Homework 11

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In all sections for big O notation I will be using the definition that

$$f(n) = O(g(n)) \Leftrightarrow \limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty$$

and for small o notation I will be using the definition that

$$f(n) = o(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

.

a) Take the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{n^{2.5}}{3n^3 + 2n^2 + 5n\log_2(n) + 1} \\
&= \lim_{n\to\infty} \frac{n^{2.5}n^{-2.5}}{(3n^3 + 2n^2 + 5n\log_2(n) + 1)n^{-2.5}} \\
&= \lim_{n\to\infty} \frac{1}{3n^{0.5} + 2n^{-0.5} + 5n^{-1.5}\log_2(n) + n^{-2.5}}
\end{aligned}$$

In the denominator of the last limit $3n^{0.5}$ goes to infinity as n goes to infinity and the rest of the terms are all positive for any positive n so the denominator goes to infinity while the numerator is finite, meaning that the limit is 0. This means that both $f(n) = O(g(n))$ and $f(n) = o(g(n))$. If we take the inverse of the fraction i nthe limit we get infinity since the numerator goes to infinity and the denominator is finite, meaning that neither $g(n) = O(f(n))$ nor $g(n) = o(f(n))$ hold.

b) Take the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{\log_2(n)\cdot\log_2\log_2(n)}{n^{\frac{1}{3}}} \\
&\overset{L'H}{=} \lim_{n\to\infty} \frac{(\log_2(n)\cdot\log_2\log_2(n))'}{(n^{\frac{1}{3}})'} \\
&= \lim_{n\to\infty} \frac{\frac{\ln(\frac{\ln n}{\ln(2)})+1}{n\ln^2(2)}}{\frac{1}{3}n^{-\frac{2}{3}}} \\
&= \lim_{n\to\infty} \frac{\ln(\frac{\ln n}{\ln(2)})+1}{\frac{1}{3}n^{\frac{1}{3}}\ln^2(2)} \\
&\overset{L'H}{=} \lim_{n\to\infty} \frac{(\ln(\frac{\ln n}{\ln(2)})+1)'}{(\frac{1}{3}n^{\frac{1}{3}}\ln^2(2))'} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n\ln(n)}}{\frac{1}{9}n^{-\frac{2}{3}}\ln^2(2)} \\
&= \lim_{n\to\infty} \frac{9}{n^{\frac{1}{3}}\ln^2(2)\ln(n)} = 0
\end{aligned}$$

which means that both $f(n) = O(g(n))$ and $f(n) = o(g(n))$ hold. If we take the inverse however, the limit goes to infinity, meaning that neither $g(n) = O(f(n))$ nor $g(n) = o(f(n))$ hold.

c) Take the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{2^{(n^2)}}{100^{5n}} \le \lim_{n\to\infty} \frac{2^{(n^2)}}{128^{5n}} \\
&= \lim_{n\to\infty} \frac{2^{(n^2)}}{2^{7\cdot 5n}} \\
&= \lim_{n\to\infty} 2^{n^2-35n} = \infty
\end{aligned}$$

Meaning that neither $f(n) = O(g(n))$ nor $f(n) = o(g(n))$ hold. If we take the inverse however, it is easy to see that the limit goes to 0, meaning that both $g(n) = O(f(n))$ and $g(n) = o(f(n))$ hold.