1 Newton's Laws of Motion

1.1 Given two vectors $\mathbf{b} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and $\mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{z}}$ find $\mathbf{b} + \mathbf{c}, 5\mathbf{b} + 2\mathbf{c}, \mathbf{b} \cdot \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$.

$$\mathbf{b} + \mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{x}} + \hat{\mathbf{z}} = 2\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

$$5\mathbf{b} + 2\mathbf{c} = 5\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{x}} + 2\hat{\mathbf{z}} = 7\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$$

$$\mathbf{b} \cdot \mathbf{c} = \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} + \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 1 + 0 + 0 + 0 = 1$$

$$\mathbf{b} \times \mathbf{c} = \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$$

1.2 Given Vectors $\mathbf{b} = (1, 2, 3),\ \mathbf{c} = (3, 2, 1)$ compute 1.1

$$\mathbf{b} + \mathbf{c} = (4, 4, 4)$$

$$5\mathbf{b} + 2\mathbf{c} = (5, 10, 15) + (6, 4, 2) = (11, 14, 17)$$

$$\mathbf{b} \cdot \mathbf{c} = 1(3) + 2(2) + 3(1) = 10$$

$$\mathbf{b} \times \mathbf{c} = \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = -4\hat{\mathbf{x}} + 8\hat{\mathbf{y}} - 4\hat{\mathbf{z}}$$

1.3 Pythagorean Theorem for three dimensions

First find the magntitude of the vector $\mathbf{a} = x + y$ made up of the x and y components

$$|\mathbf{a}| = \sqrt{x^2 + y^2}$$

Then the magnitude of the vector $\mathbf{r} = a + z$ made up of the x, y, and z components

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$
$$r^2 = x^2 + y^2 + z^2$$

1.4 Find $\angle$ between vectors $\mathbf{b} = (1, 2, 4),\ \mathbf{c} = (4, 2, 1)$ using dot product

$$\mathbf{b} \cdot \mathbf{c} = 1(4) + 2(2) + 4(1) = 12$$

$$|\mathbf{b}| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$$

$$|\mathbf{c}| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = \frac{12}{21}$$

$$\theta = \cos^{-1}\frac{12}{21} = 55^\circ$$

1.5 Angle between cube body diagonal and face diagonal

Face diagonal vector $\mathbf{P} = (1, 1, 0)$ and Body Diagonal $\mathbf{Q} = (1, 1, 1)$

$$\mathbf{P} \cdot \mathbf{Q} = 1 + 1 + 0 = 2$$

$$|\mathbf{P}| = \sqrt{2} \qquad |\mathbf{Q}| = \sqrt{3}$$

$$\cos\theta = \frac{2}{\sqrt{6}}$$

$$\theta = 35^\circ$$

1.6 Find scalar s for orthogonal vectors $\mathbf{B} = \hat{\mathbf{x}} + s\hat{\mathbf{y}}$, $\mathbf{C} = \hat{\mathbf{x}} - s\hat{\mathbf{y}}$

The dot product of orthgonal vectors is zero:

$$\mathbf{B} \cdot \mathbf{C} = 0$$
$$(1, s) \cdot (1, -s) = 1 - s^2 = 0$$
$$s^2 = 1$$
$$s = \pm 1$$

1.7 Prove the 2 definitions of scalar product are equal

Treat vector $\mathbf{r}$ stricly in the x axis: $\mathbf{r} = (x, 0, 0)$ and $\mathbf{s} = (s_x, s_y, s_z)$:

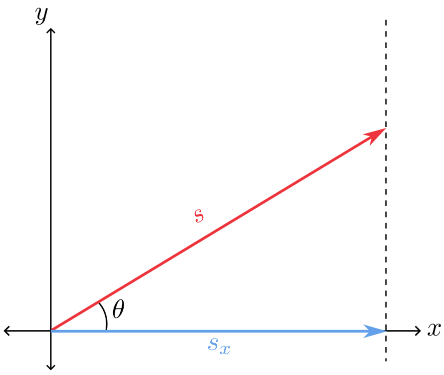

Figure 1.1

the x component of the vector s_x is equivalent to $s\cos\theta$...

$$\mathbf{r} \cdot \mathbf{s} = |\mathbf{r}||\mathbf{s}|\cos\theta \qquad = \sum_{i=1}^{3} r_i s_i$$
$$= xs\cos\theta \qquad = xs_x$$
$$= xs_x$$

1.8 Prove dot product is distributive and differentiable

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \sum_{i=1}^{3} (a_i + b_i) c_i$$
$$= \sum_{i=1}^{3} a_i c_i + \sum_{i=1}^{3} b_i c_i$$
$$= \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$$

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d}{dt} \sum_{i=1}^{3} a_i b_i$$
$$= \sum_{i=1}^{3} \frac{d}{dt} a_i b_i$$
$$= \sum_{i=1}^{3} \frac{da_i}{dt} b_i + \sum_{i=1}^{3} a_i \frac{db_i}{dt}$$
$$= \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}$$

1.9 Show law of cosines from the identity $(\mathbf{a}+\mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b}$

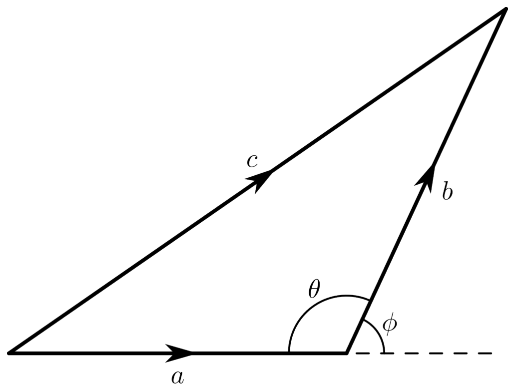

Figure 1.2: Law of Cosine: $c^2 = a^2 + b^2 - 2ab\cos\theta$

Using the identity $\cos\phi = \cos(\pi - \phi) = -\cos\theta$

$$
\begin{aligned}
\mathbf{c}^2 &= (\mathbf{a}+\mathbf{b})^2 \\
&= a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} \\
&= a^2 + b^2 + 2|\mathbf{a}||\mathbf{b}|\cos\phi \\
&= a^2 + b^2 - 2ab\cos\theta \\
&= a^2 + b^2 - 2ab\cos\theta
\end{aligned}
$$

1.11 Describe the orbit of a particle with the position function $r(t) = \hat{\mathbf{x}}b\cos\omega t + \hat{\mathbf{y}}c\sin\omega t$

This is a parametric representation of an ellipse using trigonometric functions $x = b\cos\omega t$, $y = c\sin\omega t$ equivalent to the standard ellipse equation:

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$$

The particle is moving counter-clockwise in the x-y plane with semi-major(longer) axis and semi-minor(short) axis c and b.

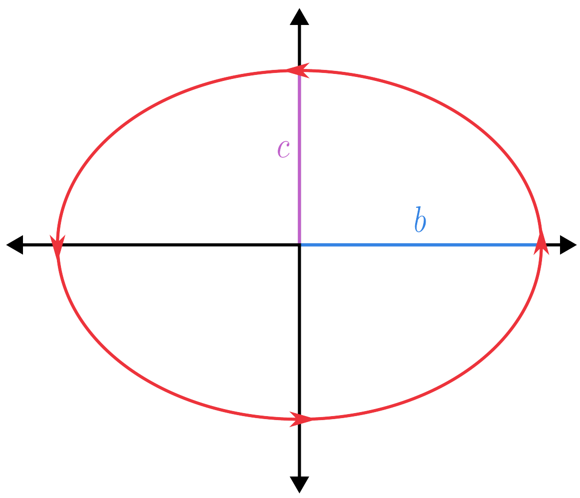

Figure 1.3: Ellipse with semi-major axis b and semi-minor axis c

1.13 For a fixed unit vector $\mathbf{u}$ show the any vector $\mathbf{b}$ satisfies $b^2 = (\mathbf{b} \cdot \mathbf{u})^2 + (\mathbf{b} \times \mathbf{u})^2$

The magnitude of a unit vector is 1

$$b^2 = (b \sin\theta)^2 + (b\cos\theta)^2$$

This is equivalent to the Pythagorean Theorem.

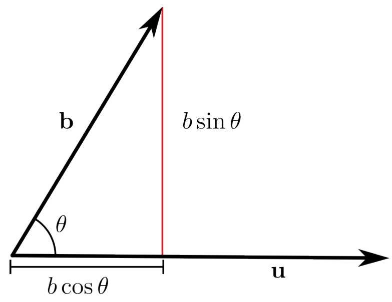

Figure 1.4

1.15 Show $\mathbf{r} \times \mathbf{s}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{s}$ with magnitude $rs\sin\theta$ given by the right-hand rule

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Choosing $\mathbf{r} = (r, 0, 0), \mathbf{s} = (s_x, s_y, 0)$ where $s_y = s\sin\theta$

$$\begin{aligned}
\mathbf{r} \times \mathbf{s} &= \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & 0 & 0 \\ s_x & s_y & 0 \end{bmatrix} \\
&= -r_x s_y \hat{\mathbf{z}} \\
&= rs\sin\theta\hat{\mathbf{z}}
\end{aligned}$$

The result is a vector stricly in the z direction, orthogonal to the x-y plane.

1.17 (a) Prove the vector product is distributive as in: $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = \mathbf{r} \times \mathbf{u} + \mathbf{r} \times \mathbf{v}$ (b) and differentiable by product rule

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \frac{d\mathbf{r}}{dt} \times \mathbf{s} + \mathbf{r} \times \frac{d\mathbf{s}}{dt}$$

(a) The components of the vector cross product $\mathbf{p} = \mathbf{r} \times \mathbf{s}$

$$\begin{aligned}
p_x &= r_y s_z - r_z s_y \\
p_y &= r_z s_x - r_x s_z \\
p_z &= r_x s_y - r_y s_x
\end{aligned} \tag{1.9}$$

Starting with the x componenet of the the resultant vector

$$\begin{aligned}
\mathbf{r} \times (\mathbf{u} + \mathbf{v}) &= \det \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ u_x + v_x & u_y + v_y & u_z + v_z \end{bmatrix} \\
&= r_y(u_z + v_z) - r_z(u_y + v_y) \\
&= (r_y u_z - r_z u_y) + (r_y v_z - r_z v_y) \\
&= (r \times u)_x + (r \times v)_x
\end{aligned}$$

The same can be done for the y and z components to show that the product is distributive.

(b) Using (1.9) starting with just the x component again

$$\begin{aligned}\frac{d}{dt}[(\mathbf{r}\times\mathbf{s})_x] &= \frac{d}{dt}[r_y s_z - r_z s_y]\\ &= \frac{dr_y}{dt}s_z + r_y\frac{ds_z}{dt} - \frac{dr_z}{dt}s_y - r_z\frac{ds_y}{dt} = \left(\frac{dr_y}{dt}s_z - \frac{dr_z}{dt}s_y\right) + \left(r_y\frac{ds_z}{dt} + r_z\frac{ds_y}{dt}\right)\\ &= \left[\frac{d\mathbf{r}}{dt}\times\mathbf{s} + \mathbf{r}\times\frac{d\mathbf{s}}{dt}\right]_x\end{aligned}$$

id. can the done in the y and z components to prove the product rule.

1.19 If $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ are the position, velocity, and acceleration vectors of a particle, prove that

$$\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] = \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r})$$

Using the product rule for the dot product

$$\begin{aligned}\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] &= \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{r})\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\frac{d\mathbf{r}}{dt}\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\mathbf{a}\times\mathbf{r} + \mathbf{a}\cdot\mathbf{v}\times\mathbf{v}\\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r})\end{aligned}$$

The cross product of a vector with itself is zero. $\mathbf{a}\times\mathbf{a} = 0$ and the dot product of orthogonal vectors is zero.

1.21 Show the volume of a parallelepiped defined by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$

In geometry, the cross product refers to the positive area of a parallelgram(directed area product) which is the base area of the parallelepiped.

$$\mathbf{b}\times\mathbf{c} = bc\sin\theta = \text{base area}$$

The dot product equates to the volume of the parallelepiped with height $\mathbf{a}\cos\phi$... scalar triple product

1.23 The unknown vector $\mathbf{b}$ satisfies $\mathbf{b}\cdot\mathbf{v} = \lambda$ and $\mathbf{b}\times\mathbf{v} = \mathbf{c}$. Find $\mathbf{v}$ in terms of λ, $\mathbf{b}$, and $\mathbf{c}$.

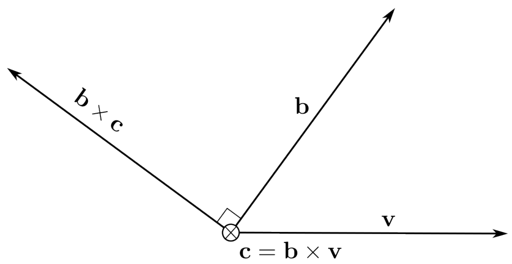

Figure 1.5: Visual example of vector $\mathbf{b}$ and $\mathbf{v}$ with vector $\mathbf{c}$ pointing into the page

$\mathbf{v}$ can be expressed as a linear combination of 2 orthogonal vectors

$$\mathbf{v} = \alpha\mathbf{b} + \beta\mathbf{b}\times\mathbf{c}$$

Taking the dot product to solve for α

$$\begin{aligned}
\mathbf{b}\cdot\mathbf{v} &= \mathbf{b}\cdot(\alpha\mathbf{b}+\beta\mathbf{b}\times\mathbf{c}) \\
&= \alpha\mathbf{b}\cdot\mathbf{b}+\beta\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) \\
\lambda &= \alpha b^2 \\
\alpha &= \frac{\lambda}{b^2}
\end{aligned}$$

1.5 shows that $\mathbf{b}\times\mathbf{c}$ is orthogonal to $\mathbf{b}$ so the dot product is zero. Solving for β

$$\begin{aligned}
\mathbf{b}\times\mathbf{v} &= \mathbf{b}\times(\alpha\mathbf{b}+\beta\mathbf{b}\times\mathbf{c}) \\
&= \alpha\mathbf{b}\times\mathbf{b}+\beta\mathbf{b}\times(\mathbf{b}\times\mathbf{c}) \\
&= \beta\mathbf{b}\times(\mathbf{b}\times\mathbf{c}) \\
&= \beta\mathbf{b}(b^2c) \\
&= \beta(-b^2c)
\end{aligned}$$

The direction of the resultant triple product is in the negative direction of $\mathbf{c}$ so $\beta=-\frac{1}{b^2}$

$$\mathbf{v}=\frac{\lambda}{b^2}\mathbf{b}-\frac{1}{b^2}\mathbf{b}\times\mathbf{c}$$

1.25 Find the general solution for the first-order differential equation $df/dt=-3f$

$$\begin{aligned}
\frac{df}{dt} &= -3f \\
\int\frac{1}{f}df &= \int -3dt \\
\ln f &= -3t+C \\
f &= e^{-3t+C} \\
f &= Ae^{-3t}
\end{aligned}$$

1.29 Go over the steps from (1.25) to (1.29) for the conservation of momentum for $N=4$ particles

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$$\text{(net force on particle)}=\mathbf{F}_\alpha=\sum_{\beta\neq\alpha}\mathbf{F}_{\alpha\beta}+\mathbf{F}_\alpha^{\text{ext}} \tag{1.25}$$

Where $\mathbf{F}_{\alpha\beta}$ denotes the force on particle α due to particle β

$$\dot{\mathbf{p}}_\alpha=\sum_{\beta\neq\alpha}\mathbf{F}_{\alpha\beta}+\mathbf{F}_\alpha^{\text{ext}} \tag{1.26}$$

This is in accordance to Newton's second law, same as the rate of change of momentum $\mathbf{p}_\alpha$. For the total momentum of the particle $\mathbf{P}$

$$\dot{\mathbf{P}}=\sum_\alpha^N\dot{\mathbf{p}}_\alpha=\sum_\alpha^N\sum_{\beta\neq\alpha}\mathbf{F}_{\alpha\beta}+\sum_\alpha^N\mathbf{F}_\alpha^{\text{ext}} \tag{1.27}$$

Reorganizing double sum

$$\sum_\alpha^N\sum_{\beta\neq\alpha}\mathbf{F}_{\alpha\beta}=\sum_\alpha^N\sum_{\beta>\alpha}(\mathbf{F}_{\alpha\beta}+\mathbf{F}_{\beta\alpha}) \tag{1.28}$$

Since the terms in double sum (1.28) is zero by Newton's third law

$$\dot{\mathbf{P}}=\sum_\alpha^N\mathbf{F}_\alpha^{\text{ext}}\equiv\mathbf{F}^{\text{ext}} \tag{1.29}$$

(1.25) and (1.26) for $N = 4$ particles

$$\begin{aligned}
\dot{\mathbf{p}}_1 &= \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}} \\
\dot{\mathbf{p}}_2 &= \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}} \\
\dot{\mathbf{p}}_3 &= \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}}
\end{aligned}$$

Summation of momentum (1.27) $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 + \dot{\mathbf{p}}_3 + \dot{\mathbf{p}}_4$

$$\begin{aligned}
\dot{\mathbf{P}} = &(\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_1^{\text{ext}}) + (\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_2^{\text{ext}}) \\
&+(\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_3^{\text{ext}}) + (\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_4^{\text{ext}})
\end{aligned}$$

Reorganizing the double sum like (1.28)

$$\begin{aligned}
\dot{\mathbf{P}} = &(\mathbf{F}_{12} + \mathbf{F}_{21}) + (\mathbf{F}_{13} + \mathbf{F}_{31}) + (\mathbf{F}_{14} + \mathbf{F}_{41}) + (\mathbf{F}_{23} + \mathbf{F}_{32}) \\
&+(\mathbf{F}_{24} + \mathbf{F}_{42}) + (\mathbf{F}_{34} + \mathbf{F}_{43}) + (\mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}})
\end{aligned}$$

By Newton's third law, the opposing forces cancel out

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} + \mathbf{F}_4^{\text{ext}} \equiv \mathbf{F}^{\text{ext}}$$

1.31 The law of conservation of momentum says that if there are no external forces on this pair of particles, then their total momentum must be constant. Use this to prove that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

$$\dot{\mathbf{P}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} \equiv \mathbf{F}^{\text{ext}}$$

For a two particle system

$$\dot{\mathbf{P}} = \mathbf{F}_{12} + \mathbf{F}_{21} + \mathbf{F}^{\text{ext}}$$

If there are no external forces, then $\mathbf{F}^{\text{ext}} = 0$ and for total momentum to be constant, $\dot{\mathbf{P}} = 0$. Therefore the interparticle forces obey the third law i.e. $\mathbf{F}_{12} = -\mathbf{F}_{21}$

1.33 Prove the magnetic forces, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$, between two steady current loops obey Newton's 3rd law

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Hints: for currents I_1 and I_2, and points r_1 and r_2. According to Bio-Savart Law, the force on the segment $d\mathbf{r}_1$ due to $d\mathbf{r}_2$ of loop 2 is

$$\frac{\mu_0}{4\pi}\frac{I_1 I_2}{s^2} d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

where $\mathbf{s} = \mathbf{r_1} - \mathbf{r_2}$. The force $\mathbf{F_{12}}$ is found integrating over both loops. The unit vector is equivalent to

$$\hat{\mathbf{s}} = \frac{\mathbf{s}}{s} = \frac{\mathbf{r_1} - \mathbf{r_2}}{|\mathbf{r_1} - \mathbf{r_2}|}$$

and the *BAC - CAB* identity for the triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Integrating over both loops the force on loop 1 due to loop 2

$$\mathbf{F}_{12} = \oint \oint \frac{\mu_0}{4\pi}\frac{I_1 I_2}{s^2} d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \hat{\mathbf{s}})$$

Using the *BAC - CAB* identity

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \left[\oint \oint \frac{d\mathbf{r}_2 (d\mathbf{r}_1 \cdot \hat{\mathbf{s}})}{s^2} - \oint \oint \frac{\hat{\mathbf{s}}(d\mathbf{r}_1 \cdot d\mathbf{r}_2)}{s^2} \right]$$

In the first term, the dot product $d\mathbf{r_1} \cdot \hat{\mathbf{s}}$ is projection of in the direction of $\mathbf{s}$ and the integral of the closed current loop is zero

$$\oint_{C_2}\oint_{C_1} \frac{d\mathbf{r}_2(d\mathbf{r}_1 \cdot \hat{\mathbf{s}})}{s^3} = \oint_{C_2} d\mathbf{r_2} \oint_{C_1} \frac{ds}{s^2} = 0$$

We end up with the force on loop 1 due to loop 2 as

$$\mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2}\oint_{C_1} \frac{\hat{\mathbf{s}}(d\mathbf{r}_1 \cdot d\mathbf{r}_2)}{s^2} = -\mathbf{F}_{21}$$

1.35 A golf ball is hit from ground level due east at a velocity v_0 at an angle θ above the horizontal. Neglecting air resistance use Newton's Second Law to find position as a function of time, the time for the ball to hit the ground, and the range of the ball. x measures east, y north, and z vertically up.

Newton's second law states $\mathbf{F} = m\ddot{\mathbf{r}}$ where $\ddot{\mathbf{r}} = \mathbf{g} = -g\hat{\mathbf{z}}$ is the gravitational force. We can find the position of the ball by integrating twice with respect to time

$$\begin{aligned} \ddot{x} &= 0 & \ddot{y} &= 0 & \ddot{z} &= -g \\ \dot{x} &= 0 & \dot{y} &= 0 & \dot{z} &= -gt + v_0 \sin\theta \\ x(t) &= 0 & y(t) &= 0 & z(t) &= -\frac{1}{2}gt^2 + v_0 t \sin\theta \end{aligned}$$

The time for the ball to hit the ground is when $z(t) = 0$

$$-\frac{1}{2}gt^2 + v_0 t \sin\theta = 0$$
$$t = \frac{2v_0 \sin\theta}{g}$$

To get the range of the ball we subsitute t from above into $x(t)$

$$\begin{aligned} x(t) &= v_0 \cos\theta \frac{2v_0 \sin\theta}{g} \\ x(t) &= \frac{2v_0^2 \sin\theta \cos\theta}{g} \\ x(t) &= \frac{v_0^2 \sin 2\theta}{g} \end{aligned}$$

1.37 A student kicks a frictionless puck with initial speed v_0, so that it slides straight up a plane that is inclined at an angle θ above the horizontal (a) Write down Newton's second law for the puck and solve to give its position as a function of time (b) How long will the puck take to return to its starting point?

(a) Having the x-axis on the plane parallel to the incline we get the force equation

$$F_x = m\ddot{x} = -mg\sin\theta$$

Solving for the position of the puck by integrating twice with respect to time and using the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$

$$\begin{aligned} \ddot{x} &= -g\sin\theta \\ \dot{x} &= g\cos\theta t + v_0 \\ x(t) &= -\frac{1}{2}g\sin\theta t^2 + v_0 t \end{aligned}$$

(b) Solving for when the puck returns to its starting point $x(t) = 0$

$$-\frac{1}{2}g\sin\theta t^2 + v_0 t = 0$$
$$t = \frac{2v_0}{g\sin\theta}$$

1.39 Show the ball lands a distance $R = 2v_o^2 \sin\theta \cos(\theta+\phi)/\left(g\cos^2\phi\right)$ and $R_{max} = v_o^2/[g(1+\sin\phi)]$

Using θ as the angle above the incline and ϕ as the angle of the incline plane the components of the inital velocity are

$$v_{ox} = v_0\cos\theta \quad v_{oy} = v_0\sin\theta \quad v_{oz} = 0$$

Newton's second law

$$\begin{aligned} F_x &= -mg\sin\phi & F_y &= -mg\cos\phi & F_z &= 0 \\ \ddot{x} &= -g\sin\phi & \ddot{y} &= -g\cos\phi & \ddot{z} &= 0 \\ \dot{x} &= -g\sin\phi t + v_{ox} & \dot{y} &= -g\cos\phi t + v_{oy} & \dot{z} &= 0 \\ x(t) &= -\frac{1}{2}g\sin\phi t^2 + v_{ox}t & y(t) &= -\frac{1}{2}g\cos\phi t^2 + v_{oy}t & z(t) &= 0 \end{aligned}$$

The range of the ball is when $y(t) = 0$ as it lands on the incline plane

$$\begin{aligned} 0 &= -\frac{1}{2}g\cos\phi t^2 + v_{oy}t \\ t &= \frac{2v_{oy}}{g\cos\phi} \end{aligned}$$

Substituting t into $x(t)$ and using the identity $\cos(\theta\pm\phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$ simplifies the range...

$$\begin{aligned} x(t) &= -\frac{1}{2}g\sin\phi\left(\frac{2v_{oy}}{g\cos\phi}\right)^2 + v_{ox}\left(\frac{2v_{oy}}{g\cos\phi}\right) \\ R &= \frac{-2v_o^2}{g\cos^2\phi}\sin^2\theta\sin\phi + \frac{2v_o^2}{g\cos\phi}\sin\theta\cos\theta \\ &= \frac{2v_o^2\sin\theta}{g\cos^2\phi}\left(-\sin\theta\sin\phi + \cos\theta\cos\phi\right) \\ R &= \frac{2v_o^2\sin\theta}{g\cos^2\phi}\cos(\theta+\phi) \end{aligned}$$

Set identity $\sin\alpha\cos\beta = 1/2[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$$\begin{aligned} \frac{dR}{d\theta} &= \frac{2v_o^2}{g\cos^2\phi}\left(\cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi)\right) \\ 0 &= \cos\theta\cos(\theta+\phi) - \sin\theta\sin(\theta+\phi) \\ 0 &= \cos(\theta + (\theta+\phi)) \\ \pi/2 &= 2\theta + \phi \\ \theta &= \frac{\pi - 2\phi}{4} \\ R_{max} &= \frac{2v_o^2\sin\left(\frac{\pi-2\phi}{4}\right)}{g\cos^2\phi}\cos\left(\frac{\pi-2\phi}{4} + \phi\right) \\ &= \frac{v_o^2}{g}\frac{\sin\left(\frac{\pi-2\phi}{2} + \phi\right) + \sin -\phi}{\cos^2\phi} \\ &= \frac{v_o^2}{g}\frac{\sin\left(\frac{\pi}{2}\right) - \sin\phi}{1 - \sin^2\phi} \\ &= \frac{v_o^2}{g}\frac{1-\sin\phi}{(1+\sin\phi)(1-\sin\phi)} \\ R_{max} &= \frac{v_o^2}{g[1+\sin\phi]} \end{aligned}$$

1.41 An astronaut in gravity-free space is twirling a mass m on the end of a string of length R in a circle, with constant angular velocity ω. Write down Newton's second law in polar coordinates and find the tension in the string.

Newton's second law in polar coordinates

$$\begin{aligned} F_r &= m\ddot{r} - mr\dot{\phi}^2 \\ F_\theta &= mr\ddot{\phi} + 2m\dot{r}\dot{\phi} \end{aligned} \tag{1.48}$$

The only force acting on the mass is the tension in the string. The tension is in the radial direction, so $F_r = -T$ and the mass is moving in a circle of radius $r = R$ so $\ddot{r} = \dot{r} = 0$. Since the angular velocity $\dot{\phi} = \omega$ is constant, $\ddot{\phi} = 0$. Newton's second law (1.48) then simplifies to $F_r = -mr\dot{\phi}^2$ and $F_\theta = 0$. Solving for tension we get

$$\begin{aligned} -T &= -mr\dot{\phi}^2 \\ T &= mr\omega^2 \end{aligned}$$

1.43 (a) Prove that the unit vector $\hat{\mathbf{r}}$ of two-dimensional polar coorddinates is equal to

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \tag{1.59}$$

and find a corresponding expression for $\hat{\boldsymbol{\phi}}$ (b) Assuming that ϕ depends on the time t, differentiate your answers in part (a) to give an alternative proof of the results (1.42) and (1.46) for the time derivatives $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$.

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\phi}\hat{\boldsymbol{\phi}} \tag{1.42}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = -\dot{\phi}\hat{\mathbf{r}} \tag{1.46}$$

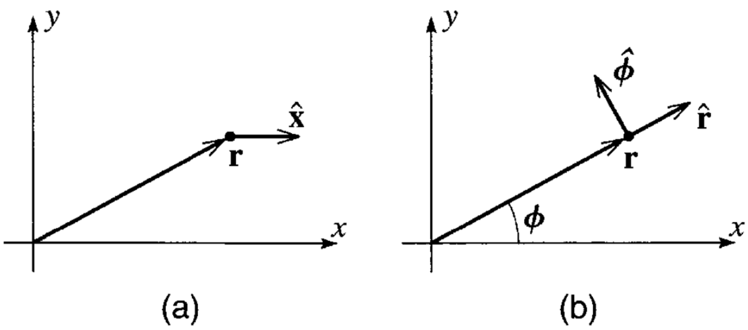

Figure 1.6: Unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ on the cartesian plane

(a) Figure 1.6 shows that the x and y component of the radial unit vector are $\hat{\mathbf{r}}_x = \cos\phi$ and $\hat{\mathbf{r}}_y = \sin\phi$. For the angular unit vector, the x and y components are $\hat{\boldsymbol{\phi}}_x = -\sin\phi$ and $\hat{\boldsymbol{\phi}}_y = \cos\phi$. The unit vector can be expressed as

$$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}_x + \hat{\boldsymbol{\phi}}_y = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \tag{1.60}$$

(b) Keep in mind that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constants. Differentiating (1.59) and (1.60) with respect to time

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} &= -\dot{\phi}\hat{\mathbf{x}}\sin\phi + \dot{\phi}\hat{\mathbf{y}}\cos\phi = \dot{\phi}\hat{\boldsymbol{\phi}} \\ \frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} &= -\dot{\phi}\hat{\mathbf{x}}\cos\phi - \dot{\phi}\hat{\mathbf{y}}\sin\phi = -\dot{\phi}\hat{\mathbf{r}} \end{aligned}$$

1.45 Prove that if $\mathbf{v}(t)$ is any vector that depends on time but which has constant magnitude, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. Prove the converse that $|\mathbf{v}(t)|$ is constant.

Hint: Consider the derivative of $\mathbf{v}^2$ Since the magnitude of $\mathbf{v}(t)$ is also $\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$, the derivative of $\mathbf{v}^2$ is tells us if the magnitude is constant.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\cdot\mathbf{v}(t))\\ &= 2\dot{\mathbf{v}}(t)\cdot\mathbf{v}(t)\end{aligned}$$

The magnitude of $\mathbf{v}(t)$ is constant if $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^2 = 0$. Since the dot product is zero, $\mathbf{v}(t)$ is orthogonal to $\dot{\mathbf{v}}(t)$. The converse is also true because having $\dot{\mathbf{v}}(t)$ orthogonal to $\mathbf{v}(t)$ means that $|\mathbf{v}(t)|$ is always constant from the definition of the dot product.

1.47 (a) Make a sketch to illustrate the three cylindrical polar coordinates ρ, ϕ, z with a position of a point P. Let P' denote the projection of P onto the xy plane. (b) Describe the three unit vectors $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ and write the expansion of the position vector $\mathbf{r} = (x, y, z)$ in terms of these unit vectors. (c) Differentiate the last answers twice to find the cylindrical component of acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ of the particle.

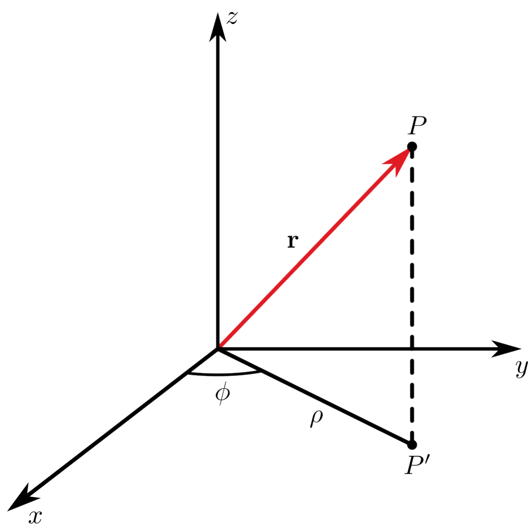

Figure 1.7: Cylindrical polar coordinates ρ, ϕ, z with a position of a point P

(a) Figure 1.7 shows the three cylindrical polar coordinates ρ, ϕ, z. $\rho = \sqrt{x^2 + y^2}$ is the distance of P from the projected point P' on the xy-plane. $\phi = \arctan y/x$ is the angle between the x-axis and the line from origin to P'. z is the height of P from the xy plane.

(b) $\hat{\boldsymbol{\rho}}$ is in the direction outward and orthogonal to the z axis. $\hat{\boldsymbol{\phi}}$ is perpendicular to $\hat{\boldsymbol{\rho}}$ and pointing counterclockwise along the tangent of a circle centered on the z axis. $\hat{\mathbf{z}}$ is in the direction of the z axis. The position vector $\mathbf{r} = (x, y, z)$ can be expressed as

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

(c) Differentiating twice with respect to time and substituting (1.42) and $\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = -\dot{\phi}\hat{\boldsymbol{\rho}}$ from (1.46)

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}\\ \ddot{\mathbf{r}} &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}}\end{aligned}$$

1.49 Imagine two concentric cylinders, centered on the vertical z axis, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from the vertical axis. If we use cylindrical polar coordinates (ρ, ϕ, z) for its position (Problem 1.47), then ρ is fixed at $\rho = R$, while ϕ and z can vary at will. Write down and solve Newton's second law for the general motion of the puck, including the effects of gravity. Describe the puck's motion.

The forces on the puck consist of the normal force and the gravitational force. The normal force is in the radial direction and the gravitational force is in the negative z direction.

$$F = N\hat{\boldsymbol{\rho}} - mg\hat{\mathbf{z}} \tag{1.49}$$

Since ρ is fixed, $\dot{\rho} = \ddot{\rho} = 0$ Newton's second law in cylindrical polar coordinates:

$$\begin{aligned} F_\rho &= m(\ddot{\rho} - \rho\dot{\phi}^2) = -mR\dot{\phi}^2 = N \\ F_\phi &= m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) = mR\ddot{\phi} = 0 \\ F_z &= m\ddot{z} = -mg \end{aligned}$$

From the F_ϕ equation, $\ddot{\phi} = 0$ so $\dot{\phi}$ or the angular velocity of the ball is constant. From the F_z equation, $\ddot{z} = -g$ and integrating twice with respect to time gives us $z(t) = -\frac{1}{2}gt^2 + v_0 t + z_0$ where v_0 is the initial velocity and z_0 is the initial height. This shows us that the puck is in free fall along the z axis. With these equations of motion we can imagine the puck tracing a helical path with a downard increasing pitch.

1.51 Solve the differential equation for the skateboard given by

$$\ddot{\phi} = -\frac{g}{R}\sin\phi$$

and make a plot of ϕ against time for two or three periods. Make a plot of the approximate solution $\phi = \phi_0 \cos\omega t$ for the same time interval, where $\omega = \sqrt{g/R}$ and using the initial value $\phi_0 = \pi/2$

Python code for solving the differential equation

```
import scipy as sp
import numpy as np
import matplotlib.pyplot as plt

# 1.51
# initial conditions
phi0 = np.pi / 2
dot_phi0 = 0

# constants
R = 5 # [m]
g = 9.8 # [m/s^2]

# differential equation
def ddot_phi(phi, t):
    return [phi[1], -g / R * np.sin(phi[0])]

# time
t = np.linspace(0, 10, 1000)

# solve the differential equation
pos, vel = sp.integrate.odeint(ddot_phi, [phi0, dot_phi0], t).T

# approximate solution
omega = np.sqrt(g / R)
pos_approx = phi0 * np.cos(omega * t)

# plot the solution
plt.plot(t, pos)

```

```python
# plot the approximate solution
plt.plot(t, pos_approx, '--')

plt.xlabel('t [s]')
plt.ylabel('$\phi$ [rad]')

# create legend
plt.legend(['$\phi(t)$', '$\phi_{approx}(t)$'])
plt.show()
```

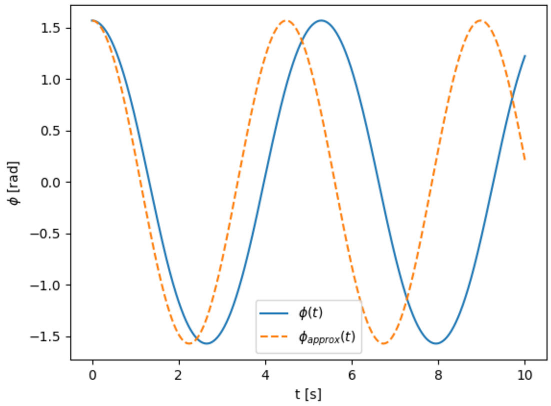

Figure 1.8: Plot of ϕ against time for two periods

Figure 1.8 shows the plot of ϕ and the approximate solution ϕ_{approx} against time for two periods. The approximate solution has a faster period than the actual solution, and the actual solution is not a perfect sinusoidal wave.