Game Theory 1: Problem Set 3

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Problem 1

(a) Backward induction:

Second stage (t=2)

$q_2 = b(q_1, q_3) = \frac{1}{2}(a - c_2 - q_1 - q_3)$
$q_3 = b(q_1, q_2) = \frac{1}{2}(a - c_3 - q_1 - q_2)$

$\hookrightarrow q_2 = b(q_1) = \frac{1}{2}(a - c_2 - q_1 - (\frac{1}{2}(a - c_3 - q_1 - q_2)))$
$\frac{3}{4}q_2 = \frac{1}{4}(a + c_3 - q_1 - 2c_2)$

$$q_2(q_1) = \begin{cases} \frac{1}{3}(a + c_3 - q_1 - 2c_2) & \text{if } q_1 \leq a - 2c_2 + c_3 \\ 0 & \text{if } q_1 > a - 2c_2 + c_3 \end{cases}$$

Similarly: $q_3(q_1) = \begin{cases} \frac{1}{3}(a + c_2 - q_1 - 2c_3) & \text{if } q_1 \leq a - 2c_3 + c_2 \\ 0 & \text{if } q_1 > a - 2c_3 + c_2 \end{cases}$

First stage (t=1)
$\pi_1(q_1) = (a - c_1 - q_2(q_1) - q_3(q_1))q_1$
$\pi_1(q_1) = (a - c_1 - \frac{1}{3}(a + c_3 - q_1 - 2c_2) - \frac{1}{3}(a + c_2 - q_1 - 2c_3))q_1$
$\pi_1(q_1) = \frac{1}{3}(a - q_1 + c_3 + c_2 - 3c_1)q_1$
$FOC : \frac{\partial \pi_1}{\partial q_1} = \frac{1}{3}(a - 2q_1 + c_3 + c_2 - 3c_1) = 0$
$\hookrightarrow q_1^* = \frac{1}{2}(a - 3c_1 + c_2 + c_3)$
This function along with the reaction functions of q_2 and q_3 as specified above is the SPNE of this Stackelberg game.

$q_2^* = \frac{1}{3}(a + c_3 - (\frac{1}{2}(a - 3c_1 + c_2 + c_3)) - 2c_2) = \frac{1}{6}(a + c_3 - 5c_2 + 3c_1)$
Similarly: $q_3^* = \frac{1}{6}(a + c_2 - 5c_3 + 3c_1)$

Thus the outcome of the SPNE is:
$(q_1, q_2, q_3) = (\frac{1}{2}(a - 3c_1 + c_2 + c_3), \frac{1}{6}(a + c_3 - 5c_2 + 3c_1), \frac{1}{6}(a + c_2 - 5c_3 + 3c_1))$

(b) Backwards induction:

Second stage (t=2)

$q_3 = b(q_1, q_2) = \begin{cases} \frac{1}{2}(a - c_3 - q_1 - q_2) & \text{if } q_1 + q_2 \leq a - c_3 \\ 0 & \text{if } q_1 + q_2 > a - c_3 \end{cases}$

First stage (t=1)

$\pi_1 = (a - c_1 - q_1 - q_2 - \frac{1}{2}(a - c_3 - q_1 - q_2))q_1 = \frac{1}{2}q_1(a - 2c_1 + c_3 - q_1 - q_2)$

$FOC: \frac{\partial \pi_1}{\partial q_1} = \frac{1}{2}(a - 2c_1 + c_3 - 2q_1 - q_2) = 0$

$q_1(q_2) = \frac{1}{2}(a - 2c_1 + c_3 - q_2)$

Similarly, $q_2(q_1) = \frac{1}{2}(a - 2c_2 + c_3 - q_1)$

Substituting for q_2 in $q_1(q_2)$ we find:

$q_1 = \frac{1}{2}(a - 2c_1 + c_3 - (\frac{1}{2}(a - 2c_2 + c_3 - q_1)))$

$q_1 = \frac{1}{4}(a - 4c_1 + c_3 + 2c_2 + q_1)$

$\frac{3}{4}q_1 = \frac{1}{4}(a - 4c_1 + c_3 + 2c_2)$

$q_1^* = \frac{1}{3}(a - 4c_1 + c_3 + 2c_2)$

Similarly: $q_2^* = \frac{1}{3}(a - 4c_2 + c_3 + 2c_1)$

These functions along with the reaction function of firm 3 as specified above is the SPNE of this Stackelberg game

$q_3^* = \frac{1}{2}(a - c_3 - \frac{1}{3}(a - 4c_1 + c_3 + 2c_2) - \frac{1}{3}(a - 4c - 2 + c_3 + c_1))$

$q_3^* = \frac{1}{6}(a - 5c_3 + 2c_1 + 2c_2)$

Thus the outcome of the SPNE is:

$(q_1, q_2, q_3) = (\frac{1}{3}(a - 4c_1 + c_3 + 2c_2), \frac{1}{3}(a - 4c_2 + c_3 + 2c_1), \frac{1}{6}(a - 5c_3 + 2c_1 + 2c_2))$

(c) At time $t = 1$, firm 1 acts as the Stackelberg leader in this market and therefore chooses her quantity q_1 first, based on the reaction functions of firm 2 and 3. To maximize profits of firm 1, we first have to find the reaction functions of both firms 2 and 3. The reaction function of firm 3 is the same as we found before and thus equals:

$$q_3 = \frac{1}{2}(a - c_3 - q_1 - q_2)$$

We can substitute this reaction function into the profit function for firm 2 in order to find the reaction function of firm 2. Than we find:

$$\pi_2 = \left(a - c_2 - q_1 - q_2 - \frac{1}{2}(a - c_3 - q_1 - q_2)\right) \cdot q_2$$

$$FOC: \frac{\partial \pi_2}{\partial q_2} = \frac{1}{2}a - c_2 + \frac{1}{2}c_3 - \frac{1}{2}q_1 - q_2 = 0$$

$$q_2 = \frac{1}{2}(a - 2c_2 + c_3 - q_1)$$

This strategy set for player 2 can be filled into the best response function for firm 3, this gives the new reaction function of firm 3:

$$q_3 = \frac{1}{4}(a - 3c_3 + 2c_2 - q_1)$$

We fill in q_3 into the profit function of firm 1 to find the first-stage decision of firm 1. This is as follows:

$$\pi_1 = \left(a - c_1 - q_1 - \frac{1}{2}\left(a - 2c_2 + c_3 - q_1\right) - \frac{1}{4}(a - 3c_3 - q_1 + 2c_2)\right) \cdot q_1$$

$$\pi_1 = \left(\frac{1}{4}a - c_1 + \frac{1}{2}c_2 + \frac{1}{4}c_3 - \frac{1}{4}q_1\right) \cdot q_1$$

$$FOC : \frac{\partial \pi_1}{\partial q_1} = \frac{1}{4}a - c_1 + \frac{1}{2}c_2 + \frac{1}{4}c_3 - \frac{1}{2}q_1 = 0$$

$$q_1^* = \frac{1}{2}(a - 4c_1 + 2c_2 + c_3)$$

This function along with the reactions function of firm 2 and 3 as specified above, gives us the SPNE.

Further we obtain the second-stage decision when filling in q_1^* into the reaction function of firm 2:

$$q_2 = \frac{1}{2}\left(a - 2c_2 + c_3 - \frac{1}{2}\left(a - 4c_1 + 2c_2 + c_3\right)\right)$$

$$q_2^* = \frac{1}{4}\left(a - 6c_2 + 4c_1 + c_3\right)$$

Ultimately, we solve the third-stage decision substituting q_1^* into the reaction function of firm 3:

$$q_3 = \frac{1}{4}\left(a - 3c_3 + 2c_2 - \frac{1}{2}\left(a - 4c_1 + 2c_2 + c_3\right)\right)$$

$$q_3^* = \frac{1}{8}(a - 7c_3 + 4c_1 + 2c_2)$$

Thus, the outcome along the SPNE path is therefore as follows:

$$(q_1,\ q_2,\ q_3) = \left(\frac{1}{2}\left(a - 4c_1 + 2c_2 + c_3\right), \frac{1}{4}\left(a - 6c_2 + 4c_1 + c_3\right),\ \frac{1}{8}\left(a - 7c_3 + 4c_1 + 2c_2\right)\right)$$

(d) $c_1 = c_2 = c_3 = c$ and $a > c$

$SPNE_{(a)} = (\frac{1}{2}(a-c), \frac{1}{6}(a-c), \frac{1}{6}(a-c)) \rightarrow Q = \frac{5}{6}(a-c)$
$SPNE_{(b)} = (\frac{1}{3}(a-c), \frac{1}{3}(a-c), \frac{1}{6}(a-c)) \rightarrow Q = \frac{5}{6}(a-c)$
$SPNE_{(c)} = (\frac{1}{2}(a-c), \frac{1}{2}(a-c), \frac{1}{8}(a-c)) \rightarrow Q = \frac{9}{8}(a-c)$

With symmetric cost levels and $a > c$ a welfare maximizing regulator would choose for the market structure defined in question (c). Total output is the greatest under this structure, therefore this is the closest to the perfectly competititve market (maximum total welfare) of all three structures (a), (b) and (c).

$TW_{(c)} > TW_{(a)} = TW_{(b)}$ as: $\frac{9}{8}(a-c) > \frac{5}{6}(a-c) = \frac{5}{6}(a-c)$

Problem 2

Firstly, we will attempt to find all the NE through backwards induction.

		Player 2	
		A	B
Player 1	S	$\underline{6}$, $\underline{1}$	0 , 0
	T	0 , 0	$\underline{4}$, $\underline{4}$

The post LR decision subgame gives the following pure strategy NE:

- (A, S)
- (B, T)

  Mixed strategy NE in the subgame:

  $u_1(S, \sigma_2) = 6q$
  $u_1(T, \sigma_2) = 4 - 4q$
  $6q = 4 - 4q$ iff $q = \frac{2}{5}$

  $u_2(\sigma_1, A) = p$
  $u_2(\sigma_1, B) = 4 - 4p$
  $p = 4 - 4p$ iff $p = \frac{4}{5}$

  Thus, the mixed strategy NE in the post LR decision subgame is:

- $(\sigma_1, \sigma_2) = ((\frac{4}{5}, \frac{1}{5}), (\frac{2}{5}, \frac{3}{5}))$ with $u_1(\sigma_1, \sigma_2) = \frac{12}{5}$

x=1	**x=5**
1. (S, A) gives player 1 a payoff of 6. Player 1 will prefer playing L over playing R as this gives a higher payoff $(6 > 1)$. This leads to the **SPNE:** (L, A, S)	**1**. Player 1 will prefer L over R as this gives a higher payoff $(6 > 5)$ given the NE of the subgame (S, A). This leads to the **SPNE:** (L, A, S)
2.(T, B) gives player 1 a payoff of 4. Player 1 will prefer L over R as this gives a higher payoff $(4 > 1)$. This leads to the **SPNE:**(L, B, T)	**2**. Player 1 will choose R over L as this gives a higher payoff $(5 > 4)$ given the NE of the subgame (T, B). This leads to the **SPNE:**(R, B, T)
3.$\left(\left(\frac{4}{5}, \frac{1}{5}\right), \left(\frac{2}{5}, \frac{3}{5}\right)\right)$ gives player 1 a payoff of $\frac{12}{5}$. Player 1 will prefer playing L over R as this gives a higher payoff $(\frac{12}{5} > 1)$. This leads to the **SPNE:** $\left(L, \left(\frac{4}{5}, \frac{1}{5}\right), \left(\frac{2}{5}, \frac{3}{5}\right)\right)$	**3**. Player 1 will prefer R over L as this gives a higher payoff $(5 > \frac{12}{5})$ given the NE in mixed strategies of the subgame. This leads to the **SPNE:**$\left(R, \left(\frac{4}{5}, \frac{1}{5}\right), \left(\frac{2}{5}, \frac{3}{5}\right)\right)$