EXPLORATION AND ANALYSIS OF RED-BLACK TREES AND MAX FLOW ALGORITHMS

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Contents

1. Abstract

We explore various properties and real-world applications of red-black trees and max-flow optimization algorithms. We explore several algorithms for the later, and provide experimental comparative analysis of their performances.

Date: December 4th, 2023.

2. Introduction

2.1. **Efficient Tree-Based Lookup Structures.** The Binary Search Tree (BST) is a vital data structure for efficient lookups in large lookup tables. However, they have their issues. While a BST can provide lookups in logarithmic time in the best-case scenario, this can quickly devolve into linear time given suboptimal insertion order. For instance, if data is inserted into a BST in monotonically ascending order, each node in the tree will have only right children. This quickly devolves into linear lookup time! This weakness motivates the need for a tree data structure which can be proven to avoid this worst-case scenario.

Several such tree-based data structures exist; The 2-3-4 tree and the red-black tree. Both of these structures can be proven to remain perfectly balanced, allowing a guaranteed logarithmic lookup time. In this paper, we will explore the links between these two data structures, and perform a deeper analysis of the red-black tree.

2.2. **Graph Flow Algorithms.** Determining the maximum flow across a directed weighted graph is vital in numerous fields, especially since many problems are convertible to graph problems. For instance, if we were to visualize a computer network as a directed graph where the nodes are computers and the weights are bandwidths, it would be desirable to determine the maximum possible flow of data between a given two nodes. Graph Flow Algorithms solve just this problem. The two we will be exploring in this paper are the Ford-Fulkerson method and the Edmonds-Karp algorithm, both of which use the underlying mechanic of augmenting paths to determine the maximal flow.

3. Red-Black Trees

3.1. **Theoretical Analysis. Discuss the theoretical aspects of Red-Black Trees, including time and space complexity, best/worst/average-case scenarios, and any important properties.**

Red-Black Trees are a special type of binary search tree that are self-balancing: they maintain their balance during insertions and deletions.

Just as for any other type of binary search trees, for each node of a Red-Black Tree, all nodes in its left subtree have a value less than the node, and all nodes in its right subtree have a value greater than the node.

Red-Black Trees have additional properties:

- Each node of the tree is either colored red or black
- The root and all leaves are black
- Having two consecutive red nodes is not allowed on any path from the root to a leaf.
- Every path from a node to any of its leaves contains the same number of black nodes.

Now, I will discuss the time complexity of Red-Black Trees.

- For the search operation, it has a time complexity of O(1). It happens when the element being searched is found at the root of the tree. It has an average and worst time complexity of O(log n), because the tree maintains balance, resulting in a logarithmic height.
- Concerning the insertion operation, it has a time complexity of O(1) in its best case, in which the new node is inserted at the root. Its average and worst time complexities are O(log n). On average, the tree needs to be balanced, keeping the height logarithmic. In the worst case, the tree needs restructuring, which takes logarithmic time.
- For the deletion operation, we have the exact same time complexities than for the insertion operation, for the same reasons.
- Finally, rotation operations are done in constant time.

Finally, I will expose the space complexity of Red-Black Trees. Each node in a Red-Black Tree requires constant space for the key, value, color information, and pointers to its children and parent. The overall space complexity is O(n), where n is the number of nodes in the tree.

Moreover, during operations like insertion and deletion, some additional space may be required for temporary variables or recursive function calls. The auxiliary space complexity is O(log n) in the worst case, where log n is the height of the tree.

3.2. **Practical Implementation Details.**

3.3. **Real-World Application.**

3.4. **Theoretical Questions. Prove that the height of a Red-Black tree with n nodes is guaranteed to be $O(logn)$ in the worst case scenario. Provide a rigorous mathematical proof.**

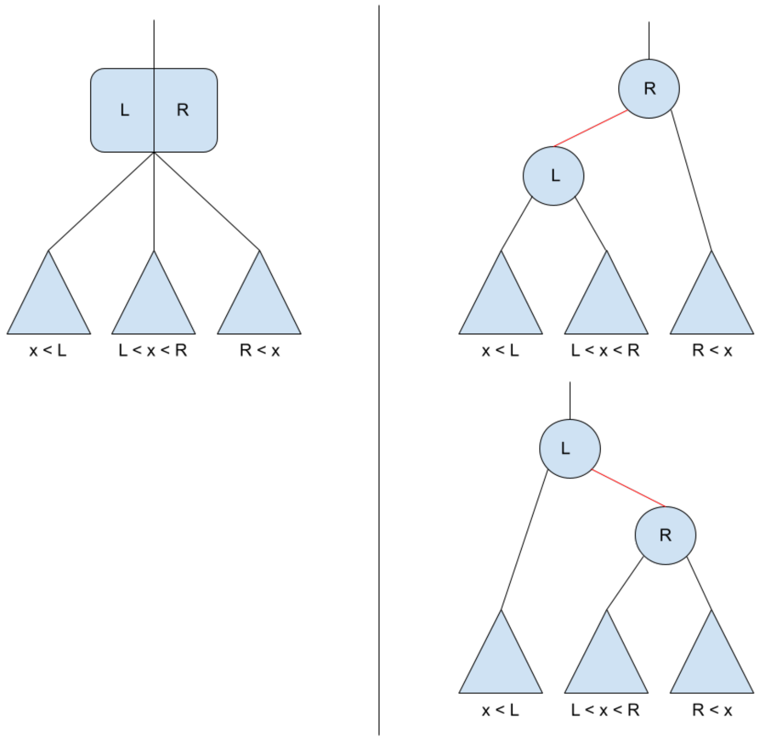
In this proof, we will first prove that the height of a 2-3-4 tree is limited by $O(logn)$. Then, we will prove that any valid red-black tree can be converted directly into a 2-3-4 tree. Finally, we will note that, so long as height is measured only in black links, tree height is maintained by this conversion.

A 2-3-4 tree, by its nature, only grows by pushing the root "upwards". The only time at which the height of such a tree increases is when a 4-node at the root splits, sending a node upwards to become the new root. In this case, the height of the tree uniformly increases by one for all leaf nodes. This means that the height of the tree is precisely equal for all leaf nodes.

Now we will examine the equivalency between red-black trees and 2-3-4 trees. We will show that each node in a 2-3-4 tree corresponds to exactly one black link in a red-black tree, and that the only additions needed are red links.

First, we will consider a 2-node. This is a node with two output links. This is equivalent to the standard node in a binary tree- no modifications are needed to modify it into red-black tree form.

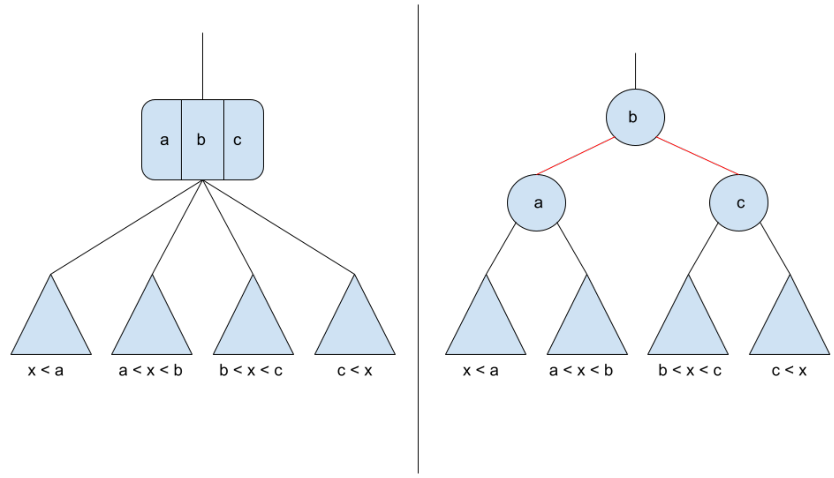
Next, we will consider a 3-node. This is a node with three output links. The leftmost represents the subtree wherein all nodes are less than the lesser item in the node. The rightmost similarly represents the subtree wherein all nodes are larger than the greater item, and the middle represents the subtree containing nodes who fit neither of these trees.



A 3-node and its possible red-black tree versions.

Since the height of a red-black tree is the number of black links the root must follow to get to a leaf, the two possible red-black subtrees above both have a height of 1: the same height as the 2-3-4 tree they came from.

The only remaining case is the 4-node. A 4-node usually only exists in a 2-3-4 tree for a moment before it is split apart. If we designate the 3 items within the node as a, b, and c, then we say that (from left to right) the child links represent the ranges $x < a$, $a < x < b$, $b < x < c$, and $c < x$ for any item x in the given child subtree. These cases, of course, can also be covered by an equivalent red-black tree, as shown below.



A 4-node and its red-black tree version.

Again, this red-black tree has the same height as its 2-3-4 tree equivalent: 1. Since we have accounted for all possible variations of red-black subtree herein, we can use the above rules to translate between red-black trees and 2-3-4 trees. Therefore, any statement we make about 2-3-4 trees holds for red-black trees.

In the best-case scenario, a 2-3-4 tree (post 4-node splitting) containing n nodes will have a height of $log_3(n)$, where every node is a 3-node. At worst case, it will have a height of $log_2(n)$, where every node is a 2-node. Since red-black and 2-3-4 trees are equivalent, we can thusly say that the worst-case height of a red-black tree of size n is limited by $log_2(n)$ black links.

Discuss how Red-Black trees are used in modern databases and file systems to maintain balanced structures. Explain the trade-offs and advantages of using Red-Black trees in these contexts.

To begin with, I will discuss the use of Red-Black trees in modern databases. Firstly, Red-Black trees provide efficient search and retrieval operations with a time complexity of O(log n). This makes them well-suited for databases, where quick access to data is crucial.

Also, the self-balancing property of Red-Black trees ensures that it remains balanced during insertions and deletions. This property is vital in databases, because obtaining an asymmetric data structure can result in degraded performances.

Finally, Red-Black trees are useful for querying a single or range elements, essential operation for a database.

Red-Black trees are also very useful in file systems. Since they are capable of indexing and maintaining the hierarchical structure of directories in file systems, they permit to efficiently search for files.

Also, as files are added or removed, the Red-Black tree's self-balancing property ensures that the directory structure remains balanced. This is important for maintaining quick and consistent file operations, such as searches and deletions.

In these two contexts, their are some advantages and trade-off of using Red-Black trees. Red-Black trees guarantee a balanced structure, which results in predictable and efficient performance for various operations, including search, insertion, and deletion.

Also, the worst-case time complexity for operations on Red-Black trees is O(log n), providing predictable and consistent performance in a variety of scenarios.

Anyway, Red-Black trees require additional space to store color information for each node, which can increase the overall space overhead compared to simpler data structures.

Moreover, the algorithms for maintaining the Red-Black tree's properties during insertions and deletions are more complex than those for simpler binary search trees, which may lead to slightly slower operations.

Ultimately, Red-Black trees are a good choice for modern databases and file systems, that require predictable performance. Anyway, the decision to use Red-Black trees involves trade-offs, such as increased space overhead and algorithmic complexity. The appropriateness of Red-Black trees depends on the specific requirements and characteristics of the application or system.

4. Max Flow Algorithms

4.1. **Theoretical Analysis.** We will begin by analyzing the time and space complexity of the **Ford-Fulkerson algorithm** for max flow, and use this to motivate the segue into the Edmonds-Karp algorithm.

The Ford-Fulkerson algorithm works by repeatedly finding an augmenting path (a path which can traverse the graph from the source node to the sink node and contains some amount of unused capacity) and adding the minimal flow across this path to the flow of each edge within. When no such path can be found, the graph has reached its maximal flow, and an answer to the problem can be returned. The pseudocode for the Ford-Fulkerson algorithm is shown below.

```
// 1) Initialize
Set all edges in flow graph to zero
Set residual graph to input graph

// 2) Iterate
While an augmenting path exists:
    Get augmenting path
```

```
    Add augmenting path to flow graph
    Subtract augmenting path from residual graph
End while

// 3) Output
Set net flow to zero
For edge exiting source node:
    Increment net flow by edge flow
End for

Return net flow
```

The most ambiguous part of this process is `get augmenting path`. This vagueness has lead some to classify this process the Ford-Fulkerson method, rather than algorithm, as the time complexity could be vastly changed by the specific implementation of this line. For clarity, we will continue to refer to it as an algorithm. In our implementation, we will simply interpret this line to mean "choose the first augmenting path"- more specifically, "at each node, if multiple augmenting paths are present, choose the one leading to the lowest-indexed node from this point". That is to say, if the paths `0 -> 1 -> 2` and `0 -> 3 -> 2` both exists, our initial algorithm would choose the first.

Using these definitions and the above pseudocode, we can begin algorithmic analyses of the algorithm. We will call the set of all vertices $\mathbb{V}$, and the number of vertices $|\mathbb{V}|$. Similarly, the set of all edges and the number thereof are $\mathbb{E}$ and $|\mathbb{E}|$, respectively.

The algorithm starts by initializing its variables, marked section 1 in the above code. These operations take $|\mathbb{E}|$ each. Next, the algorithm iterates over the augmenting paths. For each iteration, it gets an augmenting path, adds the augmenting path to the flow, and subtracts it from the residual. This means that this section runs with time proportional to the number of augmenting paths times two times the length of the current path. Each pass of this iteration is guaranteed to add at least 1 unit to the net flow, so there will be at most f^* iterations, where f^* is the maximal flow. The final section of the algorithm (section 3, which computes the calculated net flow), runs in time proportional to the number of edges leading from the source node, which is at most $|\mathbb{E}|$.

Tying everything together, we can say that the Ford-Fulkerson algorithm runs in time proportional to

$$2 \cdot |\mathbb{E}| + f^* \cdot |\mathbb{E}| + |\mathbb{E}|$$

Converting this to big-O notation, we find that the running time complexity of the Ford-Fulkerson algorithm as listed here is $O(f^*|\mathbb{E}|)$.

As written here, the Ford-Fulkerson algorithm uses several auxiliary variables of equal size to the input graph. However, were the preservation of the input graph not necessary, the operations using these variables could easily be modified to work by modifying the input graph. This means that, under proper optimization, the Ford-Fulkerson algorithm runs with space complexity $O(1)$.

Having now finished analyzing the Ford-Fulkerson algorithm, we can move on to the Edmonds-Karp algorithm. As proven above, the Ford-Fulkerson algorithm can only be proved to converge upon the solution in time proportional to the max flow value itself. Thus, it is particularly unsuited for graphs with large maximal flows. Additionally, the original specification for the Ford-Fulkerson algorithm left the specific method used to select the next augmenting path unspecified. The solution to both of these problems comes in the form of the **Edmonds-Karp algorithm**.

The Edmonds-Karp algorithm is a specific implementation of the Ford-Fulkerson method / algorithm wherein the next augmenting path to be chosen is the shortest one. This shortest path is chosen via breadth-first-search of the network. This allows the time complexity of the algorithm to become independent from $f*$. More specifically, the Edmonds-Karp algorithm runs in time complexity $O(|\mathbb{V}||\mathbb{E}|^2)$.

The pseudocode for the Edmonds-Karp algorithm is listed below.

```
// 1) Initialize
Set all edges in flow graph to zero
Set residual graph to input graph

// 2) Iterate
While an augmenting path exists:
    Get shortest augmenting path via BFS
    Add augmenting path to flow graph
    Subtract augmenting path from residual graph
End while

// 3) Output
Set net flow to zero
For edge exiting source node:
    Increment net flow by edge flow
End for

Return net flow
```

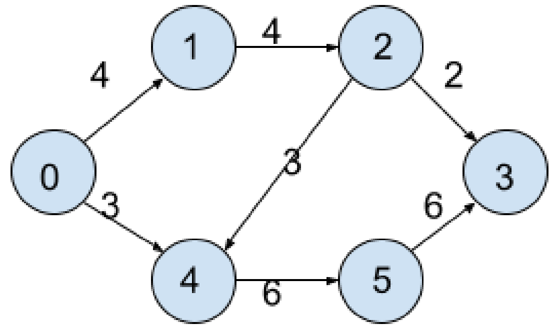
Both algorithms as described here create copies of their input graphs for the flow and residual graphs. Additionally, neither the Ford-Fulkerson nor the Edmonds-Karp method of augmenting path discovery requires space larger than the size of the input graph. Thus, we can say that both algorithms run with space complexity bounded by the size of the input graph. Expressed in big-O notation, this appears as follows:

$$O(|\mathbb{E}| + |\mathbb{V}|)$$

4.2. **Practical Implementation Details.** The pseudocode from the previous section is fairly easy to translate into a viable program. However, we must first define how we will represent a weighted graph.

We will represent a graph via a `C++` vector of graph nodes. In turn, we will represent a graph node as a struct containing a `C++` map mapping a node index to the weight on the edge associated with it. Additionally, a graph node will have a list of nodes which have edges pointing to it.

We will represent an augmenting path as a vector of pairs of integers, with the first item being the index of the next node to visit and the second integer being the weight on that edge. For this representation to work, we also need to keep track of the initial node. We will use the same type graph data structure to represent the flow graph, the capacity graph, and the residual graph. To load graphs, we will use edge lists. Specifically, we will have the number of nodes, followed by the number of edges. After this, each edge will be three integers: The source node, the destination node, and the weight.



A benchmark graph for max-flow.

Represented as an edge list, this graph would look like this:

```
6 7
0 1 4
0 4 3
1 2 4
4 5 6
2 4 3
2 3 2
5 3 6
```

Next, let's formalize the process by which we select augmenting paths for both algorithms. In the Ford-Fulkerson algorithm, we will use the following method.

```
Function to find the first augmenting path
Takes: The residual graph, start node, and sink node
Returns: The first augmenting path

    Let current = start node
    Let path = empty list
    Let visited = empty set

    While current != end node, do
        Insert current into visited

        For edge leading from current in order:
            If edge destination is in visited
                Continue
            End if

            If edge has additional flow remaining:
                Append edge onto path
                Current = edge destination
                Break from for loop
            End if
        End for
    End while

    Return path
```

```
End function
```

This returns the first available path, regardless of its length. Similarly, the Edmonds-Karp algorithm uses the pseudocode below to find the shortest available augmenting path.

```
Function to find the shortest augmenting path via BFS
Takes: The residual graph, start node, and sink node
Returns: The shortest augmenting path

    Let to_visit = queue
    Let came_from = array of size count(nodes) filled with -1

    push source node onto to_visit

    // Breadth-First Search

    While to_visit is not empty, do
        Let current = front of to_visit
        Pop front off of to_visit
        Insert current into visited

        For edge leading from current:
            If came_from[edge destination] == -1:
                came_from[edge destination] = current
                push edge destination onto to_visit

                If edge destination is sink node:
                    Break from for loop
                End if
            End if
        End for
    End while

    // Reconstruct path

    Let path = empty list
    Let current = sink node

    While current != source node, do
        If came_from[current] == -1:
            Return empty path
        End if

        Insert edge from came_from[current] to current at front of path
    End while

    Return path
```

```
End function
```

These two pseudocode functions complete our algorithm for the Ford-Fulkerson and the Edmonds-Karp algorithm. The `C++` used in this project is a translation of this.

4.3. **Real-World Application.**

4.4. **Theoretical Questions. Describe the concept of augmenting paths and their role in the Ford-Fulkerson algorithm. Prove that the algorithm terminates and converges to the maximum flow in finite time, even for non-integer capacities.**

An augmenting path is a path through the graph from the source node to the sink node such that a certain non-zero amount of flow may be added to the net flow between these nodes. An augmenting path is found in relation to a residual graph, which is a graph representing the remaining possible flow for each edge in the capacity graph. The residual graph is initialized as the capacity graph, and each additional augmenting path decreases the remaining flow in the residual graph by its net flow.

The net flow across an augmenting path is defined as the smallest residual in any edge along that path.

It is important to note that the net flow across the graph increases monotonically as a function of number of passes- That is to say that, for each additional augmenting path, the net flow must increase. In fact, the terminal condition of the algorithm is only met when no augmenting path with a net flow above zero can be found. Thus, any failure of this condition will result in the algorithm halting.

Let us assume the following:

- With zero augmenting paths, the net flow is zero
- An augmenting path must increase the net flow
- The net flow is a finite real number greater than or equal to zero
- The net flow is equal to the sum of the net flows of the augmenting paths

Mathematically:

$$p \in \mathbb{R}^n, [\forall p_i \in p]\, p_i > 0$$

$$f^* \in \mathbb{R} : 0 \leq f^* < \infty$$

$$\sum_{i=0}^{n} p_i = f^*$$

$$\therefore$$

$$\sum_{i=0}^{n} p_i < \infty, [\forall p_i \in p]\, p_i > 0$$

$$\therefore$$

$$n < \infty$$

That is to say, that since the net flow is a sum of non-zero real numbers and it is itself a finite real number, it must be a finite sum. This implies that the number of augmenting paths must be finite, implying that the algorithm converges in finite time.

Since we only assumed that the net flow across an augmenting path is a real number, this proves the convergence of this method upon both integer and non-integer weights.

Explain the Min-Cut Max-Flow Theorem and its significance in the context of network flows. How can this theorem be used to find a maximum flow and minimum cut in a flow network?

The Min-Cut Max-Flow Theorem states that the cost of the min-cut across a network is the same as the maximal flow across that network. Since we have just defined several algorithms for finding the max flow across a network, the theorem allows us to also use these algorithms to find the minimum cut.

This theorem means that, given the min-cut or the max-flow, you can easily derive the other. This makes it very significant, as it allows our FF and EK algorithms to have twice as many applications.

The min-cut of a network is vital for identifying weak points- For instance, the point in a computer network which is most likely to fail. It can also be used to identify weak points in other people's networks. For instance, the text mentioned times during the cold war where the min-cut algorithm was used to identify the method of severing soviet supply lines while costing the least and causing the lease amount of damage. In this case, nodes would be cities or railway junctions, and edges would be railway lines. The weight would be some value representing the cost to destroy a given line.

The following proof of the mincut-maxflow theorem is paraphrased from source [6].

Consider an arbitrary weighted directed graph G. Consider the solution flow value f^* computed via the FF algorithm. Let R be the final residual graph after the application of this algorithm. As usual, assume the source node is s and the sink / target node is t. We will denote the set of all nodes reachable in R from s as R_s, and the set of all nodes reachable in R from t as R_t (equivalent to $R - R_s$ or R^c). We will denote a cut across G by defining the set containing the source node S and the set containing the sink node T. These sets are disjoint, and together represent the graph.

The capacity c of a cut separating G into S and T is determined by the formula below.

From source 6:

$$c(S,T) = \Sigma_{(u,v)\in S\times T} c_{uv}$$

In English, c is equal to the sum of the capacities of the edges which connect S to T.

If the capacity of the mincut were to be less than the maxflow, the maxflow would never be able to flow beyond the "crunch point" which is the mincut. Because of this, we know that $c(R_s, R_t) \geq f^*$. From here, we must prove that the capacity of the mincut must not be greater than the maxflow.

For the capacity $c(R_s, R_t)$ to be equal to f^* (that is to say, for the min-cut to be equal to the max-flow), the following must be true.

$netflow(A) = outflow(A) - inflow(A)$ for any subset A. $netflow(G) = f^*$. Thus, for $c(R_s, R_t) = f^*$, $c(R_s, R_t) = outflow(R_s) - inflow(R_t)$. For the cut (R_s, R_t) to completely sever contact between s and t, the following must hold.

- There must be no edges from R_s to R_t with remaining capacity.
- There must be no edges from R_t to R_s with nonzero flow.

If the first condition were not true, there would exist an additional augmenting path from s to t via a forward edge, and the maxflow algorithm would have failed. Thus, the first condition must be true.

If the second condition were not true, there would exist a possible backwards edge allowing an additional augmenting path, and the maxflow algorithm would be similarly violated. Thus, the second condition must also be true.

Because of these conditions, there must not be any possible augmenting paths leading to or from the cut. This means that no additional network flow could be added to the graph. Thus, the current flow across the cut is maximal. Since the current flow across the cut is equal to the capacity of the cut, we now know that the the capacity of the mincut must not be greater than the maxflow. This allows us to narrow down our previous statement even further into the final $c(R_s, R_t) = f^*$.

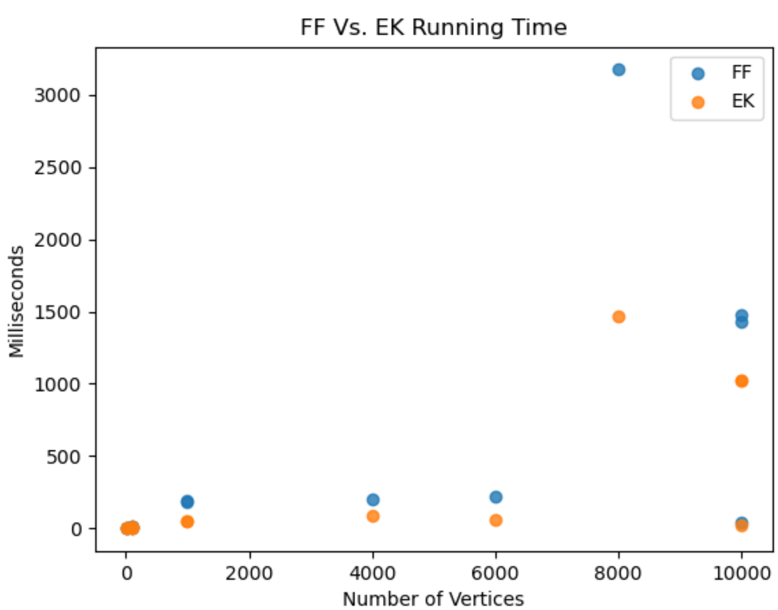
5. Real-World Problem Solving

6. Comparative Analysis and Reporting

6.1. Red-Black Trees.

6.2. Max Flow Algorithms.

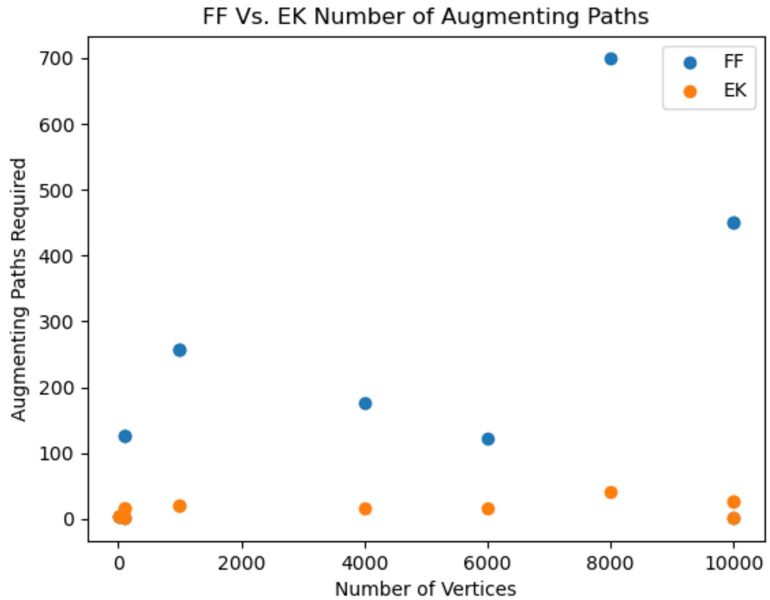
Below is a graph of running times from a series of experimental runs of the Ford-Fulkerson (FF) and Edmonds-Karp (EK) algorithms on various randomly-generated graphs.



The Ford-Fulkerson (FF) algorithm vs the Edmonds-Karp (EK) algorithm on randomly generated graphs of various size.

Clearly, the Edmonds-Karp algorithm performs reliably better than the original Ford-Fulkerson one, especially for large graphs.

The below graph shows the number of augmenting paths present in graphs of various sizes.



The Ford-Fulkerson (FF) algorithm vs the Edmonds-Karp (EK) algorithm on various graphs, by number of passes required.

Although the difference is not always extreme, the Edmonds-Karp algorithm reliably runs at worst as well as Ford-Fulkerson, and usually much better. From this data, it would seem that EK is the best choice. However, this is not always the case.

```
In a graph w/ 1000 nodes and 631216 edges
1 <= weight <= 10

FF ms:     22,950.6
FF passes: 2,732

EK ms:     151,082.0
EK passes: 1,243
```

The above output exemplifies a case in which the Ford-Fulkerson algorithm outperforms the Edmonds-Karp one. Here, EK ran in 151 seconds, while FF took only 23 seconds. This is because of the disparity between f^* (3,161), $|\mathbb{E}|$ (631,216), and $|\mathbb{V}|$ (1,000). Since FF runs in time proportional to $f^* \cdot |\mathbb{E}|$, we can expect it to run in approximately $3,161 \cdot 631,216 = 1,995,273,776$ operations. EK, on the other hand, will run in about $1,000 \cdot 631,216^2 = 398,433,638,656,000$ operations- 5 orders of magnitude more! Luckily, EK only performed 1 order of magnitude worse than FF in our experiment, but this is still a non-trivial performance cost in what is supposed to be the better algorithm.

Thus, the choice between FF and EK is more nuanced than it first would seem.

7. Conclusion

7.1. **Red-Black Trees.**

7.2. **Maxflow.** The EK algorithm was developed as an instance of the FF algorithm, specifically optimized to remove FF's complexity reliance upon f^*. Thus, in cases where f^* is very large in comparison to $|\mathbb{E}|$ and $|\mathbb{V}|$, EK will be the more efficient choice. It will require fewer augmenting paths, and each augmenting path will increase the total flow more. However, each of these augmenting paths will be more costly to compute. Thus, if f^* is very small in comparison to the number of nodes and edges, FF will run more efficiently.

It is notable that, even in the failure case described above, EK will still require fewer augmenting paths than FF. It is simply the cost of computing these paths which causes EK to fail.

Since it is rare that $f^* \cdot |\mathbb{E}|$ is smaller than $|\mathbb{V}||\mathbb{E}|^2$, EK will most often be the correct choice for the maxflow problem. However, the wisest approach would be to implement both algorithms, and switch to FF if it is computed to be more efficient.

8. References

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