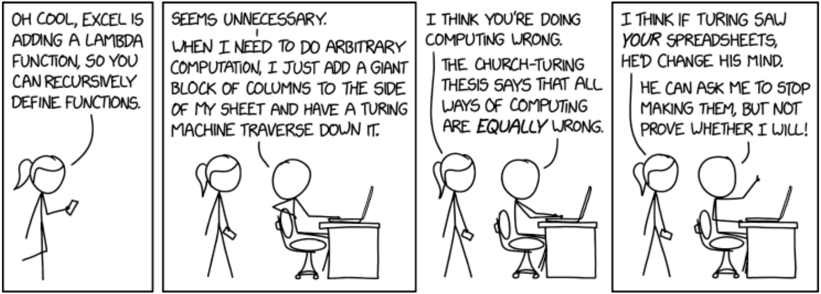
Bonus Topic: The Lambda Calculus



Intro: python lambdas

- In many languages, a "lambda" function is a function that can be treated like an object (usually with captured variables)
- However, `python` has a different interpretation!

```
a = lambda x: x ** 2
print(a(5))
# >> 25

plus = lambda lhs: lambda rhs: lhs + rhs
print(plus(123)(321))
# >> 444
```

- **Generally:** `lambda b: f(b)` is a formula which, when applied on input x, returns $f(x)$
  - b is the thing that will be replaced
  - Everything after the colon is the text in which b will be replaced

Church's Version

- Alonzo Church was Turing's PhD advisor at Princeton
- Followed the work of Gödel, Peano, Principia Mathematica in formalizing mathematics
- Also following the string-rewriting-rule formalizations of Thue
- He invented "the lambda calculus", or λ-calculus

Def: λ-calculus. Let $M[x := N]$ mean "string M, but with any occurrence of x replaced with N".

1. $\lambda x.M$ is a **lambda abstraction**
2. If M is a lambda abstraction, MN is an **application**
3. $\lambda x.M$ and $\lambda y.M[x := y]$ are equivalent (α-conversion, used to avoid collisions)
4. $(\lambda x.M)(N) = M[x := N]$ is the result of an application (β-reduction, represents computation)

Example

Ex: $(\lambda x.\sqrt{x})((\lambda x.x^2)5)$

1. Input: $(\lambda x.\sqrt{x})\underline{((\lambda x.x^2)5)}$
2. β-reduction: $\sqrt{((\lambda x.x^2)5)}$
3. β-reduction: $\sqrt{5^2}$
4. Arithmetic: 5

Currying

- What if we want to have multiple arguments?
- In `python`, `lambda x, y: f(x, y)` is legal
  - This is **not** legal in λ-calculus!
  - However, abstractions are allowed to return abstractions!
- We can make a function that takes the first argument and returns a function taking the second argument

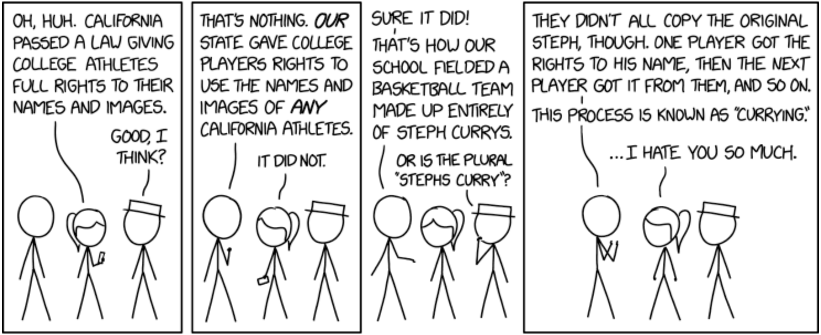
```python
mean = lambda a: lambda b: lambda c: (a + b + c) / 3

# mean(1) is `lambda b: lambda c: (1 + b + c) / 3`
# mean(1)(2) is `lambda c: (1 + 2 + c) / 3`
# mean(1)(2)(3) is `(1 + 2 + 3) / 3`

print(mean(1)(2)(3))
# >> 2
```

- This is called **Currying**
- Named after Haskell Curry (also the namesake of the Haskell language)

Currying Example

Ex: $[\lambda x.\lambda y.\lambda z.\frac{x}{y} + z](15)(3)(37)$

1. Input: $[\lambda x.\lambda y.\lambda z.\frac{x}{y} + z](15)(3)(37)$
2. β-reduction: $[\lambda y.\lambda z.\frac{15}{y} + z](3)(37)$
3. β-reduction: $[\lambda z.\frac{15}{3} + z](37)$
4. β-reduction: $\frac{15}{3} + 37$
5. Arithmetic: $\frac{15}{3} + 37 = 5 + 37 = 42$



The Y-Combinator

Consider the classical “paradox” discovered by Curry:

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

Puzzle: What does Yg β-reduce to?

Y-Combinator β-reduction

1. Yg (input)
2. $= \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))g$ (by definition of Y)
3. $= (\lambda x.g(xx))(\lambda x.g(xx))$ (by β-reduction)
4. $= g((\lambda x.g(xx))(\lambda x.g(xx)))$ (by β-reduction)
5. $= g(Yg)$ (by equivalency of steps 1 and 3)
6. $= g(g(Yg)) = g(g(g(Yg))) = \ldots$ (ad infinitum)

This can be used to create loops in λ-calculus!

Turing-Equivalency

- Church and Turing proved λ-calculus is equivalent to Turing machines
- Since both Peano arithmetic and λ-calculus are Turing-equivalent, they must be equivalent to each other!
  - They are both equally valid characterizations of unbounded arithmetic on the natural numbers