

Boundary layer expansions of the steady MHD equations in a bounded domain

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Abstract

In this paper, we investigate the validity of boundary layer expansions for the MHD system in a rectangle. We describe the solution up to the third order when the tangential magnetic field is much smaller or much larger than the tangential velocity field, thereby extending [3].

Key words: the steady MHD equations, boundary layer expansions, bounded domain.

1 Introduction

1.1 The model

We consider the following steady, incompressible magnetohydrodynamic system

$$\begin{cases} UU_X + VU_Y - HH_X - GH_Y + P_X = \nu_1\varepsilon(U_{XX} + U_{YY}), \\ UV_X + VV_Y - HG_X - GG_Y + P_Y = \nu_2\varepsilon(V_{XX} + V_{YY}), \\ UH_X + VH_Y - HU_X - GU_Y = \nu_3\varepsilon(H_{XX} + H_{YY}), \\ UG_X + VG_Y - HV_X - GV_Y = \nu_4\varepsilon(G_{XX} + G_{YY}), \\ U_X + V_Y = H_X + G_Y = 0, \end{cases} \quad (1.1)$$

in the domain

$$\Omega = \{(X, Y) \mid 0 \leq X \leq L, 0 \leq Y \leq 2\},$$

where (U, V) is the velocity field, (H, G) the magnetic field, P the pressure, $\nu_1\varepsilon$, $\nu_2\varepsilon$ the horizontal and vertical viscosities and $\nu_3\varepsilon$, $\nu_4\varepsilon$ the horizontal and vertical magnetic resistivity coefficients. We assume that Ω is limited by two plates moving with a velocity u_b , namely we consider the following boundary conditions

$$U(X, 0) = U(X, 2) = u_b > 0, \quad V(X, 0) = V(X, 2) = 0,$$

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and

$$(\partial_Y H, G)(X, 0) = (\partial_Y H, G)(X, 2) = 0. \quad (1.2)$$

The boundary conditions at $X = 0$ and $X = L$ will be made precise later.

When $\varepsilon = 0$, (1.1) reduces to the ideal magnetohydrodynamic system

$$\begin{cases} U_0 \partial_X U_0 + V_0 \partial_Y U_0 - H_0 \partial_X H_0 - G_0 \partial_Y H_0 + \partial_X P_0 = 0, \\ U_0 \partial_X V_0 + V_0 \partial_Y V_0 - H_0 \partial_X G_0 - G_0 \partial_Y G_0 + \partial_Y P_0 = 0, \\ U_0 \partial_X H_0 + V_0 \partial_Y H_0 - H_0 \partial_X U_0 - G_0 \partial_Y U_0 = 0, \\ U_0 \partial_X G_0 + V_0 \partial_Y G_0 - H_0 \partial_X V_0 - G_0 \partial_Y V_0 = 0, \\ \partial_X U_0 + \partial_Y V_0 = \partial_X H_0 + \partial_Y G_0 = 0, \end{cases} \quad (1.3)$$

together with the boundary conditions

$$V_0(X, 0) = V_0(X, 2) = 0, \quad G_0(X, 0) = G_0(X, 2) = 0. \quad (1.4)$$

We will consider particular solutions of this system, of the form

$$(U_0, V_0, H_0, G_0) = (u_e^0(Y), 0, h_e^0(Y), 0),$$

in which $u_e^0(\cdot), h_e^0(\cdot)$ are smooth given functions, satisfying $u_e^0(0) = u_e^0(2) \neq u_b$, and symmetric with respect to $Y = 1$, that is, for any $0 \leq Y \leq 1$

$$u_e^0(1 - Y) = u_e^0(1 + Y), \quad h_e^0(1 - Y) = h_e^0(1 + Y).$$

The aim of this article is to construct a sequence of solution to (1.1) which converges to (U_0, V_0, H_0, G_0) as $\varepsilon \rightarrow 0$. As (1.2) and (1.4) are different, we expect that the solution of (1.1) has a boundary layer behavior at $Y = 0$ and $Y = 2$ [30].

When the magnetic field (H, G) identically vanishes, (1.1) reduces to the classical time independent Navier-Stokes equations. For these equations, in $\Omega = [0, L] \times \mathbb{R}_+$, using energy estimates, Guo and Nguyen [15] have proved the validity of Prandtl boundary layer expansions, together with L^∞ norms on the error. This pioneering work has then be extended to the case $\Omega = [0, L] \times [0, 2]$ in [22]. For more results on moving boundaries, please refer to [17–21]. For the no-slip boundary condition, namely in the case $u_b = 0$, we in particular refer to the works of Guo and Iyer [12–14].

For the unsteady MHD equations, Xie, Luo and Li [37] proved the existence of the boundary layer under the non-characteristic boundary conditions in three dimensional case, and obtained that the solution of the viscous MHD equations converges to the solution of ideal MHD equations when the viscosity goes to zero. With the no-slip boundary condition, Wang and Xin [34] studied the initial boundary data problem of the boundary layer in two and three dimensional cases. Liu, Xie and Yang [26] proved the validity of the MHD boundary layer expansions under non-degenerate boundary conditions of the tangential magnetic field. For more results, we refer to the works [1, 2, 7, 9, 11, 16, 23–25, 33, 35, 36, 38] and the references therein.

For the steady MHD equations, Gao, Li and Yao [8] proved high regularity and asymptotic behavior of the Prandtl-Hartmann equations using energy methods in Sobolev spaces. Liu, Yang and Zhang [27] proved the validity of the MHD boundary layer expansions in the case of non-degenerate tangential magnetic field on the half plane. Under an assumption of degeneracy on the tangential magnetic field, Ding, Ji and Lin [5] obtained the stability of Prandtl layer expansions when the outer ideal MHD flow is of the form $(1, 0, \sigma, 0) (\sigma \geq 0)$. When the outer ideal MHD flow is a shear flow, Ding, Lin and Xie [3] established the validity of Prandtl boundary layer expansion

and gave the L^∞ norm estimates of the error terms. Later, Ding, Ji and Lin [4] extended these results to the case of non-shear flows. In addition, Ding and Wang [6] generalized the results of [17] to MHD flows.

In this paper, inspired by the methods used in [22] for the Navier-Stokes equations, we consider the MHD system when the tangential magnetic field is much smaller or much larger than the tangential velocity field, this last case being not studied in [3, 22]. We moreover precise the expansions of [3, 22] by going up to the third order.

We restrict ourselves to symmetric solutions, which satisfy, for $Y \in [0, 1]$,

$$\begin{aligned} U(1+Y) &= U(1-Y), & V(1+Y) &= -V(1-Y), \\ H(1+Y) &= H(1-Y), & G(1+Y) &= -G(1-Y). \end{aligned} \quad (1.5)$$

Using this symmetry, it is sufficient to study the solutions on

$$\Omega_1 = \{(X, Y) \mid 0 \leq X \leq L, 0 \leq Y \leq 1\},$$

with boundary conditions

$$(U, V, \partial_Y H, G)(X, 0) = (u_b, 0, 0, 0), \quad (\partial_Y U, V, \partial_Y H, G)(X, 1) = (0, 0, 0, 0).$$

The first step is to introduce the rescaled variable

$$x = X, \quad y = \frac{Y}{\sqrt{\varepsilon}},$$

and related unknown functions

$$\begin{aligned} U^\varepsilon(x, y) &= U(X, Y), & V^\varepsilon(x, y) &= \frac{V(X, Y)}{\sqrt{\varepsilon}}, \\ H^\varepsilon(x, y) &= H(X, Y), & G^\varepsilon(x, y) &= \frac{G(X, Y)}{\sqrt{\varepsilon}}, & P^\varepsilon(x, y) &= P(X, Y). \end{aligned} \quad (1.6)$$

This leads to

$$\begin{cases} U^\varepsilon U_x^\varepsilon + V^\varepsilon U_y^\varepsilon - H^\varepsilon H_x^\varepsilon - G^\varepsilon H_y^\varepsilon + P_x^\varepsilon = \nu_1(\varepsilon U_{xx}^\varepsilon + U_{yy}^\varepsilon), \\ U^\varepsilon V_x^\varepsilon + V^\varepsilon V_y^\varepsilon - H^\varepsilon G_x^\varepsilon - G^\varepsilon G_y^\varepsilon + \varepsilon^{-1} P_y^\varepsilon = \nu_2(\varepsilon V_{xx}^\varepsilon + V_{yy}^\varepsilon), \\ U^\varepsilon H_x^\varepsilon + V^\varepsilon H_y^\varepsilon - H^\varepsilon U_x^\varepsilon - G^\varepsilon U_y^\varepsilon = \nu_3(\varepsilon H_{xx}^\varepsilon + H_{yy}^\varepsilon), \\ U^\varepsilon G_x^\varepsilon + V^\varepsilon G_y^\varepsilon - H^\varepsilon V_x^\varepsilon - G^\varepsilon V_y^\varepsilon = \nu_4(\varepsilon G_{xx}^\varepsilon + G_{yy}^\varepsilon), \\ U_x^\varepsilon + V_y^\varepsilon = H_x^\varepsilon + G_y^\varepsilon = 0, \end{cases} \quad (1.7)$$

in the domain

$$\Omega_\varepsilon = \{(x, y) \mid 0 \leq x \leq L, 0 \leq y \leq \varepsilon^{-1/2}\},$$

together with the boundary conditions

$$(U^\varepsilon, V^\varepsilon, \partial_y H^\varepsilon, G^\varepsilon)(x, 0) = (u_b, 0, 0, 0), \quad (\partial_y U^\varepsilon, V^\varepsilon, \partial_y H^\varepsilon, G^\varepsilon)(x, \varepsilon^{-1/2}) = (0, 0, 0, 0). \quad (1.8)$$

We now expand the solutions $(U^\varepsilon, V^\varepsilon, H^\varepsilon, G^\varepsilon, P^\varepsilon)$, which leads to in ε

$$\begin{aligned} U^\varepsilon(x, y) &= u_e^0(\sqrt{\varepsilon}y) + u_p^0(x, y) + \sqrt{\varepsilon}u_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}u_p^1(x, y) + \varepsilon u_e^2(x, \sqrt{\varepsilon}y) + \varepsilon u_p^2(x, y) \\ &\quad + \sqrt{\varepsilon}^3 u_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 u_p^3(x, y) + \varepsilon^{\gamma+\frac{3}{2}} u^\varepsilon(x, y), \end{aligned}$$

$$\begin{aligned}
V^\varepsilon(x, y) &= v_p^0(x, y) + v_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}v_p^1(x, y) + \sqrt{\varepsilon}v_e^2(x, \sqrt{\varepsilon}y) + \varepsilon v_p^2(x, y) \\
&\quad + \varepsilon v_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 v_p^3(x, y) + \varepsilon^{\gamma+\frac{3}{2}} v^\varepsilon(x, y), \\
H^\varepsilon(x, y) &= h_e^0(\sqrt{\varepsilon}y) + h_p^0(x, y) + \sqrt{\varepsilon}h_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}h_p^1(x, y) + \varepsilon h_e^2(x, \sqrt{\varepsilon}y) + \varepsilon h_p^2(x, y) \\
&\quad + \sqrt{\varepsilon}^3 h_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 h_p^3(x, y) + \varepsilon^{\gamma+\frac{3}{2}} h^\varepsilon(x, y), \\
G^\varepsilon(x, y) &= g_p^0(x, y) + g_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}g_p^1(x, y) + \sqrt{\varepsilon}g_e^2(x, \sqrt{\varepsilon}y) + \varepsilon g_p^2(x, y) \\
&\quad + \varepsilon g_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 g_p^3(x, y) + \varepsilon^{\gamma+\frac{3}{2}} g^\varepsilon(x, y), \\
P^\varepsilon(x, y) &= \sqrt{\varepsilon}p_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}p_p^1(x, y) + \varepsilon p_e^2(x, \sqrt{\varepsilon}y) + \varepsilon p_p^2(x, y) + \sqrt{\varepsilon}^3 p_e^3(x, \sqrt{\varepsilon}y) \\
&\quad + \sqrt{\varepsilon}^3 p_p^3(x, y) + \varepsilon^2 p_p^4(x, y) + \varepsilon^{\gamma+\frac{3}{2}} p^\varepsilon(x, y),
\end{aligned} \tag{1.9}$$

where $(u_e^i, v_e^i, h_e^i, g_e^i, p_e^i)$, $(u_p^i, v_p^i, h_p^i, g_p^i, p_p^i)$ ($i = 0, 1, 2, 3$) denote the interior and boundary layer parts, and $(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon, p^\varepsilon)$ the error terms. These approximate solutions are constructed in Sections 2 to 5.

1.2 Boundary conditions and compatibility conditions

Let us now detail the boundary conditions and compatibility conditions of the various orders of the asymptotic expansion. First, we choose the zeroth order interior flow and magnetic field to be $(u_e^0(Y), 0, h_e^0(Y), 0)$. At $y = 0$, the zeroth order interior flow does not satisfy the boundary condition since, by assumption, $u_e^0(0) := u_e \neq u_b$. Similarly, $h_e^0(0) := h_e \neq 0$. We thus add a boundary layer u_p^0 on the velocity, and a boundary layer h_p^0 on the magnetic field. We have to prescribe the values of u_p^0 and h_p^0 at $x = 0$. For this we choose two smooth and decaying functions \tilde{u}_0 and \tilde{h}_0 .

The boundary conditions on the zeroth order velocity and magnetic boundary layer profiles are thus

$$\begin{aligned}
u_p^0(x, 0) + u_e &= u_b, \quad \partial_y h_p^0(x, 0) = 0, \\
\partial_y u_p^0(x, \varepsilon^{-1/2}) &= v_p^0(x, \varepsilon^{-1/2}) = \partial_y h_p^0(x, \varepsilon^{-1/2}) = g_p^0(x, \varepsilon^{-1/2}) = 0, \\
u_p^0(0, y) &= \tilde{u}_0(y), \quad h_p^0(0, y) = \tilde{h}_0(y).
\end{aligned} \tag{1.10}$$

For the i -th order interior magnetic profiles, the boundary conditions are, for $i = 1, 2, 3$,

$$\begin{aligned}
v_e^i(x, 0) + v_p^{i-1}(x, 0) &= 0, \quad g_e^i(x, 0) + g_p^{i-1}(x, 0) = 0, \quad u_e^i(0, Y) = u_b^i(Y), \\
v_e^i(0, Y) &= V_{b0}^i(Y), \quad h_e^i(0, Y) = h_b^i(Y), \quad g_e^i(0, Y) = G_{b0}^i(Y), \\
v_e^i(L, Y) &= V_{bL}^i(Y), \quad g_e^i(L, Y) = G_{bL}^i(Y), \\
u_{eY}^i(x, 1) &= v_e^i(x, 1) = h_{eY}^i(x, 1) = g_e^i(x, 1) = 0,
\end{aligned} \tag{1.11}$$

where $u_b^i(Y)$, $h_b^i(Y)$, $V_{b0}^i(Y)$, $G_{b0}^i(Y)$, $V_{bL}^i(Y)$ and $G_{bL}^i(Y)$ are given functions.

For the i -th order magnetic boundary layer profiles, we have, for $i = 1, 2, 3$,

$$\begin{aligned}
u_p^i(x, 0) + u_e^i(x, 0) &= 0, \quad \partial_y h_p^i(x, 0) + \partial_Y h_e^{i-1}(0) = 0, \\
\partial_y u_p^i(x, \varepsilon^{-1/2}) &= v_p^i(x, \varepsilon^{-1/2}) = \partial_y h_p^i(x, \varepsilon^{-1/2}) = g_p^i(x, \varepsilon^{-1/2}) = 0, \\
u_p^i(0, y) &= \tilde{u}_i(y), \quad h_p^i(0, y) = \tilde{h}_i(y),
\end{aligned} \tag{1.12}$$

where \tilde{u}_i and \tilde{h}_i are given smooth and rapidly decreasing functions.

Moreover, we also have $v_p^3(x, 0) = g_p^3(x, 0) = 0$ for the third order boundary layer. Finally, the boundary conditions of the remainder terms $(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon, p^\varepsilon)$ are

$$u^\varepsilon(x, 0) = 0, \quad v^\varepsilon(x, 0) = 0, \quad \partial_y h^\varepsilon(x, 0) = 0, \quad g^\varepsilon(x, 0) = 0,$$

$$\begin{aligned}
\partial_y u^\varepsilon(x, \varepsilon^{-1/2}) &= 0, \quad v^\varepsilon(x, \varepsilon^{-1/2}) = 0, \quad \partial_y h^\varepsilon(x, \varepsilon^{-1/2}) = 0, \quad g^\varepsilon(x, \varepsilon^{-1/2}) = 0, \\
u^\varepsilon(0, y) &= 0, \quad v^\varepsilon(0, y) = 0, \quad h^\varepsilon(0, y) = 0, \quad g^\varepsilon(0, y) = 0, \\
(p^\varepsilon - 2\varepsilon u_x^\varepsilon)(L, y) &= 0, \quad (u_y^\varepsilon + \varepsilon v_x^\varepsilon)(L, y) = h^\varepsilon(L, y) = \partial_x g^\varepsilon(L, y) = 0.
\end{aligned} \tag{1.13}$$

To solve this system we need the following compatibility conditions

$$\begin{aligned}
(V_{b0}^i, G_{b0}^i)(0) &= -(v_p^{i-1}, g_p^{i-1})(0, 0), \quad (V_{bL}^i, G_{bL}^i)(0) = -(v_p^{i-1}, g_p^{i-1})(L, 0), \\
V_{b0}^i(1) &= V_{bL}^i(1) = 0, \quad G_{b0}^i(1) = G_{bL}^i(1) = 0.
\end{aligned} \tag{1.14}$$

1.3 Main result

Next, we present the main result in this paper. We will assume that either the tangential velocity is larger with respect to the horizontal magnetic field or that the horizontal magnetic field dominates the tangential velocity. More precisely, we assume one of the following hypothesis, called (H1) and (H2).

- (H1) There exists $C_0 > 0$, such that, for any $0 \leq Y \leq 1$,

$$\begin{aligned}
u_e^0(Y) &\geq C_0 h_e^0(Y), \\
u_e + \tilde{u}_0(y) &> h_e + \tilde{h}_0(y) \geq \theta_0 > 0
\end{aligned}$$

and

$$|u_e^0(\sqrt{\varepsilon}y) + \tilde{u}_0(y)| \geq C_0 |h_e^0(\sqrt{\varepsilon}y) + \tilde{h}_0(y)|. \tag{1.15}$$

- (H2) There exists $C_0 > 0$, such that, for any $0 \leq Y \leq 1$,

$$\begin{aligned}
h_e^0(Y) &\geq C_0 u_e^0(Y), \\
h_e + \tilde{h}_0(y) &> u_e + \tilde{u}_0(y) \geq \theta_0 > 0
\end{aligned}$$

and

$$|h_e^0(\sqrt{\varepsilon}y) + \tilde{h}_0(y)| \geq C_0 |u_e^0(\sqrt{\varepsilon}y) + \tilde{u}_0(y)|. \tag{1.16}$$

Theorem 1.1. *Let $u_b > 0$ and let $u_e^0(Y), h_e^0(Y)$ be given smooth positive functions satisfying $u_{eY}^0(1) = h_{eY}^0(1) = 0$, together with the boundary conditions. Suppose that $\|\langle Y \rangle \partial_Y (u_e^0, h_e^0)\|_{L^\infty} \lesssim \sigma_1$ for some sufficiently small $\sigma_1 > 0$. Assume that either (H1) or (H2) holds true for some C_0 large enough. Assume moreover that, for any $0 \leq Y \leq 1$,*

$$\begin{aligned}
|\langle y \rangle^{l+1} \partial_y (u_e + \tilde{u}_0, h_e + \tilde{h}_0)(y)| &\leq \frac{1}{2} \sigma_0, \\
|\langle y \rangle^{l+1} \partial_y^2 (u_e + \tilde{u}_0, h_e + \tilde{h}_0)(y)| &\leq \frac{1}{2} \theta_0^{-1}
\end{aligned} \tag{1.17}$$

where σ_0 and θ_0 are small enough. Then there exists a constant $L_0 > 0$, which depends only on the prescribed data, such that for $0 < L \leq L_0$ and $\gamma \in (0, \frac{1}{16})$, provided ε is small enough, there exists a solution of the form (1.9) to (1.7) on Ω_ε . Moreover, $(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)$ satisfy

$$\begin{aligned}
&\|\nabla_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon v^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon h^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon g^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&+ \varepsilon^{\frac{\gamma}{8}} \|u^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{1}{2} + \frac{\gamma}{8}} \|v^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{\gamma}{8}} \|h^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{1}{2} + \frac{\gamma}{8}} \|g^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C_1,
\end{aligned} \tag{1.18}$$

where the positive constant C_1 depends only on norms of u_e^0 and h_e^0 .

1.4 Notations

Throughout this paper, we use the notations $\langle y \rangle = \sqrt{1+y^2}$ and $I_\varepsilon = [0, \varepsilon^{-1/2}]$. Moreover, $\bar{f} = f(x, 0)$ denotes the boundary value of a function f at $y = 0$ in Sections 2 to 5. For $l \in \mathbb{R}$, we introduce the following L^2 weighted space

$$L_l^2 := \left\{ f(x, y) : [0, L] \times [0, \infty) \rightarrow \mathbb{R}, \|f\|_{L_l^2}^2 = \int_0^\infty \langle y \rangle^{2l} |f(y)|^2 dy < \infty \right\}.$$

For $\alpha = (k_1, k_2) \in \mathbb{N}^2$, we define the Sobolev spaces

$$H_l^m := \left\{ f(x, y) : [0, L] \times [0, \infty) \rightarrow \mathbb{R}, \|f\|_{H_l^m}^2 < \infty \right\}$$

and denote its norm by

$$\|f\|_{H_l^m}^2 = \sum_{|\alpha| \leq m} \|\langle y \rangle^{l+k_2} D^\alpha f\|_{L_y^2(0, \infty)}^2,$$

where $D^\alpha = \partial_x^{k_1} \partial_y^{k_2}$.

2 The zeroth order magnetic boundary layer profile

In this section, we construct the zeroth order magnetic boundary layer. Let us first define

$$\left\{ \begin{array}{l} u_{app}(x, y) = u_e^0(\sqrt{\varepsilon}y) + u_p^0(x, y) + \sqrt{\varepsilon}u_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}u_p^1(x, y) + \varepsilon u_e^2(x, \sqrt{\varepsilon}y) + \varepsilon u_p^2(x, y) \\ \quad + \sqrt{\varepsilon}^3 u_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 u_p^3(x, y), \\ v_{app}(x, y) = v_p^0(x, y) + v_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}v_p^1(x, y) + \sqrt{\varepsilon}v_e^2(x, \sqrt{\varepsilon}y) + \varepsilon v_p^2(x, y) \\ \quad + \varepsilon v_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 v_p^3(x, y), \\ h_{app}(x, y) = h_e^0(\sqrt{\varepsilon}y) + h_p^0(x, y) + \sqrt{\varepsilon}h_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}h_p^1(x, y) + \varepsilon h_e^2(x, \sqrt{\varepsilon}y) + \varepsilon h_p^2(x, y) \\ \quad + \sqrt{\varepsilon}^3 h_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 h_p^3(x, y), \\ g_{app}(x, y) = g_p^0(x, y) + g_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}g_p^1(x, y) + \sqrt{\varepsilon}g_e^2(x, \sqrt{\varepsilon}y) + \varepsilon g_p^2(x, y) \\ \quad + \varepsilon g_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 g_p^3(x, y), \\ p_{app}(x, y) = \sqrt{\varepsilon}p_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}p_p^1(x, y) + \varepsilon p_e^2(x, \sqrt{\varepsilon}y) + \varepsilon p_p^2(x, y) \\ \quad + \sqrt{\varepsilon}^3 p_e^3(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}^3 p_p^3(x, y) + \varepsilon^2 P_p^4. \end{array} \right. \quad (2.1)$$

Putting (1.9) into (1.7) and matching order of ε , we obtain

$$\left\{ \begin{array}{l} R_{app}^1 + \varepsilon^{\gamma+\frac{3}{2}} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) u_{app} + (u_{app} \partial_x + v_{app} \partial_y) u^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) h_{app} + p_x^\varepsilon \\ \quad - \nu_1 \Delta_\varepsilon u^\varepsilon - (h_{app} \partial_x + g_{app} \partial_y) h^\varepsilon] + \varepsilon^{2\gamma+3} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) u^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) h^\varepsilon] = 0, \\ R_{app}^2 + \varepsilon^{\gamma+\frac{3}{2}} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) v_{app} + (u_{app} \partial_x + v_{app} \partial_y) v^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) g_{app} + \frac{p_y^\varepsilon}{\varepsilon} \\ \quad - \nu_2 \Delta_\varepsilon v^\varepsilon - (h_{app} \partial_x + g_{app} \partial_y) g^\varepsilon] + \varepsilon^{2\gamma+3} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) v^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) g^\varepsilon] = 0, \\ R_{app}^3 + \varepsilon^{\gamma+\frac{3}{2}} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) h_{app} + (u_{app} \partial_x + v_{app} \partial_y) h^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) u_{app} - \nu_3 \Delta_\varepsilon h^\varepsilon \\ \quad - (h_{app} \partial_x + g_{app} \partial_y) u^\varepsilon] + \varepsilon^{2\gamma+3} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) h^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) u^\varepsilon] = 0, \\ R_{app}^4 + \varepsilon^{\gamma+\frac{3}{2}} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) g_{app} + (u_{app} \partial_x + v_{app} \partial_y) g^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) v_{app} - \nu_4 \Delta_\varepsilon g^\varepsilon \\ \quad - (h_{app} \partial_x + g_{app} \partial_y) v^\varepsilon] + \varepsilon^{2\gamma+3} [(u^\varepsilon \partial_x + v^\varepsilon \partial_y) g^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) v^\varepsilon] = 0, \end{array} \right. \quad (2.2)$$

and the following remainder terms

$$\begin{aligned}
R_{app}^1 &:= (u_{app}\partial_x + v_{app}\partial_y)u_{app} - (h_{app}\partial_x + g_{app}\partial_y)h_{app} + \partial_x p_{app} - \nu_1 \Delta_\varepsilon u_{app}, \\
R_{app}^2 &:= (u_{app}\partial_x + v_{app}\partial_y)v_{app} - (h_{app}\partial_x + g_{app}\partial_y)g_{app} + \frac{\partial_y p_{app}}{\varepsilon} - \nu_2 \Delta_\varepsilon v_{app}, \\
R_{app}^3 &:= (u_{app}\partial_x + v_{app}\partial_y)h_{app} - (h_{app}\partial_x + g_{app}\partial_y)u_{app} - \nu_3 \Delta_\varepsilon h_{app}, \\
R_{app}^4 &:= (u_{app}\partial_x + v_{app}\partial_y)g_{app} - (h_{app}\partial_x + g_{app}\partial_y)v_{app} - \nu_4 \Delta_\varepsilon g_{app}.
\end{aligned} \tag{2.3}$$

We now construct the zeroth order terms $(u_p^0, v_p^0, h_p^0, g_p^0, 0)$. The zeroth order terms of (2.1) are

$$\begin{aligned}
R_0^1 &= [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](u_e^0 + u_p^0) \\
&\quad - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](h_e^0 + h_p^0) - \nu_1 \partial_y^2 (u_e^0 + u_p^0), \\
R_0^3 &= [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](h_e^0 + h_p^0) \\
&\quad - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](u_e^0 + u_p^0) - \nu_3 \partial_y^2 (h_e^0 + h_p^0), \\
R_0^4 &= [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](g_p^0 + g_e^1) \\
&\quad - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](v_p^0 + v_e^1) - \nu_4 \partial_y^2 (g_p^0 + g_e^1).
\end{aligned} \tag{2.4}$$

We have

$$\begin{aligned}
(v_p^0 + v_e^1)\partial_y u_e^0 &= \sqrt{\varepsilon}(v_p^0 + v_e^1)\partial_Y u_e^0, & (g_p^0 + g_e^1)\partial_y h_e^0 &= \sqrt{\varepsilon}(g_p^0 + g_e^1)\partial_Y h_e^0, \\
(v_p^0 + v_e^1)\partial_y h_e^0 &= \sqrt{\varepsilon}(v_p^0 + v_e^1)\partial_Y h_e^0, & (g_p^0 + g_e^1)\partial_y u_e^0 &= \sqrt{\varepsilon}(g_p^0 + g_e^1)\partial_Y u_e^0, \\
(v_p^0 + v_e^1)\partial_y g_e^1 &= \sqrt{\varepsilon}(v_p^0 + v_e^1)\partial_Y g_e^1, & (g_p^0 + g_e^1)\partial_y v_e^1 &= \sqrt{\varepsilon}(g_p^0 + g_e^1)\partial_Y v_e^1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&u_e^0 \partial_x u_p^0 + v_e^1 \partial_y u_p^0 - h_e^0 \partial_x h_p^0 - g_e^1 \partial_y h_p^0 \\
&= u_e \partial_x u_p^0 + \overline{v_e^1} \partial_y u_p^0 + \sqrt{\varepsilon} y (u_{eY}^0 (\sqrt{\varepsilon} y) \partial_x u_p^0 + v_{eY}^1 (x, \sqrt{\varepsilon} y) \partial_y u_p^0) \\
&\quad - h_e \partial_x h_p^0 - \overline{g_e^1} \partial_y h_p^0 - \sqrt{\varepsilon} y (h_{eY}^0 (\sqrt{\varepsilon} y) \partial_x h_p^0 + g_{eY}^1 (x, \sqrt{\varepsilon} y) \partial_y h_p^0) + E_1, \\
&u_e^0 \partial_x h_p^0 + v_e^1 \partial_y h_p^0 - h_e^0 \partial_x u_p^0 - g_e^1 \partial_y u_p^0 \\
&= u_e \partial_x h_p^0 + \overline{v_e^1} \partial_y h_p^0 + \sqrt{\varepsilon} y (u_{eY}^0 (\sqrt{\varepsilon} y) \partial_x h_p^0 + v_{eY}^1 (x, \sqrt{\varepsilon} y) \partial_y h_p^0) \\
&\quad - h_e \partial_x u_p^0 - \overline{g_e^1} \partial_y u_p^0 - \sqrt{\varepsilon} y (h_{eY}^0 (\sqrt{\varepsilon} y) \partial_x u_p^0 + g_{eY}^1 (x, \sqrt{\varepsilon} y) \partial_y u_p^0) + E_3,
\end{aligned}$$

and

$$\begin{aligned}
&u_e^0 \partial_x g_p^0 + v_e^1 \partial_y g_p^0 + u_p^0 \partial_x g_e^1 - h_e^0 \partial_x v_p^0 - g_e^1 \partial_y v_p^0 - h_p^0 \partial_x v_e^1 \\
&= u_e \partial_x g_p^0 + \overline{v_e^1} \partial_y g_p^0 + \sqrt{\varepsilon} y (u_{eY}^0 (\sqrt{\varepsilon} y) \partial_x g_p^0 + v_{eY}^1 (x, \sqrt{\varepsilon} y) \partial_y g_p^0) \\
&\quad - h_e \partial_x v_p^0 - \overline{g_e^1} \partial_y v_p^0 - \sqrt{\varepsilon} y (h_{eY}^0 (\sqrt{\varepsilon} y) \partial_x v_p^0 + g_{eY}^1 (x, \sqrt{\varepsilon} y) \partial_y v_p^0) \\
&\quad + u_p^0 \partial_x g_e^1 - h_p^0 \partial_x v_e^1 + E_4,
\end{aligned}$$

where $u_e = u_e^0(0)$, $h_e = h_e^0(0)$ and

$$\begin{aligned}
E_1 &= \varepsilon \partial_x u_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 u_e^0 (\sqrt{\varepsilon} \tau) d\tau dy_1 + \varepsilon \partial_y u_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 v_e^1 (x, \sqrt{\varepsilon} \tau) d\tau dy_1 \\
&\quad - \varepsilon \partial_x h_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 h_e^0 (\sqrt{\varepsilon} \tau) d\tau dy_1 - \varepsilon \partial_y h_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 g_e^1 (x, \sqrt{\varepsilon} \tau) d\tau dy_1,
\end{aligned}$$

$$\begin{aligned}
E_3 &= \varepsilon \partial_x h_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 u_e^0(\sqrt{\varepsilon}\tau) d\tau dy_1 + \varepsilon \partial_y h_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 v_e^1(x, \sqrt{\varepsilon}\tau) d\tau dy_1 \\
&\quad - \varepsilon \partial_x u_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 h_e^0(\sqrt{\varepsilon}\tau) d\tau dy_1 - \varepsilon \partial_y u_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 g_e^1(x, \sqrt{\varepsilon}\tau) d\tau dy_1, \\
E_4 &= \varepsilon \partial_x g_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 u_e^0(\sqrt{\varepsilon}\tau) d\tau dy_1 + \varepsilon \partial_y g_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 v_e^1(x, \sqrt{\varepsilon}\tau) d\tau dy_1 \\
&\quad - \varepsilon \partial_x v_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 h_e^0(\sqrt{\varepsilon}\tau) d\tau dy_1 - \varepsilon \partial_y v_p^0 \int_0^y \int_y^{y_1} \partial_Y^2 g_e^1(x, \sqrt{\varepsilon}\tau) d\tau dy_1 \\
&\quad + \sqrt{\varepsilon} u_p^0 \int_0^y \partial_{xY} g_e^1(x, \sqrt{\varepsilon}\tau) d\tau - \sqrt{\varepsilon} h_p^0 \int_0^y \partial_{xY} v_e^1(x, \sqrt{\varepsilon}\tau) d\tau.
\end{aligned} \tag{2.5}$$

This leads to

$$\left\{ \begin{array}{l}
(u_e + u_p^0) \partial_x u_p^0 + (v_p^0 + \overline{v_e^1}) \partial_y u_p^0 - (h_e + h_p^0) \partial_x h_p^0 - (g_p^0 + \overline{g_e^1}) \partial_y h_p^0 = \nu_1 \partial_y^2 u_p^0, \\
(u_e + u_p^0) \partial_x h_p^0 + (v_p^0 + \overline{v_e^1}) \partial_y h_p^0 - (h_e + h_p^0) \partial_x u_p^0 - (g_p^0 + \overline{g_e^1}) \partial_y u_p^0 = \nu_3 \partial_y^2 h_p^0, \\
(u_e + u_p^0) \partial_x (g_p^0 + \overline{g_e^1}) + (v_p^0 + \overline{v_e^1}) \partial_y (g_p^0 + \overline{g_e^1}) \\
\quad - (h_e + h_p^0) \partial_x (v_p^0 + \overline{v_e^1}) - (g_p^0 + \overline{g_e^1}) \partial_y (v_p^0 + \overline{v_e^1}) = \nu_4 \partial_y^2 g_p^0, \\
(v_p^0, g_p^0)(x, y) = \int_y^{\varepsilon^{-1/2}} \partial_x(u_p^0, h_p^0)(x, z) dz, \\
(v_e^1, g_e^1)(x, 0) = - \int_0^{\varepsilon^{-1/2}} \partial_x(u_p^0, h_p^0)(x, z) dz, \\
(u_p^0, \partial_y h_p^0)(x, 0) = (u_b - u_e, 0), \\
(v_p^0, g_p^0)(x, 0) = -(v_e^1, g_e^1)(x, 0), (u_p^0, h_p^0)(0, y) = (\tilde{u}_0(y), \tilde{h}_0(y)).
\end{array} \right. \tag{2.6}$$

We note that the third equality can be deduced from the second equality and the boundary conditions. Then, the zeroth order terms R_0^1, R_0^3 and R_0^4 equal

$$\left\{ \begin{array}{l}
R_0^1 = \sqrt{\varepsilon} (v_p^0 + v_e^1) \partial_Y u_e^0 + \sqrt{\varepsilon} y (\partial_Y u_e^0(\sqrt{\varepsilon}y) \partial_x u_p^0 + \partial_Y v_e^1 \partial_y u_p^0) - \nu_1 \varepsilon \partial_Y^2 u_e^0 \\
\quad - \sqrt{\varepsilon} (g_p^0 + g_e^1) \partial_Y h_e^0 - \sqrt{\varepsilon} y (\partial_Y h_e^0(\sqrt{\varepsilon}y) \partial_x h_p^0 + \partial_Y g_e^1 \partial_y h_p^0) + E_1, \\
R_0^3 = \sqrt{\varepsilon} (v_p^0 + v_e^1) \partial_Y h_e^0 + \sqrt{\varepsilon} y (\partial_Y u_e^0(\sqrt{\varepsilon}y) \partial_x h_p^0 + \partial_Y v_e^1 \partial_y h_p^0) - \nu_3 \varepsilon \partial_Y^2 h_e^0 \\
\quad - \sqrt{\varepsilon} (g_p^0 + g_e^1) \partial_Y u_e^0 - \sqrt{\varepsilon} y (\partial_Y h_e^0(\sqrt{\varepsilon}y) \partial_x u_p^0 + \partial_Y g_e^1 \partial_y u_p^0) + E_3, \\
R_0^4 = \sqrt{\varepsilon} y (\partial_Y u_e^0(\sqrt{\varepsilon}y) \partial_x g_p^0 + \partial_Y v_e^1 \partial_y g_p^0) - \nu_4 \varepsilon \partial_Y^2 g_e^1 \\
\quad - \sqrt{\varepsilon} y (\partial_Y h_e^0(\sqrt{\varepsilon}y) \partial_x v_p^0 + \partial_Y g_e^1 \partial_y v_p^0) + E_4.
\end{array} \right. \tag{2.7}$$

Step 1: We first solve (2.6) on $[0, L] \times \mathbb{R}_+$ and denote by $(u_p^{0,\infty}, v_p^{0,\infty}, h_p^{0,\infty}, g_p^{0,\infty})$ the corresponding solution.

Proposition 2.1. *Let $m \geq 9$ and $l \geq 0$ and let u_e^0 and h_e^0 be smooth functions satisfying the assumptions of Theorem 1.1. Assume moreover that $\partial_Y u_e^0, \partial_Y h_e^0$ and their derivatives decay fast at infinity. Then, for $0 < L_1 \leq L$, the solutions $u_p^{0,\infty}, v_p^{0,\infty}, h_p^{0,\infty}$ and $g_p^{0,\infty}$ to the system (2.6) satisfy*

$$\sup_{0 \leq x \leq L_1} \|\langle y \rangle^l D^\alpha(u_p^{0,\infty}, v_p^{0,\infty}, h_p^{0,\infty}, g_p^{0,\infty})\|_{L^2(0,\infty)} \leq C(l, m, \tilde{u}_0, \tilde{h}_0), \tag{2.8}$$

for any α, l, m with $|\alpha| \leq m$.

Proof. Let us introduce

$$\begin{cases} u = u_p^{0,\infty} + u_e - u_e\phi(y) - u_b(1 - \phi(y)), \\ v = v_e^1(x, 0) + v_p^{0,\infty}, \\ h = h_p^{0,\infty} + h_e - h_e\phi(y), \\ g = g_e^1(x, 0) + g_p^{0,\infty}, \end{cases} \quad (2.9)$$

in which $\phi(y)$ is the following cut-off function

$$\phi(y) = \begin{cases} 1, & y \geq 2R_0, \\ 0, & y \in [0, R_0], \end{cases} \quad (2.10)$$

where $R_0 > 0$. Putting (2.9) into (2.6), we obtain

$$\begin{cases} [(u + u_e\phi(y) + u_b(1 - \phi(y)))\partial_x + v\partial_y]u - [(h + h_e\phi(y))\partial_x + g\partial_y]h \\ \quad - \nu_1\partial_y^2u - gh_e\phi'(y) + v(u_e - u_b)\phi'(y) = a_1, \\ [(u + u_e\phi(y) + u_b(1 - \phi(y)))\partial_x + v\partial_y]h - [(h + h_e\phi(y))\partial_x + g\partial_y]u \\ \quad - \nu_3\partial_y^2h - g(u_e - u_b)\phi'(y) + vh_e\phi'(y) = a_2, \\ \partial_xu + \partial_yv = \partial_xh + \partial_yg = 0, \\ (u, v, \partial_yh, g)|_{y=0} = (0, 0, 0, 0), \\ (u, h) \rightarrow (0, 0) \text{ as } y \rightarrow \infty, \\ (u, h)|_{x=0} = (\tilde{u}_0(y) + (u_e - u_b)(1 - \phi(y)), \tilde{h}_0(y) + h_e(1 - \phi(y))) := (u_0, h_0)(y), \end{cases} \quad (2.11)$$

where

$$a_1 = \nu_1(u_b - u_e)\phi''(y), \quad a_2 = \nu_3h_e\phi''(y).$$

The proposition is a direct consequence of the following lemma. \square

Lemma 2.2. *Under the assumptions of Proposition and Theorem 1.1, there exists smooth solution (u, v, h, g) to (2.11), such that*

$$\sup_{0 \leq x \leq L_1} \|(u, h)\|_{H_t^m(0, \infty)} + \|\partial_y(u, h)\|_{L^2(0, L_1; H_t^m(0, \infty))} \leq C(l, m, u_0, h_0), \quad (2.12)$$

where $C(l, m, u_0, h_0)$ depends only on l, m, u_0, h_0 . Furthermore, for $(x, y) \in [0, L_1] \times [0, \infty)$,

a) under assumption (H1) and provided C_0 is large enough, $(u_p^{0,\infty}, h_p^{0,\infty})$ satisfies the following estimates

$$\begin{aligned} h_e + h_p^{0,\infty}(x, y) &\geq \frac{\theta_0}{2} > 0, \\ |u_e^0(\sqrt{\varepsilon}y) + u_p^{0,\infty}(x, y)| &\geq C_0 |h_e^0(\sqrt{\varepsilon}y) + h_p^{0,\infty}(x, y)|, \end{aligned}$$

b) under assumption (H2) and provided C_0 is large enough, $(u_p^{0,\infty}, h_p^{0,\infty})$ satisfies

$$\begin{aligned} u_e + u_p^{0,\infty}(x, y) &\geq \frac{\theta_0}{2} > 0, \\ |h_e^0(\sqrt{\varepsilon}y) + h_p^{0,\infty}(x, y)| &\geq C_0 |u_e^0(\sqrt{\varepsilon}y) + u_p^{0,\infty}(x, y)|. \end{aligned}$$

Moreover, in both cases,

$$\left| \langle y \rangle^{l+1} \partial_y (u_e + u_p^{0,\infty}, h_e + h_p^{0,\infty}) (x, y) \right| \leq \sigma_0,$$

$$\left| \langle y \rangle^{l+1} \partial_y^2 (u_e + u_p^{0,\infty}, h_e + h_p^{0,\infty}) (x, y) \right| \leq \theta_0^{-1}.$$

Proof. By using the divergence free conditions, (2.11)₂ may be rewritten

$$\partial_y [v(h + h_e \phi) - g(u + u_e \phi + u_b(1 - \phi))] - \nu_3 \partial_y^2 h = \nu_3 \phi''(y) h_e. \quad (2.13)$$

Integrating (2.13) from 0 to y , we obtain

$$v(h + h_e \phi) - g(u + u_e \phi + u_b(1 - \phi)) - \nu_3 \partial_y h = \nu_3 \phi'(y) h_e, \quad (2.14)$$

in which we have used the following boundary conditions

$$v|_{y=0} = g|_{y=0} = \partial_y h|_{y=0} = \phi'|_{y=0} = 0.$$

We introduce the stream function ψ , such that

$$\psi = \int_0^y h(x, \theta) d\theta, \quad \psi|_{y=0} = 0.$$

Then we can obtain via the divergence free condition

$$\partial_x \psi = -g, \quad \partial_y \psi = h.$$

The equation (2.14) may be rewritten as

$$[(u + u_e \phi + u_b(1 - \phi)) \partial_x + v \partial_y] \psi + v h_e \phi - \nu_3 \partial_y^2 \psi = \nu_3 h_e \phi'. \quad (2.15)$$

We just sketch the proof of the Lemma, and refer to [3] for more details. First, by using standard energy methods, we obtain weighted L^2 estimates on the $(m-1)$ -order tangential derivatives $D^\alpha(u, h)$ of the solutions to the system (2.11). Here, $D^\alpha = \partial_x^{k_1} \partial_y^{k_2}$, $|\alpha| = k_1 + k_2$, $|\alpha| \leq m$, $k_1 \leq m-1$. Second, we obtain weighted L^2 estimates of the m -order tangential derivatives $\partial_x^m(u, h)$. We introduce the new unknowns

$$u^{k_1} := \partial_x^{k_1} u - \frac{\partial_y u + (u_e - u_b) \phi'}{h + h_e \phi} \partial_x^{k_1} \psi, \quad h^{k_1} := \partial_x^{k_1} h - \frac{\partial_y h + h_e \phi'}{h + h_e \phi} \partial_x^{k_1} \psi.$$

By taking $\partial_x^{k_1}$ for (2.11) and (2.15) respectively and combining them, we obtain the equations of u^{k_1} and h^{k_1} , in which the terms involving $\partial_x^{k_1}(v, g)$ vanish. Hence, we obtain weighted L^2 estimates on u^{k_1} and h^{k_1} , using $h + h_e \phi \geq \frac{\theta_0}{2} > 0$ or $u + u_e \phi + u_b(1 - \phi) \geq \frac{\theta_0}{2} > 0$. To derive the estimates on $\partial_x^{k_1}(u, h)$, we need to verify the equivalence of L^2 norms between $\partial_x^{k_1}(u, h)$ and (u^{k_1}, h^{k_1}) for $k_1 = m$. We then obtain the desired estimates of $\partial_x^{k_1}(u, h)$.

Based on all the above estimates, we obtain the existence of the solution (u, v, h, g) to the problem (2.11) through the use of the classical Picard iteration and fixed point theorem. For more details, we refer to [3] and [25]. \square

Step 2: We cut-off the domain from \mathbb{R}_+ to I_ε , and obtain estimates on $[0, L] \times I_\varepsilon$.

Proposition 2.3. (Estimates on $[0, L] \times I_\varepsilon$) Under the assumptions in Theorem 1.1, there exists smooth functions $(u_p^0, v_p^0, h_p^0, g_p^0)$ on Ω_ε , which satisfy the following inhomogeneous system

$$\begin{cases} (u_e + u_p^0)u_{px}^0 + (v_p^0 + v_e^1(x, 0))u_{py}^0 - (h_e + h_p^0)h_{px}^0 - (g_p^0 + g_e^1(x, 0))h_{py}^0 - \nu_1 u_{pyy}^0 = R_p^{1,0}, \\ (u_e + u_p^0)h_{px}^0 + (v_p^0 + v_e^1(x, 0))h_{py}^0 - (h_e + h_p^0)u_{px}^0 - (g_p^0 + g_e^1(x, 0))u_{py}^0 - \nu_3 h_{pyy}^0 = R_p^{3,0}, \\ u_{px}^0 + v_{py}^0 = h_{px}^0 + g_{py}^0 = 0, \\ (u_p^0, \partial_y h_p^0)(x, 0) = (u_b - u_e, 0), u_{py}^0(x, \varepsilon^{-1/2}) = h_{py}^0(x, \varepsilon^{-1/2}) = 0, \\ (v_p^0 + v_e^1)(x, 0) = (g_p^0 + g_e^1)(x, 0) = 0, v_p^0(x, \varepsilon^{-1/2}) = g_p^0(x, \varepsilon^{-1/2}) = 0, \end{cases} \quad (2.16)$$

where $R_p^{1,0}$ and $R_p^{3,0}$ are higher order terms of $\sqrt{\varepsilon}$. and such that for any given $l, m \in \mathbb{N}$,

$$\sup_{x \in [0, L]} \|\langle y \rangle^l D^\alpha (u_p^0, v_p^0, h_p^0, g_p^0)\|_{L^2(I_\varepsilon)} \leq C(l, m, \tilde{u}_0, \tilde{h}_0). \quad (2.17)$$

Proof. $u_p^{0,\infty}$ and $h_p^{0,\infty}$ have already been constructed in Proposition 2.1, and $v_p^{0,\infty}$ and $g_p^{0,\infty}$ are directly obtained through the divergence free conditions. Next, we define

$$\begin{cases} u_p^0(x, y) := \chi(\sqrt{\varepsilon}y)u_p^{0,\infty}(x, y) - \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_y^\infty u_p^{0,\infty}(x, \theta) d\theta, \\ v_p^0(x, y) := \chi(\sqrt{\varepsilon}y)v_p^{0,\infty}(x, y), \\ h_p^0(x, y) := \chi(\sqrt{\varepsilon}y)h_p^{0,\infty}(x, y) - \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_y^\infty h_p^{0,\infty}(x, \theta) d\theta, \\ g_p^0(x, y) := \chi(\sqrt{\varepsilon}y)g_p^{0,\infty}(x, y), \end{cases} \quad (2.18)$$

where $\chi(\cdot)$ is a cut-off function with support in $[0, 1]$. It is easy to check that $(u_p^0, v_p^0, h_p^0, g_p^0)$ satisfy (2.16) with

$$\begin{aligned} R_p^{1,0} &= \sqrt{\varepsilon}\chi \int_0^y \chi' d\theta (u_p^{0,\infty}u_{px}^{0,\infty} + v_p^{0,\infty}u_{py}^{0,\infty} - h_p^{0,\infty}h_{px}^{0,\infty} - g_p^{0,\infty}h_{py}^{0,\infty}) - \sqrt{\varepsilon}\chi' \chi(u_{px}^{0,\infty} \int_y^\infty u_p^{0,\infty} d\theta \\ &\quad - h_{px}^{0,\infty} \int_y^\infty h_p^{0,\infty} d\theta) - \sqrt{\varepsilon}\chi' [v_p^{0,\infty}(u_e + \chi u_p^{0,\infty}) - g_p^{0,\infty}(h_e + \chi h_p^{0,\infty})] - 3\nu_1 \sqrt{\varepsilon}\chi' u_{py}^{0,\infty} \\ &\quad + 2\sqrt{\varepsilon}\chi' (u_p^{0,\infty} \int_0^y \chi v_{py}^{0,\infty} d\theta - h_p^{0,\infty} \int_0^y \chi g_{py}^{0,\infty} d\theta) + 2\varepsilon\chi' (u_p^{0,\infty} \int_0^y \chi' v_p^{0,\infty} d\theta - h_p^{0,\infty} \int_0^y \chi' g_p^{0,\infty} d\theta) \\ &\quad - 3\nu_1 \varepsilon \chi'' u_p^{0,\infty} + \varepsilon(\chi')^2 (v_p^{0,\infty} \int_y^\infty u_p^{0,\infty} d\theta - g_p^{0,\infty} \int_y^\infty h_p^{0,\infty} d\theta) \\ &\quad - \varepsilon\chi'' [(\chi v_p^{0,\infty} - v_p^{0,\infty}(0)) \int_y^\infty u_p^{0,\infty} d\theta - (\chi g_p^{0,\infty} - g_p^{0,\infty}(0)) \int_y^\infty h_p^{0,\infty} d\theta] + \nu_1 \varepsilon^{3/2} \chi''' \int_y^\infty u_p^{0,\infty} d\theta \\ &:= \sqrt{\varepsilon}\Delta_1 + \varepsilon\Delta_2, \\ R_p^{3,0} &= \sqrt{\varepsilon}\chi \int_0^y \chi' d\theta (u_p^{0,\infty}h_{px}^{0,\infty} + v_p^{0,\infty}h_{py}^{0,\infty} - h_p^{0,\infty}u_{px}^{0,\infty} - g_p^{0,\infty}u_{py}^{0,\infty}) - \sqrt{\varepsilon}\chi' \chi(h_{px}^{0,\infty} \int_y^\infty u_p^{0,\infty} d\theta \\ &\quad - u_{px}^{0,\infty} \int_y^\infty h_p^{0,\infty} d\theta) - \sqrt{\varepsilon}\chi' [g_p^{0,\infty}(u_e + \chi u_p^{0,\infty}) - v_p^{0,\infty}(h_e + \chi h_p^{0,\infty})] - 3\nu_3 \sqrt{\varepsilon}\chi' h_{py}^{0,\infty} \\ &\quad + 2\sqrt{\varepsilon}\chi' (h_p^{0,\infty} \int_0^y \chi v_{py}^{0,\infty} d\theta - u_p^{0,\infty} \int_0^y \chi g_{py}^{0,\infty} d\theta) + 2\varepsilon\chi' (h_p^{0,\infty} \int_0^y \chi' v_p^{0,\infty} d\theta - u_p^{0,\infty} \int_0^y \chi' g_p^{0,\infty} d\theta) \end{aligned}$$

$$\begin{aligned}
& -3\nu_3\varepsilon\chi''h_p^{0,\infty} + \varepsilon(\chi')^2(g_p^{0,\infty}\int_y^\infty u_p^{0,\infty}d\theta - v_p^{0,\infty}\int_y^\infty h_p^{0,\infty}d\theta) \\
& -\varepsilon\chi''[(\chi v_p^{0,\infty} - v_p^{0,\infty}(0))\int_y^\infty h_p^{0,\infty}d\theta - (\chi g_p^{0,\infty} - g_p^{0,\infty}(0))\int_y^\infty u_p^{0,\infty}d\theta] + \nu_3\varepsilon^{3/2}\chi''' \int_y^\infty h_p^{0,\infty}d\theta \\
& := \sqrt{\varepsilon}\Delta_3 + \varepsilon\Delta_4,
\end{aligned} \tag{2.19}$$

Finally, we get (2.17) by using (2.8) and the definition of $\chi(\cdot)$. \square

3 The first order inner and boundary layer magnetic profiles

In this section, we construct the first inner profile $(u_e^1, v_e^1, h_e^1, g_e^1, p_e^1)$ and first boundary layer profile $(u_p^1, v_p^1, h_p^1, g_p^1, p_p^1)$.

3.1 The first order inner magnetic profile

Matching the various terms of order ε in (2.2) and $R_p^{1,0}, R_p^{3,0}$, we have

$$\begin{aligned}
R_1^u &= [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](u_e^0 + u_p^0) + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](u_e^1 + u_p^1) \\
& \quad + \partial_x(p_e^1 + p_p^1) - \nu_1\partial_y^2(u_e^1 + u_p^1) + (y\partial_x u_p^0 + v_p^0 + v_e^1)\partial_Y u_e^0 + y\partial_Y v_e^1\partial_y u_p^0 \\
& \quad - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](h_e^0 + h_p^0) - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](h_e^1 + h_p^1) \\
& \quad - (y\partial_x h_p^0 + g_p^0 + g_e^1)\partial_Y h_e^0 - y\partial_Y g_e^1\partial_y h_p^0 + \Delta_1, \\
R_1^h &= [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](h_e^0 + h_p^0) + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](h_e^1 + h_p^1) \\
& \quad - \nu_3\partial_y^2(h_e^1 + h_p^1) + (v_p^0 + v_e^1)\partial_Y h_e^0 + y\partial_Y u_e^0\partial_x h_p^0 + y\partial_Y v_e^1\partial_y h_p^0 \\
& \quad - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](u_e^0 + u_p^0) - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](u_e^1 + u_p^1) \\
& \quad - (g_p^0 + g_e^1)\partial_Y u_e^0 - y\partial_Y h_e^0\partial_x u_p^0 - y\partial_Y g_e^1\partial_y u_p^0 + \Delta_3.
\end{aligned} \tag{3.1}$$

Note that the inner magnetic correctors are always evaluated at $(x, \sqrt{\varepsilon}y)$. Thus, we have

$$\begin{aligned}
(v_p^1 + v_e^2)\partial_y u_e^0 &= \sqrt{\varepsilon}(v_p^1 + v_e^2)\partial_Y u_e^0, & (g_p^1 + g_e^2)\partial_y h_e^0 &= \sqrt{\varepsilon}(g_p^1 + g_e^2)\partial_Y h_e^0, \\
(v_p^0 + v_e^1)\partial_y u_e^1 &= \sqrt{\varepsilon}(v_p^0 + v_e^1)\partial_Y u_e^1, & (g_p^0 + g_e^1)\partial_y h_e^1 &= \sqrt{\varepsilon}(g_p^0 + g_e^1)\partial_Y h_e^1, \\
(v_p^1 + v_e^2)\partial_y h_e^0 &= \sqrt{\varepsilon}(v_p^1 + v_e^2)\partial_Y h_e^0, & (g_p^1 + g_e^2)\partial_y u_e^0 &= \sqrt{\varepsilon}(g_p^1 + g_e^2)\partial_Y u_e^0, \\
(v_p^0 + v_e^1)\partial_y h_e^1 &= \sqrt{\varepsilon}(v_p^0 + v_e^1)\partial_Y h_e^1, & (g_p^0 + g_e^1)\partial_y u_e^1 &= \sqrt{\varepsilon}(g_p^0 + g_e^1)\partial_Y u_e^1, \\
\partial_y^2 u_e^1 &= \varepsilon\partial_Y^2 u_e^1, & \partial_y^2 h_e^1 &= \varepsilon\partial_Y^2 h_e^1.
\end{aligned}$$

By matching the terms of order ε , we obtain the following first order inner magnetic system

$$\begin{cases} u_e^0\partial_x u_e^1 + v_e^1\partial_Y u_e^0 - h_e^0\partial_x h_e^1 - g_e^1\partial_Y h_e^0 + \partial_x p_e^1 = 0, \\ u_e^0\partial_x h_e^1 + v_e^1\partial_Y h_e^0 - h_e^0\partial_x u_e^1 - g_e^1\partial_Y u_e^0 = 0, \end{cases} \tag{3.2}$$

and the first order magnetic boundary layer system

$$\begin{aligned}
& (u_e^1 + u_p^1)\partial_x u_p^0 + u_p^0\partial_x u_e^1 + (u_e^0 + u_p^0)\partial_x u_p^1 + (v_p^1 + v_e^2)\partial_y u_p^0 \\
& \quad + (v_p^0 + v_e^1)\partial_y u_p^1 + \partial_x p_p^1 - \nu_1\partial_y^2 u_p^1 + (y\partial_x u_p^0 + v_p^0)\partial_Y u_e^0 + y\partial_Y v_e^1\partial_y u_p^0 \\
& \quad - (h_e^1 + h_p^1)\partial_x h_p^0 - h_p^0\partial_x h_e^1 - (h_e^0 + h_p^0)\partial_x h_p^1 - (g_p^1 + g_e^2)\partial_y h_p^0
\end{aligned}$$

$$\begin{aligned}
& - (g_p^0 + g_e^1) \partial_y h_p^1 - (y \partial_x h_p^0 + g_p^0) \partial_Y h_e^0 - y \partial_Y g_e^1 \partial_y h_p^0 + \Delta_1 = 0, \\
& (u_e^1 + u_p^1) \partial_x h_p^0 + u_p^0 \partial_x h_e^1 + (u_e^0 + u_p^0) \partial_x h_p^1 + (v_p^1 + v_e^2) \partial_y h_p^0 \\
& + (v_p^0 + v_e^1) \partial_y h_p^1 - \nu_3 \partial_y^2 h_p^1 + v_p^0 \partial_Y h_e^0 + y \partial_Y u_e^0 \partial_x h_p^0 + y \partial_Y v_e^1 \partial_y h_p^0 \\
& - (h_e^1 + h_p^1) \partial_x u_p^0 - h_p^0 \partial_x u_e^1 - (h_e^0 + h_p^0) \partial_x u_p^1 - (g_p^1 + g_e^2) \partial_y u_p^0 \\
& - (g_p^0 + g_e^1) \partial_y u_p^1 - g_p^0 \partial_Y u_e^0 - y \partial_Y h_e^0 \partial_x u_p^0 - y \partial_Y g_e^1 \partial_y u_p^0 + \Delta_3 = 0.
\end{aligned} \tag{3.3}$$

Hence, the remainder R_1^u and R_1^h may be rewritten

$$\begin{aligned}
R_1^u &= \sqrt{\varepsilon} [(v_p^1 + v_e^2) \partial_Y u_e^0 + (v_p^0 + v_e^1) \partial_Y u_e^1 - (g_p^1 + g_e^2) \partial_Y h_e^0 - (g_p^0 + g_e^1) \partial_Y h_e^1] - \nu_1 \varepsilon \partial_Y^2 u_e^1, \\
R_1^h &= \sqrt{\varepsilon} [(v_p^1 + v_e^2) \partial_Y h_e^0 + (v_p^0 + v_e^1) \partial_Y h_e^1 - (g_p^1 + g_e^2) \partial_Y u_e^0 - (g_p^0 + g_e^1) \partial_Y u_e^1] - \nu_3 \varepsilon \partial_Y^2 h_e^1.
\end{aligned} \tag{3.4}$$

Next, we consider the zeroth order terms in R_{app}^2 and R_{app}^4

$$\begin{cases}
R_0^v = [(u_e^0 + u_p^0) \partial_x + (v_p^0 + v_e^1) \partial_y] (v_p^0 + v_e^1) + \partial_Y p_e^1 + \frac{p_{py}^1}{\sqrt{\varepsilon}} + \partial_y p_p^2 \\
\quad - \nu_2 \partial_y^2 (v_p^0 + v_e^1) - [(h_e^0 + h_p^0) \partial_x + (g_p^0 + g_e^1) \partial_y] (g_p^0 + g_e^1), \\
R_0^g = [(u_e^0 + u_p^0) \partial_x + (v_p^0 + v_e^1) \partial_y] (g_p^0 + g_e^1) - \nu_4 \partial_y^2 (g_p^0 + g_e^1) \\
\quad - [(h_e^0 + h_p^0) \partial_x + (g_p^0 + g_e^1) \partial_y] (v_p^0 + v_e^1).
\end{cases} \tag{3.5}$$

The leading order term in R_0^v is p_{py}^1 . We take $p_{py}^1 = 0$, and obtain that p_p^1 only depends on the variable x , that is

$$p_p^1 = p_p^1(x).$$

Next, we match the terms of order ε in $R_0^v = R_0^g = 0$. We obtain that the first order inner magnetic profile $(u_e^1, v_e^1, h_e^1, g_e^1, p_e^1)$ satisfies the following system

$$\begin{cases}
u_e^0 \partial_x v_e^1 - h_e^0 \partial_x g_e^1 + \partial_Y p_e^1 = 0, \\
u_e^0 \partial_x g_e^1 - h_e^0 \partial_x v_e^1 = 0,
\end{cases} \tag{3.6}$$

and the next order boundary layer of pressure p_p^2 takes the form

$$\begin{aligned}
p_p^2(x, y) &= \int_y^{1/\sqrt{\varepsilon}} [(u_e^0 + u_p^0) \partial_x v_p^0 + u_p^0 \partial_x v_e^1 + (v_p^0 + v_e^1) \partial_y v_p^0 \\
&\quad - (h_e^0 + h_p^0) \partial_x g_p^0 - h_p^0 \partial_x g_e^1 - (g_p^0 + g_e^1) \partial_y g_p^0 - \nu_2 \partial_y^2 v_p^0] (x, \theta) d\theta.
\end{aligned} \tag{3.7}$$

Then the error terms R_0^v and R_0^g may be rewritten as follows

$$\begin{cases}
R_0^v = \sqrt{\varepsilon} [(v_p^0 + v_e^1) \partial_Y v_e^1 - (g_p^0 + g_e^1) \partial_Y g_e^1] - \nu_2 \varepsilon \partial_Y^2 v_e^1, \\
R_0^g = \sqrt{\varepsilon} [(v_p^0 + v_e^1) \partial_Y g_e^1 - (g_p^0 + g_e^1) \partial_Y v_e^1] - \nu_4 \varepsilon \partial_Y^2 g_e^1.
\end{cases} \tag{3.8}$$

Here, we only consider the construction of $(u_e^1, v_e^1, h_e^1, g_e^1, p_e^1)$, and detail the $\sqrt{\varepsilon}$ -order inner system

$$\begin{cases}
u_e^0 \partial_x u_e^1 + v_e^1 \partial_Y u_e^0 - h_e^0 \partial_x h_e^1 - g_e^1 \partial_Y h_e^0 + \partial_x p_e^1 = 0, \\
u_e^0 \partial_x v_e^1 - h_e^0 \partial_x g_e^1 + \partial_Y p_e^1 = 0, \\
u_e^0 \partial_x h_e^1 + v_e^1 \partial_Y h_e^0 - h_e^0 \partial_x u_e^1 - g_e^1 \partial_Y u_e^0 = 0, \\
u_e^0 \partial_x g_e^1 - h_e^0 \partial_x v_e^1 = 0, \\
\partial_x u_e^1 + \partial_Y v_e^1 = \partial_x h_e^1 + \partial_Y g_e^1 = 0.
\end{cases} \tag{3.9}$$

with the following boundary conditions

$$\begin{cases} (v_e^1, g_e^1)(x, 0) = -(v_p^0, g_p^0)(x, 0), \\ (v_e^1, g_e^1)(x, 1) = (0, 0), \\ (v_e^1, g_e^1)(0, Y) = (V_{b0}^1, G_{b0}^1)(Y), \\ (v_e^1, g_e^1)(L, Y) = (V_{bL}^1, G_{bL}^1)(Y), \end{cases} \quad (3.10)$$

which enjoy the compatibility assumptions at the corners

$$\begin{cases} (V_{b0}^1, G_{b0}^1)(0) = -(v_p^0, g_p^0)(0, 0), \\ (V_{bL}^1, G_{bL}^1)(1) = (0, 0), \quad (V_{b0}, G_{b0})(1) = (0, 0). \end{cases} \quad (3.11)$$

To solve the problem (3.9)–(3.11), using the divergence free conditions of the velocity and magnetic fields, we rewrite the third and the forth equations of (3.9) respectively, and obtain

$$\partial_Y(u_e^0 g_e^1 - h_e^0 v_e^1) = 0,$$

$$\partial_x(u_e^0 g_e^1 - h_e^0 v_e^1) = 0.$$

Integrating the above two equalities from 0 to Y and from 0 to x respectively yields

$$v_e^1 = \frac{u_e^0}{h_e^0} g_e^1 + \frac{u_e \overline{g_p^0} - h_e \overline{v_p^0} - u_e^0 G_{b0}^1 + h_e^0 V_{b0}^1}{2h_e^0} := \frac{u_e^0}{h_e^0} g_e^1 + \frac{b_1^1(x, Y)}{h_e^0}, \quad (3.12)$$

and

$$g_e^1 = \frac{h_e^0}{u_e^0} v_e^1 + \frac{h_e \overline{v_p^0} - u_e \overline{g_p^0} + u_e^0 G_{b0}^1 - h_e^0 V_{b0}^1}{2u_e^0} := \frac{h_e^0}{u_e^0} v_e^1 + \frac{b_1^2(x, Y)}{u_e^0}, \quad (3.13)$$

where

$$b_1^1(x, Y) = \frac{u_e \overline{g_p^0} - h_e \overline{v_p^0} - u_e^0 G_{b0}^1 + h_e^0 V_{b0}^1}{2}$$

and $b_1^2(x, Y) = -b_1^1(x, Y)$. Using the first, second and fifth equations of (3.9), we obtain

$$-u_e^0 \Delta v_e^1 + \partial_Y^2 u_e^0 \cdot v_e^1 + (h_e^0 \Delta g_e^1 - \partial_Y^2 h_e^0 \cdot g_e^1) = 0. \quad (3.14)$$

To avoid the singularity at the corners of Ω_1 , we consider the following elliptic problem instead of (3.14)

$$-u_e^0 \Delta v_e^1 + \partial_Y^2 u_e^0 \cdot v_e^1 + (h_e^0 \Delta g_e^1 - \partial_Y^2 h_e^0 \cdot g_e^1) = E_b, \quad (3.15)$$

which satisfies the boundary conditions (3.10). We will construct E_b later. To construct E_b , we first introduce

$$\begin{cases} B_v(x, Y) := (1 - \frac{x}{L}) \frac{V_{b0}^1(Y)}{v_p^0(0,0)} v_p^0(x, 0) + \frac{x}{L} \frac{V_{bL}^1(Y)}{v_p^0(L,0)} v_p^0(x, 0), \\ B_g(x, Y) := (1 - \frac{x}{L}) \frac{G_{b0}^1(Y)}{g_p^0(0,0)} g_p^0(x, 0) + \frac{x}{L} \frac{G_{bL}^1(Y)}{g_p^0(L,0)} g_p^0(x, 0), \end{cases} \quad (3.16)$$

where all of $v_p^0(0,0), v_p^0(L,0), g_p^0(0,0), g_p^0(L,0)$ are nonzero. When $v_p^0(0,0)$ or $v_p^0(L,0)$ is zero, $\frac{V_{b0}^1(Y)}{v_p^0(0,0)} v_p^0(x, 0)$ or $\frac{V_{bL}^1(Y)}{v_p^0(L,0)} v_p^0(x, 0)$ should be substituted by $V_{b0}^1(Y) - v_p^0(x, 0)(1 - Y)$ or $V_{bL}^1(Y) -$

$v_p^0(x, 0)(1 - Y)$, respectively. For $g_p^0(0, 0) = 0$ or $g_p^0(L, 0) = 0$, we follow the same arguments. It is easy to show that $B_v(x, Y)$ and $B_g(x, Y)$ satisfy the boundary conditions (3.10). Then, we introduce the smooth function F_e

$$-u_e^0 \Delta B_v + \partial_Y^2 u_e^0 \cdot B_v + (h_e^0 \Delta B_g - \partial_Y^2 h_e^0 \cdot B_g) = F_e. \quad (3.17)$$

As $|\partial_Y^k (V_{bL}^1(Y) - V_{b0}^1(Y), G_{bL}^1(Y) - G_{b0}^1(Y))| \leq CL$, we obtain $B_v, B_g \in W^{k,p}(\Omega_1)$. Therefore, for any $k \geq 0, p \in [1, \infty]$, we have

$$\|F_e\|_{W^{k,p}(\Omega_1)} \leq C,$$

where C is a positive constant.

Let us take $E_b = \chi\left(\frac{Y}{\varepsilon^{\frac{3}{32}}}\right) F_e(x, 0)$ where $\chi(\cdot)$ is a smooth function with support in $[0, 1]$. It follows that

$$\|\partial_Y^k E_b\|_{L^p(\Omega_1)} \leq \varepsilon^{-\frac{3k}{32} + \frac{3}{32p}}.$$

Let

$$v_e^1 = B_v + \omega_1, \quad g_e^1 = B_g + \omega_2.$$

Using (3.15), we obtain the following system

$$\begin{cases} -u_e^0 \Delta \omega_1 + \partial_Y^2 u_e^0 \cdot \omega_1 + (h_e^0 \Delta \omega_2 - \partial_Y^2 h_e^0 \cdot \omega_2) = E_b - F_e, \\ \omega_1 = \frac{u_e^0}{h_e^0}(\omega_2 + B_g) + \frac{b_1^1(x, Y)}{h_e^0} - B_v, \\ \omega_2 = \frac{h_e^0}{u_e^0}(\omega_1 + B_v) + \frac{b_2^1(x, Y)}{u_e^0} - B_g, \\ \omega_i|_{\partial\Omega_1} = 0, \quad i = 1, 2. \end{cases} \quad (3.18)$$

If $u_e^0(Y) \geq C_0 h_e^0(Y)$, we consider the first and third equations of (3.18). If $h_e^0(Y) \geq C_0 u_e^0(Y)$, we consider the first and second equations of (3.18).

Proposition 3.1. *Assume the hypotheses of Theorem 1.1, for any $k \geq 0, p \geq 1$, suppose that $F_e(x, Y) \in W^{k,p}(\Omega_1)$, and that*

$$\left| \partial_Y^k (V_{bL}(Y) - V_{b0}(Y), G_{bL}(Y) - G_{b0}(Y)) \right| \leq CL.$$

Then, for $0 < L_2 \leq L_1$, there exists a unique smooth solution (ω_1, ω_2) of the boundary value problem (3.18), satisfying

$$\begin{aligned} \|(\omega_1, \omega_2)\|_{L^\infty(\Omega_1)} + \|(\omega_1, \omega_2)\|_{H^2(\Omega_1)} &\leq C, \\ \|(\omega_1, \omega_2)\|_{H^{2+j}(\Omega_1)} &\leq C\varepsilon^{-\frac{3j}{32} + \frac{3}{64}}, \quad j = 1, 2, 3, 4, 5, 6, \end{aligned} \quad (3.19)$$

where the constant $C > 0$ only depends the given boundary data and ε . Moreover, it holds that

$$\begin{aligned} \|(\omega_1, \omega_2)\|_{W^{2,q}(\Omega_1)} &\leq C, \\ \|(\omega_1, \omega_2)\|_{W^{2+j,q}(\Omega_1)} &\leq C\varepsilon^{-\frac{3j}{32} + \frac{3}{32q}}, \end{aligned} \quad (3.20)$$

for $q \in (1, \infty)$.

Proof. Case 1: Under the assumption $u_e^0(Y) \geq C_0 h_e^0(Y)$, with C_0 large enough, we first estimate the L^2 norms of the solutions ω_1, ω_2 of (3.18). Multiplying the first equation in (3.18) by $\frac{\omega_1}{u_e^0}$ and integrating by parts, we have

$$\begin{aligned} & \iint_{\Omega_1} |\nabla \omega_1|^2 dx dY + \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{u_e^0} |\omega_1|^2 dx dY \\ &= \iint_{\Omega_1} \frac{h_e^0}{u_e^0} \nabla \omega_2 \cdot \nabla \omega_1 dx dY + \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_2 \cdot \omega_1 dx dY \\ &+ \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{u_e^0} \omega_1 \cdot \omega_2 dx dY + \iint_{\Omega_1} \frac{(E_b - F_e)}{u_e^0} \omega_1 dx dY := \sum_{i=1}^4 s_{1i}. \end{aligned}$$

We obtain the following estimates

$$\begin{aligned} |s_{11}| &= \left| \iint_{\Omega_1} \frac{h_e^0}{u_e^0} \nabla \omega_2 \cdot \nabla \omega_1 dx dY \right| \lesssim \left| \frac{h_e^0}{u_e^0} \right| \|\nabla \omega_1\|_{L^2(\Omega_1)} \|\nabla \omega_2\|_{L^2(\Omega_1)}, \\ |s_{12}| &= \left| \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_2 \cdot \omega_1 dx dY \right| \\ &\lesssim \left(\left| \frac{h_e^0}{u_e^0} \right| \left| \frac{\partial_Y u_e^0}{u_e^0} \right| + \left| \frac{\partial_Y h_e^0}{u_e^0} \right| \right) \|\omega_1\|_{L^2(\Omega_1)} \|\partial_Y \omega_2\|_{L^2(\Omega_1)} \\ &\lesssim L \left(\left| \frac{h_e^0}{u_e^0} \right| \left| \frac{\partial_Y u_e^0}{u_e^0} \right| + \left| \frac{\partial_Y h_e^0}{u_e^0} \right| \right) \|\partial_x \omega_1\|_{L^2(\Omega_1)} \|\partial_Y \omega_2\|_{L^2(\Omega_1)}, \\ |s_{13}| &= \left| \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{u_e^0} \omega_1 \cdot \omega_2 dx dY \right| \\ &\lesssim \|\omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{L^2(\Omega_1)} \lesssim L^2 \|\partial_x \omega_1\|_{L^2(\Omega_1)} \|\partial_x \omega_2\|_{L^2(\Omega_1)}, \\ |s_{14}| &= \left| \iint_{\Omega_1} \frac{E_b - F_e}{u_e^0} \omega_1 dx dY \right| \lesssim L \|\partial_x \omega_1\|_{L^2(\Omega_1)} \|E_b - F_e\|_{L^2(\Omega_1)} \lesssim L \|\partial_x \omega_1\|_{L^2(\Omega_1)}. \end{aligned}$$

Combining all the above estimates, we get

$$\begin{aligned} & \iint_{\Omega_1} |\nabla \omega_1|^2 dx dY + \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{u_e^0} |\omega_1|^2 dx dY \\ &\lesssim \left(\left| \frac{h_e^0}{u_e^0} \right| + \left| \frac{\partial_Y h_e^0}{u_e^0} \right| \right) \|\nabla \omega_1\|_{L^2(\Omega_1)} \|\nabla \omega_2\|_{L^2(\Omega_1)} + L \|\nabla \omega_1\|_{L^2(\Omega_1)}. \end{aligned} \quad (3.21)$$

Using the crucial estimate obtained in [15](Page 19), we have

$$\iint_{\Omega_1} \left(|\partial_Y \omega_1|^2 + \frac{\partial_Y^2 u_e^0}{u_e^0} |\omega_1|^2 \right) dx dY = \iint_{\Omega_1} |u_e^0|^2 \left| \partial_Y \left(\frac{\omega_1}{u_e^0} \right) \right|^2 dx dY \geq \beta_0 \iint_{\Omega_1} |\partial_Y \omega_1|^2 dx dY, \quad (3.22)$$

where β_0 is positive constant. From the second equation of (3.18), we obtain

$$\nabla \omega_2 = \left(\begin{array}{c} \frac{h_e^0}{u_e^0} \partial_x \omega_1 \\ \frac{h_e^0}{u_e^0} \partial_Y \omega_1 + \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \omega_1 \end{array} \right) + \nabla \left(\frac{h_e^0}{u_e^0} B_v + \frac{b_1^2}{u_e^0} - B_g \right).$$

Thus, we have

$$\begin{aligned}
\|\nabla\omega_2\|_{L^2(\Omega_1)} &\lesssim \left\| \nabla \left(\frac{h_e^0}{u_e^0} B_v \right) \right\|_{L^2(\Omega_1)} + \left\| \nabla \left(\frac{b_1^2}{u_e^0} \right) \right\|_{L^2(\Omega_1)} + \|\nabla B_g\|_{L^2(\Omega_1)} \\
&\quad + \left| \frac{h_e^0}{u_e^0} \right| \|\nabla\omega_1\|_{L^2(\Omega_1)} + \left(\left| \frac{h_e^0}{u_e^0} \right| + \|\partial_Y h_e^0\|_{L^\infty(\Omega_1)} \right) \|\omega_1\|_{L^2(\Omega_1)} \\
&\lesssim \left\| \nabla \left(\frac{h_e^0}{u_e^0} B_v \right) \right\|_{L^2(\Omega_1)} + \left\| \nabla \left(\frac{b_1^2}{u_e^0} \right) \right\|_{L^2(\Omega_1)} + \|\nabla B_g\|_{L^2(\Omega_1)} \\
&\quad + \left| \frac{h_e^0}{u_e^0} \right| \|\nabla\omega_1\|_{L^2(\Omega_1)} + L \left(\left| \frac{h_e^0}{u_e^0} \right| + \|\partial_Y h_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x \omega_1\|_{L^2(\Omega_1)} \\
&\lesssim 1 + \sup_Y \left| \frac{h_e^0}{u_e^0} \right| \|\nabla\omega_1\|_{L^2(\Omega_1)} + L \left(\sup_Y \left| \frac{h_e^0}{u_e^0} \right| + \|\partial_Y h_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x \omega_1\|_{L^2(\Omega_1)}.
\end{aligned}$$

By putting the above estimate into (3.21) and using Young's inequality, we obtain

$$\|\omega_1\|_{H^1(\Omega_1)} \leq C,$$

in which we have used the smallness of $\sup_Y \left| \frac{h_e^0}{u_e^0} \right|$ and $|\partial_Y h_e^0|$.

In addition, we also obtain

$$\|\omega_2\|_{H^1(\Omega_1)} \leq C.$$

Now, we rewrite the first and fourth equations of (3.18) in the following form

$$\begin{cases} -\Delta\omega_1 + \frac{h_e^0}{u_e^0} \Delta\omega_2 = G_e^1, \\ \omega_i|_{\partial\Omega_1} = 0, \quad i = 1, 2, \end{cases} \quad (3.23)$$

where $G_e^1 := \frac{1}{u_e^0} (E_b - F_e - \partial_Y^2 u_e^0 \cdot \omega_1 + \partial_Y^2 h_e^0 \cdot \omega_2)$. It is clear that $\|G_e^1\|_{L^2(\Omega_1)} \leq C$. On the boundary $\{Y = 0\}$, $G_e^1(x, 0) = 0$. For the first equation of (3.23), we have

$$-\partial_Y^2 \omega_1 + \frac{h_e^0}{u_e^0} \partial_Y^2 \omega_2 = 0, \quad \text{on } Y = 0.$$

We now turn to H^2 norms of $\omega_i (i = 1, 2)$. First, taking the Y -derivative of the first equation of (3.23), multiplying it by $\partial_Y \omega_1$, and integrating by parts, we get

$$\begin{aligned}
&\iint_{\Omega_1} |\nabla \partial_Y \omega_1|^2 dx dY + \int_0^L \left(-\partial_Y^2 \omega_1 + \frac{h_e^0}{u_e^0} \partial_Y^2 \omega_2 \right) \partial_Y \omega_1(x, 0) dx \\
&\quad - \int_0^1 \partial_{xY} \omega_1 \cdot \partial_Y \omega_1|_{x=0}^{x=L} dY + \int_0^1 \frac{h_e^0}{u_e^0} \partial_{xY} \omega_2 \cdot \partial_Y \omega_1|_{x=0}^{x=L} dY \\
&= \iint_{\Omega_1} \frac{h_e^0}{u_e^0} \nabla \partial_Y \omega_1 \cdot \nabla \partial_Y \omega_2 dx dY + \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y^2 \omega_2 \cdot \partial_Y \omega_1 dx dY \\
&\quad - \int_0^L \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 \Big|_{Y=0}^{Y=1} dx + \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \nabla \omega_2 \cdot \nabla \partial_Y \omega_1 dx dY \\
&\quad + \iint_{\Omega_1} \partial_Y^2 \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 dx dY - \iint_{\Omega_1} G_e^1 \partial_Y^2 \omega_1 dx dY.
\end{aligned}$$

Using the boundary conditions, the last three terms on the left hand side all vanish and only the first term $\iint_{\Omega_1} |\nabla \partial_Y \omega_1|^2 dx dY$ persists. For the right hand side terms, we have the following estimates

$$\begin{aligned}
\left| \iint_{\Omega_1} \frac{h_e^0}{u_e^0} \nabla \partial_Y \omega_1 \cdot \nabla \partial_Y \omega_2 dx dY \right| &\lesssim \sup_Y \left| \frac{h_e^0}{u_e^0} \right| \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)}, \\
\left| \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y^2 \omega_2 \cdot \partial_Y \omega_1 dx dY \right| &\lesssim \left\| \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \right\|_{L^\infty(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} \|\partial_Y \omega_1\|_{L^2(\Omega_1)} \\
&\lesssim \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} \|\omega_1\|_{H^1(\Omega_1)}, \\
\left| \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \nabla \omega_2 \cdot \nabla \partial_Y \omega_1 dx dY \right| &\lesssim \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)}, \\
\left| \iint_{\Omega_1} \partial_Y^2 \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 dx dY \right| &\lesssim \|\omega_1\|_{H^1(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)}, \\
\left| - \iint_{\Omega_1} G_e^1 \partial_Y^2 \omega_1 dx dY \right| &\lesssim \|G_e^1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)}.
\end{aligned}$$

Using boundary conditions, we have

$$\begin{aligned}
&\left| \int_0^L \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 \Big|_{Y=0}^{Y=1} dx \right| \\
&\lesssim \left\| \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_1 \right\|_{L^2(\Omega_1)} \|\partial_Y \omega_2\|_{L^2(\Omega_1)} \\
&\quad + \left\| \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_1 \right\|_{L^2(\Omega_1)} \|\partial_Y^2 \omega_2\|_{L^2(\Omega_1)} \\
&\lesssim \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)} + \|\omega_1\|_{H^1(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)}.
\end{aligned}$$

Combining all these inequalities, we get

$$\begin{aligned}
\|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)}^2 &\lesssim \sup_Y \left| \frac{h_e^0}{u_e^0} \right| \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} \\
&\quad + \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)} + \|\omega_1\|_{H^1(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} \\
&\quad + \|\omega_1\|_{H^1(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)} + \|G_e^1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)}.
\end{aligned}$$

For the H^2 norm of ω_2 , we obtain from the third equation of (3.18)

$$\nabla \partial_Y \omega_2 = \begin{pmatrix} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_x \omega_1 + \frac{h_e^0}{u_e^0} \partial_{xY} \omega_1 \\ 2 \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y \omega_1 + \partial_Y^2 \left(\frac{h_e^0}{u_e^0} \right) \omega_1 + \frac{h_e^0}{u_e^0} \partial_Y^2 \omega_1 \end{pmatrix} + \nabla \partial_Y \left(\frac{h_e^0}{u_e^0} B_v + \frac{b_1^2}{u_e^0} - B_g \right).$$

Hence, we obtain the following estimate

$$\begin{aligned}
\|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} &\lesssim \left\| \nabla \partial_Y \left(\frac{h_e^0}{u_e^0} B_v \right) \right\|_{L^2(\Omega_1)} + \left\| \nabla \partial_Y \left(\frac{b_1^2}{u_e^0} \right) \right\|_{L^2(\Omega_1)} + \|\nabla \partial_Y B_g\|_{L^2(\Omega_1)} \\
&\quad + \sup_Y \left| \frac{h_e^0}{u_e^0} \right| \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} + \left(\left| \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \right| + \left| \partial_Y^2 \left(\frac{h_e^0}{u_e^0} \right) \right| \right) \|\omega_1\|_{H^1(\Omega_1)} \\
&\lesssim 1 + \sup_Y \left| \frac{h_e^0}{u_e^0} \right| \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} + \|\omega_1\|_{H^1(\Omega_1)}.
\end{aligned}$$

Combining all the previous estimates, we have

$$\|\nabla \partial_Y \omega_i\|_{L^2(\Omega_1)} \leq C, \quad i = 1, 2,$$

in which we used the H^1 estimates of ω_i and the smallness of $\sup_Y \left| \frac{u_e^0}{h_e^0} \right|$.

For $\partial_{xx} \omega_i$, ($i = 1, 2$), we follow similar arguments, which leads to bounds of ω_i ($i = 1, 2$) in H^2 norms. For the L^∞ norms of ω_i , we have

$$\begin{aligned} |\omega_i(x, Y)| &\leq \int_0^x |\partial_x \omega_i(s, Y)| \, ds \\ &\lesssim \int_0^x \left(\int_0^Y |\partial_x \omega_i \partial_{xY} \omega_i| (s, \theta) \, d\theta \right)^{1/2} \, ds \\ &\lesssim \sqrt{x} \|\partial_x \omega_i\|_{L^2(\Omega_1)}^{\frac{1}{2}} \|\partial_{xY} \omega_i\|_{L^2(\Omega_1)}^{\frac{1}{2}} \lesssim \sqrt{L}. \end{aligned}$$

Next, we study estimates of ω_i , ($i = 1, 2$) in H^3 and H^4 norms. Taking ∂_Y and ∂_Y^2 of the first equation of (3.23) leads to

$$\begin{cases} -\Delta \partial_Y \omega_1 + \frac{h_e^0}{u_e^0} \Delta \partial_Y \omega_2 = \partial_Y G_e^1 - \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \Delta \omega_2, \\ \omega_2 = \frac{h_e^0}{u_e^0} (\omega_1 + B_v) + \frac{b_v^2}{u_e^0} - B_g, \\ \partial_Y \omega_i|_{x=0, L} = \partial_Y^2 \omega_i|_{Y=0} = 0, \quad \partial_Y^2 \omega_i|_{Y=1} = 0, \end{cases} \quad (3.24)$$

and

$$\begin{cases} -\Delta \partial_Y^2 \omega_1 + \frac{h_e^0}{u_e^0} \Delta \partial_Y^2 \omega_2 = \partial_Y^2 G_e^1 - 2\partial_Y \left(\frac{h_e^0}{u_e^0} \right) \Delta \partial_Y \omega_2 - \Delta \omega_2 \cdot \partial_Y^2 \left(\frac{h_e^0}{u_e^0} \right), \\ \partial_Y^2 \omega_i|_{x=0, L} = \partial_Y^2 \omega_i|_{Y=0} = 0, \quad \partial_Y^2 \omega_i|_{Y=1} = 0. \end{cases} \quad (3.25)$$

Based on the H^2 estimates of ω_i ($i = 1, 2$) together with the estimates of E_b, F_e obtained earlier, we have

$$\|\partial_Y^k \omega_i\|_{H^2(\Omega_1)} \leq C \varepsilon^{-\frac{3k}{32} + \frac{3}{64}}, \quad k = 1, 2.$$

To obtain the estimates of ω_i , ($i = 1, 2$) in H^3 and H^4 norms, we need to estimate $\partial_x^3 \omega_i$ in L^2 and H^1 norms, respectively. Taking the x -derivative of the first equation of (3.23), we get

$$\begin{aligned} -\partial_x^3 \omega_1 + \frac{h_e^0}{u_e^0} \partial_x^3 \omega_2 &= \partial_{xYY} \omega_1 - \frac{h_e^0}{u_e^0} \partial_{xYY} \omega_2 + (-\Delta \partial_x \omega_1 + \frac{h_e^0}{u_e^0} \Delta \partial_x \omega_2) \\ &= \partial_{xYY} \omega_1 - \frac{h_e^0}{u_e^0} \partial_{xYY} \omega_2 + \partial_x G_e^1. \end{aligned}$$

Taking ∂_x^3 of the second equation in (3.24) and combining the above equality, using the similar arguments as the previous one, we obtain the L^2 and H^1 norms of $\partial_x^3 \omega_i$ ($i = 1, 2$). Hence, the full H^3 and H^4 estimates of ω_i ($i = 1, 2$) are completed. For the higher order estimates of ω_i ($i = 1, 2$) in H^5, H^6, H^7 and H^8 norms, we take the similar methods as above to estimate them. This completes the proof of (3.18) in the assumption of $u_e^0(Y) \geq C_0 h_e^0(Y)$.

Case 2: Under the assumption of $h_e^0(Y) \geq C_0 u_e^0(Y)$, where C_0 is large enough, we estimate the solutions ω_1, ω_2 of (3.18). Multiplying the first equation in (3.18) by $-\frac{\omega_2}{h_e^0}$ and integrating by parts, we get

$$\iint_{\Omega_1} |\nabla \omega_2|^2 \, dx \, dY + \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{h_e^0} |\omega_2|^2 \, dx \, dY$$

$$\begin{aligned}
&= \iint_{\Omega_1} \frac{u_e^0}{h_e^0} \nabla \omega_1 \cdot \nabla \omega_2 dx dY + \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \cdot \omega_2 dx dY \\
&\quad + \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{h_e^0} \omega_1 \cdot \omega_2 dx dY + \iint_{\Omega_1} \frac{(F_e - E_b)}{u_e^0} \omega_2 dx dY := \sum_{i=1}^4 s_{2i}.
\end{aligned}$$

For s_{2i} ($i = 1, 2, 3, 4$), we obtain

$$\begin{aligned}
|s_{21}| &= \left| \iint_{\Omega_1} \frac{u_e^0}{h_e^0} \nabla \omega_1 \cdot \nabla \omega_2 dx dY \right| \lesssim \left| \frac{u_e^0}{h_e^0} \right| \|\nabla \omega_1\|_{L^2(\Omega_1)} \|\nabla \omega_2\|_{L^2(\Omega_1)}, \\
|s_{22}| &= \left| \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \cdot \omega_2 dx dY \right| \\
&\lesssim \left(\left| \frac{u_e^0}{h_e^0} \right| \left| \frac{\partial_Y h_e^0}{h_e^0} \right| + \left| \frac{\partial_Y u_e^0}{h_e^0} \right| \right) \|\omega_2\|_{L^2(\Omega_1)} \|\partial_Y \omega_1\|_{L^2(\Omega_1)} \\
&\lesssim L \left(\left| \frac{u_e^0}{h_e^0} \right| \left| \frac{\partial_Y h_e^0}{h_e^0} \right| + \left| \frac{\partial_Y u_e^0}{h_e^0} \right| \right) \|\partial_x \omega_2\|_{L^2(\Omega_1)} \|\partial_Y \omega_1\|_{L^2(\Omega_1)}, \\
|s_{23}| &= \left| \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{h_e^0} \omega_1 \cdot \omega_2 dx dY \right| \\
&\lesssim \|\omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{L^2(\Omega_1)} \lesssim L^2 \|\partial_x \omega_1\|_{L^2(\Omega_1)} \|\partial_x \omega_2\|_{L^2(\Omega_1)}, \\
|s_{24}| &= \left| \iint_{\Omega_1} \frac{F_e - E_b}{h_e^0} \omega_2 dx dY \right| \lesssim L \|\partial_x \omega_2\|_{L^2(\Omega_1)} \|F_e - E_b\|_{L^2(\Omega_1)} \lesssim L \|\partial_x \omega_2\|_{L^2(\Omega_1)}.
\end{aligned}$$

Combining all the previous estimates, we get

$$\begin{aligned}
&\iint_{\Omega_1} |\nabla \omega_2|^2 dx dY + \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{h_e^0} |\omega_2|^2 dx dY \\
&\lesssim \left(\left| \frac{u_e^0}{h_e^0} \right| + \left| \frac{\partial_Y u_e^0}{h_e^0} \right| \right) \|\nabla \omega_1\|_{L^2(\Omega_1)} \|\nabla \omega_2\|_{L^2(\Omega_1)} + L \|\nabla \omega_2\|_{L^2(\Omega_1)}.
\end{aligned} \tag{3.26}$$

By applying the method of Case 1, we have

$$\iint_{\Omega_1} (|\partial_Y \omega_2|^2 + \frac{\partial_Y^2 h_e^0}{h_e^0} |\omega_2|^2) dx dY = \iint_{\Omega_1} |h_e^0|^2 \left| \partial_Y \left(\frac{\omega_2}{h_e^0} \right) \right|^2 dx dY \geq \beta_0 \iint_{\Omega_1} |\partial_Y \omega_2|^2 dx dY. \tag{3.27}$$

From the second equation of (3.18), we obtain

$$\nabla \omega_1 = \left(\begin{array}{c} \frac{u_e^0}{h_e^0} \partial_x \omega_2 \\ \frac{u_e^0}{h_e^0} \partial_Y \omega_2 + \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \omega_2 \end{array} \right) + \nabla \left(\frac{u_e^0}{h_e^0} B_g + \frac{b_1^1}{h_e^0} - B_v \right).$$

Thus, we have

$$\begin{aligned}
\|\nabla \omega_1\|_{L^2(\Omega_1)} &\lesssim \left\| \nabla \left(\frac{u_e^0}{h_e^0} B_g + \frac{b_1^1}{h_e^0} - B_v \right) \right\|_{L^2(\Omega_1)} + \left| \frac{u_e^0}{h_e^0} \right| \|\nabla \omega_2\|_{L^2(\Omega_1)} \\
&\quad + L \left(\left| \frac{u_e^0}{h_e^0} \right| + \|\partial_Y u_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x \omega_2\|_{L^2(\Omega_1)}
\end{aligned}$$

$$\lesssim 1 + \sup_Y \left| \frac{u_e^0}{h_e^0} \right| \|\nabla \omega_2\|_{L^2(\Omega_1)} + L \left(\sup_Y \left| \frac{u_e^0}{h_e^0} \right| + \|\partial_Y u_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x \omega_2\|_{L^2(\Omega_1)}.$$

Combining the above estimate into (3.26) and using Young's inequality, we obtain

$$\|\omega_2\|_{H^1(\Omega_1)} \leq C,$$

in which we have used the smallness of $\sup_Y \left| \frac{u_e^0}{h_e^0} \right|$ and $|\partial_Y u_e^0|$. In addition, we also get

$$\|\omega_1\|_{H^1(\Omega_1)} \leq C.$$

Next, (3.18) may be rewritten

$$\begin{cases} -\Delta \omega_2 + \frac{u_e^0}{h_e^0} \Delta \omega_1 = M_e^1, \\ \omega_i|_{\partial\Omega_1} = 0, \quad i = 1, 2, \end{cases} \quad (3.28)$$

where $M_e^1 := \frac{1}{h_e^0} (F_e - E_b + \partial_Y^2 u_e^0 \cdot \omega_1 - \partial_Y^2 h_e^0 \cdot \omega_2)$. We then have $\|M_e^1\|_{L^2(\Omega_1)} \leq C$. On the boundary $\{Y = 0\}$, $M_e^1(x, 0) = 0$ and

$$-\partial_Y^2 \omega_2 + \frac{u_e^0}{h_e^0} \partial_Y^2 \omega_1 = 0.$$

We follow the same approach as the previous case to estimate H^2 norms of $\omega_i (i = 1, 2)$. First, taking Y -derivative of the first equation in (3.28), multiplying it by $\partial_Y \omega_2$ and integrating the obtained results by parts, we have

$$\begin{aligned} & \iint_{\Omega_1} |\nabla \partial_Y \omega_2|^2 dx dY + \int_0^L \left(-\partial_Y^2 \omega_2 + \frac{u_e^0}{h_e^0} \partial_Y^2 \omega_1 \right) \partial_Y \omega_2(x, 0) dx \\ & - \int_0^1 \partial_{xY} \omega_2 \cdot \partial_Y \omega_2|_{x=0}^{x=L} dY + \int_0^1 \frac{u_e^0}{h_e^0} \partial_{xY} \omega_1 \cdot \partial_Y \omega_2 \Big|_{x=0}^{x=L} dY \\ & = \iint_{\Omega_1} \frac{u_e^0}{h_e^0} \nabla \partial_Y \omega_1 \cdot \nabla \partial_Y \omega_2 dx dY + \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y^2 \omega_1 \cdot \partial_Y \omega_2 dx dY \\ & - \int_0^L \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 \Big|_{Y=0}^{Y=1} dx + \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \nabla \omega_1 \cdot \nabla \partial_Y \omega_2 dx dY \\ & + \iint_{\Omega_1} \partial_Y^2 \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 dx dY - \iint_{\Omega_1} M_e^1 \partial_Y^2 \omega_2 dx dY. \end{aligned}$$

The last three terms on the left hand side of above equality vanish thanks to the boundary conditions. For the right hand side terms, we obtain the following estimates

$$\begin{aligned} & \left| \iint_{\Omega_1} \frac{u_e^0}{h_e^0} \nabla \partial_Y \omega_1 \cdot \nabla \partial_Y \omega_2 dx dY \right| \lesssim \sup_Y \left| \frac{u_e^0}{h_e^0} \right| \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)}, \\ & \left| \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y^2 \omega_1 \cdot \partial_Y \omega_2 dx dY \right| \lesssim \left\| \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \right\|_{L^\infty(\Omega_1)} \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\partial_Y \omega_2\|_{L^2(\Omega_1)} \\ & \lesssim \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)}, \\ & \left| \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \nabla \omega_1 \cdot \nabla \partial_Y \omega_2 dx dY \right| \lesssim \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} \|\omega_1\|_{H^1(\Omega_1)}, \end{aligned}$$

$$\begin{aligned} \left| \iint_{\Omega_1} \partial_Y^2 \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 dx dY \right| &\lesssim \|\omega_1\|_{H^1(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)}, \\ \left| - \iint_{\Omega_1} M_e^1 \partial_Y^2 \omega_2 dx dY \right| &\lesssim \|M_e^1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)}. \end{aligned}$$

For the boundary term in the right hand side terms, we have

$$\begin{aligned} &\left| \int_0^L \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \cdot \partial_Y \omega_2 \Big|_{Y=0}^{Y=1} dx \right| \\ &\lesssim \left\| \partial_Y \left(\partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \right) \right\|_{L^2(\Omega_1)} \|\partial_Y \omega_2\|_{L^2(\Omega_1)} \\ &\quad + \left\| \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_1 \right\|_{L^2(\Omega_1)} \|\partial_Y^2 \omega_2\|_{L^2(\Omega_1)} \\ &\lesssim \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)} + \|\omega_1\|_{H^1(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)}. \end{aligned}$$

Combining all the above inequalities, we get

$$\begin{aligned} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)}^2 &\lesssim \sup_Y \left| \frac{u_e^0}{h_e^0} \right| \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} \\ &\quad + \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)} + \|\omega_1\|_{H^1(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} \\ &\quad + \|\omega_1\|_{H^1(\Omega_1)} \|\omega_2\|_{H^1(\Omega_1)} + \|M_e^1\|_{L^2(\Omega_1)} \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)}. \end{aligned}$$

For the H^2 norm of ω_1 , we obtain from the second equation of (3.18)

$$\nabla \partial_Y \omega_1 = \begin{pmatrix} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_x \omega_2 + \frac{u_e^0}{h_e^0} \partial_{xY} \omega_2 \\ 2 \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y \omega_2 + \partial_Y^2 \left(\frac{u_e^0}{h_e^0} \right) \omega_2 + \frac{u_e^0}{h_e^0} \partial_Y^2 \omega_2 \end{pmatrix} + \nabla \partial_Y \left(\frac{u_e^0}{h_e^0} B_g + \frac{b_1^1}{h_e^0} - B_v \right).$$

Hence, we get the following estimate

$$\begin{aligned} \|\nabla \partial_Y \omega_1\|_{L^2(\Omega_1)} &\lesssim \left\| \nabla \partial_Y \left(\frac{u_e^0}{h_e^0} B_g \right) \right\|_{L^2(\Omega_1)} + \left\| \nabla \partial_Y \left(\frac{b_1^1}{h_e^0} \right) \right\|_{L^2(\Omega_1)} + \|\nabla \partial_Y B_v\|_{L^2(\Omega_1)} \\ &\quad + \sup_Y \left| \frac{u_e^0}{h_e^0} \right| \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} + \left(\left| \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \right| + \left| \partial_Y^2 \left(\frac{u_e^0}{h_e^0} \right) \right| \right) \|\omega_2\|_{H^1(\Omega_1)} \\ &\lesssim 1 + \sup_Y \left| \frac{u_e^0}{h_e^0} \right| \|\nabla \partial_Y \omega_2\|_{L^2(\Omega_1)} + \|\omega_2\|_{H^1(\Omega_1)}, \end{aligned}$$

Combining all the above estimates, we have

$$\|\nabla \partial_Y \omega_i\|_{L^2(\Omega_1)} \leq C, \quad i = 1, 2,$$

in which we use the H^1 estimates of ω_i and the smallness of $\sup_Y \left| \frac{u_e^0}{h_e^0} \right|$.

For $\partial_{xx} \omega_i$, ($i = 1, 2$), we use similar arguments, leading to estimates of ω_i ($i = 1, 2$) in H^2 norms. For the L^∞ estimates of ω_i , ($i = 1, 2$), we have the same results as in Case 1.

Next, we follow similar methods to obtain H^3 and H^4 estimates on $\omega_i (i = 1, 2)$. Taking the first and second order derivatives of the first equation of (3.28), we obtain

$$\begin{cases} -\Delta \partial_Y \omega_2 + \frac{u_e^0}{h_e^0} \Delta \partial_Y \omega_1 = \partial_Y M_e^1 - \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \Delta \omega_1, \\ \omega_1 = \frac{u_e^0}{h_e^0} (\omega_2 + B_g) + \frac{b_1^1}{h_e^0} - B_v, \\ \partial_Y \omega_i|_{x=0, L} = \partial_Y^2 \omega_i|_{Y=0} = 0, \quad \partial_Y^2 \omega_i|_{Y=1} = 0, \end{cases} \quad (3.29)$$

and

$$\begin{cases} -\Delta \partial_Y^2 \omega_2 + \frac{u_e^0}{h_e^0} \Delta \partial_Y^2 \omega_1 = \partial_Y^2 M_e^1 - 2\partial_Y \left(\frac{u_e^0}{h_e^0} \right) \Delta \partial_Y \omega_1 - \Delta \omega_1 \cdot \partial_Y^2 \left(\frac{u_e^0}{h_e^0} \right), \\ \partial_Y^2 \omega_i|_{x=0, L} = \partial_Y^2 \omega_i|_{Y=0} = 0, \quad \partial_Y^2 \omega_i|_{Y=1} = 0. \end{cases} \quad (3.30)$$

Based on the previous estimates, we have

$$\|\partial_Y^k \omega_i\|_{H^2(\Omega_1)} \leq C\varepsilon^{-\frac{3k}{32} + \frac{3}{64}}, \quad k = 1, 2.$$

Similarly, in order to obtain H^3 and H^4 estimates of ω_i , ($i = 1, 2$), we establish estimates of $\partial_x^3 \omega_i$ in L^2 and H^1 norms, respectively. Taking the x -derivative of the first equation of (3.28),

$$\begin{aligned} -\partial_x^3 \omega_2 + \frac{u_e^0}{h_e^0} \partial_x^3 \omega_1 &= \partial_{xYY} \omega_2 - \frac{u_e^0}{h_e^0} \partial_{xYY} \omega_1 + (-\Delta \partial_x \omega_2 + \frac{u_e^0}{h_e^0} \Delta \partial_x \omega_1) \\ &= \partial_{xYY} \omega_2 - \frac{u_e^0}{h_e^0} \partial_{xYY} \omega_1 + \partial_x M_e^1. \end{aligned}$$

Taking the third derivative of the second equation in (3.29), we obtain L^2 and H^1 bounds on $\partial_x^3 \omega_i$, ($i = 1, 2$). Hence, using similar arguments as the previous Case 1, we complete the full H^3 and H^4 estimates of ω_i , ($i = 1, 2$) and the higher order estimates of ω_i , ($i = 1, 2$) in H^5, H^6, H^7 and H^8 norms. This proves the estimates (3.19) in the assumption of $h_e^0(Y) \geq C_0 u_e^0(Y)$.

Based on the above two cases, we obtain the full proof of (3.19). By the standard elliptic theory [10], we obtain $W^{k,p}$ estimates of $\omega_i (i = 1, 2)$. This ends the proof of the Proposition. \square

Next, we establish the estimates of v_e^1, g_e^1 . By their definition and boundary conditions (3.10) satisfied by B_v, B_g , we obtain that $(v_e^1, g_e^1) \in W^{k,p}(\Omega_1)$ is the unique smooth solution to the system (3.15) with boundary conditions (3.10). Since $B_v, B_g \in W^{8,p}(\Omega_1)$, it follows that

$$\|(v_e^1, g_e^1)\|_{L^\infty(\Omega_1)} + \|(v_e^1, g_e^1)\|_{W^{2,q}(\Omega_1)} \leq C, \quad \|(v_e^1, g_e^1)\|_{W^{2+j,q}(\Omega_1)} \leq C\varepsilon^{-\frac{3j}{32} + \frac{3}{32q}}, \quad j = 1, 2, 3, 4, 5, 6.$$

Now, we construct the first order inner magnetic correctors u_e^1, h_e^1 and p_e^1 . Applying the equation (3.9) and the divergence free conditions, we have

$$\begin{aligned} u_e^1(x, Y) &= u_b^1(Y) - \int_0^x v_{eY}^1(s, Y) ds, \\ h_e^1(x, Y) &= h_b^1(Y) - \int_0^x g_{eY}^1(s, Y) ds, \\ p_e^1(x, y) &= \int_Y^1 (u_e^0 v_{ex}^1 - h_e^0 g_{ex}^1)(x, \theta) d\theta - \int_0^x (u_e^0(1) v_{eY}^1(s, 1) - h_e^0(1) g_{eY}^1(s, 1)) ds, \end{aligned} \quad (3.31)$$

where $u_e^1(0, Y) = u_b^1(Y)$ and $h_e^1(0, Y) = h_b^1(Y)$ satisfy $\partial_Y u_b^1(1) = \partial_Y h_b^1(1) = 0$, hence, we have $u_{eY}^1(x, 1) = h_{eY}^1(x, 1) = 0$. By definition of (u_e^1, h_e^1) in (3.31) and Proposition 3.1, we get

$$\|(u_e^1, h_e^1)\|_{L^\infty(\Omega_1)} + \|(u_e^1, h_e^1)\|_{H^{1+j}(\Omega_1)} \leq C\varepsilon^{-\frac{3j}{32} + \frac{3}{64}}, \quad j = 1, 2, 3, 4, 5, 6. \quad (3.32)$$

3.2 The first order magnetic boundary layer profile

In this subsection, our objective is to construct the first order magnetic boundary layer profile $(u_p^1, v_p^1, h_p^1, g_p^1, p_p^1)$, which solves (3.3). For convenience, we define

$$u^0 := u_e + u_p^0(x, y), \quad h^0 := h_e + h_p^0(x, y).$$

Let $p_{px}^1 = 0$, then we rewrite (3.3)

$$\left\{ \begin{array}{l} u^0 \partial_x u_p^1 + u_p^1 \partial_x u^0 + v_p^1 \partial_y u^0 + (v_p^0 + \overline{v_e^1}) \partial_y u_p^1 - \nu_1 \partial_y^2 u_p^1 \\ - h^0 \partial_x h_p^1 - h_p^1 \partial_x h^0 - g_p^1 \partial_y h^0 - (g_p^0 + \overline{g_e^1}) \partial_y h_p^1 \\ = - \overline{\partial_Y u_e^0} [y \partial_x u_p^0 + v_p^0] - y \overline{\partial_Y v_e^1} \partial_y u_p^0 - \overline{u_e^1} \partial_x u_p^0 - u_p^0 \overline{\partial_x u_e^1} - \overline{v_e^2} \partial_y u_p^0 \\ + \overline{\partial_Y h_e^0} [y \partial_x h_p^0 + g_p^0] + y \overline{\partial_Y g_e^1} \partial_y h_p^0 + \overline{h_e^1} \partial_x h_p^0 + h_p^0 \overline{\partial_x h_e^1} + \overline{g_e^2} \partial_y h_p^0 := F_{p_1}^1, \\ u^0 \partial_x h_p^1 + u_p^1 \partial_x h^0 + v_p^1 \partial_y h^0 + (v_p^0 + \overline{v_e^1}) \partial_y h_p^1 - \nu_3 \partial_y^2 h_p^1 \\ - h^0 \partial_x u_p^1 - h_p^1 \partial_x u^0 - g_p^1 \partial_y u^0 - (g_p^0 + \overline{g_e^1}) \partial_y u_p^1 \\ = - \overline{\partial_Y h_e^0} v_p^0 - y \overline{\partial_Y u_e^0} \partial_x h_p^0 - y \overline{\partial_Y v_e^1} \partial_y h_p^0 - \overline{\partial_x h_e^1} u_p^0 - \overline{u_e^1} \partial_x h_p^0 - \overline{v_e^2} \partial_y h_p^0 \\ + \overline{\partial_Y u_e^0} g_p^0 + y \overline{\partial_Y h_e^0} \partial_x u_p^0 + y \overline{\partial_Y g_e^1} \partial_y u_p^0 + \overline{h_e^1} \partial_x u_p^0 + \overline{\partial_x u_e^1} h_p^0 + \overline{g_e^2} \partial_y u_p^0 := F_{p_2}^1, \end{array} \right. \quad (3.33)$$

with boundary conditions

$$\begin{aligned} (u_p^1, h_p^1)(0, y) &= (\tilde{u}_1(y), \tilde{h}_1(y)), (u_p^1, h_{py}^1)(x, 0) = -(u_e^1, h_{eY}^0)(x, 0), \\ u_{py}^1(x, \varepsilon^{-1/2}) &= v_p^1(x, 0) = v_p^1(x, \varepsilon^{-1/2}) = 0, \\ h_{py}^1(x, \varepsilon^{-1/2}) &= g_p^1(x, 0) = g_p^1(x, \varepsilon^{-1/2}) = 0. \end{aligned} \quad (3.34)$$

Therefore, the new error terms are given by

$$\left\{ \begin{array}{l} E_{r_1}^1 := (u_e^0 - u_e) \partial_x u_p^1 + (v_e^2 - \overline{v_e^2}) \partial_y u_p^0 + (v_e^1 - \overline{v_e^1}) \partial_y u_p^1 - (h_e^0 - h_e) \partial_x h_p^1 \\ - (g_e^2 - \overline{g_e^2}) \partial_y h_p^0 - (g_e^1 - \overline{g_e^1}) \partial_y h_p^1 + (\partial_Y u_e^0 - \overline{\partial_Y u_e^0}) (y \partial_x u_p^0 + v_p^0) \\ + y (\partial_Y v_e^1 - \overline{\partial_Y v_e^1}) \partial_y u_p^0 + (u_e^1 - \overline{u_e^1}) \partial_x u_p^0 + (\partial_x u_e^1 - \overline{\partial_x u_e^1}) u_p^0 \\ - (\partial_Y h_e^0 - \overline{\partial_Y h_e^0}) (y \partial_x h_p^0 + g_p^0) - y (\partial_Y g_e^1 - \overline{\partial_Y g_e^1}) \partial_y h_p^0 \\ - (h_e^1 - \overline{h_e^1}) \partial_x u_p^0 - (\partial_x h_e^1 - \overline{\partial_x h_e^1}) h_p^0 + \triangle_1, \\ E_{r_2}^1 := (u_e^0 - u_e) \partial_x h_p^1 + (v_e^2 - \overline{v_e^2}) \partial_y h_p^0 + (v_e^1 - \overline{v_e^1}) \partial_y h_p^1 \\ - (h_e^0 - h_e) \partial_x u_p^1 - (g_e^2 - \overline{g_e^2}) \partial_y u_p^0 - (g_e^1 - \overline{g_e^1}) \partial_y u_p^1 \\ + (\partial_Y h_e^0 - \overline{\partial_Y h_e^0}) v_p^0 + y \partial_x h_p^0 (\partial_Y u_e^0 - \overline{\partial_Y u_e^0}) \\ + y \partial_y h_p^0 (\partial_Y v_e^1 - \overline{\partial_Y v_e^1}) + (\partial_x h_e^1 - \overline{\partial_x h_e^1}) u_p^0 + (u_e^1 - \overline{u_e^1}) \partial_x h_p^0 \\ - (\partial_Y u_e^0 - \overline{\partial_Y u_e^0}) g_p^0 - y (\partial_Y h_e^0 - \overline{\partial_Y h_e^0}) \partial_x u_p^0 - (h_e^1 - \overline{h_e^1}) \partial_x u_p^0 \\ - y (\partial_Y g_e^1 - \overline{\partial_Y g_e^1}) \partial_y u_p^0 - (\partial_x u_e^1 - \overline{\partial_x u_e^1}) h_p^0 + \triangle_3. \end{array} \right. \quad (3.35)$$

Next, we establish the well-posedness of the solution $(u_p^1, v_p^1, h_p^1, g_p^1)$ to the system (3.33). For this purpose, we adopt a method similar to that in Section 2 and we will omit the details here. First,

we extend the domain I_ε to \mathbb{R}_+ and establish the estimates on $[0, L] \times \mathbb{R}_+$, using the boundary conditions $u_p^{1,\infty}(x, \infty) = v_p^{1,\infty}(x, \infty) = 0$, $h_p^{1,\infty}(x, \infty) = g_p^{1,\infty}(x, \infty) = 0$ instead of $u_{py}^1(x, \varepsilon^{-1/2}) = v_p^1(x, \varepsilon^{-1/2}) = 0$, $h_p^1(x, \varepsilon^{-1/2}) = g_p^1(x, \varepsilon^{-1/2}) = 0$. Second, we cut-off the domain from \mathbb{R}_+ to I_ε .

Step1: Half space problem.

Proposition 3.2. *For $0 < L_3 \leq L_2$, there exists smooth solution $(u_p^{1,\infty}, v_p^{1,\infty}, h_p^{1,\infty}, g_p^{1,\infty})$ to the problem (3.33) in $[0, L_3] \times [0, \infty)$ such that the following estimates hold*

$$\begin{aligned} & \| (u_p^{1,\infty}, v_p^{1,\infty}, h_p^{1,\infty}, g_p^{1,\infty}) \|_{L^\infty(0, L_3; \mathbb{R}_+)} + \sup_{0 \leq x \leq L_3} \| \langle y \rangle^l \partial_{yy} (v_p^{1,\infty}, g_p^{1,\infty}) \|_{L^2(0, \infty)} \\ & + \| \langle y \rangle^l \partial_{xy} (v_p^{1,\infty}, g_p^{1,\infty}) \|_{L^2(0, L_3; \mathbb{R}_+)} \leq C(L, \xi) \varepsilon^{-\xi}, \\ & \sup_{0 \leq x \leq L_3} \| \langle y \rangle^l \partial_{yy} (v_p^{1,\infty}, g_p^{1,\infty}) \|_{H^j(0, \infty)} + \| \langle y \rangle^l \partial_{xy} (v_p^{1,\infty}, g_p^{1,\infty}) \|_{H^j(0, L_3; \mathbb{R}_+)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3j}{32}}, \end{aligned} \quad (3.36)$$

where ξ is small positive constant and $j = 1, 2, 3, 4, 5$.

We do not detail the proof of this Proposition, which follows the line of Section 2.

Step 2: Now, we cut-off the solutions from $[0, L] \times \mathbb{R}_+$ to Ω_ε by applying methods similar to those of Section 2.

Proposition 3.3. *Under the assumptions of Theorem 1.1, there exists smooth functions $(u_p^1, v_p^1, h_p^1, g_p^1)$, satisfying the following inhomogeneous system:*

$$\begin{cases} u_p^0 u_{px}^1 + u_x^0 u_p^1 + u_y^0 v_p^1 + [v_p^0 + \overline{v_e^1}] u_{py}^1 - h_p^0 h_{px}^1 - h_x^0 h_p^1 - h_y^0 g_p^1 \\ \quad - [g_p^0 + \overline{g_e^1}] h_{py}^1 - u_{pyy}^1 = R_p^{u,1}, \\ u_p^0 h_{px}^1 + h_x^0 u_p^1 + h_y^0 v_p^1 + [v_p^0 + \overline{v_e^1}] h_{py}^1 - h_p^0 u_{px}^1 - u_x^0 h_p^1 - u_y^0 g_p^1 \\ \quad - [g_p^0 + \overline{g_e^1}] u_{py}^1 - h_{pyy}^1 = R_p^{h,1}, \\ u_{px}^1 + v_{py}^1 = h_{px}^1 + g_{py}^1 = 0, \\ (u_p^1, h_p^1)(0, y) = (\tilde{u}_1(y), \tilde{h}_1(y)), (u_p^1, \partial_y h_p^1)(x, 0) = -(u_e^1, \partial_Y h_e^0)(x, 0), \\ (u_{py}^1, h_{py}^1)(x, \varepsilon^{-1/2}) = (v_p^1, g_p^1)(x, 0) = (v_p^1, g_p^1)(x, \varepsilon^{-1/2}) = 0, \end{cases} \quad (3.37)$$

such that, for any given $l \in \mathbb{N}$, it holds that

$$\begin{aligned} & \| (u_p^1, v_p^1, h_p^1, g_p^1) \|_{L^\infty(\Omega_\varepsilon)} + \sup_{0 \leq x \leq L_3} \| \langle y \rangle^l (v_{pyy}^1, g_{pyy}^1) \|_{L^2(I_\varepsilon)} + \| \langle y \rangle^l (v_{pxy}^1, g_{pxy}^1) \|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi}, \\ & \sup_{0 \leq x \leq L_3} \| \langle y \rangle^l (v_{pyy}^1, g_{pyy}^1) \|_{H^j(I_\varepsilon)} + \| \langle y \rangle^l (v_{pxy}^1, g_{pxy}^1) \|_{H^j(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3j}{32}}, \end{aligned} \quad (3.38)$$

where $R_p^{u,1}$ and $R_p^{h,1}$ are higher order terms of $\sqrt{\varepsilon}$, $j = 1, 2, 3, 4, 5$.

Proof. The solution $(u_p^{1,\infty}, v_p^{1,\infty}, h_p^{1,\infty}, g_p^{1,\infty})$ has been constructed in Proposition 3.2. Next, we define

$$\begin{aligned} (u_p^1, h_p^1)(x, y) &:= \chi(\sqrt{\varepsilon}y) (u_p^{1,\infty}, h_p^{1,\infty})(x, y) + \sqrt{\varepsilon} \chi'(\sqrt{\varepsilon}y) \int_0^y (u_p^{1,\infty}, h_p^{1,\infty})(x, \theta) d\theta, \\ (v_p^1, g_p^1)(x, y) &:= \chi(\sqrt{\varepsilon}y) (v_p^{1,\infty}, g_p^{1,\infty})(x, y). \end{aligned} \quad (3.39)$$

It is clear that $(u_p^1, v_p^1, h_p^1, g_p^1)$ satisfies (3.37). Thus, we have

$$\left\{ \begin{aligned} R_p^{u,1} &= \sqrt{\varepsilon} \chi' [u_p^0 v_p^{1,\infty} - h_p^0 g_p^{1,\infty}] + \sqrt{\varepsilon} \chi' [u_x^0 \int_0^y u_p^{1,\infty} d\theta - h_x^0 \int_0^y h_p^{1,\infty} d\theta] + 2\sqrt{\varepsilon} \chi' [v_p^0 + \overline{v_e^1}] u_p^{1,\infty} \\ &\quad - 2\sqrt{\varepsilon} \chi' [g_p^0 + \overline{g_e^1}] h_p^{1,\infty} - 3\sqrt{\varepsilon} \chi' u_{py}^{1,\infty} + \varepsilon \chi'' \{ [v_p^0 + \overline{v_e^1}] \int_0^y u_p^{1,\infty} d\theta - [g_p^0 + \overline{g_e^1}] \int_0^y h_p^{1,\infty} d\theta \} \\ &\quad + \sqrt{\varepsilon} F_{p1}^1 \int_0^y \chi' d\theta - 3\varepsilon \chi'' u_p^{1,\infty} - \varepsilon^{3/2} \chi''' \int_0^y u_p^{1,\infty} d\theta \\ &:= \sqrt{\varepsilon} \Delta_5 + \varepsilon \Delta_6, \\ R_p^{h,1} &= \sqrt{\varepsilon} \chi' [u_p^0 g_p^{1,\infty} - h_p^0 v_p^{1,\infty}] + \sqrt{\varepsilon} \chi' [h_x^0 \int_0^y u_p^{1,\infty} d\theta - u_x^0 \int_0^y h_p^{1,\infty} d\theta] + 2\sqrt{\varepsilon} \chi' [v_p^0 + \overline{v_e^1}] h_p^{1,\infty} \\ &\quad - 2\sqrt{\varepsilon} \chi' [g_p^0 + \overline{g_e^1}] u_p^{1,\infty} - 3\sqrt{\varepsilon} \chi' h_{py}^{1,\infty} + \varepsilon \chi'' \{ [v_p^0 + \overline{v_e^1}] \int_0^y h_p^{1,\infty} d\theta - [g_p^0 + \overline{g_e^1}] \int_0^y u_p^{1,\infty} d\theta \} \\ &\quad + \sqrt{\varepsilon} F_{p2}^1 \int_0^y \chi' d\theta - 3\varepsilon \chi'' h_p^{1,\infty} - \varepsilon^{3/2} \chi''' \int_0^y u_p^{1,\infty} d\theta \\ &:= \sqrt{\varepsilon} \Delta_7 + \varepsilon \Delta_8. \end{aligned} \right. \quad (3.40)$$

It is direct to check that

$$\partial_x u_p^1 + \partial_y v_p^1 = \partial_x h_p^1 + \partial_y g_p^1 = 0.$$

Next, applying the estimates (3.36), we have

$$\begin{aligned} \left| \sqrt{\varepsilon} \chi' (\sqrt{\varepsilon} y) \int_0^y u_p^{1,\infty}(x, \theta) d\theta \right| &\leq \sqrt{\varepsilon} y |\chi'(\sqrt{\varepsilon} y)| \|u_p^{1,\infty}\|_{L^\infty} \leq C(L, \xi) \varepsilon^{-\xi}, \\ \left| \sqrt{\varepsilon} \chi' (\sqrt{\varepsilon} y) \int_0^y h_p^{1,\infty}(x, \theta) d\theta \right| &\leq \sqrt{\varepsilon} y |\chi'(\sqrt{\varepsilon} y)| \|h_p^{1,\infty}\|_{L^\infty} \leq C(L, \xi) \varepsilon^{-\xi}. \end{aligned}$$

Hence, we obtain that the estimates (3.38) hold through the Proposition 3.2. This completes the proof of the proposition. \square

4 The second order ideal MHD profile and boundary layer profile

4.1 The second order ideal MHD profile

In this subsection, we consider the second order ideal MHD profile $(u_e^2, v_e^2, h_e^2, g_e^2, p_e^2)$, and have the following definition of R_2^u and R_2^h

$$\begin{aligned} R_2^u &= [(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y] (u_e^0 + u_p^0) + [(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y] (u_e^1 + u_p^1) \\ &\quad + [(u_e^0 + u_p^0) \partial_x + (v_p^0 + v_e^1) \partial_y] (u_e^2 + u_p^2) + \partial_x (p_e^2 + p_p^2) - \nu_1 \partial_y^2 (u_e^2 + u_p^2) \\ &\quad - [(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y] (h_e^0 + h_p^0) - [(h_e^1 + h_p^1) \partial_x + (g_p^1 + g_e^2) \partial_y] (h_e^1 + h_p^1) \\ &\quad - [(h_e^0 + h_p^0) \partial_x + (g_p^0 + g_e^1) \partial_y] (h_e^2 + h_p^2) - \nu_1 (\partial_Y^2 u_e^0 + \partial_x^2 u_p^0) \\ &\quad + [(v_p^1 + v_e^2) \partial_Y u_e^0 + (v_p^0 + v_e^1) \partial_Y u_e^1] - [(g_p^1 + g_e^2) \partial_Y h_e^0 + (g_p^0 + g_e^1) \partial_Y h_e^1] \\ &\quad + \frac{E_1}{\varepsilon} + \Delta_2 + \varepsilon^{-\frac{1}{2}} E_{r1}^1 + \Delta_5, \end{aligned} \quad (4.1)$$

$$\begin{aligned}
R_2^h = & [(u_e^2 + u_p^2)\partial_x + (v_p^2 + v_e^3)\partial_y](h_e^0 + h_p^0) + [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](h_e^1 + h_p^1) \\
& + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](h_e^2 + h_p^2) - \nu_3 \partial_y^2 (h_e^2 + h_p^2) \\
& - [(h_e^2 + h_p^2)\partial_x + (g_p^2 + g_e^3)\partial_y](u_e^0 + u_p^0) - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](u_e^1 + u_p^1) \\
& - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](u_e^2 + u_p^2) - \nu_3 (\partial_Y^2 h_e^0 + \partial_x^2 h_p^0) \\
& + [(v_p^1 + v_e^2)\partial_Y h_e^0 + (v_p^0 + v_e^1)\partial_Y h_e^1] - [(g_p^1 + g_e^2)\partial_Y u_e^0 + (g_p^0 + g_e^1)\partial_Y u_e^1] \\
& + \frac{E_3}{\varepsilon} + \Delta_4 + \varepsilon^{-\frac{1}{2}} E_{r_2}^1 + \Delta_7.
\end{aligned}$$

Note that the ideal MHD profiles are always evaluated at $(x, \sqrt{\varepsilon}y)$, which generates a factor of $\sqrt{\varepsilon}$.

$$\begin{aligned}
[v_p^2 + v_e^3]\partial_y u_e^0 &= \sqrt{\varepsilon}[v_p^2 + v_e^3]\partial_Y u_e^0, & [g_p^2 + g_e^3]\partial_y h_e^0 &= \sqrt{\varepsilon}[g_p^2 + g_e^3]\partial_Y h_e^0, \\
[v_p^1 + v_e^2]\partial_y u_e^1 &= \sqrt{\varepsilon}[v_p^1 + v_e^2]\partial_Y u_e^1, & [g_p^1 + g_e^2]\partial_y h_e^1 &= \sqrt{\varepsilon}[g_p^1 + g_e^2]\partial_Y h_e^1, \\
[v_p^0 + v_e^1]\partial_y u_e^2 &= \sqrt{\varepsilon}[v_p^0 + v_e^1]\partial_Y u_e^2, & [g_p^0 + g_e^1]\partial_y h_e^2 &= \sqrt{\varepsilon}[g_p^0 + g_e^1]\partial_Y h_e^2, \\
[v_p^2 + v_e^3]\partial_y h_e^0 &= \sqrt{\varepsilon}[v_p^2 + v_e^3]\partial_Y h_e^0, & [g_p^2 + g_e^3]\partial_y u_e^0 &= \sqrt{\varepsilon}[g_p^2 + g_e^3]\partial_Y u_e^0, \\
[v_p^1 + v_e^2]\partial_y h_e^1 &= \sqrt{\varepsilon}[v_p^1 + v_e^2]\partial_Y h_e^1, & [g_p^1 + g_e^2]\partial_y u_e^1 &= \sqrt{\varepsilon}[g_p^1 + g_e^2]\partial_Y u_e^1, \\
[v_p^0 + v_e^1]\partial_y h_e^2 &= \sqrt{\varepsilon}[v_p^0 + v_e^1]\partial_Y h_e^2, & [g_p^0 + g_e^1]\partial_y u_e^2 &= \sqrt{\varepsilon}[g_p^0 + g_e^1]\partial_Y u_e^2, \\
\partial_y^2 u_e^2 &= \varepsilon \partial_Y^2 u_e^2, & \partial_y^2 h_e^2 &= \varepsilon \partial_Y^2 h_e^2.
\end{aligned}$$

By matching the order of ε , we obtain the following second order ideal MHD system

$$\begin{cases} u_e^0 \partial_x u_e^2 + v_e^2 \partial_Y u_e^0 - h_e^0 \partial_x h_e^2 - g_e^2 \partial_Y h_e^0 + \partial_x p_e^2 = f_1^1, \\ u_e^0 \partial_x h_e^2 + v_e^2 \partial_Y h_e^0 - h_e^0 \partial_x u_e^2 - g_e^2 \partial_Y u_e^0 = f_3^1, \end{cases} \quad (4.2)$$

where

$$\begin{aligned}
f_1^1 &= -u_e^1 u_{ex}^1 - v_e^1 u_{eY}^1 + h_e^1 h_{ex}^1 + g_e^1 h_{eY}^1 + \nu_1 \partial_Y^2 u_e^0, \\
f_3^1 &= -u_e^1 h_{ex}^1 - v_e^1 h_{eY}^1 + h_e^1 u_{ex}^1 + g_e^1 u_{eY}^1 + \nu_3 \partial_Y^2 h_e^0,
\end{aligned}$$

and also obtain the second order boundary layer system

$$\begin{cases} (u_e^2 + u_p^2)\partial_x u_p^0 + u_p^1 \partial_x u_e^1 + (u_e^1 + u_p^1)\partial_x u_p^1 + u_p^0 \partial_x u_e^2 + (u_e^0 + u_p^0)\partial_x u_p^2 \\ + (v_p^2 + v_e^3)\partial_y u_p^0 + (v_p^1 + v_e^2)\partial_y u_p^1 + (v_p^0 + v_e^1)\partial_y u_p^2 + \partial_x p_p^2 - \nu_1 \partial_y^2 u_p^2 \\ - (h_e^2 + h_p^2)\partial_x h_p^0 - h_p^1 \partial_x h_e^1 - (h_e^1 + h_p^1)\partial_x h_p^1 - h_p^0 \partial_x h_e^2 - (h_e^0 + h_p^0)\partial_x h_p^2 \\ - (g_p^2 + g_e^3)\partial_y h_p^0 - (g_p^1 + g_e^2)\partial_y h_p^1 - (g_p^0 + g_e^1)\partial_y h_p^2 + \frac{E_1}{\varepsilon} + \Delta_2 - \nu_1 \partial_x^2 u_p^0 \\ + v_p^1 \partial_Y u_e^0 + v_p^0 \partial_Y u_e^1 - g_p^1 \partial_Y h_e^0 - g_p^0 \partial_Y h_e^1 + \Delta_5 + \varepsilon^{-\frac{1}{2}} E_{r_1}^1 = 0, \\ (u_e^2 + u_p^2)\partial_x h_p^0 + u_p^1 \partial_x h_e^1 + (u_e^1 + u_p^1)\partial_x h_p^1 + u_p^0 \partial_x h_e^2 + (u_e^0 + u_p^0)\partial_x h_p^2 \\ + (v_p^2 + v_e^3)\partial_y h_p^0 + (v_p^1 + v_e^2)\partial_y h_p^1 + (v_p^0 + v_e^1)\partial_y h_p^2 - \nu_3 \partial_y^2 h_p^2 \\ - (h_e^2 + h_p^2)\partial_x u_p^0 - h_p^1 \partial_x u_e^1 - (h_e^1 + h_p^1)\partial_y u_p^1 - h_p^0 \partial_x u_e^2 - (h_e^0 + h_p^0)\partial_x h_p^2 \\ - (g_p^2 + g_e^3)\partial_y u_p^0 - (g_p^1 + g_e^2)\partial_y u_p^1 - (g_p^0 + g_e^1)\partial_y u_p^2 + \frac{E_3}{\varepsilon} + \Delta_4 - \nu_3 \partial_x^2 h_p^0 \\ + v_p^1 \partial_Y h_e^0 + v_p^0 \partial_Y h_e^1 - g_p^1 \partial_Y u_e^0 - g_p^0 \partial_Y u_e^1 + \Delta_7 + \varepsilon^{-\frac{1}{2}} E_{r_2}^1 = 0. \end{cases} \quad (4.3)$$

Hence, the errors R_2^u and R_2^h further reduce to

$$R_2^u = \sqrt{\varepsilon}[(v_p^2 + v_e^3)\partial_Y u_e^0 + (v_p^1 + v_e^2)\partial_Y u_e^1 + (v_p^0 + v_e^1)\partial_Y u_e^2]$$

$$\begin{aligned}
& -\sqrt{\varepsilon}[(g_p^2 + g_e^3)\partial_Y h_e^0 + (g_p^1 + g_e^2)\partial_Y h_e^1 + (g_p^0 + g_e^1)\partial_Y h_e^2] - \nu_1 \varepsilon \partial_Y^2 u_e^2, \\
R_2^h &= \sqrt{\varepsilon}[(v_p^2 + v_e^3)\partial_Y h_e^0 + (v_p^1 + v_e^2)\partial_Y h_e^1 + (v_p^0 + v_e^1)\partial_Y h_e^2] \\
& - \sqrt{\varepsilon}[(g_p^2 + g_e^3)\partial_Y u_e^0 + (g_p^1 + g_e^2)\partial_Y u_e^1 + (g_p^0 + g_e^1)\partial_Y u_e^2] - \nu_3 \varepsilon \partial_Y^2 h_e^2.
\end{aligned} \tag{4.4}$$

Next, we consider the first order terms of R_{app}^2 and R_{app}^4

$$\left\{ \begin{aligned} R_1^v &= [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](v_p^0 + v_e^1) + (v_p^0 + v_e^2)\partial_Y v_e^1 - (g_p^0 + g_e^1)\partial_Y g_e^1 \\ &+ [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](v_p^1 + v_e^2) + \partial_Y p_e^2 + \frac{p_{py}^2}{\sqrt{\varepsilon}} + \partial_y p_p^3 \\ &- \nu_2 \partial_y^2 (v_p^0 + v_e^1) - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](g_p^0 + g_e^1) \\ &- [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](g_p^1 + g_e^2), \\ R_1^g &= [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](g_p^0 + g_e^1) + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](g_p^1 + g_e^2) \\ &- \nu_4 \partial_y^2 (g_p^0 + g_e^1) - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](v_p^0 + v_e^1) + (v_p^0 + v_e^1)\partial_Y g_e^1 \\ &- [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](v_p^1 + v_e^2) - (g_p^0 + g_e^1)\partial_Y v_e^1. \end{aligned} \right. \tag{4.5}$$

Matching the terms of order ε in $R_1^v = R_1^g = 0$, the second order inner magnetic profile $(u_e^2, v_e^2, h_e^2, g_e^2, p_e^2)$ satisfies

$$\left\{ \begin{aligned} u_e^0 \partial_x v_e^2 - h_e^0 \partial_x g_e^2 + \partial_Y p_e^2 &= f_2^1, \\ u_e^0 \partial_x g_e^2 - h_e^0 \partial_x v_e^2 &= f_4^1, \end{aligned} \right. \tag{4.6}$$

where

$$\begin{aligned} f_2^1 &= -v_e^1 \partial_Y v_e^1 - u_e^1 \partial_x v_e^1 + g_e^1 \partial_Y g_e^1 + h_e^1 \partial_x g_e^1, \\ f_4^1 &= -v_e^1 \partial_Y g_e^1 - u_e^1 \partial_x g_e^1 + g_e^1 \partial_Y v_e^1 + h_e^1 \partial_x v_e^1, \end{aligned}$$

and the next order boundary layer of pressure p_p^3 is

$$\begin{aligned} p_p^3(x, y) &= \int_y^{1/\sqrt{\varepsilon}} [(u_e^1 + u_p^1)\partial_x v_p^0 + u_p^1 \partial_x v_e^1 + (v_p^1 + v_e^2)\partial_y v_p^0 + u_p^0 \partial_x v_e^2 + (u_e^0 + u_p^0)\partial_x v_p^1 \\ &+ v_p^0 \partial_Y v_e^1 - g_p^0 \partial_Y g_e^1 - (h_e^1 + h_p^1)\partial_x g_p^0 - h_p^1 \partial_x g_e^1 - (g_p^1 + g_e^2)\partial_y g_p^0 \\ &- h_p^0 \partial_x g_e^2 - (h_e^0 + h_p^0)\partial_x g_p^1 - \nu_2 \partial_y^2 v_p^1] (x, \theta) d\theta. \end{aligned} \tag{4.7}$$

Then, the errors R_1^v and R_1^g may be rewritten as follows

$$\left\{ \begin{aligned} R_1^v &= \sqrt{\varepsilon}[(v_p^1 + v_e^2)\partial_Y v_e^1 + (v_p^0 + v_e^1)\partial_Y v_e^2 - (g_p^1 + g_e^2)\partial_Y g_e^1 \\ &- (g_p^0 + g_e^1)\partial_Y g_e^2] - \nu_2 \varepsilon \partial_Y^2 v_e^2, \\ R_1^g &= \sqrt{\varepsilon}[(v_p^1 + v_e^2)\partial_Y g_e^1 + (v_p^0 + v_e^1)\partial_Y g_e^2 - (g_p^1 + g_e^2)\partial_Y v_e^1 \\ &- (g_p^0 + g_e^1)\partial_Y v_e^2] - \nu_4 \varepsilon \partial_Y^2 g_e^2. \end{aligned} \right. \tag{4.8}$$

Now, we only consider the ε -order ideal MHD system

$$\left\{ \begin{aligned} u_e^0 \partial_x u_e^2 + v_e^2 \partial_Y u_e^0 - h_e^0 \partial_x h_e^2 - g_e^2 \partial_Y h_e^0 + \partial_x p_e^2 &= f_1^1, \\ u_e^0 \partial_x v_e^2 - h_e^0 \partial_x g_e^2 + \partial_Y p_e^2 &= f_2^1, \\ u_e^0 \partial_x h_e^2 + v_e^2 \partial_Y h_e^0 - h_e^0 \partial_x u_e^2 - g_e^2 \partial_Y u_e^0 &= f_3^1, \\ u_e^0 \partial_x g_e^2 - h_e^0 \partial_x v_e^2 &= f_4^1, \\ \partial_x u_e^2 + \partial_Y v_e^2 &= \partial_x h_e^2 + \partial_Y g_e^2 = 0. \end{aligned} \right. \tag{4.9}$$

The corresponding boundary conditions are

$$\begin{cases} (v_e^2, g_e^2)(x, 0) = -(v_p^1, g_p^1)(x, 0), & (v_e^2, g_e^2)(x, 1) = (0, 0), \\ (v_e^2, g_e^2)(0, Y) = (V_{b0}^2, G_{b0}^2)(Y), \\ (v_e^2, g_e^2)(L, Y) = (V_{bL}^2, G_{bL}^2)(Y). \end{cases} \quad (4.10)$$

We also have the corresponding compatibility assumptions at corners

$$\begin{cases} (V_{b0}^2, G_{b0}^2)(0) = -(v_p^1, g_p^1)(0, 0), \\ (V_{bL}^2, G_{bL}^2)(1) = (0, 0), & (V_{b0}^2, G_{b0}^2)(1) = (0, 0). \end{cases} \quad (4.11)$$

To solve the problem (4.9) – (4.11), we rewrite the third and the fourth equations in (4.9) as

$$\begin{aligned} \partial_Y(u_e^0 g_e^2 - h_e^0 v_e^2) &= -f_3^1, \\ \partial_x(u_e^0 g_e^2 - h_e^0 v_e^2) &= f_4^1, \end{aligned}$$

thus, we have

$$v_e^2 = \frac{u_e^0}{h_e^0} g_e^2 + \frac{u_e \overline{g_p^1} - h_e \overline{v_p^1} - u_e^0 G_{b0}^2 + h_e^0 V_{b0}^2}{2h_e^0} + \frac{\int_0^Y f_3^1(x, \theta) d\theta - \int_0^x f_4^1(s, Y) ds}{2h_e^0} := \frac{u_e^0}{h_e^0} g_e^2 + \frac{b_2^1(x, Y)}{h_e^0},$$

and

$$g_e^2 = \frac{h_e^0}{u_e^0} v_e^2 + \frac{h_e \overline{v_p^1} - u_e \overline{g_p^1} + u_e^0 G_{b0}^2 - h_e^0 V_{b0}^2}{2u_e^0} + \frac{\int_0^x f_4^1(s, Y) ds - \int_0^Y f_3^1(x, \theta) d\theta}{2u_e^0} := \frac{h_e^0}{u_e^0} v_e^2 + \frac{b_2^2(x, Y)}{u_e^0},$$

where

$$b_2^1(x, Y) = \frac{u_e \overline{g_p^1} - h_e \overline{v_p^1} - u_e^0 G_{b0}^2 + h_e^0 V_{b0}^2 + \int_0^Y f_3^1(x, \theta) d\theta - \int_0^x f_4^1(s, Y) ds}{2}$$

and $b_2^2(x, Y) = -b_2^1(x, Y)$. Using the divergence free conditions, it follows from the first and second equations in (4.9) that

$$-u_e^0 \Delta v_e^2 + \partial_Y^2 u_e^0 \cdot v_e^2 + (h_e^0 \Delta g_e^2 - \partial_Y^2 h_e^0 \cdot g_e^2) = f_{1Y}^1 - f_{2x}^1,$$

which satisfies the boundary conditions (4.10). Combining the above three equalities, we consider the following system:

$$\begin{cases} -u_e^0 \Delta v_e^2 + \partial_Y^2 u_e^0 \cdot v_e^2 + (h_e^0 \Delta g_e^2 - \partial_Y^2 h_e^0 \cdot g_e^2) = f_{1Y}^1 - f_{2x}^1, \\ v_e^2 = \frac{u_e^0}{h_e^0} g_e^2 + \frac{b_2^1(x, Y)}{h_e^0}, \\ g_e^2 = \frac{h_e^0}{u_e^0} v_e^2 + \frac{b_2^2(x, Y)}{u_e^0}, \\ (v_e^2, g_e^2)(x, 0) = -(v_p^1, g_p^1)(x, 0), \quad (v_e^2, g_e^2)(x, 1) = (0, 0). \end{cases} \quad (4.12)$$

If $u_e^0(Y) \geq C_0 h_e^0(Y)$, we consider the first and third equations of (4.12). If $h_e^0(Y) \geq C_0 u_e^0(Y)$, we consider the first and second equations of (4.12).

Proposition 4.1. *Suppose that these conditions in Theorem 1.1 are satisfied. Then, for $0 < L_4 \leq L_3$, there exists unique smooth solution (v_e^2, g_e^2) to the boundary value problems (4.12), satisfying*

$$\|(v_e^2, g_e^2)\|_{L^\infty(\Omega_1)} + \|(v_e^2, g_e^2)\|_{H^2(\Omega_1)} \leq C \varepsilon^{-\frac{3}{64}},$$

$$\|(v_e^2, g_e^2)\|_{H^{2+j}(\Omega_1)} \leq C\varepsilon^{-\frac{3j}{32}-\frac{3}{64}}, j = 1, 2, 3, 4. \quad (4.13)$$

where the constant $C > 0$ only depends on the given boundary data. Moreover, it holds that

$$\begin{aligned} \|(v_e^2, g_e^2)\|_{W^{2,q}(\Omega_1)} &\leq C(L)\varepsilon^{-\frac{3}{32}+\frac{3}{32q}}, \\ \|(v_e^2, g_e^2)\|_{W^{2+j,q}(\Omega_1)} &\leq C(L)\varepsilon^{-\frac{3}{32}(j+1)+\frac{3}{32q}}, \end{aligned} \quad (4.14)$$

for $q \in [1, \infty)$, $C(L) > 0$.

We use the arguments of Section 3.1 to prove this proposition.

Proof. The main goal is to establish the estimates of v_e^2 and g_e^2 .

The case $u_e^0(Y) \geq C_0 h_e^0(Y)$: multiplying the first equation in (4.12) by $\frac{v_e^2}{u_e^0}$ and integrating by parts, we obtain

$$\begin{aligned} &\iint_{\Omega_1} |\nabla v_e^2|^2 dx dY + \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{u_e^0} |v_e^2|^2 dx dY \\ &= \iint_{\Omega_1} \frac{h_e^0}{u_e^0} \nabla g_e^2 \cdot \nabla v_e^2 dx dY + \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y g_e^2 \cdot v_e^2 dx dY \\ &\quad + \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{u_e^0} v_e^2 \cdot g_e^2 dx dY + \iint_{\Omega_1} \frac{(f_{1Y}^1 - f_{2x}^1)}{u_e^0} v_e^2 dx dY := \sum_{i=1}^4 I_{1i}. \end{aligned}$$

For I_{1i} , ($i = 1, 2, 3, 4$), we have the following estimates

$$\begin{aligned} |I_{11}| &= \left| \iint_{\Omega_1} \frac{h_e^0}{u_e^0} \nabla g_e^2 \cdot \nabla v_e^2 dx dY \right| \lesssim \left| \frac{h_e^0}{u_e^0} \right| \|\nabla v_e^2\|_{L^2(\Omega_1)} \|\nabla g_e^2\|_{L^2(\Omega_1)}, \\ |I_{12}| &= \left| \iint_{\Omega_1} \partial_Y \left(\frac{h_e^0}{u_e^0} \right) \partial_Y g_e^2 \cdot v_e^2 dx dY \right| \\ &\lesssim \left(\left| \frac{h_e^0}{u_e^0} \right| \left| \frac{\partial_Y u_e^0}{u_e^0} \right| + \left| \frac{\partial_Y h_e^0}{u_e^0} \right| \right) \|v_e^2\|_{L^2(\Omega_1)} \|\partial_Y g_e^2\|_{L^2(\Omega_1)} \\ &\lesssim L \left(\left| \frac{h_e^0}{u_e^0} \right| \left| \frac{\partial_Y u_e^0}{u_e^0} \right| + \left| \frac{\partial_Y h_e^0}{u_e^0} \right| \right) \|\partial_x v_e^2\|_{L^2(\Omega_1)} \|\partial_Y g_e^2\|_{L^2(\Omega_1)}, \end{aligned}$$

and

$$\begin{aligned} |I_{13}| &= \left| \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{u_e^0} v_e^2 \cdot g_e^2 dx dY \right| \\ &\lesssim \|v_e^2\|_{L^2(\Omega_1)} \|g_e^2\|_{L^2(\Omega_1)} \\ &\lesssim L^2 \|\partial_x v_e^2\|_{L^2(\Omega_1)} \|\partial_x g_e^2\|_{L^2(\Omega_1)}, \end{aligned}$$

$$|I_{14}| = \iint_{\Omega_1} \frac{f_{1Y}^1 - f_{2x}^1}{u_e^0} v_e^2 dx dY \lesssim L \|\partial_x v_e^2\|_{L^2(\Omega_1)} \|f_{1Y}^1 - f_{2x}^1\|_{L^2(\Omega_1)} \lesssim L\varepsilon^{-\frac{3}{64}} \|\partial_x v_e^2\|_{L^2(\Omega_1)}.$$

Combining the previous inequalities, we have

$$\|\nabla v_e^2\|_{L^2(\Omega_1)} + \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{u_e^0} |v_e^2|^2 dx dY \lesssim \left| \frac{h_e^0}{u_e^0} \right| \|\nabla v_e^2\|_{L^2(\Omega_1)} \|\nabla g_e^2\|_{L^2(\Omega_1)}$$

$$+ L \left(\left| \frac{h_e^0}{u_e^0} \right| + \left| \frac{\partial_Y h_e^0}{u_e^0} \right| \right) \|\nabla v_e^2\|_{L^2(\Omega_1)} \|\nabla g_e^2\|_{L^2(\Omega_1)} + L \varepsilon^{-\frac{3}{64}} \|\nabla v_e^2\|_{L^2(\Omega_1)}. \quad (4.15)$$

From the third equation in (4.12), we obtain the gradient of g_e^2

$$\nabla g_e^2 = \left(\begin{array}{c} \frac{h_e^0}{u_e^0} \partial_x v_e^2 \\ \frac{h_e^0}{u_e^0} \partial_Y v_e^2 + \partial_Y \left(\frac{h_e^0}{u_e^0} \right) v_e^2 \end{array} \right) + \nabla \frac{b_2^2(x, Y)}{u_e^0}.$$

Thus, we have the following estimate

$$\begin{aligned} \|\nabla g_e^2\|_{L^2(\Omega_1)} &\lesssim \sup_Y \left| \frac{h_e^0}{u_e^0} \right| \|\nabla v_e^2\|_{L^2(\Omega_1)} + L \left(\sup_Y \left| \frac{h_e^0}{u_e^0} \right| + \|\partial_Y h_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x v_e^2\|_{L^2(\Omega_1)} + \left\| \nabla \left(\frac{b_2^2}{u_e^0} \right) \right\|_{L^2(\Omega_1)} \\ &\lesssim 1 + \sup_Y \left| \frac{h_e^0}{u_e^0} \right| \|\nabla v_e^2\|_{L^2(\Omega_1)} + L \left(\sup_Y \left| \frac{h_e^0}{u_e^0} \right| + \|\partial_Y h_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x v_e^2\|_{L^2(\Omega_1)}. \end{aligned}$$

Using the estimates mentioned in Section 3 and inserting the above inequality into (4.15), we obtain

$$\|v_e^2\|_{H^1(\Omega_1)} \leq C \varepsilon^{-\frac{3}{64}},$$

and

$$\|g_e^2\|_{H^1(\Omega_1)} \leq C \varepsilon^{-\frac{3}{64}}.$$

Now, we rewrite the first and fourth equations of (4.12) in the following form

$$\begin{cases} -\Delta v_e^2 + \frac{h_e^0}{u_e^0} \Delta g_e^2 = G_e^2, \\ (v_e^2, g_e^2)|_{\partial\Omega_1} = 0, \end{cases} \quad (4.16)$$

where $G_e^2 := \frac{1}{u_e^0} (f_{1Y}^1 - f_{2x}^1 - \partial_Y^2 u_e^0 \cdot v_e^2 + \partial_Y^2 h_e^0 \cdot g_e^2)$. It is easy to get that $\|G_e^2\|_{L^2(\Omega_1)} \leq C \varepsilon^{-\frac{3}{64}}$ via the previous L^2 estimates on f_1^1, f_2^1 and v_e^2, g_e^2 . Following the method used to estimate v_e^1, g_e^1 , we get the H^2 norms of v_e^2, g_e^2

$$\|(v_e^2, g_e^2)\|_{H^2(\Omega_1)} \leq C \varepsilon^{-\frac{3}{64}}.$$

Similarly, we also obtain the L^∞ norms and higher order estimates

$$\begin{aligned} \|(v_e^2, g_e^2)\|_{L^\infty(\Omega_1)} &\leq C \varepsilon^{-\frac{3}{64}}, \\ \|(v_e^2, g_e^2)\|_{H^{2+j}(\Omega_1)} &\leq C \varepsilon^{-\frac{3}{64} - \frac{3j}{32}}, \end{aligned}$$

where $j = 1, 2, 3, 4$.

The case $h_e^0(Y) \geq C_0 u_e^0(Y)$: multiplying the first equation in (4.12) by $-\frac{g_e^2}{h_e^0}$ and integrating by parts, we have

$$\begin{aligned} &\iint_{\Omega_1} |\nabla g_e^2|^2 dx dY + \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{h_e^0} |g_e^2|^2 dx dY \\ &= \iint_{\Omega_1} \frac{u_e^0}{h_e^0} \nabla g_e^2 \cdot \nabla v_e^2 dx dY + \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y v_e^2 \cdot g_e^2 dx dY \\ &\quad + \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{h_e^0} v_e^2 \cdot g_e^2 dx dY - \iint_{\Omega_1} \frac{(f_{1Y}^1 - f_{2x}^1)}{h_e^0} g_e^2 dx dY := \sum_{i=1}^4 I_{2i}. \end{aligned}$$

For I_{2i} , ($i = 1, 2, 3, 4$), we get the following estimates

$$\begin{aligned}
|I_{21}| &= \left| \iint_{\Omega_1} \frac{u_e^0}{h_e^0} \nabla g_e^2 \cdot \nabla v_e^2 dx dY \right| \lesssim \left| \frac{u_e^0}{h_e^0} \right| \|\nabla v_e^2\|_{L^2(\Omega_1)} \|\nabla g_e^2\|_{L^2(\Omega_1)}, \\
|I_{22}| &= \left| \iint_{\Omega_1} \partial_Y \left(\frac{u_e^0}{h_e^0} \right) \partial_Y v_e^2 \cdot g_e^2 dx dY \right| \\
&\lesssim \left(\left| \frac{u_e^0}{h_e^0} \right| \left| \frac{\partial_Y h_e^0}{h_e^0} \right| + \left| \frac{\partial_Y u_e^0}{h_e^0} \right| \right) \|g_e^2\|_{L^2(\Omega_1)} \|\partial_Y v_e^2\|_{L^2(\Omega_1)} \\
&\lesssim L \left(\left| \frac{u_e^0}{h_e^0} \right| \left| \frac{\partial_Y h_e^0}{h_e^0} \right| + \left| \frac{\partial_Y u_e^0}{h_e^0} \right| \right) \|\partial_x g_e^2\|_{L^2(\Omega_1)} \|\partial_Y v_e^2\|_{L^2(\Omega_1)}, \\
|I_{23}| &= \left| \iint_{\Omega_1} \frac{\partial_Y^2 u_e^0}{h_e^0} v_e^2 \cdot g_e^2 dx dY \right| \\
&\lesssim \|v_e^2\|_{L^2(\Omega_1)} \|g_e^2\|_{L^2(\Omega_1)} \lesssim L^2 \|\partial_x v_e^2\|_{L^2(\Omega_1)} \|\partial_x g_e^2\|_{L^2(\Omega_1)}, \\
|I_{24}| &= \left| - \iint_{\Omega_1} \frac{f_{1Y}^1 - f_{2x}^1}{h_e^0} g_e^2 dx dY \right| \lesssim L \|\partial_x g_e^2\|_{L^2(\Omega_1)} \|f_{1Y}^1 - f_{2x}^1\|_{L^2(\Omega_1)} \lesssim L \varepsilon^{-\frac{3}{64}} \|\partial_x g_e^2\|_{L^2(\Omega_1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\nabla g_e^2\|_{L^2(\Omega_1)} + \iint_{\Omega_1} \frac{\partial_Y^2 h_e^0}{h_e^0} |g_e^2|^2 dx dY \lesssim \left| \frac{u_e^0}{h_e^0} \right| \|\nabla v_e^2\|_{L^2(\Omega_1)} \|\nabla g_e^2\|_{L^2(\Omega_1)} \\
&\quad + L \left(\left| \frac{u_e^0}{h_e^0} \right| + \left| \frac{\partial_Y u_e^0}{h_e^0} \right| \right) \|\nabla v_e^2\|_{L^2(\Omega_1)} \|\nabla g_e^2\|_{L^2(\Omega_1)} + L \varepsilon^{-\frac{3}{64}} \|\nabla g_e^2\|_{L^2(\Omega_1)}. \tag{4.17}
\end{aligned}$$

From the second equation in (4.12), we obtain the gradient of v_e^2

$$\nabla v_e^2 = \begin{pmatrix} \frac{u_e^0}{h_e^0} \partial_x g_e^2 \\ \frac{u_e^0}{h_e^0} \partial_Y g_e^2 + \partial_Y \left(\frac{u_e^0}{h_e^0} \right) g_e^2 \end{pmatrix} + \nabla \frac{b_2^1(x, Y)}{h_e^0}.$$

Thus,

$$\begin{aligned}
\|\nabla v_e^2\|_{L^2(\Omega_1)} &\lesssim \sup_Y \left| \frac{u_e^0}{h_e^0} \right| \|\nabla g_e^2\|_{L^2(\Omega_1)} + L \left(\sup_Y \left| \frac{u_e^0}{h_e^0} \right| + \|\partial_Y u_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x g_e^2\|_{L^2(\Omega_1)} + \left\| \nabla \left(\frac{b_2^1}{h_e^0} \right) \right\|_{L^2(\Omega_1)} \\
&\lesssim 1 + \sup_Y \left| \frac{u_e^0}{h_e^0} \right| \|\nabla g_e^2\|_{L^2(\Omega_1)} + L \left(\sup_Y \left| \frac{u_e^0}{h_e^0} \right| + \|\partial_Y u_e^0\|_{L^\infty(\Omega_1)} \right) \|\partial_x g_e^2\|_{L^2(\Omega_1)}.
\end{aligned}$$

Similarly, inserting the previous inequality into (4.17), we obtain

$$\|(v_e^2, g_e^2)\|_{H^1(\Omega_1)} \leq C \varepsilon^{-\frac{3}{64}}.$$

Now, we rewrite the equations (4.12)

$$\begin{cases} -\Delta g_e^2 + \frac{u_e^0}{h_e^0} \Delta v_e^2 = M_e^2, \\ (v_e^2, g_e^2)|_{\partial\Omega_1} = 0, \end{cases} \tag{4.18}$$

where $M_e^2 := \frac{1}{h_e^0}(-f_{1Y}^1 + f_{2x}^1 + \partial_Y^2 u_e^0 \cdot v_e^2 - \partial_Y^2 h_e^0 \cdot g_e^2)$. It is obvious that $\|M_e^2\|_{L^2(\Omega_1)} \leq C\varepsilon^{-\frac{3}{64}}$. Using the same method as in the previous case, we have

$$\begin{aligned}\|(v_e^2, g_e^2)\|_{H^2(\Omega_1)} &\leq C\varepsilon^{-\frac{3}{64}}, \\ \|(v_e^2, g_e^2)\|_{L^\infty(\Omega_1)} &\leq C\varepsilon^{-\frac{3}{64}}.\end{aligned}$$

Similarly, we also obtain the higher order estimates

$$\|(v_e^2, g_e^2)\|_{H^{2+j}(\Omega_1)} \leq C\varepsilon^{-\frac{3}{64} - \frac{3j}{32}},$$

where $j = 1, 2, 3, 4$.

This leads to the desired estimates (4.13). By the standard elliptic theory, we obtain $W^{k,p}$ estimates of v_e^2 and g_e^2 . This finishes the proof of the proposition. \square

Next, we study the estimates of u_e^2, h_e^2 and p_e^2 . Using (4.9) and the divergence free conditions, we have

$$\begin{aligned}u_e^2(x, Y) &= u_b^2(Y) - \int_0^x v_{eY}^2(s, Y) ds, \\ h_e^2(x, Y) &= h_b^2(Y) - \int_0^x g_{eY}^2(s, Y) ds, \\ p_e^2(x, y) &= \int_Y^1 [u_e^0 v_{ex}^2 - h_e^0 g_{ex}^2 - f_2^1](x, \theta) d\theta - \int_0^x [u_e^0(1) v_{eY}^2(s, 1) - h_e^0(1) g_{eY}^2(s, 1) - f_1^1(s, 1)] ds,\end{aligned}\tag{4.19}$$

where $u_e^2(0, Y) = u_b^2(Y)$ and $h_e^2(0, Y) = h_b^2(Y)$ satisfy $\partial_Y u_b^2(1) = \partial_Y h_b^2(1) = 0$. Hence, we have $u_{eY}^2(x, 1) = h_{eY}^2(x, 1) = 0$. By definition of (u_e^2, h_e^2) in (4.19) and Proposition 4.1, we get

$$\|(u_e^2, h_e^2)\|_{L^\infty(\Omega_1)} + \|(u_e^2, h_e^2)\|_{H^1(\Omega_1)} \leq C\varepsilon^{-\frac{3}{64}}, \quad \|(u_e^2, h_e^2)\|_{H^{1+j}(\Omega_1)} \leq C\varepsilon^{-\frac{3}{64} - \frac{3j}{32}}, \quad j = 1, 2, 3, 4.\tag{4.20}$$

4.2 The second order MHD boundary layer profile

We now construct the second order magnetic boundary layer profile $(u_p^2, v_p^2, h_p^2, g_p^2, p_p^2)$. System (4.3) can be rewritten

$$\begin{cases} u^0 \partial_x u_p^2 + u_p^2 \partial_x u^0 + v_p^2 \partial_y u^0 + (v_p^0 + \overline{v_e^1}) \partial_y u_p^2 - \nu_1 \partial_y^2 u_p^2 + \partial_x P_p^2 \\ \quad - h^0 \partial_x h_p^2 - h_p^2 \partial_x h^0 - g_p^2 \partial_y h^0 - (g_p^0 + \overline{g_e^1}) \partial_y h_p^2 = F_{p_1}^2, \\ u^0 \partial_x h_p^2 + u_p^2 \partial_x h^0 + v_p^2 \partial_y h^0 + (v_p^0 + \overline{v_e^1}) \partial_y h_p^2 - \nu_3 \partial_y^2 h_p^2 \\ \quad - h^0 \partial_x u_p^2 - h_p^2 \partial_x u^0 - g_p^2 \partial_y u^0 - (g_p^0 + \overline{g_e^1}) \partial_y u_p^2 = F_{p_2}^2, \end{cases}\tag{4.21}$$

with the boundary conditions

$$\begin{aligned}(u_p^2, h_p^2)(0, y) &= (\tilde{u}_2(y), \tilde{h}_2(y)), (u_p^2, h_{py}^2)(x, 0) = -(u_e^2, h_{eY}^1)(x, 0), \\ u_{py}^2(x, \varepsilon^{-1/2}) &= v_p^2(x, 0) = v_p^2(x, \varepsilon^{-1/2}) = 0, \\ h_{py}^2(x, \varepsilon^{-1/2}) &= g_p^2(x, 0) = g_p^2(x, \varepsilon^{-1/2}) = 0,\end{aligned}\tag{4.22}$$

where

$$\begin{cases} F_{p_1}^2 = - \left[\varepsilon^{-\frac{1}{2}} E_{r_1}^1 + u_p^1 \partial_x u_e^1 + (u_e^1 + u_p^1) \partial_x u_p^1 + v_p^1 \partial_Y u_e^0 + v_e^2 \partial_y u_p^1 + u_p^0 \partial_x u_e^2 + u_e^2 \partial_x u_p^0 \right. \\ \quad \left. - h_p^1 \partial_x h_e^1 - (h_e^1 + h_p^1) \partial_x h_p^1 - g_p^1 \partial_Y h_e^0 - g_e^2 \partial_y h_p^1 - h_p^0 \partial_x h_e^2 - h_e^2 \partial_x h_p^0 - \nu_1 \partial_x^2 u_p^1 \right], \\ F_{p_2}^2 = - \left[\varepsilon^{-\frac{1}{2}} E_{r_2}^1 + u_p^1 \partial_x h_e^1 + (u_e^1 + u_p^1) \partial_x h_p^1 + v_p^1 \partial_Y h_e^0 + v_e^2 \partial_y h_p^1 + u_p^0 \partial_x h_e^2 + u_e^2 \partial_x h_p^0 \right. \\ \quad \left. - h_p^1 \partial_x u_e^1 - (h_e^1 + h_p^1) \partial_x u_p^1 - g_p^1 \partial_Y u_e^0 - g_e^2 \partial_y u_p^1 - h_p^0 \partial_x u_e^2 - h_e^2 \partial_x u_p^0 - \nu_3 \partial_x^2 h_p^1 \right]. \end{cases}$$

Therefore,

$$\begin{cases} E_{r_1}^2 := (u_e^0 - u_e) \partial_x u_p^2 + (v_e^1 - \overline{v_e^1}) \partial_y u_p^2 - (h_e^0 - h_e) \partial_x h_p^2 - (g_e^1 - \overline{g_e^1}) \partial_y h_p^2 \\ \quad + v_e^3 \partial_y u_p^0 + v_p^1 \partial_y u_p^1 - g_e^3 \partial_y h_p^0 - g_p^1 \partial_y h_p^1 + \frac{E_1}{\varepsilon} + \Delta_2 \\ \quad + v_p^0 \partial_Y u_e^1 - g_p^0 \partial_Y h_e^1 + \Delta_5, \\ E_{r_2}^2 := (u_e^0 - u_e) \partial_x h_p^2 + (v_e^1 - \overline{v_e^1}) \partial_y h_p^2 - (h_e^0 - h_e) \partial_x u_p^2 - (g_e^1 - \overline{g_e^1}) \partial_y u_p^2 \\ \quad + v_e^3 \partial_y h_p^0 + v_p^1 \partial_y h_p^1 - g_e^3 \partial_y u_p^0 - g_p^1 \partial_y u_p^1 + \frac{E_3}{\varepsilon} + \Delta_4 \\ \quad + v_p^0 \partial_Y h_e^1 - g_p^0 \partial_Y u_e^1 + \Delta_7. \end{cases} \quad (4.23)$$

Next, we estimate the solution $(u_p^2, v_p^2, h_p^2, g_p^2)$ to (4.21). We follow the method of Section 3.2 and omit the details. First, we extend the domain I_ε to \mathbb{R}_+ and establish the estimates on $[0, L] \times \mathbb{R}_+$. Second, we cut-off the domain from \mathbb{R}_+ to I_ε .

Step1: We apply the method used to estimate $(u_p^{1,\infty}, v_p^{1,\infty}, h_p^{1,\infty}, g_p^{1,\infty})$ to bound $(u_p^{2,\infty}, v_p^{2,\infty}, h_p^{2,\infty}, g_p^{2,\infty})$ on $[0, L] \times \mathbb{R}_+$.

Proposition 4.2. *For $0 < L_5 \leq L_4$, there exists a smooth solution $(u_p^{2,\infty}, v_p^{2,\infty}, h_p^{2,\infty}, g_p^{2,\infty})$ to the problem (4.21) in $[0, L_5] \times [0, \infty)$ such that the following estimates hold*

$$\begin{aligned} & \| (u_p^{2,\infty}, v_p^{2,\infty}, h_p^{2,\infty}, g_p^{2,\infty}) \|_{L^\infty(0, L_5; \mathbb{R}_+)} + \sup_{0 \leq x \leq L_5} \| \langle y \rangle^l \partial_{yy} (v_p^{2,\infty}, g_p^{2,\infty}) \|_{L^2(0, \infty)} \\ & + \| \langle y \rangle^l \partial_{xy} (v_p^{2,\infty}, g_p^{2,\infty}) \|_{L^2(0, L_5; \mathbb{R}_+)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{32}}, \\ & \sup_{0 \leq x \leq L_5} \| \langle y \rangle^l \partial_{yy} (v_p^{2,\infty}, g_p^{2,\infty}) \|_{H^j(0, \infty)} + \| \langle y \rangle^l \partial_{xy} (v_p^{2,\infty}, g_p^{2,\infty}) \|_{H^j(0, L_5; \mathbb{R}_+)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{32} - \frac{3j}{32}}, \end{aligned} \quad (4.24)$$

for some small positive constant ξ and for $j = 1, 2, 3$.

Step 2: Now, we cut-off the solution from $[0, L] \times \mathbb{R}_+$ to Ω_ε as in Section 3.2, which leads to

Proposition 4.3. *Under the assumptions of Theorem 1.1, there exists smooth functions $(u_p^2, v_p^2, h_p^2, g_p^2)$ satisfying the following inhomogeneous system*

$$\begin{cases} u^0 u_{px}^2 + u_x^0 u_p^2 + u_y^0 v_p^2 + [v_p^0 + \overline{v_e^1}] u_{py}^2 + \partial_x p_p^2 - \nu_1 \partial_y^2 u_p^2 \\ \quad - h^0 h_{px}^2 - h_x^0 h_p^2 - h_y^0 g_p^2 - [g_p^0 + \overline{g_e^1}] h_{py}^2 = R_p^{u,2}, \\ u^0 h_{px}^2 + h_x^0 u_p^2 + h_y^0 v_p^2 + [v_p^0 + \overline{v_e^1}] h_{py}^2 - \nu_3 \partial_y^2 h_p^2 \\ \quad - h^0 u_{px}^2 - u_x^0 h_p^2 - u_y^0 g_p^2 - [g_p^0 + \overline{g_e^1}] u_{py}^2 = R_p^{h,2}, \\ u_{px}^2 + v_{py}^2 = h_{px}^2 + g_{py}^2 = 0, \\ (u_p^2, h_p^2)(0, y) = (\tilde{u}_2(y), \tilde{h}_2(y)), (u_p^2, \partial_y h_p^2)(x, 0) = -(u_e^2, \partial_Y h_e^1)(x, 0), \\ (u_{py}^2, h_{py}^2)(x, \varepsilon^{-1/2}) = (v_p^2, g_p^2)(x, 0) = (v_p^2, g_p^2)(x, \varepsilon^{-1/2}) = 0, \end{cases} \quad (4.25)$$

such that, for any given $l \in \mathbb{N}$,

$$\begin{aligned} & \| (u_p^2, v_p^2, h_p^2, g_p^2) \|_{L^\infty(\Omega_\varepsilon)} + \sup_{0 \leq x \leq L_5} \| \langle y \rangle^l \partial_{yy} (v_p^2, g_p^2) \|_{L^2(I_\varepsilon)} + \| \langle y \rangle^l \partial_{xy} (v_p^2, g_p^2) \|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{32}}, \\ & \sup_{0 \leq x \leq L_5} \| \langle y \rangle^l \partial_{yy} (v_p^2, g_p^2) \|_{H^j(0, \infty)} + \| \langle y \rangle^l \partial_{xy} (v_p^2, g_p^2) \|_{H^j(0, L_5; \mathbb{R}_+)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{32} - \frac{3j}{32}}, \end{aligned} \quad (4.26)$$

where $R_p^{u,2}$ and $R_p^{h,2}$ are higher order terms of $\sqrt{\varepsilon}$ and $j = 1, 2, 3$.

Proof. The solution $(u_p^{2,\infty}, v_p^{2,\infty}, h_p^{2,\infty}, g_p^{2,\infty})$ has been constructed in Proposition 4.2. We define

$$\begin{aligned} (u_p^2, h_p^2)(x, y) &:= \chi(\sqrt{\varepsilon}y)(u_p^{2,\infty}, h_p^{2,\infty})(x, y) + \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_0^y (u_p^{2,\infty}, h_p^{2,\infty})(x, \theta) d\theta, \\ (v_p^2, g_p^2)(x, y) &:= \chi(\sqrt{\varepsilon}y)(v_p^{2,\infty}, g_p^{2,\infty})(x, y). \end{aligned} \quad (4.27)$$

It is clear that $(u_p^2, v_p^2, h_p^2, g_p^2)$ satisfies (4.25). Thus, we have

$$\left\{ \begin{aligned} R_p^{u,2} &:= \sqrt{\varepsilon}\chi'[u_p^0 v_p^{2,\infty} - h_p^0 g_p^{2,\infty}] + \sqrt{\varepsilon}\chi'[u_x^0 \int_0^y u_p^{2,\infty} d\theta - h_x^0 \int_0^y h_p^{2,\infty} d\theta] \\ &\quad + 2\sqrt{\varepsilon}\chi'[v_p^0 + \overline{v_e^1}]u_p^{2,\infty} - 2\sqrt{\varepsilon}\chi'[g_p^0 + \overline{g_e^1}]h_p^{2,\infty} - 3\sqrt{\varepsilon}\chi' u_{py}^{2,\infty} \\ &\quad + \varepsilon\chi''\{[v_p^0 + \overline{v_e^1}] \int_0^y u_p^{2,\infty} d\theta - [g_p^0 + \overline{g_e^1}] \int_0^y h_p^{2,\infty} d\theta\} \\ &\quad + \sqrt{\varepsilon}F_{p1}^2 \int_0^y \chi' d\theta - 3\varepsilon\chi'' u_p^{1,\infty} - \varepsilon^{3/2}\chi''' \int_0^y u_p^{1,\infty} d\theta \\ &:= \sqrt{\varepsilon}\Delta_9 + \varepsilon\Delta_{10}, \\ R_p^{h,2} &:= \sqrt{\varepsilon}\chi'[u_p^0 g_p^{2,\infty} - h_p^0 v_p^{2,\infty}] + \sqrt{\varepsilon}\chi'[h_x^0 \int_0^y u_p^{2,\infty} d\theta - u_x^0 \int_0^y h_p^{2,\infty} d\theta] \\ &\quad + 2\sqrt{\varepsilon}\chi'[v_p^0 + \overline{v_e^1}]h_p^{2,\infty} - 2\sqrt{\varepsilon}\chi'[g_p^0 + \overline{g_e^1}]u_p^{2,\infty} - 3\sqrt{\varepsilon}\chi' h_{py}^{2,\infty} \\ &\quad + \varepsilon\chi''\{[v_p^0 + \overline{v_e^1}] \int_0^y h_p^{2,\infty} d\theta - [g_p^0 + \overline{g_e^1}] \int_0^y u_p^{2,\infty} d\theta\} \\ &\quad + \sqrt{\varepsilon}F_{p2}^2 \int_0^y \chi' d\theta - 3\varepsilon\chi'' h_p^{2,\infty} - \varepsilon^{3/2}\chi''' \int_0^y u_p^{2,\infty} d\theta \\ &:= \sqrt{\varepsilon}\Delta_{11} + \varepsilon\Delta_{12}. \end{aligned} \right. \quad (4.28)$$

It is direct to check that

$$\partial_x u_p^2 + \partial_y v_p^2 = \partial_x h_p^2 + \partial_y g_p^2 = 0.$$

Next, by using Proposition 4.2, we have

$$\begin{aligned} \left| \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_0^y u_p^{2,\infty}(x, \theta) d\theta \right| &\leq \sqrt{\varepsilon}y |\chi'(\sqrt{\varepsilon}y)| \|u_p^{2,\infty}\|_{L^\infty} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{32}}, \\ \left| \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_0^y h_p^{2,\infty}(x, \theta) d\theta \right| &\leq \sqrt{\varepsilon}y |\chi'(\sqrt{\varepsilon}y)| \|h_p^{2,\infty}\|_{L^\infty} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{32}}. \end{aligned}$$

Hence, applying Proposition 4.2, we obtain (4.26). This completes the proof of the proposition. \square

5 The third order ideal MHD profile and boundary layer profile

In this section, we consider $(u_e^3, v_e^3, h_e^3, g_e^3, p_e^3)$ and $(u_p^3, v_p^3, h_p^3, g_p^3, p_p^3)$. First, we construct the third order ideal MHD flow.

5.1 The third order ideal MHD profile

Let us define R_3^u and R_3^h by

$$\left\{ \begin{aligned} R_3^u &= [(u_e^3 + u_p^3)\partial_x + v_p^3\partial_y](u_e^0 + u_p^0) + [(u_e^2 + u_p^2)\partial_x + (v_p^2 + v_e^3)\partial_y](u_e^1 + u_p^1) \\ &\quad + [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](u_e^2 + u_p^2) + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](u_e^3 + u_p^3) \\ &\quad - [(h_e^3 + h_p^3)\partial_x + g_p^3\partial_y](h_e^0 + h_p^0) - [(h_e^2 + h_p^2)\partial_x + (g_p^2 + g_e^3)\partial_y](h_e^1 + h_p^1) \\ &\quad - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](h_e^2 + h_p^2) - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](h_e^3 + h_p^3) \\ &\quad + \partial_x(p_e^3 + p_p^3) - \nu_1\partial_y^2(u_e^3 + u_p^3) + [(v_p^2 + v_e^3)\partial_Y u_e^0 + (v_p^1 + v_e^2)\partial_Y u_e^1] \\ &\quad + (v_p^0 + v_e^1)\partial_Y u_e^2 - [(g_p^2 + g_e^3)\partial_Y h_e^0 + (g_p^1 + g_e^2)\partial_Y h_e^1 + (g_p^0 + g_e^1)\partial_Y h_e^2] \\ &\quad + \nu_1\partial_x^2 u_p^1 + \Delta_6 + \varepsilon^{-\frac{1}{2}}E_{r_1}^2 + \Delta_9 - \nu_1\Delta u_e^1, \\ R_3^h &= [(u_e^3 + u_p^3)\partial_x + v_p^3\partial_y](h_e^0 + h_p^0) + [(u_e^2 + u_p^2)\partial_x + (v_p^2 + v_e^3)\partial_y](h_e^1 + h_p^1) \\ &\quad + [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](h_e^2 + h_p^2) + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](h_e^3 + h_p^3) \\ &\quad - [(h_e^3 + h_p^3)\partial_x + g_p^3\partial_y](u_e^0 + u_p^0) - [(h_e^2 + h_p^2)\partial_x + (g_p^2 + g_e^3)\partial_y](u_e^1 + u_p^1) \\ &\quad - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](u_e^2 + u_p^2) - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](u_e^3 + u_p^3) \\ &\quad - \nu_3\partial_y^2(h_e^3 + h_p^3) + [(v_p^2 + v_e^3)\partial_Y h_e^0 + (v_p^1 + v_e^2)\partial_Y h_e^1 + (v_p^0 + v_e^1)\partial_Y h_e^2] \\ &\quad - [(g_p^2 + g_e^3)\partial_Y u_e^0 + (g_p^1 + g_e^2)\partial_Y u_e^1 + (g_p^0 + g_e^1)\partial_Y u_e^2] + \nu_3\partial_x^2 h_p^1 \\ &\quad + \Delta_8 + \varepsilon^{-\frac{1}{2}}E_{r_2}^2 + \Delta_{11} - \nu_3\Delta h_e^1. \end{aligned} \right. \quad (5.1)$$

As the inner correctors are always calculated at $(x, \sqrt{\varepsilon}y)$, we have the following higher order terms

$$\begin{aligned} (v_p^2 + v_e^3)\partial_y u_e^1 &= \sqrt{\varepsilon}(v_p^2 + v_e^3)\partial_Y u_e^1, & (g_p^2 + g_e^3)\partial_y h_e^1 &= \sqrt{\varepsilon}(g_p^2 + g_e^3)\partial_Y h_e^1, \\ (v_p^2 + v_e^3)\partial_y h_e^1 &= \sqrt{\varepsilon}(v_p^2 + v_e^3)\partial_Y h_e^1, & (g_p^2 + g_e^3)\partial_y u_e^1 &= \sqrt{\varepsilon}(g_p^2 + g_e^3)\partial_Y u_e^1, \\ \partial_y^2 u_e^2 &= \varepsilon\partial_Y^2 u_e^2, & \partial_y^2 h_e^2 &= \varepsilon\partial_Y^2 h_e^2. \end{aligned}$$

The remaining higher terms of (5.1) are of the same form. Hence, they will be omitted here. By matching the terms of order ε , we obtain the following third order ideal MHD system

$$\left\{ \begin{aligned} u_e^0\partial_x u_e^3 + v_e^3\partial_Y u_e^0 - h_e^0\partial_x h_e^3 - g_e^3\partial_Y h_e^0 + \partial_x p_e^3 &= f_1^2, \\ u_e^0\partial_x h_e^3 + v_e^3\partial_Y h_e^0 - h_e^0\partial_x u_e^3 - g_e^3\partial_Y u_e^0 &= f_3^2, \end{aligned} \right. \quad (5.2)$$

where

$$\begin{aligned} f_1^2 &= -[u_e^1 u_{ex}^2 + u_e^2 u_{ex}^1 + v_e^1 u_{eY}^2 + v_e^2 u_{eY}^1 \\ &\quad - h_e^1 h_{ex}^2 - h_e^2 h_{ex}^1 - g_e^1 h_{eY}^2 - g_e^2 h_{eY}^1 - \nu_1 \Delta u_e^1], \\ f_3^2 &= -[u_e^1 h_{ex}^2 + u_e^2 h_{ex}^1 + v_e^1 h_{eY}^2 + v_e^2 h_{eY}^1 \\ &\quad - h_e^1 u_{ex}^2 - h_e^2 u_{ex}^1 - g_e^1 u_{eY}^2 - g_e^2 u_{eY}^1 - \nu_3 \Delta h_e^1], \end{aligned}$$

and also obtain the third order MHD boundary layer system

$$\left\{ \begin{array}{l} (u_e^3 + u_p^3)\partial_x u_p^0 + u_p^2\partial_x u_e^1 + (u_e^2 + u_p^2)\partial_x u_p^1 + u_p^1\partial_x u_e^2 + (u_e^1 + u_p^1)\partial_x u_p^2 \\ + u_p^0\partial_x u_e^3 + (u_e^0 + u_p^0)\partial_x u_p^3 + v_p^3\partial_y(u_e^0 + u_p^0) + (v_p^2 + v_e^3)\partial_y u_p^1 + (v_p^1 + v_e^2)\partial_y u_p^2 \\ + (v_p^0 + v_e^1)\partial_y u_p^3 + \partial_x p_p^3 - \nu_1\partial_y^2 u_p^3 - (h_e^3 + h_p^3)\partial_x h_p^0 - h_p^2\partial_x h_e^1 \\ - (h_e^2 + h_p^2)\partial_x h_p^1 - h_p^1\partial_x h_e^2 - (h_e^1 + h_p^1)\partial_x h_p^2 - h_p^0\partial_x h_e^3 - (h_e^0 + h_p^0)\partial_x h_p^3 \\ - g_p^3\partial_y(h_e^0 + h_p^0) - (g_p^2 + g_e^3)\partial_y h_p^1 - (g_p^1 + g_e^2)\partial_y h_p^2 - (g_p^0 + g_e^1)\partial_y h_p^3 \\ + v_p^2\partial_Y u_e^0 + v_p^1\partial_Y u_e^1 + v_p^0\partial_Y u_e^2 - g_p^2\partial_Y h_e^0 - g_p^1\partial_Y h_e^1 - g_p^0\partial_Y h_e^2 \\ + \Delta_6 - \nu_1\partial_x^2 u_p^1 + \Delta_9 + \varepsilon^{-\frac{1}{2}}E_{r_1}^2 = 0, \\ (u_e^3 + u_p^3)\partial_x h_p^0 + u_p^2\partial_x h_e^1 + (u_e^2 + u_p^2)\partial_x h_p^1 + u_p^1\partial_x h_e^2 + (u_e^1 + u_p^1)\partial_x h_p^2 \\ + u_p^0\partial_x h_e^3 + (u_e^0 + u_p^0)\partial_x h_p^3 + v_p^3\partial_y(h_e^0 + h_p^0) + (v_p^2 + v_e^3)\partial_y h_p^1 + (v_p^1 + v_e^2)\partial_y h_p^2 \\ + (v_p^0 + v_e^1)\partial_y h_p^3 - \nu_3\partial_y^2 h_p^3 - (h_e^3 + h_p^3)\partial_x u_p^0 - h_p^2\partial_x u_e^1 \\ - (h_e^2 + h_p^2)\partial_x u_p^1 - h_p^1\partial_x u_e^2 - (h_e^1 + h_p^1)\partial_x u_p^2 - h_p^0\partial_x u_e^3 - (h_e^0 + h_p^0)\partial_x u_p^3 \\ - g_p^3\partial_y(u_e^0 + u_p^0) - (g_p^2 + g_e^3)\partial_y u_p^1 - (g_p^1 + g_e^2)\partial_y u_p^2 - (g_p^0 + g_e^1)\partial_y u_p^3 \\ + v_p^2\partial_Y h_e^0 + v_p^1\partial_Y h_e^1 + v_p^0\partial_Y h_e^2 - g_p^2\partial_Y u_e^0 - g_p^1\partial_Y u_e^1 - g_p^0\partial_Y u_e^2 \\ + \Delta_8 - \nu_3\partial_x^2 h_p^1 + \Delta_{11} + \varepsilon^{-\frac{1}{2}}E_{r_2}^2 = 0. \end{array} \right. \quad (5.3)$$

Hence, the errors R_3^u and R_3^h equal

$$\left\{ \begin{array}{l} R_3^u = \sqrt{\varepsilon}[(v_p^2 + v_e^3)\partial_Y u_e^1 + (v_p^1 + v_e^2)\partial_Y u_e^2 + (v_p^0 + v_e^1)\partial_Y u_e^3] \\ - \sqrt{\varepsilon}[(g_p^2 + g_e^3)\partial_Y h_e^1 + (g_p^1 + g_e^2)\partial_Y h_e^2 + (g_p^0 + g_e^1)\partial_Y h_e^3] - \nu_1\varepsilon\partial_Y^2 u_e^3, \\ R_3^h = \sqrt{\varepsilon}[(v_p^2 + v_e^3)\partial_Y h_e^1 + (v_p^1 + v_e^2)\partial_Y h_e^2 + (v_p^0 + v_e^1)\partial_Y h_e^3] \\ - \sqrt{\varepsilon}[(g_p^2 + g_e^3)\partial_Y u_e^1 + (g_p^1 + g_e^2)\partial_Y u_e^2 + (g_p^0 + g_e^1)\partial_Y u_e^3] - \nu_3\varepsilon\partial_Y^2 h_e^3. \end{array} \right. \quad (5.4)$$

Next, we consider the third order terms of R_{app}^2 and R_{app}^4

$$\left\{ \begin{array}{l} R_2^v = [(u_e^2 + u_p^2)\partial_x + (v_p^2 + v_e^3)\partial_y](v_p^0 + v_e^1) + [(u_e^1 + u_p^1)\partial_x + (v_p^1 + v_e^2)\partial_y](v_p^1 + v_e^2) \\ + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](v_p^2 + v_e^3) + \partial_Y p_e^3 + \partial_Y p_p^4 - \nu_2\partial_y^2(v_p^2 + v_e^3) \\ - [(h_e^2 + h_p^2)\partial_x + (g_p^2 + g_e^3)\partial_y](g_p^0 + g_e^1) - [(h_e^1 + h_p^1)\partial_x + (g_p^1 + g_e^2)\partial_y](g_p^1 + g_e^2) \\ - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](g_p^2 + g_e^3) - \nu_2\Delta v_e^1 + (v_p^1 + v_e^2)\partial_Y v_e^1 \\ + (v_p^0 + v_e^2)\partial_Y v_e^2 - (g_p^1 + g_e^2)\partial_Y g_e^1 - (g_p^0 + g_e^1)\partial_Y g_e^2, \\ R_2^g = [(u_e^2 + u_p^2)\partial_x + (v_p^2 + v_e^3)\partial_y](g_p^0 + g_e^1) + (u_e^1 + u_p^1)\partial_x(g_p^1 + g_e^2) \\ + (v_p^1 + v_e^2)\partial_y(g_p^1 + g_e^2) + [(u_e^0 + u_p^0)\partial_x + (v_p^0 + v_e^1)\partial_y](g_p^2 + g_e^3) \\ - \nu_4\partial_y^2(g_p^2 + g_e^3) - [(h_e^2 + h_p^2)\partial_x + (g_p^2 + g_e^3)\partial_y](v_p^0 + v_e^1) - (g_p^1 + g_e^2)\partial_y(v_p^1 + v_e^2) \\ - (h_e^1 + h_p^1)\partial_x(v_p^1 + v_e^2) - [(h_e^0 + h_p^0)\partial_x + (g_p^0 + g_e^1)\partial_y](v_p^2 + v_e^3) - \nu_4\Delta g_e^1 \\ + (v_p^1 + v_e^2)\partial_Y g_e^1 + (v_p^0 + v_e^2)\partial_Y g_e^2 - (g_p^1 + g_e^2)\partial_Y v_e^1 - (g_p^0 + g_e^1)\partial_Y v_e^2. \end{array} \right. \quad (5.5)$$

Writing $R_2^v = R_2^g = 0$ and matching the various terms of order ε , the third order ideal MHD profile $(u_e^3, v_e^3, h_e^3, g_e^3, p_e^3)$ satisfies

$$\left\{ \begin{array}{l} u_e^0\partial_x v_e^3 - h_e^0\partial_x g_e^3 + \partial_Y p_e^3 = f_2^2, \\ u_e^0\partial_x g_e^3 - h_e^0\partial_x v_e^3 = f_4^2, \end{array} \right. \quad (5.6)$$

where

$$\begin{aligned} f_2^2 &= - [u_e^1 \partial_x v_e^2 + u_e^2 \partial_x v_e^1 + v_e^1 \partial_Y v_e^2 + v_e^2 \partial_Y v_e^1 \\ &\quad - h_e^1 \partial_x g_e^2 - h_e^2 \partial_x g_e^1 - g_e^1 \partial_Y g_e^2 - g_e^2 \partial_Y g_e^1 - \nu_2 \Delta v_e^1], \\ f_4^2 &= - [u_e^1 \partial_x g_e^2 + u_e^2 \partial_x g_e^1 + v_e^1 \partial_Y g_e^2 + v_e^2 \partial_Y g_e^1 \\ &\quad - h_e^1 \partial_x v_e^2 - h_e^2 \partial_x v_e^1 - g_e^1 \partial_Y v_e^2 - g_e^2 \partial_Y v_e^1 - \nu_4 \Delta h_e^1], \end{aligned}$$

and p_p^4 is obtained through the first equation of (5.5)

$$\begin{aligned} p_p^4(x, y) &= \int_y^{1/\sqrt{\varepsilon}} [(u_e^2 + u_p^2) \partial_x v_p^0 + u_p^2 \partial_x v_e^1 + (v_p^2 + v_e^3) \partial_y v_p^0 + u_p^1 \partial_x v_e^2 + (u_e^1 + u_p^1) \partial_x v_p^1 + u_p^0 \partial_x v_e^3 \\ &\quad + (u_e^0 + u_p^0) \partial_x v_p^2 + (v_p^0 + v_e^1) \partial_y v_p^2 - \nu_2 \partial_y^2 v_p^2 + v_p^1 \partial_Y v_e^1 + v_p^0 \partial_Y v_e^2 - g_p^1 \partial_Y g_e^1 - g_p^0 \partial_Y g_e^2 \\ &\quad - (h_e^2 + h_p^2) \partial_x g_p^0 - h_p^2 \partial_x g_e^1 - (g_p^2 + g_e^3) \partial_y g_p^0 - h_p^1 \partial_x g_e^2 - (h_e^1 + h_p^1) \partial_x g_p^1 - h_p^0 \partial_x g_e^3 \\ &\quad - (h_e^0 + h_p^0) \partial_x g_p^2 - (g_p^0 + g_e^1) \partial_y g_p^2] (x, \theta) d\theta. \end{aligned} \quad (5.7)$$

Then the errors R_2^v and R_2^g equal

$$\begin{cases} R_2^v = \sqrt{\varepsilon} [(v_p^2 + v_e^3) \partial_Y v_e^1 + (v_p^1 + v_e^2) \partial_Y v_e^2 + (v_p^0 + v_e^1) \partial_Y v_e^3 - (g_p^2 + g_e^3) \partial_Y g_e^1 \\ \quad - (g_p^1 + g_e^2) \partial_Y g_e^2 - (g_p^0 + g_e^1) \partial_Y g_e^3] - \nu_2 \varepsilon \partial_Y^2 v_e^3, \\ R_2^g = \sqrt{\varepsilon} [(v_p^2 + v_e^3) \partial_Y g_e^1 + (v_p^1 + v_e^2) \partial_Y g_e^2 + (v_p^0 + v_e^1) \partial_Y g_e^3 - (g_p^2 + g_e^3) \partial_Y v_e^1 \\ \quad - (g_p^1 + g_e^2) \partial_Y v_e^2 - (g_p^0 + g_e^1) \partial_Y v_e^3] - \nu_4 \varepsilon \partial_Y^2 g_e^3. \end{cases} \quad (5.8)$$

Now, we only consider the third order ideal MHD system

$$\begin{cases} u_e^0 \partial_x u_e^3 + v_e^3 \partial_Y u_e^0 - h_e^0 \partial_x h_e^3 - g_e^3 \partial_Y h_e^0 + \partial_x p_e^3 = f_1^2, \\ u_e^0 \partial_x v_e^3 - h_e^0 \partial_x g_e^3 + \partial_Y p_e^3 = f_2^2, \\ u_e^0 \partial_x h_e^3 + v_e^3 \partial_Y h_e^0 - h_e^0 \partial_x u_e^3 - g_e^3 \partial_Y u_e^0 = f_3^2, \\ u_e^0 \partial_x g_e^3 - h_e^0 \partial_x v_e^3 = f_4^2, \\ \partial_x u_e^3 + \partial_Y v_e^3 = \partial_x h_e^3 + \partial_Y g_e^3 = 0. \end{cases} \quad (5.9)$$

The boundary conditions of the above system are

$$\begin{cases} (v_e^3, g_e^3)(x, 0) = -(v_p^2, g_p^2)(x, 0), \\ (v_e^3, g_e^3)(x, 1) = (0, 0), \\ (v_e^3, g_e^3)(0, Y) = (V_{b0}^3, G_{b0}^3)(Y), \\ (v_e^3, g_e^3)(L, Y) = (V_{bL}^3, G_{bL}^3)(Y), \end{cases} \quad (5.10)$$

and the compatibility assumptions at corners are

$$\begin{cases} (V_{b0}^3, G_{b0}^3)(0) = -(v_p^2, g_p^2)(0, 0), \\ (V_{bL}^3, G_{bL}^3)(1) = (0, 0), \quad (V_{b0}^3, G_{b0}^3)(1) = (0, 0). \end{cases} \quad (5.11)$$

Using the divergence free conditions, the third and the forth equations of (5.9) may be rewritten as

$$\begin{aligned} \partial_Y (u_e^0 g_e^3 - h_e^0 v_e^3) &= -f_3^2, \\ \partial_x (u_e^0 g_e^3 - h_e^0 v_e^3) &= f_4^2, \end{aligned}$$

thus, we get

$$v_e^3 = \frac{u_e^0}{h_e^0} g_e^3 + \frac{u_e \overline{g_p^2} - h_e \overline{v_p^2} - u_e^0 G_{b0}^3 + h_e^0 V_{b0}^3}{2h_e^0} + \frac{\int_0^Y f_3^2(x, \theta) d\theta - \int_0^x f_4^2(s, Y) ds}{2h_e^0} := \frac{u_e^0}{h_e^0} g_e^3 + \frac{b_3^1(x, Y)}{h_e^0},$$

and

$$g_e^3 = \frac{h_e^0}{u_e^0} v_e^3 + \frac{h_e \overline{v_p^2} - u_e \overline{g_p^2} + u_e^0 G_{b0}^3 - h_e^0 V_{b0}^3}{2u_e^0} + \frac{\int_0^x f_4^2(s, Y) ds - \int_0^Y f_3^2(x, \theta) d\theta}{2u_e^0} := \frac{h_e^0}{u_e^0} v_e^3 + \frac{b_3^2(x, Y)}{u_e^0},$$

where

$$b_3^1(x, Y) = \frac{u_e \overline{g_p^2} - h_e \overline{v_p^2} - u_e^0 G_{b0}^3 + h_e^0 V_{b0}^3 + \int_0^Y f_3^2(x, \theta) d\theta - \int_0^x f_4^2(s, Y) ds}{2}$$

and $b_3^2(x, Y) = -b_3^1(x, Y)$. It follows from the first and second equations of (5.9) that

$$-u_e^0 \Delta v_e^3 + \partial_Y^2 u_e^0 \cdot v_e^3 + (h_e^0 \Delta g_e^3 - \partial_Y^2 h_e^0 \cdot g_e^3) = f_{1Y}^2 - f_{2x}^2,$$

which satisfies the boundary conditions (5.10). Collecting the previous equations, we now consider the following system

$$\begin{cases} -u_e^0 \Delta v_e^3 + \partial_Y^2 u_e^0 \cdot v_e^3 + (h_e^0 \Delta g_e^3 - \partial_Y^2 h_e^0 \cdot g_e^3) = f_{1Y}^2 - f_{2x}^2, \\ v_e^3 = \frac{u_e^0}{h_e^0} g_e^3 + \frac{b_3^1(x, Y)}{h_e^0}, \\ g_e^3 = \frac{h_e^0}{u_e^0} v_e^3 + \frac{b_3^2(x, Y)}{u_e^0}, \\ (v_e^3, g_e^3)(x, 0) = -(v_p^2, g_p^2)(x, 0), \quad (v_e^3, g_e^3)(x, 1) = (0, 0). \end{cases} \quad (5.12)$$

Next, we estimate the solution (v_e^3, g_e^3) of the previous system.

Proposition 5.1. *Suppose that the conditions in Theorem 1.1 are satisfied. Then, for $0 < L_6 \leq L_5$, there exists a unique smooth solution (v_e^3, g_e^3) to the boundary value problem (5.12), satisfying*

$$\begin{aligned} \|(v_e^3, g_e^3)\|_{L^\infty(\Omega_1)} + \|(v_e^3, g_e^3)\|_{H^2(\Omega_1)} &\leq C\varepsilon^{-\frac{9}{64}}, \\ \|(v_e^3, g_e^3)\|_{H^{2+j}(\Omega_1)} &\leq C\varepsilon^{-\frac{3j}{32} - \frac{9}{64}}, j = 1, 2. \end{aligned} \quad (5.13)$$

where the constant $C > 0$ only depends on the boundary data. Moreover, it holds that

$$\begin{aligned} \|(v_e^3, g_e^3)\|_{W^{2,q}(\Omega_1)} &\leq C(L)\varepsilon^{-\frac{3}{32} - \frac{3}{32q}}, \\ \|(v_e^3, g_e^3)\|_{W^{2+j,q}(\Omega_1)} &\leq C(L)\varepsilon^{-\frac{3}{32}(j+1) - \frac{3}{32q}}, \end{aligned} \quad (5.14)$$

for $q \in [1, \infty)$, $C(L) > 0$.

We follow the arguments of Section 4.1 to prove the above proposition and do not detail the proof. For more details, we refer to [3] and [22].

Next we estimate u_e^3, h_e^3 and p_e^3 . Let us first construct the inner correctors u_e^3, h_e^3 and p_e^3 . Using (5.9) and divergence free conditions, we have

$$u_e^3(x, Y) = u_b^3(Y) - \int_0^x v_{eY}^3(s, Y) ds,$$

$$h_e^3(x, Y) = h_b^3(Y) - \int_0^x g_{eY}^3(s, Y) ds, \quad (5.15)$$

$$p_e^3(x, y) = \int_Y [u_e^0 v_{ex}^3 - h_e^0 g_{ex}^3 - f_2^2](x, \theta) d\theta - \int_0^x [u_e^0(1) v_{eY}^3(s, 1) - h_e^0(1) g_{eY}^3(s, 1) - f_1^2(s, 1)] ds,$$

where $u_e^3(0, Y) = u_b^3(Y)$ and $h_e^3(0, Y) = h_b^3(Y)$ satisfy $\partial_Y u_b^3(1) = \partial_Y h_b^3(1) = 0$. Hence, we have $u_{eY}^3(x, 1) = h_{eY}^3(x, 1) = 0$. By definition of (u_e^3, h_e^3) in (5.15) and Proposition 5.1, we have

$$\|(u_e^3, h_e^3)\|_{L^\infty(\Omega_1)} + \|(u_e^3, h_e^3)\|_{H^1(\Omega_1)} \leq C\varepsilon^{-\frac{9}{64}}, \quad \|(u_e^3, h_e^3)\|_{H^{1+j}(\Omega_1)} \leq C\varepsilon^{-\frac{9}{64} - \frac{3j}{32}}, \quad j = 1, 2. \quad (5.16)$$

It follows from the previous estimates that

$$\begin{aligned} \|(u_e^3, h_e^3)\|_{L^\infty(\Omega_\varepsilon)} &\leq C\varepsilon^{-\frac{9}{64}}, \quad \|(u_e^3, h_e^3)\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{-\frac{25}{64}}, \\ \|\partial_Y^2 u_e^3\|_{L^2(\Omega_\varepsilon)} &\leq C\varepsilon^{-\frac{1}{4}} \|\partial_Y^2 u_b^3\|_{L^2(0,1)} + C\varepsilon^{-\frac{1}{4}} \|\partial_Y^3 v_e^3\|_{L^2(\Omega_1)} \leq C\varepsilon^{-\frac{31}{64}}, \\ \|(v_p^0 + v_e^1) \partial_Y u_e^3\|_{L^2(\Omega_\varepsilon)} &\leq \|v_p^0 + v_e^1\|_{L^\infty(\Omega_\varepsilon)} \|\partial_Y u_e^3\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{-\frac{25}{64}}, \\ \|(v_p^1 + v_e^2) \partial_Y u_e^2\|_{L^2(\Omega_\varepsilon)} &\leq \|v_p^1 + v_e^2\|_{L^\infty(\Omega_\varepsilon)} \|\partial_Y h_e^2\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{-\frac{11}{32}}, \\ \|(v_p^2 + v_e^3) \partial_Y u_e^1\|_{L^2(\Omega_\varepsilon)} &\leq \|v_p^2 + v_e^3\|_{L^\infty(\Omega_\varepsilon)} \|\partial_Y h_e^1\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{-\frac{25}{64}}. \end{aligned}$$

Using similar calculations for the other terms of R_3^u , we obtain

$$\|R_3^u\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{1}{16}}.$$

Using similar arguments, we have

$$\|(R_3^h, R_2^v, R_2^g)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{\frac{1}{16}}.$$

5.2 The third order MHD boundary layer profile

In this subsection, we construct the third order boundary layer profiles $(u_p^3, v_p^3, h_p^3, g_p^3, p_p^3)$, which solve (5.3). Then the system (5.3) is written as

$$\begin{cases} u^0 \partial_x u_p^3 + u_p^3 \partial_x u^0 + v_p^3 \partial_y u^0 + (v_p^0 + \overline{v_e^1}) \partial_y u_p^3 - \nu_1 \partial_y^2 u_p^3 + \partial_x P_p^3 \\ \quad - h^0 \partial_x h_p^3 - h_p^3 \partial_x h^0 - g_p^3 \partial_y h^0 - (g_p^0 + \overline{g_e^1}) \partial_y h_p^3 = F_{p_1}^3, \\ u^0 \partial_x h_p^3 + u_p^3 \partial_x h^0 + v_p^3 \partial_y h^0 + (v_p^0 + \overline{v_e^1}) \partial_y h_p^3 - \nu_3 \partial_y^2 h_p^3 \\ \quad - h^0 \partial_x u_p^3 - h_p^3 \partial_x u^0 - g_p^3 \partial_y u^0 - (g_p^0 + \overline{g_e^1}) \partial_y u_p^3 = F_{p_2}^3, \end{cases} \quad (5.17)$$

with the boundary conditions

$$\begin{aligned} (u_p^3, h_p^3)(0, y) &= (\tilde{u}_3(y), \tilde{h}_3(y)), \quad (u_p^3, h_{py}^3)(x, 0) = -(u_e^3, h_{eY}^2)(x, 0), \\ u_{py}^3(x, \varepsilon^{-1/2}) &= v_p^3(x, 0) = v_p^3(x, \varepsilon^{-1/2}) = 0, \\ h_{py}^3(x, \varepsilon^{-1/2}) &= g_p^3(x, 0) = g_p^3(x, \varepsilon^{-1/2}) = 0, \end{aligned} \quad (5.18)$$

where $F_{p_1}^3$ and $F_{p_2}^3$ are as follows

$$F_{p_1}^3 = - \left[\varepsilon^{-\frac{1}{2}} E_{r_1}^2 + u_p^2 \partial_x (u_p^1 + u_e^1) + (u_e^1 + u_p^1) \partial_x u_p^2 + v_p^2 \partial_Y u_e^0 + v_e^2 \partial_y u_p^2 + v_p^1 \partial_y u_p^2 \right]$$

$$\begin{aligned}
& -h_p^2 \partial_x (h_p^1 + h_e^1) - (h_e^1 + h_p^1) \partial_x h_p^2 - g_p^2 \partial_Y h_e^0 - g_e^2 \partial_y h_p^2 - g_p^1 \partial_y h_p^2 - \nu_1 \partial_x^2 u_p^1, \\
F_{p_2}^3 = & - \left[\varepsilon^{-\frac{1}{2}} E_{r_2}^2 + u_p^2 \partial_x (h_p^1 + h_e^1) + (u_e^1 + u_p^1) \partial_x h_p^2 + v_p^2 \partial_Y h_e^0 + v_e^2 \partial_y h_p^2 + v_p^1 \partial_y h_p^2 \right. \\
& \left. - h_p^2 \partial_x (u_p^1 + u_e^1) - (h_e^1 + h_p^1) \partial_x u_p^2 - g_p^2 \partial_Y u_e^0 - g_e^2 \partial_y u_p^2 - g_p^1 \partial_y u_p^2 - \nu_1 \partial_x^2 h_p^1 \right].
\end{aligned}$$

Therefore, the new error terms are

$$\begin{cases} E_{r_1}^3 := (u_e^0 - u_e) \partial_x u_p^3 + u_e^3 \partial_x u_p^0 + u_e^2 \partial_x u_p^1 + u_p^1 \partial_x u_e^2 + u_p^0 \partial_x u_e^3 + v_p^3 \partial_y u_e^0 \\ \quad + (v_p^2 + v_e^3) \partial_y u_p^1 - (h_e^0 - h_e) \partial_x h_p^3 - h_e^3 \partial_x h_p^0 - h_e^2 \partial_x h_p^1 - h_p^1 \partial_x h_e^2 \\ \quad - h_p^0 \partial_x h_e^3 - g_p^3 \partial_y h_e^0 - (g_p^2 + g_e^3) \partial_y h_p^1 + v_p^1 \partial_Y u_e^1 + v_p^0 \partial_Y u_e^2 - g_p^1 \partial_Y h_e^1 \\ \quad - g_p^0 \partial_Y h_e^2 + \triangle_6 + \triangle_9, \\ E_{r_2}^3 := (u_e^0 - u_e) \partial_x h_p^3 + u_e^3 \partial_x h_p^0 + u_e^2 \partial_x h_p^1 + u_p^1 \partial_x h_e^2 + u_p^0 \partial_x h_e^3 + v_p^3 \partial_y h_e^0 \\ \quad + (v_p^2 + v_e^3) \partial_y h_p^1 - (h_e^0 - h_e) \partial_x u_p^3 - h_e^3 \partial_x u_p^0 - h_e^2 \partial_x u_p^1 - h_p^1 \partial_x u_e^2 \\ \quad - h_p^0 \partial_x u_e^3 - g_p^3 \partial_y u_e^0 - (g_p^2 + g_e^3) \partial_y u_p^1 + v_p^1 \partial_Y h_e^1 + v_p^0 \partial_Y h_e^2 - g_p^1 \partial_Y u_e^1 \\ \quad - g_p^0 \partial_Y u_e^2 + \triangle_8 + \triangle_{11}. \end{cases} \quad (5.19)$$

Next, we estimate the solution $(u_p^3, v_p^3, h_p^3, g_p^3)$ of (5.17).

Step1: We establish the estimates of $(u_p^{3,\infty}, v_p^{3,\infty}, h_p^{3,\infty}, g_p^{3,\infty})$ on $[0, L] \times \mathbb{R}_+$ as in Section 4.2 and neglect the proof here.

Proposition 5.2. *For $0 < L_7 \leq L_6$, there exists smooth solution $(u_p^{3,\infty}, v_p^{3,\infty}, h_p^{3,\infty}, g_p^{3,\infty})$ to the problem (5.17) in $[0, L_7] \times [0, \infty)$ such that the following estimates hold,*

$$\begin{aligned}
& \|(u_p^{3,\infty}, v_p^{3,\infty}, h_p^{3,\infty}, g_p^{3,\infty})\|_{L^\infty(0, L_7; \mathbb{R}_+)} + \sup_{0 \leq x \leq L_7} \|\langle y \rangle^l \partial_{yy} (v_p^{3,\infty}, g_p^{3,\infty})\|_{L^2(0, \infty)} \\
& + \|\langle y \rangle^l \partial_{xy} (v_p^{3,\infty}, g_p^{3,\infty})\|_{L^2(0, L_7; \mathbb{R}_+)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{16}}, \\
& \sup_{0 \leq x \leq L_7} \|\langle y \rangle^l \partial_{xyy} (v_p^{3,\infty}, g_p^{3,\infty})\|_{L^2(0, \infty)} + \|\langle y \rangle^l \partial_{xxy} (v_p^{3,\infty}, g_p^{3,\infty})\|_{L^2(0, L_7; \mathbb{R}_+)} \leq C(L) \varepsilon^{-\frac{9}{32}},
\end{aligned} \quad (5.20)$$

where ξ some small positive constant.

Step 2: We cut-off the domain from $[0, L] \times \mathbb{R}_+$ to Ω_ε and give the following proposition to estimate the solution of the system (5.17) on Ω_ε .

Proposition 5.3. *Under the assumptions in Theorem 1.1, there exists smooth functions $(u_p^3, v_p^3, h_p^3, g_p^3)$, satisfying the following inhomogeneous system*

$$\begin{cases} u_p^0 u_{px}^3 + u_x^0 u_p^3 + u_y^0 v_p^3 + [v_p^0 + \overline{v_e^1}] u_{py}^3 + p_{px}^3 - \nu_1 \partial_y^2 u_p^3 \\ \quad - h_p^0 h_{px}^3 - h_x^0 h_p^3 - h_y^0 g_p^3 - [g_p^0 + \overline{g_e^1}] h_{py}^3 = R_p^{u,3}, \\ u_p^0 h_{px}^3 + h_x^0 u_p^3 + h_y^0 v_p^3 + [v_p^0 + \overline{v_e^1}] h_{py}^3 - \nu_3 \partial_y^2 h_p^3 \\ \quad - h_p^0 u_{px}^3 - u_x^0 h_p^3 - u_y^0 g_p^3 - [g_p^0 + \overline{g_e^1}] u_{py}^3 = R_p^{h,3}, \\ u_{px}^3 + v_{py}^3 = h_{px}^3 + g_{py}^3 = 0, \\ (u_p^3, h_p^3)(0, y) = (\tilde{u}_3(y), \tilde{h}_3(y)), (u_p^3, \partial_y h_p^3)(x, 0) = -(u_e^3, \partial_Y h_e^2)(x, 0), \\ (u_{py}^3, h_{py}^3)(x, \varepsilon^{-1/2}) = (v_p^3, g_p^3)(x, 0) = (v_p^3, g_p^3)(x, \varepsilon^{-1/2}) = 0, \end{cases} \quad (5.21)$$

such that for any given $l \in \mathbb{N}$,

$$\|(u_p^3, v_p^3, h_p^3, g_p^3)\|_{L^\infty(\Omega_\varepsilon)} + \sup_{0 \leq x \leq L_7} \|\langle y \rangle^l \partial_{yy}(v_p^3, g_p^3)\|_{L^2(I_\varepsilon)} + \|\langle y \rangle^l \partial_{xy}(v_p^3, g_p^3)\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{16}}, \quad (5.22)$$

$$\sup_{0 \leq x \leq L_7} \|\langle y \rangle^l \partial_{xyy}(v_p^3, g_p^3)\|_{L^2(I_\varepsilon)} + \|\langle y \rangle^l \partial_{xxy}(v_p^3, g_p^3)\|_{L^2(\Omega_\varepsilon)} \leq C(L) \varepsilon^{-\frac{9}{32}},$$

where $R_p^{u,3}$ and $R_p^{h,3}$ are higher order terms.

Proof. The solution $(u_p^{3,\infty}, v_p^{3,\infty}, h_p^{3,\infty}, g_p^{3,\infty})$ has been constructed in Proposition 5.2. We introduce the following definition

$$\begin{aligned} (u_p^3, h_p^3)(x, y) &:= \chi(\sqrt{\varepsilon}y)(u_p^{3,\infty}, h_p^{3,\infty})(x, y) + \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_0^y (u_p^{3,\infty}, h_p^{3,\infty})(x, \theta) d\theta, \\ (v_p^3, g_p^3)(x, y) &:= \chi(\sqrt{\varepsilon}y)(v_p^{3,\infty}, g_p^{3,\infty})(x, y). \end{aligned} \quad (5.23)$$

Clearly, $(u_p^3, v_p^3, h_p^3, g_p^3)$ satisfies (5.21) and we have

$$\left\{ \begin{aligned} R_p^{u,3} &:= \sqrt{\varepsilon}\chi'[u^0 v_p^{3,\infty} - h^0 g_p^{3,\infty}] + \sqrt{\varepsilon}\chi'[u_x^0 \int_0^y u_p^{3,\infty} d\theta - h_x^0 \int_0^y h_p^{3,\infty} d\theta] + 2\sqrt{\varepsilon}\chi'[v_p^0 + \overline{v_e^1}]u_p^{3,\infty} \\ &\quad - 2\sqrt{\varepsilon}\chi'[g_p^0 + \overline{g_e^1}]h_p^{3,\infty} - 3\sqrt{\varepsilon}\chi' u_{py}^{3,\infty} + \varepsilon\chi''\{[v_p^0 + \overline{v_e^1}] \int_0^y u_p^{3,\infty} d\theta - [g_p^0 + \overline{g_e^1}] \int_0^y h_p^{3,\infty} d\theta\} \\ &\quad + \sqrt{\varepsilon}F_{p1}^3 \int_0^y \chi' d\theta - 3\varepsilon\chi'' u_p^{3,\infty} - \varepsilon^{3/2}\chi''' \int_0^y u_p^{3,\infty} d\theta, \\ R_p^{h,3} &:= \sqrt{\varepsilon}\chi'[u^0 g_p^{3,\infty} - h^0 v_p^{3,\infty}] + \sqrt{\varepsilon}\chi'[h_x^0 \int_0^y u_p^{3,\infty} d\theta - u_x^0 \int_0^y h_p^{3,\infty} d\theta] + 2\sqrt{\varepsilon}\chi'[v_p^0 + \overline{v_e^1}]h_p^{3,\infty} \\ &\quad - 2\sqrt{\varepsilon}\chi'[g_p^0 + \overline{g_e^1}]u_p^{3,\infty} - 3\sqrt{\varepsilon}\chi' h_{py}^{3,\infty} + \varepsilon\chi''\{[v_p^0 + \overline{v_e^1}] \int_0^y h_p^{3,\infty} d\theta - [g_p^0 + \overline{g_e^1}] \int_0^y u_p^{3,\infty} d\theta\} \\ &\quad + \sqrt{\varepsilon}F_{p2}^3 \int_0^y \chi' d\theta - 3\varepsilon\chi'' h_p^{3,\infty} - \varepsilon^{3/2}\chi''' \int_0^y u_p^{3,\infty} d\theta. \end{aligned} \right. \quad (5.24)$$

It is direct to check that

$$\partial_x u_p^3 + \partial_y v_p^3 = \partial_x h_p^3 + \partial_y g_p^3 = 0.$$

By using the Proposition 5.2, we obtain

$$\begin{aligned} \left| \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_0^y u_p^{3,\infty}(x, \theta) d\theta \right| &\leq \sqrt{\varepsilon}y |\chi'(\sqrt{\varepsilon}y)| \|u_p^{3,\infty}\|_{L^\infty} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{16}}, \\ \left| \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_0^y h_p^{3,\infty}(x, \theta) d\theta \right| &\leq \sqrt{\varepsilon}y |\chi'(\sqrt{\varepsilon}y)| \|h_p^{3,\infty}\|_{L^\infty} \leq C(L, \xi) \varepsilon^{-\xi - \frac{3}{16}}. \end{aligned}$$

We thus obtain (5.22), which ends the proof of the proposition. \square

Proposition 5.4. Assume that $(u_p^3, v_p^3, h_p^3, g_p^3, p_p^3)$ is the solution to (5.21). Then

$$\begin{aligned} \|E_{r1}^3\|_{L^2(\Omega_\varepsilon)} + \|E_{r2}^3\|_{L^2(\Omega_\varepsilon)} + \|R_p^{u,3}\|_{L^2(\Omega_\varepsilon)} + \|R_p^{h,3}\|_{L^2(\Omega_\varepsilon)} &\leq C(L, \xi) \varepsilon^{-\xi + \frac{1}{16}}, \\ \|p_{px}^4\|_{L^2(\Omega_\varepsilon)} &\leq C\varepsilon^{-\frac{25}{64}}, \end{aligned} \quad (5.25)$$

for any $\xi > 0$ small enough. Here E_{ri}^3 ($i = 1, 2$) and p_p^4 have been defined in the previous section.

Proof. $E_{r_i}^3$ ($i = 1, 2$) are defined in (5.19) and, using the same method as in Section 3, we get

$$\|E_{r_i}^3\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi + \frac{1}{16}}, \quad i = 1, 2.$$

It follows from the Proposition 5.2 that

$$\begin{aligned} \|R_p^{u,3}\|_{L^2(\Omega_\varepsilon)}^2 &\lesssim \sqrt{\varepsilon} L \|(v_p^{3,\infty}, g_p^{3,\infty})\|_{L^\infty}^2 + \varepsilon (\|u_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} \langle y \rangle^2 |u_{px}^0|^2 dx dy + \|h_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} \langle y \rangle^2 |h_{px}^0|^2 dx dy) \\ &\quad + \varepsilon (\|u_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} |v_p^0 + \overline{v_e^1}|^2 dx dy + \|h_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} |g_p^0 + \overline{g_e^1}|^2 dx dy) \\ &\quad + \varepsilon \iint_{\Omega_\varepsilon} |u_{py}^{3,\infty}|^2 dx dy + \varepsilon \iint_{\Omega_\varepsilon} \langle y \rangle^2 |F_{p1}^3|^2 dx dy + \varepsilon^{\frac{3}{2}} \|u_p^{3,\infty}\|_{L^\infty}^2 \\ &\leq C(L, \xi) \varepsilon^{-2\xi + \frac{1}{8}}. \end{aligned} \tag{5.26}$$

Similarly, we have

$$\begin{aligned} \|R_p^{h,3}\|_{L^2(\Omega_\varepsilon)}^2 &\lesssim \sqrt{\varepsilon} L \|(v_p^{3,\infty}, g_p^{3,\infty})\|_{L^\infty}^2 + \varepsilon (\|u_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} \langle y \rangle^2 |h_{px}^0|^2 dx dy + \|h_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} \langle y \rangle^2 |u_{px}^0|^2 dx dy) \\ &\quad + \varepsilon (\|h_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} |v_p^0 + \overline{v_e^1}|^2 dx dy + \|u_p^{3,\infty}\|_{L^\infty}^2 \iint_{\Omega_\varepsilon} |g_p^0 + \overline{g_e^1}|^2 dx dy) \\ &\quad + \varepsilon \iint_{\Omega_\varepsilon} |h_{py}^{3,\infty}|^2 dx dy + \varepsilon \iint_{\Omega_\varepsilon} \langle y \rangle^2 |F_{p2}^3|^2 dx dy + \varepsilon^{\frac{3}{2}} \|h_p^{3,\infty}\|_{L^\infty}^2 \\ &\leq C(L, \xi) \varepsilon^{-2\xi + \frac{1}{8}}. \end{aligned} \tag{5.27}$$

By definition of p_p^4 , we have

$$\begin{aligned} p_{px}^4 &= \int_y^{1/\sqrt{\varepsilon}} \partial_x [(u_e^2 + u_p^2) v_{px}^0 + u_p^2 v_{ex}^1 + (v_p^2 + v_e^3) v_{py}^0 + u_p^1 v_{ex}^2 + (u_e^1 + u_p^1) v_{px}^1 \\ &\quad + u_p^0 v_{ex}^3 + (u_e^0 + u_p^0) v_{px}^2 + (v_p^0 + v_e^1) v_{py}^2 - \nu_2 \partial_{yy} v_p^2 + v_p^1 v_{eY}^1 + v_p^0 v_{eY}^2 \\ &\quad - g_p^1 g_{eY}^1 - g_p^0 g_{eY}^2 - (h_e^2 + h_p^2) g_{px}^0 - h_p^2 g_{ex}^1 - (g_p^2 + g_e^3) g_{py}^0 - h_p^1 g_{ex}^2 \\ &\quad - (h_e^1 + h_p^1) g_{px}^1 - h_p^0 g_{ex}^3 - (h_e^0 + h_p^0) g_{px}^2 - (g_p^0 + g_e^1) g_{py}^2] (x, \theta) d\theta. \end{aligned}$$

Note that

$$\begin{aligned} \int_y^{1/\sqrt{\varepsilon}} (u_e^2 + u_p^2) v_{pxx}^0(x, \theta) d\theta &\leq C \langle y \rangle^{-l+1} \|u_e^2 + u_p^2\|_{L^\infty} \|\langle y \rangle^l v_{pxx}^0\|_{L^2}, \\ \int_y^{1/\sqrt{\varepsilon}} u_p^2 v_{exx}^1(x, \theta) d\theta &\leq C \langle y \rangle^{-l+2} \|\langle y \rangle^l u_{py}^2\|_{L^2} \|v_{exx}^1(x, \sqrt{\varepsilon} \cdot)\|_{L^2}, \\ \int_y^{1/\sqrt{\varepsilon}} (v_p^2 + v_e^3) v_{pxy}^0(x, \theta) d\theta &\leq C \langle y \rangle^{-l+1} \|v_p^2 + v_e^3\|_{L^\infty} \|\langle y \rangle^l v_{pxy}^0\|_{L^2}, \\ \int_y^{1/\sqrt{\varepsilon}} v_{pxyy}^2(x, \theta) d\theta &\leq C \langle y \rangle^{-l+1} \|\langle y \rangle^l v_{pxyy}^2\|_{L^2}. \end{aligned}$$

Similarly, we get

$$\int_y^{1/\sqrt{\varepsilon}} (h_e^2 + h_p^2) g_{pxx}^0(x, \theta) d\theta \leq C \langle y \rangle^{-l+1} \|h_e^2 + h_p^2\|_{L^\infty} \|\langle y \rangle^l g_{pxx}^0\|_{L^2},$$

$$\begin{aligned} \int_y^{1/\sqrt{\varepsilon}} h_p^2 g_{exx}^1(x, \theta) d\theta &\leq C \langle y \rangle^{-l+2} \|\langle y \rangle^l h_{py}^2\|_{L^2} \|g_{exx}^1(x, \sqrt{\varepsilon} \cdot)\|_{L^2}, \\ \int_y^{1/\sqrt{\varepsilon}} (g_p^0 + g_e^1) g_{pxy}^2(x, \theta) d\theta &\leq C \langle y \rangle^{-l+1} \|h_e^0 + h_p^0\|_{L^\infty} \|\langle y \rangle^l g_{pxy}^2\|_{L^2}. \end{aligned}$$

The remaining terms of p_{px}^4 are treated using the same method as above. Hence, taking $l \geq 3$, we have

$$\|p_{px}^4\|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon^{-\frac{25}{64}}.$$

This completes the proof of the proposition. \square

Combining all the error terms derived from the construction of the ideal MHD profiles and boundary layer profiles above, we obtain

Proposition 5.5. *Under the assumptions of Theorem 1.1, for arbitrarily small $\xi > 0$,*

$$\|R_{app}^1\|_{L^2(\Omega_\varepsilon)} + \|R_{app}^3\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}(\|R_{app}^2\|_{L^2(\Omega_\varepsilon)} + \|R_{app}^4\|_{L^2(\Omega_\varepsilon)}) \leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}. \quad (5.28)$$

Proof. Collecting all the error terms in R_{app}^1 and R_{app}^3 , we obtain

$$\begin{aligned} R_{app}^1 := & -\nu_1 \varepsilon^2 \partial_{YY} u_e^2 + \varepsilon^2 \Delta_{10} + \varepsilon^{\frac{3}{2}}(R_3^u + R_p^{u,3} + E_{r_1}^3) + \varepsilon^2[(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y](u_e^1 + u_p^1) \\ & + \varepsilon^{\frac{5}{2}}[(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y](u_e^2 + u_p^2) + \varepsilon^3[(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y](u_e^3 + u_p^3) \\ & + \varepsilon^2[(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y](u_e^2 + u_p^2) + \varepsilon^{\frac{5}{2}}[(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y](u_e^3 + u_p^3) \\ & + \varepsilon^2[(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y](u_e^3 + u_p^3) - \varepsilon^2[(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y](h_e^1 + h_p^1) \\ & - \varepsilon^{\frac{5}{2}}[(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y](h_e^2 + h_p^2) - \varepsilon^3[(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y](h_e^3 + h_p^3) \\ & - \varepsilon^2[(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y](h_e^2 + h_p^2) - \varepsilon^{\frac{5}{2}}[(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y](h_e^3 + h_p^3) \\ & - \varepsilon^2[(h_e^1 + h_p^1) \partial_x + (g_p^1 + g_e^2) \partial_y](h_e^3 + h_p^3) + \varepsilon \partial_x p_p^4 - \varepsilon \partial_x^2[(u_p^2 + u_e^2) + \sqrt{\varepsilon}(u_e^3 + u_p^3)], \quad (5.29) \end{aligned}$$

$$\begin{aligned} R_{app}^3 := & -\nu_3 \varepsilon^2 \partial_{YY} h_e^2 + \varepsilon^2 \Delta_{12} + \varepsilon^{\frac{3}{2}}(R_3^h + R_p^{h,3} + E_{r_2}^3) + \varepsilon^2[(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y](h_e^1 + h_p^1) \\ & + \varepsilon^{\frac{5}{2}}[(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y](h_e^2 + h_p^2) + \varepsilon^3[(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y](h_e^3 + h_p^3) \\ & + \varepsilon^2[(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y](h_e^2 + h_p^2) + \varepsilon^{\frac{5}{2}}[(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y](h_e^3 + h_p^3) \\ & + \varepsilon^2[(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y](h_e^3 + h_p^3) - \varepsilon^2[(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y](u_e^1 + u_p^1) \\ & - \varepsilon^{\frac{5}{2}}[(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y](h_e^2 + h_p^2) - \varepsilon^3[(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y](u_e^3 + u_p^3) \\ & - \varepsilon^2[(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y](u_e^2 + u_p^2) - \varepsilon^{\frac{5}{2}}[(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y](u_e^3 + u_p^3) \\ & - \varepsilon^2[(h_e^1 + h_p^1) \partial_x + (g_p^1 + g_e^2) \partial_y](u_e^3 + u_p^3) - \varepsilon \partial_x^2[(h_p^2 + h_e^2) + \sqrt{\varepsilon}(h_e^3 + h_p^3)]. \quad (5.30) \end{aligned}$$

To estimate the terms $E_{r_1}^3, E_{r_2}^3, R_3^u, R_p^{u,3}, R_3^h, R_p^{h,3}$, we directly have

$$\begin{aligned} \|- \nu_1 \varepsilon^2 \partial_{YY} u_e^2 + \varepsilon^{\frac{3}{2}}(R_3^u + R_p^{u,3} + E_{r_1}^3) + \varepsilon^2 \partial_x p_p^4\|_{L^2(\Omega_\varepsilon)} &\leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}, \\ \|- \nu_3 \varepsilon^2 \partial_{YY} h_e^2 + \varepsilon^{\frac{3}{2}}(R_3^h + R_p^{h,3} + E_{r_2}^3)\|_{L^2(\Omega_\varepsilon)} &\leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}. \end{aligned}$$

Applying the estimates of $(u_e^i, v_e^i, h_e^i, g_e^i)$ and $(u_p^i, v_p^i, h_p^i, g_p^i)$ ($i = 1, 2, 3$), we obtain

$$\varepsilon^2 \|(u_e^3 + u_p^3) \partial_x (u_e^1 + u_p^1)\|_{L^2(\Omega_\varepsilon)}$$

$$\begin{aligned}
&\leq \varepsilon^2 (\|u_e^3\|_{L^\infty} + \|u_p^3\|_{L^\infty}) \times (\|\partial_x u_e^1\|_{L^2} + \|\partial_x u_p^1\|_{L^2}) \leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}, \\
&\quad \varepsilon^2 \|v_p^3 \partial_y (u_e^1 + u_p^1)\|_{L^2(\Omega_\varepsilon)} \\
&\leq \varepsilon^2 \|v_p^3\|_{L^\infty} (\sqrt{\varepsilon} \|\partial_Y u_e^1\|_{L^2} + \|\partial_y u_p^1\|_{L^2}) \leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}, \\
&\quad \varepsilon^2 \|\partial_x^2 [(u_p^2 + u_e^2) + \sqrt{\varepsilon} (u_e^3 + u_p^3)]\|_{L^2(\Omega_\varepsilon)} \\
&\leq \varepsilon^2 \|\partial_x^2 u_p^2 + \partial_x^2 u_e^2\|_{L^2} + \varepsilon^{\frac{5}{2}} (\|\partial_{xY} v_e^3\|_{L^2} + \|\partial_{xy} v_p^3\|_{L^2}) \leq C \varepsilon^{\frac{59}{32}-\xi}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\varepsilon^2 \|(h_e^3 + h_p^3) \partial_x (h_e^1 + h_p^1)\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}, \\
&\varepsilon^2 \|g_p^3 \partial_y (h_e^1 + h_p^1)\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}, \\
&\varepsilon^2 \|\partial_x^2 [(h_p^2 + h_e^2) + \sqrt{\varepsilon} (h_e^3 + h_p^3)]\|_{L^2(\Omega_\varepsilon)} \\
&\leq \varepsilon \|\partial_x^2 h_p^2 + \partial_x^2 h_e^2\|_{L^2} + \varepsilon^{\frac{5}{2}} (\|\partial_{xY} g_e^3\|_{L^2} + \|\partial_{xy} g_p^3\|_{L^2}) \leq C \varepsilon^{\frac{59}{32}-\xi}.
\end{aligned}$$

For the remaining terms of R_{app}^1 , we follow the same approach as previously. Recalling the definition of Δ_{10} , we get

$$\begin{aligned}
\|\Delta_{10}\|_{L^2(\Omega_\varepsilon)} &\lesssim \|u_p^{2,\infty}\|_{L^2} \|\langle y \rangle (v_p^0 + \overline{v_e^1})\|_{L^\infty} + \|h_p^{2,\infty}\|_{L^2} \|\langle y \rangle (g_p^0 + \overline{g_e^1})\|_{L^\infty} \\
&\quad + \|\chi''\|_{L^2} \|u_p^{2,\infty}\|_{L^2} + \|\chi''\|_{L^2} \|u_p^{2,\infty}\|_{L^2} \\
&\leq C(L, \xi) \varepsilon^{-\xi-\frac{3}{32}},
\end{aligned}$$

similarly, we have $\|\Delta_{12}\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi-\frac{3}{32}}$. Therefore, we obtain

$$\|R_{app}^1\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}.$$

Similarly, we have

$$\|R_{app}^3\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{\frac{25}{16}-\xi}.$$

Next, we turn to estimates of R_{app}^2 and R_{app}^4 . Collecting all error terms, we have

$$\begin{aligned}
R_{app}^2 &:= \varepsilon^2 R_2^v - \nu_2 \varepsilon^{\frac{3}{2}} \Delta v_e^2 + \varepsilon^{\frac{3}{2}} [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] (v_p^0 + v_e^1) + \varepsilon^2 [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] (v_p^1 + v_e^2) \\
&\quad + \varepsilon^{\frac{5}{2}} [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] (v_p^2 + v_e^3) + \varepsilon^3 [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] v_p^3 \\
&\quad + \varepsilon^{\frac{3}{2}} [(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y] (v_p^1 + v_e^2) + \varepsilon^2 [(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y] (v_p^2 + v_e^3) \\
&\quad + \varepsilon^{\frac{5}{2}} [(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y] v_p^3 + \varepsilon^{\frac{3}{2}} [(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y] (v_p^2 + v_e^3) \\
&\quad + \varepsilon^2 [(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y] v_p^3 + \varepsilon^{\frac{3}{2}} [(u_e^0 + u_p^0) \partial_x + (v_p^0 + v_e^1) \partial_y] v_p^3 \\
&\quad - \varepsilon^{\frac{3}{2}} [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] (g_p^0 + g_e^1) - \varepsilon^2 [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] (g_p^1 + g_e^2) \\
&\quad - \varepsilon^{\frac{5}{2}} [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] (g_p^2 + g_e^3) - \varepsilon^3 [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] g_p^3 \\
&\quad - \varepsilon^{\frac{3}{2}} [(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y] (g_p^1 + g_e^2) - \varepsilon^2 [(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y] (g_p^2 + g_e^3) \\
&\quad - \varepsilon^{\frac{5}{2}} [(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y] g_p^3 - \varepsilon^{\frac{3}{2}} [(h_e^1 + h_p^1) \partial_x + (g_p^1 + g_e^2) \partial_y] (g_p^2 + g_e^3) \\
&\quad - \varepsilon^2 [(h_e^1 + h_p^1) \partial_x + (g_p^1 + g_e^2) \partial_y] g_p^3 - \varepsilon^{\frac{3}{2}} [(h_e^0 + h_p^0) \partial_x + (g_p^0 + g_e^1) \partial_y] g_p^3
\end{aligned}$$

$$- \nu_2 \varepsilon^{\frac{3}{2}} \partial_y^2 v_p^3 - \nu_2 \varepsilon^{\frac{3}{2}} \partial_x^2 (v_p^1 + \sqrt{\varepsilon} v_p^2 + \varepsilon v_e^3 + \varepsilon v_p^3), \quad (5.31)$$

$$\begin{aligned} R_{app}^4 := & \varepsilon^2 R_2^g - \nu_2 \varepsilon^{\frac{3}{2}} \Delta g_e^2 + \varepsilon^{\frac{3}{2}} [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] (g_p^0 + g_e^1) + \varepsilon^2 [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] (g_p^1 + g_e^2) \\ & + \varepsilon^{\frac{5}{2}} [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] (g_p^2 + g_e^3) + \varepsilon^3 [(u_e^3 + u_p^3) \partial_x + v_p^3 \partial_y] g_p^3 \\ & + \varepsilon^{\frac{3}{2}} [(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y] (g_p^1 + g_e^2) + \varepsilon^2 [(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y] (g_p^2 + g_e^3) \\ & + \varepsilon^{\frac{5}{2}} [(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y] g_p^3 + \varepsilon^{\frac{3}{2}} [(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y] (g_p^2 + g_e^3) \\ & + \varepsilon^2 [(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y] g_p^3 + \varepsilon^{\frac{3}{2}} [(u_e^0 + u_p^0) \partial_x + (v_p^0 + v_e^1) \partial_y] g_p^3 \\ & - \varepsilon^{\frac{3}{2}} [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] (v_p^0 + v_e^1) - \varepsilon^2 [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] (v_p^1 + v_e^2) \\ & - \varepsilon^{\frac{5}{2}} [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] (v_p^2 + v_e^3) - \varepsilon^3 [(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y] v_p^3 \\ & - \varepsilon^{\frac{3}{2}} [(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y] (v_p^1 + v_e^2) - \varepsilon^2 [(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y] (v_p^2 + v_e^3) \\ & - \varepsilon^{\frac{5}{2}} [(h_e^2 + h_p^2) \partial_x + (g_p^2 + g_e^3) \partial_y] v_p^3 - \varepsilon^{\frac{3}{2}} [(h_e^1 + h_p^1) \partial_x + (g_p^1 + g_e^2) \partial_y] (v_p^2 + v_e^3) \\ & - \varepsilon^2 [(h_e^1 + h_p^1) \partial_x + (g_p^1 + g_e^2) \partial_y] v_p^3 - \varepsilon^{\frac{3}{2}} [(h_e^0 + h_p^0) \partial_x + (g_p^0 + g_e^1) \partial_y] v_p^3 \\ & - \nu_4 \varepsilon^{\frac{3}{2}} \partial_y^2 g_p^3 - \nu_4 \varepsilon^{\frac{3}{2}} \partial_x^2 (g_p^1 + \sqrt{\varepsilon} g_p^2 + \varepsilon g_e^3 + \varepsilon g_p^3). \end{aligned} \quad (5.32)$$

Using the estimates of R_2^v and R_2^g , we have

$$\|(R_2^v, R_2^g)\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{\frac{1}{16}}.$$

Next, we estimate the other terms of R_{app}^2 and get

$$\begin{aligned} & \varepsilon^{\frac{3}{2}} \|(u_e^0 + u_p^0) \partial_x + (v_p^0 + v_e^1) \partial_y\|_{L^2(\Omega_\varepsilon)} v_p^3 \\ & \leq \varepsilon^{\frac{3}{2}} \|(u_e^0, u_p^0, v_p^0, v_e^1)\|_{L^\infty} \|\nabla v_p^3\|_{L^2} \\ & \leq C(L, \xi) \varepsilon^{-\xi + \frac{17}{16}}, \\ & \varepsilon^{\frac{3}{2}} \|(u_e^1 + u_p^1) \partial_x + (v_p^1 + v_e^2) \partial_y\|_{L^2(\Omega_\varepsilon)} (v_p^2 + v_e^3) \\ & \leq \varepsilon^{\frac{3}{2}} \|(u_e^1, u_p^1, v_p^1, v_e^2)\|_{L^\infty} (\|\partial_x v_p^2 + \partial_x v_e^3\|_{L^2} + \|\partial_y v_p^2 + \sqrt{\varepsilon} \partial_Y v_e^3\|_{L^2}) \\ & \leq C(L, \xi) \varepsilon^{-\xi + \frac{17}{16}}, \\ & \varepsilon^{\frac{3}{2}} \|(u_e^2 + u_p^2) \partial_x + (v_p^2 + v_e^3) \partial_y\|_{L^2(\Omega_\varepsilon)} (v_p^1 + v_e^2) \\ & \leq \varepsilon^{\frac{3}{2}} \|(u_e^2, u_p^2, v_p^2, v_e^3)\|_{L^\infty} (\|\partial_x v_p^1 + \partial_x v_e^2\|_{L^2} + \|\partial_y v_p^1 + \sqrt{\varepsilon} \partial_Y v_e^2\|_{L^2}) \\ & \leq C(L, \xi) \varepsilon^{-\xi + \frac{17}{16}}, \end{aligned}$$

and

$$\begin{aligned} & \varepsilon^{\frac{3}{2}} \|(h_e^3 + h_p^3) \partial_x + g_p^3 \partial_y\|_{L^2(\Omega_\varepsilon)} (g_p^0 + g_e^1) \\ & \leq \varepsilon^{\frac{3}{2}} \|(h_e^3, h_p^3, g_p^3)\|_{L^\infty} (\|\partial_x g_p^0 + \partial_x g_e^1\|_{L^2} + \|\partial_y g_p^0 + \sqrt{\varepsilon} \partial_Y g_e^1\|_{L^2}) \\ & \leq C(L, \xi) \varepsilon^{-\xi + \frac{17}{16}}. \end{aligned}$$

For the last two terms of R_{app}^2 , we have

$$\|\varepsilon^{\frac{3}{2}} \partial_{yy} v_p^3 + \varepsilon^{\frac{3}{2}} \partial_x^2 (v_p^1 + \sqrt{\varepsilon} v_p^2 + \sqrt{\varepsilon} v_e^3 + \varepsilon v_p^3)\|_{L^2(\Omega_\varepsilon)} \leq C(L) \varepsilon^{-\xi + \frac{17}{16}},$$

in which we use the estimates of v_p^1, v_p^2, v_p^3 and v_e^3 . Hence, we obtain

$$\|R_{app}^2\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi + \frac{17}{16}}.$$

Similar arguments imply that

$$\|R_{app}^4\|_{L^2(\Omega_\varepsilon)} \leq C(L, \xi) \varepsilon^{-\xi + \frac{17}{16}}.$$

Combining all the previous estimates, we obtain (5.28). \square

6 The proof of the main results

Each term in the expansions (1.9) has already been constructed previously. In this section, we prove the existence of the remainder $(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon, p^\varepsilon)$. Let us define

$$\left\{ \begin{array}{l} u_s(x, y) = u_e^0(\sqrt{\varepsilon}y) + u_p^0(x, y) + \sqrt{\varepsilon}u_p^1(x, y) + \sqrt{\varepsilon}u_e^1(x, \sqrt{\varepsilon}y) \\ \quad + \varepsilon u_p^2(x, y) + \varepsilon u_e^2(x, \sqrt{\varepsilon}y) + \varepsilon^{\frac{3}{2}}u_e^3(x, \sqrt{\varepsilon}y), \\ v_s(x, y) = v_p^0(x, y) + v_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}v_p^1(x, y) \\ \quad + \sqrt{\varepsilon}v_e^2(x, \sqrt{\varepsilon}y) + \varepsilon v_p^2(x, y) + \varepsilon v_e^3(x, \sqrt{\varepsilon}y), \\ h_s(x, y) = h_e^0(\sqrt{\varepsilon}y) + h_p^0(x, y) + \sqrt{\varepsilon}h_p^1(x, y) + \sqrt{\varepsilon}h_e^1(x, \sqrt{\varepsilon}y) \\ \quad + \varepsilon h_p^2(x, y) + \varepsilon h_e^2(x, \sqrt{\varepsilon}y) + \varepsilon^{\frac{3}{2}}h_e^3(x, \sqrt{\varepsilon}y), \\ g_s(x, y) = g_p^0(x, y) + g_e^1(x, \sqrt{\varepsilon}y) + \sqrt{\varepsilon}g_p^1(x, y) \\ \quad + \sqrt{\varepsilon}g_e^2(x, \sqrt{\varepsilon}y) + \varepsilon g_p^2(x, y) + \varepsilon g_e^3(x, \sqrt{\varepsilon}y). \end{array} \right. \quad (6.1)$$

Substituting (6.1) into (1.7), we obtain that $(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon, p^\varepsilon)$ satisfy the following system

$$\left\{ \begin{array}{l} u_s u_x^\varepsilon + u^\varepsilon u_{sx} + v_s u_y^\varepsilon + v^\varepsilon u_{sy} + p_x^\varepsilon - \nu_1 \Delta_\varepsilon u^\varepsilon \\ \quad - h_s h_x^\varepsilon - h^\varepsilon h_{sx} - g_s h_y^\varepsilon - g^\varepsilon h_{sy} = R_1(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ u_s v_x^\varepsilon + u^\varepsilon v_{sx} + v_s v_y^\varepsilon + v^\varepsilon v_{sy} + \frac{p_y^\varepsilon}{\varepsilon} - \nu_2 \Delta_\varepsilon v^\varepsilon \\ \quad - h_s g_x^\varepsilon - h^\varepsilon g_{sx} - g_s g_y^\varepsilon - g^\varepsilon g_{sy} = R_2(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ u_s h_x^\varepsilon + u^\varepsilon h_{sx} + v_s h_y^\varepsilon + v^\varepsilon h_{sy} - \nu_3 \Delta_\varepsilon h^\varepsilon \\ \quad - h_s u_x^\varepsilon - h^\varepsilon u_{sx} - g_s u_y^\varepsilon - g^\varepsilon u_{sy} = R_3(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ u_s g_x^\varepsilon + u^\varepsilon g_{sx} + v_s g_y^\varepsilon + v^\varepsilon g_{sy} - \nu_4 \Delta_\varepsilon g^\varepsilon \\ \quad - h_s v_x^\varepsilon - h^\varepsilon v_{sx} - g_s v_y^\varepsilon - g^\varepsilon v_{sy} = R_4(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ u_x^\varepsilon + v_y^\varepsilon = h_x^\varepsilon + g_y^\varepsilon = 0. \end{array} \right. \quad (6.2)$$

The source terms R_i ($i = 1, 2, 3, 4$) are defined by

$$\begin{aligned} R_1(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon) &:= \varepsilon^{-\gamma - \frac{3}{2}} R_{app}^1 - \varepsilon^{\frac{3}{2}} [(u_p^3 + \varepsilon^\gamma u^\varepsilon) u_x^\varepsilon + u^\varepsilon u_{px}^3 + (v_p^3 + \varepsilon^\gamma v^\varepsilon) u_y^\varepsilon + v^\varepsilon u_{py}^3 \\ &\quad - (h_p^3 + \varepsilon^\gamma h^\varepsilon) h_x^\varepsilon - h^\varepsilon h_{px}^3 - (g_p^3 + \varepsilon^\gamma g^\varepsilon) h_y^\varepsilon - g^\varepsilon h_{py}^3], \\ R_2(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon) &:= \varepsilon^{-\gamma - \frac{3}{2}} R_{app}^2 - \varepsilon^{\frac{3}{2}} [(u_p^3 + \varepsilon^\gamma u^\varepsilon) v_x^\varepsilon + u^\varepsilon v_{px}^3 + (v_p^3 + \varepsilon^\gamma v^\varepsilon) v_y^\varepsilon + v^\varepsilon v_{py}^3 \\ &\quad - (h_p^3 + \varepsilon^\gamma h^\varepsilon) g_x^\varepsilon - h^\varepsilon g_{px}^3 - (g_p^3 + \varepsilon^\gamma g^\varepsilon) g_y^\varepsilon - g^\varepsilon g_{py}^3], \end{aligned}$$

$$\begin{aligned}
& -(h_p^3 + \varepsilon^\gamma h^\varepsilon)g_x^\varepsilon - h^\varepsilon g_{px}^3 - (g_p^3 + \varepsilon^\gamma g^\varepsilon)g_y^\varepsilon - g^\varepsilon g_{py}^3, \\
R_3(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon) &:= \varepsilon^{-\gamma-\frac{3}{2}} R_{app}^3 - \varepsilon^{\frac{3}{2}} [(u_p^3 + \varepsilon^\gamma u^\varepsilon)h_x^\varepsilon + u^\varepsilon h_{px}^3 + (v_p^3 + \varepsilon^\gamma v^\varepsilon)h_y^\varepsilon + v^\varepsilon h_{py}^3 \\
& -(h_p^3 + \varepsilon^\gamma h^\varepsilon)u_x^\varepsilon - h^\varepsilon u_{px}^3 - (g_p^3 + \varepsilon^\gamma g^\varepsilon)u_y^\varepsilon - g^\varepsilon u_{py}^3], \\
R_4(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon) &:= \varepsilon^{-\gamma-\frac{3}{2}} R_{app}^4 - \varepsilon^{\frac{3}{2}} [(u_p^3 + \varepsilon^\gamma u^\varepsilon)g_x^\varepsilon + u^\varepsilon g_{px}^3 + (v_p^3 + \varepsilon^\gamma v^\varepsilon)g_y^\varepsilon + v^\varepsilon g_{py}^3 \\
& -(h_p^3 + \varepsilon^\gamma h^\varepsilon)v_x^\varepsilon - h^\varepsilon v_{px}^3 - (g_p^3 + \varepsilon^\gamma g^\varepsilon)v_y^\varepsilon - g^\varepsilon v_{py}^3],
\end{aligned} \tag{6.3}$$

in which the error terms $R_{app}^i (i = 1, 2, 3, 4)$ have been estimated in Proposition 5.5.

By the definition of u_s, h_s in (6.1) and the estimates of $(u_p^i, v_p^i, h_p^i, g_p^i) (i = 0, 1, 2, 3)$ and $(u_e^j, v_e^j, h_e^j, g_e^j) (j = 1, 2, 3)$ obtained earlier, for small enough $\sigma_0 > 0$, we get $\|y\partial_y(u_s, h_s)\|_{L^\infty} < C\sigma_0$. In addition, under the assumption (H2), we also obtain $\|u_s/h_s\|_{L^\infty} \ll 1$. These two estimates will be used in the proof of Proposition 6.1.

Next, we state two propositions which will be used later in the proof of Theorem 1.1.

Proposition 6.1. *For any given $f_i \in L^2(\Omega_\varepsilon) (i = 1, 2, 3, 4)$, there exists $L > 0$ such that the following linear problem*

$$\begin{cases} u_s u_x + u u_{sx} + v_s u_y + v u_{sy} - (h_s h_x + h h_{sx} + g_s h_y + g h_{sy}) + p_x - \nu_1 \Delta_\varepsilon u = f_1, \\ u_s v_x + u v_{sx} + v_s v_y + v v_{sy} - (h_s g_x + h g_{sx} + g_s g_y + g g_{sy}) + \frac{p_y}{\varepsilon} - \nu_2 \Delta_\varepsilon v = f_2, \\ u_s h_x + u h_{sx} + v_s h_y + v h_{sy} - (h_s u_x + h u_{sx} + g_s u_y + g u_{sy}) - \nu_3 \Delta_\varepsilon h = f_3, \\ u_s g_x + u g_{sx} + v_s g_y + v g_{sy} - (h_s v_x + h v_{sx} + g_s v_y + g v_{sy}) - \nu_4 \Delta_\varepsilon g = f_4, \\ u_x + v_y = h_x + g_y = 0, \end{cases} \tag{6.4}$$

with boundary conditions

$$\begin{cases} (u, v, h, g)(x, 0) = 0, & (u_y, v, h, g)(x, \varepsilon^{-1/2}) = 0, \\ (u, v, h, g)(0, y) = 0, & (p - 2\varepsilon u_x, u_y + \varepsilon v_x, h, g_x)(L, y) = 0, \end{cases} \tag{6.5}$$

exists an unique solution (u, v, h, g, p) on the domain Ω_ε , which satisfies

$$\begin{aligned} & \|\nabla_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon v\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon h\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon g\|_{L^2(\Omega_\varepsilon)} \\ & \lesssim \|f_1\|_{L^2(\Omega_\varepsilon)} + \|f_3\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}(\|f_2\|_{L^2(\Omega_\varepsilon)} + \|f_4\|_{L^2(\Omega_\varepsilon)}). \end{aligned} \tag{6.6}$$

Proof. Under the assumption (H1), the previous proposition has been proven in Section 3.1 of [3]. Hence, we only consider the proof of this proposition under the assumption (H2). The proposition is then a direct consequence of the following two Lemmas. \square

We first recall the following lemma, proved in [3].

Lemma 6.2. *Let (u, v, h, g) be the solution to the linear problem (6.4), and assume that $\varepsilon \ll L$, then the following estimate holds*

$$\begin{aligned} & \nu_1 \|\nabla_\varepsilon u\|_{L^2(\Omega_\varepsilon)}^2 + \nu_3 \|\nabla_\varepsilon h\|_{L^2(\Omega_\varepsilon)}^2 + \int_{x=L} u_s (|u|^2 + \varepsilon |v|^2 + \varepsilon |g|^2) dy \\ & \lesssim L(\|\nabla_\varepsilon v\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla_\varepsilon g\|_{L^2(\Omega_\varepsilon)}^2) + \|(f_1, f_3)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon \|(f_2, f_4)\|_{L^2(\Omega_\varepsilon)}^2. \end{aligned} \tag{6.7}$$

We now turn to the proof of the following lemma.

Lemma 6.3. *Let (u, v, h, g) be the solution to the linear problem (6.4), then*

$$\begin{aligned} & \|\nabla_\varepsilon g\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla_\varepsilon v\|_{L^2(\Omega_\varepsilon)}^2 + \int_{x=0} \frac{\varepsilon^2 v_x g_x}{h_s} dy + 2\varepsilon \int_{x=L} \frac{v_y g_y}{h_s} dy - \varepsilon \int_{x=L} \frac{|v_y|^2 + |g_y|^2}{h_s} dy \\ & \leq C \left[\|(f_1, f_3)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon \|(f_2, f_4)\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla_\varepsilon u\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla_\varepsilon h\|_{L^2(\Omega_\varepsilon)}^2 \right. \\ & \quad \left. + \left(L + \left\| \frac{u_s}{h_s} \right\|_{L^\infty} + \|y \partial_y(u_s, h_s)\|_{L^\infty} \right) \left(\|\nabla_\varepsilon v\|_{L^2(\Omega_\varepsilon)}^2 + \|\nabla_\varepsilon g\|_{L^2(\Omega_\varepsilon)}^2 \right) \right]. \end{aligned} \quad (6.8)$$

Proof. We multiply (6.4.1) by $\left[-\partial_y \left(\frac{g}{h_s}\right)\right]$, (6.4.2) by $\varepsilon \partial_x \left(\frac{g}{h_s}\right)$, (6.4.3) by $\left[-\partial_y \left(\frac{v}{h_s}\right)\right]$ and (6.4.4) by $\varepsilon \partial_x \left(\frac{v}{h_s}\right)$ and sum these four equalities, which gives

$$\begin{aligned} & \iint_{\Omega_\varepsilon} \partial_y \left(\frac{g}{h_s}\right) (h_s \partial_x h + h \partial_x h_s + g_s \partial_y h + g \partial_y h_s) dx dy \\ & - \iint_{\Omega_\varepsilon} \partial_y \left(\frac{g}{h_s}\right) (u_s \partial_x u + u \partial_x u_s + v_s \partial_y u + v \partial_y u_s + \partial_x p - \nu_1 \Delta_\varepsilon u) dx dy \\ & + \iint_{\Omega_\varepsilon} \varepsilon \partial_x \left(\frac{g}{h_s}\right) \left(u_s \partial_x v + u \partial_x v_s + v_s \partial_y v + v \partial_y v_s + \frac{\partial_y p}{\varepsilon} - \nu_2 \Delta_\varepsilon v\right) dx dy \\ & - \iint_{\Omega_\varepsilon} \varepsilon \partial_x \left(\frac{g}{h_s}\right) (h_s \partial_x g + h \partial_x g_s + g_s \partial_y g + g \partial_y g_s) dx dy \\ & - \iint_{\Omega_\varepsilon} \partial_y \left(\frac{v}{h_s}\right) (u_s \partial_x h + u \partial_x h_s + v_s \partial_y h + v \partial_y h_s - \nu_3 \Delta_\varepsilon h) dx dy \\ & + \iint_{\Omega_\varepsilon} \partial_y \left(\frac{v}{h_s}\right) (h_s \partial_x u + h \partial_x u_s + g_s \partial_y u + g \partial_y u_s) dx dy \\ & + \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left(\frac{v}{h_s}\right) (u_s \partial_x g + u \partial_x g_s + v_s \partial_y g + v \partial_y g_s - \nu_4 \Delta_\varepsilon g) dx dy \\ & - \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left(\frac{v}{h_s}\right) (h_s \partial_x v + h \partial_x v_s + g_s \partial_y v + g \partial_y v_s) dx dy \\ & = - \iint_{\Omega_\varepsilon} \partial_y \left(\frac{g}{h_s}\right) f_1 dx dy + \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left(\frac{g}{h_s}\right) f_2 dx dy \\ & - \iint_{\Omega_\varepsilon} \partial_y \left(\frac{v}{h_s}\right) f_3 dx dy + \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left(\frac{v}{h_s}\right) f_4 dx dy := \mathcal{F}. \end{aligned} \quad (6.9)$$

The right hand side of (6.9) equals

$$\begin{aligned} \mathcal{F} &= \iint_{\Omega_\varepsilon} \left\{ \frac{g \partial_y h_s}{h_s^2} - \frac{g_y}{h_s} \right\} f_1 + \varepsilon \left\{ \frac{g_x}{h_s} - \frac{g \partial_x h_s}{h_s^2} \right\} f_2 dx dy \\ &+ \iint_{\Omega_\varepsilon} \left\{ \frac{v \partial_y h_s}{h_s^2} - \frac{v_y}{h_s} \right\} f_3 + \varepsilon \left\{ \frac{v_x}{h_s} - \frac{v \partial_x h_s}{h_s^2} \right\} f_4 dx dy \\ &\leq \frac{\sup_x |\langle y \rangle^{1/2} \nabla_\varepsilon h_s| + \sup \|h_s\|_{L^\infty}}{\{\min h_s\}^2} \left\{ \|f_1\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} \|f_2\|_{L^2(\Omega_\varepsilon)} \right\} \|\nabla_\varepsilon g\|_{L^2(\Omega_\varepsilon)} \\ &+ \frac{\sup_x |\langle y \rangle^{1/2} \nabla_\varepsilon h_s| + \sup \|h_s\|_{L^\infty}}{\{\min h_s\}^2} \left\{ \|f_3\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} \|f_4\|_{L^2(\Omega_\varepsilon)} \right\} \|\nabla_\varepsilon v\|_{L^2(\Omega_\varepsilon)} \end{aligned}$$

$$\lesssim (\|f_1\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|f_2\|_{L^2(\Omega_\varepsilon)})\|\nabla_\varepsilon g\|_{L^2(\Omega_\varepsilon)} + (\|f_3\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|f_4\|_{L^2(\Omega_\varepsilon)})\|\nabla_\varepsilon v\|_{L^2(\Omega_\varepsilon)}.$$

For the left hand side of (6.9), we collect the following terms, follow the approach of [3] and obtain

$$\begin{aligned} & \iint_{\Omega_\varepsilon} \partial_y \left(\frac{g}{h_s} \right) (h_s \partial_x h + h \partial_x h_s + g_s \partial_y h + g \partial_y h_s) dx dy \\ & - \iint_{\Omega_\varepsilon} \varepsilon \partial_x \left(\frac{g}{h_s} \right) (h_s \partial_x g + h \partial_x g_s + g_s \partial_y g + g \partial_y g_s) dx dy \\ & + \iint_{\Omega_\varepsilon} \partial_y \left(\frac{v}{h_s} \right) (h_s \partial_x u - u \partial_x h_s + g_s \partial_y u - v \partial_y h_s) dx dy \\ & - \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left(\frac{v}{h_s} \right) (h_s \partial_x v - u \partial_x g_s + g_s \partial_y v - v \partial_y g_s) dx dy \\ & \lesssim - \iint_{\Omega_\varepsilon} |\nabla_\varepsilon g|^2 dx dy - \iint_{\Omega_\varepsilon} |\nabla_\varepsilon v|^2 dx dy + \|\nabla_\varepsilon(u, h)\|_{L^2(\Omega_\varepsilon)}^2 + L \|\nabla_\varepsilon(v, g)\|_{L^2(\Omega_\varepsilon)}^2 \\ & + \|\nabla_\varepsilon(u, h)\|_{L^2(\Omega_\varepsilon)} \|\nabla_\varepsilon(v, g)\|_{L^2(\Omega_\varepsilon)} + \iint_{\Omega_\varepsilon} \frac{|v|^2 |\partial_y h_s|^2}{h_s^2} dx dy \\ & \lesssim - \iint_{\Omega_\varepsilon} |\nabla_\varepsilon g|^2 dx dy - \iint_{\Omega_\varepsilon} |\nabla_\varepsilon v|^2 dx dy + \|\nabla_\varepsilon(u, h)\|_{L^2(\Omega_\varepsilon)}^2 + L \|\nabla_\varepsilon(v, g)\|_{L^2(\Omega_\varepsilon)}^2 \\ & + \|\nabla_\varepsilon(u, h)\|_{L^2(\Omega_\varepsilon)} \|\nabla_\varepsilon(v, g)\|_{L^2(\Omega_\varepsilon)} + \|y \partial_y h_s\|_{L^\infty}^2 \|\partial_y v\|_{L^2(\Omega_\varepsilon)}^2, \end{aligned}$$

where we use the fact that h_s is strictly positive. Similarly, we have

$$\begin{aligned} & \iint_{\Omega_\varepsilon} \partial_y \left(\frac{g}{h_s} \right) u_s \partial_x u dx dy = \iint_{\Omega_\varepsilon} \frac{\partial_y g}{h_s} u_s \partial_x u dx dy - \iint_{\Omega_\varepsilon} \frac{\partial_y h_s g}{h_s^2} u_s \partial_x u dx dy \\ & \lesssim \left\| \frac{u_s}{h_s} \right\|_{L^\infty} \|\partial_x u\|_{L^2} \|\partial_y g\|_{L^2} \\ & + \|y \partial_y h_s\|_{L^\infty} \|\partial_y g\|_{L^2} \|\partial_x u\|_{L^2} \\ & \lesssim \left\| \frac{u_s}{h_s} \right\|_{L^\infty} \|\partial_y v\|_{L^2} \|\partial_y g\|_{L^2} \\ & + \|y \partial_y h_s\|_{L^\infty} \|\partial_y g\|_{L^2} \|\partial_y v\|_{L^2}, \\ & \iint_{\Omega_\varepsilon} \partial_y \left(\frac{g}{h_s} \right) v \partial_y u_s dx dy = \iint_{\Omega_\varepsilon} \frac{\partial_y g}{h_s} v \partial_y u_s dx dy - \iint_{\Omega_\varepsilon} \frac{\partial_y h_s g}{h_s^2} v \partial_y u_s dx dy \\ & \lesssim (\|y \partial_y u_s\|_{L^\infty} + \|y \partial_y h_s\|_{L^\infty} \|y \partial_y u_s\|_{L^\infty}) \|\partial_y v\|_{L^2} \|\partial_y g\|_{L^2}, \\ & \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left(\frac{g}{h_s} \right) u_s \partial_x v dx dy = \varepsilon \iint_{\Omega_\varepsilon} \frac{\partial_x g}{h_s} u_s \partial_x v dx dy - \varepsilon \iint_{\Omega_\varepsilon} \frac{\partial_x h_s g}{h_s^2} u_s \partial_x v dx dy \\ & \lesssim \varepsilon \left(\left\| \frac{u_s}{h_s} \right\|_{L^\infty} + L |u_s| \|\partial_x h_s\|_{L^\infty} \right) \|\partial_x g\|_{L^2} \|\partial_x v\|_{L^2}, \\ & \iint_{\Omega_\varepsilon} \partial_y \left(\frac{v}{h_s} \right) u_s \partial_x h dx dy = \iint_{\Omega_\varepsilon} \frac{\partial_y v}{h_s} u_s \partial_x h dx dy - \iint_{\Omega_\varepsilon} \frac{\partial_y h_s v}{h_s^2} u_s \partial_x h dx dy \\ & \lesssim \left\| \frac{u_s}{h_s} \right\|_{L^\infty} \|\partial_y v\|_{L^2} \|\partial_x h\|_{L^2} \\ & + |u_s| \|y \partial_y h_s\|_{L^\infty} \|\partial_y v\|_{L^2} \|\partial_x h\|_{L^2} \end{aligned}$$

$$\begin{aligned}
& \lesssim \left(\left\| \frac{u_s}{h_s} \right\|_{L^\infty} + |u_s| \|y \partial_y h_s\|_{L^\infty} \right) \|\partial_y v\|_{L^2} \|\partial_y g\|_{L^2}, \\
& \iint_{\Omega_\varepsilon} \partial_y \left(\frac{v}{h_s} \right) g \partial_y u_s dx dy = \iint_{\Omega_\varepsilon} \frac{\partial_y v}{h_s} g \partial_y u_s dx dy - \iint_{\Omega_\varepsilon} \frac{v \partial_y h_s}{h_s^2} g \partial_y u_s \\
& \lesssim (\|h_s\|_{L^\infty} + \|y \partial_y h_s\|_{L^\infty}) \|y \partial_y u_s\|_{L^\infty} \|\partial_y v\|_{L^2} \|\partial_y g\|_{L^2}, \\
& \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left(\frac{v}{h_s} \right) u_s \partial_x g dx dy = \varepsilon \iint_{\Omega_\varepsilon} \frac{\partial_x v}{h_s} u_s \partial_x g dx dy - \varepsilon \iint_{\Omega_\varepsilon} \frac{\partial_x h_s v}{h_s^2} u_s \partial_x g dx dy \\
& \lesssim \varepsilon \left(\left\| \frac{u_s}{h_s} \right\|_{L^\infty} + L |u_s| \|\partial_x h_s\|_{L^\infty} \right) \|\partial_x g\|_{L^2} \|\partial_x v\|_{L^2}.
\end{aligned}$$

For the pressure terms in (6.9), by using the condition $p = 2\varepsilon u_x$ at $x = L$, we get

$$\begin{aligned}
& \iint_{\Omega_\varepsilon} \partial_y \left\{ \frac{g}{h_s} \right\} p_x dx dy - \iint_{\Omega_\varepsilon} \partial_x \left\{ \frac{g}{h_s} \right\} p_y dx dy \\
& = \int_{x=L} \partial_y \left\{ \frac{g}{h_s} \right\} p dy = 2\varepsilon \int_{x=L} \partial_y \left\{ \frac{g}{h_s} \right\} u_x dy \\
& = -2\varepsilon \int_{x=L} \partial_y \left\{ \frac{g}{h_s} \right\} v_y dy \\
& = -2\varepsilon \int_{x=L} \frac{v_y g_y}{h_s} dy - 2\varepsilon \int_{x=L} \partial_y \left\{ \frac{1}{h_s} \right\} g v_y dy \\
& \leq -2\varepsilon \int_{x=L} \frac{v_y g_y}{h_s} dy + C(h_s, g_s) \sqrt{L} \varepsilon \|g_x\|_{L^2} \left\{ \int_{x=L} \frac{v_y^2}{h_s} dy \right\}^{1/2} \\
& \leq -2\varepsilon \int_{x=L} \frac{v_y g_y}{h_s} dy + \frac{\varepsilon}{2} \int_{x=L} \frac{v_y^2}{h_s} dy + C(h_s, g_s) L \|\nabla_\varepsilon g\|_{L^2}^2,
\end{aligned} \tag{6.10}$$

in which we use the Young inequality. We now compute the dissipative terms

$$\begin{aligned}
& \iint_{\Omega_\varepsilon} \left[-\partial_y \left\{ \frac{g}{h_s} \right\} \Delta_\varepsilon u + \partial_x \left\{ \frac{g}{h_s} \right\} \varepsilon \Delta_\varepsilon v \right] dx dy \\
& = \iint_{\Omega_\varepsilon} \left[\frac{u_{yy} h_x}{h_s} + \varepsilon \frac{u_{xx} h_x}{h_s} + \varepsilon \frac{g_x v_{yy}}{h_s} + \varepsilon^2 \frac{g_x v_{xx}}{h_s} \right] dx dy \\
& \quad + \iint_{\Omega_\varepsilon} \left[\left\{ \frac{g \partial_y h_s}{h_s^2} \right\} [u_{yy} + \varepsilon u_{xx}] - \frac{g \partial_x h_s}{h_s^2} [\varepsilon v_{yy} + \varepsilon^2 v_{xx}] \right] dx dy \\
& = - \int_{x=L} \frac{u_y h_y}{h_s} dy + \iint_{\Omega_\varepsilon} \partial_x \left\{ \frac{1}{h_s} \right\} u_y h_y dx dy - \iint_{\Omega_\varepsilon} \partial_y \left\{ \frac{1}{h_s} \right\} u_y h_x dx dy \\
& \quad + \iint_{\Omega_\varepsilon} \frac{u_{xy} h_y}{h_s} dx dy + \varepsilon \int_{x=L} \frac{u_x h_x}{h_s} dy - \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left\{ \frac{1}{h_s} \right\} u_x h_x dy - \varepsilon \iint_{\Omega_\varepsilon} \frac{u_x h_{xx}}{h_s} dx dy \\
& \quad - \varepsilon \int_{x=L} \frac{v_y g_y}{h_s} dy + \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left\{ \frac{1}{h_s} \right\} v_y g_y dx dy - \varepsilon \iint_{\Omega_\varepsilon} \partial_y \left\{ \frac{1}{h_s} \right\} v_y g_x dx dy + \varepsilon \iint_{\Omega_\varepsilon} \frac{g_y v_{xy}}{h_s} dx dy \\
& \quad + \varepsilon^2 \int_{x=L} \frac{v_x g_x}{h_s} dy - \varepsilon^2 \int_{x=0} \frac{v_x g_x}{h_s} dy - \varepsilon^2 \iint_{\Omega_\varepsilon} \partial_x \left\{ \frac{1}{h_s} \right\} v_x g_x dx dy - \varepsilon^2 \iint_{\Omega_\varepsilon} \frac{g_{xx} v_x}{h_s} dx dy \\
& \quad - \iint_{\Omega_\varepsilon} \partial_y \left\{ \frac{g \partial_y h_s}{h_s^2} \right\} u_y dx dy - \varepsilon \iint_{\Omega_\varepsilon} \partial_x \left\{ \frac{g \partial_y h_s}{h_s^2} \right\} u_x dx dy + \varepsilon \int_{x=L} \left\{ \frac{g \partial_y h_s}{h_s^2} \right\} u_x dy
\end{aligned}$$

$$+ \varepsilon \iint_{\Omega_\varepsilon} \partial_y \left\{ \frac{g \partial_x h_s}{h_s^2} \right\} v_y dx dy + \varepsilon^2 \iint_{\Omega_\varepsilon} \partial_x \left\{ \frac{g \partial_x h_s}{h_s^2} \right\} v_x dx dy - \varepsilon^2 \int_{x=L} \left\{ \frac{g \partial_x h_s}{h_s^2} \right\} v_x dy, \quad (6.11)$$

where we have chosen $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$. Using a similar method to estimate the boundary terms of (6.11) at $x = L$, we directly obtain

$$\begin{aligned} & - \int_{x=L} \frac{u_y h_y}{h_s} dy + \varepsilon \int_{x=L} \frac{u_x h_x}{h_s} dy - \varepsilon \int_{x=L} \frac{v_y g_y}{h_s} dy + \varepsilon^2 \int_{x=L} \frac{v_x g_x}{h_s} dy \\ & + \varepsilon \int_{x=L} \left\{ \frac{g \partial_y h_s}{h_s^2} \right\} u_x dy - \varepsilon^2 \int_{x=L} \left\{ \frac{g \partial_x h_s}{h_s^2} \right\} v_x dy \\ & = - \varepsilon \int_{x=L} \left\{ \frac{g \partial_y h_s}{h_s^2} \right\} v_y dy + \varepsilon \int_{x=L} \left\{ \frac{g \partial_x h_s}{h_s^2} \right\} u_y dy \\ & = - \varepsilon \int_{x=L} \left\{ \frac{g \partial_y h_s}{h_s^2} \right\} v_y dy - \varepsilon \int_{x=L} \partial_y \left\{ \frac{g \partial_x h_s}{h_s^2} \right\} u dy \\ & \leq \varepsilon \sqrt{L} \sup_x \left| \frac{\partial_y h_s}{h_s} \right| \|g_x\|_{L^2} \sqrt{\int_{x=L} \frac{v_y^2}{h_s} dy} + \varepsilon \sqrt{L} \sup_x \left| \frac{\partial_x h_s}{h_s} \right| \|u_x\|_{L^2} \sqrt{\int_{x=L} \frac{g_y^2}{h_s} dy} \\ & \quad + \varepsilon L \left\{ \sup_x \sqrt{\int_{I_\varepsilon} \frac{y |\partial_{xy} h_s|}{h_s^2} dy} + \sup_x \sqrt{\int_{I_\varepsilon} \frac{y \{ \partial_x h_s \}^2}{h_s^3} dy} \right\} \|u_x\|_{L^2} \sqrt{\int_{x=L} \frac{g_y^2}{h_s} dy} \\ & \leq \frac{\varepsilon}{2} \int_{x=L} \frac{v_y^2 + g_y^2}{h_s} dy + C(h_s, g_s) L \|(\nabla_\varepsilon v, \nabla_\varepsilon g)\|_{L^2}^2, \end{aligned}$$

where we have used (1.13) and $u_x + v_y = 0$ at $x = L$.

We apply a similar method to estimate the other terms in (6.9) and (6.11). Combining all the previous estimates and using the smallness of $\|u_s/h_s\|_{L^\infty}$ and $\|y \partial_y(u_s, h_s)\|_{L^\infty}$, we obtain (6.8). \square

The following proposition provides L^∞ estimates on the solutions for the nonlinear problem.

Proposition 6.4. *For any given $F_i \in L^2(\Omega_\varepsilon)$ ($i = 1, 2, 3, 4$), suppose that the following system on the domain Ω_ε*

$$\begin{cases} -\nu_1 \Delta_\varepsilon u + p_x = F_1, \\ -\nu_2 \Delta_\varepsilon v + \frac{p_y}{\varepsilon} = F_2, \\ -\nu_3 \Delta_\varepsilon h = F_3, \\ -\nu_4 \Delta_\varepsilon g = F_4, \\ u_x + v_y = h_x + g_y = 0, \end{cases} \quad (6.12)$$

has a solution (u, v, h, g) with the same boundary conditions as that in (6.5). Then, for any $\gamma > 0$,

$$\begin{aligned} & \|u\|_{L^\infty(\Omega_\varepsilon)} + \|h\|_{L^\infty(\Omega_\varepsilon)} + \sqrt{\varepsilon} (\|v\|_{L^\infty(\Omega_\varepsilon)} + \|g\|_{L^\infty(\Omega_\varepsilon)}) \\ & \lesssim C_{\gamma, L} \varepsilon^{-\frac{\gamma}{16}} [\|\nabla_\varepsilon u\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon v\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon h\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon g\|_{L^2(\Omega_\varepsilon)} \\ & \quad + \|F_1\|_{L^2(\Omega_\varepsilon)} + \|F_3\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} (\|F_2\|_{L^2(\Omega_\varepsilon)} + \|F_4\|_{L^2(\Omega_\varepsilon)})]. \end{aligned} \quad (6.13)$$

The detailed proof of this proposition is obtained by modifying the results of [3].

Proof of Theorem 1.1. We will apply the standard contraction mapping principle to prove the existence of the solutions of the nonlinear problem. First, we define the function space \mathbb{X} by its norm

$$\begin{aligned} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} &:= \|\nabla_\varepsilon u^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon v^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon h^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon g^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\quad + \varepsilon^{\frac{7}{8}} \|u^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{7}{8}} \|h^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{1}{2} + \frac{7}{8}} (\|v^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \|g^\varepsilon\|_{L^\infty(\Omega_\varepsilon)}), \end{aligned} \quad (6.14)$$

where $\nabla_\varepsilon := \partial_y + \sqrt{\varepsilon} \partial_x$. Next, we select a subspace of \mathbb{X}

$$\mathbb{X}_K := \{(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon) \in \mathbb{X} \mid \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} \leq K\},$$

where K will be fixed later. Regarding each $(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)$ of \mathbb{X}_K , we solve the following linear problem:

$$\begin{cases} u_s \bar{u}_x^\varepsilon + \bar{u}^\varepsilon u_{sx} + v_s \bar{u}_y^\varepsilon + \bar{v}^\varepsilon u_{sy} - (h_s \bar{h}_x^\varepsilon + \bar{h}^\varepsilon h_{sx} + g_s \bar{h}_y^\varepsilon + \bar{g}^\varepsilon h_{sy}) + \bar{p}_x^\varepsilon - \nu_1 \Delta_\varepsilon \bar{u}^\varepsilon = R_1(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ u_s \bar{v}_x^\varepsilon + \bar{u}^\varepsilon v_{sx} + v_s \bar{v}_y^\varepsilon + \bar{v}^\varepsilon v_{sy} - (h_s \bar{g}_x^\varepsilon + \bar{h}^\varepsilon g_{sx} + g_s \bar{g}_y^\varepsilon + \bar{g}^\varepsilon g_{sy}) + \frac{\bar{p}_y^\varepsilon}{\varepsilon} - \nu_2 \Delta_\varepsilon \bar{v}^\varepsilon = R_2(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ u_s \bar{h}_x^\varepsilon + \bar{u}^\varepsilon h_{sx} + v_s \bar{h}_y^\varepsilon + \bar{v}^\varepsilon h_{sy} - (h_s \bar{u}_x^\varepsilon + \bar{h}^\varepsilon u_{sx} + g_s \bar{u}_y^\varepsilon + \bar{g}^\varepsilon u_{sy}) - \nu_3 \Delta_\varepsilon \bar{h}^\varepsilon = R_3(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ u_s \bar{g}_x^\varepsilon + \bar{u}^\varepsilon g_{sx} + v_s \bar{g}_y^\varepsilon + \bar{v}^\varepsilon g_{sy} - (h_s \bar{v}_x^\varepsilon + \bar{h}^\varepsilon v_{sx} + g_s \bar{v}_y^\varepsilon + \bar{g}^\varepsilon v_{sy}) - \nu_4 \Delta_\varepsilon \bar{g}^\varepsilon = R_4(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon), \\ \bar{u}_x^\varepsilon + \bar{v}_y^\varepsilon = \bar{h}_x^\varepsilon + \bar{g}_y^\varepsilon = 0. \end{cases} \quad (6.15)$$

By using the Proposition 6.1, we have

$$\begin{aligned} &\|\nabla_\varepsilon \bar{u}^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon \bar{v}^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon \bar{h}^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon \bar{g}^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\lesssim \|R_1\|_{L^2(\Omega_\varepsilon)} + \|R_3\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} (\|R_2\|_{L^2(\Omega_\varepsilon)} + \|R_4\|_{L^2(\Omega_\varepsilon)}). \end{aligned} \quad (6.16)$$

Next, we estimate $R_i (i = 1, 2, 3, 4)$. Applying the Proposition 5.5, it is clear that

$$\varepsilon^{-\gamma - \frac{3}{2}} (\|R_{app}^1\|_{L^2(\Omega_\varepsilon)} + \|R_{app}^3\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} \|R_{app}^2\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} \|R_{app}^4\|_{L^2(\Omega_\varepsilon)}) \leq C(L, \xi) \varepsilon^{-\gamma - \xi + \frac{1}{16}}.$$

For R_1 , the other terms are estimated as follows

$$\begin{aligned} &\varepsilon^{\frac{3}{2}} \|(u_p^3 + \varepsilon^\gamma u^\varepsilon) u_x^\varepsilon + (v_p^3 + \varepsilon^\gamma v^\varepsilon) u_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq \varepsilon^{\frac{3}{2}} [(\|u_p^3\|_{L^\infty} + \varepsilon^\gamma \|u^\varepsilon\|_{L^\infty}) \|v_y^\varepsilon\|_{L^2} + (\|v_p^3\|_{L^\infty} + \varepsilon^\gamma \|v^\varepsilon\|_{L^\infty}) \|u_y^\varepsilon\|_{L^2}] \\ &\leq \varepsilon^{\frac{3}{2}} \|(u_p^3, v_p^3)\|_{L^\infty} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} + \varepsilon^{\frac{7\gamma}{8} + 1} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}}^2 \\ &\leq C(L, \xi) \varepsilon^{\frac{21}{16} - \xi} K + \varepsilon^{\frac{7\gamma}{8} + 1} K^2, \\ &\varepsilon^{\frac{3}{2}} \|(h_p^3 + \varepsilon^\gamma h^\varepsilon) h_x^\varepsilon + (g_p^3 + \varepsilon^\gamma g^\varepsilon) h_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leq \varepsilon^{\frac{3}{2}} [(\|h_p^3\|_{L^\infty} + \varepsilon^\gamma \|h^\varepsilon\|_{L^\infty}) \|g_y^\varepsilon\|_{L^2} + (\|g_p^3\|_{L^\infty} + \varepsilon^\gamma \|g^\varepsilon\|_{L^\infty}) \|h_y^\varepsilon\|_{L^2}] \\ &\leq \varepsilon^{\frac{3}{2}} \|(h_p^3, g_p^3)\|_{L^\infty} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} + \varepsilon^{\frac{7\gamma}{8} + 1} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}}^2 \\ &\leq C(L, \xi) \varepsilon^{\frac{21}{16} - \xi} K + \varepsilon^{\frac{7\gamma}{8} + 1} K^2, \end{aligned}$$

in which we use the divergence free conditions and the estimates of $(u_p^3, v_p^3, h_p^3, g_p^3)$, and

$$\varepsilon^{\frac{3}{2}} \|u^\varepsilon u_{px}^3 + v^\varepsilon u_{py}^3 + h^\varepsilon h_{px}^3 + g^\varepsilon h_{py}^3\|_{L^2(\Omega_\varepsilon)}$$

$$\begin{aligned}
&\leq C\varepsilon^{\frac{3}{2}} \left[\|u_y^\varepsilon\|_{L^2} \sup_x \|\langle y \rangle^l u_{px}^3\|_{L^2} + \|v_y^\varepsilon\|_{L^2} \sup_x \|\langle y \rangle^l u_{py}^3\|_{L^2} \right. \\
&\quad \left. + \|h_y^\varepsilon\|_{L^2} \sup_x \|\langle y \rangle^l h_{px}^3\|_{L^2} + \|g_y^\varepsilon\|_{L^2} \sup_x \|\langle y \rangle^l h_{py}^3\|_{L^2} \right] \\
&\leq C(L, \xi) \varepsilon^{\frac{21}{16}-\xi} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} \leq C(L, \xi) \varepsilon^{\frac{21}{16}-\xi} K,
\end{aligned}$$

in which we have used the fact that $|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)| \leq \sqrt{y} \|(u_y^\varepsilon, v_y^\varepsilon, h_y^\varepsilon, g_y^\varepsilon)\|_{L^2(0, \varepsilon^{1/2})}$.

Using similar methods to estimate R_2 , we obtain

$$\begin{aligned}
&\varepsilon^{\frac{3}{2}} \|(u_p^3 + \varepsilon^\gamma u^\varepsilon) v_x^\varepsilon + (v_p^3 + \varepsilon^\gamma v^\varepsilon) v_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq \varepsilon (\|u_p^3\|_{L^\infty} + \varepsilon^\gamma \|u^\varepsilon\|_{L^\infty}) \|\sqrt{\varepsilon} v_x^\varepsilon\|_{L^2} + \varepsilon^{\frac{3}{2}} (\|v_p^3\|_{L^\infty} + \varepsilon^\gamma \|v^\varepsilon\|_{L^\infty}) \|v_y^\varepsilon\|_{L^2} \\
&\leq C\varepsilon (\|u_p^3, v_p^3\|_{L^\infty}) \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} + \varepsilon^{\frac{7\gamma}{8}+1} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}}^2 \\
&\leq C(L, \xi) \varepsilon^{\frac{13}{16}-\xi} K + \varepsilon^{\frac{7\gamma}{8}+1} K^2, \\
&\varepsilon^{\frac{3}{2}} \|(h_p^3 + \varepsilon^\gamma h^\varepsilon) g_x^\varepsilon + (g_p^3 + \varepsilon^\gamma g^\varepsilon) g_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq \varepsilon (\|h_p^3\|_{L^\infty} + \varepsilon^\gamma \|h^\varepsilon\|_{L^\infty}) \|\sqrt{\varepsilon} g_x^\varepsilon\|_{L^2} + \varepsilon^{\frac{3}{2}} (\|g_p^3\|_{L^\infty} + \varepsilon^\gamma \|g^\varepsilon\|_{L^\infty}) \|g_y^\varepsilon\|_{L^2} \\
&\leq C\varepsilon (\|h_p^3, g_p^3\|_{L^\infty}) \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} + \varepsilon^{\frac{7\gamma}{8}+1} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}}^2 \\
&\leq C(L, \xi) \varepsilon^{\frac{13}{16}-\xi} K + \varepsilon^{\frac{7\gamma}{8}+1} K^2,
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon^{\frac{3}{2}} \|u^\varepsilon v_{px}^3 + v^\varepsilon v_{py}^3 + h^\varepsilon g_{px}^3 + g^\varepsilon g_{py}^3\|_{L^2(\Omega_\varepsilon)} \\
&\leq C\varepsilon^{\frac{3}{2}} \left[\|u^\varepsilon\|_{L^\infty} \|v_{px}^3\|_{L^2} + \|v_y^\varepsilon\|_{L^2} \sup_x \|\langle y \rangle^l v_{pyy}^3\|_{L^2} \right. \\
&\quad \left. + \|h^\varepsilon\|_{L^\infty} \|g_{px}^3\|_{L^2} + \|g_y^\varepsilon\|_{L^2} \sup_x \|\langle y \rangle^l g_{pyy}^3\|_{L^2} \right] \\
&\leq C(L, \xi) \varepsilon^{\frac{13}{16}-\xi-\frac{\gamma}{8}} \|(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{\mathbb{X}} \leq C(L, \xi) \varepsilon^{\frac{13}{16}-\xi-\frac{\gamma}{8}} K.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\|R_1(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} \|R_2(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{L^2(\Omega_\varepsilon)} \\
&\leq C(L, \xi) \varepsilon^{-\xi-\gamma+\frac{1}{16}} + C(L, \xi) \varepsilon^{\frac{21}{16}-\xi-\frac{\gamma}{8}} K + \varepsilon^{\frac{7\gamma}{8}+1} K^2.
\end{aligned} \tag{6.17}$$

Similarly, we have

$$\begin{aligned}
&\|R_3(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} \|R_4(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\|_{L^2(\Omega_\varepsilon)} \\
&\leq C(L, \xi) \varepsilon^{-\xi-\gamma+\frac{1}{16}} + C(L, \xi) \varepsilon^{\frac{21}{16}-\xi-\frac{\gamma}{8}} K + \varepsilon^{\frac{7\gamma}{8}+1} K^2.
\end{aligned} \tag{6.18}$$

This leads to

$$\begin{aligned}
&\|\nabla_\varepsilon \bar{u}^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon \bar{v}^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon \bar{h}^\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla_\varepsilon \bar{g}^\varepsilon\|_{L^2(\Omega_\varepsilon)} \\
&\leq C(u_s, v_s, h_s, g_s, L, \xi) \varepsilon^{-\xi-\gamma+\frac{1}{16}} + C(u_s, v_s, h_s, g_s, L, \xi) \varepsilon^{\frac{21}{16}-\xi-\frac{\gamma}{8}} K \\
&\quad + C(u_s, v_s, h_s, g_s) \varepsilon^{\frac{7\gamma}{8}+1} K^2.
\end{aligned} \tag{6.19}$$

We now give L^∞ bounds of $(\bar{u}^\varepsilon, \bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{g}^\varepsilon)$. From the equations (6.2) and (6.12), we get

$$\begin{cases} F_1 := R_1 - (u_s \bar{u}_x^\varepsilon + u_{sx} \bar{u}^\varepsilon + v_s \bar{u}_y^\varepsilon + \bar{v}^\varepsilon u_{sy}) + (h_s \bar{h}_x^\varepsilon + h_{sx} \bar{h}^\varepsilon + g_s \bar{h}_y^\varepsilon + \bar{g}^\varepsilon h_{sy}), \\ F_2 := R_2 - (u_s \bar{v}_x^\varepsilon + v_{sx} \bar{u}^\varepsilon + v_s \bar{v}_y^\varepsilon + \bar{v}^\varepsilon v_{sy}) + (h_s \bar{g}_x^\varepsilon + g_{sx} \bar{h}^\varepsilon + g_s \bar{g}_y^\varepsilon + \bar{g}^\varepsilon g_{sy}), \\ F_3 := R_3 - (u_s \bar{h}_x^\varepsilon + h_{sx} \bar{u}^\varepsilon + v_s \bar{h}_y^\varepsilon + \bar{v}^\varepsilon h_{sy}) + (h_s \bar{u}_x^\varepsilon + u_{sx} \bar{h}^\varepsilon + g_s \bar{u}_y^\varepsilon + \bar{g}^\varepsilon u_{sy}), \\ F_4 := R_4 - (u_s \bar{g}_x^\varepsilon + g_{sx} \bar{u}^\varepsilon + v_s \bar{g}_y^\varepsilon + \bar{v}^\varepsilon g_{sy}) + (h_s \bar{v}_x^\varepsilon + v_{sx} \bar{h}^\varepsilon + g_s \bar{v}_y^\varepsilon + \bar{g}^\varepsilon v_{sy}). \end{cases} \quad (6.20)$$

According to the Proposition 6.4, we obtain

$$\begin{aligned} & \varepsilon^{\frac{7}{8}} \|\bar{u}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{7}{8}} \|\bar{h}^\varepsilon(\Omega_\varepsilon) + \varepsilon^{\frac{1}{2} + \frac{7}{8}} (\|\bar{v}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \|\bar{g}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)}) \\ & \leq C_{\gamma, L} \varepsilon^{\frac{7}{16}} (\|\nabla_\varepsilon \bar{u}^\varepsilon\|_{L^2} + \|\nabla_\varepsilon \bar{v}^\varepsilon\|_{L^2} + \|\nabla_\varepsilon \bar{h}^\varepsilon\|_{L^2} + \|\nabla_\varepsilon \bar{g}^\varepsilon\|_{L^2}) \\ & \quad + C_{\gamma, L} \varepsilon^{\frac{7}{16}} (\|F_1\|_{L^2} + \|F_3\|_{L^2}) + C_{\gamma, L} \varepsilon^{\frac{1}{2} + \frac{7}{16}} (\|F_2\|_{L^2} + \|F_4\|_{L^2}). \end{aligned} \quad (6.21)$$

Using (6.19), we obtain the estimate on $(\nabla_\varepsilon \bar{u}^\varepsilon, \nabla_\varepsilon \bar{v}^\varepsilon, \nabla_\varepsilon \bar{h}^\varepsilon, \nabla_\varepsilon \bar{g}^\varepsilon)$. Next, we provide estimates of $F_i (i = 1, 2, 3, 4)$. We have already obtained the estimates of $R_i (i = 1, 2, 3, 4)$ and here we only estimate the remaining terms of (6.21). For the remaining terms of F_1 , we have

$$\begin{aligned} \|u_{sx} \bar{u}^\varepsilon + u_{sy} \bar{v}^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq (\|\bar{u}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} u_{sx}\|_{L^2} + \|\bar{v}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} u_{sy}\|_{L^2}) \\ & \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{\frac{7}{16} - \xi} \|(\nabla_\varepsilon \bar{u}^\varepsilon, \nabla_\varepsilon \bar{v}^\varepsilon)\|_{L^2}, \\ \|u_s \bar{u}_x^\varepsilon + v_s \bar{u}_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq \|(u_s, v_s)\|_{L^\infty} \|(\bar{u}_y^\varepsilon, \bar{v}_y^\varepsilon)\|_{L^2} \\ & \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{\frac{7}{16} - \xi} \|(\nabla_\varepsilon \bar{u}^\varepsilon, \nabla_\varepsilon \bar{v}^\varepsilon)\|_{L^2}, \\ \|h_{sx} \bar{h}^\varepsilon + h_{sy} \bar{g}^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq (\|\bar{h}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} h_{sx}\|_{L^2} + \|\bar{g}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} h_{sy}\|_{L^2}) \\ & \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{\frac{7}{16} - \xi} \|(\nabla_\varepsilon \bar{h}^\varepsilon, \nabla_\varepsilon \bar{g}^\varepsilon)\|_{L^2}, \\ \|h_s \bar{h}_x^\varepsilon + g_s \bar{h}_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq \|(h_s, g_s)\|_{L^\infty} \|(\bar{h}_y^\varepsilon, \bar{g}_y^\varepsilon)\|_{L^2} \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{\frac{7}{16} - \xi} \|(\nabla_\varepsilon \bar{h}^\varepsilon, \nabla_\varepsilon \bar{g}^\varepsilon)\|_{L^2}. \end{aligned} \quad (6.22)$$

Similarly, we obtain the estimates of the remaining terms in F_2

$$\begin{aligned} \|v_{sx} \bar{u}^\varepsilon + v_{sy} \bar{v}^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq (\|\bar{u}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} v_{sx}\|_{L^2} + \|\bar{v}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} v_{sy}\|_{L^2}) \\ & \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{-\frac{1}{2} - \xi} \|(\nabla_\varepsilon \bar{u}^\varepsilon, \nabla_\varepsilon \bar{v}^\varepsilon)\|_{L^2}, \\ \|u_s \bar{v}_x^\varepsilon + v_s \bar{v}_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq \|(u_s, v_s)\|_{L^\infty} \|(\bar{v}_x^\varepsilon, \bar{v}_y^\varepsilon)\|_{L^2} \\ & \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{-\frac{3}{64} - \xi} \|(\nabla_\varepsilon \bar{u}^\varepsilon, \nabla_\varepsilon \bar{v}^\varepsilon)\|_{L^2}, \\ \|g_{sx} \bar{h}^\varepsilon + g_{sy} \bar{g}^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq (\|\bar{h}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} g_{sx}\|_{L^2} + \|\bar{g}_y^\varepsilon\|_{L^2} \sup_x \|\sqrt{y} h_{sx}\|_{L^2}) \\ & \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{-\frac{1}{2} - \xi} \|(\nabla_\varepsilon \bar{h}^\varepsilon, \nabla_\varepsilon \bar{g}^\varepsilon)\|_{L^2}, \\ \|h_s \bar{g}_x^\varepsilon + g_s \bar{g}_y^\varepsilon\|_{L^2(\Omega_\varepsilon)} & \leq \|(h_s, g_s)\|_{L^\infty} \|(\bar{g}_x^\varepsilon, \bar{g}_y^\varepsilon)\|_{L^2} \leq C(L, u_s, v_s, h_s, g_s) \varepsilon^{-\frac{3}{64} - \xi} \|(\nabla_\varepsilon \bar{h}^\varepsilon, \nabla_\varepsilon \bar{g}^\varepsilon)\|_{L^2}, \end{aligned} \quad (6.23)$$

where we have used

$$\begin{aligned} \sup_x \|\sqrt{y} u_{sx}\|_{L^2(I_\varepsilon)} & \leq \sup_x \|\langle y \rangle u_{px}^0\|_{L^2} + \sup_x \|v_{eY}^1\|_{L^2} + \varepsilon^{\frac{1}{2}} \sup_x \|\langle y \rangle u_{px}^1\|_{L^2} \\ & \quad + \varepsilon^{\frac{1}{2}} \sup_x \|v_{eY}^2\|_{L^2} + \varepsilon \sup_x \|\langle y \rangle u_{px}^2\|_{L^2} + \varepsilon \sup_x \|v_{eY}^3\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_x \|\langle y \rangle u_{px}^0\|_{L^2} + \|v_{exY}^1\|_{L^2} + \|\partial_Y V_{b0}^1\|_{L^2} \\
&\quad + \varepsilon^{\frac{1}{2}} \sup_x \|\langle y \rangle u_{px}^1\|_{L^2} + \varepsilon^{\frac{1}{2}} \|v_{exY}^2\|_{L^2} + \varepsilon^{\frac{1}{2}} \|\partial_Y V_{b0}^2\|_{L^2} \\
&\quad + \varepsilon \sup_x \|\langle y \rangle u_{px}^2\|_{L^2} + \varepsilon \|v_{exY}^3\|_{L^2} + \varepsilon \|\partial_Y V_{b0}^3\|_{L^2}, \\
\sup_x \|\sqrt{y} u_{sy}\|_{L^2(I_\varepsilon)} &\leq \|u_{eY}^0\|_{L^2} + \sup_x \|\langle y \rangle u_{py}^0\|_{L^2} + \sup_x \|u_{eY}^1\|_{L^2} + \varepsilon^{\frac{1}{2}} \sup_x \|\langle y \rangle u_{py}^1\|_{L^2} \\
&\quad + \varepsilon^{\frac{1}{2}} \sup_x \|u_{eY}^2\|_{L^2} + \varepsilon \sup_x \|\langle y \rangle u_{py}^2\|_{L^2} + \varepsilon \sup_x \|u_{eY}^3\|_{L^2} \\
&\leq \|u_{eY}^0\|_{L^2} + \sup_x \|\langle y \rangle u_{py}^0\|_{L^2} + \|u_{bY}^1\|_{L^2} + \|v_{eYY}^1\|_{L^2} \\
&\quad + \varepsilon^{\frac{1}{2}} \sup_x \|\langle y \rangle u_{py}^1\|_{L^2} + \varepsilon^{\frac{1}{2}} \|u_{bY}^2\|_{L^2} + \varepsilon^{\frac{1}{2}} \|v_{eYY}^2\|_{L^2} \\
&\quad + \varepsilon \|\langle y \rangle u_{py}^2\|_{L^2} + \varepsilon \|u_{bY}^3\|_{L^2} + \varepsilon \|v_{eYY}^3\|_{L^2}, \\
\|(u_s, v_s)\|_{L^\infty(\Omega_\varepsilon)} &\leq \|u_e^0\|_{L^\infty} + \|(u_p^0, v_p^0)\|_{L^\infty} + \varepsilon^{\frac{1}{2}} \|(u_p^1, v_p^1)\|_{L^\infty} + \|(\varepsilon^{\frac{1}{2}} u_e^1, v_e^1)\|_{L^\infty} \\
&\quad + \|(\varepsilon u_p^2, \varepsilon v_p^2)\|_{L^\infty} + \|(\varepsilon u_e^2, \varepsilon^{\frac{1}{2}} v_e^2)\|_{L^\infty} + \|(\varepsilon^{\frac{3}{2}} u_e^3, \varepsilon v_e^3)\|_{L^\infty},
\end{aligned}$$

and

$$\begin{aligned}
\sup_x \|\sqrt{y} v_{sx}\|_{L^2(I_\varepsilon)} &\leq \sup_x \|\langle y \rangle v_{px}^0\|_{L^2} + \varepsilon^{-\frac{1}{2}} \sup_x \|v_{ex}^1(x, \cdot)\|_{L^2} + \varepsilon^{\frac{1}{2}} \sup_x \|\langle y \rangle v_{px}^1\|_{L^2} \\
&\quad + \sup_x \|v_{ex}^2(x, \cdot)\|_{L^2} + \varepsilon \sup_x \|\langle y \rangle v_{px}^2\|_{L^2} + \varepsilon^{\frac{1}{2}} \sup_x \|v_{ex}^3(x, \cdot)\|_{L^2} \\
&\leq \sup_x \|\langle y \rangle v_{px}^0\|_{L^2} + \varepsilon^{-\frac{1}{2}} \|v_{ex}^1(x, \cdot)\|_{L^2}^{\frac{1}{2}} \|v_{exx}^1(x, \cdot)\|_{L^2}^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \sup_x \|\langle y \rangle v_{px}^1\|_{L^2} \\
&\quad + \|v_{ex}^2(x, \cdot)\|_{L^2}^{\frac{1}{2}} \|v_{exx}^2(x, \cdot)\|_{L^2}^{\frac{1}{2}} + \varepsilon \sup_x \|\langle y \rangle v_{px}^2\|_{L^2} + \varepsilon^{\frac{1}{2}} \|v_{ex}^3\|_{L^2}^{\frac{1}{2}} \|v_{exx}^3(x, \cdot)\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

We use similar methods for the terms $\sup_x \|\langle y \rangle h_{sx}\|_{L^2(I_\varepsilon)}$, $\sup_x \|\langle y \rangle h_{sy}\|_{L^2(I_\varepsilon)}$, $\sup_x \|\langle y \rangle g_{sx}\|_{L^2(I_\varepsilon)}$ and $\|(h_s, g_s)\|_{L^\infty(\Omega_\varepsilon)}$ of the inequalities (6.22) and (6.23). Similarly, we follow the previous method to estimate F_3, F_4 . Combining all the above estimates into (6.21), we obtain

$$\begin{aligned}
&\varepsilon^{\frac{7}{8}} \|\bar{u}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{1}{2} + \frac{7}{8}} \|\bar{v}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{7}{8}} \|\bar{h}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} + \varepsilon^{\frac{1}{2} + \frac{7}{8}} \|\bar{g}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \\
&\leq C(u_s, v_s, h_s, g_s, L, \xi) \varepsilon^{\frac{1}{16} - \xi - \frac{15\gamma}{16}} + C(u_s, v_s, h_s, g_s, L, \xi) \varepsilon^{\frac{21}{16} - \xi - \frac{\gamma}{16}} K \\
&\quad + C(u_s, v_s, h_s, g_s) \varepsilon^{\frac{15\gamma}{16} + 1} K^2 + C(L, u_s, v_s, h_s, g_s) \varepsilon^{\frac{\gamma}{16} - \xi} \|(\bar{u}^\varepsilon, \bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{g}^\varepsilon)\|_{\mathbb{X}}, \tag{6.24}
\end{aligned}$$

where $\xi < \frac{\gamma}{16}$. Combining (6.19) with (6.24), taking $\frac{1}{16} - \xi - \gamma \geq 0$ and $\varepsilon \ll 1$, we have

$$\|(\bar{u}^\varepsilon, \bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{g}^\varepsilon)\|_{\mathbb{X}} \leq C(u_s, v_s, h_s, g_s, L, \xi) + C(u_s, v_s, h_s, g_s, L, \xi) \varepsilon^{\frac{5}{8}} K + C(u_s, v_s, h_s, g_s) \varepsilon^{\frac{7\gamma}{8} + 1} K^2.$$

Let $K := C(u_s, v_s, h_s, g_s, L, \xi) + 1$. If we want to obtain $\|(\bar{u}^\varepsilon, \bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{g}^\varepsilon)\|_{\mathbb{X}} \leq K$, for any small ε , we must take

$$C(u_s, v_s, h_s, g_s, L, \xi) \varepsilon^{\frac{5}{8}} K + C(u_s, v_s, h_s, g_s) \varepsilon^{\frac{7\gamma}{8} + 1} K^2 \leq 1.$$

This leads to $(\bar{u}^\varepsilon, \bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{g}^\varepsilon) \in \mathbb{X}_K$. Therefore, there exists an operator $\mathbb{T} : (u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon) \mapsto (\bar{u}^\varepsilon, \bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{g}^\varepsilon)$ that maps \mathbb{X}_K to itself. We now prove that the operator \mathbb{T} is a contraction mapping.

Indeed, for any two pairs $(u_1^\varepsilon, v_1^\varepsilon, h_1^\varepsilon, g_1^\varepsilon)$ and $(u_2^\varepsilon, v_2^\varepsilon, h_2^\varepsilon, g_2^\varepsilon)$ in \mathbb{X}_K , it follows from a similar approach that

$$\begin{aligned} & \|(\bar{u}_1^\varepsilon - \bar{u}_2^\varepsilon, \bar{v}_1^\varepsilon - \bar{v}_2^\varepsilon, \bar{h}_1^\varepsilon - \bar{h}_2^\varepsilon, \bar{g}_1^\varepsilon - \bar{g}_2^\varepsilon)\|_{\mathbb{X}} \\ & \leq C(u_s, v_s, h_s, g_s, L, \xi)(\varepsilon^{\frac{21}{16}-\xi-\frac{\gamma}{8}} + \varepsilon^{\frac{7\gamma}{8}+1}K)\|(u_1^\varepsilon - u_2^\varepsilon, v_1^\varepsilon - v_2^\varepsilon, h_1^\varepsilon - h_2^\varepsilon, g_1^\varepsilon - g_2^\varepsilon)\|_{\mathbb{X}}. \end{aligned}$$

This indicates that for any small ε , the operator \mathbb{T} is a contraction mapping. We then prove that the system (6.2) has unique solution through the standard contraction mapping theorem, which ends the proof. \square

Acknowledgments

This work is supported by NSF of China under the Grant 12271032.

References

- [1] D.X. Chen, S.Q. Ren, Y.X. Wang, Z.F. Zhang, Global well-posedness of the 2D magnetic Prandtl model in the Prandtl-Hartmann regime. *Asymptotic Analysis*, 1-21, 2020.
- [2] S.J. Ding, Z.L. Lin, D.J. Niu, Stability of the boundary layer expansion for the 3D plane parallel MHD flow. *J. Math. Phys.*, 62(2), 021510, 2021.
- [3] S.J. Ding, Z.L. Lin, F. Xie, Verification of Prandtl boundary layer ansatz for the steady electrically conducting fluids with a moving physical boundary. *SIAM J. Math. Anal.*, 53(5), 4997-5059, 2021.
- [4] S.J. Ding, Z.J. Ji, Z.L. Lin, Validity of Prandtl layer theory for steady magnetohydrodynamics over a moving plate with nonshear outer ideal MHD flows. *J. Differ. Equations*, 278, 220-293, 2021.
- [5] S.J. Ding, Z.J. Ji, Z.L. Lin, Global-in- x stability of Prandtl layer expansions for steady magnetodynamics flows over a moving plate. *arXiv:2203.00380*, 2022.
- [6] S.J. Ding, C.Y. Wang, Validity of Prandtl layer expansions for steady magnetohydrodynamics over a rotating disk. *J. Math. Phys.*, 64(2), 021501, 2023.
- [7] D. Gérard-Varet, M. Prestipino, Formal derivation and stability analysis of boundary layer models in MHD. *Z. Angew. Math. Phys.*, 68(76), 2017.
- [8] J.C. Gao, M.L. Li, Z.A. Yao, Higher regularity and asymptotic behavior of 2D magnetic Prandtl model in the Prandtl-Hartmann regime. *J. Differ. Equations*, 386, 294-367, 2024.
- [9] J.C. Gao, B.L. Guo, D.W. Huang, Local-in-time well-posedness of boundary layer system for the full incompressible MHD equations by energy methods. *arXiv:1704.06766*, 2017.
- [10] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*. Second edition. *Grundlehren der Mathematischen Wissenschaften 224*, Springer-Verlag, Berlin, 1983.
- [11] L.H. Guo, Z.J. Ji, Validity of boundary layer theory for the 3D plane-parallel nonhomogeneous electrically conducting flows. *Math. Methods Appl. Sci.*, 44(11), 8862-8882, 2021.
- [12] Y. Guo, S. Iyer, Validity of steady Prandtl layer expansions. *Comm. Pure Appl. Math.*, 76(11), 3150-3232, 2023.
- [13] Y. Guo, S. Iyer, Steady Prandtl layer expansions with external forcing. *Quart. Appl. Math.*, 81(2), 375-411, 2023.
- [14] Y. Guo, S. Iyer, Regularity and expansion for steady Prandtl equations. *Comm. Math. Phys.*, 382(3), 1403-1447, 2021.

- [15] Y. Guo, T. Nguyen, Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. *Ann. PDE*, 3(10), 1-58, 2017.
- [16] Y.T. Huang, C.J. Liu, T. Yang, Local-in-time well-posedness for compressible MHD boundary layer. *J. Differ. Equations*, 266(6), 2978-3013, 2019.
- [17] S. Iyer, Steady Prandtl boundary layer expansions over a rotating disk. *Arch. Ration. Mech. Anal.*, 224(2), 421-469, 2017.
- [18] S. Iyer, Global steady Prandtl expansion over a moving boundary I. *Peking Math. J.*, 2(2), 155-238, 2019.
- [19] S. Iyer, Global steady Prandtl expansion over a moving boundary II. *Peking Math. J.*, 2, 353-437, 2019.
- [20] S. Iyer, Global steady Prandtl expansion over a moving boundary III. *Peking Math. J.*, 3(1), 47-102, 2020.
- [21] S. Iyer, Steady Prandtl layers over a moving boundary: non-shear Euler flows. *SIAM J. Math. Anal.*, 51(3), 1657-1695, 2019.
- [22] Q.R. Li, S.J. Ding, Symmetrical Prandtl boundary layer expansions of steady Navier-Stokes equations on bounded domain. *J. Differ. Equations*, 268(4), 1771-1819, 2020.
- [23] F.H. Lin, X. Li, P. Zhang, Global small solutions of 2D incompressible MHD system. *J. Differ. Equations*, 259, 5440-5485, 2015.
- [24] X.Y. Lin, T. Zhang, Almost global existence for 2D magnetohydrodynamics boundary layer system. *Math. Methods Appl. Sci.*, 41(17), 7530-7553, 2018.
- [25] C.J. Liu, F. Xie, T. Yang, MHD boundary layers theory in Sobolev spaces without monotonicity, I. Well-posedness theory. *Comm. Pure Appl. Math.*, 72(1), 63-121, 2019.
- [26] C.J. Liu, F. Xie, T. Yang, MHD boundary layers in Sobolev spaces without monotonicity, II. Convergence theory. *arXiv:1704.00523*, 2017.
- [27] C.J. Liu, T. Yang, Z. Zhang, Validity of Prandtl expansions for steady MHD in the Sobolev framework. *SIAM J. Math. Anal.*, 55(3), 2377-2410, 2023.
- [28] Y. Maekawa, On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.*, 67(7), 1045-1128, 2014.
- [29] A. Mazzucato, M. Taylor, Vanishing viscosity plane parallel channel flow and related singular perturbation problems. *Anal. PDE*, 1(1), 35-93, 2008.
- [30] L. Prandtl, Über viskositäts-bewegung bei sehr kleiner reibung. In *Verhandlungen des III Internationalen Mathematiker-Kongresses, Heidelberg*. Teubner, Leipzig, 484-491, 1904. English Translation: "Motion of fluids with very little viscosity," Technical Memorandum No. 452 by National Advisory Committee for Aeronautics.
- [31] M. Sammartino, R.E. Caflisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half space, II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.*, 192(2), 463-491, 1998.
- [32] C. Wang, Y. Wang, Z. Zhang, Zero-viscosity limit of the Navier-Stokes equations in the analytic setting. *Arch. Ration. Mech. Anal.*, 224(2), 555-595, 2017.
- [33] J. Wang, S.X. Ma, On the steady Prandtl type equations with magnetic effects arising from 2D incompressible MHD equations in a half plane. *J. Math. Phys.*, 59(12), 121508, 2018.
- [34] S. Wang, Z.P. Xin, Boundary layer problems in the viscosity-diffusion vanishing limits for the incompressible MHD systems. *Sci. Sin. Math.*, 47(10), 1303-1326, 2017.

- [35] N. Wang, S. Wang, The boundary layer for MHD equations in a plane-parallel channel. *Acta Math. Sci.*, 39A(4), 738-760, 2019.
- [36] Z.L. Wu and S. Wang, Viscosity vanishing limit of the nonlinear pipe magnetohydrodynamic flow with diffusion. *Math. Methods Appl. Sci.*, 42(1), 161-174, 2019.
- [37] X.Q. Xie, L. Luo, C.M. Li, Boundary layer for MHD equations with the noncharacteristic boundary conditions. *Chin. Ann. Math.*, 35A(2), 171-192, 2014.
- [38] F. Xie, T. Yang, Global-in-time stability of 2D MHD boundary layer in the Prandtl-Hartmann regime. *SIAM J. Math. Anal.*, 50(6), 5749-5760, 2018.