EXPLORATION AND ANALYSIS OF RED-BLACK TREES AND MAX FLOW ALGORITHMS

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Contents

1. Abstract

We explore various properties and real-world applications of red-black trees and max-flow optimization algorithms. We explore several algorithms for the later, and provide experimental comparative analysis of their performances.

2. Introduction

3. Red-Black Trees

3.1. **Theoretical Analysis. Discuss the theoretical aspects of Red-Black Trees, including time and space complexity, best/worst/average-case scenarios, and any important properties.**

Red-Black Trees are a special type of binary search tree that are self-balancing: they maintain their balance during insertions and deletions.

Just as for any other type of binary search trees, for each node of a Red-Black Tree, all nodes in its left subtree have a value less than the node, and all nodes in its right subtree have a value greater than the node.

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Red-Black Trees have additional properties:

- Each node of the tree is either colored red or black
- The root and all leaves are black
- Having two consecutive red nodes is not allowed on any path from the root to a leaf.
- Every path from a node to any of its leaves contains the same number of black nodes.

Now, I will discuss the time complexity of Red-Black Trees.

- For the search operation, it has a time complexity of O(1). It happens when the element being searched is found at the root of the tree. It has an average and worst time complexity of O(log n), because the tree maintains balance, resulting in a logarithmic height.
- Concerning the insertion operation, it has a time complexity of O(1) in its best case, in which the new node is inserted at the root. Its average and worst time complexities are O(log n). On average, the tree needs to be balanced, keeping the height logarithmic. In the worst case, the tree needs restructuring, which takes logarithmic time.
- For the deletion operation, we have the exact same time complexities than for the insertion operation, for the same reasons.
- Finally, rotation operations are done in constant time.

Finally, I will expose the space complexity of Red-Black Trees. Each node in a Red-Black Tree requires constant space for the key, value, color information, and pointers to its children and parent. The overall space complexity is O(n), where n is the number of nodes in the tree.

Moreover, during operations like insertion and deletion, some additional space may be required for temporary variables or recursive function calls. The auxiliary space complexity is O(log n) in the worst case, where log n is the height of the tree.

3.2. **Practical Implementation Details.**

3.3. **Real-World Application.**

3.4. **Theoretical Questions. Prove that the height of a Red-Black tree with n nodes is guaranteed to be $O(logn)$ in the worst case scenario. Provide a rigorous mathematical proof.**

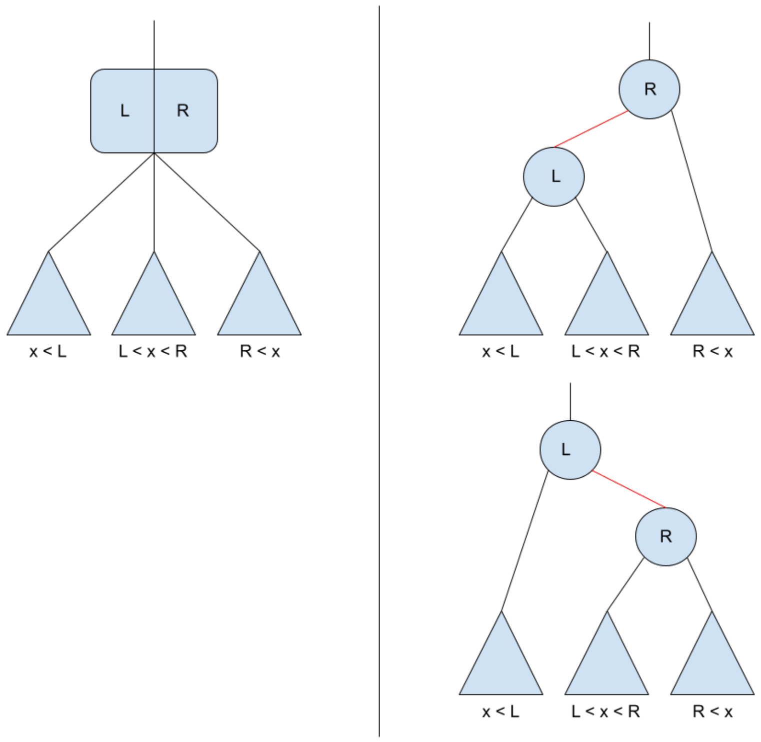
In this proof, we will first prove that the height of a 2-3-4 tree is limited by $O(logn)$. Then, we will prove that any valid red-black tree can be converted directly into a 2-3-4 tree. Finally, we will note that, so long as height is measured only in black links, tree height is maintained by this conversion.

A 2-3-4 tree, by its nature, only grows by pushing the root "upwards". The only time at which the height of such a tree increases is when a 4-node at the root splits, sending a node upwards to become the new root. In this case, the height of the tree uniformly increases by one for all leaf nodes. This means that the height of the tree is precisely equal for all leaf nodes.

Now we will examine the equivalency between red-black trees and 2-3-4 trees. We will show that each node in a 2-3-4 tree corresponds to exactly one black link in a red-black tree, and that the only additions needed are red links.

First, we will consider a 2-node. This is a node with two output links. This is equivalent to the standard node in a binary tree- no modifications are needed to modify it into red-black tree form.

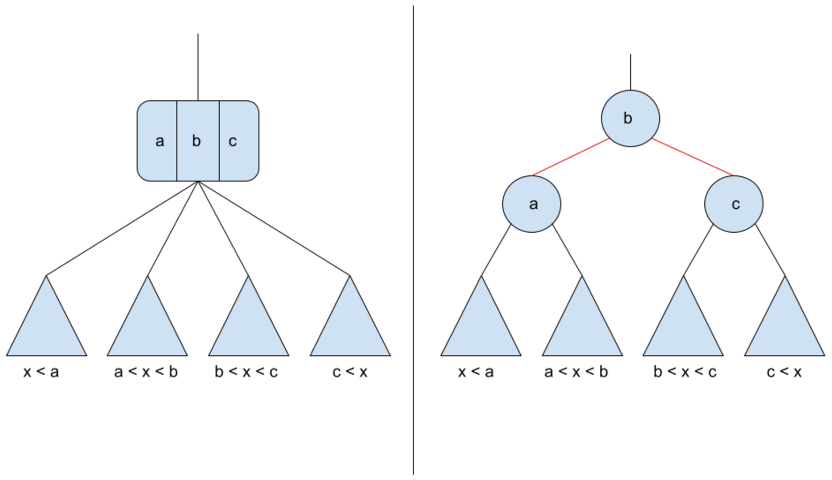
Next, we will consider a 3-node. This is a node with three output links. The leftmost represents the subtree wherein all nodes are less than the lesser item in the node. The rightmost similarly represents the subtree wherein all nodes are larger than the greater item, and the middle represents the subtree containing nodes who fit neither of these trees.



A 3-node and its possible red-black tree versions.

Since the height of a red-black tree is the number of black links the root must follow to get to a leaf, the two possible red-black subtrees above both have a height of 1: the same height as the 2-3-4 tree they came from.

The only remaining case is the 4-node. A 4-node usually only exists in a 2-3-4 tree for a moment before it is split apart. If we designate the 3 items within the node as a, b, and c, then we say that (from left to right) the child links represent the ranges $x < a$, $a < x < b$, $b < x < c$, and $c < x$ for any item x in the given child subtree. These cases, of course, can also be covered by an equivalent red-black tree, as shown below.



A 4-node and its red-black tree version.

Again, this red-black tree has the same height as its 2-3-4 tree equivalent: 1. Since we have accounted for all possible variations of red-black subtree herein, we can use the above rules to translate between red-black trees and 2-3-4 trees. Therefore, any statement we make about 2-3-4 trees holds for red-black trees.

In the best-case scenario, a 2-3-4 tree (post 4-node splitting) containing n nodes will have a height of $log_3(n)$, where every node is a 3-node. At worst case, it will have a height of $log_2(n)$, where every node is a 2-node. Since red-black and 2-3-4 trees are equivalent, we can thusly say that the worst-case height of a red-black tree of size n is limited by $log_2(n)$ black links.

Discuss how Red-Black trees are used in modern databases and file systems to maintain balanced structures. Explain the trade-offs and advantages of using Red-Black trees in these contexts.

4. Max Flow Algorithms

4.1. **Theoretical Analysis.** We will begin by analyzing the time and space complexity of the **Ford-Fulkerson algorithm** for max flow, and use this to motivate the segue into the Edmonds-Karp algorithm.

The Ford-Fulkerson algorithm works by repeatedly finding an augmenting path (a path which can traverse the graph from the source node to the sink node and contains some amount of unused capacity) and adding the minimal flow across this path to the flow of each edge within. When no such path can be found, the graph has reached its maximal flow, and an answer to the problem can be returned. The pseudocode for the Ford-Fulkerson algorithm is shown below.

```
// 1) Initialize
Set all edges in flow graph to zero
Set residual graph to input graph

// 2) Iterate
While an augmenting path exists:
    Get augmenting path
    Add augmenting path to flow graph
    Subtract augmenting path from residual graph
End while

// 3) Output
Set net flow to zero
For edge exiting source node:
    Increment net flow by edge flow
End for

Return net flow
```

The most ambiguous part of this process is `get augmenting path`. This vagueness has lead some to classify this process the Ford-Fulkerson method, rather than algorithm, as the time complexity could be vastly changed by the specific implementation of this line. For clarity, we will continue to refer to it as an algorithm. In our implementation, we will simply interpret this line to mean "choose the first augmenting path"- more specifically, "at each node, if multiple augmenting paths are present, choose the one leading to the lowest-indexed node from this point". That is to say, if the paths `0 -> 1 -> 2` and `0 -> 3 -> 2` both exists, our initial algorithm would choose the first.

Using these definitions and the above pseudocode, we can begin algorithmic analyses of the algorithm. We will call the set of all vertices $\mathbb{V}$, and the number of vertices $|\mathbb{V}|$. Similarly, the set of all edges and the number thereof are $\mathbb{E}$ and $|\mathbb{E}|$, respectively.

The algorithm starts by initializing its variables, marked section 1 in the above code. These operations take $|\mathbb{E}|$ each. Next, the algorithm iterates over the augmenting paths. For each iteration, it gets an augmenting path, adds the augmenting path to the flow, and subtracts it from the residual. This means that this section runs with time proportional to the number of augmenting paths times two times the length of the current path. Each pass of this iteration is guaranteed to add at least 1 unit to the net flow, so there will be at most f^* iterations, where f^* is the maximal flow. The final section of the algorithm (section 3, which computes the calculated net flow), runs in time proportional to the number of edges leading from the source node, which is at most $|\mathbb{E}|$.

Tying everything together, we can say that the Ford-Fulkerson algorithm runs in time proportional to

$$2 \cdot |\mathbb{E}| + f^* \cdot |\mathbb{E}| + |\mathbb{E}|$$

Converting this to big-O notation, we find that the running time complexity of the Ford-Fulkerson algorithm as listed here is $O(f^*|\mathbb{E}|)$.

As written here, the Ford-Fulkerson algorithm uses several auxiliary variables of equal size to the input graph. However, were the preservation of the input graph not necessary, the operations using these variables could easily be modified to work by modifying the input graph. This means that, under proper optimization, the Ford-Fulkerson algorithm runs with space complexity $O(1)$.

Having now finished analyzing the Ford-Fulkerson algorithm, we can move on to the Edmonds-Karp algorithm. As proven above, the Ford-Fulkerson algorithm can only be proved to converge upon the solution in time proportional to the max flow value itself. Thus, it is particularly unsuited for graphs with large maximal flows. Additionally, the original specification for the Ford-Fulkerson algorithm left the specific method used to select the next augmenting path unspecified. The solution to both of these problems comes in the form of the **Edmonds-Karp algorithm**.

The Edmonds-Karp algorithm is a specific implementation of the Ford-Fulkerson method / algorithm wherein the next augmenting path to be chosen is the shortest one. This shortest path is chosen via breadth-first-search of the network. This allows the time complexity of the algorithm to become independent from $f*$.

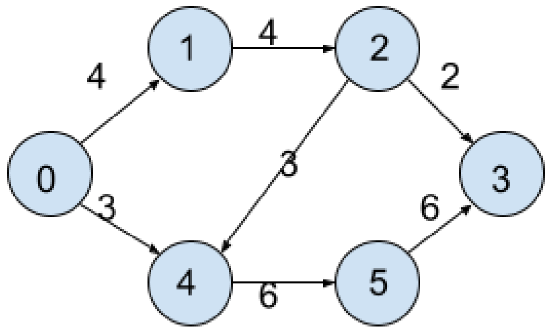
The pseudocode for the Edmonds-Karp algorithm is listed below.

```
// 1) Initialize
Set all edges in flow graph to zero
Set residual graph to input graph

// 2) Iterate
While an augmenting path exists:
    Get shortest augmenting path via BFS
    Add augmenting path to flow graph
    Subtract augmenting path from residual graph
End while

// 3) Output
Set net flow to zero
For edge exiting source node:
    Increment net flow by edge flow
End for

Return net flow
```

4.2. **Practical Implementation Details.** The pseudocode from the previous section is fairly easy to translate into a viable program. However, we must first define how we will represent a weighted graph.

We will represent a graph via a `C++` vector of graph nodes. In turn, we will represent a graph node as a struct containing a `C++` map mapping a node index to the weight on the edge associated with it. Additionally, a graph node will have a list of nodes which have edges pointing to it.

We will represent an augmenting path as a vector of pairs of integers, with the first item being the index of the next node to visit and the second integer being the weight on that edge. For this representation to work, we also need to keep track of the initial node. We will use the same type graph data structure to represent the flow graph, the capacity graph, and the residual graph. To load graphs, we will use edge lists. Specifically, we will have the number of nodes, followed by the number of edges. After this, each edge will be three integers: The source node, the destination node, and the weight.



A benchmark graph for max-flow.

Represented as an edge list, this graph would look like this:

```
6 7
0 1 4
0 4 3
1 2 4
4 5 6
2 4 3
2 3 2
5 3 6
```

Next, let's formalize the process by which we select augmenting paths for both algorithms. In the Ford-Fulkerson algorithm, we will use the following method.

```
Function to find the first augmenting path
Takes: The residual graph, start node, and sink node
Returns: The first augmenting path

    Let current = start node
    Let path = empty list
    Let visited = empty set

    While current != end node, do
        Insert current into visited

        For edge leading from current in order:
            If edge destination is in visited
                Continue
            End if

            If edge has additional flow remaining:
                Append edge onto path
                Current = edge destination
                Break from for loop
            End if
        End for
    End while

    Return path

End function
```

This returns the first available path, regardless of its length. Similarly, the Edmonds-Karp algorithm uses the pseudocode below to find the shortest available augmenting path.

```
Function to find the shortest augmenting path via BFS
Takes: The residual graph, start node, and sink node
Returns: The shortest augmenting path

    Let to_visit = queue
    Let came_from = array of size count(nodes) filled with -1

    push source node onto to_visit

    // Breadth-First Search

    While to_visit is not empty, do
        Let current = front of to_visit
        Pop front off of to_visit
        Insert current into visited

        For edge leading from current:
            If came_from[edge destination] == -1:
                came_from[edge destination] = current
                push edge destination onto to_visit

                If edge destination is sink node:
                    Break from for loop
                End if
            End if
        End for
    End while

    // Reconstruct path

    Let path = empty list
    Let current = sink node

    While current != source node, do
        If came_from[current] == -1:
            Return empty path
        End if

        Insert edge from came_from[current] to current at front of path
    End while

    Return path

End function
```

These two pseudocode functions complete our algorithm for the Ford-Fulkerson and the Edmonds-Karp algorithm. The `C++` used in this project is a translation of this.

4.3. Real-World Application.

4.4. Theoretical Questions. Describe the concept of augmenting paths and their role in the Ford-Fulkerson algorithm. Prove that the algorithm terminates and converges to the maximum flow in finite time, even for non-integer capacities.

Explain the Min-Cut Max-Flow Theorem and its significance in the context of network flows. How can this theorem be used to find a maximum flow and minimum cut in a flow network?

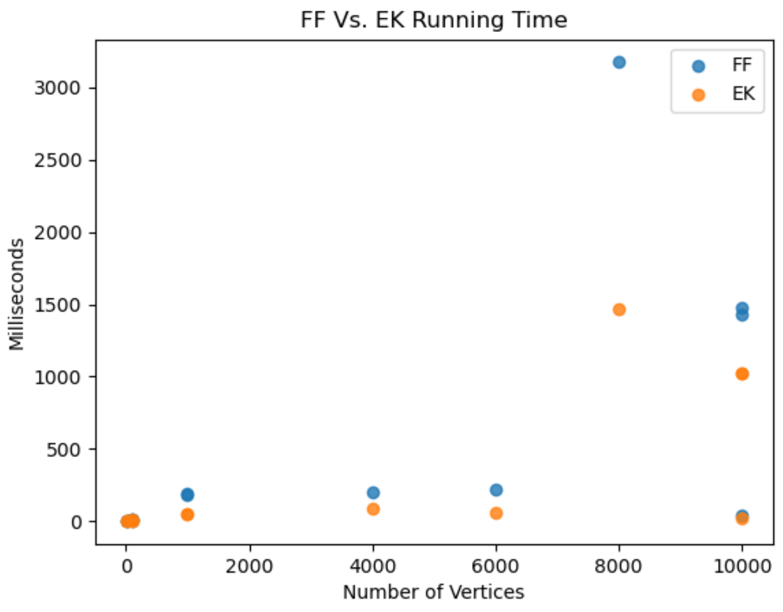
5. Real-World Problem Solving

6. Comparative Analysis and Reporting

6.1. Red-Black Trees.

6.2. Max Flow Algorithms.

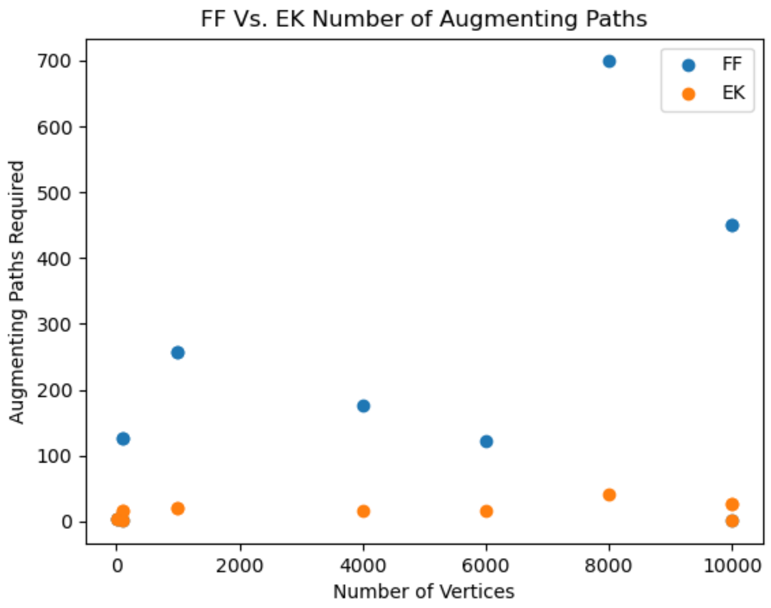
Below is a graph of running times from a series of experimental runs of the Ford-Fulkerson (FF) and Edmonds-Karp (EK) algorithms on various randomly-generated graphs.



The Ford-Fulkerson (FF) algorithm vs the Edmonds-Karp (EK) algorithm on randomly generated graphs of various size.

Clearly, the Edmonds-Karp algorithm performs reliably better than the original Ford-Fulkerson one, especially for large graphs.

The below graph shows the number of augmenting paths present in graphs of various sizes.



The Ford-Fulkerson (FF) algorithm vs the Edmonds-Karp (EK) algorithm on various graphs, by number of passes required.

7. Conclusion

8. References

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- https://brilliant.org/wiki/edmonds-karp-algorithm/