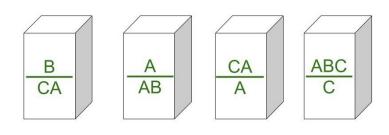
Post's Correspondence Problem (PCP): A reduction by computation history



Textbook: Chapter 5.2

Intro

- Imagine we have some set of dominos, each with a top and bottom section
- ▶ We want to arrange them (repetition allowed) so that the top and bottom match when read left to right

$$\left\{ \begin{bmatrix} \frac{ab}{a} \end{bmatrix}, \begin{bmatrix} \frac{c}{bc} \end{bmatrix} \right\}$$
$$\left[\frac{ab}{a} \end{bmatrix} \begin{bmatrix} \frac{c}{bc} \end{bmatrix} \rightarrow \frac{abc}{abc}$$

Some sets of dominos have possible matches, and some don't

For example, this set has no matches:

$$\left\{ \left[\frac{abc}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{acc}{ba}\right] \right\}$$

Definition

Def. Post's Correspondence Problem. Does a given set of dominos have a possible match?

As a language:

 $PCP = \{w : w \text{ encodes a set of dominos which match }\}$

Undecidability

Thm. *PCP* is undecidable.

Pf. By reduction from the decision problem A_{TM} . We will assume PCP is decidable and show that this implies A_{TM} is decidable (a contradiction).

- We can construct an instance of PCP which simulates a TM
 - Can design it so it matches iff the TM accepts
- Therefore a PCP decider would decide A_{TM}

MPCP

- First let's look at a simpler version: Modified PCP (MPCP)
- ▶ MPCP is PCP, but matches must begin with the first domino
- Later we will prove MPCP and PCP are equivalent

Def: Modified Post's Correspondence Problem.

 $\textit{MPCP} = \{\textit{w} \in \textit{PCP}: \textit{w's} \texttt{ match starts w/ the first domino}\}$

Thm: A_{TM} is reducible to MPCP (a decider for MPCP can be used to decide A_{TM}). Given some TM M and input w, we will construct an instance P' of MPCP which matches iff M(w) accepts by simulating M's accepting computation history.

Configurations will be delimited by #s

1. The first domino will be the starting configuration

$$\left[\frac{\#}{\#q_0w_1w_2\cdots w_n\#}\right]\in P'$$

Other dominoes will force the next configuration to appear.

- 2. **Rightward movement:** For every $a, b \in \Gamma$ (the tape alphabet) and $q, r \in Q$ where $q \neq q_{reject}$:
 - ▶ If $\delta(q, a) = (r, b, R)$ (move from state q on tape cell a to state r, writing b and moving right), then add $\left[\frac{qa}{br}\right]$ to P'
- 3. **Leftward movement:** For every $a, b, c \in \Gamma$ (tape alphabet) and $q, r \in Q$ where $q \neq q_{reject}$:
 - ▶ If $\delta(q, a) = (r, b, L)$ (move from state q on tape cell a to state r, writing b and moving left) then add $\left[\frac{cqa}{rcb}\right]$ to P'

- 4. **Tape:** For every $a \in \Gamma$, add $\left[\frac{a}{a}\right]$ to P'
- 5. **Configurations:** Add $\begin{bmatrix} \# \\ \# \end{bmatrix}$ (the configuration deliminator on the top and bottom) and $\begin{bmatrix} \# \\ \sqcup \# \end{bmatrix}$ (where " \sqcup " accounts for the infinitely many empty tape cells on the RHS of the configuration) to P'
- 6. For every $a \in \Gamma$, add to P' the dominoes:

$$\left[rac{aq_{ exttt{accept}}}{q_{ exttt{accept}}}
ight], \left[rac{q_{ exttt{accept}}a}{q_{ exttt{accept}}}
ight]$$

These allow the acceptance state to "eat" surrounding tape characters until only it remains

7. Finally we add $\left[\frac{q_{\text{accept}}\#\#}{\#}\right]$, allowing the dominoes to finally match

▶ Clearly A_{TM} is reducible to $MPCP \ P'$, but can we convert $P' \in MPCP$ to some $P \in PCP$?

"Starring" strings: Let $u = u_1 u_2 \cdots u_n$ be a string of length n. Define $\star u$, $u\star$, and $\star u\star$ as follows.

$$\star u = \star u_1 \star u_2 \cdots \star u_n$$

$$u \star = u_1 \star u_2 \star \cdots u_n \star$$

$$\star u \star = \star u_1 \star u_2 \star \cdots \star u_n \star$$

If P' is the set

$$\left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \left[\frac{t_3}{b_3}\right], \cdots, \left[\frac{t_k}{b_k}\right] \right\}$$

Then we let *P* be the set

$$\left\{ \left[\frac{\star t_1}{\star b_1 \star} \right], \left[\frac{\star t_2}{b_2 \star} \right], \left[\frac{\star t_3}{b_3 \star} \right], \cdots, \left[\frac{\star t_k}{b_k \star} \right], \left[\frac{\star \diamond}{\diamond} \right] \right\}$$

- $ightharpoonup \left[rac{\star \diamond}{\diamond}
 ight]$ allows the match to terminate with matching stars
- ▶ Thus, MPCP can be converted to PCP. Since A_{TM} is reducible to MPCP, PCP is undecidable. End of proof.

Next up: Nondeterministic Turing Machines