1 Vector Analysis

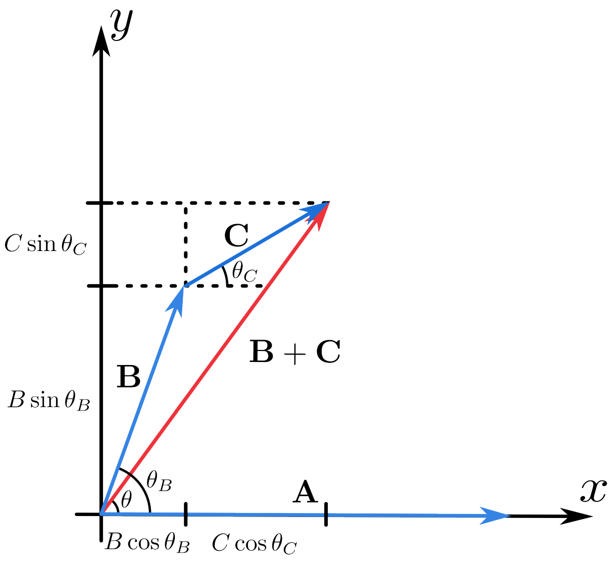

Figure 1.1: Three Coplanar Vectors

1.1 (a) When three vectors are coplanar as shown in Figure 1.1, the dot product is

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$
$$A(B+C)\cos\theta = AB\cos\theta_B + AC\cos\theta_C$$

Since $B\cos\theta_B + B\cos\theta_C = (B+C)\cos\theta$ from Figure 1.1, the distriubtive property holds true. The cross product also holds true since $B\sin\theta_B + B\sin\theta_C = (B+C)\sin\theta$, and multiplying by A on both sides gives the same result as the distriubtive property:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$A(B+C)\sin\theta = AB\sin\theta_B + AC\sin\theta_C$$

(b) In the general case in three-dimensional space, each vector has three components:
$\mathbf{A} = (A_x, A_y, A_z)$. Therefore,

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot(B_x+C_x, B_y+C_y, B_z+C_z)\\ &= A_x(B_x+C_x)+A_y(B_y+C_y)+A_z(B_z+C_z)\\ &= A_xB_x+A_yB_y+A_zB_z+A_xC_x+A_yC_y+A_zC_z\\ &= \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\end{aligned}$$

1.2 Setting $\mathbf{A}=\mathbf{B}=(1,1,1)$ and $\mathbf{C}=(1,1,-1)$:

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times\mathbf{C} &\stackrel{?}{=} \mathbf{A}\times(\mathbf{B}\times\mathbf{C})\\ 0 &\stackrel{?}{=} (1,1,1)\times[(1,1,1)\times(1,1,-1)]\\ 0 &\stackrel{?}{=} (1,1,1)\times(-2,2,0)\\ 0 &\neq (-2,-2,4)\end{aligned}$$

where the cross product of parallel vectors $\mathbf{A}\times\mathbf{B}=0$. Therefore, the cross product is not associative.

1.3 Taking the dot product of a unit cube's body diagonals $\mathbf{A} = (1, 1, 1)$, $\mathbf{B} = (1, 1, -1)$:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= AB \cos\theta \\ 1 &= 3 \cos\theta \\ \theta &= \arccos 1/3 \approx 70.53^\circ \end{aligned}$$

1.4 The cross product of two vectors coplanar to the shaded plane—$\mathbf{A} = (-1, 2, 0)$, $\mathbf{B} = (-1, 0, 3)$—is parallel to the normal unit vector $\hat{\mathbf{n}}$ of the plane:

$$\begin{gathered} \mathbf{A} \times \mathbf{B} = \mathbf{C} \\ \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6, 3, 2) \end{gathered}$$

where $\hat{\mathbf{n}} = \mathbf{C}/C$, and $C = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$. Therefore,

$$\hat{\mathbf{n}} = \frac{1}{7}(6, 3, 2)$$

1.5 Proving the "BAC–CAB" rule for three-dimensional vectors:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \times \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} \end{aligned}$$

where the x component is $A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$. Similarly,

$$\mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B}(A_x C_x + A_y C_y + A_z C_z) - \mathbf{C}(A_x B_x + A_y B_y + A_z B_z)$$

where x component simplifes to

$$B_x(\cancel{A_x C_x} + A_y C_y + A_z C_z) - C_x(\cancel{A_x B_x} + A_y B_y + A_z B_z) = A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)$$

the same is done for the y and z components. Therefore, the "BAC–CAB" rule holds true.

1.6

$$\begin{aligned} [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] &= \cancel{\mathbf{B}(\mathbf{A} \cdot \mathbf{C})} - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}(\mathbf{B} \cdot \mathbf{A}) = 0 \\ &\quad - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{A}(\mathbf{C} \cdot \mathbf{B}) - \cancel{\mathbf{B}(\mathbf{C} \cdot \mathbf{A})} \end{aligned}$$

since dot product is associative, the first and last terms cancel out, and the middle terms also cancel out with each other.

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \\ \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{A}(\mathbf{B} \cdot \mathbf{C}) \\ 0 &= -\mathbf{B}(\mathbf{C} \times \mathbf{A}) = (\mathbf{C} \times \mathbf{A}) \times \mathbf{B} \end{aligned}$$

For the relation to hold true, either the vectors $\mathbf{A}$ and $\mathbf{C}$ are parallel ($\mathbf{A} \times \mathbf{A} = 0$) or $\mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{C}$ ($\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = 0$).

1.7 Finding the seperation vector $\boldsymbol{\mathscr{r}}$:

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' = (4,6,8) - (2,8,7) = (2,-2,1) \\
\mathscr{r} &= \sqrt{2^2 + (-2)^2 + 1^2} = 3 \\
\hat{\boldsymbol{\mathscr{r}}} &= \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)
\end{aligned}$$

1.8 (a)

$$\begin{aligned}
\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z &= (A_y\cos\phi + A_z\sin\phi)(B_y\cos\phi + B_z\sin\phi) \\
&\quad + (-A_y\sin\phi + A_z\cos\phi)(-B_y\sin\phi + B_z\cos\phi) \\
&= A_yB_y\cos^2\phi + A_zB_z\sin^2\phi + \cancel{A_yB_z\sin\phi\cos\phi} + \cancel{A_zB_y\sin\phi\cos\phi} \\
&\quad + A_yB_y\sin^2\phi - \cancel{A_yB_z\sin\phi\cos\phi} - \cancel{A_zB_y\sin\phi\cos\phi} + A_zB_z\cos^2\phi \\
&= A_yB_y(\sin^2\phi + \cos^2\phi) + A_zB_z(\sin^2\phi + \cos^2\phi) \\
\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z &= A_yB_y + A_zB_z
\end{aligned}$$

(b) To preserve length $|\bar{A}| = |A|$. Squaring both sides and expanding gives

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_x^2 + A_y^2 + A_z^2$$

in summation form,

$$\sum_{i=1}^{3}\bar{A}_i\bar{A}_i = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}R_{ij}A_j\right)\left(\sum_{k=1}^{3}R_{ik}A_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}R_{ij}R_{ik}A_jA_k$$

For the length to be preserved, the indices j and k must be equal. Therefore,

$$R_{ij}R_{ik} = \delta_{jk}$$

where δ_{ij} is the Kronecker delta. Thus the length is preserved if the rotation matrix is orthogonal, i.e.

$$R_{ij}R_{ik} = (R^T)_{ji}R_{ik} = \delta_{jk} \quad \text{or} \quad R^TR = I$$

1.9 A clockwise rotation of 120° about an axis through the origin and point $(1,1,1)$ is equivalent to changing the position of the basis vectors $\hat{\mathbf{x}} \to \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$, and $\hat{\mathbf{z}} \to \hat{\mathbf{y}}$. Therefore, the rotation matrix is a permutation matrix:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1.10