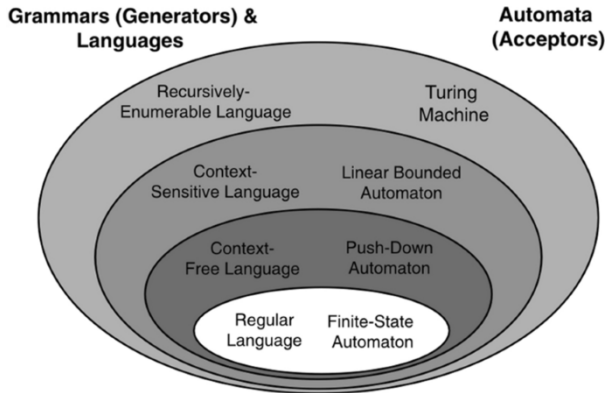


Finite-State Acceptors



The Formal Language Hierarchy

Textbook: 1.1 and 1.2

Definitions

- ▶ An **automaton** (plural automata) is a self-moving machine. In this context, it just means “small, simple, simulated machine”.
- ▶ An **acceptor** is an automaton which takes in a string and either accepts or rejects it through a series of state transitions.
Note: Many texts fail to distinguish between *automata* and *acceptors*. Acceptors are a proper subset of automata.
- ▶ A **finite-state** automaton has a finite number of states that it can transition through.
- ▶ A **deterministic** automaton has exactly one set of state transitions which follow from a given string. A nondeterministic one, on the other hand, has multiple series of state transitions which may follow.

Kleene operators

- ▶ For an alphabet Σ , the **Kleene star** is an operator creating the set of all strings of zero or more characters
- ▶ If Σ^i is the set of all strings of length i over Σ , then $\Sigma^* = \bigcup_{i=0}^{\infty} \Sigma^i$

- ▶ Ex:

$$\{0, 1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, \dots\}$$

- ▶ The **Kleene plus** gives the set of all strings of *one* or more characters over an alphabet. It is less common

$$\Sigma^* = \Sigma^+ \cup \{\epsilon\}$$

DFAs

- ▶ A **deterministic finite-state acceptor (DFA)** is an automaton built to accept strings using a finite number of states where each string corresponds to exactly one series of state transitions. Formally:

Def: Finite-state acceptor. A finite acceptor is a 5-tuple $A = (Q, \Sigma, \delta, q_0, F)$ where:

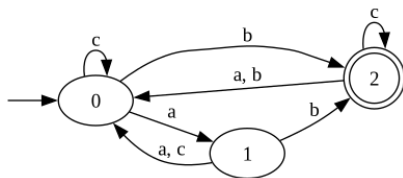
- 1) Q is a finite set called the **states**
- 2) Σ is a finite set of symbols called the **alphabet**
- 3) $\delta : (Q \times \Sigma) \rightarrow Q$ is the **transition function**
- 4) $q_0 \in Q$ is the **start state**
- 5) $F \subseteq Q$ is the set of acceptance states

Note: A DFA must define all transitions! Every state must have a specified target state for every character in the alphabet.

DFA operation

Consider the DFA A with states $Q = \{q_0, q_1, q_2\}$, starting state q_0 , alphabet $\Sigma = \{a, b, c\}$, and acceptance state set $F = \{q_2\}$. The following graph demonstrates δ .

Note: An arrow from nothing means “initial state”. In these diagrams, a double circle means “accepting state”.



This will accept any string of the forms abc^i or bc^i , where i is any nonnegative integer.

Formal definition of computation

We say that a DFA A **accepts** string $w = w_1 w_2 \dots w_n \in \Sigma^*$ if and only if there exists some series of states $r_0 r_1 r_2 \dots r_n \in Q^+$ such that the following all hold:

- 1) $r_0 = q_0$
- 2) $\delta(r_i, w_{i+1}) = r_{i+1}$ for all $i = 0, \dots, n_1$
- 3) $r_n \in F$

Def: The “language of” operator. For an automaton A over alphabet Σ , the language of A is given by the operator $\mathcal{L}(A) = \{w \in \Sigma^* : A \text{ accepts } w\}$.

Ex: If A recognizes all nonnegative even integers, then $\mathcal{L}(A) = \{0, 2, 4, 6, 8, 10, \dots\}$

NFAs

- ▶ A **nondeterministic finite-state acceptor (NFA)** is a DFA that can follow multiple series of state transitions
- ▶ The NFA accepts iff any of its branches do
- ▶ The only formal difference is in the transition function: δ **can output any number of output states**

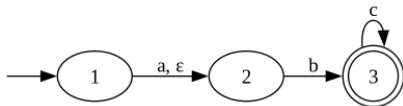
Def: Nondeterministic finite-state acceptor (NFA). An NFA is a 5-tuple $A = (Q, \Sigma, \delta, q_0, F)$ where:

- 1) Q is a finite set called the states
- 2) Σ is a finite set of symbols called the alphabet
- 3) $\delta : (Q \times \{\Sigma \cup \epsilon\}) \rightarrow \mathcal{P}(Q)$ is the **nondeterministic transition function**
- 4) $q_0 \in Q$ is the start state
- 5) $F \subseteq Q$ is the set of acceptance states

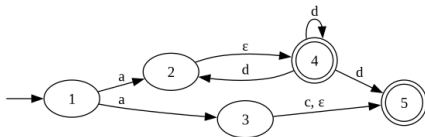
Note: If an NFA can leave transitions undefined, unlike a DFA. If it comes to a fork in the road, it duplicates and goes down both paths.

NFA operation

This NFA accepts the same language as our previous DFA:



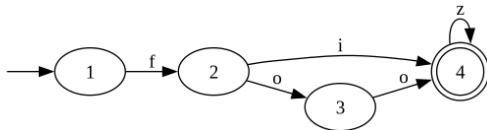
This NFA has more “weird” branches:



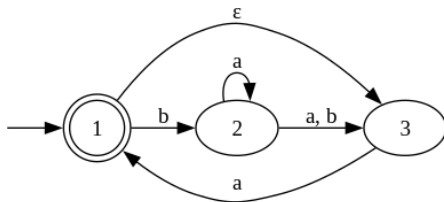
Practice problems

Do these problems on your own, then we will do them on the board.

- 1) Does the following DFA accept `fizz`? `foo`? `buzz`?



- 2) Describe the set of all strings accepted by the this NFA.



- 3) Construct an NFA **and** a DFA to recognize the set of all strings consisting of repetitions of `abc`.

Automaton equivalence

Def: Automaton equivalence. Automata A and B are equivalent if and only if they recognize the same language: That is, $(A = B) \iff (\mathcal{L}(A) = \mathcal{L}(B))$.

Ex: Let A be such that

$$\mathcal{L}(A) = \{w : w \text{ is an even nonnegative integer}\}$$

and B be such that

$$\mathcal{L}(B) = \{w : w \text{ is a non-empty string of numbers ending in } 0,\}$$

Since any nonnegative even integer is a non-empty string of numbers and ends in 0, 2, 4, 6, or 8, all items recognized by A are recognized by B . Since all non-empty strings of numbers ending in 0, 2, 4, 6, or 8 are nonnegative even integers, all items recognized by B are recognized by A . Therefore $\mathcal{L}(A) = \mathcal{L}(B)$ and $A = B$.

DFA-NFA equivalence

- ▶ Counterintuitively, **NFA and DFA are equivalent**: Every DFA has an equivalent NFA and vice versa
- ▶ To simulate an NFA, you can keep track of all the states currently held by any branch
- ▶ To advance to the next step, let the new states be the union of all valid transitions from existing states
- ▶ It turns out the set of all possible combinations of NFA states is also a valid DFA state set! We will use this in our proof

DFA-NFA equivalence proof pt. 1

We will first prove that all DFA are NFA, then prove that all NFA have equivalent DFA.

Thm: All DFA have equivalent NFA. **Pf:** The set of all DFA is a subset of the set of all NFA. Therefore, all DFA are trivially NFA. End of proof.

Thm: All NFA have equivalent DFA. **Pf:** By construction. DFA have $\delta_{\text{DFA}} : (Q \times \Sigma) \rightarrow Q$, while NFA have $\delta_{\text{NFA}} : (Q \times \{\Sigma \cup \epsilon\}) \rightarrow \mathcal{P}(Q)$. We will show that all δ_{NFA} have an equivalent modification in the form of δ_{DFA} , proving that all NFA are equivalent to DFA. Let A be an arbitrary NFA $= (Q, \Sigma, \delta, q_0, F)$.

- 1) Q vs $\mathcal{P}(Q)$. Since Q is a finite set, $\mathcal{P}(Q)$ is a finite set of size $2^{|Q|}$. Any finite set can be used as a set of DFA states, so $\mathcal{P}(Q)$ works. **Let** $Q' = \mathcal{P}(Q)$.

DFA-NFA equivalence proof pt. 2

- 2) ϵ transitions. We must eliminate any nonconsumptive transitions in order to be a valid DFA. Let $E : Q \rightarrow \mathcal{P}(Q)$ be the **ϵ -closure** function mapping a state to the set all all states reachable by ϵ transitions from it. Note that $E : Q \rightarrow Q'$. **Let** $\delta'(q, \sigma) = E(\delta_{\text{NFA}}(E(q), \sigma))$. Note that $\delta' : (Q' \times \Sigma) \rightarrow Q'$.
- 3) Starting states. The starting state may have outgoing ϵ transitions which must be accounted for. Thus, **let** $q'_0 = E(q_0)$.
- 4) Accepting states. An NFA's F is a subset of the set of its states ($F \subset Q$). However, it is not a subset of Q' yet: Thus, we must adjust it. **Let**
 $F' = \{R \in Q' : R \text{ contains an accept state from } F\}$.

Now, for an arbitrary NFA $A = (Q, \Sigma, \delta, q_0, F)$, let $A' = (Q', \Sigma, \delta', q'_0, F')$. By the above rules, A' is a valid DFA equivalent to A . End of proof.

Operation closure

Def: Closure under an operation. A set is **closed** under an operation iff that operation maps back to the set. We will study closure under unary and binary relations, but the concept generalizes to k -ary relations.

Ex: Integers. Since, for all integers a, b the results $a + b$ and ab are integers, the set of all integers is closed under addition and multiplication. Since there exist integers b, c such that b/c is not an integer, the set of all integers is *not* closed under division.

Ex: Derived closure. Since -1 is an integer, then a, b being arbitrary integers implies that $a - b = a + (-1 * b)$ is also an integer. Since addition and multiplication were previously shown to be closed, we can now say subtraction is as well.

The regular operations

- ▶ Just like in numeric math we have standard operations like $+$ and \times , in the study of regular languages we have the following “**regular**” operations

Def: The regular operations. The following three operations are known as the regular operations.

- ▶ **Union:** $A \cup B = \{x : (x \in A) \vee (x \in B)\}$
- ▶ **Concatenation:** $A \circ B = \{xy : (x \in A) \wedge (y \in B)\}$
- ▶ **Kleene (“KLAY-nee”) Star:**
 $A^* = \{x_1x_2 \dots x_k : k \geq 0, \forall i[x_i \in A]\}$

Note: Kleene plus. If a set A is closed under concatenation and Kleene star, then $A^+ = A \circ A^*$ is also closed.

State machine closure under regular operations

Thm: Finite-state-acceptor recognizable languages are closed under the regular operations: Union, concatenation, and Kleene star.

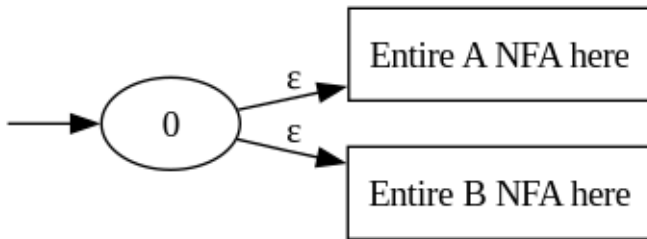
Pf: By construction in 3 parts. This can be done using only DFA (as in book), but for simplicity and without loss of generality we will instead use NFA here.

- 1) Union. Given NFA A, B , we want to construct some NFA representing $A \cup B$.
- 2) Concatenation. Given NFA A, B , we want to construct some NFA representing $A \circ B$.
- 3) Kleene star. Given NFA A , we want to construct some NFA representing A^* .

This proof will rely on graph representations over formal definitions.

1: Closure under union

Let Q_A and Q_B be the state sets Q from A and B , respectively. They are relabeled this way to avoid collisions. Let a_0 and b_0 be the initial states of their corresponding NFAs. To create an NFA for $A \cup B$, we just need to add a new initial state q'_0 with outgoing ϵ transitions to a_0 and b_0 . After this, our new state set Q' , accepting state set F' , and δ' are just trivial unions of A and B 's.

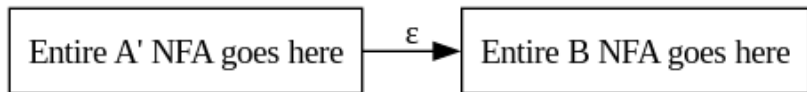


$$\delta'(q_0, \epsilon) = \{a_0, b_0\}$$

This NFA accepts a string matching A or B , therefore NFA are

2: Closure under concatenation

Let us define A' as a modification of A where all accepting states are no longer accepting and have outgoing ϵ transitions to the initial state of B . Let all states of A and B be relabelled so they do not conflict. This is represented in the diagram below.

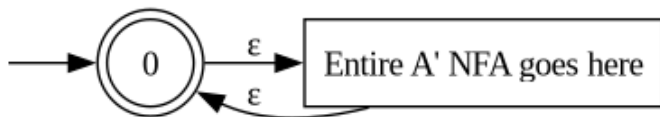


Our new NFA has the initial state of A , the state transition map of A' merged with B , the state set of relabelled states from both, and the final states of B .

This NFA accepts a string composed of a substring matching A followed by a substring matching B , therefore NFA are closed under concatenation.

3: Closure under Kleene star

Let us define a new non-conflicting initial state q_0 . All states of A will be relabelled to avoid a collision with q_0 . Let A' be the modification of A such that all accepting states are no longer accepting and have an outgoing ϵ transition to q_0 . q_0 , in turn, has an outgoing ϵ transition to the old starting node of A . The below diagram demonstrates this.



This NFA accepts zero or more instances of a substring matching A , therefore NFA are closed under Kleene star.

End of proof

Since NFA are closed under union, concatenation, and Kleene star, DFA are too. Thus, finite-state machines are closed under the regular operations. End of proof.

Next up: Regular languages and expressions

“The syntactic component of a grammar must specify, for each sentence, a deep structure that determines its semantic interpretation and a surface structure that determines its phonetic interpretation.” - **Noam Chomsky**

“Give Orange Me Give Eat Orange Me Eat Orange Give Me Eat Orange Give Me You” - **Nim Chimpsky**