Maple Homework 2

Brooke Joplin

In this homework, you will learn to use Maple in Document Mode. Before starting this homework, complete the **Maple HW_1** assignment and read through the **Maple_1_Expressions_Functions** document. You may also find the online document Maple for Physics Students helpful.

Always start your document with the "restart" command. Notice that the restart: command below is in Math mode, while descriptions are in Text mode.

$restart:$

Problem 1) Black Holes

Black holes are objects that are so compact light cannot escape their gravitational fields. The radius of a black hole is given by the Schwarschild radius:

$$r = \frac{2\,G\,M}{c^2}$$

where G is the gravitational constant, M is the object's mass and c is the speed of light.

a) Define an **expression** for the Schwarschild radius. Make sure you type your expression into a 2D Math box and use the $:=$ assignment operator.

answer:

$r := \frac{2\,G \cdot m}{c^2}$

$$r := \frac{2\,G\,m}{c^2} \quad \textbf{(1)}$$

b) Now, create an expression for the density of a homogeneous sphere given its mass M and radius R, remembering that density = mass volumne and the volume of a cube is $\frac{4}{3}\pi R^3$. If everything goes well, Maple should automatically substitute in for R and display the expression for the density in terms of G, M, c, etc. r(No R). If it doesn't, check your expression for possible errors or consult with your instructor or other students.

answer:

$\text{rho} := \frac{m}{\left(\left(\frac{4}{3}\right)\pi \cdot r^3\right)}$

$$\rho := \frac{3\,c^6}{32\,m^2\,\pi\,G^3} \quad \textbf{(2)}$$

d) Let's plug in some numbers and calculate the density of the black hole at the center of the Milky Way, which has a mass of 4 million times the mass of our Sun. Define constants for Newton's gravitational constant, the speed of light, and the black hole mass. I defined the speed of light below to get you started. Notice I used text mode to label what the constant is. Define the other variables following this format. Make sure the **Math** button on the upper menu bar is highlighted before you start typing math.

After you define all the variables in the equation for the density, simply type the ρ symbol for density and hit enter. The density of the black hole will appear in MKS units (i.e. kg/m^3). Please label your result with text. i.e. "The density of the black hole is ___ kg/m^3"

$$\frac{3\,c^6}{32\,m^2\,\pi\,G^3} \tag{3}$$

answer:

Speed of light
$c := 2.99792458\mathrm{e}8$

$$c := 2.99792458 \times 10^{8} \tag{4}$$

$G := 6.67430\mathrm{e}\text{-}11$

$$G := 6.67430 \times 10^{-11} \tag{5}$$

$m := 2\mathrm{e}30 \cdot 4\mathrm{e}6$

$$m := 8. \times 10^{36} \tag{6}$$

ρ

$$1.138540480 \times 10^{6} \tag{7}$$

Problem 2) Projectile Motion

A common problem in first-year mechanics is to find the range of a projectile given its initial velocity v_0 and its initial launch angle θ . You will use Maple to derive the result that the range is equal to $R = \frac{v_0^{\,2}}{g}\sin(2\cdot\theta)$. We'll solve it just like you would in introductory mechanics, i.e. by splitting the motion into x and y components and using the kinematic equations: $x = v_{ox}\cdot t$ and $y = v_{0y}\cdot t - \frac{1}{2}g\cdot t^2$, where $v_{0x} = v_0\cdot\cos(\theta)$ and $v_{0y} = v_0\cdot\sin(\theta)$.

We'll use the restart command to clear previously defined variables:

restart :

a) Define the initial velocity components $v_{0x} = v_0\cdot\cos(\theta)$ and $v_{0y} = v_0\cdot\sin(\theta)$.

answer:

$v_{0x} := v_0 \cdot \cos(\theta)$

$$v_{0x} := v_0 \cos(\theta) \tag{8}$$

$v_{0y} := v_0 \cdot \sin(\theta)$

$$v_{0y} := v_0 \sin(\theta) \tag{9}$$

b) Write down the vertical kinematic equation and use the $tsol := solve(\ )$ command to solve for time, where we assign the solution to the variable $tsol$

answer:

$y := v_{0y} \cdot t - \left(\frac{1}{2}\right) \cdot g \cdot t^2$

$$y := v_0 \sin(\theta)\, t - \frac{g\, t^2}{2} \tag{10}$$

$tsol := solve\left(y = v_{0y} \cdot t - \left(\frac{1}{2}\right) \cdot g \cdot t^2, t\right)$

$$tsol := t \tag{11}$$

Becuase we are solving the quadratic equation, there will be two roots. Which root will be the relevant one? To pick the first root you can write $t := [tsol][1]$. To pick the second root, you can write $t := [tsol][2]$.

answer:

$t := [tsol][1]$

$$t := t \tag{12}$$

c) Write down the horizontal kinematic equation assuming constant velocity

answer:

$x := v_{ox} \cdot t$

$$x := v_{ox}\, t \tag{13}$$

d) Your expression for the horizontal position should be a function of the vertical height y. Because we want the range on level ground, we can set the height to be $y = 0$. Use the subs() command to substitute $y = 0$ into the expression for x. Assign the final result to R (this will be the range).

answer:

$R := subs(y = 0, x)$

$$R := v_{ox}\, t \tag{14}$$

We'll want to simplify this result by specifying that both $\sin(\theta)$ and v_0 are positive. Here's the command. You'll just need to execute this command after the expression for R has been defined above.

execute this command:

$R := simplify(R)$ assuming $\sin(\theta) > 0$ **and** $v_0 > 0$

$$R := v_{ox}\, t \tag{15}$$

If everything went well, you have derived the desired result!!!

e) Let's now plot your result. To plot, you will need to assign numerical values to the gravitational acceleration and initial velocity. Thus, we'll want to substitute the following in for the range R: $g = 9.8$ and $v_0 = 20$. Use the subs() command

answer:

$Rtheta := unapply(\ subs(g = 9.8, v\_\_0 = 20, R), \text{theta})$

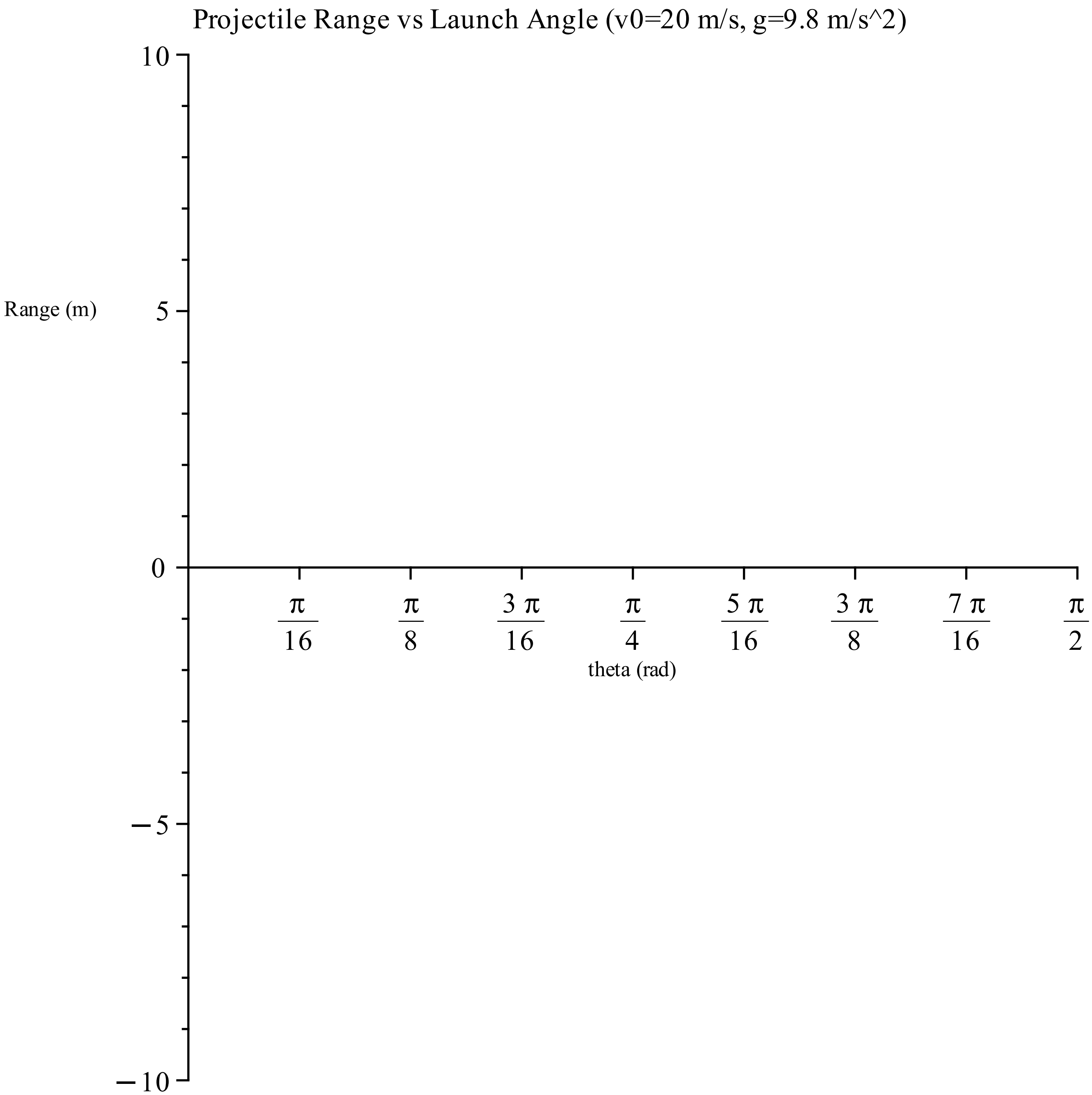
$$Rtheta := \theta \mapsto v_{ox} \cdot t \tag{16}$$

Plot the range as a function of the initial angle θ and label the axes:

answer:

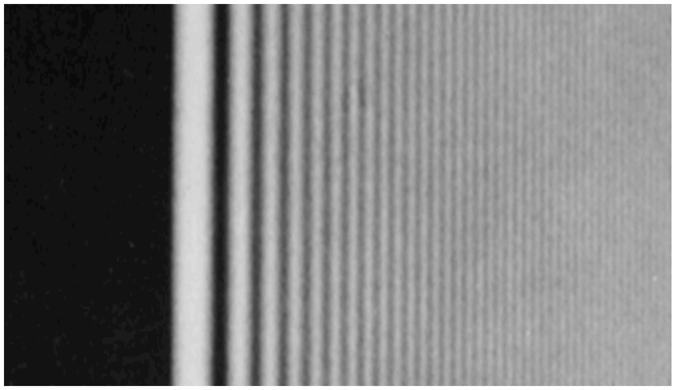
```
plot( Rtheta(theta),
    theta = 0..Pi/2,
    labels = ["theta (rad)", "Range (m)"],
    title = "Projectile Range vs Launch Angle (v0=20 m/s, g=9.8 m/s^2)")
```

Warning, expecting only range variable theta in expression v__ox*t to be plotted but found names [t, v__ox]



Problem 3) Fresnel Diffraction

When a straight edge casts a shadow, it produces a pattern of constructive and destructive interference zones when examined at scales close to the wavelength of the illuminating light. The pattern of light and dark bands is called Fresnel diffraction. Here's a photograph:



The intensity of the diffraction pattern close to the straight edge is approximately given by

$$F(z) = \frac{1}{2}\left(C(z) + \frac{1}{2}\right)^2 + \frac{1}{2}\left(S(z) + \frac{1}{2}\right)^2 \quad \text{where}$$

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2} t^2\right) \mathrm{d}t$$

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2} t^2\right) \mathrm{d}t$$

are known as the Fresnel integrals and z is a measure of the distance out from the straight edge.

Define two functions to calculate $C(z)$ and $S(z)$ and then create a third function to calculate the intensity $F(z)$. Plot $F(z)$ for $-5 < z < 10$.

answer:

```
restart;
C := unapply( int( cos(Pi/2*t^2), t = 0..z ), z )
```

$$C := z \mapsto \mathrm{FresnelC}(z) \tag{17}$$

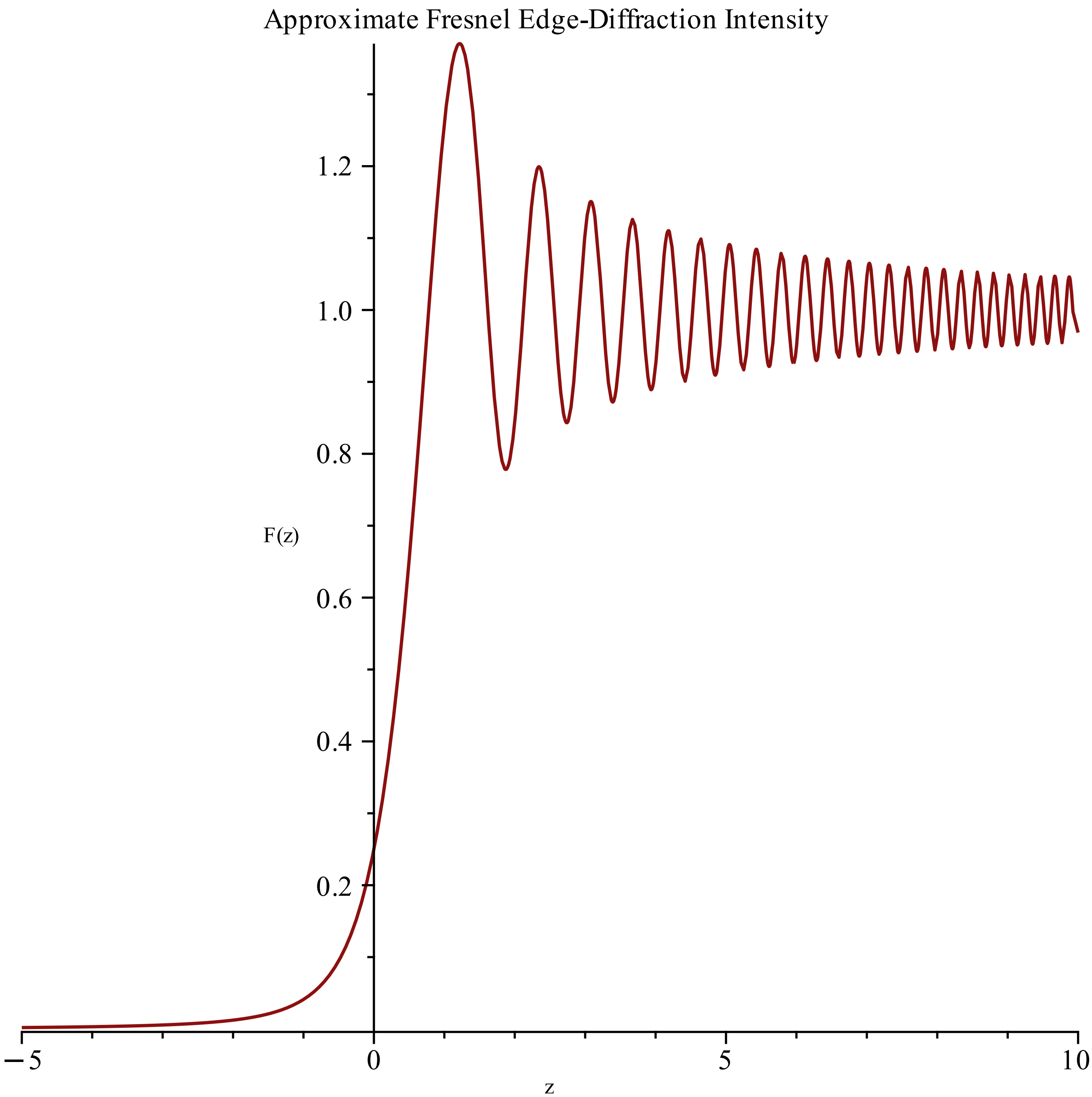
```
S := unapply( int( sin(Pi/2*t^2), t = 0..z ), z )
```

$$S := z \mapsto \mathrm{FresnelS}(z) \tag{18}$$

```
F := z -> 1/2*(C(z) + 1/2)^2 + 1/2*(S(z) + 1/2)^2
```

$$F := z \mapsto \frac{\left(C(z) + \frac{1}{2}\right)^2}{2} + \frac{\left(S(z) + \frac{1}{2}\right)^2}{2} \tag{19}$$

```
plot( F(z), z = -5..10,
    labels = ["z", "F(z)"],
    title = "Approximate Fresnel Edge-Diffraction Intensity" )
```



Approximate Fresnel Edge-Diffraction Intensity
1.2
1.0
0.8
F(z)
0.6
0.4
0.2
−5
0
5
10
z