1 Vector Analysis

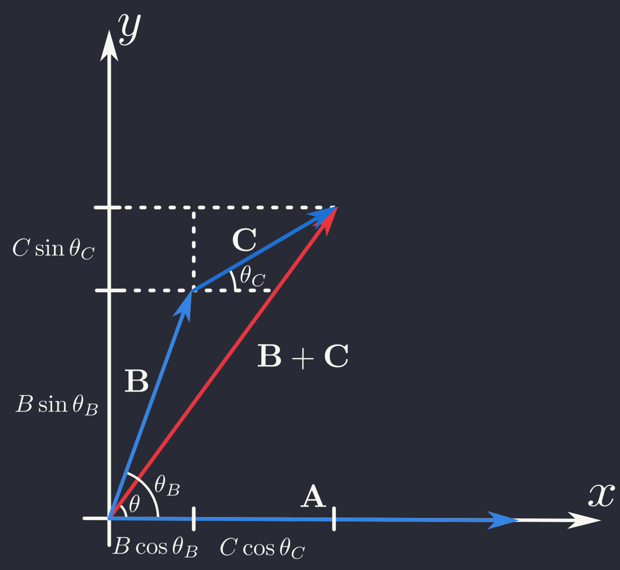

Figure 1.1: Three Coplanar Vectors

1.1 (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\\ A(B+C)\cos\theta &= AB\cos\theta_B + AC\cos\theta_C\end{aligned}$$

Since $B\cos\theta_B + B\cos\theta_C = (B+C)\cos\theta$ from Figure 1.1, the distriubtive property holds true. The cross product also holds true since $B\sin\theta_B + B\sin\theta_C = (B+C)\sin\theta$, and multiplying by A on both sides gives the same result as the distriubtive property:

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}\\ A(B+C)\sin\theta &= AB\sin\theta_B + AC\sin\theta_C\end{aligned}$$

(b) In the general case in three-dimensional space, each vector has three components:
$\mathbf{A} = (A_x, A_y, A_z)$. Therefore,

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot(B_x+C_x, B_y+C_y, B_z+C_z)\\ &= A_x(B_x+C_x)+A_y(B_y+C_y)+A_z(B_z+C_z)\\ &= A_xB_x+A_yB_y+A_zB_z+A_xC_x+A_yC_y+A_zC_z\\ &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\end{aligned}$$

1.2 Setting $\mathbf{A}=\mathbf{B}=(1,1,1)$ and $\mathbf{C}=(1,1,-1)$:

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times\mathbf{C} &\stackrel{?}{=} \mathbf{A}\times(\mathbf{B}\times\mathbf{C})\\ 0 &\stackrel{?}{=} (1,1,1)\times[(1,1,1)\times(1,1,-1)]\\ 0 &\stackrel{?}{=} (1,1,1)\times(-2,2,0)\\ 0 &\neq (-2,-2,4)\end{aligned}$$

where the cross product of parallel vectors $\mathbf{A}\times\mathbf{B}=0$. Therefore, the cross product is not associative.

1.3 Taking the dot product of a unit cube's body diagonals $\mathbf{A} = (1, 1, 1)$, $\mathbf{B} = (1, 1, -1)$:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= AB\cos\theta \\ 1 &= 3\cos\theta \\ \theta &= \arccos 1/3 \approx 70.53^\circ \end{aligned}$$

1.4 The cross product of two vectors coplanar to the shaded plane—$\mathbf{A} = (-1, 2, 0)$, $\mathbf{B} = (-1, 0, 3)$—is parallel to the normal unit vector $\hat{\mathbf{n}}$ of the plane:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \mathbf{C} \\ \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} &= (6, 3, 2) \end{aligned}$$

where $\hat{\mathbf{n}} = \mathbf{C}/C$, and $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$. Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

1.5 Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix} \end{aligned}$$

where the x component is $A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$. Similarly,

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z)$$

where x component simplifes to

$$B_x(\cancel{A_xC_x} + A_yC_y + A_zC_z) - C_x(\cancel{A_xB_x} + A_yB_y + A_zB_z) = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)$$

the same is done for the y and z components. Therefore, the "BAC–CAB" rule holds true.

1.6

$$\begin{aligned} [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) = 0 \\ &\quad - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C} \cdot \mathbf{A})} \end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \\ \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \times \mathbf{B} \end{aligned}$$

For the relation to hold true, either the vectors $\mathbf{A}$ and $\mathbf{C}$ are parallel $(\mathbf{A} \times \mathbf{A} = 0)$ or $\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{C}$ $(\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = 0)$.

1.7 Finding the seperation vector $\boldsymbol{\mathscr{r}}$:

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' = (4, 6, 8) - (2, 8, 7) = (2, -2, 1) \\
\mathscr{r} &= \sqrt{2^2 + (-2)^2 + 1^2} = 3 \\
\hat{\boldsymbol{\mathscr{r}}} &= \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)
\end{aligned}$$

1.8 (a)

$$\begin{aligned}
\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z = (A_y\cos\phi + A_z\sin\phi)(B_y\cos\phi + B_z\sin\phi) \\
+ (-A_y\sin\phi + A_z\cos\phi)(-B_y\sin\phi + B_z\cos\phi) \\
= A_yB_y\cos^2\phi + A_zB_z\sin^2\phi + \cancel{A_yB_z\sin\phi\cos\phi} + \cancel{A_zB_y\sin\phi\cos\phi} \\
+A_yB_y\sin^2\phi - \cancel{A_yB_z\sin\phi\cos\phi} - \cancel{A_zB_y\sin\phi\cos\phi} + A_zB_z\cos^2\phi \\
= A_yB_y(\sin^2\phi + \cos^2\phi) + A_zB_z(\sin^2\phi + \cos^2\phi) \\
\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z = A_yB_y + A_zB_z
\end{aligned}$$

(b) To preserve length $|\bar{A}| = |A|$. Squaring both sides and expanding gives

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_x^2 + A_y^2 + A_z^2$$

in summation form,

$$\sum_{i=1}^{3}\bar{A}_i\bar{A}_i = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}R_{ij}A_j\right)\left(\sum_{k=1}^{3}R_{ik}A_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}R_{ij}R_{ik}A_jA_k$$

For the length to be preserved, the indices j and k must be equal. Therefore,

$$R_{ij}R_{ik} = \delta_{jk}$$

where δ_{ij} is the Kronecker delta. Thus the length is preserved if the rotation matrix is orthogonal, i.e.

$$R_{ij}R_{ik} = (R^T)_{ji}R_{ik} = \delta_{jk} \quad \text{or} \quad R^TR = I$$

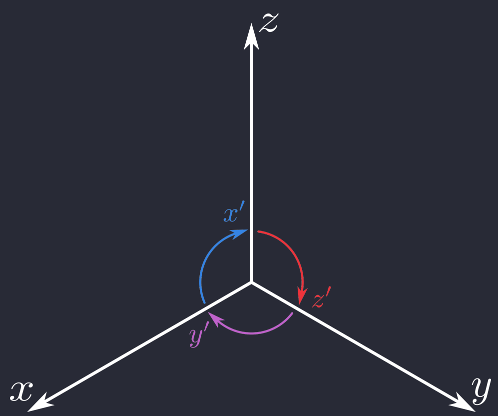

Figure 1.2: Rotation of 120° about an axis through the origin and point $(1, 1, 1)$

1.9 From Figure 1.2, the rotation is equivalent to changing the position of the basis vectors $\hat{\mathbf{x}} \to \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \to \hat{\mathbf{y}}$. Therefore, the rotation matrix is a permutation matrix:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1.10 (a) Under a **translation** of coordinates $\bar{y} = y - a$, the origin O and terminus $A = (x, y, z)$ of some vector are translated to

$$\begin{aligned} O &\to O' = (0, -a, 0) \\ A &\to A' = (x, y - a, z) \end{aligned}$$

therefore, the translated vector is

$$\overline{O'A'} = (x, y - a, z) - (0, -a, 0) = (x, y, z) = \overline{OA} = \mathbf{A}$$

which is the same as the original vector, so

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) Under **inversion** of coordinates, only the terminus changes

$$\begin{aligned} O &\to O' = (0, 0, 0) \\ A &\to A' = (-x, -y, -z) \end{aligned}$$

therefore, the inverted vector is

$$\overline{O'A'} = (-x, -y, -z) \quad \text{or} \quad \begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

(c) The components of cross product under inversion

$$\bar{\mathbf{A}} \times \bar{\mathbf{B}} = \begin{pmatrix} \bar{A}_y \bar{B}_z - \bar{A}_z \bar{B}_y \\ \bar{A}_z \bar{B}_x - \bar{A}_x \bar{B}_z \\ \bar{A}_x \bar{B}_y - \bar{A}_y \bar{B}_x \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}$$

which is the same as the original cross product $\mathbf{A} \times \mathbf{B}$. The cross product of two pseudovectors is also a pseudovector. Torque $\tau = \mathbf{r} \times \mathbf{F}$ and magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ are examples of pseudovectors.
(d) Scalar triple product under inversion

$$\begin{aligned} \bar{A} \cdot (\bar{B} \times \bar{C}) &= -\mathbf{A} \cdot (-\mathbf{B} \times -\mathbf{C}) \\ &= -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \end{aligned}$$

the scalar triple product changes sign under inversion.

1.11 (a) Finding gradient of $f(x, y, z) = x^2 + y^3 + z^4$:

$$\begin{aligned} \boldsymbol{\nabla} f &= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\ &= 2x\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + 4z^3\hat{\mathbf{z}} \end{aligned}$$

(b) Gradient of $f(x, y, z) = x^2 y^3 z^4$:

$$\boldsymbol{\nabla} f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$$

(c) Gradient of $f(x, y, z) = e^x \sin(y) \ln(z)$:

$$\boldsymbol{\nabla} f = e^x\hat{\mathbf{x}} + e^x \cos(y) \ln(z)\hat{\mathbf{y}} + \frac{e^x \sin(y)}{z}\hat{\mathbf{z}}$$

1.12 The height of the hill (in feet) is given by the function

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is north and x is east in miles. The gradient of h is

$$\boldsymbol{\nabla} h = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10(2x - 8y + 28)\hat{\mathbf{y}}$$

(a) The top of the hill is a stationary point, so the summit is found by setting the gradient to zero which gives the system of equations

$$0 = 2y - 6x - 18 \qquad\qquad 0 = 2x - 8y + 28$$

adding the first equation to 3 times the second equation gives

$$\begin{aligned} 0 &= 2y - 6x - 18 + 3(2x - 8y + 28) \\ 0 &= -22y + 66 \\ y &= 3 \end{aligned}$$

substituting $y = 3$ into the first equation

$$0 = 2(3) - 6x - 18 \rightarrow x = -2$$

Therefore, the top of the hill is at $(-2, 3)$ or 2 miles west and 3 miles north of the origin.
(b) The height of the hill is simply $h(-2, 3) = 10(12) = 720$ feet.
(c) The steepness of the hill at $h(1, 1)$ is given by the magnitude of the gradient

$$\begin{aligned} |\boldsymbol{\nabla} h| &= 10\sqrt{(2y - 6x - 18)^2 + (2x - 8y + 28)^2} \\ &= 10\sqrt{(2 - 6 - 18)^2 + (2 - 8 + 28)^2} \\ &= 10\sqrt{(-22)^2 + (22)^2} = 220\sqrt{2} \approx 311\,\mathrm{ft/mi} \end{aligned}$$

The direction of the steepest slope is given by the vector in the direction of the gradient at the point $\boldsymbol{\nabla} h(1, 1) = 220(-\mathbf{x} + \mathbf{y})$, or simply northwest.

1.13 Given the seperation vector

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(a) Show that $\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}} = 2\boldsymbol{\mathscr{r}}$$

(b)

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{z}}$$

looking at the x component,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right) &= -\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial x}(\mathscr{r}) \\ &= -\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial x}\left(\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\right) \\ &= -\frac{1}{\mathscr{r}^2}\frac{1}{2}\frac{2(x - x')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ &= -\frac{x - x'}{\mathscr{r}^3} \end{aligned}$$

therefore,

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}^3}[(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}] = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(c) The general formula is

$$\nabla(\mathscr{r}^n) = n\,\mathscr{r}^{n-1}\,\hat{\boldsymbol{\mathscr{r}}}$$

1.14 Given the rotation matrix

$$\begin{pmatrix}\bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix}$$

or the two equations

$$\bar{y} = y\cos\phi + z\sin\phi$$
$$\bar{z} = -y\sin\phi + z\cos\phi$$

differentiating with respect to $\bar{y}$ and $\bar{z}$ respectively gives

$$1 = \frac{\partial y}{\partial \bar{y}}\cos\phi + \frac{\partial z}{\partial \bar{y}}\sin\phi$$
$$1 = -\frac{\partial y}{\partial \bar{z}}\sin\phi + \frac{\partial z}{\partial \bar{z}}\cos\phi$$

this can only be true if

$$\frac{\partial y}{\partial \bar{y}} = \cos\phi, \quad \frac{\partial z}{\partial \bar{y}} = \sin\phi \quad \text{and} \quad \frac{\partial y}{\partial \bar{z}} = -\sin\phi, \frac{\partial z}{\partial \bar{z}} = \cos\phi$$

which satisfies the trig identity $\sin^2\phi + \cos^2\phi = 1$. Differentiating f with respect to the rotated coordinates $\bar{y}$ and $\bar{z}$ is given by

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}} = -\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi$$

therefore, the gradient of f transforms as a vector under rotations given by

$$\overline{\nabla f} = \frac{\partial f}{\partial \bar{y}}\hat{\bar{\mathbf{y}}} + \frac{\partial f}{\partial \bar{z}}\hat{\bar{\mathbf{z}}} = \left(\frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi\right)\hat{\bar{\mathbf{y}}} + \left(-\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi\right)\hat{\bar{\mathbf{z}}}$$

or in matrix form

$$\overline{\nabla f} = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\nabla f$$

where the gradient is a column vector

$$\nabla f = \begin{pmatrix}\frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{pmatrix}$$

1.15 (a) Calculating divergence of $v_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\nabla\cdot v_a = \frac{\partial v_{ax}}{\partial x} + \frac{\partial v_{ay}}{\partial y} + \frac{\partial v_{az}}{\partial z}$$
$$= 2x + 0 - 2x = 0$$

(b) $v_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\nabla\cdot v_b = y + 2z + 3x$$

(c) $v_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\nabla\cdot v_c = 0 + 2x + 2y = 2(x+y)$$

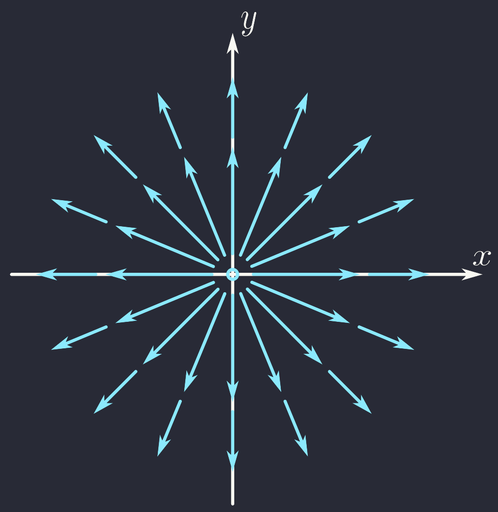

Figure 1.3: Sketch of the vector field $\mathbf{v} = \hat{\mathbf{r}}/r^2$

1.16 Given

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is the position vector. The vector functions is

$$v = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

where the components are

$$v_x = \frac{x}{r^3} \quad \text{and} \quad v_y = \frac{y}{r^3} \quad \text{and} \quad v_z = \frac{z}{r^3}$$

Looking at the x component of the divergence,

$$\begin{aligned}
[\nabla \cdot \mathbf{v}]_x &= \frac{\partial v_x}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) \quad \text{using chain rule...} \\
&= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{1}{r^3} - \frac{3x^2}{r^5}
\end{aligned}$$

therefore, the divergence of $\mathbf{v}$ is

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) \\
&= \frac{3}{r^3} - 3\frac{x^2 + y^2 + z^2}{r^5} \\
&= \frac{3}{r^3} - 3\frac{r^2}{r^5} = 0
\end{aligned}$$

This is consistent with the sketch in Figure 1.3 because the vector field is not ‘sourcing’ or ‘sinking’.

1.17 Given

$$\bar{v}_y = v_y \cos\phi + v_z \sin\phi \quad \text{and} \quad \bar{v}_z = -v_y \sin\phi + v_z \cos\phi$$

Calculating the derivatives

$$\begin{aligned}
\frac{\partial \bar{v}_y}{\partial \bar{y}} &= \frac{\partial v_y}{\partial \bar{y}} \cos\phi + \frac{\partial v_z}{\partial \bar{y}} \sin\phi \\
\frac{\partial \bar{v}_z}{\partial \bar{z}} &= -\frac{\partial v_y}{\partial \bar{z}} \sin\phi + \frac{\partial v_z}{\partial \bar{z}} \cos\phi
\end{aligned}$$

from Problem 1.14,

$$\begin{aligned}
\frac{\partial f}{\partial \bar{y}} &= \frac{\partial f}{\partial y} \cos\phi + \frac{\partial f}{\partial z} \sin\phi \\
\frac{\partial f}{\partial \bar{z}} &= -\frac{\partial f}{\partial y} \sin\phi + \frac{\partial f}{\partial z} \cos\phi
\end{aligned}$$

therefore, the derivatives are rewritten as

$$\begin{aligned}
\frac{\partial \bar{v}_y}{\partial \bar{y}} &= \left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \bar{y}}\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial v_z}{\partial y}\frac{\partial z}{\partial \bar{z}}\right)\sin\phi \\
&= \left(\frac{\partial v_y}{\partial y}\cos\phi + \frac{\partial v_y}{\partial z}\sin\phi\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi\right)\sin\phi
\end{aligned}$$

and likewise,

$$\frac{\partial \bar{v}_z}{\partial \bar{z}} = -\left(-\frac{\partial v_y}{\partial y}\sin\phi + \frac{\partial v_y}{\partial z}\cos\phi\right)\sin\phi + \left(-\frac{\partial v_z}{\partial y}\sin\phi + \frac{\partial v_z}{\partial z}\cos\phi\right)\cos\phi$$

Finally adding the two equations together gives

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\bar{\mathbf{v}} &= \frac{\partial \bar{v}_y}{\partial \bar{y}} + \frac{\partial \bar{v}_z}{\partial \bar{z}} \\
&= \frac{\partial v_y}{\partial y}\cos^2\phi + \frac{\partial v_y}{\partial z}\sin\phi\cos\phi + \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi^2 \\
&+ \frac{\partial v_y}{\partial y}\sin^2\phi - \frac{\partial v_y}{\partial z}\sin\phi\cos\phi - \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\cos^2\phi \\
&= (\sin\phi^2 + \cos\phi^2)\left[\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] \\
&= \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}
\end{aligned}$$

which shows that the divergence transforms as a scalar under rotations.

1.18 Curl of vector functions from Problem 1.15: (a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v}_a &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0) \\
&= -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}
\end{aligned}$$

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v}_b &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x) \\
&= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}
\end{aligned}$$

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v}_c &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\
&= \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) \\
&= 0
\end{aligned}$$

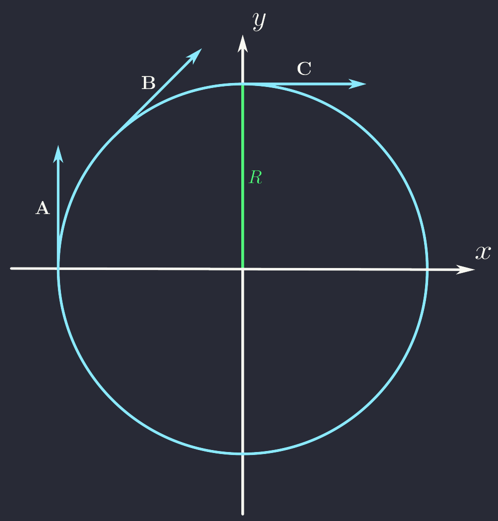

Figure 1.4: Sketch of the vector field pointing clockwise around a circle of radius R

1.19 From Figure 1.4, the sign of $\partial v_x/\partial y$ is positive, and the sign of $\partial v_y/\partial x$ is negative. Therefore, the curl

$$\boldsymbol{\nabla} \times \mathbf{v} = \hat{\mathbf{z}}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial z}\right)$$

is in the negative z direction, or into the page. This is consistent with the right-hand rule as the thumb points into the page.

1.20 Proof that the vector function

$$\mathbf{g} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3}$$

has zero divergence and curl given

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{and} \quad \frac{\partial x}{\partial y} = \frac{y}{x} = 0$$

From Problem 1.16, the divergence of $\mathbf{g}$ is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{g} &= \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} \\
&= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) \\
&= \frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5} \\
&= \frac{3}{r^3} - \frac{3}{r^3} = 0
\end{aligned}$$

and the curl is

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{g} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} \\
&= \hat{\mathbf{x}}(0-0) - \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-0) = 0
\end{aligned}
$$

1.21 Proving product rule for (i)

$$
\begin{aligned}
\boldsymbol{\nabla}(fg) &= \frac{\partial(fg)}{\partial x}\hat{\mathbf{x}} + \frac{\partial(fg)}{\partial y}\hat{\mathbf{y}} + \frac{\partial(fg)}{\partial z}\hat{\mathbf{z}} \\
&= \left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right)\hat{\mathbf{x}} + \left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right)\hat{\mathbf{y}} + \left(\frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\right)\hat{\mathbf{z}} \\
&= f\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\right) + g\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) \\
&= f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f
\end{aligned}
$$

(iv)

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= \boldsymbol{\nabla} \cdot [(A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}] \\
&= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x) \\
&= \left(\frac{\partial A_y}{\partial x}B_z + A_y\frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x}B_y - A_z\frac{\partial B_y}{\partial x}\right) + \left(\frac{\partial A_z}{\partial y}B_x + A_z\frac{\partial B_x}{\partial y} - \frac{\partial A_x}{\partial y}B_z - A_x\frac{\partial B_z}{\partial y}\right) \\
&+ \left(\frac{\partial A_x}{\partial z}B_y + A_x\frac{\partial B_y}{\partial z} - \frac{\partial A_y}{\partial z}B_x - A_y\frac{\partial B_x}{\partial z}\right) \\
&= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&+ A_x\left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right) + A_y\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + A_z\left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) \\
&= \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) + \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})
\end{aligned}
$$

(v)

$$
\begin{aligned}
\boldsymbol{\nabla} \times (f\mathbf{A}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix} \\
&= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y)\right) - \hat{\mathbf{y}}\left(\frac{\partial}{\partial x}(fA_z) - \frac{\partial}{\partial z}(fA_x)\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x)\right) \\
&= \hat{\mathbf{x}}\left(f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right) - \hat{\mathbf{y}}\left(f\frac{\partial A_z}{\partial x} + A_z\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial z} - A_x\frac{\partial f}{\partial z}\right) \\
&+ \hat{\mathbf{z}}\left(f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right) \\
&= f\left[\hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right] \\
&- \hat{\mathbf{x}}\left(A_y\frac{\partial f}{\partial z} - A_z\frac{\partial f}{\partial y}\right) + \hat{\mathbf{y}}\left(A_z\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial z}\right) - \hat{\mathbf{z}}\left(A_y\frac{\partial f}{\partial x} - A_x\frac{\partial f}{\partial y}\right) \\
&= f(\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \times (\boldsymbol{\nabla} f)
\end{aligned}
$$

1.22 (a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, then

$$\begin{aligned}
(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} &= \left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}) \\
&= \hat{\mathbf{x}}\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right)+\hat{\mathbf{y}}\left(A_x\frac{\partial B_y}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_y}{\partial z}\right) \\
&+\hat{\mathbf{z}}\left(A_x\frac{\partial B_z}{\partial x}+A_y\frac{\partial B_z}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right)
\end{aligned}$$

This means that the direction of $\mathbf{A}$ points in the direction of where $\mathbf{B}$ moves fastest.
(b)

$$(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} = \frac{1}{r}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\frac{(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})}{r}$$

looking at the x component,

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\mathbf{r}}{r}\right) &= \frac{1}{r}\frac{\partial}{\partial x}(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}})+\mathbf{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \\
&= \frac{\hat{\mathbf{x}}}{r}-\mathbf{r}\frac{x^2}{r^3}
\end{aligned}$$

therefore,

$$\begin{aligned}
(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\hat{\mathbf{r}} &= \frac{1}{r}\left[x\left(\frac{\hat{\mathbf{x}}}{r}-\mathbf{r}\frac{x}{r^3}\right)+y\left(\frac{\hat{\mathbf{y}}}{r}-\mathbf{r}\frac{y}{r^3}\right)+z\left(\frac{\hat{\mathbf{z}}}{r}-\mathbf{r}\frac{z}{r^3}\right)\right] \\
&= \frac{1}{r}\left[\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{r}-\mathbf{r}\frac{x^2+y^2+z^2}{r^3}\right] \\
&= \frac{1}{r}\left[\frac{\mathbf{r}}{r}-\frac{\mathbf{r}}{r}\right]=0
\end{aligned}$$

(c)

$$\begin{aligned}
(v_a\cdot\boldsymbol{\nabla})v_b &= \left(x^2\frac{\partial}{\partial x}+3xz^2\frac{\partial}{\partial y}-2xz\frac{\partial}{\partial z}\right)(xy\hat{\mathbf{x}}+2yz\hat{\mathbf{y}}+3zx\hat{\mathbf{z}}) \\
&= x^2(y\hat{\mathbf{x}}+0+3z\hat{\mathbf{z}})+3xz^2(x\hat{\mathbf{x}}+2z\hat{\mathbf{y}}+0)-2xz(0+2y\hat{\mathbf{y}}+3x\hat{\mathbf{z}}) \\
&= (x^2y+3x^2z^2)\hat{\mathbf{x}}+(6xz^3-4xyz)\hat{\mathbf{y}}+(3x^2z-6x^2z)\hat{\mathbf{z}} \\
&= x^2(y+3z^2)\hat{\mathbf{x}}+2xz(3z^2-2y)\hat{\mathbf{y}}-3x^2z\hat{\mathbf{z}}
\end{aligned}$$

1.23 Proving the product rule for (ii) given the x component of the left hand side is

$$\begin{aligned}
[\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})]_x &= \frac{\partial(\mathbf{A}\cdot\mathbf{B})}{\partial x}\hat{\mathbf{x}} \\
&= \frac{\partial}{\partial x}(A_xB_x+A_yB_y+A_zB_z)\hat{\mathbf{x}} \\
&= A_x\frac{\partial B_x}{\partial x}+B_x\frac{\partial A_x}{\partial x}+A_y\frac{\partial B_y}{\partial x}+B_y\frac{\partial A_y}{\partial x}+A_z\frac{\partial B_z}{\partial x}+B_z\frac{\partial A_z}{\partial x}
\end{aligned}$$

Finding the x component of the right hand side of (ii)

$$
\begin{aligned}
[\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B})]_x &= \left[\mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \right]_x \\
&= \left[\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ () & -\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{vmatrix} \right]_x \\
&= A_y \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) - A_z \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right)
\end{aligned}
$$

and

$$
[\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A})]_x = B_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - B_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)
$$

From Problem 1.22:

$$
[(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}] = A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}
$$

and likewise,

$$
[(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}] = B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z}
$$

Adding all four equations gives

$$
\begin{aligned}
&[\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A}]_x = \\
&\quad A_y \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) - A_z \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + B_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - B_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\
&\quad + A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\
&= A_x \frac{\partial B_x}{\partial x} + A_y \left(\frac{\partial B_y}{\partial x} - \cancel{\frac{\partial B_x}{\partial y}} + \cancel{\frac{\partial B_x}{\partial y}} \right) + A_z \left(\frac{\partial B_z}{\partial x} - \cancel{\frac{\partial B_x}{\partial z}} + \cancel{\frac{\partial B_x}{\partial z}} \right) \\
&\quad + B_x \frac{\partial A_x}{\partial x} + B_y \left(\frac{\partial A_y}{\partial x} - \cancel{\frac{\partial A_x}{\partial y}} + \cancel{\frac{\partial A_x}{\partial y}} \right) + B_z \left(\frac{\partial A_z}{\partial x} - \cancel{\frac{\partial A_x}{\partial z}} + \cancel{\frac{\partial A_x}{\partial z}} \right) \\
&= A_x \frac{\partial B_x}{\partial x} + B_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + B_y \frac{\partial A_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + B_z \frac{\partial A_x}{\partial z} \\
&= [\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B})]_x
\end{aligned}
$$

and likewise for the y and z components.
For (vi), the x on the left hand side is

$$
\begin{aligned}
[\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B})]_x &= \left[\boldsymbol{\nabla} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \right]_x \\
&= \left[\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ () & -(A_x B_z - A_z B_x) & A_x B_y - A_y B_x \end{vmatrix} \right]_x \\
&= \frac{\partial}{\partial y}(A_x B_y - A_y B_x) - \frac{\partial}{\partial z}(A_z B_x - A_x B_z) \\
&= A_x \frac{\partial B_y}{\partial y} + B_y \frac{\partial A_x}{\partial y} - A_y \frac{\partial B_x}{\partial y} - B_x \frac{\partial A_y}{\partial y} \\
&- A_z \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_z}{\partial z} + A_x \frac{\partial B_z}{\partial z} + B_z \frac{\partial A_x}{\partial z} + \\
&= A_x \left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} - B_x \left(\frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z}
\end{aligned}
$$

On the right hand side, first we find the x component of the two new operations:

$$\begin{aligned}
[A(\boldsymbol{\nabla}\cdot\mathbf{B})]_x &= \left[A\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)\right]_x \\
&= A_x\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)
\end{aligned}$$

and likewise,

$$[B(\boldsymbol{\nabla}\cdot\mathbf{A})]_x = B_x\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)$$

Therefore, $[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}-(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}+A(\boldsymbol{\nabla}\cdot\mathbf{B})-B(\boldsymbol{\nabla}\cdot\mathbf{A})]_x =$

$$\begin{aligned}
&B_x\frac{\partial A_x}{\partial x}+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z}-\left(A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right) \\
&+A_x\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)-\left(B_x\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\right) \\
&= A_x\left(\cancel{\frac{\partial B_x}{\partial x}}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}-\cancel{\frac{\partial B_x}{\partial x}}\right)-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z} \\
&-B_x\left(\cancel{\frac{\partial A_x}{\partial x}}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}-\cancel{\frac{\partial A_x}{\partial x}}\right)+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z} \\
&= A_x\left(\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)-A_y\frac{\partial B_x}{\partial y}-A_z\frac{\partial B_x}{\partial z}-B_x\left(\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)+B_y\frac{\partial A_x}{\partial y}+B_z\frac{\partial A_x}{\partial z} \\
&= [\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})]_x
\end{aligned}$$

and likewise for the y and z components.

1.24 Deriving the three quotient rules from the product rule: The gradient is

$$\begin{aligned}
\boldsymbol{\nabla}\left(\frac{f}{g}\right) &= \boldsymbol{\nabla}\left(fg^{-1}\right) = f\boldsymbol{\nabla}\left(g^{-1}\right)+g^{-1}\boldsymbol{\nabla}(f) \\
&= f(-g^{-2}\boldsymbol{\nabla}(g))+g^{-1}\boldsymbol{\nabla}(f) \\
&= -\frac{f}{g^2}\boldsymbol{\nabla}(g)+\frac{g}{g}\frac{1}{g}\boldsymbol{\nabla}(f) \\
&= \frac{g\boldsymbol{\nabla}f-f\boldsymbol{\nabla}g}{g^2}
\end{aligned}$$

the divergence

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\left(\frac{A}{g}\right) &= \boldsymbol{\nabla}\cdot\left(Ag^{-1}\right) = A(\boldsymbol{\nabla}\cdot g^{-1})+g^{-1}(\boldsymbol{\nabla}\cdot A) \\
&= A(-g^{-2}(\boldsymbol{\nabla}\cdot g))+\frac{g}{g}g^{-1}(\boldsymbol{\nabla}\cdot A) \\
&= \frac{g(\boldsymbol{\nabla}\cdot A)-A\boldsymbol{\nabla}\cdot g}{g^2}
\end{aligned}$$

and the curl

$$\begin{aligned}
\left[\boldsymbol{\nabla}\times\left(\frac{\mathbf{A}}{g}\right)\right]_x &= \frac{\partial}{\partial y}\left(\frac{A_z}{g}\right)-\frac{\partial}{\partial z}\left(\frac{A_y}{g}\right) \\
&= \frac{g\frac{\partial A_z}{\partial y}-A_z\frac{\partial g}{\partial y}}{g^2}-\frac{g\frac{\partial A_y}{\partial z}-A_y\frac{\partial g}{\partial z}}{g^2} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\left(A_y\frac{\partial g}{\partial z}-A_z\frac{\partial g}{\partial y}\right)\right] \\
&= \frac{g[\boldsymbol{\nabla}\times\mathbf{A}]_x-\mathbf{A}\times[\boldsymbol{\nabla}g]_x}{g^2}
\end{aligned}$$

and likewise for the y and z components. Therefore,

$$\boldsymbol{\nabla}\times\left(\frac{\mathbf{A}}{g}\right)=\frac{g(\boldsymbol{\nabla}\times\mathbf{A})-\mathbf{A}\times(\boldsymbol{\nabla}g)}{g^2}$$

1.25 (a) Calculating (iv) for the functions

$$\mathbf{A}=x\hat{\mathbf{x}}+2y\hat{\mathbf{y}}+3z\hat{\mathbf{z}};\qquad \mathbf{B}=3y\hat{\mathbf{x}}-2x\hat{\mathbf{y}}$$

LHS:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B})&=\boldsymbol{\nabla}\cdot\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\ x&2y&3z\\ 3y&-2x&0\end{vmatrix}\\&=\boldsymbol{\nabla}\cdot\left[(0+6xz)\hat{\mathbf{x}}-(0-9yz)\hat{\mathbf{y}}+(-2x^2-6y^2)\hat{\mathbf{z}}\right]\\&=\frac{\partial}{\partial x}(6xz)+\frac{\partial}{\partial y}(9yz)+\frac{\partial}{\partial z}\left(-2x^2-6y^2\right)\\&=6z+9z+0=15z\end{aligned}$$

RHS:

$$\begin{aligned}\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})&=\mathbf{B}\cdot\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x&2y&3z\end{vmatrix}\\&=\mathbf{B}\cdot[(0)\hat{\mathbf{x}}-(0)\hat{\mathbf{y}}+(0)\hat{\mathbf{z}}]=0\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})&=\mathbf{A}\cdot\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ 3y&-2x&0\end{vmatrix}\\&=\mathbf{A}\cdot[(0)\hat{\mathbf{x}}-(0)\hat{\mathbf{y}}+(-2-3)\hat{\mathbf{z}}]\\&=3z(-5)=-15z\end{aligned}$$

therefore,

$$\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A})-\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})=0-(-15z)=15z$$

(b) Checking (ii): LHS

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B})&=\boldsymbol{\nabla}(x(3y)+2y(-2x)+3z(0))\\&=\boldsymbol{\nabla}(3xy-4xy)=\boldsymbol{\nabla}(-xy)\\&=-y\hat{\mathbf{x}}-x\hat{\mathbf{y}}\end{aligned}$$

RHS:

$$\begin{aligned}\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})&=\mathbf{A}\times\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ 3y&-2x&0\end{vmatrix}\\&=\mathbf{A}\times[(0)\hat{\mathbf{x}}-(0)\hat{\mathbf{y}}+(-5)\hat{\mathbf{z}}]\\&=\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\ x&2y&3z\\ 0&0&-5\end{vmatrix}=-10y\hat{\mathbf{x}}+5x\hat{\mathbf{y}}\end{aligned}$$

and

$$\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \mathbf{B}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z\end{vmatrix} = \mathbf{B}\times[(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0$$

and

$$\begin{aligned}(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} &= \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right)(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) \\ &= 6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}\end{aligned}$$

and

$$\begin{aligned}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} &= \left(3y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}\right)(x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \\ &= 3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}\end{aligned}$$

therefore,

$$\begin{aligned}\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} &= (-10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}) + (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) \\ &= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}\end{aligned}$$

(c) For rule (vi), the left hand side is

$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) &= \boldsymbol{\nabla}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0\end{vmatrix} \\ &= \boldsymbol{\nabla}\times\left[6xz\hat{\mathbf{x}} + 9yz\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}\right] \\ &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & -2x^2 - 6y^2\end{vmatrix} \\ &= \hat{\mathbf{x}}(-12y - 9y) - \hat{\mathbf{y}}(-4x - 6x) + \hat{\mathbf{z}}(0) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}\end{aligned}$$

and on the right hand side, the new terms are

$$\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) = \mathbf{A}[0+0] = 0$$

and

$$\begin{aligned}\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) &= \mathbf{B}[1+2+3] = 6\mathbf{B} \\ &= 6(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) = 18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}\end{aligned}$$

combining these with the terms from (iv) gives

$$\begin{aligned}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} - (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} &= (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) - (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + 0 - (18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}\end{aligned}$$

1.26 Given the Laplacian of a scalar function T is

$$\nabla^2 T = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

(a) $T_a = x^2 + 2xy + 3z + 4$:

$$\nabla^2 T_a = 2 + 0 + 0 = 2$$

(b) $T_b = \sin x \sin y \sin z$:

$$\frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -\sin x \sin y \sin z = -T_b$$

$$\nabla^2 T_b = -3T_b$$

(c) $T_c = e^{-5x} \sin 4y \cos 3z$: The components are

$$\begin{aligned}
\frac{\partial^2 T_c}{\partial x^2} &= 25e^{-5x} \sin 4y \cos 3z = 25T_c \\
\frac{\partial^2 T_c}{\partial y^2} &= -16e^{-5x} \sin 4y \cos 3z = -16T_c \\
\frac{\partial^2 T_c}{\partial z^2} &= -9e^{-5x} \sin 4y \cos 3z = -9T_c
\end{aligned}$$

therefore

$$\nabla^2 T_c = T_c(25 - 16 - 9) = 0$$

(d) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$: The laplacian of a vector function is

$$\nabla^2 \mathbf{v} = (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$$

and the components are

$$\begin{aligned}
\nabla^2 v_x &= \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = 2 + 0 + 0 = 2 \\
\nabla^2 v_y &= 0 + 0 + 6x = 6x \\
\nabla^2 v_z &= 0
\end{aligned}$$

therefore

$$\nabla^2 \mathbf{v} = 2\hat{\mathbf{x}} + 6x\hat{\mathbf{y}}$$

1.27 The divergence of curl is always zero:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\
&= \boldsymbol{\nabla} \cdot \left(\hat{\mathbf{x}} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) - \hat{\mathbf{y}} \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + \hat{\mathbf{z}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right) \\
&= \frac{\partial}{\partial x} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\
&= \left[\frac{\partial}{\partial x} \left(\frac{\partial v_z}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial v_z}{\partial x} \right) \right] + \left[\frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial v_x}{\partial y} \right) \right] + \left[\frac{\partial}{\partial z} \left(\frac{\partial v_y}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial v_y}{\partial z} \right) \right] \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) &= 0
\end{aligned}$$

where the last step hinges on the equality of cross derivatives:

$$\frac{\partial}{\partial x_i} \left(\frac{\partial v}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left(\frac{\partial v}{\partial x_i} \right)$$

Checking for $v_a = x^2\hat{\mathbf{x}} + 2xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times v_a) &= \boldsymbol{\nabla} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xz^2 & -2xz \end{vmatrix} \\
&= \boldsymbol{\nabla} \cdot \big[\hat{\mathbf{x}}(0 - 4xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}\big(2z^2 - 0\big)\big] \\
&= \boldsymbol{\nabla} \cdot \left[\frac{\partial}{\partial x}(-4xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}\big(2z^2\big)\right] \\
&= -4z + 0 + 4z = 0
\end{aligned}
$$

1.28 The curl of gradient is always zero:

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) &= \boldsymbol{\nabla} \times \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} \\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial y}\right)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial x}\right)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)\right] \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) &= 0
\end{aligned}
$$

where the last step uses the equality of cross derivatives again. Checking for $T = x^2y^3z^4$:

$$
\frac{\partial T}{\partial x} = 2xy^3z^4, \quad \frac{\partial T}{\partial y} = 3x^2y^2z^4, \quad \text{and} \quad \frac{\partial T}{\partial z} = 4x^2y^3z^3
$$

and

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) &= \boldsymbol{\nabla} \times \big(2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}\big) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix} \\
&= \hat{\mathbf{x}}\big(12x^2y^2z^4 - 12x^2y^2z^4\big) - \hat{\mathbf{y}}\big(8x^2y^3z^3 - 8x^2y^3z^3\big) + \hat{\mathbf{z}}\big(6x^2y^3z^3 - 6x^2y^3z^3\big) \\
&= 0
\end{aligned}
$$

1.29 Calculating the line integral of the function $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$: from the origin to point $(1, 1, 1)$ along three different paths:
(a) $a = (0, 0, 0) \to b = (1, 0, 0) \to c = (1, 1, 0) \to d = (1, 1, 1)$ split to three paths:
(i) From $a \to b$: $\mathrm{d}l = \mathrm{d}x\,\hat{\mathbf{x}}$ and $\mathbf{v} = x^2\hat{\mathbf{x}}$.
(ii) From $b \to c$: $\mathrm{d}l = \mathrm{d}y\,\hat{\mathbf{y}}$ and $\mathbf{v} = 2yz\hat{\mathbf{y}} = 0$ since $z = 0$.
(iii) From $c \to d$: $\mathrm{d}l = \mathrm{d}z\,\hat{\mathbf{z}}$ and $\mathbf{v} = y^2\hat{\mathbf{z}} = 1\hat{\mathbf{z}}$ since $y = 1$.

$$
\begin{aligned}
\int_a^b \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3} \\
\int_b^c \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 0\,\mathrm{d}y = 0 \\
\int_c^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \int_0^1 1\,\mathrm{d}z = 1 \\
\int_a^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} &= \frac{1}{3} + 0 + 1 = \frac{4}{3}
\end{aligned}
$$

(b) $a = (0, 0, 0) \to b = (0, 0, 1) \to c = (0, 1, 1) \to d = (1, 1, 1)$ split to three paths:

(i) From $a \to b$: $\mathrm{d}l = \mathrm{d}z\,\hat{\mathbf{z}}$ and $\mathbf{v} = y^2\hat{\mathbf{z}} = 0$ since $y = 0$.
(ii) From $b \to c$: $\mathrm{d}l = \mathrm{d}y\,\hat{\mathbf{y}}$ and $\mathbf{v} = 2yz\hat{\mathbf{y}} = 2y\hat{\mathbf{z}}$ since $y = 1$.
(iii) From $c \to d$: $\mathrm{d}l = \mathrm{d}x\,\hat{\mathbf{x}}$ and $\mathbf{v} = x^2\hat{\mathbf{x}}$.

$$\int_a^b \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^1 0\,\mathrm{d}z = 0$$
$$\int_b^c \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^1 2y\,\mathrm{d}y = 1$$
$$\int_c^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$$
$$\int_a^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} = 0 + 1 + \frac{1}{3} = \frac{4}{3}$$

(c) A straight line: Since $x = y = z$ and $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z$,
$\mathrm{d}l = \mathrm{d}x\,(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$ and $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}} = 4x^2\hat{\mathbf{x}}$.

$$\int_a^d \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^1 4x^2\,\mathrm{d}x = \frac{4}{3}$$

(d) The line integral around the closed loop:

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_{(a)} \mathbf{v} \cdot \mathrm{d}\mathbf{l} - \int_{(b)} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \frac{4}{3} - \frac{4}{3} = 0$$

1.30 Surface integral of $\mathbf{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ over the bottom of the box:
$z = 0$, $\mathrm{d}\mathbf{A} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}$ $\mathbf{v} \cdot \mathrm{d}\mathbf{A} = y(z^2-3)\,\mathrm{d}x\,\mathrm{d}y = -3y\,\mathrm{d}x\,\mathrm{d}y$, so

$$\int \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_0^2 \mathrm{d}x \int_0^2 -3y\,\mathrm{d}y = -12$$

From Example 1.7 (v) has the same boundary line but a different surface integral, so the surface integral does not depend on just the boundary line. The surface over the closed surface of the box is

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 20 + 12 = 32$$

since the direction of $\mathrm{d}\mathbf{A}$ on the bottom side is in the negative z direction for it to point 'outward'.

1.31 Calculating the volume integral of $T = z^2$ over the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$:
The equation of the plane containing the three vertices $A = (1,0,0)$, $B = (0,1,0)$, and $C(0,0,1)$:
The vector normal to this plane $\mathbf{n} = (a,b,c)$ is the cross product of two vectors in the plane given by $\mathbf{AB} = (-1,1,0)$ and $\mathbf{AC} = (-1,0,1)$:

$$\mathbf{n} = \begin{vmatrix} x & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1,1,1)$$

the equation of the plane is therefore

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_o) &= 0 \\ (1,1,1) \cdot [(x,y,z) - A] &= 0 \\ x + y + z &= 1 \end{aligned}$$

therefore, the boundary for x is $x=0$ and $x=1-y-z$; for y is $y=0$ and $y=1-z$; and for z is $z=0$ and $z=1$. The volume integral is therefore

$$\begin{aligned}
\int T\,\mathrm{d}V &= \int_0^1 z^2\,\mathrm{d}z \int_0^{1-z} \mathrm{d}y \int_0^{1-y-z} \mathrm{d}x \\
&= \int_0^1 z^2\,\mathrm{d}z \int_0^{1-z} (1-y-z)\,\mathrm{d}y \\
&= \int_0^1 z^2\,\mathrm{d}z \left(y - y^2/2 - yz \Big|_0^{1-z} \right) \\
&= \int_0^1 z^2[(1-z)-(1-z)^2/2 - z(1-z)]\,\mathrm{d}z \\
&= \int_0^1 z^2(1-z-1/2-z^2/2+z-z+z^2)\,\mathrm{d}z \\
&= \int_0^1 z^2(1/2-z+z^2/2)\,\mathrm{d}z \\
&= \int_0^1 (z^2/2-z^3+z^4/2)\,\mathrm{d}z \\
&= z^3/6-z^4/4+z^5/10\Big|_0^1 \\
&= 1/6-1/4+1/10 = 1/60
\end{aligned}$$

1.32 Given $T = x^2+4xy+2yz^3$,

$$\frac{\partial T}{\partial x} = 2x+4y, \quad \frac{\partial T}{\partial y} = 4x+2z^3, \quad \text{and} \quad \frac{\partial T}{\partial z} = 6yz^2$$

therefore,

$$\boldsymbol{\nabla} T = \hat{\mathbf{x}}(2x+4y)+\hat{\mathbf{y}}(4x+2z^3)+\hat{\mathbf{z}}(6yz^2)$$

Checking the fundamental theorem for gradients using the points $a=(0,0,0)\to b=(1,1,1)$:

$$\int_a^b \boldsymbol{\nabla} T\cdot \mathrm{d}\mathbf{l} = T(b)-T(a) = 1^2+4(1)(1)+2(1)(1)^3-0 = 7$$

For the three paths: (a) $a\to c=(1,0,0)\to d=(1,1,0)\to d$;
(i) $a\to c$:

$$y=z=\mathrm{d}y=\mathrm{d}z=0; \quad \mathrm{d}\mathbf{l}=\mathrm{d}x\,\hat{\mathbf{x}}; \quad \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{l} = 2x\,\mathrm{d}x$$

and

$$\int_a^c \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{l} = \int_0^1 2x\,\mathrm{d}x = 1$$

(ii) $c\to d$:

$$x=1, \quad z=\mathrm{d}x=\mathrm{d}z=0; \quad \mathrm{d}\mathbf{l}=\mathrm{d}y\,\hat{\mathbf{y}}; \quad \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{l} = 4\,\mathrm{d}y$$

and

$$\int_c^d \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{l} = \int_0^1 4\,\mathrm{d}y = 4$$

(iii) $d\to b$:

$$x=y=1, \quad \mathrm{d}x=\mathrm{d}y=0; \quad \mathrm{d}\mathbf{l}=\mathrm{d}z\,\hat{\mathbf{z}}; \quad \boldsymbol{\nabla} T\cdot\mathrm{d}\mathbf{l} = 6z^2\,\mathrm{d}z$$

and

$$\int_d^b \nabla T \cdot d\mathbf{l} = \int_0^1 6z^2\, dz = 2$$

therefore,

$$\int_a^b \nabla T \cdot d\mathbf{l} = 1 + 4 + 2 = 7$$

(b) $a \to c = (0, 0, 1) \to d = (0, 1, 1) \to b$;
(i) $a \to c$:

$$x = y = dx = dy = 0; \quad d\mathbf{l} = dz\, \hat{\mathbf{z}}; \quad \nabla T \cdot d\mathbf{l} = 0$$

and

$$\int_a^c \nabla T \cdot d\mathbf{l} = \int_0^1 0 = 0$$

(ii) $c \to d$:

$$z = 1, \quad x = dx = dz = 0; \quad d\mathbf{l} = dy\, \hat{\mathbf{y}}; \quad \nabla T \cdot d\mathbf{l} = 2\, dy$$

and

$$\int_c^d \nabla T \cdot d\mathbf{l} = \int_0^1 2\, dy = 2$$

(iii) $d \to b$:

$$y = z = 1, \quad dy = dz = 0; \quad d\mathbf{l} = dx\, \hat{\mathbf{x}}; \quad \nabla T \cdot d\mathbf{l} = (2x + 4)\, dx$$

and

$$\int_d^b \nabla T \cdot d\mathbf{l} = \int_0^1 (2x + 4)\, dx = 5$$

therefore,

$$\int_a^b \nabla T \cdot d\mathbf{l} = 0 + 2 + 5 = 7$$

(c) the parabolic path $z = x^2$; $y = x$:

$$dx = dy\,, \quad \text{and} \quad dz = 2x\, dx\,; \quad d\mathbf{l} = dx\, \hat{\mathbf{x}} + dx\, \hat{\mathbf{y}} + 2x\, dx\, \hat{\mathbf{z}}$$

and

$$\begin{aligned} \nabla T \cdot d\mathbf{l} &= (2x + 4y)\, dx + (4x + 2z^3)\, dx + (6yz^2) 2x\, dx \\ &= 6x\, dx + (4x + 2x^6)\, dx + (12x^6)\, dx \\ &= 10x\, dx + 14x^6\, dx \end{aligned}$$

therefore,

$$\begin{aligned} \int_a^b \nabla T \cdot d\mathbf{l} &= \int_0^1 (10x + 14x^6)\, dx \\ &= 5x^2 + 2x^7 \Big|_0^1 = 7 \end{aligned}$$

1.33