

Turing-Completeness of $Th(\mathcal{N}, +, \times)$ / Peano Arithmetic

An interjection not found in the book

Peano Arithmetic

- ▶ Axioms of natural arithmetic made by Giuseppe Peano
- ▶ The natural numbers with addition and multiplication
 - ▶ AKA $Th(\mathcal{N}, +, \times)$
- ▶ Later used by Gödel, Church, and Turing to cause mathematical mayhem

Implicit Functions and Iteration in Peano Arithmetic

- ▶ We are limited to two relations: $+$ and \times
- ▶ However, we can use these to construct more!
 - ▶ **Even recursive ones**
 - ▶ They just have to be *implicit*

Ex:

$$\forall a, b[(\text{Square}(a) = b) \leftrightarrow (b = a \cdot a)]$$

- ▶ Similar to “we say . . .” as a colloquialism
- ▶ “We say b is the square of a iff $b = a \cdot a$ ”

Preliminary Functions

- ▶ We can add and multiply, but we need a lot more!
- ▶ We can implicitly define subtraction, exponentiation, division, modulo
- ▶ As well as some special encoding functions Get_2 and Set_2
- ▶ These can get and set bits from an arbitrary-length binary integer

Let $\phi_{preliminary}$ be the equation describing these functions.

Preliminary Functions (Formally)

$$\begin{aligned}\phi_{\text{preliminary}} = & \forall a, b, c ([(a - b = c) \leftrightarrow (a = c + b)] \\ & \wedge [\left(\frac{a}{b} = c \right) \leftrightarrow (a = bc)] \\ & \wedge [(\text{Rem}(a, b) = c) \leftrightarrow \exists d [db + c = a]] \\ & \wedge [(a^0 = 1) \wedge (b > 0 \rightarrow (a^b = a \cdot a^{b-1}))] \\ & \wedge [\text{Log}(b^a, b) = a] \\ & \wedge \left[\text{Get}_2(a, b) = \frac{\text{Rem}(a, 2^{b+1}) - \text{Rem}(a, 2^{b-1})}{2^b} \right] \\ & \wedge [\text{Set}_2(a, b, c) = a + (c - \text{Get}_2(a, b)) \cdot 2^b]\end{aligned}$$

- We could bring this to prenex normal form, but it doesn't really matter

Gödel Numberings of TMs

- ▶ We previously gave a configuration as $w_l q_i w_r$ for some state q_i and tape content $w_l w_r$
- ▶ This will be too clunky for Gödel numbering!
- ▶ Let $n = |w_l|$, let w be a binary integer encoding $w_l w_r$

Def: A Gödel TM encoding. For configuration $C = w_l q_i w_r$ where $n = |w_l|$, let w be a positive binary integer encoding of $w_l w_r$. Then, a Gödel encoding of C is given by $2^i 3^n 5^w$.

Encoding the Transition Function

- ▶ We want to create some implicit function T which takes us from one configuration to the next
- ▶ For all state indices i, j , R/W head position n , string content encoding w , binary tape character z , and $D \in \{-1, 1\}$:
 - ▶ If $\delta(q_i, \text{Get}_2(w, n)) = (q_j, z, D)$, **then**
 - ▶ $T(2^i 3^n 5^w) = 2^j 3^{n+D} 5^{\text{Set}_2(w, n, z)}$
 - ▶ T can be encoded in Peano arithmetic as a **large** piecewise function

Let ϕ_T be the formula describing T .

An Algorithm for ϕ_T

Given TM T , the following algorithm creates ϕ_T .

1. Let p be the logic string $\forall a \exists i, n, w [[a = 2^i 3^n 5^w]$
2. For each $q_i \in Q$
 - 2.1 For each $\sigma \in \Sigma$
 - 2.1.1 Let $\delta(q_i, \sigma) = (q_j, \gamma, D)$ (for $q_j \in Q, \gamma \in \Sigma, D \in \{-1, 1\}$, where $D = -1$ represents leftward movement and $D = 1$ rightward)
 - 2.1.2 Append to p the string
$$\wedge [[[i = \%_1] \wedge [Get(w, n) = \%_2]] \rightarrow [T(a) = 2^{\%_3} 3^{n+\%_4} 5^{Set_2(w, n, \%_5)}]]$$
Where $\%_1$ is replaced by i , $\%_2$ by σ , $\%_3$ by j , $\%_4$ by D , and $\%_5$ by γ
3. Append the string “]” to p
4. Return p as ϕ_T

Encoding TM Simulation

- ▶ Here's where loops come in
- ▶ For every configuration C_t :

$$\dots \wedge \forall t [C_{t+1} = T(C_t)]$$

- ▶ We can then specify some initial configuration on the input string encoding in binary as w : $C_0 = 2^0 3^0 5^w$

The Acceptance Problem in Peano Arithmetic

- If we let the index of the accepting state q_A be A , we can ask “Does this TM accept?” as:

$$\begin{aligned}\phi_A = & \\ & \wedge \phi_{\text{preliminary}} \\ & \wedge \phi_T \\ & \wedge \forall t [C_{t+1} = T(C_t)] \\ & \wedge [C_0 = 2^0 3^0 5^w] \\ & \wedge \exists h, \mu, \omega [C_h = 2^A 3^\mu 5^\omega]\end{aligned}$$

$Th(\mathcal{N}, +, \times)$ is **Turing complete!**

Next up: Whatever we were doing before

Sing us a song you're the Peano man

Sing us a song tonight

Cuz we're all in the mood for arithmetic

And you've got us feelin alright

-Billy Joel