Tärnülesanne nr. 60

Joosep Näks

Olgu $m, n \in \mathbb{N}$. Leida piirväärtus $\lim\limits_{x\to 1}\left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right)$

Lahendus:

Teisendan kasutades l'Hôptiali reeglit:

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right) &= \lim_{x\to 1}\left(\frac{m(1-x^n)-n(1-x^m)}{(1-x^m)(1-x^n)}\right)\\
&= \lim_{x\to 1}\left(\frac{m-mx^n-n+nx^m}{1-x^m-x^n+x^{n+m}}\right)\\
&\overset{\text{l'Hôptial}}{=} \lim_{x\to 1}\left(\frac{-nmx^{n-1}+mnx^{m-1}}{-mx^{m-1}-nx^{n-1}+(m+n)x^{n+m-1}}\right)\\
&= \lim_{x\to 1}\left(\frac{-nmx^{n}+mnx^{m}}{-mx^{m}-nx^{n}+(m+n)x^{n+m}}\right)\\
&\overset{\text{l'Hôptial}}{=} \lim_{x\to 1}\left(\frac{-n^2mx^{n-1}+m^2nx^{m-1}}{-m^2x^{m-1}-n^2x^{n-1}+(m+n)^2x^{n+m-1}}\right)\\
&= \lim_{x\to 1}\left(\frac{-n^2mx^{n}+m^2nx^{m}}{-m^2x^{m}-n^2x^{n}+(m+n)^2x^{n+m}}\right)\\
&= \lim_{x\to 1}\left(\frac{nm(mx^m-nx^n)}{-m^2x^{m}-n^2x^{n}+(m+n)^2x^{n+m}}\right)\\
&= \frac{nm(m-n)}{-m^2-n^2+(m+n)^2}\\
&= \frac{nm(m-n)}{-m^2-n^2+m^2+2mn+n^2}\\
&= \frac{nm(m-n)}{2mn}\\
&= \frac{m-n}{2}
\end{aligned}$$

Piirväärtuse väärtus on $\frac{m-n}{2}$.