_0^2 (2y + 4z + 6) \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_0^2 (4 + 8z + 12) \, \mathrm{d}z \\
&= 8 + 16 + 24 = 48
\end{aligned}$$

The surface integral is evaluated over the six faces of the cube noted by Figure 1.29:
(i) $x = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}y \, \mathrm{d}z \, \hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 2y \, \mathrm{d}y \, \mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 2y \, \mathrm{d}y \, \mathrm{d}z = 8$$

(ii) $x = 0$, $\mathrm{d}\mathbf{A} = -\, \mathrm{d}y \, \mathrm{d}z \, \hat{\mathbf{x}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 0 \, \mathrm{d}y \, \mathrm{d}z = 0$$

(iii) $y = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x \, \mathrm{d}z \, \hat{\mathbf{y}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 4z \, \mathrm{d}x \, \mathrm{d}z$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 4z \, \mathrm{d}x \, \mathrm{d}z = 16$$

(iv) $y = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

(v) $z = 2$, $\mathrm{d}\mathbf{A} = \mathrm{d}x \, \mathrm{d}y \, \hat{\mathbf{z}}$, $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = 6x \, \mathrm{d}x \, \mathrm{d}y$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \int_0^2 6x \, \mathrm{d}x \, \mathrm{d}y = 24$$

(vi) $z = 0$;

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0$$

So, the total flux is

$$\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 8 + 0 + 16 + 0 + 24 + 0 = 48$$

therefore, the divergence theorem is verified.

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{v}) \, \mathrm{d}V = \oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{A}$$

1.34 Testing Stokes' theorem for the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

using the triangular shaded area bounded by the vertices $O = (0,0,0)$, $A = (0,2,0)$, and $B = (0,0,2)$:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{v} &= (0 - 2y)\hat{\mathbf{x}} - (3z - 0)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}} \quad \text{and} \quad \mathrm{d}\mathbf{A} = \mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{x}} \\ &= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}} \end{aligned}$$

$x = 0$ on this surface, and the limits of integration are $z = 0$ and $z = 2 - y$:

$$(\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = -2y\,\mathrm{d}y\,\mathrm{d}z$$

Thus, the flux of the curl through the surface is

$$\begin{aligned} \int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} &= \int_0^2 \mathrm{d}y \int_0^{2-y} -2y\,\mathrm{d}z \\ &= \int_0^2 -2y(2-y)\,\mathrm{d}y = -8/3 \end{aligned}$$

The line integral is evaluated over the three sides of the triangle:
(i) On the path OA: $x = z = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 2yz\,\mathrm{d}y = 0$;

$$\int_{OA} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

(ii) On the path AB:
$x = 0$, $y = 2 - z$; $\mathrm{d}y = -\,\mathrm{d}z$; $\mathrm{d}\mathbf{l} = -\,\mathrm{d}z\,(\hat{\mathbf{y}} - \hat{\mathbf{z}})$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = -2yz\,\mathrm{d}z = -2(2-z)z\,\mathrm{d}z = (2z^2 - 4z)\,\mathrm{d}z$;

$$\int_{AB} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^2 (2z^2 - 4z)\,\mathrm{d}z = -8/3$$

(iii) On the path BO:
$x = y = 0$; $\mathrm{d}\mathbf{l} = \mathrm{d}z\,\hat{\mathbf{z}}$; $\mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$;

$$\int_{BO} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0$$

So, the circulation of $\mathbf{v}$ around the triangle is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0 + -8/3 + 0 = -8/3$$

thus,

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

1.35 By Corollary 1, the function

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$$

with curl

$$\boldsymbol{\nabla} \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}$$

the flux of the curl through the flat surface from Ex. 1.11

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \frac{4}{3}$$

is the same as a different surface shown by Figure 1.35 which has the same boundary. Integrating over the five faces:

(i) $x = 1, \quad \mathrm{d}\mathbf{A} = \mathrm{d}y\,\mathrm{d}z\,\hat{\mathbf{x}}; \quad (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = (4z^2-2)\,\mathrm{d}y\,\mathrm{d}z\,;$

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = \int_0^1\int_0^1(4z^2-2)\,\mathrm{d}y\,\mathrm{d}z = -\frac{2}{3}$$

(ii) $z = 0, \quad \mathrm{d}\mathbf{A} = -\,\mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}; \quad (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = 0;$

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = 0$$

(iii) $y = 1, \quad \mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}z\,\hat{\mathbf{y}}; \quad (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = 0;$

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = 0$$

(iv) Similar to (iii), $(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = 0$;

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = 0$$

(v) $z = 1, \quad \mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}; \quad (\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = 2\,\mathrm{d}x\,\mathrm{d}y\,;$

$$\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = \int_0^1\int_0^1 2\,\mathrm{d}x\,\mathrm{d}y = 2$$

So,

$$\int_S(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{A} = -\frac{2}{3}+0+0+0+2 = \frac{4}{3}$$

Thus the flux of the curl through a surface depends only on the boundary line.

1.36 (a) From the product rule

$$\boldsymbol{\nabla}\times(f\mathbf{A}) = f(\boldsymbol{\nabla}\times\mathbf{A}) - A\times(\boldsymbol{\nabla}f)$$

and integrating over a surface

$$\int_S\boldsymbol{\nabla}\times(f\mathbf{A})\cdot\mathrm{d}\mathbf{a} = \int_S f(\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathrm{d}\mathbf{a} - \int_S[\mathbf{A}\times(\boldsymbol{\nabla}f)]\cdot\mathrm{d}\mathbf{a}$$

or rewritten as

$$\int_S f(\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathrm{d}\mathbf{a} = \int_S\boldsymbol{\nabla}\times(f\mathbf{A})\cdot\mathrm{d}\mathbf{a} + \int_S[\mathbf{A}\times(\boldsymbol{\nabla}f)]\cdot\mathrm{d}\mathbf{a}$$

invoking Stokes' theorem $\int_S(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathrm{d}\mathbf{a} = \oint_P\mathbf{v}\cdot\mathrm{d}\mathbf{l}$:

$$\int_S f(\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathrm{d}\mathbf{a} = \oint_P f(\mathbf{A})\cdot\mathrm{d}\mathbf{l} + \int_S[\mathbf{A}\times(\boldsymbol{\nabla}f)]\cdot\mathrm{d}\mathbf{a}$$

(b) From the product rule for divergence:

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

or rewritten as

$$\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) + \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$$

integrating both sides over a volume:

$$\int_V\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})\,\mathrm{d}V = \int_V\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B})\,\mathrm{d}V + \int_V\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,\mathrm{d}V$$

Using the divergence theorem $\int_V(\boldsymbol{\nabla}\cdot\mathbf{v})\,\mathrm{d}V = \oint_S\mathbf{v}\cdot\mathrm{d}\mathbf{a}$:

$$\int_V\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})\,\mathrm{d}V = \oint_S(\mathbf{A}\times\mathbf{B})\cdot\mathrm{d}\mathbf{a} + \int_V\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})\,\mathrm{d}V$$

1.37 Given the relation of Cartesian to spherical coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad \text{and} \quad z = r\cos\theta$$

To find the formula for r, take the sum of the squares of the three equations; Solve for θ using the third equation; and solve for ϕ by dividing the second equation by the first:

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos\frac{z}{r}, \quad \text{and} \quad \phi = \arctan\frac{y}{x}$$

1.38 From the position vector

$$\begin{aligned}
\mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\
\mathbf{r} &= r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}} \\
\mathbf{r} &= r\hat{\mathbf{r}}(\theta, \phi)
\end{aligned}$$

where the unit vector $\hat{\mathbf{r}}(\theta, \phi)$ is dependent on θ and ϕ. The new basis vectors are in the same direction as the partial derivatives with respect to r, θ, and ϕ, so

$$\hat{\mathbf{r}} = \frac{e_r}{|e_r|}, \quad \hat{\boldsymbol{\theta}} = \frac{e_\theta}{|e_\theta|}, \quad \text{and} \quad \hat{\boldsymbol{\phi}} = \frac{e_\phi}{|e_\phi|}$$

The partial derivatives are

$$\begin{aligned}
e_r &= \frac{\partial\mathbf{r}}{\partial r} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
e_\theta &= \frac{\partial\mathbf{r}}{\partial\theta} = r\cos\theta\cos\phi\hat{\mathbf{x}} + r\cos\theta\sin\phi\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}} \\
e_\phi &= \frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}}
\end{aligned}$$

and the magnitude

$$\begin{aligned}
|e_r| &= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} \\
&= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} \\
&= \sqrt{\sin^2\theta + \cos^2\theta} = 1 \\
|e_\theta| &= \sqrt{r^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)} \\
&= \sqrt{r^2(\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta)} \\
&= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r \\
|e_\phi| &= \sqrt{r^2(\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi)} \\
&= \sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} \\
&= \sqrt{r^2\sin^2\theta} = r\sin\theta
\end{aligned}$$

thus, the unit vectors are:

$$\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}
\end{aligned}$$

or in matrix form:

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

where $a = Qx$ is an orthogonal matrix, so $Q^T = Q^{-1}$ and $Q^TQ = I$. Multiplying both sides by Q^T:

$$Q^Ta = Q^TQx \to x = Q^Ta$$

thus, the inverse formula is

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}$$

or in vector form:

$$\begin{aligned} \hat{\mathbf{x}} &= \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} &= \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \end{aligned}$$

1.39 (a) Divergence theorem for $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$ using a volume of a sphere of radius R centered at the origin: The divergence is

$$\boldsymbol{\nabla}\cdot\mathbf{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(r^2)\right) = 4r$$

and the volume and surface elements are

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \quad \text{and} \quad \mathrm{d}\mathbf{a}_1 = (r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}$$

So

$$\begin{aligned} \int_V (\boldsymbol{\nabla}\cdot\mathbf{v}_1)\cdot\mathrm{d}V &= \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 4r(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\ &= \int_0^R 4r^3\,\mathrm{d}r\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\ &= (R^4)(2)(2\pi) = 4\pi R^4 \end{aligned}$$

and

$$\begin{aligned} \oint_S \mathbf{v}_1\cdot\mathrm{d}\mathbf{a}_1 &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= r^4\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\ &= (r^4)(2)(2\pi) = 4\pi r^4 \end{aligned}$$

where $r = R$ on the surface of the sphere. Therefore,

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{v}_1)\cdot\mathrm{d}V = \oint_S \mathbf{v}_1\cdot\mathrm{d}\mathbf{a}_1$$

(b) For $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(1/r^2)\right) = 0$$

So

$$\int_V (\boldsymbol{\nabla}\cdot\mathbf{v}_2)\cdot\mathrm{d}V = 0$$

and

$$\begin{aligned} \oint_S \mathbf{v}_2\cdot\mathrm{d}\mathbf{a}_1 &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{1}{r^2}(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi = 4\pi \end{aligned}$$

1.40 Given the function

$$\mathbf{v} = (r\cos\theta)\hat{\mathbf{r}} + (r\sin\theta)\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\hat{\boldsymbol{\phi}}$$

the divergence in spherical coordinates is

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(v_\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin^2\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) \\
&= 3\cos\theta + 2\cos\theta - \sin\phi \\
&= 5\cos\theta - \sin\phi
\end{aligned}$$

Checking the divergence theorem using a volume of a inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin:
The volume and surface elements are

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \quad \text{and} \quad \mathrm{d}\mathbf{a}_1 = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}$$

The volume integral for the hemisphere is

$$\begin{aligned}
\int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\cdot\mathrm{d}V &= \int_{r=0}^{R}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(5\cos\theta - \sin\phi)(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\
&= \int_0^R r^2\,\mathrm{d}r\int_0^{\pi/2}\sin\theta\int_0^{2\pi}(5\cos\theta - \sin\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{R^3}{3}\int_0^{\pi/2} 5\phi\cos\theta + \cos\phi\Big|_0^{2\pi}\,\mathrm{d}\theta \\
&= \frac{R^3}{3}(10\pi)\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta \\
\text{using} \quad u = \sin\theta, &\quad \mathrm{d}u = \cos\theta\,\mathrm{d}\theta\,; \quad \int u\,\mathrm{d}u = \frac{u^2}{2} \\
&= \frac{5\pi R^3}{3}\sin^2\theta\Big|_0^{\pi/2} = \frac{5\pi R^3}{3}
\end{aligned}$$

The surface integral is split into two parts: the top surface of the hemisphere and the circular base.
(i) The top surface of the hemisphere where $r = R$:

$$\begin{aligned}
\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a}_1 &= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)(r^2\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= r^3\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\
&= \pi r^3 = \pi R^3
\end{aligned}$$

(ii) The circular base of the hemisphere where $\theta = \pi/2$ and $\mathbf{a}_2 = r\,\mathrm{d}r\,\mathrm{d}\phi\,\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}
\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a}_2 &= \int_{r=0}^{R}\int_{\phi=0}^{2\pi}(r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\phi \\
&= \sin(\pi/2)\int_0^R r^2\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi \\
&= (1)\frac{R^3}{3}(2\pi) = \frac{2\pi R^3}{3}
\end{aligned}$$

So, the total surface integral is

$$\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a}_1 + \oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a}_2 = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

1.41

$$T = r(\cos\theta + \sin\theta\cos\phi)$$

The partial derivatives are:

$$\begin{aligned}\frac{\partial T}{\partial r} &= \cos\theta + \sin\theta\cos\phi\\ \frac{\partial T}{\partial \theta} &= r(-\sin\theta + \cos\theta\cos\phi)\\ \frac{\partial T}{\partial \phi} &= -r\sin\theta\sin\phi\end{aligned}$$

thus, the gradient of T in spherical is

$$\begin{aligned}\boldsymbol{\nabla} T &= \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}\\ &= (\cos\theta + \sin\theta\cos\phi)\hat{\mathbf{r}} + (-\sin\theta + \cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - (\sin\phi)\hat{\boldsymbol{\phi}}\end{aligned}$$

and partial derivative in the laplacian are:

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) &= \frac{\partial}{\partial r}\left(r^2(\cos\theta + \sin\theta\cos\phi)\right) = 2r(\cos\theta + \sin\theta\cos\phi)\\ \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) &= r\frac{\partial}{\partial \theta}(\sin\theta(-\sin\theta + \cos\theta\cos\phi)) = r\frac{\partial}{\partial \theta}\left(-\sin^2\theta + \sin\theta\cos\theta\cos\phi\right)\\ &= -2r\sin\theta\cos\theta + r\cos^2\theta\cos\phi - r\sin^2\theta\cos\phi\\ \frac{\partial^2 T}{\partial \phi^2} &= -r\sin\theta\cos\phi\end{aligned}$$

The laplacian of T in spherical is

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

Simplifying each term: (i) The first term:

$$\frac{2}{r}(\cos\theta + \sin\theta\cos\phi)$$

(ii) The second term:

$$-\frac{2}{r}\cos\theta + \frac{\cos^2\theta\cos\phi}{r\sin\theta} - \frac{1}{r}\sin\theta\cos\phi$$

(iii) The third term:

$$-\frac{\cos\phi}{r\sin\theta}$$

adding all three terms:

$$\begin{aligned}\nabla^2 T &= \frac{2}{r}(\cos\theta + \sin\theta\cos\phi) - \frac{2}{r}\cos\theta + \frac{\cos^2\theta\cos\phi}{r\sin\theta} - \frac{1}{r}\sin\theta\cos\phi - \frac{\cos\phi}{r\sin\theta}\\ &= \frac{2}{r}(\sin\theta\cos\phi) + \frac{\cos^2\theta\cos\phi - \cos\phi}{r\sin\theta} - \frac{1}{r}\sin\theta\cos\phi\\ &= \frac{2}{r\sin\theta}(\sin^2\theta\cos\phi) + \frac{\cos^2\theta\cos\phi - \cos\phi}{r\sin\theta} - \frac{1}{r\sin\theta}\sin^2\theta\cos\phi\\ &= \frac{1}{r\sin\theta}(\sin^2\theta\cos\phi + \cos^2\theta\cos\phi - \cos\phi)\\ &= \frac{1}{r\sin\theta}(\cos\phi)(\sin^2\theta + \cos^2\theta - 1) = 0\end{aligned}$$

Converting T to Cartesian coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad \text{and} \quad z = r\cos\theta$$

So

$$T = z + x$$

The laplacian of T in Cartesian coordinates is

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Testing the gradient theorem using the path $O \to A = (2,0,0) \to B = (0,2,0) \to C = (0,0,2)$: Given the general infinitesimal displacement

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}$$

and the gradient of T in spherical coordinates

$$\boldsymbol{\nabla}T = (\cos\theta + \sin\theta\cos\phi)\hat{\mathbf{r}} + (-\sin\theta + \cos\theta\cos\phi)\hat{\boldsymbol{\theta}} + (-\sin\phi)\hat{\boldsymbol{\phi}}$$

(i) On the path OA:

$$\theta = \pi/2, \quad \phi = 0, \quad \mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}}; \quad (\boldsymbol{\nabla}T)\cdot\mathrm{d}\mathbf{l} = 1\,\mathrm{d}r$$

So

$$\int_{OA}(\boldsymbol{\nabla}T)\cdot\mathrm{d}\mathbf{l} = \int_0^2 1\,\mathrm{d}r = 2$$

(ii) On the path AB:

$$r = 2, \quad \theta = \pi/2, \quad \mathrm{d}\mathbf{l} = 2\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}; \quad (\boldsymbol{\nabla}T)\cdot\mathrm{d}\mathbf{l} = -2\sin\phi\,\mathrm{d}\phi$$

So

$$\int_{AB}(\boldsymbol{\nabla}T)\cdot\mathrm{d}\mathbf{l} = \int_0^{\pi/2} -2\sin\phi\,\mathrm{d}\phi = -2$$

(iii) On the path BC:

$$r = 2, \quad \phi = \pi/2, \quad \mathrm{d}\mathbf{l} = 2\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}}; \quad (\boldsymbol{\nabla}T)\cdot\mathrm{d}\mathbf{l} = -2\sin\theta\,\mathrm{d}\theta$$

So

$$\int_{BC}(\boldsymbol{\nabla}T)\cdot\mathrm{d}\mathbf{l} = \int_{\pi/2}^{0} -2\sin\theta\,\mathrm{d}\theta = 2$$

therefore the total line integral is

$$\int_{OC}(\boldsymbol{\nabla}T)\cdot\mathrm{d}\mathbf{l} = 2 + -2 + 2 = 2$$

For the left hand side of the gradient theorem:
At C:

$$r = 2, \quad \theta = 0, \quad \phi = 0; \quad T = 2(\cos 0 + \sin 0\cos 0) = 2$$

At O:

$$r = 0; \quad T = 0$$

So

$$T(C) - T(O) = 2 + 0 = 2$$

which is the same as the total line integral, so the gradient theorem holds.

1.42 Cylindrical coordinates are related to Cartesian coordinates by

$$x = s\cos\phi, \quad y = s\sin\phi, \quad \text{and} \quad z = z$$

and the position vector is

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$
$$\mathbf{r} = s\cos\phi\hat{\mathbf{x}} + s\sin\phi\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

The unit vectors are in the same direction as the partial derivatives with respect to s, ϕ, and z, so

$$\hat{\mathbf{s}} = \frac{e_s}{|e_s|}, \quad \hat{\boldsymbol{\phi}} = \frac{e_\phi}{|e_\phi|}, \quad \text{and} \quad \hat{\mathbf{z}} = \frac{e_z}{|e_z|}$$

where e_u is the new basis vector given by

$$e_u = \frac{\partial \mathbf{r}}{\partial u}$$

The partial derivatives are

$$e_s = \frac{\partial \mathbf{r}}{\partial s} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$
$$e_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = -s\sin\phi\hat{\mathbf{x}} + s\cos\phi\hat{\mathbf{y}}$$
$$e_z = \frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{z}}$$

and the magnitude

$$|e_s| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$
$$|e_\phi| = \sqrt{s^2\sin^2\phi + s^2\cos^2\phi} = s$$
$$|e_z| = 1$$

thus, the unit vectors are:

$$\hat{\mathbf{s}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

The cylindrical unit vectors in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in matrix form:

$$\begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

Which is an orthogonal matrix $a = Qx$, so the Cartesian unit vectors is found by multiplying a by the transpose of Q:

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

or in vector form:

$$\hat{\mathbf{x}} = \cos\phi\hat{\mathbf{s}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{y}} = \sin\phi\hat{\mathbf{s}} + \cos\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

1.43 (a) Finding the divergence of the function

$$\mathbf{v} = s(2+\sin^2\phi)\hat{\mathbf{s}} + s\sin\phi\cos\phi\hat{\boldsymbol{\phi}} + 3z\hat{\mathbf{z}}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
&= \frac{1}{s}\frac{\partial}{\partial s}\big(s(s(2+\sin^2\phi))\big) + \frac{1}{s}\frac{\partial}{\partial \phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z) \\
&= 2(2+\sin^2\phi) + \cos^2\phi - \sin^2\phi + 3 \\
&= 7 + \sin^2\phi + \cos^2\phi = 8
\end{aligned}$$

(b) Testing divergence theorem using a quarter cylinder of radius 2 and height 5 in quadrant I:
LHS: The volume elements is

$$\mathrm{d}V = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z\,,$$

so the volume integral is

$$\begin{aligned}
\int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\cdot\mathrm{d}V &= \int_{s=0}^{2}\int_{\phi=0}^{\pi/2}\int_{z=0}^{5} 8(s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z) \\
&= 8\int_0^2 s\,\mathrm{d}s \int_0^{\pi/2}\mathrm{d}\phi\int_0^5\mathrm{d}z \\
&= 8(2)(\pi/2)(5) = 40\pi
\end{aligned}$$

RHS: There are 5 surfaces: the top, bottom, and 3 sides.
(i) The top surface:

$$z=5, \quad \mathrm{d}a = s\,\mathrm{d}s\,\mathrm{d}\phi\,\hat{\mathbf{z}}; \quad \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 15s\,\mathrm{d}s\,\mathrm{d}\phi$$

So

$$\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a} = \int_{\phi=0}^{\pi/2}\int_{s=0}^{2} 15s\,\mathrm{d}s\,\mathrm{d}\phi = 15\int_0^{\pi/2}\mathrm{d}\phi\int_0^2 s\,\mathrm{d}s = 15\pi$$

(ii) The bottom surface:

$$z=0, \quad \mathrm{d}a = s\,\mathrm{d}s\,\mathrm{d}\phi\,\hat{\mathbf{z}}; \quad \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 0$$

(iii) The surface on the xy plane:

$$\phi=\pi/2 \quad \mathrm{d}a = \mathrm{d}s\,\mathrm{d}z\,\hat{\boldsymbol{\phi}}; \quad \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 0$$

(iv) The surface on the xz plane:

$$\phi=0, \quad \mathrm{d}a = -\,\mathrm{d}s\,\mathrm{d}z\,\hat{\boldsymbol{\phi}}; \quad \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 0$$

(v) The curved surface:

$$s=2, \quad \mathrm{d}a = 2\,\mathrm{d}\phi\,\mathrm{d}z\,\hat{\mathbf{s}}; \quad \mathbf{v}\cdot\mathrm{d}\mathbf{a} = 4(2+\sin^2\phi)\,\mathrm{d}\phi\,\mathrm{d}z$$

So

$$
\begin{aligned}
\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} &= \int_{z=0}^{5} \int_{\phi=0}^{\pi/2} 4(2 + \sin^2\phi)\,\mathrm{d}\phi\,\mathrm{d}z \\
&= 4\int_0^5 \mathrm{d}z \int_0^{\pi/2} (4(1/2) + \sin^2\phi)\,\mathrm{d}\phi \\
\text{using} \quad \sin^2\phi &= \frac{1-\cos(2\phi)}{2} \\
&= 4\int_0^5 \mathrm{d}z \int_0^{\pi/2} (2 + (1-\cos(2\phi))/2)\,\mathrm{d}\phi \\
&= 10\int_0^{\pi/2} (5 - \cos(2\phi))\,\mathrm{d}\phi \\
&= 10\left(5\phi - \frac{\sin(2\phi)}{2}\bigg|_0^{\pi/2} \right) = 25\pi
\end{aligned}
$$

So the total surface integral is

$$
\oint_S \mathbf{v} \cdot \mathrm{d}\mathbf{a} = 15\pi + 0 + 0 + 0 + 25\pi = 40\pi
$$

which is the same as the volume integral, so the divergence theorem holds.
(c) The curl of $\mathbf{v}$ in cylindrical coordinates is

$$
\begin{aligned}
\nabla \times \mathbf{v} &= \frac{1}{s}\begin{vmatrix} \hat{\mathbf{s}} & s\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ v_s & (s)v_\phi & v_z \end{vmatrix} \\
&= \frac{1}{s}\begin{vmatrix} \hat{\mathbf{s}} & s\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ s(2+\sin^2\phi) & (s)s\sin\phi\cos\phi & 3z \end{vmatrix} \\
&= 0\hat{\mathbf{s}} - \frac{1}{s}(0-0) + \frac{1}{s}(2s\sin\phi\cos\phi - 2s\sin\phi\cos\phi) \\
&= 0
\end{aligned}
$$

1.44 (a)

$$
\int_2^6 (3x^2 - 2x - 1)\delta(x-3)\,\mathrm{d}x = (3(3)^2 - 2(3) - 1) = 20
$$

(b)

$$
\int_0^5 \cos x\,\delta(x-\pi)\,\mathrm{d}x = \cos\pi = -1
$$

(c)

$$
\int_0^3 x^3\delta(x+1)\,\mathrm{d}x = 0
$$

since $\delta(x+1) = 0$ when $x = -1$ from the bounds of the integral $x = 0 \to 3$.
(d)

$$
\int_\infty^{-\infty} \ln(x+3)\delta(x+2)\,\mathrm{d}x = \ln(-2+3) = 0
$$

1.45 (a)

$$\int_{-2}^{2}(2x+3)\delta(3x)\,\mathrm{d}x = \int_{-2}^{2}(2x+3)\frac{1}{3}\delta(x)\,\mathrm{d}x = \frac{1}{3}(2(0)+3) = 1$$

(b) Since $\delta(1-x) = \delta(-(x-1)) = \delta(x-1)$:

$$\int_{0}^{2}(x^3+3x+2)\delta(1-x)\,\mathrm{d}x = \int_{0}^{2}(x^3+3x+2)\delta(x-1)\,\mathrm{d}x = (1)^3+3(1)+2 = 6$$

(c) Using $\delta(3x-1) = \frac{1}{3}\delta(x-1/3)$:

$$\int_{-1}^{1}9x^2\delta(3x-1)\,\mathrm{d}x = \int_{-1}^{1}3x^2\delta(x-1/3)\,\mathrm{d}x = 3(1/3)^2 = 1/3$$

(d)

$$\int_{-\infty}^{a}\delta(x-b)\,\mathrm{d}x = \begin{cases} 1 & \text{if } a > b \\ 0 & \text{if } a < b \end{cases}$$

1.46 (a) Using integration by parts:

$$\begin{aligned}
\int_{-a}^{a} x\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x))\,\mathrm{d}x &= x\delta(x)\Big|_{a}^{a} - \int_{a}^{a} x\delta(x)\,\mathrm{d}x \\
&= x\delta(x)\Big|_{a}^{a} - (0) = 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}(x\delta(x)) &= x\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x)) + \delta(x) = 0 \\
\text{thus}\quad x\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x)) &= -\delta(x)
\end{aligned}$$

where $\delta(a) = \delta(-a) = 0$

(b) Using integration by parts again:

$$\begin{aligned}
\int_{-a}^{a} f(x)\frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x &= f(x)\theta(x)\Big|_{-a}^{a} - \int_{-a}^{a}\frac{\mathrm{d}f}{\mathrm{d}x}\theta(x)\,\mathrm{d}x \\
&= f(a) - \int_{0}^{a}\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \\
&= f(a) - (f(a) - f(0)) = f(0) \\
\int_{-a}^{a} f(x)\frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x &= f(0)
\end{aligned}$$

which is the same as the definition of the delta function, so

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \delta(x)$$

1.47 (a) The expression for volume charge density of a point charge q at $\mathbf{r}'$ where

$$\int_V \rho(\mathbf{r})\,\mathrm{d}V = q \quad \text{at} \quad \mathbf{r} = \mathbf{r}'$$

is

$$\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}')$$

(b) The volume charge density of an electric dipole:

$$\begin{aligned}\rho(\mathbf{r}_d) &= \rho(\mathbf{r})\Big|_{\mathbf{r}'=\mathbf{r}} - \rho(\mathbf{r})\Big|_{\mathbf{r}'=0} \\ &= q\delta^3(\mathbf{r}-\mathbf{a}) - q\delta^3(\mathbf{r})\end{aligned}$$

where $\mathbf{r}_d$ is the position vector of the dipole. (c) The volume charge density of a spherical shell of radius R and total charge Q:

$$\int_V f\delta^3(r-R)\,\mathrm{d}V = f(R) = Q$$

where V is all space, so

$$Q = \int_0^\infty \int_0^\pi f\delta(r-R)(r^2\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

and since the charge density is uniform in θ and ϕ, f only depends on r; $f = f(r)$.

$$\begin{aligned}Q &= \int_0^\infty f(r)r^2\delta(r-R)\,\mathrm{d}r\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\ &= R^2[f(R)](2)(2\pi) = 4\pi R^2 f(R)\end{aligned}$$

and since f is constant,

$$f = \frac{Q}{4\pi R^2}$$

thus, the volume charge density is

$$\rho(r) = \frac{Q}{4\pi R^2}\delta^3(r-R)$$

1.48 (a)

$$\int(r^2+\mathbf{r}\cdot\mathbf{a}+a^2)\delta^3(\mathbf{r}-\mathbf{a})\,\mathrm{d}V = a^2+\mathbf{a}\cdot\mathbf{a}+a^2 = 3a^2$$

(b) Given V is a cube of side 2 centered at the origin and $\mathbf{b} = 4\hat{\mathbf{y}}+3\hat{\mathbf{z}}$:

$$\begin{aligned}\int_V |\mathbf{r}-\mathbf{b}|^2\delta^3(5\mathbf{r})\,\mathrm{d}V &= \int_0^1\int_0^1\int_0^1 |\mathbf{r}-\mathbf{b}|^2\delta(5x)\delta(5y)\delta(5z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{1}{5^3}|-\mathbf{b}|^2 = \frac{1}{5^3}(4^2+3^2) = \frac{1}{5}\end{aligned}$$

(c) Given V is a sphere of radius 6 centered at the origin and $\mathbf{c} = 5\hat{\mathbf{x}}+3\hat{\mathbf{y}}+2\hat{\mathbf{z}}$:
The magnitude $c = \sqrt{5^2+3^2+2^2} = \sqrt{38}$, is outside the sphere (magnitude $r = \sqrt{36}$). Therefore,

$$\int_V [r^4+r^2(\mathbf{r}\cdot\mathbf{c})+c^4]\delta^3(\mathbf{r}-\mathbf{c})\,\mathrm{d}V = 0$$

(d) Given V is a sphere of radius 1.5 centered at $(2,2,2)$ and

$$\mathbf{d} = (1,2,3), \quad \mathbf{e} = (3,2,1)$$

Checking if the delta function is inside the sphere:

$$|\mathbf{e}-(2,2,2)| = \sqrt{1+0+1} = \sqrt{2} < 1.5$$

so the delta function is inside the sphere and the integral is

$$\begin{aligned}\int_V \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})\,\mathrm{d}V &= \int_V \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{r}-\mathbf{e})\,\mathrm{d}V \\ &= \mathbf{e}\cdot(\mathbf{d}-\mathbf{e}) \\ &= (3,2,1)\cdot(-2,0,2) = -4\end{aligned}$$

1.49 Evaluating the integral

$$J = \int_V e^{-r} \left(\boldsymbol{\nabla} \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) \mathrm{d}V$$

where V is a sphere of radius R centered at the origin.
(i)

$$J = \int_V e^{-r} 4\pi \delta^3(\mathbf{r}) \, \mathrm{d}V = 4\pi e^0 = 4\pi$$

(ii) Using the product rule of divergence:

$$\boldsymbol{\nabla} \cdot (f\mathbf{A}) = f(\boldsymbol{\nabla} \cdot \mathbf{A}) + \mathbf{A} \cdot (\boldsymbol{\nabla} f)$$

integrating over a volume and using the divergence theorem:

$$\int \boldsymbol{\nabla} \cdot (f\mathbf{A}) \, \mathrm{d}V = \int f(\boldsymbol{\nabla} \cdot \mathbf{A}) \, \mathrm{d}V + \int \mathbf{A} \cdot (\boldsymbol{\nabla} f) \, \mathrm{d}V = \oint f\mathbf{A} \cdot \mathrm{d}\mathbf{a}$$

or

$$\int_V f(\boldsymbol{\nabla} \cdot \mathbf{A}) \, \mathrm{d}V = -\int_V \mathbf{A} \cdot (\boldsymbol{\nabla} f) \, \mathrm{d}V + \oint f\mathbf{A} \cdot \mathrm{d}\mathbf{a}$$

where

$$f = e^{-r} \quad \text{and} \quad \mathbf{A} = \frac{\hat{\mathbf{r}}}{r^2}$$

(a) Computing the first term:

$$\boldsymbol{\nabla} f = -e^{-r} \hat{\mathbf{r}}; \quad \mathbf{A} \cdot (\boldsymbol{\nabla} f) = -\frac{e^{-r}}{r^2}$$

So

$$\begin{aligned}
\int_V \frac{e^{-r}}{r^2} \, \mathrm{d}V &= \int_0^R \int_0^\pi \int_0^{2\pi} \frac{e^{-r}}{r^2} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \int_0^R \int_0^\pi \int_0^{2\pi} e^{-r} \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= 4\pi(1 - e^{-R})
\end{aligned}$$

(b) For the second term: the surface element is on the boundary of the sphere $r = R$.

$$\mathrm{d}\mathbf{a} = R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \hat{\mathbf{r}}; \quad f\mathbf{A} \cdot \mathrm{d}\mathbf{a} = e^{-R} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

So

$$e^{-R} \int_0^\pi \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi = 4\pi e^{-R}$$

adding (a) and (b) gives the total volume integral:

$$J = 4\pi(1 - e^{-R}) + 4\pi e^{-R} = 4\pi$$

1.50 (a)

$$\mathbf{F}_1 = x^2 \hat{\mathbf{z}} \quad \text{and} \quad \mathbf{F}_2 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

The divergence of the two vector fields:

$$\boldsymbol{\nabla} \cdot \mathbf{F}_1 = 0 \quad \text{and} \quad \boldsymbol{\nabla} \cdot \mathbf{F}_2 = 3$$

and curl:

$$\nabla \times \mathbf{F}_1 = -2x\hat{\mathbf{y}} \quad \text{and} \quad \nabla \times \mathbf{F}_2 = 0$$

Thus $\mathbf{F}_2$ can be written as a gradient of some scalar potential

$$\mathbf{F}_2 = -\nabla V \quad \text{where} \quad V = -\frac{1}{2}(x^2 + y^2 + z^2)$$

and $\mathbf{F}_1$ can be written as a curl of some vector potential

$$\mathbf{F}_1 = \nabla \times \mathbf{A}$$

where

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, \quad \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = 0, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = x^2$$

so one solution can be found setting

$$A_y = x^3/3, \quad A_y = 0, \quad A_z = 0$$

thus $A = x^3/3\hat{\mathbf{x}}$.
(b) Finding the vector and scalar potentials of the vector field

$$\mathbf{F}_3 = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$$

The gradient of a scalar:

$$\mathbf{F}_3 = -\nabla V_3$$

so

$$\frac{\partial V_3}{\partial x} = -yz, \quad \frac{\partial V_3}{\partial y} = -zx, \quad \frac{\partial V_3}{\partial z} = -xy$$

A particular solution is

$$V_3 = -xyz$$

The curl of a vector (a little harder):

$$\mathbf{F}_3 = \nabla \times \mathbf{V}_3$$

so

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, \quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = zx, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy$$

integrating the first equation with respect to y:

$$\frac{\partial A_z}{\partial y} = yz + \frac{\partial A_y}{\partial z}$$
$$\int \frac{\partial A_z}{\partial y}\,\mathrm{d}y = \int yz\,\mathrm{d}y + \int \frac{\partial A_y}{\partial z}\,\mathrm{d}y$$
$$A_z = \frac{1}{2}y^2 z + f$$

where the function f is an arbitrary function. Integrating the first equation again but with respect to z gives:

$$A_y = -\frac{1}{2}yz^2 + g$$

Repeating for all three equations:

$$\begin{aligned}
A_z &= \frac{1}{2}y^2z + f \quad \text{and} \quad A_y = -\frac{1}{2}yz^2 + g \\
A_x &= \frac{1}{2}z^2x + h \quad \text{and} \quad A_z = -\frac{1}{2}zx^2 + i \\
A_y &= \frac{1}{2}x^2y + j \quad \text{and} \quad A_x = -\frac{1}{2}xy^2 + k
\end{aligned}$$

a particular solution can be found by setting $f = g = h = i = j = k = 0$, and each component is a linear combination of the two possible solutions:

$$\begin{aligned}
2A_x &= \frac{1}{2}z^2x + f - \frac{1}{2}xy^2 + k \\
2A_x &= \frac{1}{2}z^2x - \frac{1}{2}xy^2 \\
A_x &= \frac{1}{4}(z^2x - xy^2)
\end{aligned}$$

and similarly for A_y and A_z gives the vector potential:

$$\mathbf{A} = \frac{1}{4}[x(z^2 - y^2)\hat{\mathbf{x}} + y(x^2 - z^2)\hat{\mathbf{y}} + z(y^2 - x^2)\hat{\mathbf{z}}]$$

1.51 $(d) \to (a)$: Given $\mathbf{F} = -\boldsymbol{\nabla}V$, the curl of the gradient is always zero;

$$\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} \times (-\boldsymbol{\nabla}V) = 0$$

$(a) \to (c)$: From Stokes' theorem;

$$\int_S (\boldsymbol{\nabla} \times F)\,\mathrm{d}\mathbf{a} = \oint F \cdot \mathrm{d}\mathbf{l} = 0$$

$(c) \to (b)$:

$$\oint F \cdot \mathrm{d}\mathbf{l} = \int_a^b F \cdot \mathrm{d}\mathbf{l} - \int_a^b F \cdot \mathrm{d}\mathbf{l} = 0$$

where the integrals are two different paths but equal, and thus independent of path. $(b) \to (c)$ and $(c) \to (a)$ are just the same steps in reverse.

1.52 $(d) \to (a)$: Given $\mathbf{F} = \boldsymbol{\nabla} \times A$, the divergence of curl is always zero;

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times A) = 0$$

$(a) \to (c)$: From the divergence theorem;

$$\int_V (\boldsymbol{\nabla} \cdot F)\,\mathrm{d}V = \oint F \cdot \mathrm{d}\mathbf{a} = 0$$

$(c) \to (b)$:

$$\oint F \cdot \mathrm{d}\mathbf{a} = \int F \cdot \mathrm{d}\mathbf{a}_1 + \int F \cdot \mathrm{d}\mathbf{a}_2 = 0$$

where $\mathbf{a}_2 = -\mathbf{a}_1$ and the integrals are two different surfaces but equal, and thus depend only on the boundary. $(b) \to (c)$ and $(c) \to (a)$ are just the same steps in reverse.

1.53 (a) From Problem 1.15,

$$\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$$
$$\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$$
$$\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$$

and the curl of each vector field:

$$\boldsymbol{\nabla} \times \mathbf{v}_a \neq 0$$
$$\boldsymbol{\nabla} \times \mathbf{v}_b \neq 0$$
$$\boldsymbol{\nabla} \times \mathbf{v}_c = 0$$

Thus $\mathbf{v}_c$ can be written as a gradient of some scalar potential $\mathbf{v}_c = -\boldsymbol{\nabla} V$:

$$\frac{\partial V}{\partial x} = -y^2; \quad V = -y^2 x + f$$
$$\frac{\partial V}{\partial y} = -(2xy + z^2); \quad V = -y^2 x - yz^2 + g$$
$$\frac{\partial V}{\partial z} = -2yz; \quad V = -yz^2 + h$$

where f, g, and h are arbitrary functions, thus one solution can be found setting $g = 0$ and solving for f or h;

$$V = -y^2 x - yz^2$$

(b) Given $\boldsymbol{\nabla} \cdot \mathbf{v}_a = 0$ thus $\mathbf{v}_a$ can be written as a curl of some vector potential $\mathbf{v}_a = \boldsymbol{\nabla} \times \mathbf{A}$:

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = x^2; \qquad A_z = x^2 y + f, \qquad A_y = -x^2 z + g$$
$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 3xz^2; \quad A_x = xz^3 + h, \quad A_z = -\frac{3}{2}x^2 z^2 + i$$
$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -2xz; \quad A_y = -x^2 z + j, \qquad A_x = 2xyz + k$$

where f, g, h, i, j, and k are arbitrary functions where a solution can be found by setting one function to zero and solving for the others: e.g., $h = 0$;

$$A_x = xz^3; \quad A_z = 0; \quad A_y = -x^2 z$$

or

$$\mathbf{A} = xz^3\hat{\mathbf{x}} - x^2 z\hat{\mathbf{y}}$$

Checking the solution

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -x^2 z & 0 \end{vmatrix}$$
$$= x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$$

Another solution can be found by setting $f = 0$:

$$A_z = x^2 y; \quad A_y = 0; \quad A_x = xz^3 + 2xyz$$

or

$$\mathbf{A} = 2xyz\hat{\mathbf{x}} + \frac{1}{3}x^3y\hat{\mathbf{z}}$$

Checking the solution

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 + 2xyz & 0 & x^2y \end{vmatrix} \\ &= x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}\end{aligned}$$

Another solution can be found by setting all arbitrary functions to zero and taking the linear combination of the two solutions:

$$2A_x = xz^3 + 2xyzA_x = \frac{1}{2}(xz^3 + 2xyz)$$

and thus

$$A_y = \frac{1}{2} - x^2z; \quad A_z = \frac{1}{2}x^2y - \frac{3}{4}x^2z^2$$

or

$$\mathbf{A} = \frac{1}{2}(xz^3 + 2xyz)\hat{\mathbf{x}} - \frac{1}{2}x^2z\hat{\mathbf{y}} + \frac{1}{2}x^2y - \frac{3}{4}x^2z^2\hat{\mathbf{z}}$$

Checking the solution

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}(xz^3 + 2xyz) & -\frac{1}{2}x^2z & \frac{1}{2}x^2y - \frac{3}{4}x^2z^2 \end{vmatrix} \\ &= x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}\end{aligned}$$