1 Energy

4.2 From the origin O to point $P = (1, 1)$ a two dimensional force $\mathbf{F} = (x^2, 2xy)$ moves a point along three paths where the work done by the force is

$$W = \int_O^P \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_O^P F_x \,\mathrm{d}x + F_y \,\mathrm{d}y$$

(a) Splitting the path into two parts $O \to Q = (1, 0)$ and $Q \to P$, we have two integrals

$$W = \int_O^Q F_x \,\mathrm{d}x + \int_Q^P F_y \,\mathrm{d}y$$

where the first integral accounts for just the x component of force $F_x = x^2$ and the second integral accounts for just the y component of force when $x = 1$; $F_y = 2(1)y$. Thus

$$W = \int_0^1 x^2 \,\mathrm{d}x + \int_0^1 2y \,\mathrm{d}y = \frac{4}{3}$$

(b) The path follows the parabola $y = x^2$ from $O \to P$. From $\mathrm{d}y = 2x \,\mathrm{d}x$ the integral can be rewritten in terms of just x

$$W = \int_0^1 x^2 \,\mathrm{d}x + \int_0^1 2x(x^2) \,\mathrm{d}y = \frac{1}{3} + \int_0^1 4x^4 \,\mathrm{d}x = \frac{17}{15}$$

(c) Path follows the parametric curve $x = t^3$ and $y = t^2$ where the differentials are: $dx = 3t^2 \,\mathrm{d}t$ and $dy = 2t \,\mathrm{d}t$. Thus the work done on the path is

$$W = \int_0^1 (t^6)(3t^2 \,\mathrm{d}t) + \int_0^1 (2t^3)(2t \,\mathrm{d}t) = \frac{1}{3} + \frac{4}{5} = \frac{19}{15}$$

4.3 Same as Problem 4.2 but with a force $\mathbf{F} = (-y, x)$ and three different paths from $P = (1, 0) \to Q = (0, 1)$.
(a) This path follows a straight line $y = 0$ from $P \to O$ and then $x = 0$ from $O \to Q$. Thus the work done is

$$W = \int_P^O F_x \,\mathrm{d}x + \int_O^Q F_y \,\mathrm{d}y = 0$$

(b) A straight line from $P \to Q$ is given by $y = -x + 1$ and the differential $dy = -dx$. Thus the work done is

$$W = \int_P^Q F_x \,\mathrm{d}x + F_y \,\mathrm{d}y = \int_1^0 (-(-x + 1)) \,\mathrm{d}x + (x)(-\,\mathrm{d}x) = \int_1^0 -1 \,\mathrm{d}x = 1$$

(c) The path of a quarter circle centered on the origin in polar coordinates is given by

$$x = r \cos\phi \qquad y = r \sin\phi$$

where $r = 1$, $\phi = 0 \to \pi/2$ and the differentials are

$$dx = \cos\phi dr - r \sin\phi \,\mathrm{d}\phi = -\sin\phi \,\mathrm{d}\phi \qquad dy = \sin\phi dr + r \cos\phi \,\mathrm{d}\phi = \cos\phi \,\mathrm{d}\phi$$

Thus the work done is

$$W = \int_P^Q F_x \,\mathrm{d}x + F_y \,\mathrm{d}y = \int_0^{\pi/2} (-\sin\phi)(-\sin\phi \,\mathrm{d}\phi) + (\cos\phi)(\cos\phi \,\mathrm{d}\phi) = \int_0^{\pi/2} \mathrm{d}\phi = \frac{\pi}{2}$$

4.5 (a) Given the force of gravity $\mathbf{F} = -mg\hat{\mathbf{y}}$ and vertical height from 1 to 2 $h = y_2 - y_1$, the work done by gravity is

$$W_g(1 \to 2) = \int_1^2 \mathbf{F} \cdot \mathrm{d}r = \int_0^h -mg\,\mathrm{d}y = -mgh$$

Since the force $\mathbf{F}$ depends only on position and the work done by is independent of the path taken, the force is conservative.
(b) The gravitational potential energy of the particle is

$$U_g(\mathbf{r}) = -W_g(0 \to \mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_0^{\mathbf{r}} -mg\,\mathrm{d}y = mgy$$

where $\mathbf{r} = y\hat{\mathbf{y}}$ is the position vector of the particle.

4.7 (a) Given the gravitational force has magnitude $F_y = -m\gamma y^2$, the work done by gravity is

$$W = \int_1^2 F_y\,\mathrm{d}y = \int_1^2 m\gamma y^2\,\mathrm{d}y = \frac{1}{3}m\gamma(y_2^3 - y_1^3)$$

The gravity is still conservative since the work done by gravity is independent of the path taken and the force depends only on position. Hence, the corresponding potential energy is

$$U_g(\mathbf{r}) = -W(0 \to \mathbf{r}) = -\int_0^y F_y \cdot \mathrm{d}y' = \frac{1}{3}m\gamma y^3$$

(b)

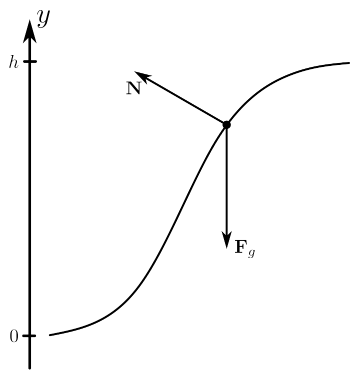

Figure 1.1: A threaded bead on a wire with two forces acting on it; The force of gravity $\mathbf{F}_g$ is conservative and the normal force $\mathbf{N}$ is non-conservative.

(c) The bead is initially released from rest at a height h. From conservation of energy:

$$E_i = E_f \tag{1.1}$$
$$\frac{1}{3}m\gamma h^3 = \frac{1}{2}mv^2 \tag{1.2}$$
$$v = \sqrt{\frac{2}{3}\gamma h^3} \tag{1.3}$$

where v is the speed of the bead at the bottom of the wire.

4.9 (a) Assuming the force of a one-dimensional spring $F = -kx$ is conservative, potential energy is

$$U(x) = -\int_0^x F\,\mathrm{d}x' = \frac{1}{2}kx^2$$

where x is the displacement of the spring from its equilibrium position.
(b) From Newton's second law, the new equilibrium position x_o is found when the spring force and gravity are equal.

$$0 = F + F_g = -kx_o + mg \implies x_o = \frac{mg}{k}$$

When $y = 0$, $U = 0$. Thus the potential energy is zero at position $x = x_o$:

$$U(x_o) = \frac{1}{2}k(x_o)^2 - mg(x_o) = 0$$

The total potential energy of the system at position $x = y + x_o$ is

$$\begin{aligned} U(x) = U_{sp} + U_g &= \frac{1}{2}k(y + x_o)^2 - mg(y + x_o) \\ &= \frac{1}{2}ky^2 + kyx_o - mgy + \frac{1}{2}kx_o^2 - mgx_o \end{aligned}$$

Since $kyx_o - mgy = 0$ and the last two terms are the potential energy at the new equilibrium $U(x_o) = 0$, the total potential energy is $U(x) = \frac{1}{2}ky^2$.

4.11 Finding the partial derivatives of the functions with constants a, b, c:
(a) $f(x, y, z) = ax^2 + bxy + cy^2$:

$$\frac{\partial f}{\partial x} = 2ax + by \qquad \frac{\partial f}{\partial y} = bx + 2cy \qquad \frac{\partial f}{\partial z} = 0$$

(b) $g(x, y, z) = \sin\big(axyz^2\big)$:

$$\frac{\partial g}{\partial x} = ayz^2 \cos\big(axyz^2\big) \qquad \frac{\partial g}{\partial y} = axz^2 \cos\big(axyz^2\big) \qquad \frac{\partial g}{\partial z} = 2axyz \cos\big(axyz^2\big)$$

(c) $h(x, y, z) = ar$ where $r = \sqrt{x^2 + y^2 + z^2}$: Since

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

The partial derivatives of h are

$$\frac{\partial h}{\partial x} = \frac{ax}{r} \qquad \frac{\partial h}{\partial y} = \frac{ay}{r} \qquad \frac{\partial h}{\partial z} = \frac{az}{r}$$

4.13 Calculating the gradient $\boldsymbol{\nabla} f$ of
(a) $f(x, y, z) = \ln(r) = \ln\Big(\sqrt{x^2 + y^2 + z^2}\Big)$:

$$\frac{\partial f}{\partial x} = \frac{x}{r^2} \qquad \frac{\partial f}{\partial y} = \frac{y}{r^2} \qquad \frac{\partial f}{\partial z} = \frac{z}{r^2}$$
$$\boldsymbol{\nabla} f = \frac{x}{r^2}\hat{\mathbf{x}} + \frac{y}{r^2}\hat{\mathbf{y}} + \frac{z}{r^2}\hat{\mathbf{z}} = \frac{\hat{\mathbf{r}}}{r}$$

(b) $f = r^n = (x^2 + y^2 + z^2)^{n/2}$ where n is a constant:

$$\frac{\partial f}{\partial x} = nr^{n-1}\frac{x}{r} = nr^{n-2}x \qquad \frac{\partial f}{\partial y} = nr^{n-2}y \qquad \frac{\partial f}{\partial z} = nr^{n-2}z$$
$$\boldsymbol{\nabla} f = nr^{n-2}x\hat{\mathbf{x}} + nr^{n-2}y\hat{\mathbf{y}} + nr^{n-2}z\hat{\mathbf{z}} = nr^{n-1}\hat{\mathbf{r}}$$

(c) $f = g(r)$ where $g(r)$ is some unspecified function of r:

$$\frac{\partial f}{\partial x} = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r} \qquad \frac{\partial f}{\partial y} = g'(r)\frac{y}{r} \qquad \frac{\partial f}{\partial z} = g'(r)\frac{z}{r}$$
$$\boldsymbol{\nabla} f = g'(r)\frac{x}{r}\hat{\mathbf{x}} + g'(r)\frac{y}{r}\hat{\mathbf{y}} + g'(r)\frac{z}{r}\hat{\mathbf{z}} = g'(r)\hat{\mathbf{r}}$$

4.15 Using the approximate formula for the change in f:

$$df = \boldsymbol{\nabla} f \cdot d\mathbf{r} \tag{4.35}$$

For $f(\mathbf{r}) = x^2 + 2y^2 + 3z^2$, The approximation of moving from $\mathbf{r} = (1, 1, 1)$ to $(1.01, 1.03, 1.05)$:

$$\begin{aligned} df &= \boldsymbol{\nabla} f \cdot d\mathbf{r} = (2x\hat{\mathbf{x}} + 4y\hat{\mathbf{y}} + 6z\hat{\mathbf{z}}) \cdot (0.01\hat{\mathbf{x}} + 0.03\hat{\mathbf{y}} + 0.05\hat{\mathbf{z}}) \\ &= 0.02 + 0.12 + 0.30 = 0.44 \end{aligned}$$

The exact change in f is

$$\Delta f = f(1.01, 1.03, 1.05) - f(1, 1, 1) = 0.4494$$

4.17 A charge q experiences a constant force $\mathbf{F} = q\mathbf{E}_o$ where $\mathbf{E}_o$ is a uniform electric field.
(a) The work done by the force from point 1 to 2

$$W = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = q\mathbf{E}_o \cdot (\mathbf{r}_1 - \mathbf{r}_2)$$

which is independent of the path hence it is a conservative force. Thus the potential energy is

$$U(\mathbf{r}) = -W(0 \to \mathbf{r}) = -q\mathbf{E}_o \cdot \mathbf{r}$$

(b) Checking that $\mathbf{F}$ is derivable from potential energy U:

$$\begin{aligned} \mathbf{F} = -\boldsymbol{\nabla} U &= -\frac{\partial U}{\partial x}\hat{\mathbf{x}} - \frac{\partial U}{\partial y}\hat{\mathbf{y}} - \frac{\partial U}{\partial z}\hat{\mathbf{z}} \\ &= -\frac{\partial}{\partial x}(-q\mathbf{E}_o \cdot \mathbf{x})\hat{\mathbf{x}} - \frac{\partial}{\partial y}(-q\mathbf{E}_o \cdot \mathbf{y})\hat{\mathbf{y}} - \frac{\partial}{\partial z}(-q\mathbf{E}_o \cdot \mathbf{z})\hat{\mathbf{z}} \\ &= q\mathbf{E}_o(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}) = q\mathbf{E}_o \end{aligned}$$

4.18 (a) If the vector $\boldsymbol{\nabla} f$ is perpendicular to the surface through r, then (4.35) becomes

$$df = \boldsymbol{\nabla} f \cdot d\mathbf{r} = 0 \tag{1.4}$$

since the dot product of perpendicular vectors is zero. Thus f is constant on the surface.
(b) Choosing a small displacement $d\mathbf{r} = \epsilon\mathbf{u}$:

$$df = \boldsymbol{\nabla} f \cdot (\epsilon\mathbf{u}) = \epsilon\boldsymbol{\nabla} f \cdot \mathbf{u} = \epsilon|\boldsymbol{\nabla} f||\mathbf{u}|\cos\theta \tag{1.5}$$

the corresponding maximum value of df is when $\theta = 0$ where $\mathbf{u}$ is in the same direction as $\boldsymbol{\nabla} f$.

4.19 (a) For a surface of constant f, $f = x^2 + 4y^2$ is an ellipse in the xy plane cenetered at the origin with semi-major axis $a = \sqrt{f}$ and semi-minor axis $b = \sqrt{f}/2$. Since z is unspecified, the surface is an infinitely long elliptical cylinder.
(b) The gradient of f is

$$\boldsymbol{\nabla} f = 2x\hat{\mathbf{x}} + 8y\hat{\mathbf{y}}$$

For a surface $f = 5$ at the point $(1, 1, 1)$, the gradient is $\boldsymbol{\nabla} f = 2\hat{\mathbf{x}} + 8\hat{\mathbf{y}}$. From Problem 4.18, $0 = \boldsymbol{\nabla} f \cdot d\mathbf{r}$ describes that $\boldsymbol{\nabla} f$ is normal to this surface. Thus the unit normal vector is

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{2\hat{\mathbf{x}} + 8\hat{\mathbf{y}}}{\sqrt{68}} = \frac{1\hat{\mathbf{x}} + 4\hat{\mathbf{y}}}{\sqrt{17}}$$

or $-\hat{\mathbf{n}}$ corresponding to the opposite direction. Moving along the direction of $\mathbf{n}$ maximizes the rate of change of f.

4.20 Finding the curl, $\boldsymbol{\nabla} \times \mathbf{F}$, for the forces:
(a) $\mathbf{F} = k\mathbf{r}$

$$\begin{aligned}\boldsymbol{\nabla} \times k\mathbf{r} &= \boldsymbol{\nabla} \times (kx\hat{\mathbf{x}} + ky\hat{\mathbf{y}} + kz\hat{\mathbf{z}}) \\ &= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} = 0\end{aligned}$$

(b) $\mathbf{F} = (Ax, By^2, Cz^3)$ where A, B, C are constants:

$$\boldsymbol{\nabla} \times (Ax, By^2, Cz^3) = 0$$

(c) $\mathbf{F} = (Ay^2, Bx, Cz)$:

$$\begin{aligned}\boldsymbol{\nabla} \times (Ay^2, Bx, Cz) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ay^2 & Bx & Cz \end{vmatrix} \\ &= (0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (B - 2Ay)\hat{\mathbf{z}} = (B - 2Ay)\hat{\mathbf{z}}\end{aligned}$$

4.21 Given the gravitational force

$$\mathbf{F} = -\frac{GmM}{r^2}\hat{\mathbf{r}} = -\frac{GmM}{r^3}\mathbf{r}$$

The curl of $\mathbf{F}$ is

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \boldsymbol{\nabla} \times \frac{GmM}{r^3}\mathbf{r} \\ &= \frac{GmM}{r^3}\boldsymbol{\nabla} \times \mathbf{r} \\ &= \frac{GmM}{r^3}\boldsymbol{\nabla} \times (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \\ &= \frac{GmM}{r^3}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \frac{GmM}{r^3}(0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) = 0\end{aligned}$$

Thus the gravitational force is conservative.