1 Electrostatics

2.1 (a) Given twelve equal charges, q situated on corners of a regular 12-sided polygon, the net force is

$$\vec{F}_a = \sum_{i=1}^{12} \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_i^2} \hat{\mathscr{r}} = 0$$

since the forces on each pair of charges (e.g., 12 and 6 o' clock) opposite to each other cancel out.
(b) If one of the charges is removed at 6 o' clock, the net force is strictly due to the the source charge at 12 o' clock:

$$\mathbf{F}_b = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_{12}^2} \hat{\mathscr{r}}_{12}$$

where $\mathbf{F}_b$ points from 12 to 6 o' clock.
(c) For 13 equal charges, the net force is still $\mathbf{F}_c = 0$ because the symmetry of the arrangement is preserved.
(d) Removing one of the charges $\mathbf{r}'_i$ is equivalent to the superposition of a source charge, $-q$, at $\mathbf{r}'_i$ and the original configuration. The net force is then

$$\mathbf{F}_d = \mathbf{F}_c - \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i = -\frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$

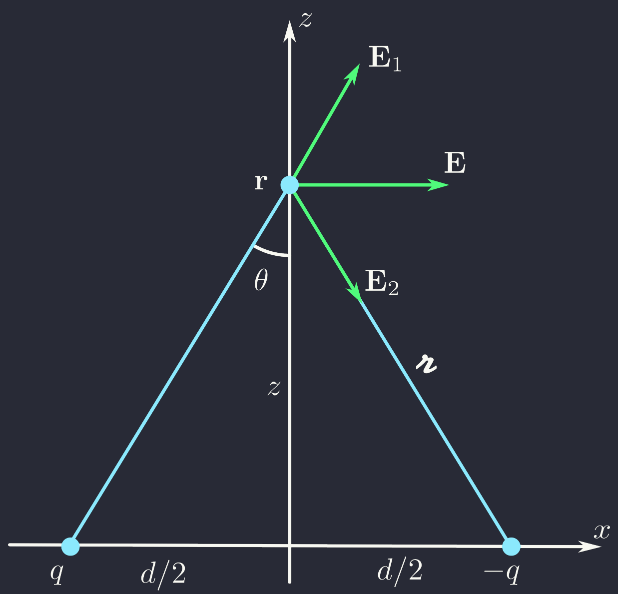

Figure 1.1: An electric field at a distance z from the midpoint between equal and opposite charges ($\pm q$) separated by a distance d. The charge at $x = d/2$ is $-q$.

2.2 The vertical componets of the electric field cancel out and the horizontal components add up:

$$E_x = 2\frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}^2} \sin\theta$$

where $E_x = E\cos\theta$, $\mathscr{r} = \sqrt{z^2 + (d/2)^2}$, and $\sin\theta = d/(2\mathscr{r})$, so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{qd}{[z^2 + (d/2)^2]^{3/2}} \hat{\mathbf{x}}$$

2.3

$$\mathbf{r} = z\hat{\mathbf{z}}, \quad \mathbf{r}' = x\hat{\mathbf{x}}, \quad \mathrm{d}l = \mathrm{d}x\,;$$
$$\boldsymbol{\mathscr{r}} = z\hat{\mathbf{z}} - x\hat{\mathbf{x}}, \quad \mathscr{r} = \sqrt{z^2 + x^2}, \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}$$

With uniform line charge λ and the limts of integration $[0, L]$,

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda\,\mathrm{d}l}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \\
&= \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{[z^2 + x^2]^{3/2}}\,\mathrm{d}x \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\hat{\mathbf{z}} \int_0^L \frac{1}{(z^2 + x^2)^{3/2}} - \hat{\mathbf{x}} \int_0^L \frac{x}{(z^2 + x^2)^{3/2}} \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\hat{\mathbf{z}} \left(\frac{x}{z^2\sqrt{z^2 + x^2}} \right) \Bigg|_0^L + \hat{\mathbf{x}} \left(\frac{1}{\sqrt{z^2 + x^2}} \right) \Bigg|_0^L \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[z\hat{\mathbf{z}} \left(\frac{L}{z^2\sqrt{z^2 + L^2}} - \frac{0}{z^2\sqrt{z^2 + 0^2}} \right) + \hat{\mathbf{x}} \left(\frac{1}{\sqrt{z^2 + L^2}} - \frac{1}{\sqrt{z^2 + 0^2}} \right) \right] \\
&= \frac{\lambda}{4\pi\epsilon_0} \left[\hat{\mathbf{z}} \left(\frac{L}{z\sqrt{z^2 + L^2}} \right) + \hat{\mathbf{x}} \left(\frac{1}{\sqrt{z^2 + L^2}} - \frac{1}{z} \right) \right] \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[\hat{\mathbf{x}} \left(\frac{z}{\sqrt{z^2 + L^2}} - 1 \right) + \hat{\mathbf{z}} \left(\frac{L}{\sqrt{z^2 + L^2}} \right) \right]
\end{aligned}$$

For $z \gg L$,

$$\mathbf{E} \approx \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z^2} \hat{\mathbf{z}}$$

From far away, the line looks like a point charge $q = \lambda L$.

2.4 One segment of the square loop is equivalent to Ex. 2.2, but with line segment length $2L \to a$ and electric field distance $z_o \to \sqrt{z_o^2 + a^2/4}$. So, the magnitude of the electric field from one segment is

$$\begin{aligned}
E &= \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z_o\sqrt{z_o^2 + L^2}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{z^2 + a^2/4}\sqrt{z^2 + a^2/2}}
\end{aligned}$$

Due to the symmetry of the loop, the electric field components in the x-direction cancel out, and the electric field components in the z-direction add up:

$$\begin{aligned}
\mathbf{E} &= 4E\cos\theta\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{z^2 + a^2/4}\sqrt{z^2 + a^2/2}} \frac{z}{\sqrt{z^2 + a^2/4}} \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda a z}{(z^2 + a^2/4)\sqrt{z^2 + a^2/2}} \hat{\mathbf{z}}
\end{aligned}$$

2.5 The horizontal components of the electric field cancel out, and the vertical components conspire:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda}{\mathscr{r}^2} \cos\theta\hat{\mathbf{z}}\,\mathrm{dl}$$

where geometrically $\mathscr{r} = \sqrt{z^2 + r^2}$ and $\cos\theta = z/\mathscr{r}$. So,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}}\, \mathrm{d}l$$

and the line integral is over the circumference of the circle, so $\mathrm{d}\mathbf{l} = r\,\mathrm{d}\theta$ and the limits of integration are $[0, 2\pi]$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{\lambda z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} \int_0^{2\pi} r\,\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0} \frac{\lambda z(2\pi r)}{(z^2 + r^2)^{3/2}}
\end{aligned}$$

2.6 Similar to Prob. 2.6 the electric field is only in the z-direction:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{\mathscr{r}^2} \cos\theta \hat{\mathbf{z}}\, \mathrm{d}\mathbf{a} \\
&= \frac{1}{4\pi\epsilon_0} \int \frac{\sigma z}{(z^2 + r^2 2)^{3/2}} \hat{\mathbf{z}}\, \mathrm{d}\mathbf{a}
\end{aligned}$$

since $\mathrm{d}\mathbf{a} = r\,\mathrm{d}r\,\mathrm{d}\theta$

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^R \frac{\sigma z}{(z^2 + r^2)^{3/2}} \hat{\mathbf{z}} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{4\pi\epsilon_0} \sigma z(2\pi) \hat{\mathbf{z}} \int_0^R \frac{r}{(z^2 + r^2)^{3/2}}\,\mathrm{d}r \\
&= \frac{1}{4\pi\epsilon_0} \sigma z(2\pi) \hat{\mathbf{z}} \left[-\frac{1}{\sqrt{z^2 + r^2}} \right]_0^R \\
&= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left[\frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}} \right] \hat{\mathbf{z}}
\end{aligned}$$

when $R \to \infty$,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma \hat{\mathbf{z}} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}}$$

for $z \gg R$,

$$-\frac{1}{\sqrt{z^2 + R^2}} = -\frac{1}{z}\left(1 + \frac{R^2}{z^2}\right)^{-1/2} \approx -\frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right) = -\frac{1}{z} + \frac{1}{2}\frac{R^2}{z^3}$$

where the binomial theorem approximation $(1 + x)^n \approx 1 + nx$ is used. So,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left[\frac{1}{z} - \frac{1}{z} + \frac{1}{2}\frac{R^2}{z^3} \right] \hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0} \frac{\pi R^2 \sigma}{z^2} \hat{\mathbf{z}}$$

or a point charge $q = \pi R^2 \sigma$ from far away.

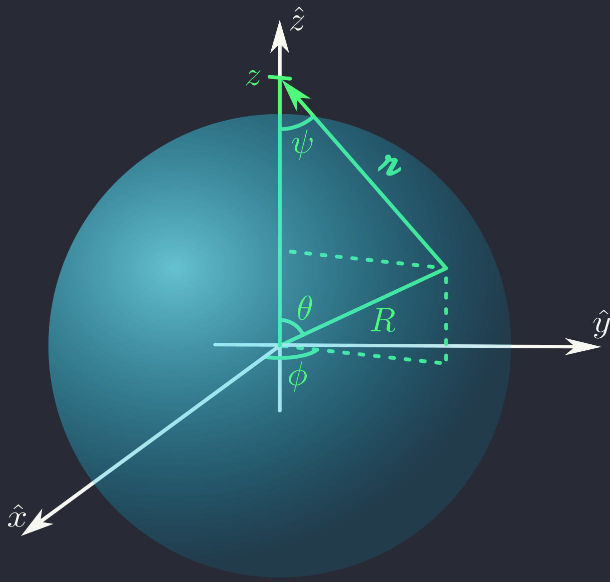

Figure 1.2: An electric field a distance z from the center of a spherical surface of radius R that carries a charge density σ.

2.7 Once again, the electric field is in the z-direction:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}^2}\cos\psi\,\hat{\mathbf{z}}\,\mathrm{d}\mathbf{a} \tag{1.1}$$

From the law of cosines, $\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$; Geometrically, $\cos\psi = \dfrac{z - R\cos\theta}{\mathscr{r}}$; the surface area element is $\mathrm{d}\mathbf{a} = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\theta$:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{\sigma R^2(z - R\cos\theta)}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}(2\pi\sigma R^2)\int_0^{\pi}\frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}}\sin\theta\,\mathrm{d}\theta\,\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}(2\pi\sigma R^2)f(\theta)\hat{\mathbf{z}}
\end{aligned}$$

using the substitution $u = \cos\theta$: $\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$, and the limits of integration are $[\cos 0, \cos\pi]$. So,

$$f(\theta) = \int_{-1}^{1}\frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}}\,\mathrm{d}u = f(u)$$

substituting again with $v = \sqrt{z^2 + R^2 - 2zRu}$; $\mathrm{d}v = -\dfrac{zR}{v}\,\mathrm{d}u$; and $u = \dfrac{1}{2zR}(z^2 + R^2 - v^2)$:

$$\begin{aligned}
f(v) &= -\frac{1}{zR}\int\frac{z - \frac{1}{2z}(z^2 + R^2 - v^2)}{v^3}v\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{2z^2 - (z^2 + R^2 - v^2)}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\frac{v^2 + z^2 - R^2}{v^2}\,\mathrm{d}v \\
&= -\frac{1}{2z^2R}\int\left(1 + \frac{z^2 - R^2}{v^2}\right)\mathrm{d}v \\
&= -\frac{1}{2z^2R}\left(v - \frac{z^2 - R^2}{v}\right)
\end{aligned}$$

back substituting $v = \sqrt{z^2 + R^2 - 2zRu}$,

$$
\begin{aligned}
f(u) &= -\frac{1}{2z^2R}\left(\frac{z^2 + R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}} - \frac{z^2 - R^2}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= -\frac{1}{2z^2R}\left(\frac{2R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{zu - R}{\sqrt{z^2 + R^2 - 2zRu}}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{z^2}\left(\frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} - \frac{-z - R}{\sqrt{z^2 + R^2 + 2zR}}\right)
\end{aligned}
$$

Taking the positive square root: $\sqrt{z^2 + R^2 - 2zR} = (R - z)$ if $R > z$, but $(z - R)$ if $R < z$. So, for the case $z < R$ (inside the sphere) the electric field is

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} - \frac{-z - R}{R + z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + \frac{z + R}{R + z}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + 1\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{R - z} + \frac{R - z}{R - z}\right)\hat{\mathbf{z}} \\
&= 0
\end{aligned}
$$

For the case $z > R$ (outside the sphere) the electric field is

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma R^2}{z^2}\left(\frac{z - R}{z - R} + \frac{z + R}{z + R}\right)\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{4\pi\sigma R^2}{z^2}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{\mathbf{z}}
\end{aligned}
$$

This makes sense: From outside the sphere, the point charge q is the charge-per-area σ times the surface area of the sphere $4\pi R^2$, or simply $q = 4\pi R^2\sigma$.

2.8 Finding the field inside and outside a solid sphere of radius R with a uniform volume charge density ρ is similar to Prob. 2.7. Outside the solid sphere the total charge q contributes to the electric field as if it were a point charge:

$$
\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}
$$

Inside the solid sphere, only the volume of the solid sphere less than r contributes to the electric field. The volume of the total sphere is $V = \frac{4}{3}\pi R^3$, and the volume of the sphere less than r is $V' = \frac{4}{3}\pi r^3$. So, electric field inside the solid sphere is

$$
\begin{aligned}
\mathbf{E}_{in} &= \frac{V'}{V}\mathbf{E}_{out} \\
&= \frac{r^3}{R^3}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}
\end{aligned}
$$

or

$$
\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\mathbf{r}
$$

2.9 (a) The electric field in some region is $\mathbf{E} = kr^3\hat{\mathbf{r}}$ in spherical coordinates, where k is a constant. The differential form of Gauss's law is

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

and the radial component of divergence in spherical coordinates is

$$\begin{aligned}[\boldsymbol{\nabla} \cdot \mathbf{E}]_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r^3\right) \\ &= 5kr^2\end{aligned}$$

So, the charge density is

$$\rho = 5\epsilon_0 k r^2$$

(b) The total charge inside a sphere of radius R is found using Gauss's law:

$$\begin{aligned}\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \frac{Q}{\epsilon_o} \\ Q &= \epsilon_o \oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} \\ &= \epsilon_o \int (kR^3\hat{\mathbf{r}}) \cdot (R^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi\,\hat{\mathbf{r}}) \\ &= \epsilon_o \int_0^{2\pi}\int_0^{\pi} (kR^5 \sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= 4\pi\epsilon_o k R^5\end{aligned}$$

or using Gauss's theorem:

$$\begin{aligned}\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} &= \int (\boldsymbol{\nabla} \cdot \mathbf{E})\,\mathrm{d}\tau \\ Q &= \epsilon_o \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 5kr^2(r^2\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= 4\pi\epsilon_o k R^5\end{aligned}$$

2.10 For simplicity, using a cube of length 1:

$$y = 1, \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{r^3}, \quad \mathrm{d}a = \mathrm{d}x\,\mathrm{d}z\,\hat{\mathbf{y}}; \quad \mathbf{E} \cdot \mathrm{d}a = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}$$

the limits of integration are $x = [0, 1]$ and $z = [0, 1]$:

$$\begin{aligned}\oint \mathbf{E} \cdot \mathrm{d}a &= \frac{1}{4\pi\epsilon_0} q \int \frac{1}{r^3}\,\mathrm{d}x\,\mathrm{d}z \\ &= \frac{1}{4\pi\epsilon_0} q \int_0^1\int_0^1 \frac{1}{(x^2+1+z^2)^{3/2}}\,\mathrm{d}x\,\mathrm{d}z \\ &= \frac{1}{4\pi\epsilon_0} q \int_0^1 \left[\frac{x}{(1+z^2)\sqrt{x^2+1+z^2}}\Bigg|_0^1\right]\mathrm{d}z \\ &= \frac{1}{4\pi\epsilon_0} q \int_0^1 \frac{1}{(1+z^2)\sqrt{2+z^2}}\,\mathrm{d}z \\ &= \frac{1}{4\pi\epsilon_0} q \arctan\left(\frac{z}{\sqrt{z^2+2}}\right)\Bigg|_0^1 \\ &= \frac{1}{4\pi\epsilon_0} q \left(\frac{\pi}{6}\right) = \frac{q}{24\epsilon_o}\end{aligned}$$

where the first integral is solved using the trig identity $x = \tan(u)\sqrt{z^2+1}$, and similarly, the second integral uses $z = \tan(u)\sqrt{2}$.

The simpler solution is though the superposition of 8 cubes with the charge in the center of the larger cube, and the surface that encloses the larger cube is made of 24 squares equivalent to the shaded region. Therefore, the flux through the shaded region is $\frac{1}{24}$ of the total flux $\frac{q}{\epsilon_o}$.