Turing-Completeness of $Th(\mathcal{N}, +, \times)$ / Peano Arithmetic

An interjection not found in the book

Peano Arithmetic

- Axioms of natural arithmetic made by Giuseppe Peano
- The natural numbers with addition and multiplication
  - AKA $Th(\mathcal{N}, +, \times)$
- Later used by Gödel, Church, and Turing to cause mathematical mayhem

Implicit Functions and Iteration in Peano Arithmetic

- We are limited to two relations: $+$ and $\times$
- However, we can use these to construct more!
  - **Even recursive ones**
  - They just have to be *implicit*

Ex:

$$\forall a, b[(Square(a) = b) \leftrightarrow (b = a \cdot a)]$$

- Similar to "we say..." as a colloquialism
- "We say b is the square of a iff $b = a \cdot a$"

Preliminary Functions

- We can add and multiply, but we need a lot more!
- We can implicitly define subtraction, exponentiation, division, modulo
- As well as some special encoding functions Get_2 and Set_2
- These can get and set bits from an arbitrary-length binary integer

Let $\phi_{preliminary}$ be the equation describing these functions.

Preliminary Functions (Formally)

$$\begin{aligned}
\phi_{preliminary} = \forall a, b, c (& [(a - b = c) \leftrightarrow (a = c + b)] \\
& \wedge \left[\left(\frac{a}{b} = c \right) \leftrightarrow (a = bc) \right] \\
& \wedge [(Rem(a, b) = c) \leftrightarrow \exists d [db + c = a]] \\
& \wedge \left[(a^0 = 1) \wedge (b > 0 \rightarrow (a^b = a \cdot a^{b-1})) \right] \\
& \wedge [Log(b^a, b) = a] \\
& \wedge \left[Get_2(a, b) = \frac{Rem(a, 2^{b+1}) - Rem(a, 2^{b-1})}{2^b} \right] \\
& \wedge \left[Set_2(a, b, c) = a + 2^b (c - Get_2(a, b)) \right])
\end{aligned}$$

- We could bring this to prenex normal form, but it doesn't really matter

Gödel Numberings of TMs

- We previously gave a configuration as $w_l q_i w_r$ for some state q_i and tape content $w_l w_r$
- This will be too clunky for Gödel numbering!
- Let $n = |w_l|$, let w be a binary integer encoding $w_l w_r$

Def: *A* Gödel TM encoding. For configuration $C = w_l q_i w_r$ where $n = |w_l|$, let w be a positive binary integer encoding of $w_l w_r$. Then, a Gödel encoding of C is given by $2^i 3^n 5^w$.

Encoding the Transition Function

- We want to create some implicit function T which takes us from one configuration to the next
- For all state indices i, j, R/W head position n, string content encoding w, binary tape character z, and $D \in \{-1, 1\}$:
  - If $\delta(q_i, Get_2(w, n)) = (q_j, z, D)$, **then**
  - $T(2^i 3^n 5^w) = 2^j 3^{n+D} 5^{Set_2(w,n,z)}$
  - T can be encoded in Peano arithmetic as a **large** piecewise function

Let ϕ_T be the formula describing T.

An Algorithm for ϕ_T

Given TM T, the following algorithm creates ϕ_T.

1. Let p be the logic string $\forall a \exists i, n, w[[a = 2^i 3^n 5^w]$
2. For each $q_i \in Q$
   2.1 For each $\sigma \in \Sigma$
      2.1.1 Let $\delta(q_i, \sigma) = (q_j, \gamma, D)$ (for $q_j \in Q, \gamma \in \Sigma, D \in \{-1, 1\}$, where $D = -1$ represents leftward movement and $D = 1$ rightward)
      2.1.2 Append to p the string

      $$\wedge[[[i = \%_1] \wedge [Get(w, n) = \%_2]] \rightarrow [T(a) = 2^{\%_3} 3^{n+\%_4} 5^{Set_2(w,n,\%_5)}]]$$

      Where $\%_1$ is replaced by i, $\%_2$ by σ, $\%_3$ by j, $\%_4$ by D, and $\%_5$ by γ
3. Append the string "]" to p
4. Return p as ϕ_T

Encoding TM Simulation

- Here's where loops come in
- For every configuration C_t:

$$\ldots \wedge \forall t[C_{t+1} = T(C_t)]$$

- We can then specify some initial configuration on the input string encoding in binary as w: $C_0 = 2^0 3^0 5^w$

$Th(\mathcal{N}, +, \times)$ **is Turing complete!**

The Acceptance Problem in Peano Arithmetic

- If we let the index of the accepting state q_A be A, we can ask "Does this TM T accept binary string w?" as:

$$
\begin{aligned}
\phi_{T,w} = &\phi_{preliminary} \\
&\wedge \phi_T \\
&\wedge \forall t [C_{t+1} = T(C_t)] \\
&\wedge [C_0 = 2^0 3^0 5^w] \\
&\wedge \exists h, \mu, \omega [C_h = 2^A 3^\mu 5^\omega]
\end{aligned}
$$

$\phi_{P,0}$ vs $\exists c[\phi_{P,0}]$

- $\phi_{P,0}$ means "$\phi_{P,0}$ is true$
- $\exists c[\phi_{P,0}]$ means "there is a proof for $\phi_{P,0}$"

Note: Peano arithmetic without implicit function definition is called **Presburger arithmetic** and is equivalent to DFA.

Next up: Whatever we were doing before

Sing us a song you're the Peano man
Sing us a song tonight

Cuz we're all in the mood for arithmetic
And you've got us feelin alright
-Billy Joel