Homogenization of non-local energies
on disconnected sets

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Abstract

We consider the problem of the homogenization of non-local quadratic energies defined on δ-periodic disconnected sets defined by a double integral, depending on a kernel concentrated at scale ε. For kernels with unbounded support we show that we may have three regimes: (i) $\varepsilon \ll \delta$, for which the Γ-limit even in the strong topology of L^2 is 0; (ii) $\frac{\varepsilon}{\delta} \to \kappa$, in which the energies are coercive with respect to a convergence of interpolated functions, and the limit is governed by a non-local homogenization formula parameterized by κ; (iii) $\delta \ll \varepsilon$, for which the Γ-limit is computed with respect to a coarse-grained convergence and exhibits a separation-of-scales effect; namely, it is the same as the one obtained by formally first letting $\delta \to 0$ (which turns out to be a pointwise weak limit, thanks to an iterated use of Jensen's inequality), and then, noting that the outcome is a nonlocal energy studied by Bourgain, Brezis and Mironescu, letting $\varepsilon \to 0$. A slightly more complex description is necessary for case (ii) if the kernel is compactly supported.

Keywords: homogenization, non-local functionals, perforated domains, separation of scales, Gamma-convergence.

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1   Introduction

In this paper we study the asymptotic behaviour of a periodic system of disconnected regions that interact through long-range potentials and can therefore be considered 'energetically' connected in the sense of V.V. Zhikov's *p-connectedness* [17]. The study of such a type of geometrical objects falls within the analysis of those commonly referred to as 'perforated domains', where the domain of integration is a portion of a scaled periodic set E contained in a given region; that is, a set of the form $\Omega \cap \delta E$, with $\delta > 0$ a small parameter. The 'classical' case is obtained by choosing as energy e.g. the Dirichlet integral, for which the functionals take the form

$$F_\delta(u) = \int_{\Omega \cap \delta E} |\nabla u|^2 \, dx. \tag{1}$$

If E is a connected open set, then the energies possess a limit, e.g. in the sense of Γ-convergence [3, 10, 6], as $\delta \to 0$ [7] and it is a coercive homogeneous quadratic form on $H^1(\Omega)$. This result

can be obtained by using Extension Lemmas, which allow to construct uniformly continuous operators from $H^1(\Omega \cap \delta E)$ to $H^1_{\text{loc}}(\Omega)$ [1], and then regard all the functionals as defined in that common space. For less regular sets it is convenient to substitute the topological notion of connectedness with a more analytic one: loosely speaking, if $p > 1$, a set A is *p-connected* if every function u such that $\int_A |\nabla u|^p dx = 0$ is a constant [17]. More in general this notion can be given for integrals with respect to a measure, of which the restriction of the Lebesgue measure to A is a particular case.

Following a seminal result of Bourgain et al. [5] (see also [15], and [4, 14] for applications) the Dirichlet integral on an open set $U \subset \mathbb{R}^d$ can itself be approximated by introducing another parameter $\varepsilon > 0$ and considering energies

$$F^\varepsilon(u) = \frac{1}{\varepsilon^d} \int_{U \times U} \varphi\left(\frac{x-y}{\varepsilon}\right) \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \, dy, \tag{2}$$

with φ a non-negative integrable kernel. Such energies have also been systematically studied within a variational theory for convolution-type functionals [2] in the equivalent form

$$F^\varepsilon(u) = \frac{1}{\varepsilon^{d+2}} \int_{U \times U} \varphi\left(\frac{x-y}{\varepsilon}\right) |u(x) - u(y)|^2 \, dx \, dy, \tag{3}$$

up to using the kernel $\varphi(\xi)|\xi|^2$ in the place of $\varphi(\xi)$. We will use this latter form. Note that if the support of φ is unbounded then any set U is in a sense '2-connected' for such functionals, in the sense that if $F^\varepsilon(u) = 0$ then $u(x) - u(y) = 0$ for each $x, y \in U$ so that u is constant on U.

This observation suggests that, contrary to the local case, for non-local energies we might study the behaviour of perforated domains as above even when E is not topologically connected. In this paper, we consider non-local functionals of the form

$$F_{\varepsilon,\delta}(u) = \frac{1}{\varepsilon^{d+2}} \int_{\Omega \cap \delta E} \int_{\Omega \cap \delta E} \varphi\left(\frac{x-y}{\varepsilon}\right) |u(x) - u(y)|^2 \, dx \, dy, \tag{4}$$

under the prototypical assumption that

$$E = K + \mathbb{Z}^d$$

is composed of a periodic array of disconnected sets; that is, K is a (topologically) connected compact set and $(K + k) \cap (K + k') = \emptyset$ if $k, k' \in \mathbb{Z}^d$ and $k \neq k'$. The behaviour of $F_{\varepsilon,\delta}$ will be driven by the mutual behaviour of the two parameters. In the notation, we will tacitly suppose that $\delta = \delta_\varepsilon$ is an infinitesimal family as $\varepsilon \to 0$. Functionals of the form (4) have been dealt with in [9] (see also [2]) when E is a topologically connected periodic Lipschitz set, using an extension theorem which cannot be applied in our cases.

The first issue that we examine for such energies is their coerciveness. Since the domain of $F_{\varepsilon,\delta}$ is composed of functions defined on a disconnected set, we have to specify the type of convergence with respect to which they are studied. In the case of φ with support the whole $\mathbb{R}^d$ we obtain the three cases:

(i) $\varepsilon \ll \delta$. In this case we have a loss of coerciveness, and in particular any function in $L^2(\Omega)$ is a limit in $L^2(\Omega)$ of a sequence of functions with vanishing energy;

(ii) $\varepsilon \sim \delta$. In this case we consider the convergence $u_\varepsilon \to u$ defined as the $L^2_{\mathrm{loc}}(\Omega)$ strong convergence of the piecewise-constant functions u^ε defined by

$$u^\varepsilon(x) = \frac{1}{|K|\delta^d} \int_{\delta K + \delta k} u_\varepsilon(y)\, dy \text{ for } x \in \delta(k + [0,1)^d). \tag{5}$$

The functionals are equicoercive with respect to this convergence, and the limit u belongs to $H^1(\Omega)$;

(iii) $\delta \ll \varepsilon$. In this case the functions u_ε must be 'coarse grained', considering the strong $L^2_{\mathrm{loc}}(\Omega)$-convergence of the piecewise-constant interpolations u^ε defined by

$$u^\varepsilon(x) = \frac{1}{|[0,\varepsilon]^d \cap \delta E|} \int_{\varepsilon(k+[0,1)^d) \cap \delta E} u_\varepsilon(x) \text{ for } x \in \varepsilon(k + [0,1)^d). \tag{6}$$

Also in this case, the limit u is in $H^1(\Omega)$.

If the support of φ is bounded then we may have loss of coerciveness also if $\varepsilon \sim \delta$. More precisely, if the support of φ is the closed ball of radius s, then the functionals are equicoercive with respect to the convergence above if and only if $s\varepsilon > \delta D$, where $D = \inf\{r : E + B_{r/2} \text{ is topologically connected}\}$. In this case again the limit u belongs to $H^1(\Omega)$.

We also compute the Γ-limit with respect to the convergences above. If the support of φ is the whole $\mathbb{R}^d$ we have:

(i) (*degenerate limit*) if $\varepsilon \ll \delta$ then the Γ-limit with respect to the strong $L^2(\Omega)$-convergence is identically 0;

(ii) (*homogenization*) if $\frac{\varepsilon}{\delta} \to \kappa$, then the Γ-limit is the quadratic form

$$F^\kappa(u) = \int_\Omega \langle A^\kappa_{\mathrm{hom}} \nabla u, \nabla u \rangle\, dx,$$

where the symmetric matrix A^κ_{hom} satisfies

$$\langle A^\kappa_{\mathrm{hom}} \xi, \xi \rangle = \min\Big\{ \int_{[0,1]^d \cap E} \int_E \varphi(\tfrac{x-y}{\kappa})(\langle \xi, x - y \rangle + u(x) - u(y))^2\, dx\, dy : u \text{ 1-periodic} \Big\}.$$

This result can be seen as following from the results in [9] (see also [2]) using the techniques therein combined with the coerciveness result above;

(iii) (*separation of scales*) if $\delta \ll \varepsilon$ and φ is radially symmetric, then the Γ-limit is given by

$$F^\infty(u) = C_\infty \int_\Omega |\nabla u|^2 dx, \text{ where } C_\infty = |K|^2 \frac{1}{d} \int_{\mathbb{R}^d} \varphi(\xi)|\xi|^2 d\xi.$$

Note that $\frac{1}{d} \int_{\mathbb{R}^d} \varphi(\xi)|\xi|^2 d\xi$ is the constant appearing in the Γ-limit by Bourgain et al. [5], so that this limit can be obtained by first letting $\delta \to 0$, noting that, upon writing

$$F_{\varepsilon,\delta}(u) = \frac{1}{\varepsilon^{d+2}} \int_{\Omega \times \Omega} \chi_{E \times E}(\tfrac{x}{\delta}, \tfrac{y}{\delta}) \varphi\big(\tfrac{x-y}{\varepsilon}\big) |u(x) - u(y)|^2\, dx\, dy,$$

the corresponding Γ-limit is simply

$$F^\infty_\varepsilon(u) = \frac{1}{\varepsilon^{d+2}} |K|^2 \int_{\Omega \times \Omega} \varphi\big(\tfrac{x-y}{\varepsilon}\big) |u(x) - u(y)|^2\, dx\, dy,$$

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whose successive Γ-limit as $\varepsilon \to 0$ is F^∞. This result is obtained using the Fonseca Müller blow-up method combined by a convexity argument for the lower bound. These arguments allow first to reduce to test functions which are δ-periodic perturbations of affine functions, and then to obtain the desired inequality by a double Jensen's inequality. The upper bound is obtained by a direct computation when the target function u is C^2. In this case we can take $u_\varepsilon = u$, in which case a discretization argument allows to write $F_{\varepsilon,\delta}(u)$ as an approximation of a Riemann integral. As a technical remark, we note that it is sufficient to treat the case $\varphi = \chi_{B_r}$ since a general φ can be approximated by linear combinations of this type of energies. The lower bound then follows by the superadditivity of the lower limit, while the upper bound is proved by the pointwise convergence on C^2-functions.

Finally, for φ with support the closed ball of radius s the computation in (ii) also holds provided $s\kappa > D$, so that the functionals are equi-coercive.

2 Notation and statement of the results

We consider a radial convolution kernel in $\mathbb{R}^d$; that is, a function $\varphi : \mathbb{R}^d \to [0, +\infty)$ such that a decreasing function $\varphi_0 : [0, +\infty) \to [0, +\infty)$ exists satisfying $\varphi(\xi) = \varphi_0(|\xi|)$. We further assume that

$$\int_{\mathbb{R}^d} \varphi(\xi)(1 + |\xi|^2)dx < +\infty, \tag{7}$$

and for each $\varepsilon > 0$ we define the scaled kernel φ_ε by

$$\varphi_\varepsilon(\xi) = \tfrac{1}{\varepsilon^d}\varphi(\tfrac{\xi}{\varepsilon}).$$

A simple kernel, which will be used as a comparison for general kernels, is

$$\varphi_\varepsilon(\xi) = \begin{cases} \varepsilon^{-d} & \text{if } |\xi| < \varepsilon, \\ 0 & \text{elsewhere,} \end{cases}$$

obtained with $\varphi_0 = \chi_{[0,1]}$.

We consider a periodic set $E \subset \mathbb{R}^d$, and we fix a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. For all $\varepsilon > 0$ and $\delta > 0$ we define the functional $F_{\delta,\varepsilon} : L^2(\Omega) \to [0, +\infty)$ by

$$F_{\delta,\varepsilon}(u) = \frac{1}{\varepsilon^2} \int_{(\Omega \cap \delta E) \times (\Omega \cap \delta E)} \varphi_\varepsilon(x - y)|u(x) - u(y)|^2 dx dy. \tag{8}$$

Note that if $E = \mathbb{R}^d$ then $F_{\delta,\varepsilon}$ is independent of δ and the Γ-limit with respect to the weak and strong convergence in $L^2(\Omega)$ of $F_\varepsilon(u) = F_{\delta,\varepsilon}(u)$ has been shown to be equal to

$$C_\varphi \int_\Omega |\nabla u|^2 dx, \text{ where } C_\varphi := \frac{1}{d} \int_{\mathbb{R}^d} \varphi(\xi)|\xi|^2 dx. \tag{9}$$

We will consider a set E composed of disconnected components; more precisely, a 1-periodic set E in $\mathbb{R}^d$ of the form

$$E = \sum_{k \in \mathbb{Z}^d} (k + K), \tag{10}$$

where K is the closure of a connected open set with boundary of zero measure, and is such that $(k + K) \cap K = \emptyset$ if $k \in \mathbb{Z}^d$ and $k \neq 0$. We also define

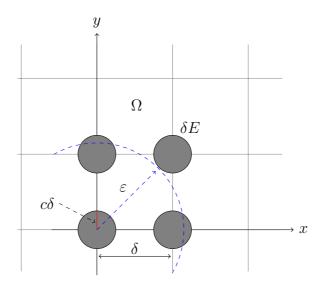$$D = \inf\{r : E + B_{r/2} \text{ is (topologically) connected}\}. \tag{11}$$

Figure 1: a simple example of non-connected domain

Example 2.1. The simplest example of such a geometry is by taking $K = \overline{B_c}$ the closed ball of center 0 and radius $c < \frac{1}{2}$ (see Fig 1), for which $D = 1 - 2c$. Note that for this choice of E $\operatorname{dist}(K, K+k) \geq D$ for all $k \in \mathbb{R}^d$, which is not the case in general. For example, if we take as K the closed ellipse given by the relation $\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} \leq 1$ with $c_2 < c_1$ in $\mathbb{R}^2$, we have $\operatorname{dist}(K, K+e_1) = 1 - 2c_1$ and $\operatorname{dist}(K, K+e_2) = 1 - 2c_2 = D$.

2.1 Definition of convergence and coerciveness

We first consider the cases in which we do not have coerciveness. The first one is in the regime $\varepsilon \ll \delta$, for which the Γ-limit, even if computed with respect to the strong $L^2(\Omega)$ convergence is 0. This is a consequence of the following result.

Theorem 2.2. *Let $\delta = \delta_\varepsilon$ be such that $\delta \to 0$ and*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta_\varepsilon} = 0.$$

then for all $u \in L^2(\Omega)$ there exists a sequence u_ε converging strongly to u in $L^2(\Omega)$ and such that

$$\lim_{\varepsilon \to 0} F_{\delta,\varepsilon}(u_\varepsilon) = 0.$$

Proof. It is sufficient to prove that the claim holds for a strongly dense subclass in $L^2(\Omega)$. We then take a Lipschitz continuous function u, and consider the function u_ε equal to the constant

$$u_\varepsilon(x) = u_k^\delta = \frac{1}{\delta^d |K|} \int_{\delta(k+K)} u(y)dy \tag{12}$$

in $\delta(k+K)$, and to u on $\Omega \setminus \delta E$. We have $u_\varepsilon \to u$ in $L^2(\Omega)$, and we can estimate

$$F_{\delta,\varepsilon}(u_\varepsilon) \leq C \frac{1}{\varepsilon^{d+2}} \int_{(\Omega \times \Omega) \cap \{|x-y| > D_0 \delta\}} \varphi\left(\frac{|x-y|}{\varepsilon}\right) |x-y|^2 \, dx \, dy,$$

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where $D_0 = \min\{\operatorname{dist}(K, k + K) : k \neq 0\}$. The change of variables $y = x + \varepsilon\xi$ gives

$$F_{\delta,\varepsilon}(u_\varepsilon) \leq C' \int_{\{|\xi| > D_0 \frac{\delta}{\varepsilon}\}} \varphi(\xi)|\xi|^2 d\xi,$$

with C' depending only on Ω and the Lipschitz constant of u. This latter integral tends to 0 by (7), since $\frac{\delta}{\varepsilon} \to +\infty$. $\qquad\square$

In the case $\delta \sim \varepsilon$ we havea lack of coerciveness in the case of kernels with compact support. Up to scaling, we can suppose that the support of φ_0 is $[0, 1]$. We then have the following result.

Proposition 2.3. *Let the support of φ_0 be $[0, 1]$, and let D be defined by (11).*
(a) If $\delta = \delta_\varepsilon$ is such that $\varepsilon < \delta D_0$ then for all $u \in L^2(\Omega)$ there exists a sequence with $F_{\delta,\varepsilon}(u_\varepsilon) = 0$ and converging to u strongly in $L^2(\Omega)$;
(b) If $\delta = \delta_\varepsilon$ is such that $\varepsilon < \delta D$ then there exists a sequence u_ε with $F_{\delta,\varepsilon}(u_\varepsilon) = 0$ and such that $u_\varepsilon \chi_{\delta E}$ does not converge weakly in $L^2(\Omega)$.

Proof. Case (a) is dealt with exactly as in the proof of Theorem 2.2. In case (b) we have that the set $\delta E + B_{\varepsilon/2}$ is composed of infinitely many disconnected components. We may suppose for simplicity that each connected component is not bounded since otherwise we are in case (a), and let $\overline{k} \in \mathbb{Z}^d$ be such that $\delta(\overline{k} + K)$ does not belong to the same connected component as δK. We may set $u_\varepsilon(x) = m$ if x belongs to the same connected component as $\delta(2m\overline{k} + K)$, and for example $u_\varepsilon(x) = 0$ elsewhere. Since $\varepsilon < \delta D$, if $\varphi(\frac{|x-y|}{\varepsilon}) > 0$ and $x, y \in \delta E$ then they belong to the same connected component. This shows that $F_{\delta,\varepsilon}(u_\varepsilon) = 0$. $\qquad\square$

Remark 2.4. Claim (a) in Proposition 2.3 shows that the Γ-limit in the strong L^2 topology is 0. In the second case, if $\delta D_0 < \varepsilon < \delta D$ then the sequence $F_{\delta,\varepsilon}$ retains some form of coerciveness, which gives a non-trivial Γ-limit in the weak L^2 topology. For example, in the case of ellipsoidal sets as in Example 2.1, the domain of the Γ-limit will be functions in $L^2(\Omega)$ whose distributional derivative in the x_1-direction is an L^2 function. Since this issue is not central in our discussion we omit the details of this case.

The following theorems will be proved in Section 3. They involve piecewise-affine interpolations obtained using *Kuhn's decomposition* [12] of cubes of $\mathbb{R}^d$ into $d!$ simplexes which are uniquely determined by a permutation of the indices $\{1, ..., d\}$. Since the actual form of the piecewise-affine interpolations is not relevant in our context we refer e.g. to [16] for their use in nonlocal interaction problems.

Theorem 2.5. *Let Ω be a connected set, let*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta} = \kappa,$$

and suppose either that φ_0 has support $[0, 1]$ and $\kappa > D$, or that the support of φ be the whole $\mathbb{R}^d$. Let u_ε be such that $\sup_\varepsilon F_{\delta,\varepsilon}(u_\varepsilon) < +\infty$. Let the functions $\overline{u}_\varepsilon$ be defined as the piecewise-affine interpolations of the functions $\delta k \mapsto u_k^\delta$ in (5) if $k \in \mathbb{Z}^d$ and $\delta(k + K) \subset \Omega$. Then, up to subsequences and addition of constants $\overline{u}_\varepsilon \to u$ in $L^2_{\text{loc}}(\Omega)$ for some $u \in H^1(\Omega)$.

The second result combines the interpolation argument with 'coarse graining'; that is, we consider averages of functions not on the characteristic period δ of the geometry, but on the larger 'mesoscopic' scale ε.

Theorem 2.6. *Let Ω be a connected set, let*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta} = +\infty.$$

Let u_ε be such that $\sup_\varepsilon F_{\delta,\varepsilon}(u_\varepsilon) < +\infty$. Let the functions $\overline{u}_\varepsilon$ be defined as the piecewise-affine interpolations of the functions $\varepsilon k \mapsto u_k^\varepsilon$ defined on $\varepsilon \mathbb{Z}^d$ by

$$u_k^\varepsilon = \frac{1}{|\delta E \cap \varepsilon(k + [0,1]^d)|} \int_{\delta E \cap \varepsilon(k + [0,1]^d)} u_\varepsilon dx. \tag{13}$$

Then, up to subsequences and addition of constants, $\overline{u}_\varepsilon \to u$ in $L^2_{\mathrm{loc}}(\Omega)$ for some $u \in H^1(\Omega)$.

2.2   Γ-convergence

The compactness results in the previous section allow to consider Γ-convergence with respect to the convergence of the interpolations as defined therein. We can then compute the Γ-limits in the hypotheses of Theorem 2.5 and Theorem 2.6, respectively, the degenerate cases being dealt with in Theorem 2.2 when $\varepsilon \ll \delta$, in which case the Γ-limit is 0 with respect to any topology weaker than the strong $L^2(\Omega)$ one, and in Proposition 2.3 for the case $\varepsilon < \delta D$, when φ_0 has support $[0,1]$.

Theorem 2.7. *Let $\delta = \delta_\varepsilon$ satisfy*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta} = \kappa,$$

and suppose either that φ_0 has support $[0,1]$ and $\kappa > D$, or that the support of φ be the whole $\mathbb{R}^d$. Then the Γ-limit of $F_{\delta,\varepsilon}$ as $\varepsilon \to 0$ with respect to the convergence of the piecewise-affine interpolations as in Theorem 2.5 is the quadratic form

$$F^\kappa(u) = \int_\Omega \langle A^\kappa_{\mathrm{hom}} \nabla u, \nabla u \rangle \, dx, \tag{14}$$

where the symmetric matrix A^κ_{hom} satisfies

$$\langle A^\kappa_{\mathrm{hom}} \xi, \xi \rangle = \min \left\{ \int_{[0,1]^d \cap E} \int_E \varphi(\tfrac{x-y}{\kappa})(\langle \xi, x - y \rangle + u(x) - u(y))^2 \, dx \, dy : u \text{ 1-periodic} \right\}. \tag{15}$$

Proof. Once the compactness in Theorem 2.5 is proved, the proof of the claim follows very closely that of the homogenization theorem for perforated domains in [9], where functionals of the same form as $F_{\delta,\varepsilon}$ are dealt with, with $\varepsilon = \kappa\delta$ and E a periodic (topologically) connected Lipschitz domain. We refer to that paper for details. $\qquad\square$

The second convergence result highlights a separation of scales, in which we may formally first let δ tend to 0 and note that the characteristic functions $\chi_{\delta E \times \delta E}$ weakly* converge to the constant $|K|^2$ in $L^\infty(\Omega \times \Omega)$.

Theorem 2.8. *Let $\delta = \delta_\varepsilon$ satisfy*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta} = +\infty.$$

Then the Γ-limit of $F_{\delta,\varepsilon}$ as $\varepsilon \to 0$ with respect to the convergence of the piecewise-affine interpolations as in Theorem 2.6 is

$$F^\infty(u) = |K|^2 C_\varphi \int_\Omega |\nabla u|^2 \, dx, \tag{16}$$

where C_φ is given by (9)

Remark 2.9. We note that both the convergences described in the theorems above can be restated as a local strong L^2-convergence; namely, that we have

$$\lim_{\varepsilon \to 0} \int_{\Omega' \cap \delta E} |u_\varepsilon - u|^2 dx = 0. \tag{17}$$

This convergence has been extensively used by Zhikov [17].

To check this claim, we note that by the Poincaré inequality for double integrals, for which we do not need the connectedness of the domain (see e.g. the proof of [13, Theorem 6.33]), taking into account the definition in Theorem 2.6. we have

$$\int_{\delta E \cap \varepsilon(k+[0,1]^d)} |u_\varepsilon - u_k^\varepsilon|^2 dx \leq \frac{1}{\varepsilon^d} \int_{(\delta E \cap \varepsilon(k+[0,1]^d))^2} |u_\varepsilon(x) - u_\varepsilon(y)|^2 dx\,dy,$$

so that, summing on k

$$\int_{\Omega' \cap \delta E} |u_\varepsilon - \widehat{u}_\varepsilon|^2 dx \leq \varepsilon^2 C F_{\delta,\varepsilon}(u_\varepsilon),$$

where $\widehat{u}_\varepsilon$ is the piecewise-constant interpolation of $\{u_k^\varepsilon\}$. The claim then follows noting that $\widehat{u}_\varepsilon - \overline{u}_\varepsilon$ tends to 0 locally in $L^2(\Omega)$. For the definition in Theorem 2.5 the argument is the same.

As a consequence of this remark, we can suppose that the sequence u_ε tends to u locally in $L^2(\Omega)$, upon substituting u_ε with the function defined by u_ε on $\Omega \cap \delta E$, and u otherwise.

3 Compactness

In this section we prove Theorems 2.5 and 2.6. We make the choice of the kernel $\varphi = \chi_{B_r}$, and we may suppose that $r = 1$ up to a scaling argument. With this choice functional (8) becomes

$$F_{\delta,\varepsilon}(u) = \frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|\} < \varepsilon}} |u(x) - u(y)|^2 dx dy. \tag{18}$$

Note that this φ is a lower bound for any other positive kernel satisfying the condition $\varphi(\xi) \geq c > 0$ for $\xi \in B_{r_0}$, so that it is sufficient to state the compactness result for families of functions u_ε such that (18) is bounded.

Case 1: $\delta \sim \varepsilon$ (Theorem 2.5). With fixed $D < \kappa' < \kappa$, we can assume $\frac{\varepsilon}{\delta} > \kappa'$. By definition of D, there exist $r > 0$ and a finite number of vectors $k_1, ..., k_N \in \mathbb{Z}^d$ generating $\mathbb{Z}^d$ on $\mathbb{Z}$ such that $\text{dist}(K, K + k_i) < r < \kappa'$ for all $i \in \{1, ..., N\}$.

For each $i \in \{1, ..., N\}$, let $z_i \in \partial K$ and $w_i \in \partial K + k_i$ be such that $|z_i - w_i| = \text{dist}(K, K + k_i)$. Moreover, set

$$A_i := \{x \in K : |x - z_i| < \tfrac{1}{2}(\kappa' - r)\} \quad \text{and} \quad B_i := \{x \in K + k_i : |x - w_i| < \tfrac{1}{2}(\kappa' - r)\}.$$

By the assumptions on K, we have $|A_i|, |B_i| > 0$. Moreover, if $x \in \delta A_i$ and $y \in \delta B_i$

$$\frac{1}{\delta}|x - y| \leq |\tfrac{1}{\delta}x - z_i| + |z_i - w_i| + |w_i - \tfrac{1}{\delta}y| < \kappa' - r + \text{dist}(K, K + k_i) < \frac{\varepsilon}{\delta};$$

that is, $\delta A_i \times \delta B_i \subseteq (\delta K \times \delta(K + k_i)) \cap \{|x - y| < \varepsilon\}$. With this observation in mind, let $u \in L^2(\Omega)$ and recall the notation

$$u_k = \frac{1}{\delta^d |K|} \int_{\delta K + \delta k} u(x) dx.$$

We compute

$$|u_0 - u_{k_i}|^2 = \left| \frac{1}{\delta^d |K|} \int_{\delta K} (u(x) - u(x + \delta k_i)) dx \right|^2$$

$$= \left| \frac{1}{\delta^d |K| |\delta A_i| |\delta B_i|} \int_{\delta K \times \delta A_i \times \delta B_i} (u(x) - u(x + \delta k_i) + u(y) - u(y) + u(z) - u(z)) dx dy dz \right|^2$$

$$\leq 3 \left(\left| \frac{1}{\delta^{2d} |K| |A_i|} \int_{\delta K \times \delta A_i} (u(x) - u(y)) dx dy \right|^2 + \left| \frac{1}{\delta^{2d} |A_i| |B_i|} \int_{\delta A_i \times \delta B_i} (u(y) - u(z)) dy dz \right|^2 \right.$$

$$\left. + \left| \frac{1}{\delta^{2d} |K| |B_i|} \int_{\delta(K+k_i) \times \delta B_i} (u(z) - u(w)) dw dz \right|^2 \right)$$

$$\leq \frac{C}{\delta^{2d}} \left(\int_{\delta K \times \delta A_i} |u(x) - u(y)|^2 dx dy + \int_{\delta A_i \times \delta B_i} |u(y) - u(z)|^2 dy dz \right.$$

$$\left. + \int_{\delta(K+k_i) \times \delta B_i} |u(z) - u(w)|^2 dw dz \right)$$

$$\leq \frac{C}{\delta^{2d}} \left(\int_{\delta K \times \delta K} |u(x) - u(y)|^2 dx dy + \int_{\substack{\delta K \times \delta(K+k_i) \\ \{|x-y| < \varepsilon\}}} |u(y) - u(z)|^2 dy dz \right.$$

$$\left. + \int_{\delta(K+k_i) \times \delta(K+k_i)} |u(z) - u(w)|^2 dw dz \right),$$

where in the first inequality we used the relation $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, and in the second one Jensen's inequality and the fact that $|\delta A_i| = \delta^d |A_i|$ and $|\delta B_i| = \delta^d |B_i|$.
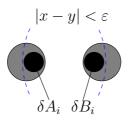

Figure 2: interacting portions of inclusions

Example 3.1. In the prototypical case $K = \overline{B}_c$, we can consider the largest balls fully contained in the region of interaction, as depicted in Fig. 2

In order to take into account the first and the third integral of the sum above, we need to control long-range interactions with short-range interactions as specified in the following lemma.

Lemma 3.2. *There exists a constant C depending on the ratio δ/ε such that*

$$\int_{\delta K \times \delta K} |u(x) - u(y)|^2 dx dy \leq C \int_{\substack{\delta K \times \delta K \\ \{|x-y| < \varepsilon\}}} |u(x) - u(y)|^2 dx dy$$

Proof. The proof follows the one of Lemma 6.1 in [2]. Since K is a connected Lipschitz bounded set, there exists $\varepsilon_1 \in \left(0, \frac{\varepsilon}{3}\right)$ such that for every two points $\eta', \eta'' \in \delta K$ there exists a discrete path from η' to η''; that is, a set of points $\eta' = \eta_0, \eta_1, \ldots, \eta_N, \eta_{N+1} = \eta''$ for which:

- $|\eta_{j+1} - \eta_j| \le \varepsilon_1$ for $j \in \{0, 1, \ldots, N\}$;

- for all $j = 1, ..., N$, the ball $B_{\varepsilon_1}(\eta_j)$ is contained in δK (note that the two extremes are excluded);

- $N \le C \left\lfloor \frac{\delta}{\varepsilon} \right\rfloor$ for all $\eta', \eta'' \in \delta K$. The constant C depends only on $\operatorname{diam}(K)$ and its Lipschitz constant;

Writing

$$u(x_0) - u(x_{N+1}) = u(x_0) - u(x_1) + u(x_1) - \cdots - u(x_N) + u(x_N) - u(x_{N+1}),$$

where $x_0 \in \delta K \cap B_{\varepsilon_1}(\eta')$, $x_{N+1} \in \delta K \cap B_{\varepsilon_1}(\eta'')$, and x_j is a point in $B_{\varepsilon_1}(\eta_j)$ for $j = 1, .., N$. By integrating in all variables we get:

$$\int_{(\delta K \cap B_{\varepsilon_1}(\eta_0)) \times (\delta K \cap B_{\varepsilon_1}(\eta_{N+1}))} (u(x_0) - u(x_{N+1}))^2 dx_0 dx_{N+1}$$

$$= (\varepsilon_1)^{-Nd} \int_{B_{\varepsilon_1}(\eta_1)} \cdots \int_{B_{\varepsilon_1}(\eta_N)} \int_{(\delta K \cap B_{\varepsilon_1}(\eta_0)) \times (\delta K \cap B_{\varepsilon_1}(\eta_{N+1}))} \left(u(x_0) - u(x_1) + u(x_1) - \cdots \right.$$

$$\left. \cdots - u(x_N) + u(x_N) - u(x_{N+1}) \right)^2 dx_0 dx_{N+1} dx_1 ... dx_N$$

$$\le (N+1)(\varepsilon_1)^{-Nd} \int_{\delta K \cap B_{\varepsilon_1}(\eta_0)} \cdots \int_{\delta K \cap B_{\varepsilon_1}(\eta_{N+1})} \sum_{j=1}^{N+1} (u(x_j) - u(x_{j-1}))^2 dx_0 \cdots dx_{N+1}$$

$$\le (N+1) \sum_{j=1}^{N+1} \int_{(\delta K \cap B_{\varepsilon_1}(\eta_j)) \times (\delta K \cap B_{\varepsilon_1}(\eta_{j-1}))} (u(x_j) - u(x_{j-1}))^2 dx_j dx_{j-1}$$

$$\le \left(C \left\lfloor \frac{\delta}{\varepsilon} \right\rfloor + 1 \right)^2 \int_{\substack{\delta K \times \delta K \\ \{|x-y|<\varepsilon\}}} |u(x) - u(y)|^2 dx dy,$$

where in the last line by set inclusion we used that for any $(x_j, x_{j-1}) \in B_{\varepsilon_1}(\eta_j) \times B_{\varepsilon_1}(\eta_{j-1})$ holds

$$|x_j - x_{j-1}| \le |x_j - \eta_j| + |\eta_j - \eta_{j-1}| + |\eta_{j-1} - x_{j-1}| \le 3\varepsilon_1 < \varepsilon$$

and that N is at most $C \left\lfloor \frac{\delta}{\varepsilon} \right\rfloor$. We can obtain the final estimate by covering δK with a finite number of balls of radius ε_1 (this number depends on the ratio δ/ε only) and then summing up the previous inequality applied to all possible pairs of these balls. $\square$

By applying Lemma 3.2, we deduce that there exists a constant C such that, for $i = 1, \ldots, d$,

$$|u_0 - u_{k_i}|^2 \le \frac{C}{\delta^{2d}} \left(\int_{\substack{\delta K \times \delta K \\ \{|x-y|<\varepsilon\}}} |u(x) - u(y)|^2 dx dy + \int_{\substack{\delta K \times \delta(K+k_i) \\ \{|x-y|\}<\varepsilon}} |u(y) - u(z)|^2 dy dz \right.$$

$$\left. + \int_{\substack{\delta(K+k_i) \times \delta(K+k_i) \\ \{|x-y|<\varepsilon\}}} |u(z) - u(w)|^2 dw dz \right).$$

Since we need an estimate of $\delta^d \left(\frac{u_0 - u_{k_i}}{\delta}\right)^2$, we multiply by δ^{d-2} and use that $\frac{C}{\delta^{2d}}\delta^{d-2} = \frac{C}{\varepsilon^{d+2}}$ (by $\delta \sim \varepsilon$) to find

$$\delta^d \left(\frac{u_0 - u_{k_i}}{\delta}\right)^2 \leq \frac{C}{\varepsilon^{d+2}} \int_{\substack{(\delta K \cup \delta(K+k_i))^2 \\ \{|x-y|\}<\varepsilon}} |u(x) - u(y)|^2 dx dy$$

By periodicity it also holds

$$\delta^d \left(\frac{u_k - u_{k+k_i}}{\delta}\right)^2 \leq \frac{C}{\varepsilon^{d+2}} \int_{\substack{(\delta(K+k) \cup \delta(K+(k+k_i)))^2 \\ \{|x-y|\}<\varepsilon}} |u(x) - u(y)|^2 dx dy.$$

Finally, since each vector of the canonical basis can be expressed as a linear combination of the k_i's with integer coefficients, via triangle inequality we bound

$$\delta^d \left(\frac{u_k - u_{k+e_i}}{\delta}\right)^2 \leq \frac{C}{\varepsilon^{d+2}} \int_{\substack{(\delta(K+k) \cup \delta(K+(k+k_i)))^2 \\ \{|x-y|\}<\varepsilon}} |u(x) - u(y)|^2 dx dy,$$

and thus, summing over nearest neighbours $k, k' \in \mathbb{Z}^d$ such that $k\delta + \delta K \subseteq \Omega$, and the same for k', we get

$$\sum_{\substack{k, k' \\ |k-k'|=1}} \delta^d \left(\frac{u_k - u_{k'}}{\delta}\right)^2 \leq \frac{C}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|\}<\varepsilon}} |u(x) - u(y)|^2 dx dy.$$

Therefore, from the equiboundedness of the functional along the sequence u_ε we get a uniform bound on the piecewise-affine interpolation $\overline{u}_\varepsilon$, as defined in Theorem 2.5, and thus the existence of a converging subsequence in the sense of (5).

Case 2: $\varepsilon \gg \delta$ (Theorem 2.6). Set $C = C(d) := \frac{1}{1+\sqrt{d}}$, $Q = [0,1]^d$ and

$$\mathcal{Z}_\varepsilon = \left\{k \in \mathbb{Z}^d : \overline{Q_\varepsilon^k} := C\varepsilon(k + \overline{Q}) \subseteq \Omega\right\}.$$

Note that if $(x, y) \in Q_\varepsilon^k \times Q_\varepsilon^{k'}$ with $|k - k'| \leq 1$; that is $x = C\varepsilon(k + w)$ and $y = C\varepsilon(k' + z)$ with $w, z \in Q$, it automatically holds

$$|x - y| \leq C\varepsilon|k - k'| + C\varepsilon|w - z| \leq C\varepsilon (1 + \text{diam}(Q)) = \varepsilon.$$

Moreover, for each $k \in Z_\varepsilon$, consider

$$Z_k := \left\{j \in \mathbb{Z}^d : \delta(j + K) \subseteq Q_\varepsilon^k\right\} = \left\{j \in \mathbb{Z}^d : j + K \subseteq \frac{\varepsilon}{\delta}(k + Q)\right\}.$$

Note that $\delta E \cap Q_\varepsilon^k \supseteq \bigcup_{j \in Z_k} \delta(j + K) \cap Q_\varepsilon^k = \bigcup_{j \in Z_k} \delta(j + K)$. Moreover, assume $K \subseteq B_L$ and fix $j \in \frac{\varepsilon}{\delta}k + B(0, \frac{\varepsilon}{2\delta} - L)$ and $x = j + w \in j + K$; then for all $i \in \{1, \ldots, d\}$ we have

$$\left|x_i - \tfrac{\varepsilon}{\delta}k_i\right| \leq \left|x - \tfrac{\varepsilon}{\delta}k\right| \leq \left|j - \tfrac{\varepsilon}{\delta}k\right| + L \leq \frac{\varepsilon}{2\delta};$$

that is, $j + K \subseteq \frac{\varepsilon}{\delta}(k + Q)$. We deduce that $\frac{\varepsilon}{\delta}k + \{-\lfloor\frac{\varepsilon}{2\delta} - L\rfloor, ..., 0, ..., \lfloor\frac{\varepsilon}{2\delta} - L\rfloor\}^d \subseteq Z_k$ and consequently $\#Z_k \geq (2(\frac{\varepsilon}{\delta} - L - 1) + 1)^d \geq \left(\frac{\varepsilon}{\delta}\right)^d$, provided $\frac{\varepsilon}{2\delta} \geq L + 1$. In particular

$$|\delta E \cap Q_\varepsilon^k| \geq \sum_{j \in Z_k} |\delta(j + K)| = \#Z_k |K|\delta^d \geq |K|\varepsilon^d.$$

We now write

$$F_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|\} < \varepsilon}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dxdy$$

$$\geq \frac{1}{\varepsilon^{d+2}} \sum_{\substack{|k-k'|=1 \\ k,k' \in \mathcal{Z}_\varepsilon}} \int_{(Q_\varepsilon^k \times Q_\varepsilon^{k'}) \cap (\delta E)^2} (u_\varepsilon(x) - u_\varepsilon(y))^2 dxdy$$

$$= \frac{1}{\varepsilon^{d+2}} \sum_{\substack{|k-k'|=1 \\ k,k' \in \mathcal{Z}_\varepsilon}} |\delta E \cap Q_\varepsilon^k||\delta E \cap Q_\varepsilon^{k'}| \fint_{(Q_\varepsilon^k \times Q_\varepsilon^{k'}) \cap (\delta E)^2} (u_\varepsilon(x) - u_\varepsilon(y))^2 dxdy$$

$$\geq \frac{|K|}{\varepsilon^{d+2}} \sum_{\substack{|k-k'|=1 \\ k,k' \in \mathcal{Z}_\varepsilon}} \varepsilon^{2d} \left| \fint_{(Q_\varepsilon^k \cap \delta E) \times (Q_\varepsilon^{k'} \cap \delta E)} (u_\varepsilon(x) - u_\varepsilon(y))\, dxdy \right|^2$$

$$= \frac{|K|}{\varepsilon^{d+2}} \sum_{\substack{|k-k'|=1 \\ k,k' \in \mathcal{Z}_\varepsilon}} \varepsilon^{2d} \left| u_\varepsilon^k - u_\varepsilon^{k'} \right|^2$$

$$= |K| \sum_{\substack{|k-k'|=1 \\ k,k' \in \mathcal{Z}_\varepsilon}} \varepsilon^d \frac{|u_\varepsilon^k - u_\varepsilon^{k'}|^2}{\varepsilon^2},$$

with the averages u_ε^k defined by

$$u_\varepsilon^k = \frac{1}{|Q_\varepsilon^k \cap \delta E|} \int_{Q_\varepsilon^k \cap \delta E} u_\varepsilon(x) dx.$$

As in the previous case, we get a uniform bound on the piecewise-affine interpolation $\overline{u}_\varepsilon$, as defined in Theorem 2.6, so we can conclude that there exists a subsequence converging in the sense of (6).

4 Asymptotic analysis

This section is devoted to the proof of Theorem 2.8, subdividing it in a lower and an upper bound.

4.1 Liminf inequality

We first prove a lower bound. To that end, let $u_\varepsilon \to u$ in the sense of (6) with $F_\varepsilon(u_\varepsilon) \leq M < +\infty$. Using the conclusion in Remark 2.9 we can suppose that the sequence u_ε tends to u locally in $L^2(\Omega)$, and actually that $u_\varepsilon = u$ outside δE.

In order to estimate the energy we use Fonseca-Müller's blow-up technique [11], as adapted to homogenization problems in [8]. To that end, we restrict $x \in A \subset \Omega$, where A is an open subset, defining

$$F_\varepsilon(u_\varepsilon, A) = \frac{1}{\varepsilon^{d+2}} \int_{\substack{(A \times \Omega) \cap (\delta E)^2 \\ \{|x-y| < \varepsilon\}}} |u_\varepsilon(x) - u_\varepsilon(y)|^2 dxdy = \mu_\varepsilon(A) \tag{19}$$

Since the measures μ_ε are equibounded in mass, we can take a converging subsequence $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, so that by the lower semicontinuity of weak* convergence on open sets it holds

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \mu(\Omega) \geq \int_\Omega \frac{d\mu}{dx} dx.$$

From now on, we denote $Q := (-\frac{1}{2}, \frac{1}{2})^d$ and, for all $r > 0$ and $x_0 \in \mathbb{R}^d$, $Q_\rho(x_0) := x_0 + \rho Q$. We fix a Lebesgue point $x_0 \in \Omega$ for u and ∇u, such that there exists

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho \to 0} \frac{\mu(Q_\rho(x_0))}{\rho^d}.$$

Since $\mu(\partial Q_\rho(x_0)) = 0$ for almost all $\rho > 0$, we can construct $\rho_\varepsilon \to 0$, with $\rho_\varepsilon \gg \varepsilon$, such that

$$\frac{d\mu}{dx}(x_0) = \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(Q_{\rho_\varepsilon}(x_0))}{\rho_\varepsilon^d}. \tag{20}$$

From now on we will tacitly suppose that $\rho = \rho_\varepsilon$. Now we prove a classical lemma that allows to match boundary data.

Lemma 4.1. *Let $\{v_\varepsilon\} \subseteq L^2(Q)$ such that $v_\varepsilon \to v$ for some $v \in H^1(Q)$. For all $\eta > 0$, there exists another sequence $\{v_\varepsilon^\eta\} \subseteq L^2(Q)$, such that*

- $v_\varepsilon^\eta \to v$ *in $L^2(Q)$*

- $v_\varepsilon^\eta = v$ *in the set $\{z \in Q : \text{dist}(z, \partial Q) < \eta\}$*

- $v_\varepsilon^\eta = v_\varepsilon$ *in the set $\{z \in Q : \text{dist}(z, \partial Q) > 2\eta\}$*

- *if we define*

$$F_\varepsilon'(v_\varepsilon) = \frac{1}{\varepsilon^{d+2}} \int_{\substack{(Q \times Q) \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v_\varepsilon(z) - v_\varepsilon(w))^2 \, dz dw,$$

  then it holds

$$\limsup_{\varepsilon \to 0} (F_\varepsilon'(v_\varepsilon^\eta) - F_\varepsilon'(v_\varepsilon)) \leq o(1),$$

  as $\eta \to 0$.

Proof. Fix $N \in \mathbb{N}$ and for $k \in \{1, ..., N\}$ define $Q_k^N := \{z \in Q : \text{dist}(z, \partial Q) > \frac{\eta}{N}(N+k)\}$ and $S_k := Q_{k-1}^N \setminus \overline{Q_k^N}$. Let ϕ_k be a cut-off function such that $\phi_k = 1$ in Q_k^N, $\phi_k = 0$ in $\mathbb{R}^d \setminus \overline{Q_{k-1}^N}$ and $|\nabla \phi_k| \leq \frac{N}{\eta}$. Define

$$v_\varepsilon^k := \phi_k v_\varepsilon + (1 - \phi_k)v.$$

Note that $v_\varepsilon^k(z) - v_\varepsilon^k(w) = \phi_k(z)(v_\varepsilon(z) - v_\varepsilon(w)) + (1 - \phi_k(z))(v(z) - v(w))(\phi_k(z) - \phi_k(w))(v_\varepsilon(w) - v(w))$ for all $z, w \in Q$.

We decompose $Q \times Q$ as follows:

$$A_1 := Q_k^N \times Q_k^N, \qquad \phi_k(z) = 1, \; \phi_k(w) = 1 \; \forall (z, w) \in A_1;$$
$$A_2 := (Q \setminus \overline{Q_{k-1}^N}) \times (Q \setminus \overline{Q_{k-1}^N}), \qquad \phi_k(z) = 0, \; \phi_k(w) = 0 \; \forall (z, w) \in A_2$$
$$A_3 := S_k \times Q, \qquad |\nabla \phi_k(z)| \leq 1$$
$$A_3' := (Q \times S_k) \setminus A_3$$
$$A_4 := Q_k^N \times (Q \setminus \overline{Q_{k-1}^N}), \qquad |z - w| \geq \frac{\eta}{N}, \forall (z, w) \in A_2$$
$$A_4' := (Q \setminus \overline{Q_{k-1}^N}) \times Q_k^N.$$

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The integral in A_1 is estimated by:

$$\frac{1}{\varepsilon^{d+2}} \int_{\substack{A_1 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v_\varepsilon^k(z) - v_\varepsilon^k(w)\right)^2 dzdw$$

$$= \frac{1}{\varepsilon^{d+2}} \int_{\substack{A_1 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v_\varepsilon(z) - v_\varepsilon(w)\right)^2 dzdw \leq \frac{1}{\varepsilon^{d+2}} \int_{\substack{(Q \times Q) \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v_\varepsilon(z) - v_\varepsilon(w)\right)^2 dzdw = F'_\varepsilon(v_\varepsilon)$$

Since Q has a Lipschitz boundary and $v \in H^1(Q)$, we can extend it to the whole $\mathbb{R}^d$. Denote the set

$$S_\eta = \{z \in Q : \operatorname{dist}(z, \partial Q) < 2\eta\}.$$

We have

$$\frac{1}{\varepsilon^{d+2}} \int_{\substack{A_2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v_\varepsilon^k(z) - v_\varepsilon^k(w)\right)^2 dzdw = \frac{1}{\varepsilon^{d+2}} \int_{\substack{A_2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v(z) - v(w)\right)^2 dzdw$$

$$\leq \frac{1}{\varepsilon^{d+2}} \int_{\substack{(S_\eta \times S\eta) \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v(z) - v(w)\right)^2 dzdw \leq \frac{1}{\varepsilon^{d+2}} \int_{\substack{(S_\eta \times S_\eta) \\ \{|z-w|<\varepsilon\}}} \left(v(z) - v(w)\right)^2 dzdw$$

$$= \int_{|\xi|<1} |\xi|^2 \int_{S_{\eta,\xi}} \left(\frac{v(z + \varepsilon\xi) - v(z)}{\varepsilon|\xi|}\right)^2 dzd\xi,$$

where $S_{\eta,\xi} = \{z \in S_\eta : z + \varepsilon\xi \in S_\eta\}$. Note that, for all $|\xi| < 1$ and ε small enough, the set $S_{\eta,\xi}$ is well included in $S_\eta + B_\eta(0)$. Thus we can use a well-known characterization of Sobolev space H^1 to estimate

$$\int_{|\xi|<1} |\xi|^2 \int_{S_{\eta,\xi}} \left(\frac{v(z + \varepsilon\xi) - v(z)}{\varepsilon|\xi|}\right)^2 dzd\xi \leq \int_{|\xi|<1} |\xi|^2 \|\nabla v\|_{L^2(S_\eta + B_\eta(0))} d\xi = \|\nabla v\|_{L^2(S_\eta + B_\eta(0))},$$

which tends to 0 as $\eta \to 0$ independently of ε. As for A_3, we have

$$\frac{1}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v_\varepsilon^k(z) - v_\varepsilon^k(w)\right)^2 dzdw \tag{21}$$

$$= \frac{1}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(\phi_k(z)\left(v_\varepsilon(z) - v_\varepsilon(w)\right) + (1 - \phi_k(z))\left(v(z) - v(w)\right)\right.$$

$$\left. + (\phi_k(z) - \phi_k(w))(v_\varepsilon(w) - v(w))\right)^2 dzdw$$

$$\leq \frac{3}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v_\varepsilon(z) - v_\varepsilon(w)\right)^2 dzdw \tag{22}$$

$$+ \frac{3}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v(z) - v(w)\right)^2 dzdw \tag{23}$$

$$+ \frac{3}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(\phi_k(z) - \phi_k(w)\right)^2 \left(v_\varepsilon(w) - v(w)\right)^2 dzdw \tag{24}$$

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Recalling that $A_3 = S_k \times Q$, we estimate (24) with

$$
\int_{\substack{(S_k \times Q) \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (\phi_k(z) - \phi_k(w))^2 (v_\varepsilon(w) - v(w))^2 dz dw
$$

$$
\leq \quad \frac{N^2}{\eta^2} \int_{\substack{S_k \times Q \\ \{|z-w|<\varepsilon\}}} |z-w|^2 (v_\varepsilon(w) - v(w))^2 dz dw
$$

$$
= \quad \frac{N^2}{\eta^2} \int_{S_k} \left(\int_{Q \cap B(w,\varepsilon)} |z-w|^2 dz \right) (v_\varepsilon(w) - v(w))^2 dw
$$

$$
\leq \quad \frac{N^2}{\eta^2} \int_{S_k} \varepsilon^2 |B_\varepsilon| (v_\varepsilon(w) - v(w))^2 dw
$$

$$
= \quad \varepsilon^{d+2} \frac{N^2}{\eta^2} \int_{S_k} (v_\varepsilon(w) - v(w))^2 dw.
$$

Note that $v_\varepsilon \to v$ in $L^2(Q)$ by assumption, so the last term goes to 0 as $\varepsilon \to 0$. Moreover, when we sum over $k = 1, ..., N$ the terms (22) and (23), we find

$$
\sum_{k=1}^{N} \left(\frac{1}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v_\varepsilon(z) - v_\varepsilon(w))^2 dz dw + \frac{1}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v(z) - v(w))^2 dz dw \right)
$$

$$
\leq \frac{1}{\varepsilon^{d+2}} \int_{\substack{Q^2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v_\varepsilon(z) - v_\varepsilon(w))^2 dz dw + \frac{1}{\varepsilon^{d+2}} \int_{\substack{Q^2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v(z) - v(w))^2 dz dw.
$$

Thus we may find $k \in \{1, \ldots, N\}$ such that

$$
\frac{1}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v_\varepsilon(z) - v_\varepsilon(w))^2 dz dw + \frac{1}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v(z) - v(w))^2 dz dw
$$

$$
\leq \frac{1}{N} \left(\frac{1}{\varepsilon^{d+2}} \int_{\substack{Q^2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v_\varepsilon(z) - v_\varepsilon(w))^2 dz dw + \frac{1}{\varepsilon^{d+2}} \int_{\substack{Q^2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v(z) - v(w))^2 dz dw \right).
$$

With such a choice of k, and setting $v_\varepsilon^\eta = v_\varepsilon^k$, (21) is estimated by

$$
\frac{1}{\varepsilon^{d+2}} \int_{\substack{A_3 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v_\varepsilon^\eta(z) - v_\varepsilon^\eta(w))^2 dz dw
$$

$$
\leq \quad \frac{3}{N} \left(\frac{1}{\varepsilon^{d+2}} \int_{\substack{Q^2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v_\varepsilon(z) - v_\varepsilon(w))^2 dz dw + \frac{1}{\varepsilon^{d+2}} \int_{\substack{Q^2 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} (v(z) - v(w))^2 dz dw \right)
$$

$$
+ 3 \frac{N^2}{\eta^2} \int_{S_k} (v_\varepsilon(w) - v(w))^2 dw = \frac{3}{N} (F_\varepsilon'(v_\varepsilon) + F_\varepsilon'(v)) + 3 \frac{N^2}{\eta^2} \int_{S_k} (v_\varepsilon(w) - v(w))^2 dw
$$

$$
\leq \quad \frac{C}{N} + 3 \frac{N^2}{\eta^2} \int_{S_k} (v_\varepsilon(w) - v(w))^2 dw.
$$

The estimate on A_3' is the same with the roles of z, w reversed.

As for A_4, as already noticed for $(z, w) \in A_4$ we have $|z - w| \geq \frac{\eta}{N}$, so that, for η, N fixed and ε small enough (indeed we let it go to 0 before arguing on N), $\frac{\eta}{N} < |z - w| < \varepsilon$ cannot

happen simultaneously and thus we simply have

$$\frac{1}{\varepsilon^{d+2}} \int_{\substack{A_4 \cap (\delta E)^2 \\ \{|z-w|<\varepsilon\}}} \left(v_\varepsilon^k(z) - v_\varepsilon^k(w) \right)^2 dz dw = 0 \quad \text{for } \varepsilon \text{ small enough.}$$

This works the same for A_4' as well. Finally, gathering all the estimates together, we find

$$\limsup_{\varepsilon \to 0} (F_\varepsilon'(v_\varepsilon^\eta) - F_\varepsilon'(v_\varepsilon)) \leq \frac{C}{N} + \|\nabla v\|_{L^2(S_\eta + B_\eta(0))}.$$

Sending $N \to \infty$, we prove the claim since $\|\nabla v\|_{L^2(S_\eta + B_\eta(0))} = \omega(\eta) = o(1)$ as $\eta \to 0$. $\square$

Recalling (20), we consider a Lebesgue point x_0 and a sequence $\rho = \rho_\varepsilon$ for which the limit

$$\frac{d\mu}{dx}(x_0) = \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d}.$$

exists, with μ_ε is defined as in (19).

Proposition 4.2. *Let $\{u_\varepsilon\} \subseteq L^2(\Omega)$, $u_\varepsilon \to u \in H^1(\Omega)$, and let $\eta > 0$ be fixed. There exists $\tilde{u}_\varepsilon \to u$ such that*

- $\tilde{u}_\varepsilon(x) = u_\varepsilon(x_0) + \nabla u(x_0) \cdot (x - x_0)$, *for all $x \in Q_\rho(x_0) \setminus Q_{\rho(1-2\eta)}(x_0)$;*

- $\tilde{u}_\varepsilon(x) = u_\varepsilon(x)$ *for all $x \in Q_{\rho(1-4\eta)}(x_0)$;*

- *denoting for simplicity*

$$F_\varepsilon'(u) = \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\rho(x_0) \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u(x) - u(y))^2 dx dy,$$

  it holds

$$\limsup_{\varepsilon \to 0} (F_\varepsilon'(u_\varepsilon) - F_\varepsilon'(\tilde{u}_\varepsilon)) \leq o(1)$$

  as $\eta \to 0$.

Proof. The proof of this proposition is obtained by applying Lemma 4.1 to the difference quotient v_ε defined by

$$v_\varepsilon(z) := \frac{u_\varepsilon(x_0 + \rho z) - u_\varepsilon(x_0)}{\rho}, \quad z \in Q.$$

We may choose $\rho = \rho_\varepsilon$ such that also $v_\varepsilon \to \nabla u(x_0) \cdot z$ in $L^2(Q)$. We rewrite the quotient

$\frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d}$ in terms of the functions v_ε:

$$
\begin{aligned}
\frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d} &= \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\rho(x_0)\times\Omega)\cap(\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x)-u_\varepsilon(y))^2 dx dy \\
&\geq \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\rho(x_0)\times Q_\rho(x_0))\cap(\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x)-u_\varepsilon(y))^2 dx dy \\
&= \frac{\rho^d}{\varepsilon^{d+2}} \int_{\substack{(Q\times Q)\cap(\frac{\delta}{\rho}E)^2 \\ \{|z-w|<\frac{\varepsilon}{\rho}\}}} (u_\varepsilon(x_0+\rho z)-u_\varepsilon(x_0+\rho w))^2 dz dw \\
&= \frac{\rho^{d+2}}{\varepsilon^{d+2}} \int_{\substack{(Q\times Q)\cap(\frac{\delta}{\rho}E)^2 \\ \{|z-w|<\frac{\varepsilon}{\rho}\}}} \left(\frac{u_\varepsilon(x_0+\rho z)-u_\varepsilon(x_0+\rho w)}{\rho} \right)^2 dz dw \\
&= \frac{\rho^{d+2}}{\varepsilon^{d+2}} \int_{\substack{(Q\times Q)\cap(\frac{\delta}{\rho}E)^2 \\ \{|z-w|<\frac{\varepsilon}{\rho}\}}} (v_\varepsilon(z)-v_\varepsilon(w))^2 dz dw.
\end{aligned}
$$

By Lemma 4.1, there exists $v_\varepsilon^\eta \to v$ in $L^2(Q)$ which satisfies the properties stated therein, in particular $v_\varepsilon^\eta(z) = \nabla u(x_0) \cdot z$ in the set $\{z \in Q : \operatorname{dist}(z, \partial Q) < \eta\}$. By setting

$$
\tilde{u}_\varepsilon(x_0+\rho z) := \rho v_\varepsilon^\eta(z) + u_\varepsilon(x_0) \text{ for all } z \in Q,
$$

the function $\tilde{u}_\varepsilon$ satisfies the properties in the statement; in particular, $\tilde{u}_\varepsilon(x) = u_\varepsilon(x_0) + \nabla u(x_0) \cdot (x-x_0)$ in the set $\{x \in Q_\rho(x_0) : \operatorname{dist}(z, \partial Q_\rho(x_0)) < \rho\eta\} = Q_\rho(x_0) \setminus Q_{\rho(1-2\eta)}(x_0)$, and by the same change of variable of above to write the functional back in terms of $\tilde{u}_\varepsilon$ it holds

$$
\limsup_{\varepsilon\to 0} \left(\frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\rho(x_0)\cap\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x)-u_\varepsilon(y))^2 dx dy \right.
$$
$$
\left. - \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\rho(x_0)\cap\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (\tilde{u}_\varepsilon(x)-\tilde{u}_\varepsilon(y))^2 dx dy \right) \leq o(1),
$$

as $\eta \to 0$. $\qquad\square$

To ease the notation we still call u_ε the sequence which satisfies $u_\varepsilon(x) = u_\varepsilon(x_0) + \nabla u(x_0) \cdot (x-x_0)$ for $x \in Q_\rho(x_0) \setminus Q_{\rho(1-2\eta)}(x_0)$ of Proposition 4.2.

Now we would like to have $\frac{\rho}{\delta} \in \mathbb{N}$, so we take a slightly smaller cube than $Q_\rho(x_0)$ that gives a negligible error at the limit. Set $N_\varepsilon := \lfloor \frac{1}{2}(\frac{\rho}{\delta}-1) \rfloor$, $l_\varepsilon := (1+2N_\varepsilon)\delta$ and

$$
Q_\varepsilon := Q_{l_\varepsilon}(x_0) = \bigcup_{k\in\{-N_\varepsilon,\dots,0,\dots,N_\varepsilon\}^d} Q_\delta(x_0+\delta k), \qquad |Q_\varepsilon| = l_\varepsilon^d,
$$

In the hypothesis $\varepsilon \gg \delta$ and $\eta \sim \frac{\varepsilon}{\rho}$, we have $Q_{\rho(1-\eta)}(x_0) \subseteq Q_\varepsilon \subseteq Q_\rho(x_0)$. In particular, we can extend $u_\varepsilon - u_\varepsilon(x_0) - \nabla u(x_0) \cdot (\cdot - x_0)$ periodically of period l_ε to the whole $\mathbb{R}^d$.

Note that for all $x \in Q_\varepsilon \subseteq Q_\rho(x_0)$ and $y \in B(x;\varepsilon)$, choosing $\eta = 2\frac{\varepsilon}{\rho}$,

$$
|y_i - (x_0)_i| \leq |x_i - (x_0)_i| + |x-y| \leq \frac{\rho}{2} + \varepsilon = \frac{\rho(1+\eta)}{2} \text{ for all } i \in \{1,\dots,d\};
$$

that is, $y \in Q_{\rho(1+\eta)}(x_0)$. Thus we have

$$(Q_\varepsilon \times \mathbb{R}^d) \cap \{|x - y| < \varepsilon\} \subset (Q_\varepsilon \times Q_{\rho(1+\eta)}(x_0)) \cap \{|x - y| < \varepsilon\}.$$

We further split the right-hand side in

$$((Q_\varepsilon \times Q_\varepsilon) \cap \{|x - y| < \varepsilon\}) \cup ((Q_\varepsilon \times (Q_{\rho(1+\eta)}(x_0) \setminus Q_\varepsilon)) \cap \{|x - y| < \varepsilon\}). \qquad (25)$$

For $y \in Q_{\rho(1+\eta)}(x_0) \setminus Q_\varepsilon \subset Q_{\rho(1+\eta)}(x_0) \setminus Q_{\rho(1-\eta)}(x_0)$, and $x \in Q_\varepsilon \cap B(y, \varepsilon)$ we have

$$\frac{\rho(1-\eta)}{2} < |y_i - (x_0)_i| \le |x_i - (x_0)_i| + |x - y| < |x_i - (x_0)_i| + \varepsilon = |x_i - (x_0)_i| + \frac{\eta\rho}{2}$$

for some index $i \in \{1, \ldots, d\}$; that is, $x \in Q_\varepsilon \setminus Q_{\rho(1-2\eta)}(x_0) \subset Q_\rho(x_0) \setminus Q_{\rho(1-2\eta)}(x_0)$. By Proposition 4.2, it follows that for $(x, y) \in (Q_\varepsilon \times (Q_{\rho(1+\eta)}(x_0) \setminus Q_\varepsilon)) \cap \{|x - y| < \varepsilon\}$, we have simultaneously

$$u_\varepsilon(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0),$$

$$u_\varepsilon(y) = u(x_0) + \nabla u(x_0) \cdot (y - x_0),$$

(the latter by periodicity in y). Therefore, by (25), we can estimate

$$\frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \times \mathbb{R}^d) \cap (\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy \le \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \times Q_\varepsilon) \cap (\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy$$

$$+ \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \times (Q_{\rho(1+\eta)}(x_0) \setminus Q_\varepsilon)) \cap (\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (\nabla u(x_0) \cdot (x - y))^2 dx dy$$

$$\le \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy + \frac{|\nabla u(x_0)|^2 \varepsilon^2}{\rho^d \varepsilon^{d+2}} \int_{Q_{\rho(1+\eta)}(x_0) \setminus Q_\varepsilon} \left(\int_{Q_\varepsilon \cap B(y,\varepsilon)} dx \right) dy$$

$$\le \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy + \frac{|\nabla u(x_0)|^2 \varepsilon^2}{\rho^d \varepsilon^{d+2}} |Q_{\rho(1+\eta)} \setminus Q_{\rho(1-\eta)}| \varepsilon^d$$

$$\le \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy + \frac{|\nabla u(x_0)|^2}{\rho^d} 4 d \varepsilon \rho^{d-1}$$

$$= \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy + C \frac{\varepsilon}{\rho}$$

since $|Q_{\rho(1+\eta)} \setminus Q_\varepsilon| \le |Q_{\rho(1+\eta)} \setminus Q_{\rho(1-\eta)}| = [\rho(1+\eta)]^d - [\rho(1-\eta)]^d \le 2(\eta\rho) d \rho^{d-1} = 4 d \varepsilon \rho^{d-1}$. We can read the estimate above as

$$\frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy \ge \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon \times \mathbb{R}^d) \cap (\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy - C \frac{\varepsilon}{\rho}$$

and $C\frac{\varepsilon}{\rho} = o(1)$ as $\varepsilon \to 0$.

Set $w_\varepsilon(z) := u_\varepsilon(x_0 + l_\varepsilon z) - \nabla u(x_0) \cdot l_\varepsilon z$ for all $z \in Q$, which is $\mathbb{Z}^d$-periodic by our assumption. For simplicity denote $M_\varepsilon = 1 + 2N_\varepsilon$, so that $l_\varepsilon = (1 + 2N_\varepsilon)\delta = M_\varepsilon \delta$ and $|Q_\varepsilon|^d = l_\varepsilon^d = M_\varepsilon^d \delta^d$.

We have

$$\frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d} = \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\rho(x_0)\times\Omega)\cap(\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy$$

$$\geq \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon\cap\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy$$

$$\geq \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon\times\mathbb{R}^d)\cap(\delta E)^2 \\ \{|x-y|<\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy + o(1)$$

$$= \frac{1}{\rho^d \varepsilon^{d+2}} \int_{\substack{(Q_\varepsilon\times\mathbb{R}^d)\cap(\delta E)^2 \\ \{|x-y|<\varepsilon\}}} \left(w_\varepsilon\left(\frac{x-x_0}{l_\varepsilon}\right) - w_\varepsilon\left(\frac{y-x_0}{l_\varepsilon}\right) + \nabla u(x_0)\cdot(x-y) \right)^2 dx dy + o(1)$$

$$\geq \frac{1}{\rho^d \varepsilon^{d+2}} \inf_{w\, \mathbb{Z}^d\text{-per.}} \left\{ \int_{\substack{(Q_\varepsilon\times\mathbb{R}^d)\cap(\delta E)^2 \\ \{|x-y|<\varepsilon\}}} \left(w\left(\frac{x-x_0}{l_\varepsilon}\right) - w\left(\frac{y-x_0}{l_\varepsilon}\right) + \nabla u(x_0)\cdot(x-y) \right)^2 dx dy \right\}.$$

We work on the infimum by change of variables, writing

$$\inf_{w\, \mathbb{Z}^d\text{-per.}} \int_{\substack{(Q_\varepsilon\times\mathbb{R}^d)\cap(\delta E)^2 \\ \{|x-y|<\varepsilon\}}} \left(w\left(\frac{x-x_0}{l_\varepsilon}\right) - w\left(\frac{y-x_0}{l_\varepsilon}\right) + \nabla u(x_0)\cdot(x-y) \right)^2 dx dy$$

$$= l_\varepsilon^{2d} \inf_{w\, \mathbb{Z}^d\text{-per.}} \int_{\substack{(Q_1(\frac{x_0}{l_\varepsilon})\times\mathbb{R}^d)\cap(\frac{E}{M_\varepsilon})^2 \\ \{|x-y|<\frac{\varepsilon}{l_\varepsilon}\}}} \left(w\left(x-\frac{x_0}{l_\varepsilon}\right) - w\left(y-\frac{x_0}{l_\varepsilon}\right) + \nabla u(x_0)\cdot l_\varepsilon(x-y) \right)^2 dx dy$$

$$= l_\varepsilon^{2d+2} \inf_{w\, \mathbb{Z}^d\text{-per.}} \int_{\substack{(Q_1(\frac{x_0}{l_\varepsilon})\times\mathbb{R}^d)\cap(\frac{E}{M_\varepsilon})^2 \\ \{|x-y|<\frac{\varepsilon}{l_\varepsilon}\}}} \left(w(x) - w(y) + \nabla u(x_0)\cdot(x-y) \right)^2 dx dy.$$

The last equality comes from the fact that, for each w 1-periodic, the transformation $\tilde w(x) = l_\varepsilon w\left(x+\frac{x_0}{l_\varepsilon}\right)$ (which is also 1-periodic) gives the same integral, and the infimum is the same by the one-to-one correspondence. Now we use the periodicity of E to get rid of the center $\frac{x_0}{l_\varepsilon}$, indeed it holds

$$\int_{\substack{(Q_1(\frac{x_0}{l_\varepsilon})\times\mathbb{R}^d)\cap(\frac{E}{M_\varepsilon})^2 \\ \{|x-y|<\frac{\varepsilon}{l_\varepsilon}\}}} \left(w(x) - w(y) + \nabla u(x_0)\cdot(x-y) \right)^2 dx dy$$

$$= \int_{\substack{(Q_1(\frac{x_0}{l_\varepsilon})\times\mathbb{R}^d) \\ \{|x-y|<\frac{\varepsilon}{l_\varepsilon}\}}} \chi_{\left(\frac{E}{M_\varepsilon}\times\frac{E}{M_\varepsilon}\right)}(x,y) \left(w(x) - w(y) + \nabla u(x_0)\cdot(x-y) \right)^2 dx dy$$

$$= \int_{\substack{(Q_1(0)\times\mathbb{R}^d) \\ \{|x-y|<\frac{\varepsilon}{l_\varepsilon}\}}} \chi_{\left(\frac{E}{M_\varepsilon}\times\frac{E}{M_\varepsilon}\right)}(x,y) \left(w(x) - w(y) + \nabla u(x_0)\cdot(x-y) \right)^2 dx dy,$$

given that the integrand is jointly 1-periodic in both x and y, thanks to the 1-periodicity of E (and in particular of $\frac{E}{M_\varepsilon}$), of w and that y runs in $\mathbb{R}^d$ (so that, when x and y is changed with $x+k$ and $y+k'$, the change of variable $y' = y+k'-k$ works).

Let w be $\mathbb{Z}^d$-periodic and define a $\frac{1}{M_\varepsilon}\mathbb{Z}^d$-periodic function

$$\tilde w(x) := \frac{1}{M_\varepsilon^d} \sum_{k\in\{-N_\varepsilon,\ldots,0,\ldots,N_\varepsilon\}^d} w\left(x + \frac{k}{M_\varepsilon}\right)$$

Via a standard convexity argument, calling $\mathcal{I}_\varepsilon = \{-N_\varepsilon, ..., 0, ..., N_\varepsilon\}^d$,

$$\int_{\substack{\left(Q_{\frac{1}{M_\varepsilon}} \times \mathbb{R}^d\right) \cap \left(\frac{E}{M_\varepsilon}\right)^2 \\ \{|x-y| < \frac{\varepsilon}{l_\varepsilon}\}}} (\tilde{w}(x) - \tilde{w}(y) + \nabla u(x_0) \cdot (x - y))^2 \, dxdy$$

$$= \frac{1}{M_\varepsilon^d} \int_{\substack{(Q_1 \times \mathbb{R}^d) \cap \left(\frac{E}{M_\varepsilon}\right)^2 \\ \{|x-y| < \frac{\varepsilon}{l_\varepsilon}\}}} (\tilde{w}(x) - \tilde{w}(y) + \nabla u(x_0) \cdot (x - y))^2 \, dxdy$$

$$= \frac{1}{M_\varepsilon^d} \int_{\substack{(Q_1 \times \mathbb{R}^d) \cap \left(\frac{E}{M_\varepsilon}\right)^2 \\ \{|x-y| < \frac{\varepsilon}{l_\varepsilon}\}}} \left(\frac{1}{M_\varepsilon^d} \sum_{k \in \mathcal{I}_\varepsilon} \left(w\left(x + \frac{k}{M_\varepsilon}\right) - w\left(y + \frac{k}{M_\varepsilon}\right) + \nabla u(x_0) \cdot (x - y)\right)\right)^2 dxdy$$

$$\leq \frac{1}{M_\varepsilon^{2d}} \sum_{k \in \mathcal{I}_\varepsilon} \int_{\substack{(Q_1 \times \mathbb{R}^d) \cap \left(\frac{E}{M_\varepsilon}\right)^2 \\ \{|x-y| < \frac{\varepsilon}{l_\varepsilon}\}}} \left(w\left(x + \frac{k}{M_\varepsilon}\right) - w\left(y + \frac{k}{M_\varepsilon}\right) + \nabla u(x_0) \cdot (x - y)\right)^2 dxdy$$

$$= \frac{1}{M_\varepsilon^d} \int_{\substack{(Q_1 \times \mathbb{R}^d) \cap \left(\frac{E}{M_\varepsilon}\right)^2 \\ \{|x-y| < \frac{\varepsilon}{l_\varepsilon}\}}} (w(x) - w(y) + \nabla u(x_0) \cdot (x - y))^2 \, dxdy.$$

Thus by taking the infimum for every $\tilde{w}$ $\frac{1}{M_\varepsilon}\mathbb{Z}^d$-periodic on the left-hand side of the last inequality, by the arbitrariness of w 1-periodic, in (26) we get

$$\frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d}$$

$$\geq \frac{l_\varepsilon^{2d+2}}{\rho^d \varepsilon^{d+2}} M_\varepsilon^d \inf_{w \, \frac{1}{M_\varepsilon}\mathbb{Z}^d\text{-per.}} \int_{\substack{\left(Q_{\frac{1}{M_\varepsilon}} \times \mathbb{R}^d\right) \cap \left(\frac{E}{M_\varepsilon}\right)^2 \\ \{|x-y| < \frac{\varepsilon}{l_\varepsilon}\}}} (w(x) - w(y) + \nabla u(x_0) \cdot (x - y))^2 \, dxdy$$

$$= \underbrace{\frac{l_\varepsilon^d}{\rho^d}}_{\sim 1 \text{ as } \varepsilon \to 0} \frac{\delta^{d+2}}{\varepsilon^{d+2}} \inf_{w \, \frac{1}{M_\varepsilon}\mathbb{Z}^d\text{-per.}} \int_{\substack{\left(Q_{\frac{1}{M_\varepsilon}} \times \mathbb{R}^d\right) \cap \left(\frac{E}{M_\varepsilon}\right)^2 \\ \{|x-y| < \frac{\varepsilon}{l_\varepsilon}\}}} M_\varepsilon^{2d+2} (w(x) - w(y) + \nabla u(x_0) \cdot (x - y))^2 \, dxdy$$

$$= \frac{\delta^{d+2}}{\varepsilon^{d+2}} \inf_{w \, \frac{1}{M_\varepsilon}\mathbb{Z}^d\text{-per.}} \int_{\substack{(Q_1 \times \mathbb{R}^d) \cap (E)^2 \\ \{|x-y| < \frac{\varepsilon}{\delta}\}}} \left(w\left(\frac{x}{M_\varepsilon}\right) - w\left(\frac{y}{M_\varepsilon}\right) + \nabla u(x_0) \cdot (x - y)\right)^2 dxdy$$

$$\geq \frac{\delta^{d+2}}{\varepsilon^{d+2}} \inf_{w \, \mathbb{Z}^d\text{-per.}} \int_{\substack{(Q_1 \times \mathbb{R}^d) \cap (E)^2 \\ \{|x-y| < \frac{\varepsilon}{\delta}\}}} (w(x) - w(y) + \nabla u(x_0) \cdot (x - y))^2 \, dxdy,$$

where the second equality holds from a change of variable and the same value of the inf with $M_\varepsilon w$, and the last inequality comes from the fact that $x \mapsto w\left(\frac{x}{M_\varepsilon}\right)$ is 1 periodic if w is $\frac{1}{M_\varepsilon}$ periodic, and the infimum decreases. Finally, with fixed $\eta > 0$, for ε small enough we have $\eta > \frac{\delta}{\varepsilon}\text{diam}(K)$, and we set $Z_\varepsilon = \{k \in \mathbb{Z}^d : \frac{\delta}{\varepsilon}|k| < 1 - \eta\}$. Note that, when $k \in Z_\varepsilon$ we have

$$|x - y - k| \leq \text{diam}(K) + |k| < \frac{\varepsilon}{\delta} - \frac{\varepsilon}{\delta}\eta + \text{diam(K)} < \frac{\varepsilon}{\delta}$$

so that $(x, y) \in (K \times E) \cap \{|x - y| < \frac{\varepsilon}{\delta}\} = (K \times (K + k)) \cap \{|x - y| < \frac{\varepsilon}{\delta}\}$. By a double Jensen's

inequality we have

$$\inf_{\substack{w\ \mathbb{Z}^d\text{-per.}}} \int_{\substack{(Q_1\times\mathbb{R}^d)\cap(E)^2 \\ \{|x-y|<\frac{\varepsilon}{\delta}\}}} (w(x) - w(y) + \nabla u(x_0)\cdot(x-y))^2 dx dy$$

$$= \inf_{w\ \mathbb{Z}^d\text{-per.}} \int_{(K\times E)\cap\{|x-y|<\frac{\varepsilon}{\delta}\}} (w(x) - w(y) + \nabla u(x_0)\cdot(x-y))^2 dx dy$$

$$\geq \inf_{w\ \mathbb{Z}^d\text{-per.}} \sum_{k\in Z_\varepsilon} \int_{K\times(K+k)} (w(x) - w(y) + \nabla u(x_0)\cdot(x-y))^2 dx dy$$

$$\geq \inf_{w\ \mathbb{Z}^d\text{-per.}} \sum_{k\in Z_\varepsilon} |K|^2 \left(\frac{1}{|K|^2}\int_{K\times(K+k)} w(x) - w(y) + \nabla u(x_0)\cdot(x-y) dx dy\right)^2$$

$$= \sum_{k\in Z_\varepsilon} |K|^2 \left(\frac{1}{|K|^2}\int_{K\times K} \nabla u(x_0)\cdot k\, dx dy\right)^2 = |K^2| \sum_{k\in Z_\varepsilon} (\nabla u(x_0)\cdot k)^2 ,$$

which is independent of w 1-periodic. Thus

$$\frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d} \geq \frac{\delta^{d+2}}{\varepsilon^{d+2}}|K^2| \sum_{k\in Z_\varepsilon} (\nabla u(x_0)\cdot k)^2$$

and

$$\liminf_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q_\rho(x_0))}{\rho^d} \geq |K^2| \liminf_{\varepsilon\to 0} \sum_{k\in Z_\varepsilon} \left(\frac{\delta}{\varepsilon}\right)^d \left(\nabla u(x_0)\cdot\frac{\delta}{\varepsilon}k\right)^2 = |K|^2 \int_{B_{1-\eta}} |\nabla u(x_0)\cdot\xi|^2 d\xi.$$

Using the lower semicontinuity of the mass,

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \mu(\Omega) \geq \int_\Omega \frac{d\mu}{dx}dx,$$

we get

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq |K|^2 \int_\Omega \int_{B_1} |\nabla u(x)\cdot\xi|^2 d\xi dx = |K|^2 C(d) \int_\Omega |\nabla u(x)|^2 dx,$$

with $C_d := \frac{1}{d}\int_{B_1} |\xi|^2 d\xi$ as in [5] (for us $\varphi(z)$ was the characteristic function of the unit ball).

4.2 Limsup inequality

Let $u \in C^2(\overline{\Omega})$ and let $\eta > 0$ be fixed. Let $\Omega \Subset \Omega'$ and set $Z_\delta := \{k\in\mathbb{Z}^d : Q_\delta^k \cap \Omega \neq \emptyset\}$. For $\delta \ll 1$ we have $\bigcup_{k\in Z_\delta} Q_\delta^k \subseteq \{x\in\mathbb{R}^d : \text{dist}(x,\Omega) \leq 2\delta\} \subseteq \Omega'$. Moreover, without loss of generality we can assume $u \in C^2(\Omega')$.

By the regularity assumption on u and a weighted Cauchy-Schwarz inequality:

$$|u(x) - u(y)|^2 = |\nabla u(x)\cdot(x-y) + R(x,y)|^2$$

$$\leq (1+\eta)|\nabla u(x)\cdot(x-y)|^2 + \left(1+\frac{1}{\eta}\right)|R(x,y)|^2,$$

with $|R(x,y)| \leq C|x-y|^2$. Moreover, in the assumption $\varepsilon \gg \delta$, it also holds

$$|k\delta - k'\delta| \leq |k\delta - x| + |k'\delta - y| + \varepsilon \leq \varepsilon + 2\delta \leq \varepsilon(1+\eta)$$

for all $k, k' \in Z_\delta$ and $x, y \in Q_\delta^k \times Q_\delta^{k'}$ such that $|x - y| < \varepsilon$, which implies

$$
\begin{aligned}
F_\varepsilon(u) \;\leq\; & (1+\eta)\frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} |\nabla u(x) \cdot (x-y)|^2 dx dy + C_\eta \frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|<\varepsilon\}}} |x-y|^4 dx dy \\
\leq\; & (1+\eta)\frac{1}{\varepsilon^{d+2}} \sum_{k,k' \in Z_\delta} \int_{\substack{(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2 \\ \{|x-y|<\varepsilon\}}} |\nabla u(x) \cdot (x-y)|^2 dx dy + C_\eta \varepsilon^2 \\
\leq\; & (1+\eta)\frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k' \in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} \int_{(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2} |\nabla u(x) \cdot (x-y)|^2 dx dy + C_\eta \varepsilon^2.
\end{aligned}
$$

Note that for all $k, k' \in Z_\delta$ and for all $x = \delta k + z$ and $y = \delta k' + w$, with $z, w \in \delta Q$,

$$
\begin{aligned}
|\nabla u(x) \cdot (x - y)| \;=\; & |\nabla u(x) \cdot (\delta k - \delta k') + \nabla u(x) \cdot (z - w)| \\
\leq\; & |\nabla u(x) \cdot (\delta k - \delta k')| + \|\nabla u\|_\infty \delta \\
\leq\; & |\nabla u(k\delta) \cdot (\delta k - \delta k')| + \|D^2 u\|_\infty \delta + \|\nabla u\|_\infty \delta \\
\leq\; & |\nabla u(k\delta) \cdot (\delta k - \delta k')| + C\delta,
\end{aligned}
$$

so that $|\nabla u(x) \cdot (x-y)|^2 \leq (1+\eta)|\nabla u(k\delta) \cdot (\delta k - \delta k')|^2 + C_\eta \delta^2$. Therefore we have

$$
\begin{aligned}
& \frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k' \in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} \int_{(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2} |\nabla u(x) \cdot (x-y)|^2 dx dy \\
\leq\; & (1+\eta)\frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k' \in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} \int_{(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2} |\nabla u(k\delta) \cdot (k\delta - k'\delta)|^2 dx dy \\
& + C_\eta \sum_{\substack{k,k' \in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} \frac{1}{\varepsilon^{d+2}} \delta^{2d} \delta^2 \\
=\; & (1+\eta)\frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k' \in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} \int_{(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2} |\nabla u(k\delta) \cdot (k\delta - k'\delta)|^2 dx dy \\
& + C_\eta \sum_{k \in Z_\delta} \sum_{|j|<\frac{(1+\eta)\varepsilon}{\delta}} \frac{1}{\varepsilon^{d+2}} \delta^{2d} \delta^2 \\
\leq\; & (1+\eta)\frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k' \in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} \int_{(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2} |\nabla u(k\delta) \cdot (k\delta - k'\delta)|^2 dx dy \\
& + C_\eta \frac{1}{\delta^d} \left(\frac{\varepsilon}{\delta}\right)^d \frac{1}{\varepsilon^{d+2}} \delta^{2d} \delta^2 \\
\leq\; & (1+\eta)\frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k' \in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} |(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2| |\nabla u(k\delta) \cdot (k\delta - k'\delta)|^2 + C_\eta \left(\frac{\delta}{\varepsilon}\right)^2.
\end{aligned}
$$

Finally, by the periodicity of the set E

$$
\begin{aligned}
|(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2| \;=\; & |Q_\delta^k \cap \delta E| |Q_\delta^{k'} \cap \delta E| \\
=\; & \delta^{2d} |(k+Q) \cap E| |(k'+Q) \cap E| = \delta^{2d} |Q \cap E|^2 = \delta^{2d} |K|^2,
\end{aligned}
$$

which implies

$$\frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k'\in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} |(Q_\delta^k \times Q_\delta^{k'}) \cap (\delta E)^2||\nabla u(k\delta)\cdot(k\delta-k'\delta)|^2$$

$$= \frac{1}{\varepsilon^{d+2}} \sum_{\substack{k,k'\in Z_\delta \\ \delta|k-k'|<(1+\eta)\varepsilon}} |K|^2\delta^{2d}|\nabla u(k\delta)\cdot(k\delta-k'\delta)|^2$$

$$\leq |K|^2 \sum_{k\in Z_\delta} \delta^d \sum_{|j|<\frac{(1+\eta)\varepsilon}{\delta}} \left(\frac{\delta}{\varepsilon}\right)^d \left|\nabla u(k\delta)\cdot\frac{\delta}{\varepsilon}j\right|^2.$$

It follows

$$F_\varepsilon(u) \leq (1+\eta)^2|K|^2 \sum_{k\in Z_\delta} \delta^d \sum_{\frac{\delta}{\varepsilon}|j|<1+\eta} \left(\frac{\delta}{\varepsilon}\right)^d \left|\nabla u(k\delta)\cdot\frac{\delta}{\varepsilon}j\right|^2 + C_\eta\left(\left(\frac{\delta}{\varepsilon}\right)^2 + \varepsilon^2\right),$$

and, thus,

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u) \leq (1+\eta)^2|K|^2 \int_{\Omega'\times B_{1+2\eta}} |\nabla u(x)\cdot\xi|^2 dx d\xi.$$

Sending $\eta \to 0$ and $\Omega' \downarrow \Omega$ we conclude

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u) \leq |K|^2 \int_{\Omega\times B_1} |\nabla u(x)\cdot\xi|^2 dx d\xi = |K|^2 C(d) \int_\Omega |\nabla u(x)|^2 dx,$$

with $C_d = \frac{1}{d}\int_{B_1} |\xi|^2 d\xi$.

4.3   Γ-convergence with general convolution kernel φ_ε

We recall that the general form of the functional we consider is

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{d+2}} \int_{(\Omega\cap\delta E)\times(\Omega\cap\delta E)} \varphi\left(\frac{x-y}{\varepsilon}\right)(u(x)-u(y))^2 dx dy \tag{26}$$

and our preliminary case considered is $\varphi(\xi) = \chi_{(0,1)}(|\xi|)$. In the case of a general φ we can proceed by approximation. As stated at the beginning of Section 2, we consider a radial convolution kernel in $\mathbb{R}^d$; that is, a function $\varphi : \mathbb{R}^d \to [0,+\infty)$ such that a decreasing function $\varphi_0 : [0,+\infty) \to [0,+\infty)$ exists satisfying $\varphi(\xi) = \varphi_0(|\xi|)$. We further assume that

$$\int_{\mathbb{R}^d} \varphi(\xi)(1+|\xi|^2)dx < +\infty,$$

and for each $\varepsilon > 0$ we define the scaled kernel φ_ε by $\varphi_\varepsilon(\xi) = \frac{1}{\varepsilon^d}\varphi(\frac{\xi}{\varepsilon})$.

Since φ_0 is decreasing, we can write a 'staircase' approximation made of characteristic functions of nested intervals $(0, r_k)$ (where r_k are defined below), that is

$$\varphi_{0,n}(t) = \sum_{k=0}^{n2^n} \chi_{(0,r_k)}(t) \tag{27}$$

where for each $k = 0, \ldots, n2^n$

$$r_k = \sup \left\{ t : \varphi_0(t) \geq (k+1)2^{-n} \right\}$$

and $r_k = 0$ if the set of such t is empty. With this choice, r_j is decreasing by the monotonicity of φ_0. Accordingly, for each $n \in \mathbb{N}$ define the simple function

$$\varphi_n(\xi) = \sum_{k=0}^{n2^n} \chi_{(0,r_k)}(|\xi|).$$

Note that $\varphi_n(\xi) \leq \varphi(\xi)$ for every $\xi \in \mathbb{R}^d$. By standard approximation through simple functions, $\varphi_n(\xi) \to \varphi(\xi)$ pointwise almost everywhere, in $L^1(\mathbb{R}^d)$ by dominated convergence and, since φ is in particular bounded, it converges also uniformly.

We define as F_ε^n the same functional (26) with φ_n in place of φ. Since φ is an upper bound for any φ_n, we have for any $u_\varepsilon \rightharpoonup u$ and $n \in \mathbb{N}$ fixed

$$
\begin{aligned}
\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon \to 0} F_\varepsilon^n(u_\varepsilon) = \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^{d+2}} \int_{(\Omega \cap \delta E)^2} \varphi_n\left(\frac{x-y}{\varepsilon}\right) (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy \\
&= \liminf_{\varepsilon \to 0} \left(\sum_{k=0}^{n2^n} \frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|<r_k\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy \right) \\
&\geq \sum_{k=0}^{n2^n} \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|<r_k\varepsilon\}}} (u_\varepsilon(x) - u_\varepsilon(y))^2 dx dy \\
&\geq \sum_{k=0}^{n2^n} |K|^2 \int_\Omega \int_{B_{r_k}} |\nabla u(x) \cdot \xi|^2 d\xi dx = \\
&= |K|^2 \int_\Omega \int_{\mathbb{R}^d} \varphi_n(\xi) |\nabla u(x) \cdot \xi|^2 d\xi dx,
\end{aligned}
$$

by the result of Section 4.1. We can pass to the limit for $n \to \infty$ by monotone convergence and we get

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq |K|^2 \int_\Omega \int_{\mathbb{R}^d} \varphi(\xi) |\nabla u(x) \cdot \xi|^2 d\xi dx = C_\infty \int_\Omega |\nabla u(x)|^2 dx,$$

with $C_\infty = |K|^2 \frac{1}{d} \int_{\mathbb{R}^d} \varphi(\xi) |\xi|^2 d\xi$. Hence, the lower bound is satisfied.

As for the upper bound, we claim that it is enough to consider the function φ with compact support. Indeed, for any $R > 0$ we can split the functional

$$
\begin{aligned}
F_\varepsilon(u) &= \frac{1}{\varepsilon^{d+2}} \int_{(\Omega \cap \delta E)^2} \varphi\left(\frac{x-y}{\varepsilon}\right) (u(x) - u(y))^2 dx dy \\
&= \frac{1}{\varepsilon^{d+2}} \int_{(\Omega \cap \delta E)^2} \varphi\left(\frac{x-y}{\varepsilon}\right) \chi_{B_R}\left(\frac{x-y}{\varepsilon}\right) (u(x) - u(y))^2 dx dy \\
&\quad + \frac{1}{\varepsilon^{d+2}} \int_{(\Omega \cap \delta E)^2} \varphi\left(\frac{x-y}{\varepsilon}\right) \chi_{\mathbb{R}^d \setminus B_R}\left(\frac{x-y}{\varepsilon}\right) (u(x) - u(y))^2 dx dy.
\end{aligned}
$$

In the second term of the sum we can perform the change of variables $y = x + \varepsilon\xi$, so that, setting $\Omega(\xi) = \{x \in \Omega \cap \delta E : x + \varepsilon\xi \in \Omega \cap \delta E\} \subset \Omega$ for each $\xi \in \mathbb{R}^d$, it holds

$$
\frac{1}{\varepsilon^{d+2}} \int_{(\Omega \cap \delta E)^2} \varphi\Big(\frac{x-y}{\varepsilon}\Big) \chi_{\mathbb{R}^d \setminus B_R}\Big(\frac{x-y}{\varepsilon}\Big) (u(x) - u(y))^2 dxdy
$$
$$
= \int_{\mathbb{R}^d \setminus B_R} \varphi(\xi) \int_{\Omega(\xi)} \Big(\frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon}\Big)^2 dxd\xi
$$
$$
\leq \int_{\mathbb{R}^d \setminus B_R} \varphi(\xi) \int_{\Omega} \big(\|\nabla u\|_{L^\infty(\Omega)} |\xi|\big)^2 dxd\xi
$$
$$
= |\Omega| \|\nabla u\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d \setminus B_R} \varphi(\xi) |\xi|^2 d\xi = o(1)
$$

as $R \to \infty$, uniformly in ε. Thus we are left with the first term where $\varphi\chi_{B_R}$ has compact support, so that the claim holds. For a general φ_0 bounded, decreasing and with compact support, we can approximate it from above in the same way of (27), since for $\varphi_{0,n}$ it holds

$$
\varphi_{0,n} \leq \varphi_0 \leq \varphi_{0,n} + 2^{-n}
$$

for each $t \in \mathbb{R}$, so it it enough to take $\tilde{\varphi}_{0,n}(t) = \varphi_{0,n}(t) + 2^{-n}$ for $t \in \text{supp}(\varphi_0) = (0, R)$, and 0 otherwise. Accordingly for each $n \in \mathbb{N}$ define

$$
\tilde{\varphi}_n(\xi) = \tilde{\varphi}_{0,n}(|\xi|) = \sum_{k=0}^{n2^n} \chi_{(0,r_k)}(|\xi|) + 2^{-n}\chi_{(0,R)}(|\xi|) = \sum_{k \in \mathcal{I}_n} \chi_{(0,r_k)}(|\xi|)
$$

($\mathcal{I}_n$ is just a finite set of indices for each n) for which $\tilde{\varphi}_n \geq \varphi$ and it converges to φ uniformly and in $L^1(\mathbb{R}^d)$. Finally, for $u \in C^2(\overline{\Omega})$ and $n \in \mathbb{N}$ fixed, by the result of Section 4.2

$$
\limsup_{\varepsilon \to 0} F_\varepsilon(u) \leq \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^{d+2}} \int_{(\Omega \cap \delta E)^2} \tilde{\varphi}_n\Big(\frac{x-y}{\varepsilon}\Big) (u(x) - u(y))^2 dxdy
$$
$$
= \limsup_{\varepsilon \to 0} \Big(\sum_{k \in \mathcal{I}_n} \frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|<r_k\varepsilon\}}} (u(x) - u(y))^2 dxdy\Big)
$$
$$
\leq \sum_{k \in \mathcal{I}_n} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{d+2}} \int_{\substack{(\Omega \cap \delta E)^2 \\ \{|x-y|<r_k\varepsilon\}}} (u(x) - u(y))^2 dxdy
$$
$$
= \sum_{k \in \mathcal{I}_n} |K|^2 \int_\Omega \int_{B_{r_k}} |\nabla u(x) \cdot \xi|^2 d\xi dx =
$$
$$
= |K|^2 \int_\Omega \int_{\mathbb{R}^d} \tilde{\varphi}_n(\xi) |\nabla u(x) \cdot \xi|^2 d\xi dx,
$$

and finally pass to the limit as $n \to \infty$ to get the upper bound

$$
\limsup_{\varepsilon \to 0} F_\varepsilon(u) \leq |K|^2 \int_\Omega \int_{\mathbb{R}^d} \varphi(\xi) |\nabla u(x) \cdot \xi|^2 d\xi dx = C_\infty \int_\Omega |\nabla u(x)|^2 dx.
$$

References

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso, and D. Percivale. An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Analysis*, 18:481–496, 1992.

[2] R. Alicandro, N. Ansini, A. Braides, A. Piatnitski, and A. Tribuzio. *A Variational Theory of Convolution-Type Functionals*. Springer Nature, 2023.

[3] H. Attouch. *Variational Convergence for Functions and Operators*. Pitman, Boston, 1984.

[4] J. C. Bellido, C. Mora-Corral, and P. Pedregal. Hyperelasticity as a Γ-limit of peridynamics when the horizon goes to zero. *Calc. Var. Partial Differential Equations*, 54(2):1643–1670, 2015.

[5] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. In *Optimal control and partial differential equations*, pages 439–455. IOS, Amsterdam, 2001.

[6] A. Braides. Γ-*convergence for Beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[7] A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals*, volume 12 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 1998.

[8] A. Braides, M. Maslennikov, and L. Sigalotti. Homogenization by blow-up. *Applicable Anal.*, 87:1341–1356, 2008.

[9] A. Braides and A. Piatnitski. Homogenization of quadratic convolution energies in periodically perforated domains. *Adv. Calc. Var.*, 15(3):351–368, 2022.

[10] G. Dal Maso. *An Introduction to Γ-convergence*. Birkhäuser, Boston, 1993.

[11] I. Fonseca and S. Müller. Quasiconvex integrands and lower semicontinuity in L^1. *SIAM J. Math. Anal.*, 23:1081–1098, 1992.

[12] H. W. Kuhn. Some combinatorial lemmas in topology. *IBM Journal of Research and Development*, 4(5):518–524, 1960.

[13] G. Leoni. *A First Course in Fractional Sobolev Spaces*, volume 229 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2023.

[14] T. Mengesha and Q. Du. On the variational limit of a class of nonlocal functionals related to peridynamics. *Nonlinearity*, 28(11):3999–4035, 2015.

[15] A. C. Ponce. A new approach to Sobolev spaces and connections to Γ-convergence. *Calc. Var. Partial Differential Equations*, 19(3):229–255, 2004.

[16] M. Solci. Nonlocal-interaction vortices. *SIAM J. Math. Anal.*, 56(3):3430–3451, 2024.

[17] V. V. Zhikov. Connectedness and homogenization. Examples of fractal conductivity. *Sbornik: Mathematics*, 187(8):1109, 1996.