# Ch3 Dimensional Analysis In Ecology

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## Fundamental principles

- Dimensional analysis can be used to simplify equations into dimensionless variables.
- This type of analysis only works on dimensionally homogeous equations -> have the same dimensions.

## Example:

Let's estimate the drag experimenced by an object immersed in a current.

Assumptions: - The immersed object is a smooth sphere. - The velocity of the current V is constant.

The drag force F is a function of:

- velocity $V$
- diameter of the sphere $D$
- density of water $\rho$
- dynamic viscosity $\eta$

$F = f(V, D, \rho, \eta)$

*Dimensional analysis allows one to find the form of the equation that relates F to the variables identified as governing the drag*

Some of these variables can be combined to produce dimensionless products:

Reynold's Number =

$$Re = \frac{VL\rho}{\eta} \tag{1}$$

and

Newton Number =

$$Ne = \frac{F}{L^2V^2\rho} \tag{2}$$

Each dimensionless product contains one exclusive variable.

These dimensionless products can be related in unitless relationships. For example, relating Newton's and Reynold's number you end up with an equation for the drag coefficient:

$$C_x = \frac{8}{\pi} f(Re) \tag{3}$$

This equation is valis for any density or dynamic viscocity of any fluid under consideration.

Both of these numbers are dimensionless products.

**Dimensionless product**: A sufficient condition for an equation to be dimensionally homogenous is that it could be resudec to an equation combining dimensionless products -> the dimensions of all the variables cancel each other.

$\Pi$ **Theorem**: If an equation is dimensionally homogenous, it can be resudec to a relationship among the members of a complete set of dimensionless products.

This was relatively easy with the relationship of five variables, but as the number of variables increases, it becomes harder. Here we present a systematic way to calcualte the dimensionless products between a several variables.

Doing the same exaple in a systematic way:

$$\begin{array}{c c} & \begin{array}{ccccc} F & \eta & \rho & L & V \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -3 & 1 & 1 \\ -2 & -1 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Note that the variables are the columns and the units for each variable are the rows. The numbers inside the matrix are the exponents of each variable. So for example, the units of Force $F$ are $ML/T^2$.

In each product, the exponents given to the variables must be such that the result is dimensionless.

According to $\Pi$ theorem then:

$$\Pi = F^{x_1}\eta^{x_2}\rho^{x_3}L^{x_4}V^{x_5} \tag{4}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -3 & 1 & 1 \\ -2 & -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

Following matrix multiplication, we end up with a series of equations:

$$x_1 + x_2 + x_3 = 0 \tag{5}$$

$$x_1 - x_2 - 3x_3 + x_4 + x_5 = 0 \tag{6}$$

$$-2x_1 - x_2 - x_5 = 0 \tag{7}$$

Then what we need to do is to solve these equations. However, they are a set of indeterminate equations since we have 3 equations and 5 unknowns.

We can find the general solution interms of $x_1$ and $x_2$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Using matrix inversion:

$$- \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -2 & -1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Then the simplest approach is to set one unkown to 0 and the other to 1.

$$\begin{array}{c c} & \begin{array}{ccccc} F & \eta & \rho & L & V \end{array} \\ \begin{array}{c} \\ \Pi_1 \\ \Pi_2 \end{array} & \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & -1 & -1 & -1 \end{pmatrix} \end{array}$$

Then the dimensionless products are:

$$\Pi_1 = \frac{F}{L^2 V^2 \rho} \tag{8}$$

= Newton Number

and

$$Re = \frac{\eta}{V L \rho} \tag{9}$$

= inverse of Reynold's Number

## Conclusions

1. Because the left-hand part of the solution matrix is an identity matrix ($I$) the dimensionless products $\Pi$ are independent
2. When partitioning the dimensional matrix, one must isolate the right hand side a matrix that can be inverted (ie. a matrix whose determinand is non-zero)
3. The number of dimensionless products in the complete set is equal to the number of variables isolated on the left-hand side of the dimensional matrix. -> The number of dimensionless producs = total number of variables - rank of the dimensional matrix
4. When the last $n$ columns of a dimensional matrix of order $r$ do not lead to a non-zero determinant, the columns of the matrix must be rearranged so as to obtain a non-zero determinant.