# 1 Vector Analysis

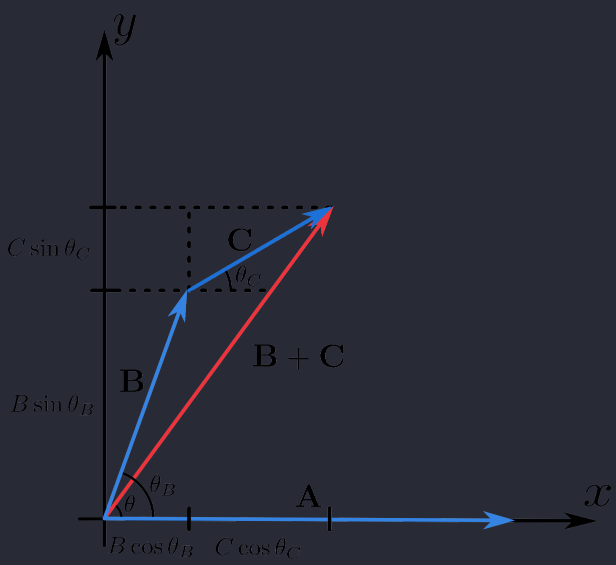


Figure 1.1: Three Coplanar Vectors

**1.1** (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$
$$A(B+C)\cos\theta = AB\cos\theta_B + AC\cos\theta_C$$

Since $B\cos\theta_B + B\cos\theta_C = (B+C)\cos\theta$ from Figure 1.1, the distriubtive property holds true. The cross product also holds true since $B\sin\theta_B + B\sin\theta_C = (B+C)\sin\theta$, and multiplying by $A$ on both sides gives the same result as the distriubtive property:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$A(B+C)\sin\theta = AB\sin\theta_B + AC\sin\theta_C$$

(b) In the general case in three-dimensional space, each vector has three components: $\mathbf{A} = (A_x, A_y, A_z)$. Therefore,

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot(B_x+C_x, B_y+C_y, B_z+C_z)\\
&= A_x(B_x+C_x)+A_y(B_y+C_y)+A_z(B_z+C_z)\\
&= A_xB_x+A_yB_y+A_zB_z+A_xC_x+A_yC_y+A_zC_z\\
&= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}
\end{aligned}$$

**1.2** Setting $\mathbf{A}=\mathbf{B}=(1,1,1)$ and $\mathbf{C}=(1,1,-1)$:

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\times\mathbf{C} &\stackrel{?}{=} \mathbf{A}\times(\mathbf{B}\times\mathbf{C})\\
0 &\stackrel{?}{=} (1,1,1)\times[(1,1,1)\times(1,1,-1)]\\
0 &\stackrel{?}{=} (1,1,1)\times(-2,2,0)\\
0 &\neq (-2,-2,4)
\end{aligned}$$

where the cross product of parallel vectors $\mathbf{A}\times\mathbf{B}=0$. Therefore, the cross product is not associative.

**1.3** Taking the dot product of a unit cube's body diagonals $\mathbf{A} = (1, 1, 1)$, $\mathbf{B} = (1, 1, -1)$:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= AB \cos\theta \\ 1 &= 3 \cos\theta \\ \theta &= \arccos 1/3 \approx 70.53^\circ \end{aligned}$$

**1.4** The cross product of two vectors coplanar to the shaded plane—$\mathbf{A} = (-1, 2, 0)$, $\mathbf{B} = (-1, 0, 3)$—is parallel to the normal unit vector $\hat{\mathbf{n}}$ of the plane:

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6, 3, 2)$$

where $\hat{\mathbf{n}} = \mathbf{C}/C$, and $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$. Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

**1.5** Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} \end{aligned}$$

where the $x$ component is $A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$. Similarly,

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}(A_x C_x + A_y C_y + A_z C_z) - \mathbf{C}(A_x B_x + A_y B_y + A_z B_z)$$

where $x$ component simplifes to

$$B_x(\cancel{A_x C_x} + A_y C_y + A_z C_z) - C_x(\cancel{A_x B_x} + A_y B_y + A_z B_z) = A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

the same is done for the $y$ and $z$ components. Therefore, the "BAC–CAB" rule holds true.

**1.6**

$$\begin{aligned} [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) = 0 \\ &\quad - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C} \cdot \mathbf{A})} \end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \\ \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \times \mathbf{B} \end{aligned}$$

For the relation to hold true, either the vectors $\mathbf{A}$ and $\mathbf{C}$ are parallel ($\mathbf{A} \times \mathbf{A} = 0$) or $\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{C}$ ($\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = 0$).

**1.7** Finding the seperation vector $\boldsymbol{\mathscr{r}}$:

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' = (4,6,8) - (2,8,7) = (2,-2,1) \\
\mathscr{r} &= \sqrt{2^2 + (-2)^2 + 1^2} = 3 \\
\hat{\boldsymbol{\mathscr{r}}} &= \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)
\end{aligned}$$

**1.8** (a)

$$\begin{aligned}
\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z &= (A_y\cos\phi + A_z\sin\phi)(B_y\cos\phi + B_z\sin\phi) \\
&\quad + (-A_y\sin\phi + A_z\cos\phi)(-B_y\sin\phi + B_z\cos\phi) \\
&= A_yB_y\cos^2\phi + A_zB_z\sin^2\phi + \cancel{A_yB_z\sin\phi\cos\phi} + \cancel{A_zB_y\sin\phi\cos\phi} \\
&\quad + A_yB_y\sin^2\phi - \cancel{A_yB_z\sin\phi\cos\phi} - \cancel{A_zB_y\sin\phi\cos\phi} + A_zB_z\cos^2\phi \\
&= A_yB_y(\sin^2\phi + \cos^2\phi) + A_zB_z(\sin^2\phi + \cos^2\phi) \\
\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z &= A_yB_y + A_zB_z
\end{aligned}$$

(b) To preserve length $|\bar{A}| = |A|$. Squaring both sides and expanding gives

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_x^2 + A_y^2 + A_z^2$$

in summation form,

$$\sum_{i=1}^{3} \bar{A}_i\bar{A}_i = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} R_{ij}A_j\right)\left(\sum_{k=1}^{3} R_{ik}A_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} R_{ij}R_{ik}A_jA_k$$

For the length to be preserved, the indices $j$ and $k$ must be equal. Therefore,

$$R_{ij}R_{ik} = \delta_{jk}$$

where $\delta_{ij}$ is the Kronecker delta. Thus the length is preserved if the rotation matrix is orthogonal, i.e.

$$R_{ij}R_{ik} = (R^T)_{ji}R_{ik} = \delta_{jk} \quad \text{or} \quad R^TR = I$$

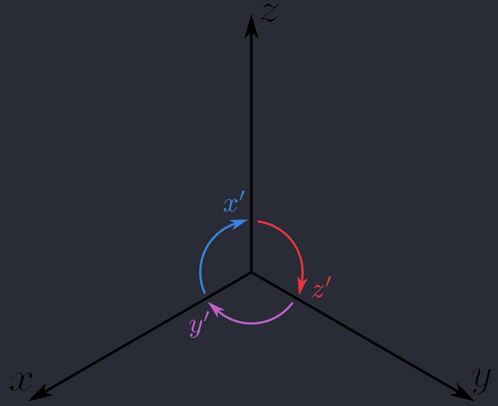


Figure 1.2: Rotation of 120° about an axis through the origin and point $(1,1,1)$

**1.9** From Figure 1.2, the rotation is equivalent to changing the position of the basis vectors $\hat{\mathbf{x}} \to \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \to \hat{\mathbf{y}}$. Therefore, the rotation matrix is a permutation matrix:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

**1.10** (a) Under a **translation** of coordinates $\bar{y} = y - a$, the origin $O$ and terminus $A = (x, y, z)$ of some vector are translated to

$$O \to O' = (0, -a, 0)$$
$$A \to A' = (x, y - a, z)$$

therefore, the translated vector is

$$\overline{O'A'} = (x, y - a, z) - (0, -a, 0) = (x, y, z) = \overline{OA} = \mathbf{A}$$

which is the same as the original vector, so

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) Under **inversion** of coordinates, only the terminus changes

$$O \to O' = (0, 0, 0)$$
$$A \to A' = (-x, -y, -z)$$

therefore, the inverted vector is

$$\overline{O'A'} = (-x, -y, -z) \quad \text{or} \quad \begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

(c) The components of cross product under inversion

$$\bar{\mathbf{A}} \times \bar{\mathbf{B}} = \begin{pmatrix} \bar{A}_y \bar{B}_z - \bar{A}_z \bar{B}_y \\ \bar{A}_z \bar{B}_x - \bar{A}_x \bar{B}_z \\ \bar{A}_x \bar{B}_y - \bar{A}_y \bar{B}_x \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}$$

which is the same as the original cross product $\mathbf{A} \times \mathbf{B}$. The cross product of two pseudovectors is also a pseudovector. Torque $\tau = \mathbf{r} \times \mathbf{F}$ and magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ are examples of pseudovectors.
(d) Scalar triple product under inversion

$$\begin{aligned} \bar{A} \cdot (\bar{B} \times \bar{C}) &= -\mathbf{A} \cdot (-\mathbf{B} \times -\mathbf{C}) \\ &= -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \end{aligned}$$

the scalar triple product changes sign under inversion.

**1.11** (a) Finding gradient of $f(x, y, z) = x^2 + y^3 + z^4$:

$$\begin{aligned} \boldsymbol{\nabla} f &= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\ &= 2x\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + 4z^3\hat{\mathbf{z}} \end{aligned}$$

(b) Gradient of $f(x, y, z) = x^2 y^3 z^4$:

$$\boldsymbol{\nabla} f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$$

(c) Gradient of $f(x, y, z) = e^x \sin(y) \ln(z)$:

$$\boldsymbol{\nabla} f = e^x\hat{\mathbf{x}} + e^x \cos(y) \ln(z)\hat{\mathbf{y}} + \frac{e^x \sin(y)}{z}\hat{\mathbf{z}}$$

**1.12** The height of the hill (in feet) is given by the function

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where $y$ is north and $x$ is east in miles. The gradient of $h$ is

$$\boldsymbol{\nabla} h = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10(2x - 8y + 28)\hat{\mathbf{y}}$$

(a) The top of the hill is a stationary point, so the summit is found by setting the gradient to zero which gives the system of equations

$$0 = 2y - 6x - 18 \qquad\qquad 0 = 2x - 8y + 28$$

adding the first equation to 3 times the second equation gives

$$\begin{aligned} 0 &= 2y - 6x - 18 + 3(2x - 8y + 28) \\ 0 &= -22y + 66 \\ y &= 3 \end{aligned}$$

substituting $y = 3$ into the first equation

$$0 = 2(3) - 6x - 18 \to x = -2$$

Therefore, the top of the hill is at $(-2, 3)$ or 2 miles west and 3 miles north of the origin.
(b) The height of the hill is simply $h(-2, 3) = 10(12) = 720$ feet.
(c) The steepness of the hill at $h(1, 1)$ is given by the magnitude of the gradient

$$\begin{aligned} |\boldsymbol{\nabla} h| &= 10\sqrt{(2y - 6x - 18)^2 + (2x - 8y + 28)^2} \\ &= 10\sqrt{(2 - 6 - 18)^2 + (2 - 8 + 28)^2} \\ &= 10\sqrt{(-22)^2 + (22)^2} = 220\sqrt{2} \approx 311\,\mathrm{ft/mi} \end{aligned}$$

The direction of the steepest slope is given by the vector in the direction of the gradient at the point $\boldsymbol{\nabla} h(1, 1) = 220(-\mathbf{x} + \mathbf{y})$, or simply northwest.

**1.13** Given the seperation vector

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(a) Show that $\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\nabla}\left(\mathscr{r}^2\right) = 2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}} = 2\boldsymbol{\mathscr{r}}$$

(b)

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{\mathscr{r}}\right)\hat{\mathbf{z}}$$

looking at the $x$ component,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{\mathscr{r}}\right) &= -\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial x}(\mathscr{r}) \\ &= -\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial x}\left(\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\right) \\ &= -\frac{1}{\mathscr{r}^2}\frac{1}{2}\frac{2(x - x')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ &= -\frac{x - x'}{\mathscr{r}^3} \end{aligned}$$

therefore,

$$\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}^3}[(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}] = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(c) The general formula is

$$\boldsymbol{\nabla}(\mathscr{r}^n) = n\,\mathscr{r}^{n-1}\,\hat{\boldsymbol{\mathscr{r}}}$$

**1.14** Given the rotation matrix

$$\begin{pmatrix}\bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix}$$

or the two equations

$$\bar{y} = y\cos\phi + z\sin\phi$$
$$\bar{z} = -y\sin\phi + z\cos\phi$$

differentiating with respect to $\bar{y}$ and $\bar{z}$ respectively gives

$$1 = \frac{\partial y}{\partial \bar{y}}\cos\phi + \frac{\partial z}{\partial \bar{y}}\sin\phi$$
$$1 = -\frac{\partial y}{\partial \bar{z}}\sin\phi + \frac{\partial z}{\partial \bar{z}}\cos\phi$$

this can only be true if

$$\frac{\partial y}{\partial \bar{y}} = \cos\phi, \quad \frac{\partial z}{\partial \bar{y}} = \sin\phi \quad \text{and} \quad \frac{\partial y}{\partial \bar{z}} = -\sin\phi, \frac{\partial z}{\partial \bar{y}} = \cos\phi$$

which satisfies the trig identity $\sin^2\phi + \cos^2\phi = 1$. Differentiating $f$ with respect to the rotated coordinates $\bar{y}$ and $\bar{z}$ is given by

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}} = -\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi$$

therefore, the gradient of $f$ transforms as a vector under rotations given by

$$\overline{\boldsymbol{\nabla} f} = \frac{\partial f}{\partial \bar{y}}\hat{\bar{\mathbf{y}}} + \frac{\partial f}{\partial \bar{z}}\hat{\bar{\mathbf{z}}} = \left(\frac{\partial f}{\partial y}\cos\phi + \frac{\partial f}{\partial z}\sin\phi\right)\hat{\bar{\mathbf{y}}} + \left(-\frac{\partial f}{\partial y}\sin\phi + \frac{\partial f}{\partial z}\cos\phi\right)\hat{\bar{\mathbf{z}}}$$

or in matrix form

$$\overline{\boldsymbol{\nabla} f} = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\boldsymbol{\nabla} f$$

where the gradient is a column vector

$$\boldsymbol{\nabla} f = \begin{pmatrix}\frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{pmatrix}$$