Methods: Game Theory 1

Fall 2017

Wieland Müller

Tilburg University

Aim of this Course

- Introduction to game theory and some applications.
- Applications mostly taken from Industrial Organization:
    - competition in imperfect markets
    - resulting lessons for market design and competition policy
- Preparation for the course “Methods: Game Theory 2” and various other courses of the M.Sc.

Grading

The components:

- Final exam (counting 70 percent toward the final grade)
- Homework (counting 30 percent toward the final grade)

Homework, I

- Form groups of n students (n will be announced at the beginning of the first lecture) and self-enroll on blackboard until the end of the first lecture week.
    - Click "Group enroll" in our course on Blackboard. (As of August 29, 2017 at 7:00 pm.)
- There will be three homework problem sets, which you will find in the folder "Course documents".
- All groups need to hand in their (clearly hand) written solutions of all problems of a given problem set at a date that will be announced ahead of time.
    - Though the exact date of the submission deadline will be announced for each problem set, it will always be on a Friday at 4:30pm.
    - Homeworks will be graded by the course's tutor Hasan Tahsin Apakan (*h.t.apakan@tilburguniversity.edu*).

Homework, II

  - Hand in your solutions in room P1.208 and put them into the mail box of Hasan Tahsin Apakan, which is labelled “Apakan, drs. H.T. P1.210” at the latest at the announced date and time. No excuse for a missed deadline will be accepted.
- Solutions to the home work will be posted on blackboard (probably one week after the submission deadline).
- You can pick up your graded problem sets and ask questions during the office hour of Hasan Tahsin Apakan (Fridays, 13.30 - 14.30 hrs, P1.210).
- Grading of homework:
  - Each problem set consists of several problems. Each problem will be graded on a 0-10 scale. The grade of a problem set will be the average score of all problems of a problem set.

Homework, III

“Hands On Problem”

- In the folder “Course documents” you will also find the document “Hands On Problem”. This document contains a problem that I ask you to prepare for the last lecture on Thursday September 21, 2017.
- In this last lecture, some groups will be randomly selected to present their results on the blackboard.
- I do not expect the most comprehensive solution of the problem, but I hope you give it serious thought as it will enable you to apply the knowledge acquired in this course to a practical problem.
- If time permits, in this lecture you will also be able to ask questions you might still have about the material of the course.

Repeaters and resit policy

- For repeaters: Those who repeat the course do not have to be part of groups again and do not have to hand in solutions. Their homework grades from the previous year will be used and count for 30%.
- Repeaters should send me an E-mail infoming me that they repeat the course.
- Resit policy: Homework grades remain valid for students who take the resit.

My office hour

- Until September 20: Wednesdays 15:00 - 16:00 hrs in room P 2.134
- After September 20: Tuesdays, 14:30 - 15:30 hrs via Skype. In case you want to talk to me via Skype, please send me your Skype username ahead of time.

Literature

- Gibbons, Robert (1992), *A Primer in Game Theory*, Harvester Wheatsheaf (Prentice Hall).
- Osborne, Martin J. (2004), *An Introduction to Game Theory*, Oxford University Press.
    - (Have a look at: http://www.economics.utoronto.ca/osborne/igt/index.html)
- Church, Jeffrey and Roger Ware (2000), *Industrial Organization: A Strategic Approach*, McGraw-Hill.
    - (Earlier version is available at: http://works.bepress.com/jeffrey_church/23/)
- Study one game theory book (+ slides + exercises, ...)

Remarks

- The lecture slides draw on material of these books.
- The lecture slides are subject to change. So if you print them, always only print a “low” number of slides for the next lecture.
- Probably, we will not be able to cover all material on the slides in class. Some material will be left for self study.

A few questions and answers I

Q: What is game theory?
A: Game theory is the analysis of multi-person decision problems; so-called games.

Q: What is a game?
A: A game is a situation in which at least two individuals (or players) who have at least partially diverging interests interact with one another. Here, the ‘payoff’ of one player does not only depend on the player’s own actions but also on the other players’ actions.

Q: What does game theory do with these games?
A: The task of game theory is to model and to solve such games.

Q: What does it mean ‘to solve’ a game?
A: To solve a game means to determine how a rational player would act in it.

A few questions and answers II

Q: What is a rational player?
A: A rational player is an omnipotent and egoistic payoff maximizer.

Game Theory versus Decision Theory

- Decision Theory:= analysis of how one person makes decisions (e.g., when faced with uncertainty).
- Game:= situation with “interdependent” decisions of $n \geq 2$ players, where one player’s actions influence other players’ payoffs
- Game Theory:= conceptual toolkit to formalize and analyze games
- Example: monopoly vs. duopoly

Game Theory in Economics

- Because of the definition above (interdependence), game theory is in particular useful to analyze markets with imperfect competition (where players have market power)
- These markets are studied in the field “Industrial Organization”
- Game theory is also applied in:
    - monetary theory (central bank and investors)
    - corporate finance (investors and lenders)
    - international trade (various countries)
    - labour relations (employer - employee)
    - ...

The Structure of the Course (Game Theory 1 and 2)

		Information of Agents	
		Complete	Incomplete
Timing of Moves	Static	**Nash Equilibrium** (Chapter 1)	**Bayesian Nash Equilibrium** (Chapter 3)
	Dynamic	**Subgame-perfect Nash Equilibrium** (Chapter 2)	**Perfect Bayesian Nash Equilibrium** (Chapter 4)

Static games with complete information

- **static**: Each player only ‘moves’ once.
- **complete information:** Each player knows the payoff function of all players.
- These games are also called games in **normal form** or **strategic form**.

Example

of a game with finitely many players and finitely many strategies:
prisoner's dilemma

- Two suspects are held in two different cells.
- If none of the two confesses, they will be sent to jail for 2 years. If both confess, they will be sent to jail for 5 years. If only one confesses and the other doesn't, the former will get 1 year and the other 10 years in prison.
- Both prisoners independently decide between "don't confess" and "confess."
- Representation using a bi-matrix:

		Prisoner 2	
		don't confess	confess
Prisoner 1	don't confess	$-2, -2$	$-10, -1$
	confess	$-1, -10$	$-5, -5$

Example

of a game with finitely many players, but infinitely many strategies:
Cournot oligopoly

- $I \geq 2$ firms who choose non-negative quantities $q_i \in [0, \infty)$
- Inverse demand: $p(Q) = max\{a - bQ, 0\}$, $Q = \sum_{i=1}^{I} q_i$
- Cost functions: $C_i(q_i) = cq_i$ (constant unit costs)
- These elements define the following game:
    - Player set: $\{1, ..., I\}$
    - Strategy sets: $S_i = [0, \infty)$, $i = 1, ..., I$
    - Payoff functions: $u_i(q_1, ..., q_I) = (a - b(q_1 + ... + q_I))q_i - cq_i$, $i = 1, ..., I$.

The components of a game and notation

Generally, a game in normal form has the following components:

- the participating players,
- the players' possible actions, and
- the payoffs for each player for each combination of actions.

Notation:

- The *players* will be denoted by $1, ..., I$.
- S_i denotes the set of possible strategies (or the strategy space) of player i.
- s_i denotes a typical element of the set S_i $(s_i \in S_i)$.
- A vector $(s_1, ..., s_I)$ that consists of one strategy for each player will be called **strategy profile** *(*or: **strategy vector**).
- For each player $i = 1, ..., I,$ u_i will be the **payoff function** of player i. That is, $u_i = u_i(s_1, ..., s_I)$ is player i's payoff if the players choose the strategy profile $(s_1, ..., s_I)$.

Formal definition of a game in nomalform

Definition

For a game with I players, the **normal form representation** Γ_N specifies for each player i a set of strategies S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1, ...s_I)$ giving the von Neumann-Morgenstern utility levels associated with the (possibly random) outcome arising from strategies $(s_1, ..., s_I)$. Formally, we write $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$.

The above examples reconsidered:

The prisoner's dilemma

- Player set: $\{1, 2\}$
- Strategy sets: $S_1 = S_2 = \{$don't confess, confess$\}$
- Payoff functions:
    - $u_1($don't confess, don't confess$) = -2$,
    - $u_1($don't confess, confess$) = -10$,
    - etc.

Cournot oligopoly

- Player set: $\{1, ..., I\}$
- Strategy sets: $S_i = [0, \infty)$, $i = 1, ..., I$
- Payoff functions: $u_i(s_1, ..., s_I) = (a - b(s_1 + ... + s_I))s_i - cs_i$, $i = 1, ..., I$.

More notation

- Let

$$S = S_1 \times ... \times S_I = \{(s_1, ..., s_I) \mid s_i \in S_i,\ i = 1, ..., I\}$$

  be the *Cartesian product* of the sets S_i (typical element: $s = (s_1, ..., s_I) \in S$).

- Using this notation, we can write

$$\begin{array}{rcl} u_i & : & S \to \mathbb{R} \\ s & \mapsto & u_i(s). \end{array}$$

- Furthermore, $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$ denotes the strategy profile of i's rivals.
- Similarly: $S_{-i} = S_1 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_I$
- We will use the notation $s = (s_1, ..., s_I)$ or $s = (s_i, s_{-i})$.

Dominant strategy I

- Is there an "obvious" prediction of how players will behave?
- Consider again the prisoners' dilemma:

		Prisoner 2	
		don't confess	confess
Prisoner 1	don't confess	−2, −2	−10, −1
	confess	−1, −10	−5, −5

- For each player, the strategy "confess" is the best strategy *no matter* what the other does.

Dominant strategy II

Definition

A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player i in game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}.$$

A strategy is a **weakly dominant strategy** for player i in game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$ if it weakly dominates every other strategy in S_i.

Example: As noted above, the strategy "confess" is a strictly dominant strategy for each player in the prisoner's dilemma. (Note: Egoistic and rational behavior need not lead to the social optimum.)

How would you check for a dominant strategy in a game that can be represented in a bi-matrix?

Dominant strategy III

Another example:

		Player 2		
		Left	Center	Right
Player 1	Top	1, 0	1, 2	0, 1
	Bottom	0, 3	0, 1	2, 0

- In this game, none of the players has a dominant strategy.
- Obviously, the concept of a dominant strategy cannot generally be employed to solve a game.

Dominated strategy I

- Are there strategies that will never be played?
- Consider again the following example:

		Player 2		
		Left	Center	Right
Player 1	Top	1, 0	1, 2	0, 1
	Bottom	0, 3	0, 1	2, 0

- For player 2, “Right” is always worse than “Center” no matter what player 1 does. (Note though that “Center” is not a dominant strategy.)
- In this case we say that player 2’s strategy “Right” is strictly dominated by the strategy “Center.”

Dominated strategy II

Definition

A strategy $s_i \in S_i$ is **strictly dominated** for player i in game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

(In this case we say that strategy s_i' strictly dominates strategy s_i.)

Dominated strategy III

Definition

A strategy $s_i \in S_i$ is **weakly dominated** in game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

and

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

for at least one $s_{-i} \in S_{-i}$. (In this case we say that strategy s_i' weakly dominates strategy s_i.

How would you check for a dominated strategy in a game that can be represented in a bi-matrix?

More examples of dominated strategies I

- In the Prisoners' dilemma the strategy "don't confess" is strictly dominated by the strategy "confess."
- In the game

		Player 2	
		Left	Right
Player 1	Top	2, 2	0, 2
	Bottom	0, 1	2, 0

  the strategy "Right" of player 2 is weakly dominated by the strategy "Left."

- **Rationality requirement: Rational individuals will not use (strictly) dominated strategies.**

Procedure of iteratively deleting dominated strategies I

- Consider once again the above example:

			Player 2	
		Left	Center	Right
Player 1	Top	1, 0	1, 2	0, 1
	Bottom	0, 3	0, 1	2, 0

- If player 2 is rational, she will not choose strategy Right.
- If player 1 knows that player 2 is rational, then player 1 can delete strategy Right from player 2's strategy set. Thus, we can consider the following game:

		Player 2	
		Left	Center
Player 1	Top	1, 0	1, 2
	Bottom	0, 3	0, 1

Procedure of iteratively deleting dominated strategies II

- In this game, the strategy Bottom of player 1 is strictly dominated by strategy Top.
- If player 1 knows that player 2 is rational *and* player 2 knows that player 1 knows that player 2 is rational, such that we can consider the above game, then player 2 can delete the strategy Bottom from the strategy set of player 1. Thus we can consider the following game:

		Player 2	
		Left	Center
Player 1	Top	1, 0	1, 2

- In this game Left is strictly dominated by Center for player 2.
- Thus, (Top, Center) is the solution of the game.

Procedure of iteratively deleting dominated strategies III

- We just applied what is called **"Procedure of iteratively deleting dominated strategies."**

- More precise description of this procedure:

Step 1: Mark all dominated strategies of player 1. Then, mark all dominated strategies of player 2 (without deleting the dominated strategies of player 1). Then do the same for all other players. Finally, delete all marked strategies.

Step 2: Apply step 1 to the resulting game.

And so on, until you get a game in which no player has a dominated strategy anymore.

Procedure of iteratively deleting dominated strategies IV

- **Another example:**

		Player 2		
		L	C	R
Player 1	T	2, 2	0, 2	0, 1
	M	2, 0	1, 1	0, 2
	B	1, 0	2, 0	0, 0

- Step 1: Player 1: M dominates T; Player 2: C dominates L
- Step 2: Player 1: B dominates M; Player 2: R dominates C
- As a result, (B, R) is the only strategy vector that survives and is, thus, the unique solution of this game.

Procedure of iteratively deleting dominated strategies V

- If we consider finite games, then this procedure stops after finitely many steps.
- If after application of the procedure of iteratively deleting dominated strategies, all players are indifferent between all strategies that have survived this procedure, then the game is called **dominance solvable**.
- If we want to apply this procedure for an arbitrary number of steps, then we have to assume that it is **common knowledge** that players are rational. That is, we have to assume that
  - players are rational, and
  - all players know that all players are rational, and
  - all players know that all players know that all players are rational and so on and so forth ad infinitum ...

Procedure of iteratively deleting dominated strategies VI

Remarks:

- The set of strategies that survive the procedure of iterative deletion of *strictly* dominated strategies does neither depend on the order in which strategies are deleted nor on the number of strategies that are deleted in each step.
- This is *not* true with regard to the elimination of *weakly* dominated strategies. That is, in this case the result can depend for instance on whether one deletes all or only some dominated strategies per step. (Exercise!)

The Nash Equilibrium I

- Consider the following game:

		Player 2		
		L	M	R
Player 1	T	0, 4	4, 0	5, 3
	M	4, 0	0, 4	5, 3
	B	3, 5	3, 5	6, 6

- In this game none of the players has a dominant or dominated strategy. What now?
- **Idea:** If game theory is to make a useful prediction, then it must be "strategically stable" or "self-enforcing" (in case of a unique prediction).
- Or: The strategies chosen have to be mutually best replies.

Definition of a Nash equilibrium I

Definition

A strategy profile $s^* = (s_1^*, ..., s_n^*)$ constitutes a **Nash equilibrium** of the game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$ if for every $i = 1, ..., n$,

$$u_i((s_1^*, ..., s_i^*, ..., s_n^*)) \geq u_i(s_1^*, ..., s_i, ..., s_n^*)$$

for all $s_i \in S_i$.

That is, s_i^* is a best reply to $(s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_n^*)$ or s_i^* is a solution of the problem

$$\max_{s_i \in S_i} u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*) \quad \text{for all } i = 1, ..., n.$$

Best-response correspondence

The correspondence $b_i : S_{-i} \rightrightarrows S_i$,

$$b_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\},$$

that assigns to every player i and every $s_{-i} \in S_{-i}$ the set of best replies of this player to s_{-i}, is called **best-response correspondence** of player i.

Examples:

- Prisoners' dilemma: $b_i(\text{don't confess}) = b_i(\text{confess}) = \{\text{confess}\}$ for $i = 1, 2$.
- Cournot oligopoly: inverse market demand: $p(Q) = \max\{a - Q, 0\}$; cost functions: $C_i(q_i) = cq_i$ for $i = 1, ..., I$ $(c < a)$.

$$b_i(q_{-i}) = \begin{cases} \{\frac{1}{2}(a - c - \Sigma_{j \neq i} q_j)\} & \text{if} \quad \Sigma_{j \neq i} q_j \leq a - c \\ \{0\} & \text{if} \quad \Sigma_{j \neq i} q_j > a - c. \end{cases}$$

Alternative characterization of the Nash equilibrium

Proposition

The strategy profile $s = (s_1, ..., s_n)$ *is a Nash equilibrium of the normal form game* $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$ *if and only if for all* $i = 1, ..., I$ *we have* $s_i \in b_i(s_{-i})$.

Remark:

- In case each player has a unique best reply to each s_{-i}, we can write these conditions as a system of simultaneous equations.
- Let $\bar{b}_i(s_{-i})$ be the only element of the set $b_i(s_{-i})$ (i.e.: $b_i(s_{-i}) = \{\bar{b}_i(s_{-i})\}$).
- Then the above conditions are equivalent to

  $$s_i = \bar{b}_i(s_{-i}) \text{ for all } i = 1, ..., n, \qquad (1)$$

  which is a system of n simultaneous equations with the n unknowns s_i (where n is the number of players).

Application of Nash equilibrium: Cournot duopoly I

- Inverse demand function: $p(Q) = \max\{a - Q, 0\}$
- Cost functions: $C_i(q_i) = cq_i$
- Number of players: $I = 2$
- Strategy sets: $S_i = [0, \infty)$, $i = 1, 2$
- Payoff functions: $\pi_i(q_i, q_j) = (a - (q_i + q_j))q_i - cq_i$, $i = 1, 2$; $i \neq j$.
- Let us determine the best response correspondence of each player i.
- If $q_j \leq a - c$, we obtain the best response of firm i from the FOC and the SOC as follows:

$$\max_{q_i} \pi_i(q_i, q_j) = \max_{q_i}(a - (q_i + q_j) - c)q_i$$

- The FOC gives:

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = a - c - q_j - 2q_i \overset{!}{=} 0 \Leftrightarrow q_i = \frac{1}{2}(a - c - q_j).$$

Application of Nash equilibrium: Cournot duopoly II

- Note that

$$\frac{\partial^2 \pi_i(q_i, q_{-i}^*)}{\partial q_i^2} = -2 < 0.$$

- That is, the second order condition is satisfied. Note that the profit function of each player i is strictly concave.
- If $q_j > a - c$, we find that $\pi_i(q_i, q_j) = (a - (q_i + q_j) - c)q_i < 0$ for all $q_i > 0$. Hence in this case the best response of firm i is $q_i = 0$.
- Summarizing these arguments, leads to firm i's best-response correspondence (actually a function in this case):

$$q_i = b_i(q_j) = \begin{cases} \frac{1}{2}(a - c - q_j) & \text{if} \quad q_j \leq a - c \\ \\ 0 & \text{if} \quad q_j > a - c. \end{cases}$$

Application of Nash equilibrium: Cournot duopoly III

- The intersection of the two best-response correspondences gives the Nash equilibrium of the Cournot game. It turns out that the two best-response correspondences intersect when $q_i, q_j \leq a - c$.
- Hence, the Nash equilibrium (q_1^*, q_2^*) is given by the solution of the two simultaneous equations:

$$q_1^* = \frac{1}{2}(a - q_2^* - c) \quad \text{and} \quad q_2^* = \frac{1}{2}(a - q_1^* - c).$$

- The solution of this system is

$$q_1^* = q_2^* = \frac{a - c}{3}.$$

Application of Nash equilibrium: Cournot duopoly IV

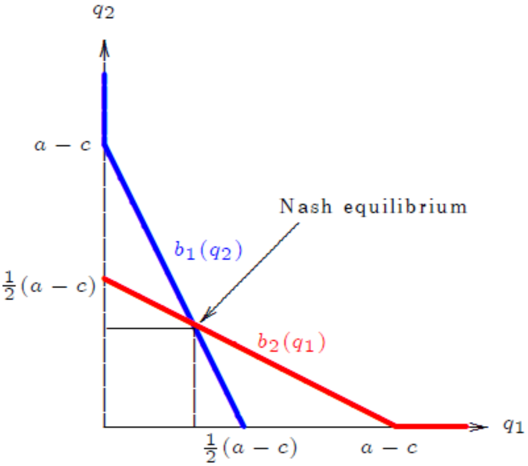


Figure: The best-reponse correspondences in the Cournot duopoly

Application of Nash equilibrium: Bertrand duopoly I

- Number of players: $I = 2$
- Strategy sets: $S_i = [0, \infty)$, $i = 1, 2$ (non-negative prices)
- Payoff functions: Let $D(p)$ be a downward sloping demand function and c the per-unit costs. Then:

$$\pi_i(p_i, p_j) = \begin{cases} 0 & \text{if} \quad p_i > p_j \\ \frac{1}{2}(p_i - c)D(p_i) & \text{if} \quad p_i = p_j \\ (p_i - c)D(p_i) & \text{if} \quad p_i < p_j \end{cases}$$

- Assume that $D(p)$ is "well behaved" so that p^m maximizes the profit function of firm i when it is a monopolist.

Application of Nash equilibrium: Bertrand duopoly II

- Best response to a price of the rival:
  - If firm 1 expects firm 2 to price above the monopoly price, then firm 1's optimal pricing strategy is to price at the monopoly level. Firm 1 then gets all of the market demand and makes monopoly profit.
  - If firm 1 expects firm 2 to set a price below the monopoly price but above MC, then firm 1's optimal pricing strategy is to set a price just below firm 2's price. Pricing above would lead to zero demand and zero profits.
  - If firm 1 expects firm 2 to set a price below MC, then firm 1's optimal strategy is to price above firm 2. In this way it gets zero profit.

Application of Nash equilibrium: Bertrand duopoly III

- Summarizing these arguments, leads to firm 1's best-response function (similarly for firm 2):

$$p_1 = b_1(p_2) = \begin{cases} \{p^m\} & \text{if} \quad p_2 > p^m \\ \varnothing & \text{if} \quad c < p_2 \leq p^m \\ \{p_1 \mid p_1 \geq p_2\} & \text{if} \quad p_2 = c \\ \{p_1 \mid p_1 > p_2\} & \text{if} \quad 0 \leq p_2 < c \end{cases}$$

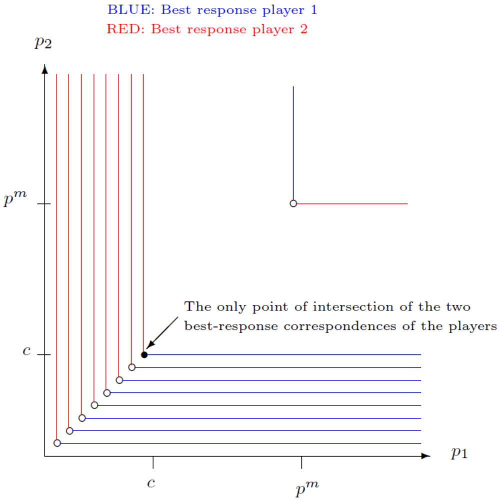
- The intersection of these two best-response correspondences gives the Nash equilibrium of the Bertrand game.
- A Nash equilibrium is a pair of prices such that no firm can increase profits by unilaterally changing price.

Application of Nash equilibrium: Bertrand duopoly IV

- If both firms were to set a price $p > MC$, each firm would earn $\frac{1}{2}(p - c)D(p)$.
- However, by setting a slightly smaller price, one of the firms would be able to almost double its profits to $(p - \varepsilon - c)D(p - \varepsilon)$.
- Thus, the only possible equilibrium price is $p_1 = p_2 = c$.

Application of Nash equilibrium: Bertrand duopoly V

The two players' best-response correspondences in the Bertrand duopoly:



Alternative method to find Bertrand-Nash equilibrium I

- One can find the Nash equilibrium of the Bertrand game by first constructing the best response correspondences of both firms and then finding the equilibrium by determining the point at which the two best response correspondences intersect (as we did above).
- However, sometimes one can make direct arguments that avoid to construct the entire best response correspondences.
- That is, sometimes one can find the solution by trial and error.
- Making a complete case distinction and checking whether or not the given constellation is a Nash equilibrium.
- In the above Bertrand duopoly there are six possible equilibrium configurations:

  1. $p_1 > p_2 > c$: This is not an equilibrium. At these prices firm 1's sales and profits are both zero. Firm 1 could profitably deviate by setting $p_1 = p_2 - \varepsilon$ where ε is very small. Firm 1's profits would increase to $\pi_1 = D(p_2 - \varepsilon)(p_2 - \varepsilon - c) > 0$ for small ε.

Alternative method to find Bertrand-Nash equilibrium II

2. $p_2 > p_1 > c$. Similar to case 1.
3. $p_1 > p_2 = c$: This is not an equilibrium. Firm 2 captures the entire market, but its profits are zero. Firm 2 could profitably deviate by setting $p_2 = p_1 - \varepsilon$ where ε is very small. Firm 2's profits would increase to $\pi_2 = D(p_1 - \varepsilon)(p_1 - \varepsilon - c) > 0$ for small ε.
4. $p_2 > p_1 = c$. Similar to case 3.
5. $p_1 = p_2 > c$. This is not an equilibrium since either firm (say, firm 1) could profitably deviate by setting $p_1 = p_2 - \varepsilon$ where ε is very small. Then, instead of sharing the market equally with firm 2 and earning profits of $\pi_1 = \frac{1}{2}D(p_1)(p_1 - c)$, firm 1 would capture the entire market, with sales of $D(p_2 - \varepsilon)$ and profits of $\pi_1 = D(p_2 - \varepsilon)(p_2 - \varepsilon - c) > 0$. For small ε this almost doubles firm 1's sales and increases profits.
6. $p_1 = p_2 = c$. This is a Nash equilibrium. Neither firm can profitably deviate and earn greater profits even though in equilibrium profits are zero. If a firm raises its price, its sales falls to zero and its profits remain at zero. Charging a lower price increases sales and ensures a market share of 100%, but it also reduces profits since price falls below unit costs.

Conclusions on the Bertrand model I

- Even with two firms we get the competitive outcome ($P = MC$).
- High-cost firms cannot survive.
- Allocative and productive efficiency are maximized.
- Price-competition gives less market power than quantity-competition.
  - This holds (quite) generally.
- The prediction of the Bertrand model (i.e., $P = MC$) does not seem very realistic though.
- Factors that make the “Bertrand paradox” (i.e., $P = MC$ with just two firms) disappear:
  - Product differentiation: undercutting price does not guarantee a firm total market demand.
  - Capacity constraints: by undercutting the rival, a firm receives all the market demand. But what if it does not have sufficient capacity to satisfy all of this demand?
  - Asymmetry (e.g., $c_1 < c_2$).

Price Competition with Differentiated Products I

Representative consumer model of product differentiation

- A standard way of modeling inverse demand for two "horizontally" differentiated products:

$$\begin{aligned} p_1 &= a - b(q_1 + \theta q_2) && (2) \\ p_2 &= a - b(q_2 + \theta q_1) && (3) \end{aligned}$$

where $a, b > 0$, and the product differentiation parameter $\theta \in [0, 1]$ measures the degree of substitutability between the two products.[1]

  - For $\theta = 0$, the two products are unrelated to each other (completely different products).
  - For $\theta = 1$, the two demand equations become identical and reduce to a homogeneous product demand equation (the products are perfect substitutes).
  - For $\theta \in (0, 1)$, the closer θ is to one, the greater the degree of substitutability between the two products.

Price Competition with Differentiated Products II

- A specific example: Let $a = 100$, $b = 1$, $\theta \in (0, 1)$.
  - Then, the inverse demand functions read

$$p_1 = 100 - (q_1 + \theta q_2) \quad \text{and} \quad p_2 = 100 - (q_2 + \theta q_1) \,. \tag{4}$$

  - The direct demand functions are (solve system (4) for quantities):

$$q_1 = \left(\frac{100(1-\theta) - p_1 + \theta p_2}{1-\theta^2} \right) \quad \text{and} \quad q_2 = \left(\frac{100(1-\theta) - p_2 + \theta p_1}{1-\theta^2} \right) . \tag{5}$$

    - Lower price by the competitor does not imply zero sales.

Competition in prices (Bertrand):

- Let firms' cost functions be $C_i(q_i) = 10q_i$, $(i = 1, 2)$.
- From (5) we derive firm 1's profit function:

$$\pi_1 = (p_1 - 10)q_1 = (p_1 - 10) \left(\frac{100(1-\theta) - p_1 + \theta p_2}{1-\theta^2} \right) .$$

Price Competition with Differentiated Products III

- The FOC reads

$$\frac{\partial \pi_1}{\partial p_1} = (1) \times \left(\frac{100(1-\theta) - p_1 + \theta p_2}{1-\theta^2} \right) + (p_1 - 10) \left(-\frac{1}{1-\theta^2} \right)$$

$$= \left(\frac{110 - 100\theta - 2p_1 + \theta p_2}{1-\theta^2} \right) \overset{!}{=} 0.$$

- Solving the FOC for p_1 gives firm 1's best-response function (likewise for firm 2):

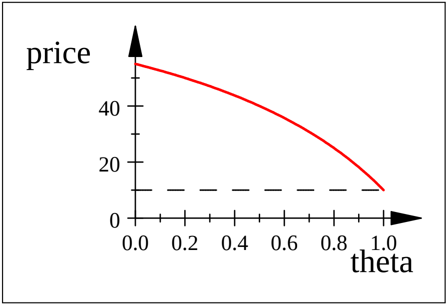
$$p_1 = \frac{1}{2}\left(110 - 100\theta + \theta p_2\right) \quad \text{and} \quad p_2 = \frac{1}{2}\left(110 - 100\theta + \theta p_1\right). \quad (6)$$

- Hence, reaction functions slope upward ("strategic complements").

Price Competition with Differentiated Products IV

- From the system of reaction functions (6), we get the Nash equilibrium prices of this Bertrand game:

$$p_1^B = p_2^B = \frac{1}{2-\theta}\left(110 - 100\theta\right). \tag{7}$$



Equilbrium prices of the Bertrand game as a function of θ.

Price Competition with Differentiated Products V

- Note that for $\theta \rightarrow 1$, the two products become perfect substitutes and we get $p_1^B = p_2^B = 10 = MC$.
- Hence, as before, price equals marginal costs when products are homogenous.
- Inserting the Nash equilibrium prices into the demand functions (5), gives equilibrium quantities

$$q_1^B = q_2^B = \frac{90}{(\theta + 1)(2 - \theta)}. \tag{8}$$

---

[1] The inverse demand equations (2) and (3) can be derived from an individual's utility function of the form $U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}b(q_1^2 + 2\theta q_1 q_2 + q_2^2)$, subject to a budget constraint.

Location model of product differentiation I

Hotelling's "Beach" model

- Two ice-cream vendors, A and B, are on a beach of fixed length (one mile).
- Bathers are uniformly distributed over the beach.
- Bathers buy from the nearest vendor.

Questions

- Taking prices as exogenously given, where do A & B locate?
- Taking location as exogenously given, what prices do A & B charge?
- What happens when A & B can choose both prices and locations endogenously?

Taking prices as exogenously given, where do A and B locate? I

- Bathers are uniformly distributed over the beach ranging from 0 to 1 (mile) and have mass 1.
- Assume vendors can only charge the (regulated) price $\overline{p}$ for the ice cream bars they sell.
- Therefore, vendors have to decide where to locate on the beach. They choose locations l_A, $l_B \in [0, 1]$.
- Bathers incur a linear transportation cost of $t|x - l_i|$, where x is the consumer's location and $l_i \in [0, 1]$ the location of vendor $i = A, B$.
- Bather located at x derives utility $V - t|x - l_i| - \overline{p}$, where V is the reservation value of a consumer for a product located at l_i, and $\overline{p}$ is the regulated price.
- We assume that V is sufficiently large such that all bathers purchase.
- Vendors have no production costs.

Taking prices as exogenously given, where do A and B locate? II

- Vendors decide where to locate on the beach and bathers then decide from which vendor to buy.
- For any locations $l_A < l_B$, there is a bather who is indifferent between the two vendors (locations).
- This bather is located in the center between the two vendors, $\widehat{x} = (l_A + l_B)/2$.
- All bathers to the left of the indifferent bather buy from vendor A, all bathers to the right of the indifferent bather buy from vendor B.
- Thus, the demand for vendor A is

$$Q_A(l_A, l_B) = (l_A + l_B)/2$$

and the demand for vendor B is

$$Q_B(l_A, l_B) = 1 - (l_A + l_B)/2.$$

Taking prices as exogenously given, where do A and B locate? III

- Vendors maximize profits with respect to their location given the location of the other vendor.
- Hence, we can define the following normal form game:
    - Set of players: $\{A, B\}$, the two vendors
    - Strategy sets: $S_i = [0, 1]$, $i = 1, 2$ (the set of possible locations)
    - Profits of vendor $i = A, B$ are given as

$$\pi_i(l_i, l_j) = \begin{cases} \overline{p}(l_i + l_j)/2 & \text{if} \quad l_i < l_j \\ \\ \frac{1}{2}\overline{p} & \text{if} \quad l_i = l_j \\ \\ \overline{p}\,(1 - (l_i + l_j)/2) & \text{if} \quad l_i > l_j. \end{cases}$$

- The unique Nash equilibrium of this location game is $l_A = l_B = \frac{1}{2}$.
    - In this candidate equilibrium, each vendor serves half of the market.

Taking prices as exogenously given, where do A and B locate? IV

- Any vendor that deviates will serve less than half of the market.
- Note also that for any different locations $l_A \neq 1/2$ and $l_B \neq 1/2$, at least one vendor has an incentive to deviate.
    - For instance, if vendor A is located to the left of 1/2, vendor B has an incentive to move slightly to the right of vendor A.
    - But this cannot be an equilibrium either because vendor A increases its profit by locating at the same place as vendor B.
    - Hence, the only location equilibrium is that both vendors locate at the center.

- The main insight of this model is that although firms can choose to (spatially) differentiate their products, they choose not to do so.
- This is what is called "principle of minimum differentiation."
- Note that in terms of total transportation cost, it would be more efficient if the two vendors would locate at 1/4 and 3/4.

Taking prices as exogenously given, where do A and B locate? V

- Food for thought: Under otherwise unchanged assumptions of this pure location model, what would be the location equilibrium in case there are $3, 4, 5$ or more vendors?

Interpretations of the location model ("transportation costs") I

- Literal interpretation:
  - Geographic dimension of a product.
  - What is the nearest location where a consumer can buy.
- "Figurative" interpretation:
  - Firms have to decide to which consumer tastes to tailor their product to.
  - A consumer's location describes his ideal point in the product space ranging
    - from early departure of a flight (location 0) to a late departure of a flight (location 1),
    - from sweet (location 0) to bitter (location 1),
    - from neutral (location 0) to very spicy (location 1),
    - from a left political orientation (location 0) of a newspaper to a right political orientation (location 1).

Interpretations of the location model ("transportation costs") II

- Consumers prefer products that are "close" to their preferred types in space, or time or characteristics.
- The "location" x of a consumer denotes his ideal taste and $t|x - l_i|$ his dis-utility from having to consume a less than ideal product.

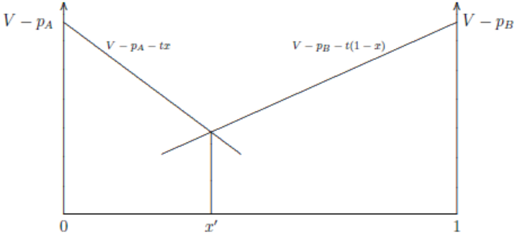
Taking location as exogenously given, what prices do A & B charge? I

- Assume the vendors are at either end of the beach, say, vendor A at 0 and vendor B at 1.
- Production costs are zero.
- Value of (ideal) product (ice cream) for a consumer is $V > 0$.
- Consumer at location x, when prices of the two vendors A and B are p_A and p_B, will buy from A if

$$V - p_A - tx > V - p_B - t(1 - x)$$

where, again, t is the per-unit-of-distance cost of being away from your ideal.

Taking location as exogenously given, what prices do A & B charge? II



The loaction x' of the indifferent consumer

- Given prices p_A and p_B the consumer at location x' is indifferent between A and B if $V - p_A - tx' = V - p_B - t(1 - x')$, that is, $x' = (t + p_B - p_A)/2t$.

Taking location as exogenously given, what prices do A & B charge? III

- All consumers to the left of x' buy from firm A; all consumers to the right of x' buy from firm B.
- Hence, profit for firm A is:

$$\pi(p_A) = p_A \times x' = p_A \times (t + p_B - p_A)/2t.$$

- FOC:

$$\frac{\partial \pi(p_A)}{\partial p_A} = t + p_B - 2p_A = 0.$$

- Firm A's best response function is

$$p_A = (t + p_B)/2.$$

- Similarly: profit for firm B is:

$$\pi(p_B) = p_B \times (1 - x') = p_B \times (1 - (t + p_B - p_A)/2t).$$

Taking location as exogenously given, what prices do A & B charge? IV

- FOC:

$$\frac{\partial \pi(p_B)}{\partial p_B} = 1 - (t + 2p_B - p_A)/2t = \left(t - 2p_B + p_A\right)/2t = 0$$

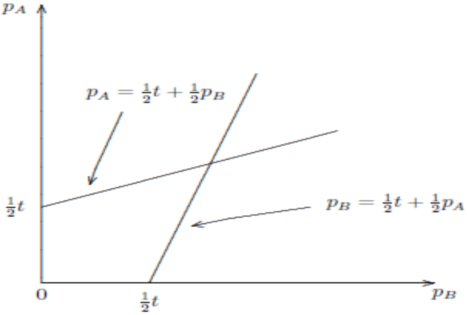
- Firm B's best response function is

$$p_B = (t + p_A)/2.$$

- Nash equilibrium:

$$p_A = p_B = t, \quad \pi_A = \pi_B = \frac{1}{2}t > 0.$$

Taking location as exogenously given, what prices do A & B charge? V



The best-response functions

Wrap up: How to find a Nash equilibrium of a normal form game?

- Determine best replies / reaction functions (use FOCs) and determine intersections. (Example: Cournot duopoly).
- Trial and error (case distinction). Ask whether certain configurations can be a Nash equilibrium. (Example: homogenous-good Bertrand duopoly with constant unit costs).
- Guess an equilibrium and confirm. Check for (non-)uniqueness.

The relation between iteratively dominated strategies and the Nash equilibrium of a game I

(From the Gibbons book, page 12ff.)

Proposition

In the normal-form game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$, *if the strategies* $(s_1^*, ..., s_I^*)$ *are a Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.*

Proof: Suppose that the strategies $(s_1^*, ..., s_I^*)$ are a Nash equilibrium of game Γ_N, but suppose that (perhaps after some strategies other than $(s_1^*, ..., s_I^*)$ have been eliminated) s_i^* is the first of the strategies $(s_1^*, ..., s_I^*)$ to be eliminated for being strictly dominated. Then there must exist a strategy s_i'' that has not yet been eliminated from S_i that strictly dominates s_i^*. That is, it must hold that $u_i(s_i^*, s_{-i}) < u_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$ that have not yet been eliminated from the other players' strategy spaces. Since s_i^* is the first of the equilibrium strategies to be

The relation between iteratively dominated strategies and the Nash equilibrium of a game II

eliminated, the other players' equilibrium strategies have not yet been eliminated, so that the last inequality implies that $u_i(s_i^*, s_{-i}^*) < u_i(s_i'', s_{-i}^*)$. But this contradicts $(s_1^*, ..., s_I^*)$ being a Nash equilibrium of game Γ_N.

Proposition

In the normal-form game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$, *if iterated elimination of strictly dominated strategies eliminates all but the strategies* $(s_1^*, ..., s_I^*)$, *then these strategies are the unique Nash equilibrium of the game.*

Proof: Uniqueness follows from the previous Proposition as any other Nash equilibrium would have survived as well. So, we only have to show that if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, ..., s_I^*)$, then these strategies are a Nash equilibrium.

The relation between iteratively dominated strategies and the Nash equilibrium of a game III

Suppose to the contrary that iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, ..., s_I^*)$, but these strategies are not a Nash equilibrium. Then there must exist some player i and some strategy $s_i \in S_i$ such that

$$u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*) \tag{9}$$

and (since s_i does not survive the process) another strategy s_i' such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \tag{10}$$

for all $s_{-i} \in S_{-i}$ remaining in the other players' strategy sets at this stage of the process. Since the other players' strategies s_{-i}^* are never eliminated, (10) implies that

$$u_i(s_i', s_{-i}^*) > u_i(s_i, s_{-i}^*). \tag{11}$$

The relation between iteratively dominated strategies and the Nash equilibrium of a game IV

If $s_i' = s_i^*$, then (11) contradicts (9) and the proof is complete. If $s_i' \neq s_i^*$, then some other strategy s_i'' must later strictly dominate s_i', since s_i' does not survive the process. Thus, inequalities analogous to (10) and (11) hold with s_i' and s_i'' replacing s_i and s_i', respectively. Once again, if $s_i'' = s_i^*$, then the proof is complete; otherwise, two more analogous inequalities can be constructed. Since s_i^* is the only strategy from S_i to survive the process, repeating this argument in a finite game eventually completes the proof.

The relation between iteratively dominated strategies and the Nash equilibrium of a game V

Remarks:

- Note that the first of the last two propositions is wrong if we consider weakly dominated strategies. Counter example:

		Player 2		
		l	m	r
Player 1	o	1, 1	0, 1	0, 1
	m	0, 0	1, 0	0, 1
	u	1, 0	0, 1	1, 0

- Here, (o, l) is the unique Nash equilibrium of the game.
- However, it is an equilibrium in weakly dominated strategies and, hence, the equilibrium strategies would be eliminated if iterated elimination of weakly dominated strategies would be applied to this game.

Mixed strategies I

- Consider the following game ("Matching Pennies"):

		Player 2	
		H(eads)	T(ails)
Player 1	H(eads)	-1, 1	1, -1
	T(ails)	1, -1	-1, 1

- This game has no Nash equilibrium in pure strategies.
- What now? Is the Nash equilibrium not a useful solution concept for general normal form games?
- Note that so far players chose a specific strategy with probability 1. What if we allow strategies to be played with a certain probability smaller than 1? $\curvearrowright$ *mixed strategies.*

Mixed strategies II

Definition

Given player i's (finite) pure strategy set S_i, a **mixed strategy** for player i, $\sigma_i : S_i \to [0,1]$, assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played with $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

- The set of all mixed strategies of player i will be denoted $\Delta(S_i)$.
- Let $\Delta(S) = \Delta(S_1) \times ... \times \Delta(S_I)$ be the space of all mixed strategy *profiles*. Typical element $\sigma \in \Delta(S)$.
- **Example:** In the above Matching Pennies game, $\sigma_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and $\sigma_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$ are mixed strategies, among many others.
- **Remark:** Every pure strategy is a (trivial) mixed strategy according to which the pure strategy considered is played with probability 1 and all others with probability 0.

Mixed strategies III

Example:

- Assume that in the Matching Pennies game players use the mixed strategies $\sigma_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and $\sigma_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$. The strategies induce the following probability distribution over the set of strategy profiles:

		Player 2	
		H	T
Player 1	H	$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$
	T	$\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$	$\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

- Thus, player 1's expected payoff is:

$$\begin{aligned} u_1(\sigma) &= \left[\frac{1}{3} \cdot \frac{2}{3}\right](-1) + \left[\frac{1}{3} \cdot \frac{1}{3}\right](1) + \left[\frac{2}{3} \cdot \frac{2}{3}\right](1) + \left[\frac{2}{3} \cdot \frac{1}{3}\right](-1) \\ &= \frac{2}{9}(-1) + \frac{1}{9}(1) + \frac{4}{9}(1) + \frac{2}{9}(-1) = \frac{1}{9}. \end{aligned}$$

Mixed strategies IV

- If we deal with mixed strategies we have to consider expected payoffs. Let $\sigma = (\sigma_1, ..., \sigma_I)$ be a mixed strategy profile. Then

$$u_i(\sigma) = \sum_{s \in S} [\sigma_1(s_1)\sigma_2(s_2)...\sigma_I(s_I)] u_i(s)$$

  is the expected payoff of player i.

Generalization of the Nash equilibrium concept I

A mixed strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_I^*)$ constitutes a **Nash equilibrium** of game $\Gamma_N = [I; \{\Delta(S_i)\}; \{u_i(\cdot)\}]$ if for every $i = 1, ..., I$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \text{ for all } \sigma_i' \in \Delta(S_i).$$

[That is, σ_i^* is a best reply to σ_{-i}^* or σ_i^* is a solution to the problem $\max\limits_{\sigma_i' \in \Delta(S_i)} u_i(\sigma_i', \sigma_{-i}^*)$.]

Generalization of the Nash equilibrium concept II

Definition

The correspondence
$b_i : \Delta(S_1) \times ... \times \Delta(S_{i-1}) \times \Delta(S_{i+1}) \times ... \times \Delta(S_I) \rightrightarrows \Delta(S_i)$,

$$b_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) \mid u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(S_i)\},$$

that assigns to every player i and every $\sigma_{-i} \in \times\Delta(S_j)_{j \neq i}$ the set of best replies of this player to σ_{-i}, is called **best-response correspondence** of player i.

Alternative characterization of the Nash equilibrium in mixed strategies:

Proposition

The strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ *is a Nash equilibrium of the normal form game* $\Gamma_N = [I; \{\Delta(S_i)\}; \{u_i(\cdot)\}]$ *if and only if for all* $i = 1, ..., I$ *we have* $\sigma_i \in b_i(\sigma_{-i})$.

Finding Nash equilibria in mixed strategies I

- Consider the following game:

		Player 2	
		F	C
Player 1	N	0, 2	0, 2
	E	−1, −1	1, 1

- For notational simplicity, represent any mixed strategy $\sigma_1 = (\sigma_{1N}, \sigma_{1E})$ of player 1 by $\sigma_{1N} \in [0, 1]$ — the weight associated with pure strategy N $(\sigma_{1E} = 1 - \sigma_{1N})$.
- Similarly, let any mixed strategy $\sigma_2 = (\sigma_{2F}, \sigma_{2C})$ of player 2 be represented by $\sigma_{2F} \in [0, 1]$.
- Then, each player i's best-response correspondence can be defined as a one-dimensional mapping $b_i : [0, 1] \rightrightarrows [0, 1]$.

Finding Nash equilibria in mixed strategies II

- For player 1, it is immediate to compute that

$$\begin{aligned} u_1(N, \sigma_2) & \gtreqless u_1(E, \sigma_2) \\ \sigma_{2F} \cdot 0 + (1 - \sigma_{2F}) \cdot 0 & \gtreqless \sigma_{2F} \cdot (-1) + (1 - \sigma_{2F}) \cdot 1 \\ \sigma_{2F} & \gtreqless \frac{1}{2}, \end{aligned}$$

and, therefore,

$$\sigma_{1N} = b_1(\sigma_2) = \begin{cases} 0 & \text{if} \quad \sigma_{2F} < 1/2 \\ [0,1] & \text{if} \quad \sigma_{2F} = 1/2 \\ 1 & \text{if} \quad \sigma_{2F} > 1/2. \end{cases}$$

Finding Nash equilibria in mixed strategies III

- On the other hand, for player 2 we have

$$\begin{aligned} \sigma_{1N} &< 1 \Rightarrow u_2(\sigma_1, F) < u_2(\sigma_1, C) \\ \sigma_{1N} &= 1 \Rightarrow u_2(\sigma_1, F) = u_2(\sigma_1, C), \end{aligned}$$

which implies that

$$\sigma_{2F} = b_2(\sigma_1) = \begin{cases} 0 & \text{if} \quad \sigma_{1N} < 1 \\ [0,1] & \text{if} \quad \sigma_{1N} = 1. \end{cases}$$

- From the Proposition above, a Nash equilibrium is a pair of mixed strategies $\sigma^* = ((\sigma^*_{1N}, 1 - \sigma^*_{1N}), (\sigma^*_{2F}, 1 - \sigma^*_{2F}))$ such that

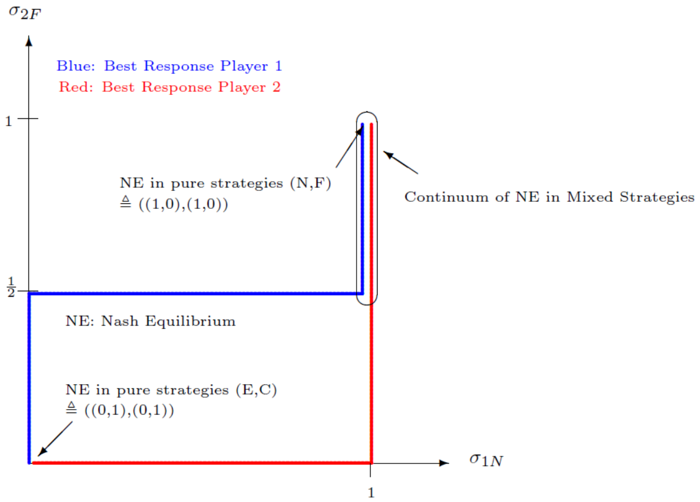
$$\sigma^*_{1N} \in b_1(\sigma^*_{2F}) \text{ and } \sigma^*_{2F} \in b_2(\sigma^*_{1N}).$$

Finding Nash equilibria in mixed strategies IV

- An easy way of identifying those equilibrium configurations is to plot the two best-response correspondences on the same $\sigma_{2F} - \sigma_{1N}$ plane (with, say, b_2 "rotated" 90 degrees) and look for points of intersection. (See the next slide.)
- Doing so, we find
  - a Nash equilibrium in pure strategies: $((0,1),(0,1))$ and
  - a continuum of Nash equilibria in mixed strategies: $\{((\sigma_{1N},\sigma_{1E}),(\sigma_{2F},\sigma_{2C})) \mid \sigma_{1N}=1,\ \sigma_{2F}\geq 1/2\}$.
    - Note that this continuum also contains the equilibrium in pure strategies: $((1,0),(1,0))$.

Finding Nash equilibria in mixed strategies V

The best-response correspondences of the game on slide 82:



A proposition characterizing mixed Nash equilibria I

Proposition

Let $S_i^+ \subseteq S_i$ *denote the set of pure strategies that player* i *plays with positive probability in mixed strategy profile* $\sigma = (\sigma_1, ..., \sigma_I)$. *Strategy profile* $\sigma = (\sigma_1, ..., \sigma_I)$ *is a Nash equilibrium in game* $\Gamma_N = [I; \{\Delta(S_i)\};$ $\{u_i(\cdot)\}]$ *if and only if for all* $i = 1, ..., I$,
(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ *for all* $s_i, s_i' \in S_i^+$;
(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ *for all* $s_i \in S_i^+$ *and for all* $s_i' \notin S_i^+$.

Proof: "$\Rightarrow$": Assume $\sigma = (\sigma_1, ..., \sigma_I)$ is a Nash equilibrium and the conditions in (i) and (ii) are violated for some player i. Then there is a strategy $s_i \in S_i^+$ and a strategy $s_i' \in S_i$ with $u(s_i', \sigma_{-i}) > u(s_i, \sigma_{-i})$. However, in this case player i can increase her expected payoff by playing strategy s_i' whenever she would have played strategy s_i. Contradiction to σ being a NE.

A proposition characterizing mixed Nash equilibria II

"$\Leftarrow$": Suppose (i) and (ii) hold but σ is not a Nash equilibrium. Then there is some player i who has a strategy σ_i' with $u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$. But if so, then there must be some pure strategy s_i' with $\sigma_i'(s_i') > 0$ for which $u_i(s_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$. Since $u_i(\sigma_i, \sigma_{-i}) = u_i(s_i, \sigma_{-i})$ for all $s_i \in S_i^+$, it would follow that $u_i(s_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i})$ for all $s_i \in S_i^+$. But this contradicts conditions (i) and (ii) being satisfied.

Significance of this Proposition:

- In a NE, the "mixing" of player i's rivals is such that player i is indifferent between choosing any of his pure strategies to which player i assigns positive probability in the NE.

Attention: In the following, I refer to this Proposition as Proposition "CMS-NE", short for "characterizing mixed-strategy Nash equilibria."

Applying Proposition CMS-NE: Example 1, I

- Let's find all Nash equilibria of the "Matching Pennies" game.

		Player 2	
		H(eads)	T(ails)
Player 1	H(eads)	$-1, 1$	$1, -1$
	T(ails)	$1, -1$	$-1, 1$

- There are no pure-strategy Nash equilibria, and no equilibria in which one player plays a pure strategy and the other mixes.
- So, let's look for a Nash equilibrium in mixed strategies.
- Let player 2's strategy be $\sigma_2 = (q, 1-q)$, $0 \leq q \leq 1$. It holds that:

$$
\begin{aligned}
u_1(H, \sigma_2) &= -1q + 1(1-q) = -2q + 1 \\
u_1(T, \sigma_2) &= 1q - 1(1-q) = 2q - 1
\end{aligned}
$$

Applying Proposition CMS-NE: Example 1, II

- If there is a Nash equilibrium in which player 1 randomizes between both of her pure strategies, she must be indifferent between the two:

$$-2q + 1 = 2q - 1 \quad \text{iff} \quad q = 1/2.$$

- Let player 1's strategy be $\sigma_1 = (p, 1-p)$, $0 \leq p \leq 1$. It holds that:

$$\begin{aligned} u_2(\sigma_1, H) &= 1p - 1(1-p) = 2p - 1 \\ \\ u_2(\sigma_1, T) &= -1p + 1(1-p) = -2p + 1 \end{aligned}$$

- If there is a Nash equilibrium in which player 2 randomizes between both of his pure strategies, he must be indifferent between the two:

$$2p - 1 = -2p + 1 \quad \text{iff} \quad p = 1/2.$$

- Hence, $\sigma = (\sigma_1, \sigma_2) = ((1/2, 1/2), (1/2, 1/2))$ is the (unique) Nash equilibrium of the "Matching Pennies" game.

Applying Proposition CMS-NE: Example 2, I

- Let's find the set of Nash equilibria of the following game:

		Player 2		
		L	C	R
Player 1	T	2, 0	1, 1	2, 2
	M	1, 3	0, 2	1, 0
	B	1, 4	2, 1	3, 3

- We know already that strategies that are iteratively strictly dominated, are never played in a Nash equilibrium. Therefore, always get rid of those strategies first.
- In the above game, strategy M of player 1 is strictly dominated by strategy T. After deleting this strategy, strategy C of player 2 is strictly dominated by strategy R of this player.

Applying Proposition CMS-NE: Example 2, II

- In the resulting game, namely

		Player 2	
		L	R
Player 1	T	2, 0	2, 2
	B	1, 4	3, 3

  no strategy is strictly dominated for any player. Therefore, the set of Nash equilibria of the original game, can be found by finding the Nash equilibria of this game.

- It can be easily seen that the 2×2 game has no Nash equilibrium in pure strategies. By using the same procedure as in the "Matching Pennies" game, one can verify that the above 2×2 game has a Nash equilibrium in mixed strategies in which player 1 plays her strategy T with probability $1/3$ and her strategy B with probability $2/3$ whereas player 2 plays both of his strategies with probability $1/2$.

Applying Proposition CMS-NE: Example 2, III

- Thus, in the original 3×3 game the unique Nash equilibrium in mixed strategies is $\sigma = (\sigma_1, \sigma_2) = ((1/3, 0, 2/3), (1/2, 0, 1/2))$.

Applying Proposition CMS-NE: Example 3, I

- Let's consider yet another example:

		Player2	
		L	R
Player 1	T	$2, \quad 2$	$2, 2$
	B	$3, -1$	$0, 0$

- One can easily verify that in this game (A, E) is the only Nash equilibrium in pure strategies.

Applying Proposition CMS-NE: Example 3, II

- Note that player 2 is indifferent between his two pure strategies as long as player 1 plays T with probability 1. According to Proposition CMS-NE, player 1 plays her pure strategy T as long as the expected payoff from playing T is larger or equal to the expected payoff from playing B. Let player 2's strategy be $\sigma_1 = (q, 1-q)$, $0 \leq p \leq 1$.

$$U_1(T, \sigma_2) = 2q + 2(1-q) = 2$$

$$U_1(B, \sigma_2) = 3q + 0 \cdot (1-q) = 3q$$

- The above condition translates into $U_1(T, \sigma_2) \geq U_1(B, \sigma_2)$ iff $2 \geq 3q$ iff $q \leq 2/3$.
- Thus, the set of all Nash equilibria of the above game is $\sigma = (\sigma_1, \sigma_2) = ((1, 0), (q, 1-q))$ with $q \leq 2/3$. (In other words, the above game has a continuum of Nash equilibria in mixed strategies.)

Applying Proposition CMS-NE: A systematic procedure I

The following is a systematic procedure to find all mixed strategy Nash equilibria of a finite bi-matrix game. It relies on Proposition CMS-NE:

- For each player i, choose a subset A_i of her set S_i of pure strategies.
- Check whether there exists a mixed strategy profile σ such that (i) the set of pure strategies to which each strategy σ_i assigns positive probability is A_i and (ii) σ satisfies the conditions in Proposition CMS-NE.
- Repeat the analysis for **every** collection of subsets of the players' sets of pure strategies.

Applying Proposition CMS-NE: A systematic procedure II

- Consider the following variant of what is pompously called a “Battle of the Sexes” (BoS) game:

		Player 2		
		B	S	X
	B	4, 2	0, 0	0, 1
Player 1	S	0, 0	2, 4	1, 3

- First, by inspection we see that the game has two pure strategy Nash equilibria, namely (B, B) and (S, S).
- Now consider the possibility of an equilibrium in which player 1’s strategy is pure, whereas player 2’s strategy assigns positive probability to two or more actions.
  - If player 1’s strategy is B then player 2’s payoffs to her three actions (2, 0, and 1) are all different, so the first condition in Proposition CMS-NE is not satisfied. Thus there is no equilibrium of this type.

Applying Proposition CMS-NE: A systematic procedure III

  - Similar reasoning rules out an equilibrium in which player 1's strategy is S and player 2's strategy assigns positive probability to more than one action, and also an equilibrium in which player 2's strategy is pure and player 1's strategy assigns positive probability to both of her actions.
- Next consider the possibility of an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to two of her three actions. Denote by p the probability player 1's strategy assigns to B, so $\sigma_1 = (p, 1-p)$. There are three possibilities for the pair of player 2's actions that have positive probability.

Applying Proposition CMS-NE: A systematic procedure IV

- *B and S:* For the conditions in Proposition CMS-NE to be satisfied we need player 2's expected payoff to B to be equal to her expected payoff to S and at least her expected payoff to X. That is, we need

  $$\begin{array}{rcl} u_2(\sigma_1, B) & = & u_2(\sigma_1, S) \geq u_2(\sigma_1, X) \\ & \Leftrightarrow & \\ 2p & = & 4(1-p) \geq p + 3(1-p). \end{array}$$

  The equation implies that $p = 2/3$, which does not satisfy the inequality. (That is, if p is such that B and S yield the same expected payoff, then X yields a higher expected payoff.) Thus there is no equilibrium of this type.

Applying Proposition CMS-NE: A systematic procedure V

- *B and X:* For the conditions in Proposition CMS-NE to be satisfied we need player 2's expected payoff to B to be equal to her expected payoff to X and at least her expected payoff to S. That is, we need

$$\begin{aligned} u_2(\sigma_1, B) &= u_2(\sigma_1, X) \geq u_2(\sigma_1, S) \\ &\Leftrightarrow \\ 2p &= p + 3(1-p) \geq 4(1-p). \end{aligned}$$

  The equation implies that $p = 3/4$, which satisfies the inequality. For the first condition in Proposition CMS-NE to be satisfied for player 1 we need player 1's expected payoffs to B and S to be equal, assuming player 2 uses the strategy $\sigma_2 = (q, 0, 1-q)$: The condition $u_1(B, \sigma_2) = u_2(S, \sigma_2)$ is equivalent to $4q = 1 - q$, where q is the probability player 2 assigns to B, or $q = 1/5$. Thus the pair of mixed strategies $((3/4, 1/4), (1/5, 0, 4/5))$ is a mixed strategy equilibrium.
- *S and X:* For every strategy of player 2 that assigns positive probability only to S and X, player 1's expected payoff to S exceeds her expected payoff to B. Thus there is no equilibrium of this sort.

Applying Proposition CMS-NE: A systematic procedure VI

- The final possibility is that there is an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to all three of her actions. Let p be the probability player 1's strategy assigns to B, so $\sigma_1 = (p, 1-p)$. Then for player 2's expected payoffs to her three actions to be equal we need

  $$\begin{array}{rcl} u_2(\sigma_1, B) & = & u_2(\sigma_1, S) = u_2(\sigma_1, X) \\ & \Leftrightarrow & \\ 2p & = & 4(1-p) = p + 3(1-p). \end{array}$$

  For the first equality we need $p = 2/3$, violating the second equality. That is, there is no value of p for which player 2's expected payoffs to her three actions are equal, and thus no equilibrium in which she chooses each action with positive probability.

Applying Proposition CMS-NE: A systematic procedure VII

- We conclude that the game has three mixed strategy equilibria:
    - $((1,0),(1,0,0))$ (i.e. the pure strategy equilibrium (B,B)),
    - $((0,1),(0,1,0))$ (i.e. the pure strategy equilibrium (S,S)), and
    - $((3/4,1/4),(1/5,0,4/5))$.

Existence of Nash equilibria I

Proposition

Every game $\Gamma_N = [I; \{\Delta(S_i)\}; \{u_i(\cdot)\}]$ *in which the sets* $S_1, ..., S_I$ *have a finite number of elements has a mixed strategy Nash equilibrium.*

Proposition

A Nash equilibrium exists in game $\Gamma_N = [I; \{S_i\}; \{u_i(\cdot)\}]$ *if for all* $i = 1, ..., I,$
(i) S_i *is a nonempty, convex, and compact subset of some Euclidean space* $\mathbb{R}^M$.
(ii) $u_i(s_1, ..., s_I)$ *is continuous in* $(s_1, ..., s_I)$ *and quasiconcave in* s_i.

Existence of Nash equilibria II

Definition: Let $X \subseteq R^n$. A function $f : X \rightarrow \mathbb{R}$ is **quasiconcave** in x, if for all $\alpha \in \mathbb{R}$ the set $\{x \in X \mid f(x) \geq \alpha\}$ is convex. Examples of quasiconcave functions: linear functions, concave functions, all monotonically increasing functions of one variable $f : \mathbb{R} \rightarrow \mathbb{R}$.

Definition: A set is **compact** if it is closed (it contains all its limit points) and bounded (it has all its elements within some fixed and finite distance of each other).

Remarks:

- Note that the second proposition makes assumptions on the set of pure strategies S_i and the payoff functions u_i, that, if fulfilled, assure the existence of a Nash equilibrium in (pure) strategies.
- Note also that these propositions don't tell us how to find Nash equilibria. However, they provide conditions (that often can be checked easily) under which the attempt of finding a Nash equilibrium is in principle not in vain.

Dynamic games of complete information I

- Also called games in extensive form
- A normal form game consists of (1) the set of players, (2) the strategy sets, and (3) the payoff functions. Furthermore, each player only moves once.
- In a dynamic game or a game in *extensive form*, players move sequentially and possibly more than once. Now we also have to specify
  - when players move,
  - what players know when they move, and
  - what exactly they can do each time they move.
- Whereas we often used bi-matrices to represent normal form games, we will often use **game trees** to represent games in extensive form.

Informal introduction of game trees I

- Nodes; branches; initial, decision and terminal nodes; payoffs (see Figure 2.1).

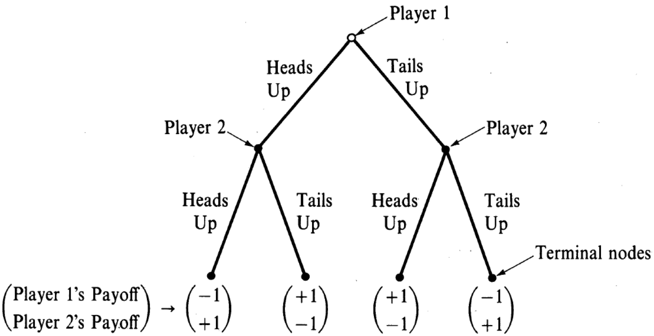


Figure 2.1: Extensive Form for Matching Pennies Version B. (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Information sets I

- An **information set** is a collection of decision nodes satisfying the following two conditions:
  - all decision nodes in this set belong to the same player, and
  - when the play of the game reaches a node in the information set, the player with the move does not know which node in the information set has (or has not) been reached.
- Information sets will be indicated in game trees by including all decision nodes belonging to an information set in a "circle". (See Figure 2.2)

Definition

A game is one of **perfect information** if each information set contains a single decision node. Otherwise, it is a game of **imperfect information**.

Information sets II

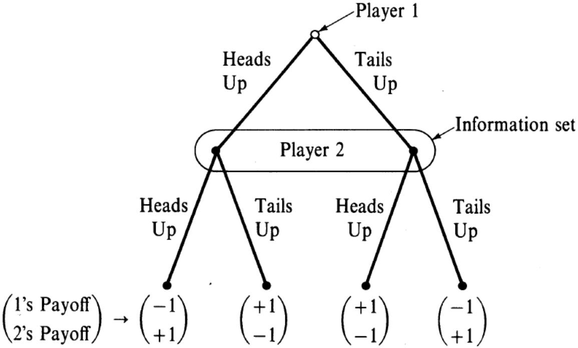


Figure 2.2: Extensive form for Matching Pennies Version C (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Games with imperfect recall I

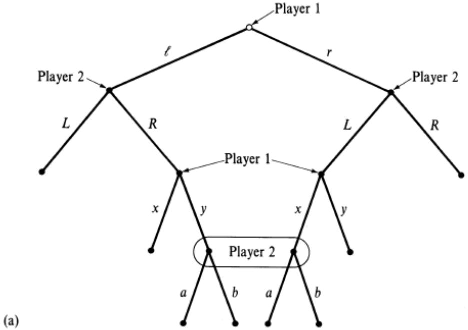


Figure 2.3 (a): A game not satisfying perfect recall (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Games with imperfect recall II

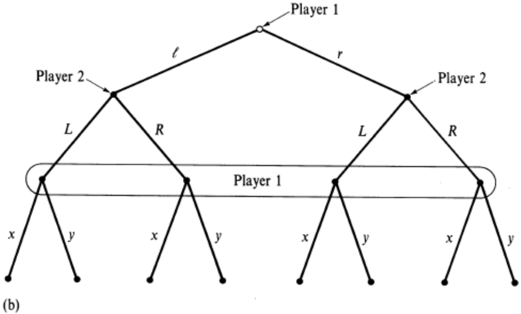


Figure 2.3 (b): Another game not satisfying perfect recall (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Definition

(informal): A game in which none of the players ever forgets what she once knew (including her own actions), is called a game with **perfect recall**.

Surely, a game in extensive form can be described much more formally ...

The concept of a strategy in an extensive form game I

- Question: In general, where does a player make a decision in a sequential game? Answer: At information sets, not necessarily at decision nodes. (The latter applies if an information set consists of only one decision node.)
- Informally: A **strategy** is a *complete* plan of action that specifies what a player will do **at each (!)** of her information sets (even at those information sets that will not be reached due to earlier moves by the player).
- Hence, a **strategy** of a player in an extensive form game is function that assigns to **each** of a player's information sets an action available to this player at this information set.
- Although it sounds quite innocent, this is a very demanding concept.
- In particular, a player's strategy must include a plan even for contingencies that her own strategy make irrelevant.

The concept of a strategy in an extensive form game II

- However, this is necessary if we (later) want to check the rationality of the strategy choice by checking whether or not the player has an incentive to deviate from his strategy.

The concept of a strategy in an extensive form game III

Example: Version B of the "matching pennies" game, Figure 2.1. What are the players' strategies in this game?

- Player 1 has only one information set. Thus, a strategy for this player is simply to specify a move at the initial node: heads (H) or tail (T).
- Player 2 has two information sets. Thus, a strategy for this player specifies how he moves at each of his two information sets: Since at each information set player 2 has two possible actions, there are four different strategies for this player:
  - *Strategy 1*: Play H if player 1 plays H; play H if player 1 plays T.
  - *Strategy 2*: Play H if player 1 plays H; play T if player 1 plays T.
  - *Strategy 3*: Play T if player 1 plays H; play H if player 1 plays T.
  - *Strategy 4*: Play T if player 1 plays H; play T if player 1 plays T.

The concept of a strategy in an extensive form game IV

Another example: Lets determine the set of strategies of both players in the game of Figure 2.3 (a):

- Player 1 has three information sets: a first at the beginning of the game, a second after his move "l" and player 2's move "R", and a third after his move "r" and player 2's move "L". At each of these information sets, player 1 has two possible actions. Hence, player 1 has the following 8 strategies:
    - (l, x, x), (l, x, y), (l, y, x), (l, y, y), (r, x, x), (r, x, y), (r, y, x), (r, y, y).
- Player 2 also has three information sets: a first after player 1's move "l", a second after player 1's move "r", and a third one further down the game tree. As player 1, at each of her information sets, player 2 has two possible actions. Hence, player 2 has the following 8 strategies:
    - (L, L, a), (L, L, b), (L, R, a), (L, R, b), (R, L, a), (R, L, b), (R, R, a), (R, R, b).

Subgames I

Definition

A **subgame** of an extensive form game Γ_E is a subset of the game having the following two properties:
(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* these nodes.
(ii) If decision node x is in the subgame, then every decision node x' that belongs to the same information set as node x is also in the subgame. (That is, there are no "broken" information sets.)

Key feature of a subgame: Each subgame is a game in its own right and we can apply to it the Nash equilibrium concept.

Subgames II

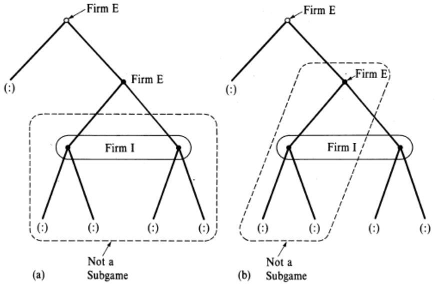


Figure 2.5 (a) and (b): Two parts of a game that are not subgames (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Subgames III

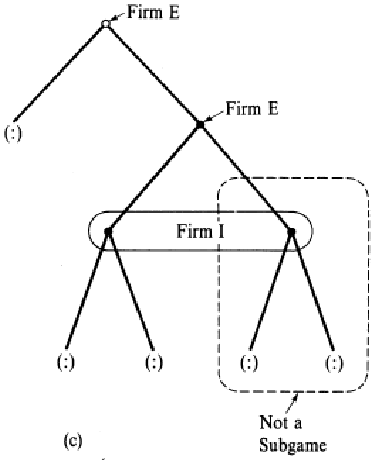


Figure 2.5 (c): A third part of the game that is not a subgame (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Analysis of games in extensive form I

Example: A market entry game (see Figure 2.6)

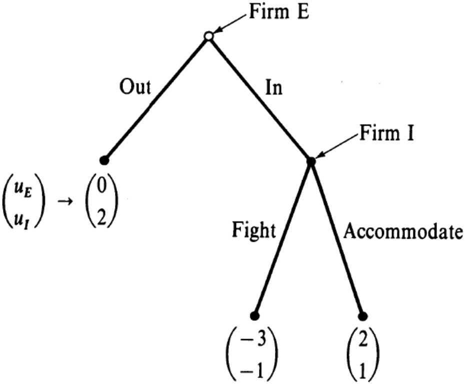


Figure 2.6: Extensive form of a market-entry game (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Analysis of games in extensive form II

How to solve a game in extensive form?

- First idea: Transform the game in extensive form into a normal form game and determine the Nash equilibria of this game.
- The normal form of the game in Figure 2.6:

		Firm I	
		Fight if firm E plays “In”	Accommodate if firm E plays “In”
Firm E	Out	0, 2	0, 2
	In	−3, −1	2, 1

- Pure-strategy equilibria of the game in Figure 2.6:
  - (Out, Fight) and (In, Accommodate)

Analysis of games in extensive form III

- The equilibrium (Out, Fight) is not a convincing prediction because it rests on an **incredible threat**:
  - In case firm I had to move, it would choose “Accommodate.”
- Problem: Behavior in information sets that are not reached when equilibrium strategies are played. The behavior in these unreached information sets does not have payoff consequences but it does influence the behavior of other players.
- To exclude incredible threats, we require that equilibrium strategies are **sequentially rational:** At each information set, the action taken by the player with the move (and the player’s subsequent strategy) must be optimal given the other players’ subsequent strategies.
- A **subsequent strategy** is a complete plan of action covering every information set that might arise after the given information set has been reached.

Backward induction I

The procedure that identifies the sequentially rational Nash equilibria in a game with perfect information is **backward induction**:

- **Step 1:** First determine the optimal actions at the last stage of the game.
- **Step 2:** Then determine the optimal actions at the second to last stage of the game given optimal behavior thereafter (as determined in Step 1).
- And so on and so forth.
- Finally, determine the optimal behavior at the first stage of the game given optimal behavior thereafter.

Backward induction II

- Example: Game in Figure 2.6 and 2.7.

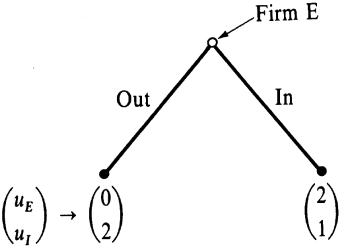
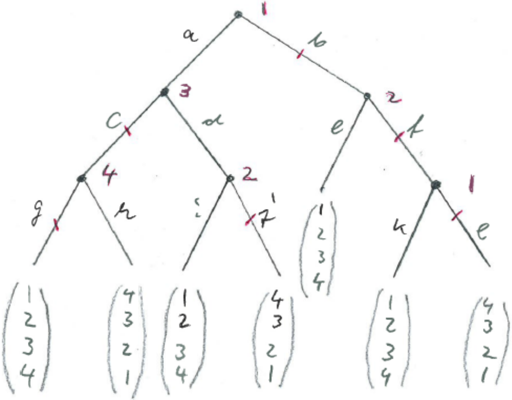


Figure 2.7: Reduced game after solving for post-entry behavior in the market-entry game (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

- (In, Accommodate) is the unique sequentially rational Nash equilibrium of the game in Figure 2.6.

Example: backward induction

Backward induction solution (as indicated in the game tree):
$((b, l), (f, j), c, g)$



Note that every subgame of a game with perfect information is a game of perfect information.

Zermelo's Theorem

Theorem

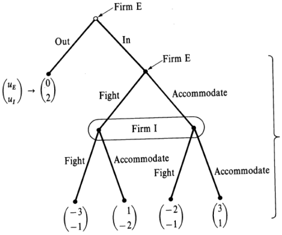
Every finite game of perfect information Γ_E *has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.*

Remark: This Proposition establishes also the existence of a pure strategy Nash equilibrium in all finite games of perfect information.

Example of how to identify sequential rational I

Nash equilibria in extensive form games with imperfect information

Consider the following game:



A Simultaneous-Move Game:

		Firm I	
		Accommodate	Fight
Firm E	Accommodate	3, 1	−2, −1
	Fight	1, −2	−3, −1

		Firm I	
		Accommodate if E Plays “In”	Fight if E Plays “In”
Firm E	Out, Accommodate if In	0, 2	0, 2
	Out, Fight if In	0, 2	0, 2
	In, Accommodate if In	3, 1	−2, −1
	In, Fight if In	1, −2	−3, −1

Figure 2.8: Extensive and normal forms of a game. A sequentially rational Nash equilibirum must have both firms play “accommodate” after entry (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Example of how to identify sequential rational II

Nash equilibria in extensive form games with imperfect information

- Unique Nash equilibrium of the subgame: (Accommodate, Accommodate).
- Anticipating this outcome in the subgame, Firm E will choose In at the beginning of the game.
- So, ((In, Accommodate), Accommodate) is the unique sequentially rational equilibrium of the game in Figure 2.8.

The subgame perfect Nash equilibrium I

- The requirement of sequential rationality will be captured by the concept of a subgame perfect Nash equilibrium.
- We say that a strategy profile σ in extensive form game Γ_E **induces** a Nash equilibrium in a particular subgame of Γ_E if the moves specified in σ for information sets within the subgame constitute a Nash equilibrium when this subgame is considered in isolation.

Definition

A profile of strategies $\sigma = (\sigma_1, ..., \sigma_I)$ in an I-player extensive form game Γ_E is a **subgame perfect Nash equilibrium** (SPNE) if it induces a Nash equilibrium in every subgame of Γ_E.

The subgame perfect Nash equilibrium II

Remarks:

- Note, every SPNE is a NE but not every NE is a SPNE.
- If the only subgame is the game as a whole, then every Nash equilibrium is subgame perfect.

Proposition

Every finite game of perfect information Γ_E *has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.*

- This Proposition follows from Zermelo's Theorem: Just as the strategy profile derived using the backward induction procedure constitutes a Nash equilibrium in the game as a whole, it also induces a Nash equilibrium in every subgame.

The subgame perfect Nash equilibrium III

To determine the set of subgame perfect Nash equilibria in general finite games in extensive form, Γ_E, one can use the following **generalized backward induction procedure:**

- **Step 1**. Start at the end of the game tree and identify the Nash equilibria for each of the final subgames (i.e., those that have no other subgames nested within them).
- **Step 2**. Select one Nash equilibrium in each of these final subgames, and derive the reduced extensive form game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.
- **Step 3**. Repeat steps 1 and 2 for the reduced game. Continue the procedure until every move in Γ_E is determined. This collection of moves at the various information sets of Γ_E constitutes a profile of SPNE strategies.

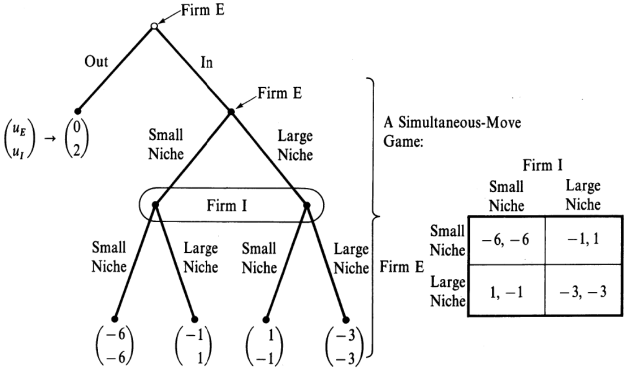
The subgame perfect Nash equilibrium IV

- **Step 4**. If multiple equilibria are never encountered in any step of this process, this profile of strategies is the unique SPNE. If multiple equilibria are encountered, the full set of SPNE is identified by repeating the procedure for each possible equilibrium that could occur for the subgames in question.

Remark: This procedure reduces to the one given earlier in the case of games with perfect information. But it also applies to games of imperfect information, which is illustrated in Figures 2.8 and 2.9.

The subgame perfect Nash equilibrium V

Another Example (Figure 2.9):



		Firm I	
		Small Niche	Large Niche
Firm E	Small Niche	−6, −6	−1, 1
	Large Niche	1, −1	−3, −3

Figure 2.9: Extensive form for the Niche Choice game. The post-entry subgame has multiple Nash equilibria. (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

The subgame perfect Nash equilibrium VI

- The post-entry subgame in Figure 2.9 has three Nash equilibria:
  - (Large Niche, Small Niche),
  - (Small Niche, Large Niche), and
  - the mixed-strategy Nash equilibrium ($(2/9, 7/9)$, $(2/9, 7/9)$) with expected payoffs of ≈ -2.1 for both players.
- Hence, the entire game in Figure 2.9 has the following three SPNE:
  - ((In, Large Niche), Small Niche),
  - ((Out, Small Niche), Large Niche), and
  - ((Out, $(2/9, 7/9)$), $(2/9, 7/9)$).

Application: Stackelberg game I

- Quantity competition (like in Cournot), but one dominant firm (Stackelberg leader) and one weaker firm (Stackelberg follower)
- Players: firm 1 (leader) and firm 2 (follower)
- Order of play:
  - Stage 1: Firm 1 chooses its quantity $q_1 \geq 0$.
  - Stage 2: After observing firm 1's choice, firm 2 chooses its quantity $q_2 \geq 0$.
- Costs: $C_i(q_i) = cq_i$
- Total quantity sold: $Q = q_1 + q_2$
- Inverse demand: $P(Q) = a - Q$
- Payoffs: $\pi_i(q_i, q_j) = P(Q)q_i - cq_i = (a - c - q_i - q_j)q_i$ for $i, j = 1, 2$; $i \neq j$

Application: Stackelberg game II

Solving the Stackelberg Game

- The subgame-perfect Nash equilibrium (SPNE) of the above Stackelberg duopoly game consists of a single quantity for firm 1 and a function $q_2(q_1)$ for firm 2 prescribing a subgame perfect Nash equilibrium choice for every quantity choice by firm 1.
- We solve the game by backward induction.

Solving the 2nd stage:

- Firm 2 will adjust to 1's quantity by using its reaction function (see the derivation of the Nash equilibrium in a Cournot duopoly):

$$q_2 = b_2(q_1) = \begin{cases} \frac{1}{2}(a - c - q_1) & \text{if} \quad q_1 \leq a - c \\ \\ 0 & \text{if} \quad q_1 > a - c. \end{cases} \tag{12}$$

Solving the 1st stage:

Application: Stackelberg game III

- Firm 1, anticipating optimal behavior by firm 2, could produce some $q_1 > a - c$ and keep firm 2 out of the market. However, this is not profit maximizing.
- Hence, firm 1 will produce q_1 to maximize

$$\begin{aligned} \pi_1(q_1) &= (a - c - q_1 - q_2(q_1))q_1 = \\ &= \left(a - c - q_1 - \frac{1}{2}(a - c - q_1)\right) q_1 = \frac{1}{2}(a - c - q_1)\, q_1, \end{aligned}$$

  which is strictly concave in q_1.
- Firm 1's FOC is: $a - c - 2q_1 = 0$, or, equivalently, $q_1 = \frac{1}{2}(a - c)$.
- Hence, $q_1^* = \frac{1}{2}(a - c)$ along with the reaction function (12) of firm 2 is the **subgame-perfect Nash equilibrium** of the Stackelberg game.
- Note that $q_1^* = \frac{1}{2}(a - c)$, and $q_2^* = \frac{1}{4}(a - c)$ is the **outcome along the subgame-perfect Nash equilibrium path**.

Application: Stackelberg game IV

- Compare this with the Cournot output: $q^*_{Cournot} = q^*_1 = q^*_2 = \frac{1}{3}(a - c)$

Application: Stackelberg game V

Some Notes on the Stackelberg Game

- $Q_{Stackelberg} > Q_{Cournot} \rightarrow P_{Stackelberg} < P_{Cournot}$
- It pays to be a leader
  - profit as Stackelberg leader: $\frac{1}{8}(a-c)^2$
  - profit as Stackelberg follower: $\frac{1}{16}(a-c)^2$
  - profit as Cournot duopolist: $\frac{1}{9}(a-c)^2$)
- But: aggregate profits are lower
  - Stackelberg: $\frac{1}{8}(a-c)^2 + \frac{1}{16}(a-c)^2 = \frac{3}{16}(a-c)^2$
  - Cournot: $\frac{2}{9}(a-c)^2$).
- In single-decision problems, having more information is always better; not necessarily in multi-decision problems!
- Firm 1 knows in $t=1$ that firm 2 knows in $t=2$ what firm 1 did in $t=1 \rightarrow$ Firm 2 is hurt by its own knowledge

Application: Strategic delegation I

- Consider a duopoly market for a homogeneous product. Each of the two firms consists of an owner and a manager.
- The inverse market demand is given by $p(Q) = \max\{a - Q, 0\}$ where $Q = q_1 + q_2$ and $a > 0$.
- Both firms have constant unit costs of c with $a > c > 0$.
- The payoff function M_i of the manager $i = 1, 2$ is a convex combination of a firm's profits and revenue, that is

  $$M_i(q_i, q_j, \alpha_i) = \alpha_i \pi_i + (1 - \alpha_i) p(Q) q_i,$$

  where

  $$\pi_i(q_i, q_j) = p(Q) q_i - c q_i$$

  is the profit of firm $i = 1, 2$; $p(Q)q_i$ is the revenue of firm $i = 1, 2$, and $\alpha_i \in [0, 1]$ is the "incentive" parameter for manager i.

Application: Strategic delegation II

- This means that if $\alpha_i = 1$, then manager i is incentivized to maximize firm profits; if $\alpha_i < 1$, then manager i is incentivized to maximize a combination of profit and revenue.[2]
- Consider now the following two-stage game:
  - **Stage 1:** The owners independently and simultaneously choose an incentive scheme for their respective managers. More precisely, owners independently and simultaneously decide about α_1 and α_2, respectively. (They are assumed to do this to maximize firm profits π_1 and π_2, respectively.)
  - **Stage 2:** Knowing both α_1 and α_2, the managers independently and simultaneously decide about their firm's quantities q_1 and q_2, respectively (in order to maximize M_1 and M_2, respectively).
- Think about the structure of this game! Does this game have proper subgames? If yes, what determines a subgame?
- Let's determine the SPNE of this strategic delegation game by backward induction:

Application: Strategic delegation III

Solving the 2nd stage:

- Here the managers simultaneously decide about their quantities, knowing α_1 and α_2.

$$\begin{aligned} M_i(q_i, q_j, \alpha_i) &= \alpha_i \pi_i + (1-\alpha_i) p(Q) q_i \\ &= \text{... (some simple algebra)} \\ &= (a - (q_i + q_j) - c\alpha_i) q_i. \end{aligned}$$

- The FOC $\frac{\partial M_i}{\partial q_i} = 0$ yields

$$q_i = \frac{a - c\alpha_i - q_j}{2} \text{ for } i, j = 1, 2;\ i \neq j. \qquad (13)$$

- Note that in (13) q_i decreases as α_i increases. That is, the more owner i incentivises his manager to maximize firm profits (by increasing α_i), the less aggressive manager i behaves.

Application: Strategic delegation IV

- Solving system (13) gives the SPNE behavior of the 2nd stage:

$$q_i^*(\alpha_i, \alpha_j) = \frac{a - c(2\alpha_i - \alpha_j)}{3} \text{ for } i, j = 1, 2;\ i \neq j. \qquad (14)$$

Application: Strategic delegation V

Solving the 1st stage:

- Here the owners simultaneously decide about their incentive parameters (anticipating the SPNE behavior of the managers thereafter). Using (14) to substitute for q_1 and q_2 in the owners' profit functions, we get:

$$\begin{aligned} \pi_i(\alpha_i, \alpha_j) &= p(Q)q_i - cq_i \\ &= \text{... (some simple algebra)} \\ &= \frac{1}{9}(a - c(3 - \alpha_i - \alpha_j))(a - c(2\alpha_i - \alpha_j)). \end{aligned}$$

- The FOC $\frac{\partial \pi_i}{\partial \alpha_i} = 0$ yields

$$\alpha_i = \frac{c(6 - \alpha_j) - a}{4c} \text{ for } i, j = 1, 2;\ i \neq j. \tag{15}$$

Application: Strategic delegation VI

- Note that in (15) α_i decreases as α_j increases. That is, the more the rival owner j incentivises his manager j to maximize firm j profits (by increasing α_j), the more owner i incentivises his manager to care for sales.
- Solving the system (15) gives

$$\alpha_i^* = \frac{6c - a}{5c} \text{ for } i = 1, 2. \tag{16}$$

- Note that $\alpha_i^* < 1$ as $a - c > 0$.
- Hence, in the SPNE of the game, the owners give their managers incentives **not** to maximize firm profits, but a convex combination of profits and sales.
- Plugging (16) into (14) gives

$$q_i^* = \frac{2(a - c)}{5} \text{ for } i = 1, 2.$$

Application: Strategic delegation VII

- To see which quantity owners would produce themselves (without managers), set $\alpha_i = \alpha_j = 1$ in (14), to get

$$q_i^{owner} = \frac{a-c}{3} \text{ for } i = 1, 2,$$

which is smaller than $q_i^* = \frac{2(a-c)}{5}$, the quantity managers produce with strategic delegation.

---

[2] More precisely, the salary of a manager is not M_i, but $A_i + B_i M_i$, where A_i and B_i are real numbers and $B_i > 0$. A_i and B_i are chosen by the owner so that $A_i + B_i M_i$ is equal to the opportunity costs of manager i. Since A_i and B_i are constant, maximization of $A_i + B_i M_i$ is equivalent to the maximization of M_i. Hence, the salary of manager i is part of the fixed costs of owner i and is therefore irrelavant for the maximization problem of owner i.

Repeated games I

Finitely often repeated games

A special class of games with imperfect information:

Proposition

Consider an I*-player extensive game* Γ_E *involving successive play of* T *simultaneous-move games,* $\Gamma_N^t = \left[I; \left\{\Delta(S_i^t)\right\}; \left\{u_i^t(\cdot)\right\}\right]$ *for* $t = 1, ..., T$*, with the players observing the pure strategies played in each game immediately after its play is concluded. Assume that each player's payoff is equal to the sum of her payoffs in the plays of the* T *games. If there is a unique Nash equilibrium in each game* Γ_N^t*, say* $\sigma^t = (\sigma_1^t, ..., \sigma_I^t)$*, then there is a unique SPNE of* Γ_E *and it consists of each player* i *playing strategy* σ_i^t *in each game* Γ_N^t *regardless of what has happened previously.*

Repeated games II

Finitely often repeated games

Proof: The proof is by induction. The result is clearly true for $T = 1$. Now suppose it is true for all $T \leq n-1$. We will show that it is true for $T = n$. We know by hypothesis that in any SPNE of the overall game, after play of Γ_N^t the play in the remaining $n-1$ simultaneous-move games must simply involve play of the Nash equilibrium of each game (since any SPNE of the overall game induces an SPNE in each of its subgames). Let player i earn G_i from this equilibrium play in these $n-1$ games. Then in the reduced game that replaces all the subgames that follow Γ_N^1 with their equilibrium payoffs, player i earns an overall payoff of $u_i(s_1^1, ..., s_n^1) + G_i$ if $(s_1^1, ..., s_I^1)$ is the profile of pure strategies played in game Γ_N^1. The unique Nash equilibrium of this reduced game is clearly σ^1. Hence, the result is also true for $T = n$. $\square$

Repeated games III

Finitely often repeated games

- Note that the last Proposition says that there is no history dependence of strategies in this class of games.
- This is not the case if the “base game” has multiple Nash equilibria.
- To see this, consider the following game:

Player 2

		L_2	M_2	R_2
	L_1	1, 1	5, 0	0, 0
Player 1	M_1	0, 5	4, 4	0, 0
	R_1	0, 0	0, 0	3, 3

- Nash equilibria in pure strategies: (L_1, L_2), (R_1, R_2).
- Suppose this game is played twice.

Repeated games IV

Finitely often repeated games

Claim: There is a SPNE in which (M_1, M_2) is played in the first stage, although (M_1, M_2) is not a NE of the base game.

Proof:

- Clearly, in the second stage a Nash equilibrium of the base game will be played. Which one? Assume it is $(R_1,\ R_2)$, if (M_1, M_2) was played in the first stage. Otherwise it is (L_1, L_2). Then, the so-called **reduced game** in the first stage is

Player 2

		L_2	M_2	R_2
	L_1	2, 2	6, 1	1, 1
Player 1	M_1	1, 6	7, 7	1, 1
	R_1	1, 1	1, 1	4, 4

Repeated games V

Finitely often repeated games

- To obtain this reduced game, the payoffs (3,3) have been added to the cell (M_1, M_2) and the payoffs (1,1) to all the other cells of the base game.
- The Nash equilibria of this reduced game are (L_1, L_2), (M_1, M_2) and (R_1, R_2).
- Clearly, the Nash equilibria of the reduced game correspond with the SPNE of the repeated game.
- For instance, the NE (L_1, L_2) of the reduced game corresponds to the following SPNE of the repeated game:
    - **Period 1:** (L_1, L_2)
    - **Period 2:** $(R_1,\ R_2)$ if (M_1, M_2) was played in the first stage, otherwise (L_1, L_2).
- Similarly, the NE (R_1, R_2) of the reduced game corresponds to the following SPNE of the repeated game:

Repeated games VI

Finitely often repeated games

  - **Period 1:** (R_1, R_2)
  - **Period 2:** $(R_1,\ R_2)$ if (M_1, M_2) was played in the first stage, otherwise (L_1, L_2).
- But the NE (M_1, M_2) of the reduced game corresponds to the following SPNE of the repeated game:
  - **Period 1:** (M_1, M_2)
  - **Period 2:** (R_1, R_2) if (M_1, M_2) was played in the first stage, otherwise (L_1, L_2).
- And this although (M_1, M_2) is not a NE of the base game. Thus, cooperation in an earlier stage is possible in a SPNE.
- More generally: Assume the base game has more than one NE. Then the finitely often repeated play of the base game can have a SPNE in which in each stage $t < T$ a strategy profile is played that is not a NE of the base game.

Repeated games VII

Finitely often repeated games

- Credible threats or promises regarding future play can then influence present behavior.

Repeated games VIII

Finitely often repeated games

Application: The Chainstore Paradox

- Let's consider again the base game of Figure 2.6:

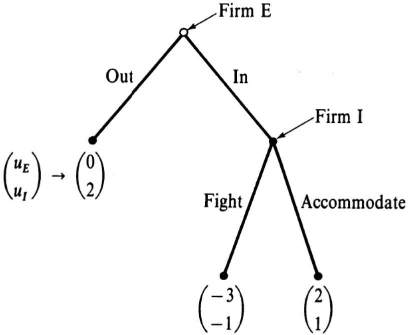


Figure 2.6: Extensive form of a market-entry game (Source: Andreu Mas-Colell/Michael D. Whinston/Jerrry R. Green: Microeconomic Theory, Oxford University Press)

Repeated games IX

Finitely often repeated games

- We have seen in the one-shot version of the game that entry would be accommodated.
- But now assume that the game is repeated 20 times in the context of a chainstore that tries to deter entry into 20 of its markets where it has outlets.
- What happens now? Would the chainstore fight entry of the first entrant to deter the next 19 entrants?
- The answer is no. The subgame perfect Nash equilibrium outcome is that entry occurs and is accommodated in all 20 markets!
- The logic is similar as in the proof of Proposition 2.3. As the SPNE of the base game is unique, the SPNE of the 20 times repeated game is also unique. There is just no room for the chainstore to build a reputation for being tough.

Repeated games X

Finitely often repeated games

- This is called the **Chainstore Paradox** because the “mighty” incumbent loses all markets.
- Note, however, that there are other Nash equilibria of the chainstore game. They are just not subgame perfect Nash equilibria.

Repeated games I

Infinitely often repeated games

Definition

Let $\Gamma_N = (I; \{S_i\}; \{u_i\})$ be a game in normal form. Then, the infinitely repeated game of Γ_N has the following components:

- Set of players as in Γ_N.
- In each stage of the repeated game, the players choose an action in S_i.
- Payoffs: Let $\pi_{i,1}, \pi_{i,2}, \pi_{i,3}, ...$ be the payoffs of player i in each stage $t = 1, 2, 3, ...$ of the game. Then, the payoff from the entire game is $\pi_{i,1} + \delta\pi_{i,2} + \delta^2\pi_{i,3} + ... = \Sigma_{t=1}^{\infty}\delta^{t-1}\pi_{i,t}$, where δ is the discount factor.

Repeated games II

Infinitely often repeated games

- Example: Consider the following prisoner's dilemma type game:

Player 2

		D	C
Player 1	D	1, 1	5, 0
	C	0, 5	4, 4

(17)

- Clearly, (D, D) is the unique Nash equilibrium of this game.
- **Question:** Could cooperation (i.e., playing (C, C)) occur in each stage of the infinitely repeated game, although (D, D) is the unique Nash equilibrium of the base game?
- **Answer:** Yes.

Repeated games III

Infinitely often repeated games

Claim: Mutual play of the following **grim-trigger** strategy is a SPNE of the above infinitely repeated prisoner's dilemma game.

- Grim trigger strategy for player i: Play C *in the first period. Play* C *in period* $t \geq 2$ if the outcome of all $t-1$ *earlier periods was* (C, C); otherwise play D.

Proof:
1st Step:

- There is a δ "close" to 1, so that mutual play of this strategy is a Nash equilibrium of the infinitely repeated game.

Repeated games IV

Infinitely often repeated games

- If both players adhere to the "grim-trigger" strategy, then the outcome in each period is (C, C), so that each player receives the series of payoffs 4, 4, ... The present value of this income stream is

$$\sum_{t=1}^{\infty} \delta^{t-1} \cdot 4 = 4 \sum_{t=1}^{\infty} \delta^{t-1} = \frac{4}{1-\delta}.$$

- Suppose player 1 sticks to the grim-trigger strategy, but player 2 plans to deviate.
- Then there is a least one period in which player 2 plays "D".
- But then player 1 will play D in all coming periods (as player 1 plays according to the grim-trigger strategy.
- The best deviation for player 2 is then to always play D.
- So, if player 2 can increase his payoffs by deviating, then by deviating already in period 1.

Repeated games V

Infinitely often repeated games

- In this case, (C, D), (D, D), (D, D), ... will be played.
- Player 2 then earns the stream of payoffs 5, 1, 1, ..., which has the following present value:

$$5 + \delta + \delta^2 + ... = 5 + \delta(1 + \delta + ...) = 5 + \frac{\delta}{1 - \delta}.$$

- So, deviating doesn't pay for player 2 if

$$\frac{4}{1 - \delta} \geq 5 + \frac{\delta}{1 - \delta} \Leftrightarrow \delta \geq \frac{1}{4}.$$

2nd step:

- To show that the grim-trigger strategy is a SPNE in the infinitely repeated game, we have to show that the grim-trigger- strategy induces a Nash equilibrium in each subgame.

Repeated games VI

Infinitely often repeated games

- Clearly, each subgame of the infinitely repeated game is identical to the entire game. The subgames can be grouped into two categories:
    - (i) subgames in which the results of all previous periods was (C, C).
    - (ii) subgames in which the outcome of at least one earlier period was not (C, C).
- In subgames of type (i), the strategies of both players are again grim-trigger strategies, of which we know from step 1 that they induce a Nash equilibrium in theses subgames.
- In subgames of type (ii), the strategies of the players prescribe to play the Nash equilibrium of the base game, (D, D), in each period of the game. This is also a Nash equilibrium of the entire game.

Repeated games VII

Infinitely often repeated games

What payoffs can be obtained in a SPNE in a general infinitely repeated game (a static base game with full information)?

Definition

The payoffs $(x_1, ..., x_n)$ in stage game Γ_N are called **feasible** if they are a convex combination of the pure strategy payoffs in Γ_N.

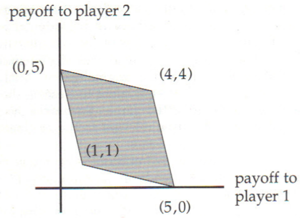


Figure 2.7: The feasible payoffs for the prisoner's dilemma game in (17) (Source: Robert Gibbons (1992): A Primer in Game Theory, Harvester Wheatsheaf (Prentice Hall))

Repeated games VIII

Infinitely often repeated games

Definition

Given the discount factor δ, the **average payoff** of the infinite sequence of payoffs π_1, π_2, π_3, ... is

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi_t.$$

- The average payoff is just a rescaling of the present value.
- Therefore the maximization of the average payoff is equivalent to the maximization of the present value.
- The advantage of the average payoff over the present value is that the former is directly comparable to the payoffs of the stage game.
- Example: sequence of payoffs 4, 4, 4, ... $\curvearrowright$ present value: $\frac{4}{1-\delta}$; average payoff: 4.

Repeated games IX

Infinitely often repeated games

Proposition

(Folk-Theorem, Friedman, 1971): Let Γ_N *be a finite, static game of complete information. Let* $(e_1, .., e_n)$ *denote the payoffs from a Nash equilibrium of* Γ_N*, and let* $(x_1, .., x_n)$ *denote any other feasible payoffs from* Γ_N*. If* $x_i > e_i$ *for every player* i *and if* δ *is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game that achieves* $(x_1, ..., x_n)$ *as the average payoff.*

Repeated games X

Infinitely often repeated games

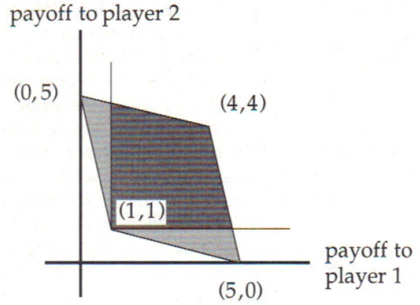


Figure 2.8: The dark shaded region can be achieved as the average payoff in a SPNE of the repeated game (Friedman's Theorem) (Source: Robert Gibbons (1992): A Primer in Game Theory, Harvester Wheatsheaf (Prentice Hall))

Idea of the proof: Use trigger strategies as described above.

Repeated games XI

Infinitely often repeated games

Application: Infinitely repeated Cournot game

- Ingredients of the stage game:
  - Number of players: $I = 2$
  - Strategy sets: $S_i = [0, \infty)$, $i = 1, 2$
  - Payoff functions: $\pi_i(q_i, q_j) = (a - b(q_i + q_j))q_i - cq_i$, $i = 1, 2$; $i \neq j$.
- Let us first determine some outcomes of the stage game:
  - Individual Nash equilibrium quantity and profit: $q^N = \frac{a-c}{3b}$, $\pi^N = \frac{(a-c)^2}{9b}$
  - Individual Cartel quantity and profit: $q^C = \frac{a-c}{4b}$, $\pi^C = \frac{(a-c)^2}{8b}$
  - One firm cheats on the other firm, when the latter plays q^C (individual Deviation quantity and profit): $q^D = \frac{3(a-c)}{8b}$, $\pi^D = \frac{9(a-c)^2}{64b}$.

Repeated games XII

Infinitely often repeated games

- Prisoner's dilemma structure:

Player 2

Player 1

		cheat	cartel
Player 1	cheat	$\frac{(a-c)^2}{9b}, \frac{(a-c)^2}{9b}$	$\frac{9(a-c)^2}{64b}, \frac{3(a-c)^2}{32b}$
	cartel	$\frac{3(a-c)^2}{32b}, \frac{9(a-c)^2}{64b}$	$\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{8b}$

Repeated games XIII

Infinitely often repeated games

Let us now consider the infinitely repeated Cournot game.

- Is there a SPNE of this game in which both players cooperate (play cartel) in each period? The answer is yes.
- Indeed, if the discount factor δ is "sufficiently" close to one, mutual play of the following grim trigger strategy constitutes a SPNE of the infinitely repeated Cournot game.
  - Let q_{1t} be the quantity of firm 1 in period t. The grim trigger strategy is defined as

$$q_{1t} = \begin{cases} q^C = \frac{a-c}{4b} & \text{if } t = 1 \text{ or if firm 2 chose } q^c \text{ in all previous periods} \\ q^N = \frac{a-c}{3b} & \text{otherwise} \end{cases}$$

  - Similarly for firm 2.
  - That is, firm 1 cooperates as long as it sees firm 2 cooperating. Once firm 2 is observed to cheat, firm 1 produces the Nash equilibrium quantity for ever after.

Repeated games XIV

Infinitely often repeated games

- Let's (briefly) show that mutual play of the grim-trigger strategy constitutes a SPNE.
- Again, there are two possible types of subgames:
  - (i) A subgame after which cheating by yourself or the other firm has occurred.
    - The grim-trigger strategy prescribes to play the Cournot-Nash quantity q^N for ever after, given that the other firm does this as well. This is a Nash equilibrium of the subgame, as playing q^N forever is a best response of a firm to the other firm playing q^N forever. So, the SPNE condition is fulfilled.
  - (ii) A subgame after which no cheating by either firm has occurred.
    - The grim-trigger strategy prescribes cooperating and playing q^C, which has a present value of payoffs equal to $\pi^c/(1-\delta)$.

Repeated games XV

Infinitely often repeated games

  - The best possible deviation is to play $q_1^D = BR(q_2^C) = \frac{3(a-c)}{8b}$ in the current period, but then playing q^N for ever after. This generates an income stream of π^D, π^N, π^N, ..., whose present value is equal to $\pi^D + \delta(\pi^N/(1-\delta))$.
  - In order for the grim trigger strategy to be a Nash equilibrium of this subgame, we need $\pi^c/(1-\delta) \geq \pi^D + \delta(\pi^N/(1-\delta))$, so that the profits from cooperating are larger than those from deviating. This condition is fulfilled if $\delta \geq 9/17$. (Please check!)

- Therefore, the grim trigger strategy induces a Nash equilibrium in both kinds of subgames if $\delta \geq 9/17$. Hence, in this case the grim trigger strategy is a SPNE of the infinitely repeated Cournot game.

Suggestion for further reading:

Robert Gibbons (1997): "An Introduction to Applicable Game Theory," *Journal of Economic Perspectives* 11(1), 127-149.
(Can be downloaded at www.jstor.org)