

# Problems for Griffiths' Electrodynamics

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# 1 Vector Analysis

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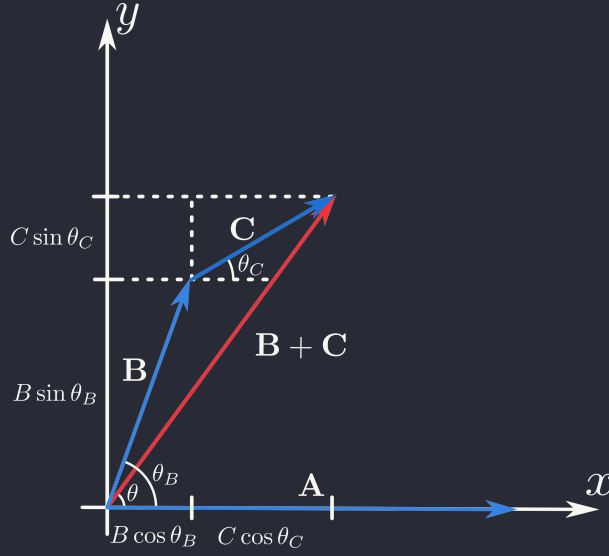


Figure 1.1: Three Coplanar Vectors

1.1 (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\ A(B + C) \cos \theta &= AB \cos \theta_B + AC \cos \theta_C\end{aligned}$$

Since  $B \cos \theta_B + C \cos \theta_C = (B + C) \cos \theta$  from Figure 1.1, the distributive property holds true. The cross product also holds true since  $B \sin \theta_B + C \sin \theta_C = (B + C) \sin \theta$ , and multiplying by  $A$  on both sides gives the same result as the distributive property:

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ A(B + C) \sin \theta &= AB \sin \theta_B + AC \sin \theta_C\end{aligned}$$

(b) In the general case in three-dimensional space, each vector has three components:  $\mathbf{A} = (A_x, A_y, A_z)$ . Therefore,

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot (B_x + C_x, B_y + C_y, B_z + C_z) \\ &= A_x(B_x + C_x) + A_y(B_y + C_y) + A_z(B_z + C_z) \\ &= A_x B_x + A_y B_y + A_z B_z + A_x C_x + A_y C_y + A_z C_z \\ &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}\end{aligned}$$

1.2 Setting  $\mathbf{A} = \mathbf{B} = (1, 1, 1)$  and  $\mathbf{C} = (1, 1, -1)$ :

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &\stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \\ 0 &\stackrel{?}{=} (1, 1, 1) \times [(1, 1, 1) \times (1, 1, -1)] \\ 0 &\stackrel{?}{=} (1, 1, 1) \times (-2, 2, 0) \\ 0 &\neq (-2, -2, 4)\end{aligned}$$

where the cross product of parallel vectors  $\mathbf{A} \times \mathbf{B} = 0$ . Therefore, the cross product is not associative.

1.3 Taking the dot product of a unit cube's body diagonals  $\mathbf{A} = (1, 1, 1)$ ,  $\mathbf{B} = (1, 1, -1)$ :

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$$

$$1 = 3 \cos \theta$$

$$\theta = \arccos 1/3 \approx 70.53^\circ$$

1.4 The cross product of two vectors coplanar to the shaded plane— $\mathbf{A} = (-1, 2, 0)$ ,  $\mathbf{B} = (-1, 0, 3)$ —is parallel to the normal unit vector  $\hat{\mathbf{n}}$  of the plane:

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6, 3, 2)$$

where  $\hat{\mathbf{n}} = \mathbf{C}/C$ , and  $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$ . Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

1.5 Proving the “BAC–CAB” rule for three-dimensional vectors:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} \end{aligned}$$

where the  $x$  component is

$$A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

From the “BAC–CAB” rule,

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}(A_x C_x + A_y C_y + A_z C_z) - \mathbf{C}(A_x B_x + A_y B_y + A_z B_z)$$

where  $x$  component simplifies to

$$B_x(A_x C_x + A_y C_y + A_z C_z) - C_x(A_x B_x + A_y B_y + A_z B_z) = A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

So,

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_x = [\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})]_x$$

and similiary for the  $y$  and  $z$  components. Therefore, the “BAC–CAB” rule holds true.

1.6

$$\begin{aligned} [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) = 0 \\ &= -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C} \cdot \mathbf{A})} \end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \\ \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \times \mathbf{B} \end{aligned}$$

For the relation to hold true, either the vectors  $\mathbf{A}$  and  $\mathbf{C}$  are parallel ( $\mathbf{A} \times \mathbf{C} = 0$ ) or  $\mathbf{B}$  is perpendicular to both  $\mathbf{A}$  and  $\mathbf{C}$  ( $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = 0$ ).

### 1.7 Finding the separation vector $\mathbf{z}$ :

$$\begin{aligned}\mathbf{z} &= \mathbf{r} - \mathbf{r}' = (4, 6, 8) - (2, 8, 7) = (2, -2, 1) \\ z &= \sqrt{2^2 + (-2)^2 + 1^2} = 3 \\ \hat{\mathbf{z}} &= \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\end{aligned}$$

### 1.8 (a)

$$\begin{aligned}\bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z &= (A_y \cos \phi + A_z \sin \phi)(B_y \cos \phi + B_z \sin \phi) \\ &\quad + (-A_y \sin \phi + A_z \cos \phi)(-B_y \sin \phi + B_z \cos \phi) \\ &= A_y B_y \cos^2 \phi + A_z B_z \sin^2 \phi + \cancel{A_y B_z \sin \phi \cos \phi} + \cancel{A_z B_y \sin \phi \cos \phi} \\ &\quad + A_y B_y \sin^2 \phi - \cancel{A_y B_z \sin \phi \cos \phi} - \cancel{A_z B_y \sin \phi \cos \phi} + A_z B_z \cos^2 \phi \\ &= A_y B_y (\sin^2 \phi + \cos^2 \phi) + A_z B_z (\sin^2 \phi + \cos^2 \phi) \\ \bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z &= A_y B_y + A_z B_z\end{aligned}$$

(b) To preserve length  $|\bar{\mathbf{A}}| = |\mathbf{A}|$ . Squaring both sides and expanding gives

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_x^2 + A_y^2 + A_z^2$$

in summation form,

$$\sum_{i=1}^3 \bar{A}_i \bar{A}_i = \sum_{i=1}^3 \left( \sum_{j=1}^3 R_{ij} A_j \right) \left( \sum_{k=1}^3 R_{ik} A_k \right) = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 R_{ij} R_{ik} A_j A_k$$

For the length to be preserved, the indices  $j$  and  $k$  must be equal. Therefore,

$$R_{ij} R_{ik} = \delta_{jk}$$

where  $\delta_{ij}$  is the Kronecker delta. Thus the length is preserved if the rotation matrix is orthogonal, i.e.

$$R_{ij} R_{ik} = (R^T)_{ji} R_{ik} = \delta_{jk} \quad \text{or} \quad R^T R = I$$

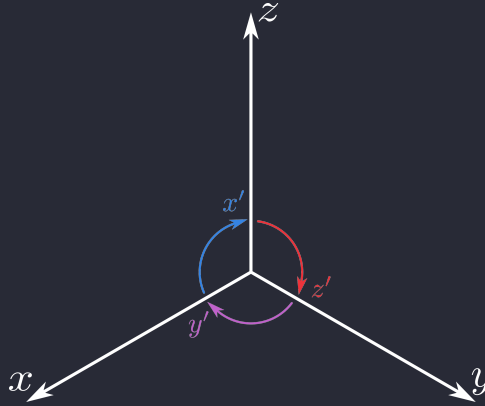


Figure 1.2: Rotation of  $120^\circ$  about an axis through the origin and point  $(1, 1, 1)$

**1.9** From Figure 1.2, the rotation is equivalent to changing the position of the basis vectors  $\hat{\mathbf{x}} \rightarrow \hat{\mathbf{z}}$ ,  $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ , and  $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{y}}$ . Therefore, the rotation matrix is a permutation matrix:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

**1.10** (a) Under a **translation** of coordinates  $\bar{y} = y - a$ , the origin  $O$  and terminus  $A = (x, y, z)$  of some vector are translated to

$$\begin{aligned} O &\rightarrow O' = (0, -a, 0) \\ A &\rightarrow A' = (x, y - a, z) \end{aligned}$$

therefore, the translated vector is

$$\overline{O'A'} = (x, y - a, z) - (0, -a, 0) = (x, y, z) = \overline{OA} = \mathbf{A}$$

which is the same as the original vector, so

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) Under **inversion** of coordinates, only the terminus changes

$$\begin{aligned} O &\rightarrow O' = (0, 0, 0) \\ A &\rightarrow A' = (-x, -y, -z) \end{aligned}$$

therefore, the inverted vector is

$$\overline{O'A'} = (-x, -y, -z) \quad \text{or} \quad \begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} -A_x \\ -A_y \\ -A_z \end{pmatrix}$$

(c) The components of cross product under inversion

$$\bar{\mathbf{A}} \times \bar{\mathbf{B}} = \begin{pmatrix} \bar{A}_y \bar{B}_z - \bar{A}_z \bar{B}_y \\ \bar{A}_z \bar{B}_x - \bar{A}_x \bar{B}_z \\ \bar{A}_x \bar{B}_y - \bar{A}_y \bar{B}_x \end{pmatrix} = \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}$$

which is the same as the original cross product  $\mathbf{A} \times \mathbf{B}$ . The cross product of two pseudovectors is also a pseudovector. Torque  $\tau = \mathbf{r} \times \mathbf{F}$  and magnetic force  $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$  are examples of pseudovectors.

(d) Scalar triple product under inversion

$$\begin{aligned} \bar{A} \cdot (\bar{B} \times \bar{C}) &= -\mathbf{A} \cdot (-\mathbf{B} \times -\mathbf{C}) \\ &= -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \end{aligned}$$

the scalar triple product changes sign under inversion.

**1.11** (a) Finding gradient of  $f(x, y, z) = x^2 + y^3 + z^4$ :

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \\ &= 2x \hat{\mathbf{x}} + 3y^2 \hat{\mathbf{y}} + 4z^3 \hat{\mathbf{z}} \end{aligned}$$

(b) Gradient of  $f(x, y, z) = x^2 y^3 z^4$ :

$$\nabla f = 2xy^3z^4 \hat{\mathbf{x}} + 3x^2y^2z^4 \hat{\mathbf{y}} + 4x^2y^3z^3 \hat{\mathbf{z}}$$

(c) Gradient of  $f(x, y, z) = e^x \sin(y) \ln(z)$ :

$$\nabla f = e^x \hat{\mathbf{x}} + e^x \cos(y) \ln(z) \hat{\mathbf{y}} + \frac{e^x \sin(y)}{z} \hat{\mathbf{z}}$$

**1.12** The height of the hill (in feet) is given by the function

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where  $y$  is north and  $x$  is east in miles. The gradient of  $h$  is

$$\nabla h = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10(2x - 8y + 28)\hat{\mathbf{y}}$$

(a) The top of the hill is a stationary point, so the summit is found by setting the gradient to zero which gives the system of equations

$$0 = 2y - 6x - 18 \qquad 0 = 2x - 8y + 28$$

adding the first equation to 3 times the second equation gives

$$\begin{aligned} 0 &= 2y - 6x - 18 + 3(2x - 8y + 28) \\ 0 &= -22y + 66 \\ y &= 3 \end{aligned}$$

substituting  $y = 3$  into the first equation

$$0 = 2(3) - 6x - 18 \rightarrow x = -2$$

Therefore, the top of the hill is at  $(-2, 3)$  or 2 miles west and 3 miles north of the origin.

(b) The height of the hill is simply  $h(-2, 3) = 10(12) = 720$  feet.

(c) The steepness of the hill at  $h(1, 1)$  is given by the magnitude of the gradient

$$\begin{aligned} |\nabla h| &= 10\sqrt{(2y - 6x - 18)^2 + (2x - 8y + 28)^2} \\ &= 10\sqrt{(2 - 6 - 18)^2 + (2 - 8 + 28)^2} \\ &= 10\sqrt{(-22)^2 + (22)^2} = 220\sqrt{2} \approx 311 \text{ ft/mi} \end{aligned}$$

The direction of the steepest slope is given by the vector in the direction of the gradient at the point  $\nabla h(1, 1) = 220(-\hat{\mathbf{x}} + \hat{\mathbf{y}})$ , or simply northwest.

**1.13** Given the separation vector

$$\mathbf{z} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}} \quad \text{and} \quad z = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

(a) Show that  $\nabla(z^2) = 2\mathbf{z}$ :

$$z^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

and each component of the gradient is

$$\begin{aligned} \frac{\partial}{\partial x}(z^2) &= 2(x - x') \\ \frac{\partial}{\partial y}(z^2) &= 2(y - y') \\ \frac{\partial}{\partial z}(z^2) &= 2(z - z') \end{aligned}$$

so

$$\nabla(z^2) = 2\mathbf{z}$$

(b) Show  $\nabla(1/z) = -\hat{\mathbf{z}}/z^2$ :

$$\frac{1}{z} = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2} = [(\quad)]^{-1/2}$$

looking at the  $x$  component of the gradient (using chain rule),

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) &= -\frac{1}{2}[(\ )]^{-3/2} \cdot 2(x-x') \\ &= -\frac{(x-x')}{r^3}\end{aligned}$$

and similarly for the  $y$  and  $z$  components:

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{\mathbf{z}} \\ &= -\frac{1}{r^3}[(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}] = -\frac{\mathbf{r}}{r^3}\end{aligned}$$

Finally, substituting the unit vector  $\hat{\mathbf{r}} = \mathbf{r}/r$  gives us

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2}$$

(c) The general formula for  $\nabla(r^n)$ :

$$\nabla(r^n) = n r^{n-1} \cdot \nabla(r)$$

where

$$\begin{aligned}\nabla(r) &= \frac{\partial}{\partial x}(r)\hat{\mathbf{x}} + \frac{\partial}{\partial y}(r)\hat{\mathbf{y}} + \frac{\partial}{\partial z}(r)\hat{\mathbf{z}} \\ &= \frac{1}{2}[(\ )]^{-1/2} \cdot 2(x-x')\hat{\mathbf{x}} + \dots \quad [\text{similar to part (b)}] \\ &= \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}\end{aligned}$$

So the general formula is

$$\nabla(r^n) = n r^{n-1} \hat{\mathbf{r}}$$

**1.14** Given the rotation matrix

$$\begin{pmatrix} \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

or the two equations

$$\begin{aligned}\bar{y} &= y \cos \phi + z \sin \phi \\ \bar{z} &= -y \sin \phi + z \cos \phi\end{aligned}$$

differentiating with respect to  $\bar{y}$  and  $\bar{z}$  respectively gives

$$\begin{aligned}1 &= \frac{\partial y}{\partial \bar{y}} \cos \phi + \frac{\partial z}{\partial \bar{y}} \sin \phi \\ 1 &= -\frac{\partial y}{\partial \bar{z}} \sin \phi + \frac{\partial z}{\partial \bar{z}} \cos \phi\end{aligned}$$

this can only be true if

$$\frac{\partial y}{\partial \bar{y}} = \cos \phi, \quad \frac{\partial z}{\partial \bar{y}} = \sin \phi \quad \text{and} \quad \frac{\partial y}{\partial \bar{z}} = -\sin \phi, \quad \frac{\partial z}{\partial \bar{z}} = \cos \phi$$

which satisfies the trig identity  $\sin^2 \phi + \cos^2 \phi = 1$ . Differentiating  $f$  with respect to the rotated coordinates  $\bar{y}$  and  $\bar{z}$  is given by

$$\begin{aligned}\frac{\partial f}{\partial \bar{y}} &= \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y} \cos \phi + \frac{\partial f}{\partial z} \sin \phi \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \bar{z}} = -\frac{\partial f}{\partial y} \sin \phi + \frac{\partial f}{\partial z} \cos \phi\end{aligned}$$

therefore, the gradient of  $f$  transforms as a vector under rotations given by

$$\overline{\nabla f} = \frac{\partial f}{\partial \bar{y}} \hat{\mathbf{y}} + \frac{\partial f}{\partial \bar{z}} \hat{\mathbf{z}} = \left( \frac{\partial f}{\partial y} \cos \phi + \frac{\partial f}{\partial z} \sin \phi \right) \hat{\mathbf{y}} + \left( -\frac{\partial f}{\partial y} \sin \phi + \frac{\partial f}{\partial z} \cos \phi \right) \hat{\mathbf{z}}$$

or in matrix form

$$\overline{\nabla f} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \nabla f$$

where the gradient is a column vector

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

**1.15** (a) Calculating divergence of  $\mathbf{v}_a = x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}}$ :

$$\begin{aligned} \nabla \cdot \mathbf{v}_a &= \frac{\partial v_{ax}}{\partial x} + \frac{\partial v_{ay}}{\partial y} + \frac{\partial v_{az}}{\partial z} \\ &= 2x + 0 - 2x = 0 \end{aligned}$$

(b)  $\mathbf{v}_b = xy \hat{\mathbf{x}} + 2yz \hat{\mathbf{y}} + 3zx \hat{\mathbf{z}}$ :

$$\nabla \cdot \mathbf{v}_b = y + 2z + 3x$$

(c)  $\mathbf{v}_c = y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}}$ :

$$\nabla \cdot \mathbf{v}_c = 0 + 2x + 2y = 2(x + y)$$

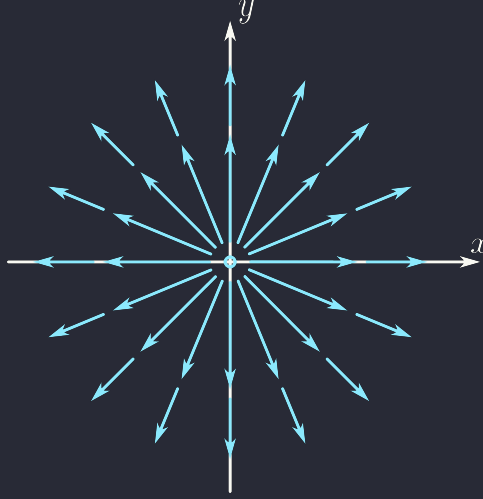


Figure 1.3: Sketch of the vector field  $\mathbf{v} = \hat{\mathbf{r}}/r^2$

**1.16** Given

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

where  $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$  is the position vector. The vector functions is

$$v = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$





Finally adding the two equations together gives

$$\begin{aligned}
\nabla \cdot \bar{\mathbf{v}} &= \frac{\partial \bar{v}_y}{\partial y} + \frac{\partial \bar{v}_z}{\partial z} \\
&= \frac{\partial v_y}{\partial y} \cos^2 \phi + \frac{\partial v_y}{\partial z} \sin \phi \cos \phi + \frac{\partial v_z}{\partial y} \sin \phi \cos \phi + \frac{\partial v_z}{\partial z} \sin^2 \phi \\
&\quad + \frac{\partial v_y}{\partial y} \sin^2 \phi - \frac{\partial v_y}{\partial z} \sin \phi \cos \phi - \frac{\partial v_z}{\partial y} \sin \phi \cos \phi + \frac{\partial v_z}{\partial z} \cos^2 \phi \\
&= (\sin^2 \phi + \cos^2 \phi) \left[ \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] \\
&= \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}
\end{aligned}$$

which shows that the divergence transforms as a scalar under rotations.

**1.18** Curl of vector functions from Problem 1.15: (a)  $\mathbf{v}_a = x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}}$ :

$$\begin{aligned}
\nabla \times \mathbf{v}_a &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 6xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(3z^2 - 0) \\
&= -6xz \hat{\mathbf{x}} + 2z \hat{\mathbf{y}} + 3z^2 \hat{\mathbf{z}}
\end{aligned}$$

(b)  $\mathbf{v}_b = xy \hat{\mathbf{x}} + 2yz \hat{\mathbf{y}} + 3zx \hat{\mathbf{z}}$ :

$$\begin{aligned}
\nabla \times \mathbf{v}_b &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} \\
&= \hat{\mathbf{x}}(0 - 2y) - \hat{\mathbf{y}}(3z - 0) + \hat{\mathbf{z}}(0 - x) \\
&= -2y \hat{\mathbf{x}} - 3z \hat{\mathbf{y}} - x \hat{\mathbf{z}}
\end{aligned}$$

(c)  $\mathbf{v}_c = y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}}$ :

$$\begin{aligned}
\nabla \times \mathbf{v}_c &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} \\
&= \hat{\mathbf{x}}(2z - 2z) - \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) \\
&= 0
\end{aligned}$$

**1.19** From Figure 1.4, the sign of  $\partial v_x / \partial y$  is positive, and the sign of  $\partial v_y / \partial x$  is negative. Therefore, the curl

$$\nabla \times \mathbf{v} = \hat{\mathbf{z}} \left( \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right)$$

is in the negative  $z$  direction, or into the page. This is consistent with the right-hand rule as the thumb points into the page.

**1.20** Proof that the vector function

$$\mathbf{g} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} = \frac{x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}}{r^3}$$

has zero divergence and curl given

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{and} \quad \frac{\partial x}{\partial y} = \frac{y}{x} = 0$$



(iv)

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= \nabla \cdot [(A_y B_z - A_z B_y)\hat{\mathbf{x}} + (A_z B_x - A_x B_z)\hat{\mathbf{y}} + (A_x B_y - A_y B_x)\hat{\mathbf{z}}] \\
&= \frac{\partial}{\partial x}(A_y B_z - A_z B_y) + \frac{\partial}{\partial y}(A_z B_x - A_x B_z) + \frac{\partial}{\partial z}(A_x B_y - A_y B_x) \\
&= \left( \frac{\partial A_y}{\partial x} B_z + A_y \frac{\partial B_z}{\partial x} - \frac{\partial A_z}{\partial x} B_y - A_z \frac{\partial B_y}{\partial x} \right) + \left( \frac{\partial A_z}{\partial y} B_x + A_z \frac{\partial B_x}{\partial y} - \frac{\partial A_x}{\partial y} B_z - A_x \frac{\partial B_z}{\partial y} \right) \\
&\quad + \left( \frac{\partial A_x}{\partial z} B_y + A_x \frac{\partial B_y}{\partial z} - \frac{\partial A_y}{\partial z} B_x - A_y \frac{\partial B_x}{\partial z} \right) \\
&= B_x \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + B_y \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + B_z \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
&\quad + A_x \left( \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \right) + A_y \left( \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \right) + A_z \left( \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) \\
&= \mathbf{B} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot (\nabla \times \mathbf{B})
\end{aligned}$$

(v)

$$\begin{aligned}
\nabla \times (f\mathbf{A}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix} \\
&= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y}(fA_z) - \frac{\partial}{\partial z}(fA_y) \right) - \hat{\mathbf{y}} \left( \frac{\partial}{\partial x}(fA_z) - \frac{\partial}{\partial z}(fA_x) \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x) \right) \\
&= \hat{\mathbf{x}} \left( f \frac{\partial A_z}{\partial y} + A_z \frac{\partial f}{\partial y} - f \frac{\partial A_y}{\partial z} - A_y \frac{\partial f}{\partial z} \right) - \hat{\mathbf{y}} \left( f \frac{\partial A_z}{\partial x} + A_z \frac{\partial f}{\partial x} - f \frac{\partial A_x}{\partial z} - A_x \frac{\partial f}{\partial z} \right) \\
&\quad + \hat{\mathbf{z}} \left( f \frac{\partial A_y}{\partial x} + A_y \frac{\partial f}{\partial x} - f \frac{\partial A_x}{\partial y} - A_x \frac{\partial f}{\partial y} \right) \\
&= f \left[ \hat{\mathbf{x}} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \\
&\quad - \hat{\mathbf{x}} \left( A_y \frac{\partial f}{\partial z} - A_z \frac{\partial f}{\partial y} \right) + \hat{\mathbf{y}} \left( A_z \frac{\partial f}{\partial x} - A_x \frac{\partial f}{\partial z} \right) - \hat{\mathbf{z}} \left( A_x \frac{\partial f}{\partial y} - A_y \frac{\partial f}{\partial x} \right) \\
&= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)
\end{aligned}$$

1.22 (a) If  $\mathbf{A}$  and  $\mathbf{B}$  are two vector functions, then

$$\begin{aligned}
(\mathbf{A} \cdot \nabla)\mathbf{B} &= \left( A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}) \\
&= \hat{\mathbf{x}} \left( A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right) + \hat{\mathbf{y}} \left( A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z} \right) \\
&\quad + \hat{\mathbf{z}} \left( A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z} \right)
\end{aligned}$$

This means that the direction of  $\mathbf{A}$  points in the direction of where  $\mathbf{B}$  moves fastest.

(b)

$$(\hat{\mathbf{r}} \cdot \nabla)\hat{\mathbf{r}} = \frac{1}{r} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \frac{(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{r}$$

looking at the  $x$  component,

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\mathbf{r}}{r}\right) &= \frac{1}{r}\frac{\partial}{\partial x}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) + \mathbf{r}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \\ &= \frac{\hat{\mathbf{x}}}{r} - \mathbf{r}\frac{x^2}{r^3}\end{aligned}$$

therefore,

$$\begin{aligned}(\hat{\mathbf{r}} \cdot \nabla)\hat{\mathbf{r}} &= \frac{1}{r}\left[x\left(\frac{\hat{\mathbf{x}}}{r} - \mathbf{r}\frac{x}{r^3}\right) + y\left(\frac{\hat{\mathbf{y}}}{r} - \mathbf{r}\frac{y}{r^3}\right) + z\left(\frac{\hat{\mathbf{z}}}{r} - \mathbf{r}\frac{z}{r^3}\right)\right] \\ &= \frac{1}{r}\left[\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r} - \mathbf{r}\frac{x^2 + y^2 + z^2}{r^3}\right] \\ &= \frac{1}{r}\left[\frac{\mathbf{r}}{r} - \frac{\mathbf{r}}{r}\right] = 0\end{aligned}$$

(c)

$$\begin{aligned}(v_a \cdot \nabla)v_b &= \left(x^2\frac{\partial}{\partial x} + 3xz^2\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}\right)(xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}) \\ &= x^2(y\hat{\mathbf{x}} + 0 + 3z\hat{\mathbf{z}}) + 3xz^2(x\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 0) - 2xz(0 + 2y\hat{\mathbf{y}} + 3x\hat{\mathbf{z}}) \\ &= (x^2y + 3x^2z^2)\hat{\mathbf{x}} + (6xz^3 - 4xyz)\hat{\mathbf{y}} + (3x^2z - 6x^2z)\hat{\mathbf{z}} \\ &= x^2(y + 3z^2)\hat{\mathbf{x}} + 2xz(3z^2 - 2y)\hat{\mathbf{y}} - 3x^2z\hat{\mathbf{z}}\end{aligned}$$

**1.23** Proving the product rule for (ii) given the  $x$  component of the left hand side is

$$\begin{aligned}[\nabla(\mathbf{A} \cdot \mathbf{B})]_x &= \frac{\partial(\mathbf{A} \cdot \mathbf{B})}{\partial x}\hat{\mathbf{x}} \\ &= \frac{\partial}{\partial x}(A_xB_x + A_yB_y + A_zB_z)\hat{\mathbf{x}} \\ &= A_x\frac{\partial B_x}{\partial x} + B_x\frac{\partial A_x}{\partial x} + A_y\frac{\partial B_y}{\partial x} + B_y\frac{\partial A_y}{\partial x} + A_z\frac{\partial B_z}{\partial x} + B_z\frac{\partial A_z}{\partial x}\end{aligned}$$

Finding the  $x$  component of the right hand side of (ii)

$$\begin{aligned}[\mathbf{A} \times (\nabla \times \mathbf{B})]_x &= \left[\mathbf{A} \times \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{bmatrix}\right]_x \\ &= \left[\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ () & -(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}) & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{bmatrix}\right]_x \\ &= A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\end{aligned}$$

and

$$[\mathbf{B} \times (\nabla \times \mathbf{A})]_x = B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - B_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

From Problem 1.22:

$$[(\mathbf{A} \cdot \nabla)\mathbf{B}] = A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}$$

and likewise,

$$[(\mathbf{B} \cdot \nabla)\mathbf{A}] = B_x\frac{\partial A_x}{\partial x} + B_y\frac{\partial A_x}{\partial y} + B_z\frac{\partial A_x}{\partial z}$$

Adding all four equations gives

$$\begin{aligned}
[\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}]_x = \\
A_y \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) - A_z \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + B_y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - B_z \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\
+ A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\
= A_x \frac{\partial B_x}{\partial x} + A_y \left( \frac{\partial B_y}{\partial x} - \cancel{\frac{\partial B_x}{\partial y}} + \cancel{\frac{\partial B_x}{\partial y}} \right) + A_z \left( \frac{\partial B_z}{\partial x} - \cancel{\frac{\partial B_x}{\partial z}} + \cancel{\frac{\partial B_x}{\partial z}} \right) \\
+ B_x \frac{\partial A_x}{\partial x} + B_y \left( \frac{\partial A_y}{\partial x} - \cancel{\frac{\partial A_x}{\partial y}} + \cancel{\frac{\partial A_x}{\partial y}} \right) + B_z \left( \frac{\partial A_z}{\partial x} - \cancel{\frac{\partial A_x}{\partial z}} + \cancel{\frac{\partial A_x}{\partial z}} \right) \\
= A_x \frac{\partial B_x}{\partial x} + B_x \frac{\partial A_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + B_y \frac{\partial A_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + B_z \frac{\partial A_x}{\partial z} \\
= [\nabla(\mathbf{A} \cdot \mathbf{B})]_x
\end{aligned}$$

and likewise for the  $y$  and  $z$  components.

For (vi), the  $x$  on the left hand side is

$$\begin{aligned}
[\nabla \times (\mathbf{A} \times \mathbf{B})]_x &= \left[ \nabla \times \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} \right]_x \\
&= \left[ \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ () & -(A_x B_z - A_z B_x) & A_x B_y - A_y B_x \end{bmatrix} \right]_x \\
&= \frac{\partial}{\partial y} (A_x B_y - A_y B_x) - \frac{\partial}{\partial z} (A_z B_x - A_x B_z) \\
&= A_x \frac{\partial B_y}{\partial y} + B_y \frac{\partial A_x}{\partial y} - A_y \frac{\partial B_x}{\partial y} - B_x \frac{\partial A_y}{\partial y} \\
&\quad - A_z \frac{\partial B_x}{\partial z} - B_x \frac{\partial A_z}{\partial z} + A_x \frac{\partial B_z}{\partial z} + B_z \frac{\partial A_x}{\partial z} + \\
&= A_x \left( \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} - B_x \left( \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z}
\end{aligned}$$

On the right hand side, first we find the  $x$  component of the two new operations:

$$\begin{aligned}
[A(\nabla \cdot \mathbf{B})]_x &= \left[ A \left( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) \right]_x \\
&= A_x \left( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right)
\end{aligned}$$

and likewise,

$$[B(\nabla \cdot \mathbf{A})]_x = B_x \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

Therefore,  $[(\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + A(\nabla \cdot \mathbf{B}) - B(\nabla \cdot \mathbf{A})]_x =$

$$\begin{aligned}
& B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} - \left( A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right) \\
& + A_x \left( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - \left( B_x \left( \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \right) \\
& = A_x \left( \cancel{\frac{\partial B_x}{\partial x}} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} - \cancel{\frac{\partial B_x}{\partial x}} \right) - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} \\
& - B_x \left( \cancel{\frac{\partial A_x}{\partial x}} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} - \cancel{\frac{\partial A_x}{\partial x}} \right) + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\
& = A_x \left( \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) - A_y \frac{\partial B_x}{\partial y} - A_z \frac{\partial B_x}{\partial z} - B_x \left( \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\
& = [\nabla \times (\mathbf{A} \times \mathbf{B})]_x
\end{aligned}$$

and likewise for the  $y$  and  $z$  components.

**1.24** Deriving the three quotient rules from the product rule: The gradient is

$$\begin{aligned}
\nabla \left( \frac{f}{g} \right) &= \nabla (fg^{-1}) = f\nabla(g^{-1}) + g^{-1}\nabla(f) \\
&= f(-g^{-2}\nabla(g)) + g^{-1}\nabla(f) \\
&= -\frac{f}{g^2}\nabla(g) + \frac{g}{g} \frac{1}{g}\nabla(f) \\
&= \frac{g\nabla f - f\nabla g}{g^2}
\end{aligned}$$

the divergence

$$\begin{aligned}
\nabla \cdot \left( \frac{\mathbf{A}}{g} \right) &= \nabla \cdot (Ag^{-1}) = A(\nabla \cdot g^{-1}) + g^{-1}(\nabla \cdot \mathbf{A}) \\
&= A(-g^{-2}(\nabla \cdot g)) + \frac{g}{g}g^{-1}(\nabla \cdot \mathbf{A}) \\
&= \frac{g(\nabla \cdot \mathbf{A}) - A\nabla \cdot g}{g^2}
\end{aligned}$$

and the curl

$$\begin{aligned}
\left[ \nabla \times \left( \frac{\mathbf{A}}{g} \right) \right]_x &= \frac{\partial}{\partial y} \left( \frac{A_z}{g} \right) - \frac{\partial}{\partial z} \left( \frac{A_y}{g} \right) \\
&= \frac{g \frac{\partial A_z}{\partial y} - A_z \frac{\partial g}{\partial y}}{g^2} - \frac{g \frac{\partial A_y}{\partial z} - A_y \frac{\partial g}{\partial z}}{g^2} \\
&= \frac{1}{g^2} \left[ g \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \left( A_y \frac{\partial g}{\partial z} - A_z \frac{\partial g}{\partial y} \right) \right] \\
&= \frac{g[\nabla \times \mathbf{A}]_x - \mathbf{A} \times [\nabla g]_x}{g^2}
\end{aligned}$$

and likewise for the  $y$  and  $z$  components. Therefore,

$$\nabla \times \left( \frac{\mathbf{A}}{g} \right) = \frac{g(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla g)}{g^2}$$

**1.25** (a) Calculating (iv) for the functions

$$\mathbf{A} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}; \quad \mathbf{B} = 3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}$$

LHS:

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\
&= \nabla \cdot [(0 + 6xz)\hat{\mathbf{x}} - (0 - 9yz)\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}] \\
&= \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(9yz) + \frac{\partial}{\partial z}(-2x^2 - 6y^2) \\
&= 6z + 9z + 0 = 15z
\end{aligned}$$

RHS:

$$\begin{aligned}
\mathbf{B} \cdot (\nabla \times \mathbf{A}) &= \mathbf{B} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} \\
&= \mathbf{B} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A} \cdot (\nabla \times \mathbf{B}) &= \mathbf{A} \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= \mathbf{A} \cdot [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-2 - 3)\hat{\mathbf{z}}] \\
&= 3z(-5) = -15z
\end{aligned}$$

therefore,

$$\mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = 0 - (-15z) = 15z$$

(b) Checking (ii): LHS

$$\begin{aligned}
\nabla(\mathbf{A} \cdot \mathbf{B}) &= \nabla(x(3y) + 2y(-2x) + 3z(0)) \\
&= \nabla(3xy - 4xy) = \nabla(-xy) \\
&= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}
\end{aligned}$$

RHS:

$$\begin{aligned}
\mathbf{A} \times (\nabla \times \mathbf{B}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 0 \end{vmatrix} \\
&= \mathbf{A} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (-5)\hat{\mathbf{z}}] \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = -10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}
\end{aligned}$$

and

$$\mathbf{B} \times (\nabla \times \mathbf{A}) = \mathbf{B} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3z \end{vmatrix} = \mathbf{B} \times [(0)\hat{\mathbf{x}} - (0)\hat{\mathbf{y}} + (0)\hat{\mathbf{z}}] = 0$$

and

$$\begin{aligned}
(\mathbf{A} \cdot \nabla)\mathbf{B} &= \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right)(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) \\
&= 6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}
\end{aligned}$$



and

$$\begin{aligned}(\mathbf{B} \cdot \nabla) \mathbf{A} &= \left( 3y \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y} \right) (x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}) \\ &= 3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}\end{aligned}$$

therefore,

$$\begin{aligned}\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} &= (-10y\hat{\mathbf{x}} + 5x\hat{\mathbf{y}}) + (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) \\ &= -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}\end{aligned}$$

(c) For rule (vi), the left hand side is

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \nabla \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} \\ &= \nabla \times [6xz\hat{\mathbf{x}} + 9yz\hat{\mathbf{y}} + (-2x^2 - 6y^2)\hat{\mathbf{z}}] \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xz & 9yz & -2x^2 - 6y^2 \end{vmatrix} \\ &= \hat{\mathbf{x}}(-12y - 9y) - \hat{\mathbf{y}}(-4x - 6x) + \hat{\mathbf{z}}(0) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}\end{aligned}$$

and on the right hand side, the new terms are

$$\mathbf{A}(\nabla \cdot \mathbf{B}) = \mathbf{A}[0 + 0] = 0$$

and

$$\begin{aligned}\mathbf{B}(\nabla \cdot \mathbf{A}) &= \mathbf{B}[1 + 2 + 3] = 6\mathbf{B} \\ &= 6(3y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) = 18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}\end{aligned}$$

combining these with the terms from (iv) gives

$$\begin{aligned}(\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{A} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{A} &= (3y\hat{\mathbf{x}} - 4x\hat{\mathbf{y}}) - (6y\hat{\mathbf{x}} - 2x\hat{\mathbf{y}}) + 0 - (18y\hat{\mathbf{x}} - 12x\hat{\mathbf{y}}) \\ &= -21y\hat{\mathbf{x}} + 10x\hat{\mathbf{y}}\end{aligned}$$

**1.26** Given the Laplacian of a scalar function  $T$  is

$$\nabla^2 T = \nabla \cdot (\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

(a)  $T_a = x^2 + 2xy + 3z + 4$ :

$$\nabla^2 T_a = 2 + 0 + 0 = 2$$

(b)  $T_b = \sin x \sin y \sin z$ :

$$\frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial y^2} = \frac{\partial^2 T_b}{\partial z^2} = -\sin x \sin y \sin z = -T_b$$

$$\nabla^2 T_b = -3T_b$$

(c)  $T_c = e^{-5x} \sin 4y \cos 3z$ : The components are

$$\frac{\partial^2 T_c}{\partial x^2} = 25e^{-5x} \sin 4y \cos 3z = 25T_c$$

$$\frac{\partial^2 T_c}{\partial y^2} = -16e^{-5x} \sin 4y \cos 3z = -16T_c$$

$$\frac{\partial^2 T_c}{\partial z^2} = -9e^{-5x} \sin 4y \cos 3z = -9T_c$$

therefore

$$\nabla^2 T_c = T_c(25 - 16 - 9) = 0$$

(d)  $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$ : The laplacian of a vector function is

$$\nabla^2 \mathbf{v} = (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$$

and the components are

$$\nabla^2 v_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = 2 + 0 + 0 = 2$$

$$\nabla^2 v_y = 0 + 0 + 6x = 6x$$

$$\nabla^2 v_z = 0$$

therefore

$$\nabla^2 \mathbf{v} = 2\hat{\mathbf{x}} + 6x\hat{\mathbf{y}}$$

**1.27** The divergence of curl is always zero:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{v}) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \nabla \cdot \left( \hat{\mathbf{x}} \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) - \hat{\mathbf{y}} \left( \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + \hat{\mathbf{z}} \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right) \\ &= \frac{\partial}{\partial x} \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\ &= \left[ \frac{\partial}{\partial x} \left( \frac{\partial v_z}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial v_z}{\partial x} \right) \right] + \left[ \frac{\partial}{\partial y} \left( \frac{\partial v_x}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial v_x}{\partial y} \right) \right] + \left[ \frac{\partial}{\partial z} \left( \frac{\partial v_y}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial v_y}{\partial z} \right) \right] \\ \nabla \cdot (\nabla \times \mathbf{v}) &= 0 \end{aligned}$$

where the last step hinges on the equality of cross derivatives:

$$\frac{\partial}{\partial x_i} \left( \frac{\partial v}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\partial v}{\partial x_i} \right)$$

Checking for  $v_a = x^2\hat{\mathbf{x}} + 2xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$ :

$$\begin{aligned} \nabla \cdot (\nabla \times v_a) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xz^2 & -2xz \end{vmatrix} \\ &= \nabla \cdot [\hat{\mathbf{x}}(0 - 4xz) - \hat{\mathbf{y}}(-2z - 0) + \hat{\mathbf{z}}(2z^2 - 0)] \\ &= \nabla \cdot \left[ \frac{\partial}{\partial x}(-4xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(2z^2) \right] \\ &= -4z + 0 + 4z = 0 \end{aligned}$$

**1.28** The curl of gradient is always zero:

$$\begin{aligned} \nabla \times (\nabla T) &= \nabla \times \left( \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \right) \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix} \\ &= \hat{\mathbf{x}} \left[ \frac{\partial}{\partial y} \left( \frac{\partial T}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial T}{\partial y} \right) \right] - \hat{\mathbf{y}} \left[ \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial T}{\partial x} \right) \right] + \hat{\mathbf{z}} \left[ \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial T}{\partial x} \right) \right] \\ \nabla \times (\nabla T) &= 0 \end{aligned}$$

where the last step uses the equality of cross derivatives again. Checking for  $T = x^2y^3z^4$ :

$$\frac{\partial T}{\partial x} = 2xy^3z^4, \quad \frac{\partial T}{\partial y} = 3x^2y^2z^4, \quad \text{and} \quad \frac{\partial T}{\partial z} = 4x^2y^3z^3$$

and

$$\begin{aligned} \nabla \times (\nabla T) &= \nabla \times (2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}) \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix} \\ &= \hat{\mathbf{x}}(12x^2y^2z^4 - 12x^2y^2z^4) - \hat{\mathbf{y}}(8x^2y^3z^3 - 8x^2y^3z^3) + \hat{\mathbf{z}}(6x^2y^3z^3 - 6x^2y^3z^3) \\ &= 0 \end{aligned}$$

**1.29** Calculating the line integral of the function  $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$ : from the origin to point  $(1, 1, 1)$  along three different paths:

(a)  $a = (0, 0, 0) \rightarrow b = (1, 0, 0) \rightarrow c = (1, 1, 0) \rightarrow d = (1, 1, 1)$  split to three paths:

- (i) From  $a \rightarrow b$ :  $dl = dx\hat{\mathbf{x}}$  and  $\mathbf{v} = x^2\hat{\mathbf{x}}$ .
- (ii) From  $b \rightarrow c$ :  $dl = dy\hat{\mathbf{y}}$  and  $\mathbf{v} = 2yz\hat{\mathbf{y}} = 0$  since  $z = 0$ .
- (iii) From  $c \rightarrow d$ :  $dl = dz\hat{\mathbf{z}}$  and  $\mathbf{v} = y^2\hat{\mathbf{z}} = 1\hat{\mathbf{z}}$  since  $y = 1$ .

$$\begin{aligned} \int_a^b \mathbf{v} \cdot d\mathbf{l} &= \int_0^1 x^2 dx = \frac{1}{3} \\ \int_b^c \mathbf{v} \cdot d\mathbf{l} &= \int_0^1 0 dy = 0 \\ \int_c^d \mathbf{v} \cdot d\mathbf{l} &= \int_0^1 1 dz = 1 \\ \int_a^d \mathbf{v} \cdot d\mathbf{l} &= \frac{1}{3} + 0 + 1 = \frac{4}{3} \end{aligned}$$

(b)  $a = (0, 0, 0) \rightarrow b = (0, 0, 1) \rightarrow c = (0, 1, 1) \rightarrow d = (1, 1, 1)$  split to three paths:

- (i) From  $a \rightarrow b$ :  $dl = dz\hat{\mathbf{z}}$  and  $\mathbf{v} = y^2\hat{\mathbf{z}} = 0$  since  $y = 0$ .
- (ii) From  $b \rightarrow c$ :  $dl = dy\hat{\mathbf{y}}$  and  $\mathbf{v} = 2yz\hat{\mathbf{y}} = 2y\hat{\mathbf{y}}$  since  $z = 1$ .
- (iii) From  $c \rightarrow d$ :  $dl = dx\hat{\mathbf{x}}$  and  $\mathbf{v} = x^2\hat{\mathbf{x}}$ .

$$\begin{aligned} \int_a^b \mathbf{v} \cdot d\mathbf{l} &= \int_0^1 0 dz = 0 \\ \int_b^c \mathbf{v} \cdot d\mathbf{l} &= \int_0^1 2y dy = 1 \\ \int_c^d \mathbf{v} \cdot d\mathbf{l} &= \int_0^1 x^2 dx = \frac{1}{3} \\ \int_a^d \mathbf{v} \cdot d\mathbf{l} &= 0 + 1 + \frac{1}{3} = \frac{4}{3} \end{aligned}$$

(c) A straight line: Since  $x = y = z$  and  $dx = dy = dz$ ,  $dl = dx(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$  and  $\mathbf{v} = x^2\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}} = 4x^2\hat{\mathbf{x}}$ .

$$\int_a^d \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4x^2 dx = \frac{4}{3}$$

(d) The line integral around the closed loop:

$$\oint \mathbf{v} \cdot d\mathbf{l} = \int_{(a)} \mathbf{v} \cdot d\mathbf{l} - \int_{(b)} \mathbf{v} \cdot d\mathbf{l} = \frac{4}{3} - \frac{4}{3} = 0$$

**1.30** Surface integral of  $\mathbf{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$  over the bottom of the box:  $z=0$ ,  $d\mathbf{A} = dx dy \hat{\mathbf{z}}$   $\mathbf{v} \cdot d\mathbf{A} = y(z^2-3) dx dy = -3y dx dy$ , so

$$\int \mathbf{v} \cdot d\mathbf{A} = \int_0^2 dx \int_0^2 -3y dy = -12$$

From Example 1.7 (v) has the same boundary line but a different surface integral, so the surface integral does not depend on just the boundary line. The surface over the closed surface of the box is

$$\oint \mathbf{v} \cdot d\mathbf{A} = 20 + 12 = 32$$

since the direction of  $d\mathbf{A}$  on the bottom side is in the negative  $z$  direction for it to point ‘outward’.

**1.31** Calculating the volume integral of  $T = z^2$  over the tetrahedron with vertices  $(0,0,0)$ ,  $(1,0,0)$ ,  $(0,1,0)$ , and  $(0,0,1)$ :

The equation of the plane containing the three vertices  $A = (1,0,0)$ ,  $B = (0,1,0)$ , and  $C(0,0,1)$ : The vector normal to this plane  $\mathbf{n} = (a,b,c)$  is the cross product of two vectors in the plane given by  $\mathbf{AB} = (-1,1,0)$  and  $\mathbf{AC} = (-1,0,1)$ :

$$\mathbf{n} = \begin{vmatrix} x & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1,1,1)$$

the equation of the plane is therefore

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_o) &= 0 \\ (1,1,1) \cdot [(x,y,z) - A] &= 0 \\ x + y + z &= 1 \end{aligned}$$

therefore, the boundary for  $x$  is  $x=0$  and  $x=1-y-z$ ; for  $y$  is  $y=0$  and  $y=1-z$ ; and for  $z$  is  $z=0$  and  $z=1$ . The volume integral is therefore

$$\begin{aligned} \int T dV &= \int_0^1 z^2 dz \int_0^{1-z} dy \int_0^{1-y-z} dx \\ &= \int_0^1 z^2 dz \int_0^{1-z} (1-y-z) dy \\ &= \int_0^1 z^2 dz \left( y - y^2/2 - yz \right) \Big|_0^{1-z} \\ &= \int_0^1 z^2 [(1-z) - (1-z)^2/2 - z(1-z)] dz \\ &= \int_0^1 z^2 (1-z-1/2-z^2/2+z-z+z^2) dz \\ &= \int_0^1 z^2 (1/2-z+z^2/2) dz \\ &= \int_0^1 (z^2/2 - z^3 + z^4/2) dz \\ &= z^3/6 - z^4/4 + z^5/10 \Big|_0^1 \\ &= 1/6 - 1/4 + 1/10 = 1/60 \end{aligned}$$

**1.32** Given  $T = x^2 + 4xy + 2yz^3$ ,

$$\frac{\partial T}{\partial x} = 2x + 4y, \quad \frac{\partial T}{\partial y} = 4x + 2z^3, \quad \text{and} \quad \frac{\partial T}{\partial z} = 6yz^2$$



(iii)  $(0, 1, 1) \rightarrow b$ :

$$z : 0 \rightarrow 1; \quad y = z = 1, \quad dy = dz = 0; \quad dl = dx \hat{\mathbf{x}}; \quad \nabla T \cdot d\mathbf{l} = (2x + 4) dx$$

and

$$\int_d^b \nabla T \cdot d\mathbf{l} = \int_0^1 (2x + 4) dx = 5$$

therefore

$$\int_a^b \nabla T \cdot d\mathbf{l} = 0 + 2 + 5 = 7$$

(c) the parabolic path  $z = x^2$ ;  $y = x$ :

$$dx = dy, \quad \text{and} \quad dz = 2x dx; \quad d\mathbf{l} = dx \hat{\mathbf{x}} + dx \hat{\mathbf{y}} + 2x dx \hat{\mathbf{z}}$$

and

$$\begin{aligned} \nabla T \cdot d\mathbf{l} &= (2x + 4y) dx + (4x + 2z^3) dx + (6yz^2) 2x dx \\ &= 6x dx + (4x + 2x^6) dx + (12x^6) dx \\ &= 10x dx + 14x^6 dx \end{aligned}$$

therefore

$$\begin{aligned} \int_a^b \nabla T \cdot d\mathbf{l} &= \int_0^1 (10x + 14x^6) dx \\ &= 5x^2 + 2x^7 \Big|_0^1 = 7 \end{aligned}$$

**1.33** Checking the divergence theorem: For the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

the divergence is

$$\nabla \cdot \mathbf{v} = y + 2z + 3x$$

so the volume integral is

$$\begin{aligned} \int_V \nabla \cdot \mathbf{v} d\tau &= \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x) dx dy dz \\ &= \int_0^2 \int_0^2 (2y + 4z + 6) dy dz \\ &= \int_0^2 (4 + 8z + 12) dz \\ &= 8 + 16 + 24 \\ \int_V \nabla \cdot \mathbf{v} d\tau &= 48 \end{aligned}$$

The surface integral is evaluated over the six faces of the cube noted by Figure 1.29:

(i)  $x = 2$ ,  $d\mathbf{A} = dy dz \hat{\mathbf{x}}$ ,  $\mathbf{v} \cdot d\mathbf{A} = 2y dy dz$ ;

$$\int \mathbf{v} \cdot d\mathbf{A} = \int_0^2 \int_0^2 2y dy dz = 8$$



**1.34** Testing Stokes' theorem for the function

$$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$$

using the triangular shaded area bounded by the vertices  $O = (0, 0, 0)$ ,  $A = (0, 2, 0)$ , and  $B = (0, 0, 2)$ :

$$\begin{aligned}\nabla \times \mathbf{v} &= (0 - 2y)\hat{\mathbf{x}} - (3z - 0)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}} \quad \text{and} \quad d\mathbf{A} = dz dy \hat{\mathbf{x}} \\ &= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}\end{aligned}$$

$x = 0$  on this surface, and the limits of integration are  $y : 0 \rightarrow 2$  and  $z = 0 \rightarrow z = 2 - y$ :

$$(\nabla \times \mathbf{v}) \cdot d\mathbf{A} = -2y dz dy$$

Thus, the flux of the curl through the surface is

$$\begin{aligned}\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{A} &= \int_0^2 \int_0^{2-y} -2y dz dy \\ &= \int_0^2 -2y(2 - y) dy \\ &= -2y^2 + \frac{2}{3}y^3 \Big|_0^2 = -8/3\end{aligned}$$

The line integral is evaluated over the three sides of the triangle:

(i) On the path  $OA$ :

$$x = z = 0; d\mathbf{l} = dy \hat{\mathbf{y}}; \mathbf{v} \cdot d\mathbf{l} = 2yz dy = 0;$$

$$\int_{OA} \mathbf{v} \cdot d\mathbf{l} = 0$$

(ii) On the path  $AB$ :

$$x = 0, y = 2 - z; dy = -dz; d\mathbf{l} = -dz (\hat{\mathbf{y}} - \hat{\mathbf{z}}); \mathbf{v} \cdot d\mathbf{l} = -2yz dz = -2(2 - z)z dz = (2z^2 - 4z) dz;$$

$$\int_{AB} \mathbf{v} \cdot d\mathbf{l} = \int_0^2 (2z^2 - 4z) dz = -8/3$$

(iii) On the path  $BO$ :

$$x = y = 0; d\mathbf{l} = dz \hat{\mathbf{z}}; \mathbf{v} \cdot d\mathbf{l} = 0;$$

$$\int_{BO} \mathbf{v} \cdot d\mathbf{l} = 0$$

So, the circulation of  $\mathbf{v}$  around the triangle is

$$\oint \mathbf{v} \cdot d\mathbf{l} = 0 + -8/3 + 0 = -8/3$$

thus,

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{A} = \oint \mathbf{v} \cdot d\mathbf{l}$$

**1.35** By Corollary 1, the function

$$\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$$

with curl

$$\nabla \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}$$





or rewritten as

$$\mathbf{B} \cdot (\nabla \times \mathbf{A}) = \nabla \cdot (\mathbf{A} \times \mathbf{B}) + \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

integrating both sides over a volume:

$$\int_V \mathbf{B} \cdot (\nabla \times \mathbf{A}) dV = \int_V \nabla \cdot (\mathbf{A} \times \mathbf{B}) dV + \int_V \mathbf{A} \cdot (\nabla \times \mathbf{B}) dV$$

Using the divergence theorem  $\int_V (\nabla \cdot \mathbf{v}) dV = \oint_S \mathbf{v} \cdot d\mathbf{a}$ :

$$\int_V \mathbf{B} \cdot (\nabla \times \mathbf{A}) dV = \oint_S (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} + \int_V \mathbf{A} \cdot (\nabla \times \mathbf{B}) dV$$

**1.37** Given the relation of Cartesian to spherical coordinates:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad \text{and} \quad z = r \cos \theta$$

To find the formula for  $r$ , take the sum of the squares of the three equations; Solve for  $\theta$  using the third equation; and solve for  $\phi$  by dividing the second equation by the first:

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos \frac{z}{r}, \quad \text{and} \quad \phi = \arctan \frac{y}{x}$$

**1.38** From the position vector

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \\ \mathbf{r} &= r \sin \theta \cos \phi \hat{\mathbf{x}} + r \sin \theta \sin \phi \hat{\mathbf{y}} + r \cos \theta \hat{\mathbf{z}} \\ \mathbf{r} &= r \hat{\mathbf{r}}(\theta, \phi) \end{aligned}$$

where the unit vector  $\hat{\mathbf{r}}(\theta, \phi)$  is dependent on  $\theta$  and  $\phi$ . The new basis vectors are in the same direction as the partial derivatives with respect to  $r$ ,  $\theta$ , and  $\phi$ , so

$$\hat{\mathbf{r}} = \frac{e_r}{|e_r|}, \quad \hat{\boldsymbol{\theta}} = \frac{e_\theta}{|e_\theta|}, \quad \text{and} \quad \hat{\boldsymbol{\phi}} = \frac{e_\phi}{|e_\phi|}$$

The partial derivatives are

$$\begin{aligned} e_r &= \frac{\partial \mathbf{r}}{\partial r} = \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ e_\theta &= \frac{\partial \mathbf{r}}{\partial \theta} = r \cos \theta \cos \phi \hat{\mathbf{x}} + r \cos \theta \sin \phi \hat{\mathbf{y}} - r \sin \theta \hat{\mathbf{z}} \\ e_\phi &= \frac{\partial \mathbf{r}}{\partial \phi} = -r \sin \theta \sin \phi \hat{\mathbf{x}} + r \sin \theta \cos \phi \hat{\mathbf{y}} \end{aligned}$$

and the magnitude

$$\begin{aligned} |e_r| &= \sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta} \\ &= \sqrt{\sin^2 \theta + \cos^2 \theta} = 1 \\ |e_\theta| &= \sqrt{r^2 (\cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \theta)} \\ &= \sqrt{r^2 (\cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + \sin^2 \theta)} \\ &= \sqrt{r^2 (\cos^2 \theta + \sin^2 \theta)} = r \\ |e_\phi| &= \sqrt{r^2 (\sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi)} \\ &= \sqrt{r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi)} \\ &= \sqrt{r^2 \sin^2 \theta} = r \sin \theta \end{aligned}$$

thus, the unit vectors are:

$$\begin{aligned}\hat{\mathbf{r}} &= \sin \theta \cos \phi \hat{\mathbf{x}} + \sin \theta \sin \phi \hat{\mathbf{y}} + \cos \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos \theta \cos \phi \hat{\mathbf{x}} + \cos \theta \sin \phi \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}\end{aligned}$$

or in matrix form:

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

where  $a = Qx$  is an orthogonal matrix, so  $Q^T = Q^{-1}$  and  $Q^T Q = I$ . Multiplying both sides by  $Q^T$ :

$$Q^T a = Q^T Q x \rightarrow x = Q^T a$$

thus, the inverse formula is

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}$$

or in vector form:

$$\begin{aligned}\hat{\mathbf{x}} &= \sin \theta \cos \phi \hat{\mathbf{r}} + \cos \theta \cos \phi \hat{\boldsymbol{\theta}} - \sin \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} &= \sin \theta \sin \phi \hat{\mathbf{r}} + \cos \theta \sin \phi \hat{\boldsymbol{\theta}} + \cos \phi \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} &= \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}\end{aligned}$$

**1.39** (a) Divergence theorem for  $\mathbf{v}_1 = r^2 \hat{\mathbf{r}}$  using a volume of a sphere of radius  $R$  centered at the origin: The divergence is

$$\nabla \cdot \mathbf{v}_1 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2(r^2)) = 4r$$

and the volume and surface elements are

$$dV = r^2 \sin \theta \, dr \, d\theta \, d\phi, \quad \text{and} \quad d\mathbf{a}_1 = (r^2 \sin \theta) \, d\theta \, d\phi \, \hat{\mathbf{r}}$$

So

$$\begin{aligned}\int_V (\nabla \cdot \mathbf{v}_1) \cdot dV &= \int_{r=0}^R \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 4r(r^2 \sin \theta) \, d\theta \, d\phi \, dr \\ &= \int_0^R 4r^3 \, dr \int_0^{\pi} \sin \theta \, d\theta \int_0^{2\pi} d\phi \\ &= (R^4)(2)(2\pi) = 4\pi R^4\end{aligned}$$

and

$$\begin{aligned}\oint_S \mathbf{v}_1 \cdot d\mathbf{a}_1 &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2(r^2 \sin \theta) \, d\theta \, d\phi \\ &= r^4 \int_0^{\pi} \sin \theta \, d\theta \int_0^{2\pi} d\phi \\ &= (r^4)(2)(2\pi) = 4\pi r^4\end{aligned}$$

where  $r = R$  on the surface of the sphere. Therefore,

$$\int_V (\nabla \cdot \mathbf{v}_1) \cdot dV = \oint_S \mathbf{v}_1 \cdot d\mathbf{a}_1$$

(b) For  $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$ :

$$\nabla \cdot \mathbf{v}_2 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 (1/r^2)) = 0$$

So

$$\int_V (\nabla \cdot \mathbf{v}_2) \cdot dV = 0$$

and

$$\begin{aligned} \oint_S \mathbf{v}_2 \cdot d\mathbf{a}_1 &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{r^2} (r^2 \sin \theta) d\theta d\phi \\ &= \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi = 4\pi \end{aligned}$$

**1.40** Given the function

$$\mathbf{v} = (r \cos \theta)\hat{\mathbf{r}} + (r \sin \theta)\hat{\boldsymbol{\theta}} + (r \sin \theta \cos \phi)\hat{\boldsymbol{\phi}}$$

the divergence in spherical coordinates is

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (v_\phi) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin^2 \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (r \sin \theta \cos \phi) \\ &= 3 \cos \theta + 2 \cos \theta - \sin \phi \\ &= 5 \cos \theta - \sin \phi \end{aligned}$$

Checking the divergence theorem using a volume of a inverted hemispherical bowl of radius  $R$ , resting on the  $xy$  plane and centered at the origin:

The volume and surface elements are

$$dV = r^2 \sin \theta dr d\theta d\phi, \quad \text{and} \quad d\mathbf{a}_1 = r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}$$

The volume integral for the hemisphere is

$$\begin{aligned} \int_V (\nabla \cdot \mathbf{v}) \cdot dV &= \int_{r=0}^R \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (5 \cos \theta - \sin \phi) (r^2 \sin \theta) d\theta d\phi dr \\ &= \int_0^R r^2 dr \int_0^{\pi/2} \sin \theta \int_0^{2\pi} (5 \cos \theta - \sin \phi) d\phi d\theta \\ &= \frac{R^3}{3} \int_0^{\pi/2} 5\phi \cos \theta + \cos \phi \Big|_0^{2\pi} d\theta \\ &= \frac{R^3}{3} (10\pi) \int_0^{\pi/2} \sin \theta \cos \theta d\theta \end{aligned}$$

$$\begin{aligned} \text{using } u &= \sin \theta, \quad du = \cos \theta d\theta; \quad \int u du = \frac{u^2}{2} \\ &= \frac{5\pi R^3}{3} \sin^2 \theta \Big|_0^{\pi/2} = \frac{5\pi R^3}{3} \end{aligned}$$

The surface integral is split into two parts: the top surface of the hemisphere and the circular base.

(i) The top surface of the hemisphere where  $r = R$ :

$$\begin{aligned} \oint_S \mathbf{v} \cdot d\mathbf{a}_1 &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (r \cos \theta) (r^2 \sin \theta) d\theta d\phi \\ &= r^3 \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi \\ &= \pi r^3 = \pi R^3 \end{aligned}$$

(ii) The circular base of the hemisphere where  $\theta = \pi/2$  and  $\mathbf{a}_2 = r \, dr \, d\phi \, \hat{\boldsymbol{\theta}}$ :

$$\begin{aligned}\oint_S \mathbf{v} \cdot d\mathbf{a}_2 &= \int_{r=0}^R \int_{\phi=0}^{2\pi} (r \sin \theta) r \, dr \, d\phi \\ &= \sin(\pi/2) \int_0^R r^2 \, dr \int_0^{2\pi} d\phi \\ &= (1) \frac{R^3}{3} (2\pi) = \frac{2\pi R^3}{3}\end{aligned}$$

So, the total surface integral is

$$\oint_S \mathbf{v} \cdot d\mathbf{a}_1 + \oint_S \mathbf{v} \cdot d\mathbf{a}_2 = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

1.41

$$T = r(\cos \theta + \sin \theta \cos \phi)$$

The partial derivatives are:

$$\begin{aligned}\frac{\partial T}{\partial r} &= \cos \theta + \sin \theta \cos \phi \\ \frac{\partial T}{\partial \theta} &= r(-\sin \theta + \cos \theta \cos \phi) \\ \frac{\partial T}{\partial \phi} &= -r \sin \theta \sin \phi\end{aligned}$$

thus, the gradient of  $T$  in spherical is

$$\begin{aligned}\nabla T &= \frac{\partial T}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \hat{\boldsymbol{\phi}} \\ &= (\cos \theta + \sin \theta \cos \phi) \hat{\mathbf{r}} + (-\sin \theta + \cos \theta \cos \phi) \hat{\boldsymbol{\theta}} - (\sin \phi) \hat{\boldsymbol{\phi}}\end{aligned}$$

and partial derivative in the laplacian are:

$$\begin{aligned}\frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) &= \frac{\partial}{\partial r} (r^2 (\cos \theta + \sin \theta \cos \phi)) = 2r(\cos \theta + \sin \theta \cos \phi) \\ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) &= r \frac{\partial}{\partial \theta} (\sin \theta (-\sin \theta + \cos \theta \cos \phi)) = r \frac{\partial}{\partial \theta} (-\sin^2 \theta + \sin \theta \cos \theta \cos \phi) \\ &= -2r \sin \theta \cos \theta + r \cos^2 \theta \cos \phi - r \sin^2 \theta \cos \phi \\ \frac{\partial^2 T}{\partial \phi^2} &= -r \sin \theta \cos \phi\end{aligned}$$

The laplacian of  $T$  in spherical is

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2}$$

Simplifying each term: (i) The first term:

$$\frac{2}{r} (\cos \theta + \sin \theta \cos \phi)$$

(ii) The second term:

$$-\frac{2}{r} \cos \theta + \frac{\cos^2 \theta \cos \phi}{r \sin \theta} - \frac{1}{r} \sin \theta \cos \phi$$

(iii) The third term:

$$-\frac{\cos \phi}{r \sin \theta}$$

adding all three terms:

$$\begin{aligned}\nabla^2 T &= \frac{2}{r}(\cos \theta + \sin \theta \cos \phi) - \frac{2}{r} \cos \theta + \frac{\cos^2 \theta \cos \phi}{r \sin \theta} - \frac{1}{r} \sin \theta \cos \phi - \frac{\cos \phi}{r \sin \theta} \\ &= \frac{2}{r}(\sin \theta \cos \phi) + \frac{\cos^2 \theta \cos \phi - \cos \phi}{r \sin \theta} - \frac{1}{r} \sin \theta \cos \phi \\ &= \frac{2}{r \sin \theta}(\sin^2 \theta \cos \phi) + \frac{\cos^2 \theta \cos \phi - \cos \phi}{r \sin \theta} - \frac{1}{r \sin \theta} \sin^2 \theta \cos \phi \\ &= \frac{1}{r \sin \theta}(\sin^2 \theta \cos \phi + \cos^2 \theta \cos \phi - \cos \phi) \\ &= \frac{1}{r \sin \theta}(\cos \phi)(\sin^2 \theta + \cos^2 \theta - 1) = 0\end{aligned}$$

Converting  $T$  to Cartesian coordinates:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad \text{and} \quad z = r \cos \theta$$

So

$$T = z + x$$

The laplacian of  $T$  in Cartesian coordinates is

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Testing the gradient theorem using the path  $O \rightarrow A = (2, 0, 0) \rightarrow B = (0, 2, 0) \rightarrow C = (0, 0, 2)$ : Given the general infinitesimal displacement

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}$$

and the gradient of  $T$  in spherical coordinates

$$\nabla T = (\cos \theta + \sin \theta \cos \phi) \hat{\mathbf{r}} + (-\sin \theta + \cos \theta \cos \phi) \hat{\boldsymbol{\theta}} + (-\sin \phi) \hat{\boldsymbol{\phi}}$$

(i) On the path  $OA$ :

$$\theta = \pi/2, \quad \phi = 0, \quad d\mathbf{l} = dr \hat{\mathbf{r}}; \quad (\nabla T) \cdot d\mathbf{l} = 1 dr$$

So

$$\int_{OA} (\nabla T) \cdot d\mathbf{l} = \int_0^2 1 dr = 2$$

(ii) On the path  $AB$ :

$$r = 2, \quad \theta = \pi/2, \quad d\mathbf{l} = 2 d\phi \hat{\boldsymbol{\phi}}; \quad (\nabla T) \cdot d\mathbf{l} = -2 \sin \phi d\phi$$

So

$$\int_{AB} (\nabla T) \cdot d\mathbf{l} = \int_0^{\pi/2} -2 \sin \phi d\phi = -2$$

(iii) On the path  $BC$ :

$$r = 2, \quad \phi = \pi/2, \quad d\mathbf{l} = 2 d\theta \hat{\boldsymbol{\theta}}; \quad (\nabla T) \cdot d\mathbf{l} = -2 \sin \theta d\theta$$



thus, the unit vectors are:

$$\begin{aligned}\hat{\mathbf{s}} &= \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \\ \hat{\phi} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}\end{aligned}$$

The cylindrical unit vectors in terms of  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\mathbf{z}}$  in matrix form:

$$\begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\phi} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

Which is an orthogonal matrix  $a = Qx$ , so the Cartesian unit vectors is found by multiplying  $a$  by the transpose of  $Q$ :

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\phi} \\ \hat{\mathbf{z}} \end{pmatrix}$$

or in vector form:

$$\begin{aligned}\hat{\mathbf{x}} &= \cos \phi \hat{\mathbf{s}} - \sin \phi \hat{\phi} \\ \hat{\mathbf{y}} &= \sin \phi \hat{\mathbf{s}} + \cos \phi \hat{\phi} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}\end{aligned}$$

**1.43** (a) Finding the divergence of the function

$$\mathbf{v} = s(2 + \sin^2 \phi) \hat{\mathbf{s}} + s \sin \phi \cos \phi \hat{\phi} + 3z \hat{\mathbf{z}}$$

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\ &= \frac{1}{s} \frac{\partial}{\partial s} (s(s(2 + \sin^2 \phi))) + \frac{1}{s} \frac{\partial}{\partial \phi} (s \sin \phi \cos \phi) + \frac{\partial}{\partial z} (3z) \\ &= 2(2 + \sin^2 \phi) + \cos^2 \phi - \sin^2 \phi + 3 \\ &= 7 + \sin^2 \phi + \cos^2 \phi = 8\end{aligned}$$

(b) Testing divergence theorem using a quarter cylinder of radius 2 and height 5 in quadrant I:  
LHS: The volume elements is

$$dV = s \, ds \, d\phi \, dz,$$

so the volume integral is

$$\begin{aligned}\int_V (\nabla \cdot \mathbf{v}) \cdot dV &= \int_{s=0}^2 \int_{\phi=0}^{\pi/2} \int_{z=0}^5 8(s \, ds \, d\phi \, dz) \\ &= 8 \int_0^2 s \, ds \int_0^{\pi/2} d\phi \int_0^5 dz \\ &= 8(2)(\pi/2)(5) = 40\pi\end{aligned}$$

RHS: There are 5 surfaces: the top, bottom, and 3 sides.

(i) The top surface:

$$z = 5, \quad da = s \, ds \, d\phi \, \hat{\mathbf{z}}; \quad \mathbf{v} \cdot d\mathbf{a} = 15s \, ds \, d\phi$$



So

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \int_{\phi=0}^{\pi/2} \int_{s=0}^2 15s \, ds \, d\phi = 15 \int_0^{\pi/2} d\phi \int_0^2 s \, ds = 15\pi$$

(ii) The bottom surface:

$$z = 0, \quad da = s \, ds \, d\phi \, \hat{\mathbf{z}}; \quad \mathbf{v} \cdot d\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v} \cdot d\mathbf{a} = 0$$

(iii) The surface on the  $xy$  plane:

$$\phi = \pi/2 \quad da = ds \, dz \, \hat{\phi}; \quad \mathbf{v} \cdot d\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v} \cdot d\mathbf{a} = 0$$

(iv) The surface on the  $xz$  plane:

$$\phi = 0, \quad da = -ds \, dz \, \hat{\phi}; \quad \mathbf{v} \cdot d\mathbf{a} = 0 \quad \text{thus} \quad \oint_S \mathbf{v} \cdot d\mathbf{a} = 0$$

(v) The curved surface:

$$s = 2, \quad da = 2 \, d\phi \, dz \, \hat{\mathbf{s}}; \quad \mathbf{v} \cdot d\mathbf{a} = 4(2 + \sin^2 \phi) \, d\phi \, dz$$

So

$$\begin{aligned} \oint_S \mathbf{v} \cdot d\mathbf{a} &= \int_{z=0}^5 \int_{\phi=0}^{\pi/2} 4(2 + \sin^2 \phi) \, d\phi \, dz \\ &= 4 \int_0^5 dz \int_0^{\pi/2} (4(1/2) + \sin^2 \phi) \, d\phi \\ \text{using } \sin^2 \phi &= \frac{1 - \cos(2\phi)}{2} \\ &= 4 \int_0^5 dz \int_0^{\pi/2} (2 + (1 - \cos(2\phi))/2) \, d\phi \\ &= 10 \int_0^{\pi/2} (5 - \cos(2\phi)) \, d\phi \\ &= 10 \left( 5\phi - \frac{\sin(2\phi)}{2} \Big|_0^{\pi/2} \right) = 25\pi \end{aligned}$$

So the total surface integral is

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = 15\pi + 0 + 0 + 0 + 25\pi = 40\pi$$

which is the same as the volume integral, so the divergence theorem holds.

(c) The curl of  $\mathbf{v}$  in cylindrical coordinates is

$$\begin{aligned} \nabla \times \mathbf{v} &= \frac{1}{s} \begin{vmatrix} \hat{\mathbf{s}} & s\hat{\phi} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ v_s & (s)v_\phi & v_z \end{vmatrix} \\ &= \frac{1}{s} \begin{vmatrix} \hat{\mathbf{s}} & s\hat{\phi} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ s(2 + \sin^2 \phi) & (s)s \sin \phi \cos \phi & 3z \end{vmatrix} \\ &= 0\hat{\mathbf{s}} - \frac{1}{s}(0 - 0) + \frac{1}{s}(2s \sin \phi \cos \phi - 2s \sin \phi \cos \phi) \\ &= 0 \end{aligned}$$





1.48 (a)

$$\int (r^2 + \mathbf{r} \cdot \mathbf{a} + a^2) \delta^3(\mathbf{r} - \mathbf{a}) dV = a^2 + \mathbf{a} \cdot \mathbf{a} + a^2 = 3a^2$$

(b) Given  $V$  is a cube of side 2 centered at the origin and  $\mathbf{b} = 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$ :

$$\begin{aligned} \int_V |\mathbf{r} - \mathbf{b}|^2 \delta^3(5\mathbf{r}) dV &= \int_0^1 \int_0^1 \int_0^1 |\mathbf{r} - \mathbf{b}|^2 \delta(5x) \delta(5y) \delta(5z) dx dy dz \\ &= \frac{1}{5^3} |\mathbf{b}|^2 = \frac{1}{5^3} (4^2 + 3^2) = \frac{1}{5} \end{aligned}$$

(c) Given  $V$  is a sphere of radius 6 centered at the origin and  $\mathbf{c} = 5\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$ :

The magnitude  $c = \sqrt{5^2 + 3^2 + 2^2} = \sqrt{38}$ , is outside the sphere (magnitude  $r = \sqrt{36}$ ). Therefore,

$$\int_V [r^4 + r^2(\mathbf{r} \cdot \mathbf{c}) + c^4] \delta^3(\mathbf{r} - \mathbf{c}) dV = 0$$

(d) Given  $V$  is a sphere of radius 1.5 centered at  $(2, 2, 2)$  and

$$\mathbf{d} = (1, 2, 3), \quad \mathbf{e} = (3, 2, 1)$$

Checking if the delta function is inside the sphere:

$$|\mathbf{e} - (2, 2, 2)| = \sqrt{1 + 0 + 1} = \sqrt{2} < 1.5$$

so the delta function is inside the sphere and the integral is

$$\begin{aligned} \int_V \mathbf{r} \cdot (\mathbf{d} - \mathbf{r}) \delta^3(\mathbf{e} - \mathbf{r}) dV &= \int_V \mathbf{r} \cdot (\mathbf{d} - \mathbf{r}) \delta^3(\mathbf{r} - \mathbf{e}) dV \\ &= \mathbf{e} \cdot (\mathbf{d} - \mathbf{e}) \\ &= (3, 2, 1) \cdot (-2, 0, 2) = -4 \end{aligned}$$

1.49 Evaluating the integral

$$J = \int_V e^{-r} \left( \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) dV$$

where  $V$  is a sphere of radius  $R$  centered at the origin.

(i)

$$J = \int_V e^{-r} 4\pi \delta^3(\mathbf{r}) dV = 4\pi e^0 = 4\pi$$

(ii) Using the product rule of divergence:

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

integrating over a volume and using the divergence theorem:

$$\int \nabla \cdot (f\mathbf{A}) dV = \int f(\nabla \cdot \mathbf{A}) dV + \int \mathbf{A} \cdot (\nabla f) dV = \oint f\mathbf{A} \cdot d\mathbf{a}$$

or

$$\int_V f(\nabla \cdot \mathbf{A}) dV = - \int_V \mathbf{A} \cdot (\nabla f) dV + \oint f\mathbf{A} \cdot d\mathbf{a}$$

where

$$f = e^{-r} \quad \text{and} \quad \mathbf{A} = \frac{\hat{\mathbf{r}}}{r^2}$$



The gradient of a scalar:

$$\mathbf{F}_3 = -\nabla V_3$$

so

$$\frac{\partial V_3}{\partial x} = -yz, \quad \frac{\partial V_3}{\partial y} = -zx, \quad \frac{\partial V_3}{\partial z} = -xy$$

A particular solution is

$$V_3 = -xyz$$

The curl of a vector (a little harder):

$$\mathbf{F}_3 = \nabla \times \mathbf{V}_3$$

so

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = yz, \quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = zx, \quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = xy$$

integrating the first equation with respect to  $y$ :

$$\begin{aligned} \frac{\partial A_z}{\partial y} &= yz + \frac{\partial A_y}{\partial z} \\ \int \frac{\partial A_z}{\partial y} dy &= \int yz dy + \int \frac{\partial A_y}{\partial z} dy \\ A_z &= \frac{1}{2}y^2z + f \end{aligned}$$

where the function  $f$  is an arbitrary function. Integrating the first equation again but with respect to  $z$  gives:

$$A_y = -\frac{1}{2}yz^2 + g$$

Repeating for all three equations:

$$\begin{aligned} A_z &= \frac{1}{2}y^2z + f \quad \text{and} \quad A_y = -\frac{1}{2}yz^2 + g \\ A_x &= \frac{1}{2}z^2x + h \quad \text{and} \quad A_z = -\frac{1}{2}zx^2 + i \\ A_y &= \frac{1}{2}x^2y + j \quad \text{and} \quad A_x = -\frac{1}{2}xy^2 + k \end{aligned}$$

a particular solution can be found by setting  $f = g = h = i = j = k = 0$ , and each component is a linear combination of the two possible solutions:

$$\begin{aligned} 2A_x &= \frac{1}{2}z^2x + f - \frac{1}{2}xy^2 + k \\ 2A_x &= \frac{1}{2}z^2x - \frac{1}{2}xy^2 \\ A_x &= \frac{1}{4}(z^2x - xy^2) \end{aligned}$$

and similarly for  $A_y$  and  $A_z$  gives the vector potential:

$$\mathbf{A} = \frac{1}{4}[x(z^2 - y^2)\hat{\mathbf{x}} + y(x^2 - z^2)\hat{\mathbf{y}} + z(y^2 - x^2)\hat{\mathbf{z}}]$$

**1.51** (d)  $\rightarrow$  (a): Given  $\mathbf{F} = -\nabla V$ , the curl of the gradient is always zero;

$$\nabla \times \mathbf{F} = \nabla \times (-\nabla V) = 0$$

(a)  $\rightarrow$  (c): From Stokes' theorem;

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint \mathbf{F} \cdot d\mathbf{l} = 0$$

(c)  $\rightarrow$  (b):

$$\oint \mathbf{F} \cdot d\mathbf{l} = \int_a^b \mathbf{F} \cdot d\mathbf{l} - \int_a^b \mathbf{F} \cdot d\mathbf{l} = 0$$

where the integrals are two different paths but equal, and thus independent of path. (b)  $\rightarrow$  (c) and (c)  $\rightarrow$  (a) are just the same steps in reverse.

**1.52** (d)  $\rightarrow$  (a): Given  $\mathbf{F} = \nabla \times A$ , the divergence of curl is always zero;

$$\nabla \cdot (\nabla \times A) = 0$$

(a)  $\rightarrow$  (c): From the divergence theorem;

$$\int_V (\nabla \cdot \mathbf{F}) dV = \oint \mathbf{F} \cdot d\mathbf{a} = 0$$

(c)  $\rightarrow$  (b):

$$\oint \mathbf{F} \cdot d\mathbf{a} = \int \mathbf{F} \cdot d\mathbf{a}_1 + \int \mathbf{F} \cdot d\mathbf{a}_2 = 0$$

where  $\mathbf{a}_2 = -\mathbf{a}_1$  and the integrals are two different surfaces but equal, and thus depend only on the boundary. (b)  $\rightarrow$  (c) and (c)  $\rightarrow$  (a) are just the same steps in reverse.

**1.53** (a) From Problem 1.15,

$$\begin{aligned}\mathbf{v}_a &= x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}} \\ \mathbf{v}_b &= xy \hat{\mathbf{x}} + 2yz \hat{\mathbf{y}} + 3zx \hat{\mathbf{z}} \\ \mathbf{v}_c &= y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}}\end{aligned}$$

and the curl of each vector field:

$$\begin{aligned}\nabla \times \mathbf{v}_a &\neq 0 \\ \nabla \times \mathbf{v}_b &\neq 0 \\ \nabla \times \mathbf{v}_c &= 0\end{aligned}$$

Thus  $\mathbf{v}_c$  can be written as a gradient of some scalar potential  $\mathbf{v}_c = -\nabla V$ :

$$\begin{aligned}\frac{\partial V}{\partial x} &= -y^2; & V &= -y^2 x + f \\ \frac{\partial V}{\partial y} &= -(2xy + z^2); & V &= -y^2 x - yz^2 + g \\ \frac{\partial V}{\partial z} &= -2yz; & V &= -yz^2 + h\end{aligned}$$

where  $f$ ,  $g$ , and  $h$  are arbitrary functions, thus one solution can be found setting  $g = 0$  and solving for  $f$  or  $h$ ;

$$V = -y^2 x - yz^2$$





1.61 (a) Show that

$$\int_V (\nabla T) dV = \oint_S T d\mathbf{a}$$

Letting  $\mathbf{v} = \mathbf{c}T$  where  $\mathbf{c}$  is a constant: using the product rule of divergence:

$$\begin{aligned}\nabla \cdot (f\mathbf{A}) &= f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f) \\ \nabla \cdot (T\mathbf{c}) &= T(\nabla \cdot \mathbf{c}) + \mathbf{c} \cdot (\nabla T) \\ \nabla \cdot (T\mathbf{c}) &= \mathbf{c} \cdot (\nabla T)\end{aligned}$$

where  $\nabla \cdot \mathbf{c} = 0$  since  $\mathbf{c}$  is a constant. Integrating over a volume and using the divergence theorem:

$$\int_V \nabla \cdot (T\mathbf{c}) dV = \int_V \mathbf{c} \cdot (\nabla T) dV = \oint_S T\mathbf{c} \cdot d\mathbf{a}$$

or moving the constant  $\mathbf{c}$  outside the integral:

$$\begin{aligned}\mathbf{c} \int_V \nabla T dV &= \mathbf{c} \oint_S T \cdot d\mathbf{a} \\ \int_V \nabla T dV &= \oint_S T \cdot d\mathbf{a}\end{aligned}$$

(b) Show that

$$\int_V (\nabla \times \mathbf{v}) dV = - \oint_S \mathbf{v} \times d\mathbf{a}$$

replacing  $\mathbf{v}$  with  $\mathbf{v} \times \mathbf{c}$  in the divergence theorem:

$$\int_V \nabla \cdot (\mathbf{v} \times \mathbf{c}) dV = \oint_S (\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a} \quad (1.1)$$

and using the product rule of divergence:

$$\begin{aligned}\nabla \cdot (\mathbf{v} \times \mathbf{c}) &= \mathbf{c} \cdot (\nabla \times \mathbf{v}) - \mathbf{v} \cdot (\nabla \times \mathbf{c}) \\ \nabla \cdot (\mathbf{v} \times \mathbf{c}) &= \mathbf{c} \cdot (\nabla \times \mathbf{v})\end{aligned}$$

where  $\nabla \times \mathbf{c} = 0$  since  $\mathbf{c}$  is a constant. So subbing into the divergence theorem and moving the constant  $\mathbf{c}$  outside the integral:

$$\mathbf{c} \int_V (\nabla \times \mathbf{v}) dV = \oint_S (\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a}$$

and using the scalar triple product identity:

$$(\mathbf{v} \times \mathbf{c}) \cdot d\mathbf{a} = \mathbf{v} \cdot (\mathbf{c} \times d\mathbf{a}) = \mathbf{c} \cdot (d\mathbf{a} \times \mathbf{v}) = -\mathbf{c} \cdot (\mathbf{v} \times d\mathbf{a})$$

so

$$\begin{aligned}\mathbf{c} \int_V (\nabla \times \mathbf{v}) dV &= -\mathbf{c} \oint_S (\mathbf{v} \times d\mathbf{a}) \\ \int_V (\nabla \times \mathbf{v}) dV &= - \oint_S (\mathbf{v} \times d\mathbf{a})\end{aligned}$$

(c) [Green's identity] Show that

$$\int_V [T\nabla^2 U dV + (\nabla T) \cdot (\nabla U)] dV = \oint_S (T\nabla U) \cdot d\mathbf{a}$$

letting  $\mathbf{v} = T\nabla U$  in the divergence theorem:

$$\int_V \nabla \cdot (T\nabla U) dV = \oint_S (T\nabla U) \cdot d\mathbf{a}$$

using the product rule of divergence:

$$\nabla \cdot (T\nabla U) = T(\nabla \cdot \nabla U) + (\nabla U) \cdot (\nabla T)$$

$$\nabla \cdot (T\nabla U) = T(\nabla^2 U) + (\nabla T) \cdot (\nabla U)$$

where  $\nabla \cdot \nabla U = \nabla^2 U$  and the dot product is commutative. Subbing into the divergence theorem:

$$\int_V [T\nabla^2 U dV + (\nabla T) \cdot (\nabla U)] dV = \oint_S (T\nabla U) \cdot d\mathbf{a}$$

(d) [Green's second identity] Show that

$$\int_V (T\nabla^2 U - U\nabla^2 T) dV = \oint_S (T\nabla U - U\nabla T) \cdot d\mathbf{a}$$

using the product rule of divergence:

$$\nabla \cdot (U\nabla T) = U\nabla^2 T + (\nabla T) \cdot (\nabla U)$$

and letting  $\mathbf{v} = U\nabla T$  in the divergence theorem:

$$\int_V \nabla \cdot (U\nabla T) dV = \oint_S (U\nabla T) \cdot d\mathbf{a}$$

$$\int_V [U\nabla^2 T + (\nabla T) \cdot (\nabla U)] dV = \oint_S (U\nabla T) \cdot d\mathbf{a}$$

and taking the difference with the result from (c):

$$\begin{aligned} \oint_S (T\nabla U) \cdot d\mathbf{a} - \oint_S (U\nabla T) \cdot d\mathbf{a} &= \oint_S (T\nabla U - U\nabla T) \cdot d\mathbf{a} \\ \int_V [T\nabla^2 U dV - U\nabla^2 T] dV &= \oint_S (T\nabla U - U\nabla T) \cdot d\mathbf{a} \end{aligned}$$

where  $(\nabla T) \cdot (\nabla U)$  cancels out on the left hand side.

(e) Show that

$$\int_S \nabla T \times d\mathbf{a} = - \oint_P T d\mathbf{l}$$

letting  $\mathbf{v} = \mathbf{c}T$  in Stokes' theorem:

$$\int_S \nabla \times (\mathbf{c}T) \cdot d\mathbf{a} = \oint_P (\mathbf{c}T) \cdot d\mathbf{l}$$

using the product rule of curl:

$$\nabla \times (\mathbf{c}T) = T(\nabla \times \mathbf{c}) - \mathbf{c} \times (\nabla T)$$

$$\nabla \times (\mathbf{c}T) = -\mathbf{c} \times (\nabla T)$$

where  $\nabla \times \mathbf{c} = 0$  since  $\mathbf{c}$  is a constant. And subbing into Stokes' theorem:

$$\int_S (-\mathbf{c} \times (\nabla T)) \cdot d\mathbf{a} = \oint_P (\mathbf{c}T) \cdot d\mathbf{l}$$

and using the scalar triple product identity

$$(-\mathbf{c} \times (\nabla T)) \cdot d\mathbf{a} = -\mathbf{c} \cdot (\nabla T \times d\mathbf{a})$$

thus the constant  $\mathbf{c}$  can move out of the integral and cancel out giving the result

$$\int_S \nabla T \times d\mathbf{a} = - \oint_P T d\mathbf{l}$$



Thus the vector area of the two surfaces are equal as long as the boundary is the same.

(d) Show that the integral around the boundary line

$$\mathbf{a} = \frac{1}{2} \oint_S \mathbf{r} \times d\mathbf{l}$$

A cone subtended by the loop at the origin (vertex at the origin) can be split into infinitesimal triangular wedges with base  $d\mathbf{l}$  and height  $\mathbf{r}$ , and the area of each wedge is

$$dA = \frac{1}{2} \mathbf{r} \times d\mathbf{l}$$

since the cross product is the area of the parallelogram formed by the two vectors or in this case the area of the triangle. Summing over all the wedges along the closed loop gives the total area

$$\frac{1}{2} \oint_S \mathbf{r} \times d\mathbf{l} = \mathbf{a} = \oint_S dA$$

(e) Show that

$$\oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} = \mathbf{a} \cdot \mathbf{c}$$

for any constant vector  $\mathbf{c}$ . Letting  $T = \mathbf{c} \cdot \mathbf{r}$  from the result in 1.61(e):

$$\begin{aligned} \int \nabla T \times d\mathbf{a} &= - \oint T d\mathbf{l} \\ \int \nabla(\mathbf{c} \cdot \mathbf{r}) \times d\mathbf{a} &= - \oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} \end{aligned}$$

using the product rule of gradients:

$$\nabla(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c} \times (\nabla \times \mathbf{r}) + \mathbf{r} \times (\nabla \times \mathbf{c}) + (\mathbf{c} \cdot \nabla) \mathbf{r} + (\mathbf{r} \cdot \nabla) \mathbf{c}$$

where  $\mathbf{r} = (x, y, z)$ ;  $\nabla \times \mathbf{r} = 0$ ,  $\nabla \times \mathbf{c} = 0$ ,  $\nabla \mathbf{c} = 0$ , and

$$(\mathbf{c} \cdot \nabla) \mathbf{r} = \mathbf{c}$$

So

$$\nabla(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c}$$

therefore

$$\begin{aligned} - \oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} &= \int \mathbf{c} \times d\mathbf{a} \\ &= \mathbf{c} \times \int d\mathbf{a} \\ &= \mathbf{c} \times \mathbf{a} \\ &= -\mathbf{a} \times \mathbf{c} \\ \oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} &= \mathbf{a} \times \mathbf{c} \end{aligned}$$

**1.63** (a) Finding the divergence of

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{r} \right) = \frac{1}{r^2}$$



## 2 Electrostatics

2.1 (a) Given twelve equal charges,  $q$  situated on corners of a regular 12-sided polygon, the net force is

$$\vec{F}_a = \sum_{i=1}^{12} \frac{1}{4\pi\epsilon_0} \frac{qQ}{r_i^2} \hat{\mathbf{r}}_i = 0$$

since the forces on each pair of charges (e.g., 12 and 6 o' clock) opposite to each other cancel out.

(b) If one of the charges is removed at 6 o' clock, the net force is strictly due to the the source charge at 12 o' clock:

$$\mathbf{F}_b = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r_{12}^2} \hat{\mathbf{r}}_{12}$$

where  $\mathbf{F}_b$  points from 12 to 6 o' clock.

(c) For 13 equal charges, the net force is still  $\mathbf{F}_c = 0$  because the symmetry of the arrangement is preserved.

(d) Removing one of the charges  $\mathbf{r}'_i$  is equivalent to the superposition of a source charge,  $-q$ , at  $\mathbf{r}'_i$  and the original configuration. The net force is then

$$\mathbf{F}_d = \mathbf{F}_c - \frac{1}{4\pi\epsilon_0} \frac{qQ}{r_i^2} \hat{\mathbf{r}}_i = -\frac{1}{4\pi\epsilon_0} \frac{qQ}{r_i^2} \hat{\mathbf{r}}_i$$

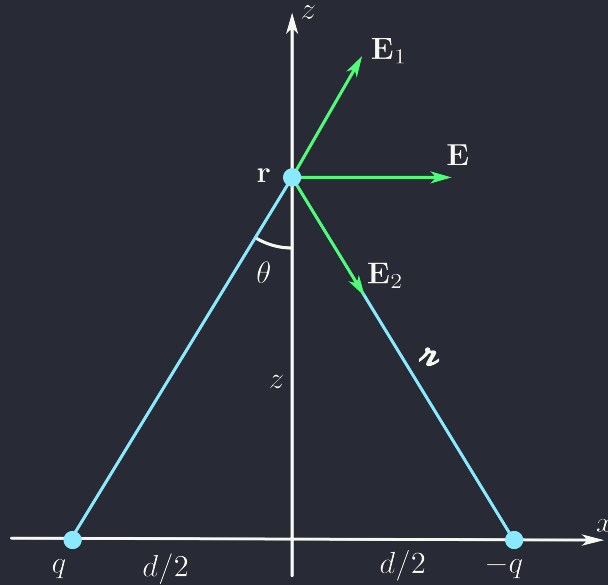


Figure 2.1: An electric field at a distance  $z$  from the midpoint between equal and opposite charges ( $\pm q$ ) separated by a distance  $d$ . The charge at  $x = d/2$  is  $-q$ .

2.2 The vertical componets of the electric field cancel out and the horizontal components add up:

$$E_x = 2 \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \sin \theta$$

where  $E_x = E \cos \theta$ ,  $r = \sqrt{z^2 + (d/2)^2}$ , and  $\sin \theta = d/(2r)$ , so

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{qd}{[z^2 + (d/2)^2]^{3/2}} \hat{\mathbf{x}}$$

### 2.3

$$\begin{aligned}\mathbf{r} &= z\hat{\mathbf{z}}, \quad \mathbf{r}' = x\hat{\mathbf{x}}, \quad d\mathbf{l} = dx; \\ \mathbf{z} &= z\hat{\mathbf{z}} - x\hat{\mathbf{x}}, \quad z = \sqrt{z^2 + x^2}, \quad \hat{\mathbf{z}} = \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}\end{aligned}$$

With uniform line charge  $\lambda$  and the limits of integration  $[0, L]$ ,

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int \frac{\lambda d\mathbf{l}}{z^2} \hat{\mathbf{z}} \\ &= \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{z\hat{\mathbf{z}} - x\hat{\mathbf{x}}}{[z^2 + x^2]^{3/2}} dx \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[ z\hat{\mathbf{z}} \int_0^L \frac{1}{(z^2 + x^2)^{3/2}} - \hat{\mathbf{x}} \int_0^L \frac{x}{(z^2 + x^2)^{3/2}} \right] \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[ z\hat{\mathbf{z}} \left( \frac{x}{z^2\sqrt{z^2 + x^2}} \right) \Big|_0^L + \hat{\mathbf{x}} \left( \frac{1}{\sqrt{z^2 + x^2}} \right) \Big|_0^L \right] \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[ z\hat{\mathbf{z}} \left( \frac{L}{z^2\sqrt{z^2 + L^2}} - \frac{0}{z^2\sqrt{z^2 + 0^2}} \right) + \hat{\mathbf{x}} \left( \frac{1}{\sqrt{z^2 + L^2}} - \frac{1}{\sqrt{z^2 + 0^2}} \right) \right] \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[ \hat{\mathbf{z}} \left( \frac{L}{z\sqrt{z^2 + L^2}} \right) + \hat{\mathbf{x}} \left( \frac{1}{\sqrt{z^2 + L^2}} - \frac{1}{z} \right) \right] \\ \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z} \left[ \hat{\mathbf{x}} \left( \frac{z}{\sqrt{z^2 + L^2}} - 1 \right) + \hat{\mathbf{z}} \left( \frac{L}{\sqrt{z^2 + L^2}} \right) \right]\end{aligned}$$

For  $z \gg L$ ,

$$\mathbf{E} \approx \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z^2} \hat{\mathbf{z}}$$

From far away, the line looks like a point charge  $q = \lambda L$ .

**2.4** One segment of the square loop is equivalent to Ex. 2.2, but with line segment length  $2L \rightarrow a$  and electric field distance  $z_o \rightarrow \sqrt{z_o^2 + a^2/4}$ . So, the magnitude of the electric field from one segment is

$$\begin{aligned}E &= \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z_o\sqrt{z_o^2 + L^2}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{z^2 + a^2/4}\sqrt{z^2 + a^2/2}}\end{aligned}$$

Due to the symmetry of the loop, the electric field components in the  $x$ -direction cancel out, and the electric field components in the  $z$ -direction add up:

$$\begin{aligned}\mathbf{E} &= 4E \cos \theta \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\lambda a}{\sqrt{z^2 + a^2/4}\sqrt{z^2 + a^2/2}} \frac{z}{\sqrt{z^2 + a^2/4}} \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\lambda a z}{(z^2 + a^2/4)\sqrt{z^2 + a^2/2}} \hat{\mathbf{z}}\end{aligned}$$

**2.5** The horizontal components of the electric field cancel out, and the vertical components conspire:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda}{z^2} \cos \theta \hat{\mathbf{z}} d\mathbf{l}$$





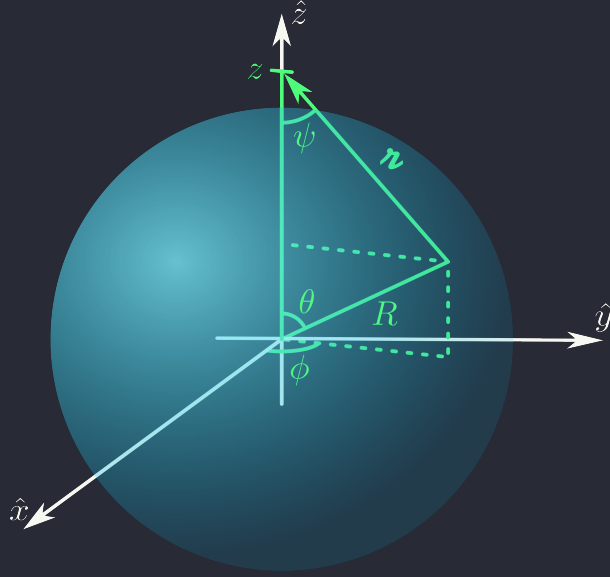


Figure 2.2: An electric field a distance  $z$  from the center of a spherical surface of radius  $R$  that carries a charge density  $\sigma$ .

**2.7** Once again, the electric field is in the  $z$ -direction:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{r^2} \cos \psi \hat{\mathbf{z}} \, d\mathbf{a} \quad (2.1)$$

From the law of cosines,  $r^2 = z^2 + R^2 - 2zR \cos \theta$ ; Geometrically,  $\cos \psi = \frac{z - R \cos \theta}{r}$ ; the surface area element is  $d\mathbf{a} = R^2 \sin \theta \, d\theta \, d\phi$ :

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \frac{\sigma R^2 (z - R \cos \theta)}{(z^2 + R^2 - 2zR \cos \theta)^{3/2}} \sin \theta \, d\theta \, d\phi \, \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} (2\pi\sigma R^2) \int_0^\pi \frac{z - R \cos \theta}{(z^2 + R^2 - 2zR \cos \theta)^{3/2}} \sin \theta \, d\theta \, \hat{\mathbf{z}} \\ &= \frac{1}{4\pi\epsilon_0} (2\pi\sigma R^2) f(\theta) \hat{\mathbf{z}} \end{aligned}$$

using the substitution  $u = \cos \theta$ :  $du = -\sin \theta \, d\theta$ , and the limits of integration are  $[\cos 0, \cos \pi]$ . So,

$$f(\theta) = \int_{-1}^1 \frac{z - Ru}{(z^2 + R^2 - 2zRu)^{3/2}} \, du = f(u)$$

substituting again with  $v = \sqrt{z^2 + R^2 - 2zRu}$ ;  $dv = -\frac{zR}{v} \, du$ ; and  $u = \frac{1}{2zR}(z^2 + R^2 - v^2)$ :

$$\begin{aligned} f(v) &= -\frac{1}{zR} \int \frac{z - \frac{1}{2z}(z^2 + R^2 - v^2)}{v^3} v \, dv \\ &= -\frac{1}{2z^2R} \int \frac{2z^2 - (z^2 + R^2 - v^2)}{v^2} \, dv \\ &= -\frac{1}{2z^2R} \int \frac{v^2 + z^2 - R^2}{v^2} \, dv \\ &= -\frac{1}{2z^2R} \int \left( 1 + \frac{z^2 - R^2}{v^2} \right) \, dv \\ &= -\frac{1}{2z^2R} \left( v - \frac{z^2 - R^2}{v} \right) \end{aligned}$$

back substituting  $v = \sqrt{z^2 + R^2 - 2zRu}$ ,

$$\begin{aligned}
f(u) &= -\frac{1}{2z^2R} \left( \frac{z^2 + R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}} - \frac{z^2 - R^2}{\sqrt{z^2 + R^2 - 2zRu}} \right) \Big|_{-1}^1 \\
&= -\frac{1}{2z^2R} \left( \frac{2R^2 - 2zRu}{\sqrt{z^2 + R^2 - 2zRu}} \right) \Big|_{-1}^1 \\
&= \frac{1}{z^2} \left( \frac{zu - R}{\sqrt{z^2 + R^2 - 2zRu}} \right) \Big|_{-1}^1 \\
&= \frac{1}{z^2} \left( \frac{z - R}{\sqrt{z^2 + R^2 - 2zR}} - \frac{-z - R}{\sqrt{z^2 + R^2 + 2zR}} \right)
\end{aligned}$$

Taking the positive square root:  $\sqrt{z^2 + R^2 - 2zR} = (R - z)$  if  $R > z$ , but  $(z - R)$  if  $R < z$ . So, for the case  $z < R$  (inside the sphere) the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma R^2}{z^2} \left( \frac{z - R}{R - z} - \frac{-z - R}{R + z} \right) \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma R^2}{z^2} \left( \frac{z - R}{R - z} + \frac{z + R}{R + z} \right) \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma R^2}{z^2} \left( \frac{z - R}{R - z} + 1 \right) \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma R^2}{z^2} \left( \frac{z - R}{R - z} + \frac{R - z}{R - z} \right) \hat{\mathbf{z}} \\
&= 0
\end{aligned}$$

For the case  $z > R$  (outside the sphere) the electric field is

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma R^2}{z^2} \left( \frac{z - R}{z - R} + \frac{z + R}{z + R} \right) \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{4\pi\sigma R^2}{z^2} \hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{\mathbf{z}}
\end{aligned}$$

This makes sense: From outside the sphere, the point charge  $q$  is the charge-per-area  $\sigma$  times the surface area of the sphere  $4\pi R^2$ , or simply  $q = 4\pi R^2 \sigma$ .

**2.8** Finding the field inside and outside a solid sphere of radius  $R$  with a uniform volume charge density  $\rho$  is similar to Prob. 2.7. Outside the solid sphere the total charge  $q$  contributes to the electric field as if it were a point charge:

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

Inside the solid sphere, only the volume of the solid sphere less than  $r$  contributes to the electric field. The volume of the total sphere is  $V = \frac{4}{3}\pi R^3$ , and the volume of the sphere less than  $r$  is  $V' = \frac{4}{3}\pi r^3$ . So, electric field inside the solid sphere is

$$\begin{aligned}
\mathbf{E}_{in} &= \frac{V'}{V} \mathbf{E}_{out} \\
&= \frac{r^3}{R^3} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r \hat{\mathbf{r}}
\end{aligned}$$

or

$$\mathbf{E}_{in} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \mathbf{r}$$

**2.9** (a) The electric field in some region is  $\mathbf{E} = kr^3\hat{\mathbf{r}}$  in spherical coordinates, where  $k$  is a constant. The differential form of Gauss's law is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

and the radial component of divergence in spherical coordinates is

$$\begin{aligned} [\nabla \cdot \mathbf{E}]_r &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 kr^3) \\ &= 5kr^2 \end{aligned}$$

So, the charge density is

$$\rho = 5\epsilon_0 kr^2$$

(b) The total charge inside a sphere of radius  $R$  is found using Gauss's law:

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{Q}{\epsilon_o} \\ Q &= \epsilon_o \oint \mathbf{E} \cdot d\mathbf{a} \\ &= \epsilon_o \int (kR^3\hat{\mathbf{r}}) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) \\ &= \epsilon_o \int_0^{2\pi} \int_0^\pi (kR^5 \sin\theta) \, d\theta \, d\phi \\ &= 4\pi\epsilon_o kR^5 \end{aligned}$$

or using Gauss's theorem:

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \int (\nabla \cdot \mathbf{E}) \, d\tau \\ Q &= \epsilon_o \int_0^{2\pi} \int_0^\pi \int_0^R 5kr^2(r^2 \sin\theta) \, dr \, d\theta \, d\phi \\ &= 4\pi\epsilon_o kR^5 \end{aligned}$$

**2.10** For simplicity, using a cube of length 1:

$$y = 1, \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{r^3}, \quad d\mathbf{a} = dx \, dz \, \hat{\mathbf{y}}; \quad \mathbf{E} \cdot d\mathbf{a} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3}$$

the limits of integration are  $x = [0, 1]$  and  $z = [0, 1]$ :

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{4\pi\epsilon_0} q \int \frac{1}{r^3} \, dx \, dz \\ &= \frac{1}{4\pi\epsilon_0} q \int_0^1 \int_0^1 \frac{1}{(x^2 + 1 + z^2)^{3/2}} \, dx \, dz \\ &= \frac{1}{4\pi\epsilon_0} q \int_0^1 \left[ \frac{x}{(1 + z^2)\sqrt{x^2 + 1 + z^2}} \right]_0^1 \, dz \\ &= \frac{1}{4\pi\epsilon_0} q \int_0^1 \frac{1}{(1 + z^2)\sqrt{2 + z^2}} \, dz \\ &= \frac{1}{4\pi\epsilon_0} q \arctan\left(\frac{z}{\sqrt{z^2 + 2}}\right) \Big|_0^1 \\ &= \frac{1}{4\pi\epsilon_0} q \left(\frac{\pi}{6}\right) = \frac{q}{24\epsilon_o} \end{aligned}$$

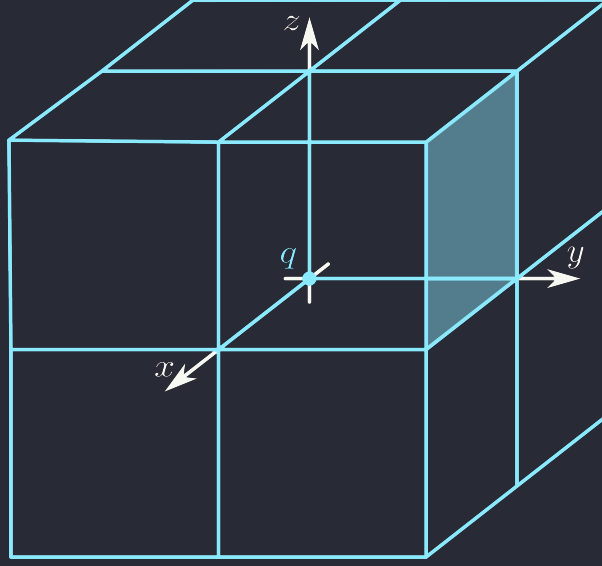


Figure 2.3: 8 cubes with a charge  $q$  at the center.

Where the first integral is solved using the trig identity  $x = \tan(u)\sqrt{z^2 + 1}$ , and similarly, the second integral uses  $z = \tan(u)\sqrt{2}$ .

The simpler solution is through the superposition of 8 cubes with the charge in the center of the larger cube, and the surface that encloses the larger cube is made of 24 squares equivalent to the shaded region as shown in Figure 2.3. Therefore, the flux through the shaded region is  $\frac{1}{24}$  of the total flux  $\frac{q}{\epsilon_o}$ .

**2.11** For a spherical shell of radius  $R$  with a uniform surface charge density  $\sigma$ , the enclosed charge inside the sphere is  $Q_{enc} = 0$ , thus the electric field inside the sphere is

$$\mathbf{E}_i = 0$$

and using the spherical symmetry of a Gaussian surface, the electric field outside the sphere is

$$\begin{aligned}\oint \mathbf{E}_o \cdot d\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}_o| \int d\mathbf{a} &= \frac{1}{\epsilon_o} (4\pi\sigma R^2) \\ \mathbf{E}_o (4\pi r^2) &= \frac{1}{\epsilon_o} (4\pi\sigma R^2) \hat{\mathbf{r}} \\ \mathbf{E}_o &= \frac{\sigma R^2}{\epsilon_o r^2} \hat{\mathbf{r}}\end{aligned}$$

**2.12** Inside a solid sphere, the total charge enclosed in the Gaussian surface is

$$Q_{enc} = V' \rho = \frac{4}{3} \pi r^3 \rho$$

where  $V'$  is the volume of the sphere enclosed by the Gaussian surface. Using Gauss's law,

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_o} Q_{enc} = \frac{1}{\epsilon_o} \frac{4}{3} \pi r^3 \rho$$

using the spherical symmetry of the Gaussian surface, the electric field is

$$\oint \mathbf{E} \cdot d\mathbf{a} = |\mathbf{E}| \int da = |\mathbf{E}| (4\pi r^2)$$

Thus

$$|\mathbf{E}|(4\pi r^2) = \frac{1}{\epsilon_o} \frac{4}{3}\pi r^3 \rho$$

or

$$\mathbf{E} = \frac{1}{3\epsilon_o} r \rho \hat{\mathbf{r}} = \frac{1}{3\epsilon_o} \rho \mathbf{r}$$

Since the total charge of the solid sphere is  $q = \frac{4}{3}\pi R^3 \rho$ , the electric field can be rewritten as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_o} \frac{q}{R^3} \mathbf{r}$$

which is the same as Prob. 2.8.

**2.13** Finding the electric field a distance  $s$  from an infinitely long straight wire that carries a uniform line charge  $\lambda$ . Using a Gaussian cylinder of radius  $s$  and length  $L$ , enclosed charge is  $Q_{enc} = \lambda L$ . Using Gauss's law and the symmetry of the cylinder,

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int ds' d\phi' dz' &= \frac{1}{\epsilon_o} \lambda L \\ E(2\pi s L) &= \frac{1}{\epsilon_o} \lambda L \end{aligned}$$

or

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o s} \hat{\mathbf{s}} = \frac{1}{4\pi\epsilon_o} \frac{2\lambda}{s} \hat{\mathbf{s}}$$

which is similar to Eq. 2.9.

**2.14** Find the electric field inside a sphere that carries a charge density proportional to the distance from the origin,  $\rho = kr$ , where  $k$  is a constant: The enclosed charge is

$$Q_{enc} = \int \rho d\tau = \int_0^{2\pi} \int_0^\pi \int_0^r kr(r^2 \sin \theta) dr d\theta d\phi = \pi k r^4$$

Using Gauss's law and the spherical symmetry of the Gaussian surface,

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int da &= \frac{1}{\epsilon_o} \pi k r^4 \\ E(4\pi r^2) &= \frac{1}{\epsilon_o} \pi k r^4 \end{aligned}$$

or

$$\mathbf{E} = \frac{1}{4\pi\epsilon_o} \pi k r^2 \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_o} k r \mathbf{r}$$

**2.15** A thick spherical shell with charge density

$$\rho = \frac{k}{r^2} \quad (a \leq r \leq b)$$

The electric field in the three regions:

(i)  $r < a$

$$Q_{enc} = 0; \mathbf{E} = 0$$

(ii)  $a \leq r \leq b$

$$Q_{enc} = \int_0^{2\pi} \int_0^\pi \int_a^r \rho(r^2 \sin \theta) dr d\theta d\phi = 4\pi \int_a^r \frac{k}{r^2}(r^2) dr = 4\pi k(r - a)$$

And from Gauss's law,

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int da &= \frac{1}{\epsilon_o} 4\pi k(r - a) \\ E(4\pi r^2) &= \frac{1}{\epsilon_o} 4\pi k(r - a) \end{aligned}$$

or

$$\mathbf{E} = \frac{k(r - a)}{\epsilon_o r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{4\pi k(r - a)}{r^3} \mathbf{r}$$

(iii)  $r > b$

$$Q_{enc} = \int_0^{2\pi} \int_0^\pi \int_a^b \rho(r^2 \sin \theta) dr d\theta d\phi = 4\pi k(b - a)$$

And from Gauss's law,

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int da &= \frac{1}{\epsilon_o} 4\pi k(b - a) \\ E(4\pi r^2) &= \frac{1}{\epsilon_o} 4\pi k(b - a) \end{aligned}$$

or

$$\mathbf{E} = \frac{k(b - a)}{\epsilon_o r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{4\pi k(b - a)}{r^3} \mathbf{r}$$

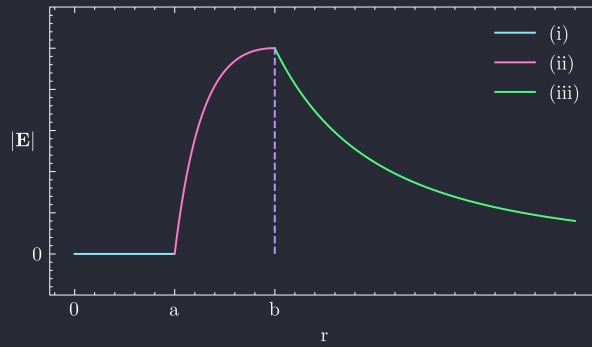


Figure 2.4: Plot of  $|\mathbf{E}|$  as a function of  $r$ , for the case  $b = 2a$ .

**2.16** A long coaxial cable carries a uniform *volume* charge density  $\rho$  on the inner cylinder (radius  $a$ ), and a uniform *surface* charge density  $\sigma$  on the outer cylindrical shell (radius  $b$ ). This surface charge is negative, and the cable as a whole is electrically neutral. Find the electric field in the three regions:  
(i) Inside the inner cylinder  $r < a$ : The enclosed charge is

$$Q_{enc} = \rho \pi s^2 l$$

where  $l$  is the length of the Gaussian cylinder. Using Gauss's law,

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int da &= \frac{1}{\epsilon_o} \rho \pi s^2 l \\ E(2\pi sl) &= \frac{1}{\epsilon_o} \rho \pi s^2 l \\ \mathbf{E} &= \frac{\rho s}{2\epsilon_o} \hat{\mathbf{s}}\end{aligned}$$

(ii) Between the cylinders  $a \leq r \leq b$ : The enclosed charge is

$$Q_{enc} = \rho \pi a^2 l$$

thus the electric field is

$$\begin{aligned}E(2\pi sl) &= \frac{1}{\epsilon_o} \rho \pi a^2 l \\ \mathbf{E} &= \frac{\rho a^2}{2\epsilon_o s} \hat{\mathbf{s}}\end{aligned}$$

(iii) Outside the cable  $r > b$ : The enclosed charge is

$$Q_{enc} = \rho \pi a^2 l - \sigma \pi b l = 0$$

thus the electric field is  $\mathbf{E} = 0$

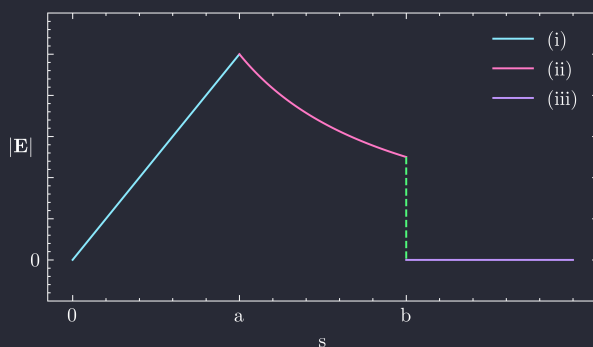


Figure 2.5: Plot of  $|\mathbf{E}|$  as a function of  $r$ , for the case  $b = 2a$ .

**2.17** Finding the electric field, as a function of  $y$ , where  $y = 0$  is the center of an infinite plane slab, of thickness  $2d$ , carrying a uniform volume charge density  $\rho$ . For the case  $y > 2d$  The enclosed charge is

$$Q_{enc} = \rho(2d)A = 2\rho Ad$$

where  $A$  is the area of the Gaussian pillbox. Using Gauss's law,

$$\begin{aligned}\oint \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_o} Q_{enc} \\ |\mathbf{E}| \int da &= \frac{1}{\epsilon_o} 2\rho Ad \\ E(2A) &= \frac{1}{\epsilon_o} 2\rho Ad \\ \mathbf{E} &= \frac{\rho d}{\epsilon_o} \hat{\mathbf{y}}\end{aligned}$$

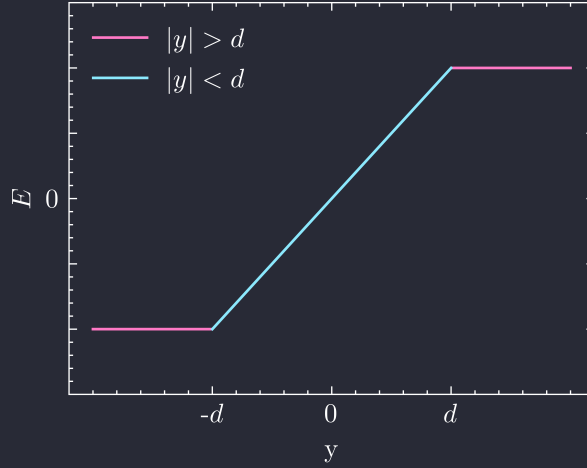


Figure 2.6: Plot of  $|\mathbf{E}|$  as a function of  $y$

For the case  $0 < y < 2d$ , the enclosed charge is

$$Q_{enc} = 2\rho yA$$

and the electric field is

$$E(2A) = \frac{1}{\epsilon_o} \rho yA$$

$$\mathbf{E} = \frac{\rho y}{\epsilon_o} \hat{\mathbf{y}}$$

In the  $-y$  direction,  $E$  is negative as shown in Figure 2.6.

**2.18** For two spheres of radius  $R$  and charge density  $+\rho$  and  $-\rho$ , respectively, are partially overlapping. From Prob 2.12, the electric field inside a sphere of radius  $R$  with a uniform volume charge density  $\rho$  is

$$\mathbf{E} = \frac{1}{3\epsilon_o} \rho \mathbf{r}$$

where  $\mathbf{r}$  is the position vector from the center of the sphere. The electric field for each sphere is

$$\mathbf{E}_1 = \frac{1}{3\epsilon_o} \rho \mathbf{r}_1$$

$$\mathbf{E}_2 = -\frac{1}{3\epsilon_o} \rho \mathbf{r}_2$$

Thus the total electric field is

$$\mathbf{E} = \frac{1}{3\epsilon_o} \rho (\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{1}{3\epsilon_o} \rho \mathbf{d}$$

where  $\mathbf{d}$  is the vector from the positive center to the negative center. Thus the electric field is constant inside the overlapping region.

**2.19** The electric field inside a sphere of radius  $R$  with a uniform volume charge density  $\rho$  is

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\mathbf{r}}}{r^2} \rho(\mathbf{r}') d\tau' \quad (2.8)$$



The curl of (2.8) is

$$\nabla \times \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \times \left( \frac{\hat{\mathbf{z}}}{z^2} \right) \rho(\mathbf{r}') d\tau'$$

From Prob 1.63,  $\nabla \times \frac{\hat{\mathbf{z}}}{z^2} = 0$ , thus

$$\nabla \times \mathbf{E} = 0$$

**2.20** From the conservative nature of the electric field, the curl of the electric field is zero: (a)  $\mathbf{E} = k[xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}]$ ;

$$\nabla \times \mathbf{E}_a = k \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = k[-2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}] \neq 0$$

Thus (a) is not a possible electric field.

(b)  $\mathbf{E}_b = k[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}]$ ;

$$\nabla \times \mathbf{E}_b = k \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix} = 0$$

So (b) is a possible electric field. Finding a potential using the origin as the reference point:

$$V = - \int_O^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \quad (2.21)$$

Setting the path of integration into three parts:

(I) From  $O \rightarrow A = (x, 0, 0)$ ;

$$d\mathbf{l} = dx \hat{\mathbf{x}}; \quad \mathbf{E} \cdot d\mathbf{l} = ky^2 dx = 0; \quad \int_O^A \mathbf{E} \cdot d\mathbf{l} = 0$$

(II) From  $A \rightarrow B = (x, y, 0)$ ;

$$d\mathbf{l} = dy \hat{\mathbf{y}}; \quad \mathbf{E} \cdot d\mathbf{l} = k(2xy + z^2) dy = 2kxy dy; \quad \int_A^B \mathbf{E} \cdot d\mathbf{l} = 2kx \int_0^y y' dy' = kxy^2$$

(III) From  $B \rightarrow C = (x, y, z)$ ;

$$d\mathbf{l} = dz \hat{\mathbf{z}}; \quad \mathbf{E} \cdot d\mathbf{l} = 2kyz; \quad \int_B^C \mathbf{E} \cdot d\mathbf{l} = 2ky \int_0^z z' dz' = kyz^2$$

So the potential is

$$V = - \left( \int_O^A \mathbf{E} \cdot d\mathbf{l} + \int_A^B \mathbf{E} \cdot d\mathbf{l} + \int_B^C \mathbf{E} \cdot d\mathbf{l} \right) = -k(xy^2 + yz^2)$$

Checking the potential function using the gradient:

$$-\nabla V = k(y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}) = \mathbf{E}_b$$

**2.21** Find the potential inside and outside a uniformly charged solid sphere whose radius is  $R$  and whose total charge is  $q$ . Use infinity as your reference point. Compute the gradient of  $V$  in each region, and check that it yields the correct field. Sketch  $V(r)$ .

---

The electric field outside the sphere is

$$\mathbf{E}_{out} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$



where  $\lambda$  is the linear charge density of the wire. Setting the reference point at an arbitrary point  $s_0$ ,

$$\begin{aligned} V(s) &= - \int_{s_0}^s \mathbf{E} \cdot d\mathbf{l} \\ &= - \frac{1}{4\pi\epsilon_0} \int_{s_0}^s \frac{2\lambda}{s'} ds' \\ &= - \frac{1}{4\pi\epsilon_0} 2\lambda \ln\left(\frac{s}{s_0}\right) \end{aligned}$$

And the gradient of  $V$  is

$$-\nabla V = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{s} \hat{\mathbf{s}} = \mathbf{E}$$

**2.23** From Problem 2.15, the electric field in the three regions are:

(i) Inside the inner sphere,  $r < a$ :

$$\mathbf{E}_1 = 0$$

(ii) Between the spheres,  $a \leq r \leq b$ :

$$\mathbf{E}_2 = \frac{k(r-a)}{\epsilon_o r^2} \hat{\mathbf{r}}$$

(iii) Outside the outer sphere,  $r > b$ :

$$\mathbf{E}_3 = \frac{k(b-a)}{\epsilon_o r^2} \hat{\mathbf{r}}$$

To find the potential at the center, the reference point is set to  $\infty$ , and the line element is  $d\mathbf{l} = dr \hat{\mathbf{r}}$ :

$$\begin{aligned} V(r) &= - \int_{\infty}^O \mathbf{E} \cdot d\mathbf{l} \\ &= - \int_{\infty}^b \mathbf{E}_3 \cdot d\mathbf{l} - \int_b^a \mathbf{E}_2 \cdot d\mathbf{l} - \int_a^0 \mathbf{E}_1 \cdot d\mathbf{l} \\ &= - \int_{\infty}^b \frac{k(b-a)}{\epsilon_o r^2} dr - \int_b^a \frac{k(r-a)}{\epsilon_o r^2} dr - \int_a^0 0 dr \\ &= \frac{k}{\epsilon_o} \frac{b-a}{b} - \frac{k}{\epsilon_o} \left( \ln\left(\frac{a}{b}\right) + 1 - \frac{a}{b} \right) \\ &= - \frac{k}{\epsilon_o} \ln\left(\frac{a}{b}\right) \end{aligned}$$

**2.24** From Problem 2.16, the potential difference between a point on the axis ( $s = 0$ ) and a point on the outside cylinder ( $s = b$ ) goes through two distinct electric fields:

(i) Inside the inner cylinder,  $s < a$ :

$$\mathbf{E}_1 = \frac{\rho s}{2\epsilon_o} \hat{\mathbf{s}}$$

(ii) Between the cylinders,  $a \leq s \leq b$ :

$$\mathbf{E}_2 = \frac{\rho a^2}{2\epsilon_o s} \hat{\mathbf{s}}$$

So the potential difference is (using the line element  $d\mathbf{l} = ds \hat{\mathbf{s}}$ ):

$$\begin{aligned}
 V(b) - V(0) &= - \int_0^b \mathbf{E} \cdot d\mathbf{l} \\
 &= - \int_0^a \mathbf{E}_1 \cdot d\mathbf{l} - \int_a^b \mathbf{E}_2 \cdot d\mathbf{l} \\
 &= - \int_0^a \frac{\rho s}{2\epsilon_o} ds - \int_a^b \frac{\rho a^2}{2\epsilon_o s} ds \\
 &= - \frac{\rho}{2\epsilon_o} \left( \int_0^a s ds + a^2 \int_a^b \frac{1}{s} ds \right) \\
 &= - \frac{\rho}{2\epsilon_o} \left( \frac{a^2}{2} + a^2 \ln\left(\frac{b}{a}\right) \right) \\
 &= - \frac{\rho a^2}{4\epsilon_o} \left( 1 + 2 \ln\left(\frac{b}{a}\right) \right)
 \end{aligned}$$