

EXPLORATION AND ANALYSIS OF RED-BLACK TREES AND MAX FLOW ALGORITHMS

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1. ABSTRACT

We explore various properties and real-world applications of red-black trees and max-flow optimization algorithms. We explore several algorithms for the later, and provide experimental comparative analysis of their performances.

2. INTRODUCTION

3. RED-BLACK TREES

3.1. Theoretical Analysis.

3.2. Practical Implementation Details.

3.3. Real-World Application.

3.4. Theoretical Questions. Prove that the height of a Red-Black tree with n nodes is guaranteed to be $O(\log n)$ in the worst case scenario. Provide a rigorous mathematical proof.

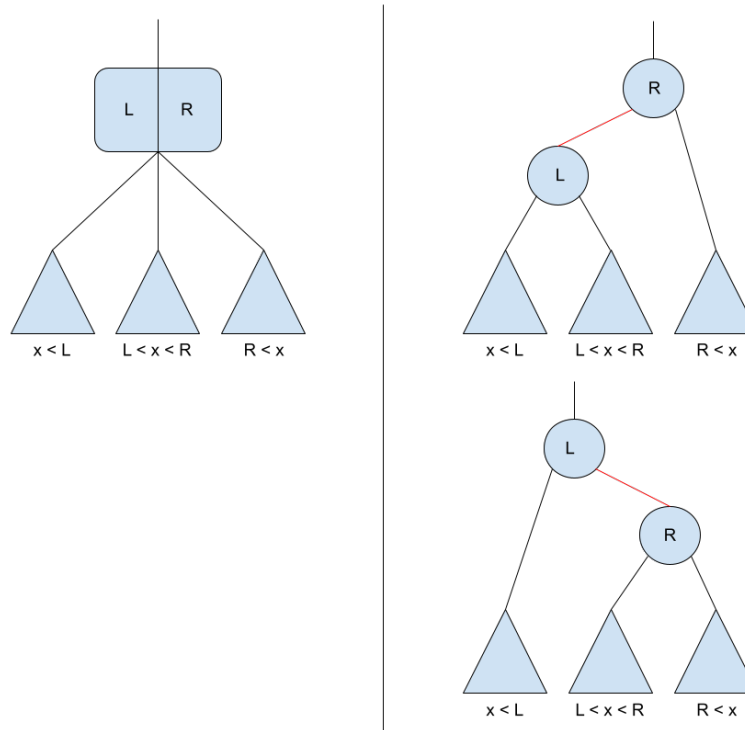
In this proof, we will first prove that the height of a 2-3-4 tree is limited by $O(\log n)$. Then, we will prove that any valid red-black tree can be converted directly into a 2-3-4 tree. Finally, we will note that, so long as height is measured only in black links, tree height is maintained by this conversion.

A 2-3-4 tree, by its nature, only grows by pushing the root "upwards". The only time at which the height of such a tree increases is when a 4-node at the root splits, sending a node upwards to become the new root. In this case, the height of the tree uniformly increases by one for all leaf nodes. This means that the height of the tree is precisely equal for all leaf nodes.

Now we will examine the equivalency between red-black trees and 2-3-4 trees. We will show that each node in a 2-3-4 tree corresponds to exactly one black link in a red-black tree, and that the only additions needed are red links.

First, we will consider a 2-node. This is a node with two output links. This is equivalent to the standard node in a binary tree- no modifications are needed to modify it into red-black tree form.

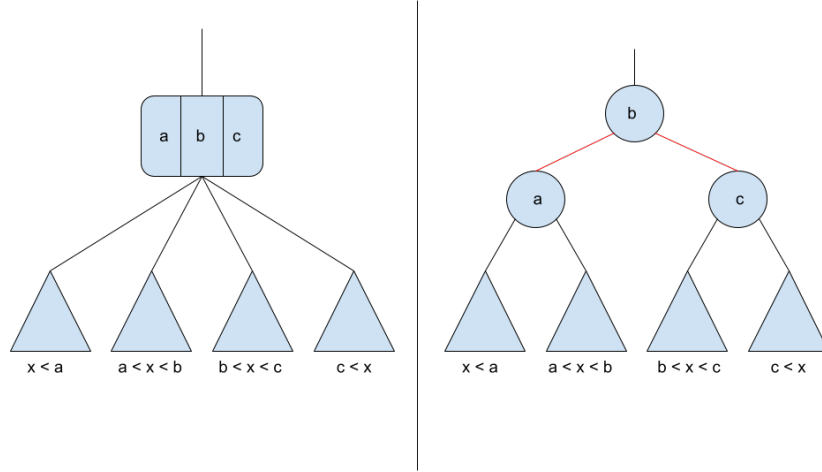
Next, we will consider a 3-node. This is a node with three output links. The leftmost represents the subtree wherein all nodes are less than the lesser item in the node. The rightmost similarly represents the subtree wherein all nodes are larger than the greater item, and the middle represents the subtree containing nodes who fit neither of these trees.



A 3-node and its possible red-black tree versions.

Since the height of a red-black tree is the number of black links the root must follow to get to a leaf, the two possible red-black subtrees above both have a height of 1: the same height as the 2-3-4 tree they came from.

The only remaining case is the 4-node. A 4-node usually only exists in a 2-3-4 tree for a moment before it is split apart. If we designate the 3 items within the node as a , b , and c , then we say that (from left to right) the child links represent the ranges $x < a$, $a < x < b$, $b < x < c$, and $c < x$ for any item x in the given child subtree. These cases, of course, can also be covered by an equivalent red-black tree, as shown below.



A 4-node and its red-black tree version.

Again, this red-black tree has the same height as its 2-3-4 tree equivalent: 1. Since we have accounted for all possible variations of red-black subtree herein, we can use the above rules to translate between red-black trees and 2-3-4 trees. Therefore, any statement we make about 2-3-4 trees holds for red-black trees.

In the best-case scenario, a 2-3-4 tree (post 4-node splitting) containing n nodes will have a height of $\log_3(n)$, where every node is a 3-node. At worst case, it will have a height of $\log_2(n)$, where every node is a 2-node. Since red-black and 2-3-4 trees are equivalent, we can thusly say that the worst-case height of a red-black tree of size n is limited by $\log_2(n)$ black links.

Discuss how Red-Black trees are used in modern databases and file systems to maintain balanced structures. Explain the trade-offs and advantages of using Red-Black trees in these contexts.

4. MAX FLOW ALGORITHMS

4.1. Theoretical Analysis. We will begin by analyzing the time and space complexity of the **Ford-Fulkerson algorithm** for max flow, and use this to motivate the segue into the Edmonds-Karp algorithm.

The Ford-Fulkerson algorithm works by repeatedly finding an augmenting path (a path which can traverse the graph from the source node to the sink node and contains some amount of unused capacity) and adding the minimal flow across this path to the flow of each edge within. When no such path can be found, the graph has reached its maximal flow, and an answer to the problem can be returned. The pseudocode for the Ford-Fulkerson algorithm is shown below.

```
// 1) Initialize
Set all edges in flow graph to zero
Set residual graph to input graph

// 2) Iterate
While an augmenting path exists:
    Get augmenting path
    Add augmenting path to flow graph
    Subtract augmenting path from residual graph
End while

// 3) Output
Set net flow to zero
For edge exiting source node:
    Increment net flow by edge flow
End for

Return net flow
```

The most ambiguous part of this process is `get augmenting path`. This vagueness has lead some to classify this process the Ford-Fulkerson method, rather than algorithm, as the time complexity could be vastly changed by the specific implementation of this line. For clarity, we will continue to refer to it as an algorithm. In our implementation, we will simply interpret this line to mean "choose the first augmenting path"- more specifically, "at each node, if multiple augmenting paths are present, choose the one leading to the lowest-indexed node from this point". That is to say, if the paths $0 \rightarrow 1 \rightarrow 2$ and $0 \rightarrow 3 \rightarrow 2$ both exists, our initial algorithm would choose the first.

Using these definitions and the above pseudocode, we can begin algorithmic analyses of the algorithm. We will call the set of all vertices \mathbb{V} , and the number of vertices $|\mathbb{V}|$. Similarly, the set of all edges and the number thereof are \mathbb{E} and $|\mathbb{E}|$, respectively.

The algorithm starts by initializing its variables, marked section 1 in the above code. These operations take $|\mathbb{E}|$ each. Next, the algorithm iterates over the augmenting paths. For each iteration, it gets an augmenting path, adds the augmenting path to the flow, and subtracts it from the residual. This means that this section runs with time proportional to the number of augmenting paths times two times the length of the current path. Each pass of this iteration is guaranteed to add at least 1 unit to the net flow, so there will be at most f^* iterations, where f^* is the maximal flow. The

final section of the algorithm (section 3, which computes the calculated net flow), runs in time proportional to the number of edges leading from the source node, which is at most $|\mathbb{E}|$.

Tying everything together, we can say that the Ford-Fulkerson algorithm runs in time proportional to

$$2 \cdot |\mathbb{E}| + f^* \cdot |\mathbb{E}| + |\mathbb{E}|$$

Converting this to big-O notation, we find that the running time complexity of the Ford-Fulkerson algorithm as listed here is $O(f^*|\mathbb{E}|)$.

As written here, the Ford-Fulkerson algorithm uses several auxiliary variables of equal size to the input graph. However, were the preservation of the input graph not necessary, the operations using these variables could easily be modified to work by modifying the input graph. This means that, under proper optimization, the Ford-Fulkerson algorithm runs with space complexity $O(1)$.

Having now finished analyzing the Ford-Fulkerson algorithm, we can move on to the Edmonds-Karp algorithm. As proven above, the Ford-Fulkerson algorithm can only be proved to converge upon the solution in time proportional to the max flow value itself. Thus, it is particularly unsuited for graphs with large maximal flows. Additionally, the original specification for the Ford-Fulkerson algorithm left the specific method used to select the next augmenting path unspecified. The solution to both of these problems comes in the form of the **Edmonds-Karp algorithm**.

The Edmonds-Karp algorithm is a specific implementation of the Ford-Fulkerson method / algorithm wherein the next augmenting path to be chosen is the shortest one. This shortest path is chosen via breadth-first-search of the network. This allows the time complexity of the algorithm to become independent from f^* .

The pseudocode for the Edmonds-Karp algorithm is listed below.

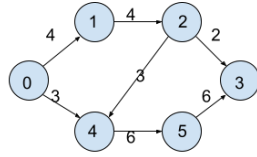
```
// 1) Initialize
Set all edges in flow graph to zero
Set residual graph to input graph

// 2) Iterate
While an augmenting path exists:
    Get shortest augmenting path via BFS
    Add augmenting path to flow graph
    Subtract augmenting path from residual graph
End while

// 3) Output
Set net flow to zero
For edge exiting source node:
    Increment net flow by edge flow
End for

Return net flow
```

4.2. Practical Implementation Details.



A benchmark graph for max-flow.

4.3. Real-World Application.

4.4. Theoretical Questions. Describe the concept of augmenting paths and their role in the Ford-Fulkerson algorithm. Prove that the algorithm terminates and converges to the maximum flow in finite time, even for non-integer capacities.

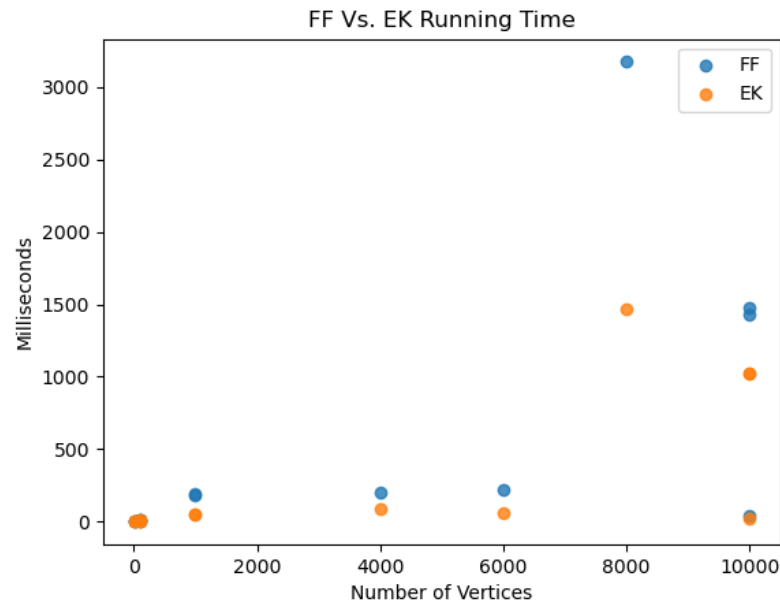
Explain the Min-Cut Max-Flow Theorem and its significance in the context of network flows. How can this theorem be used to find a maximum flow and minimum cut in a flow network?

5. REAL-WORLD PROBLEM SOLVING

6. COMPARATIVE ANALYSIS AND REPORTING

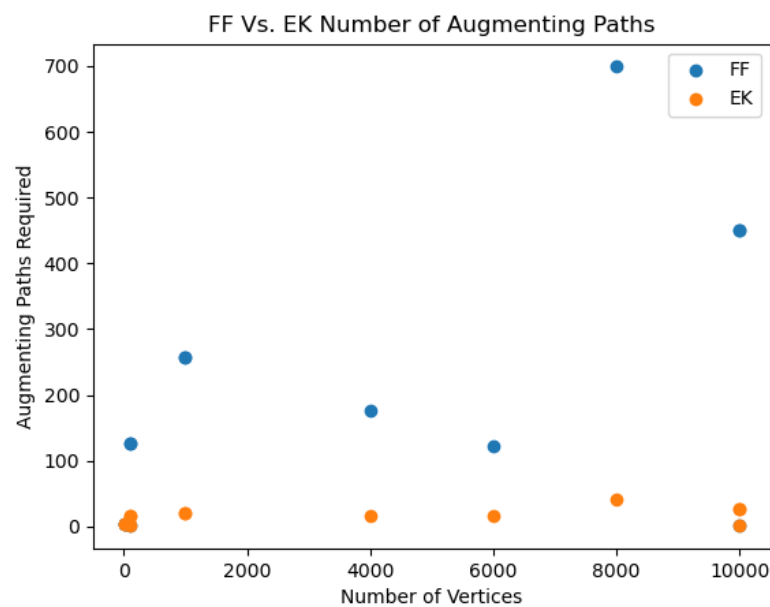
6.1. Red-Black Trees.

6.2. Max Flow Algorithms. Below is a graph of running times from a series of experimental runs of the Ford-Fulkerson (FF) and Edmonds-Karp (EK) algorithms on various randomly-generated graphs.



The Ford-Fulkerson (FF) algorithm vs the Edmonds-Karp (EK) algorithm on randomly generated graphs of various size.

Similarly, the below graph shows the number of augmenting paths present in graphs of various sizes.



The Ford-Fulkerson (FF) algorithm vs the Edmonds-Karp (EK) algorithm on various graphs, by number of passes required.

7. CONCLUSION

8. REFERENCES

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- <https://brilliant.org/wiki/ford-fulkerson-algorithm/>
- <https://brilliant.org/wiki/edmonds-karp-algorithm/>