

Advanced computability theory

Textbook: Chapter 6

Quines

- ▶ A **Quine** is a self-producing program
- ▶ A Quine takes no input (files, command-line, cin, etc) and outputs only its own source code

In C:

```
/* (formatted to fit on slide) */  
char*s="char*s=%c%s%c;main(){printf(s,34,s,34);}";  
main(){printf(s,34,s,34);};
```

In English:

Print out this sentence.

Can we build a TM to do this?

Kleene's Recursion Theorem

- ▶ By the same Kleene from section 1
- ▶ Allows any TM M to access its own description $\langle M \rangle$

The idea:

“Print the previous with quotes and then without”
Print the previous with quotes and then without

Kleene Recursion pt. 2

Lemma: Computable printing. There exists a computable function $q : \Sigma^* \rightarrow \Sigma^*$ where, for any string w , $q(w)$ is the description of a Turing machine P_w that prints out w and then halts.

Pf: The following TM Q computes $q(w)$.

$Q =$ "On input string w :

1. Construct the following TM P_w : $P_w =$ "On any input:
 - 1.1 Erase input
 - 1.2 Write w on the tape
 - 1.3 Halt."
2. Output $\langle P_w \rangle$."

The TM *SELF*

- ▶ *SELF* will be a Quine TM
- ▶ We will make it 2 parts: *A* and *B*
 - ▶ $\langle SELF \rangle = \langle AB \rangle$
- ▶ *A* will print out $\langle B \rangle$ and *B* will print out $\langle A \rangle$
- ▶ *A* is easy! Just use $q(\langle B \rangle)$
 - ▶ ***A* leaves $\langle B \rangle$ on the tape**
- ▶ **How do we create *B*?**
 - ▶ We can't use q , or we would get a circulate definition!
- ▶ *B* can look at the contents of the tape as its input
- ▶ We already know $\langle A \rangle = q(\langle B \rangle)$, so we can compute $\langle A \rangle$ given only $\langle B \rangle$

SELF pt. 2

$B =$ "On input $\langle M \rangle$:

1. Compute $q(\langle M \rangle)$ (the description of a TM that prints out the description of M)
2. Erase the tape
3. Write $q(\langle M \rangle) \langle M \rangle$ "

$$\langle SELF \rangle = q(\langle B \rangle) \langle B \rangle$$

Running *SELF*

- ▶ What happens if we run *SELF*?
 1. Start with some input tape
 2. Start running the TM $q(\langle B \rangle)$
 - 2.1 Erases tape
 - 2.2 Writes $\langle B \rangle$
 - 2.3 Halts
 3. Start running the TM B on the contents of the tape
 - 3.1 The input $\langle M \rangle$ is $\langle B \rangle$!
 - 3.2 Computes $q(\langle B \rangle)$
 - 3.3 Erases the tape
 - 3.4 Writes $q(\langle B \rangle) \langle B \rangle$
 - 3.5 Halts
 4. Halts
- ▶ The contents of the tape are now $q(\langle B \rangle) \langle B \rangle = \langle SELF \rangle$
- ▶ **Self takes any input and prints its own description**

Recursion theorem

Def: Kleene's recursion theorem. Let T be a TM that computes a function $t : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$. There is a TM R that computes some $r : \Sigma^* \rightarrow \Sigma^*$ where, for every w ,

$$r(w) = t(\langle R \rangle, w)$$

- ▶ **A TM can always obtain its own description!**

Recursion theorem pf

► Let $\langle R \rangle = \langle ABT \rangle$

$A =$ “On input w :

1. Compute $q(\langle BT \rangle)$
2. Erase tape
3. Write $q(\langle BT \rangle)w$ ”

$B =$ “On input $q(\langle MH \rangle)w$:

1. Create the TM N :

$N =$ “On input w :

- 1.1 Compute $q(\langle MH \rangle)$
- 1.2 Erase tape
- 1.3 Write $q(\langle MH \rangle)w$ ”

2. Let $\langle R \rangle = \langle NMH \rangle$
3. Simulate H on input $\langle R, w \rangle$ ”

The Minimality problem

Def: For a TM M , M is **minimal** if there exist no TMs equivalent to M with a smaller description. Let

$$MIN_{TM} = \{M : M \text{ is a minimal TM}\}$$

Thm: MIN_{TM} is not Turing-recognizable.

The Minimality problem pt. 2

Pf: We will assume some enumerator E for MIN_{TM} exists and obtain a contradiction.

$C =$ “On input w :

1. Obtain, via the recursion theorem, own description $\langle C \rangle$
2. Run E until it yields a TM D such that $|\langle C \rangle| < |\langle D \rangle|$. Since MIN_{TM} is infinite, this is guaranteed to happen
3. Simulate D on input w .”

All items yielded by E are definitionally in MIN_{TM} . However, C simulates D and is thus equivalent to it. We know C is shorter than D , but D is minimal: **Contradiction**.

Decidability of logical theories

Is logic decidable? Given some statement, can we ever know whether or not it is true?

Three logic problems of increasing difficulty:

1. $\forall q \exists p \forall x, y [p > q \wedge (x, y > 1 \rightarrow xy \neq p)]$
▶ (Infinitely many primes exist: Proven by Euclid)
2. $\forall a, b, c, n [(a, b, c > 0 \wedge n > 2) \rightarrow a^n + b^n \neq c^n]$
▶ (Fermat's last theorem: Only recently proven)
3. $\forall q \exists p \forall x, y [p > q \wedge (x, y > 1 \rightarrow (xy \neq p \wedge xy \neq p + 2))]$
▶ (Twin prime conjecture: Unproven)

Formal definition of logic

Let the alphabet of logic be:

$$\{\wedge, \vee, \neg, (,), \forall, x, \exists, R_1, \dots, R_k\}$$

- ▶ \wedge, \vee , and \neg are **Boolean operations**
- ▶ “(” and “)” are the **parenthesis**
- ▶ \forall and \exists are the **quantifiers**
- ▶ x represents the infinite sets of **variables**
- ▶ R_1, \dots, R_k denote **relations**

Def: A string ϕ is a **formula** if:

1. ϕ has the form $R_i(x_1, \dots, x_j)$ **or**
2. ϕ has the form $\phi_1 \wedge \phi_2$ or $\phi_1 \vee \phi_2$ or $\neg\phi_1$, where ϕ_1 and ϕ_2 are formulas **or**
3. ϕ has the form $\exists x_i(\phi_1)$ or $\forall x_i(\phi_1)$, where ϕ_1 is a formula

Mathematical models and theories

- ▶ A formula is in **prenex normal form** iff all quantifiers occur at the beginning
- ▶ An unquantified variable is called a **free variable**
- ▶ A formula with no free variables is called a **sentence** or **statement**
- ▶ The **universe** is the set of values that variables may take
- ▶ A **model** \mathcal{M} is a tuple (U, P_1, \dots, P_k) , where U is the universe and P_1, \dots, P_k are the relations assigned to symbols R_1 through R_k
- ▶ The **theory** of \mathcal{M} , written $Th(\mathcal{M})$, is the set of all true sentences on model \mathcal{M}

Ex: $(\mathcal{N}, +, \times)$ is a model where variables can take any value $\in \mathcal{N}$ and the relations $+$ and \times can be used. Note: $+$ corresponds to a relation $R_+(x_1, x_2, x_3) \iff (x_1 + x_2 = x_3)$, with a similar relation for \times .

Proving Gödel's incompleteness theorem

Def: Formal proof. The **formal proof** π of a statement ϕ is a sequence of statements S_1, S_2, \dots, S_l where $S_l = \phi$. Each statement must follow simply and directly from the previous statements.

Assume the following are true:

1. The correctness of a proof of a statement can be checked by a machine. Formally, $\{\langle \phi, \pi \rangle : \pi \text{ is a proof of } \phi\}$ is decidable
 - ▶ Given a proof, we can check if it implies a given statement
2. If a statement is provable, it is true

Gödel pt. 2

Thm 1: $Th(\mathcal{N}, +, \times)$ is Turing-recognizable.

Pf: Use the implied proof-checker from property 1 to enumerate the set of all proofs. If a given proof proves the statement in question, accept. This is a recognizer, but not a decider. End of proof.

- ▶ Corollary: There is an algorithm to decide if a given statement is provable. Let this algorithm be called P

Lemma 1.1: A_{TM} is reducible to $Th(\mathcal{N}, +, \times)$.

Pf: You can use $+$ and \times to extract and encode symbols in very large integers such that they simulate the evolution of computation histories. Therefore, $Th(\mathcal{N}, +, \times)$ is Turing-complete and therefore undecidable.

- ▶ Consider encoding a TM in binary, then representing binary as a series of additions and multiplications

Gödel pt. 3

Thm 2: Some (true) statement in $Th(\mathcal{N}, +, \times)$ is not provable.

Pf: We will assume all true statements are provable and derive an algorithm to decide $Th(\mathcal{N}, +, \times)$, a contradiction of lemma 1.1.

Take in some statement ϕ . Simulate P on both ϕ and $\neg\phi$. Since a proof exists for ϕ , one of these two instances will halt. Therefore, our system decides the truth value of ϕ (A_{TM}). Contradiction! End of proof.

Gödel pt. 4

Thm 3: The sentence $\psi_{\text{unprovable}} = \text{“this statement has no proof within the system”}$ is true.

Pf: Let $\phi_{M,w}$ be the statement “TM M accepts input w ”. This is implied to exist for any TM M and input w by lemma 1.1.

We will construct the statement “This statement is not provable **within the proof system**” using the recursion theorem.

$S = \text{“On any input (including 0):”}$

1. Obtain own description $\langle S \rangle$ via the recursion theorem
2. Construct the sentence $\psi_{\text{unprovable}} = \neg \exists c[\phi_{S,0}]$ using lemma 1.1. (‘this TM never accepts 0’)
 - If $\psi_{\text{unprovable}}$ is true, this TM never accepts 0
3. Run P (the theorem prover) on $\psi_{\text{unprovable}}$
4. If P accepts (there is a proof for $\psi_{\text{unprovable}}$), **accept**. If P halts and rejects (there is no proof for $\psi_{\text{unprovable}}$), **reject**.”

Gödel pt. 5

$$\begin{aligned} S &= \begin{cases} \text{always accept if there is a proof for } \psi_{\text{unprovable}} \\ \text{reject or loop if there is no proof for } \psi_{\text{unprovable}} \end{cases} \\ &= \begin{cases} \text{accept if proof exists that } S \text{ never accepts } 0 \\ \text{reject or loop otherwise} \end{cases} \end{aligned}$$

Run S on input 0

Gödel pt. 6

S must accept, reject, or loop forever on 0

- ▶ If S accepts 0, proof exists that S never accepts 0
 - ▶ Since all proven statements are true, this can't happen!
 - ▶ Therefore, **this can never be the case**
 - ▶ S must reject or loop on 0
- ▶ “ S rejects or loops on 0” is the same as $\psi_{\text{unprovable}}$
 - ▶ Only happens if no proof exists
 - ▶ “This statement has no proof within the proof system”
- ▶ $\psi_{\text{unprovable}}$ being false causes a contradiction
 - ▶ **Therefore, $\psi_{\text{unprovable}}$ must be true**
- ▶ We have used a **more powerful proof system** to prove that a **less powerful proof system** (Peano arithmetic) contains unprovable truths

Turing reducibility pt. 1

- ▶ Is mapping reducibility the best explanation? **No!**
- ▶ Consider: Could we prove that $\overline{A_{TM}}$ is undecidable by mapping reducibility?
 - ▶ Remember, $\overline{A_{TM}}$ is not Turing-recognizable
 - ▶ Therefore, we can't even identify instances of it, let alone map them!
- ▶ However, $\overline{A_{TM}}$ is intuitively reducible to A_{TM} and vice versa since a solution to one solves the other
- ▶ **Solution:** Oracle Turing Machines. An Oracle TM has a “magic oracle” decider that it can call without cost at any time
 - ▶ Assume the “oracle” is magic: It can decide even undecidable and unrecognizable languages

Turing reducibility pt. 2

Def: A TM M with access to a decider for **language** B is written M^B .

- ▶ Ex: A TM T with access to a decider for A_{TM} would be written $T^{A_{TM}}$

Def: If M^B decides A , we say A is **decidable relative** to B . - Given a decider for B , we can decide A

- ▶ **Much more intuitive than mapping reducibility**

Def: If A is decidable relative to B , then A is **Turing reducible** to B , written $A \leq_T B$.

Ex: $E_{TM} \leq_T A_{TM}$

Thm: $E_{TM} \leq_T A_{TM}$ (“given a decider for A_{TM} , we can decide E_{TM} ”)

Pf: By construction. Let $\text{accepts}(M, w)$ be the oracle procedure deciding A_{TM} .

```
def is_empty(M):           # Decides emptiness problem
    def T(_):              # Define a helper TM T
        for every string s: # Enumerate every possible string
            if accepts(M, s): # If M accepts this random string
                accept
    if accepts(T, ''):      # T is empty iff not M(s)
        reject
    else:
        accept
```

This decides E_{TM} . End of proof.

Ex: $A_{TM} \leq_T E_{TM}$

Thm: $A_{TM} \leq_T E_{TM}$ (“given a decider for E_{TM} , we can decide A_{TM} ”)

Pf: By construction. Let `is_empty(M)` be an **oracle** (not the fn from the previous: That would be circular).

```
def accepts(M, w): # Decides acceptance problem
    def T(x):      # Helper TM
        if x == w:
            if M(w): # Simulates M on w
                accept
            reject   # Rejects all non-w
    if is_empty(T): # Is empty iff M rejects w
        reject
    else:
        accept      # Nonempty iff M accepts w
```

This decides A_{TM} . End of proof.

The *other* meaning of Turing-Equivalence

- ▶ $E_{TM} \leq_T A_{TM}$ **and** $A_{TM} \leq_T E_{TM}$
- ▶ In this case we say that E_{TM} and A_{TM} are **Turing-equivalent**
- ▶ Written $E_{TM} \equiv_T A_{TM}$

This is an **equivalence relation**

Not to be confused with the Turing-equivalence of a system S , where a TM can simulate S and vice versa.

A definition of information

- ▶ There is no singular definition of information
- ▶ *One* definition is the size of the minimal representation

$A = 10101010101010101010$

$B = 10111001011111101000$

- ▶ A is just 01 10 times
- ▶ B has “more information” than A , since it doesn’t appear to follow a pattern

Def: Minimal descriptions. Let x be a binary string. The **minimal description** of x , written $d(x)$, is the shortest string $\langle M, w \rangle$ where TM M on input w halts with x on its tape. The **descriptive complexity** of x , written $K(x)$, is

$$K(x) = |d(x)|$$

Note: x ’s descriptive complexity **is not** its length! It can be **longer!**

Compression

- ▶ $K(x)$ might be longer than x
- ▶ If it's shorter, we can treat it as the compressed version, running M on w to uncompress it

Def: Let x be a string. x is said to be c -**compressible** for some natural number c if

$$K(x) \leq |x| - c$$

That is, if $|\langle M, w \rangle| \leq |x| - c$ for minimal $\langle M, w \rangle$.

If x is not compressible by 1, we say it is **incompressible**.

Incompressible strings

Thm: There are incompressible strings of every length.

Pf: If a string x is compressible, there exists a description $d(x)$ such that $|d(x)| < |x|$. Let n be any arbitrary integer.

There are 2^n binary string of length n : However, there are only $\sum_{i=1}^{n-1} 2^i = 2^n - 1$ descriptions of length $< n$. Therefore, there must exist at least one incompressible string of length n . End of proof.

What do incompressible strings look like?

- ▶ It can be shown that incompressible strings look like series of random coin tosses
- ▶ K is **incomputable**, so no examples exist
- ▶ If we had an example, we couldn't prove it was incompressible

Next up: Intro to complexity and asymptotic analysis

End of part 2 out of 3!

Notation	Common name	Limit test (note limit may not exist)
$f(n) \in O(g(n))$	Asymptotic upper bound	$\lim_{x \rightarrow \infty} \left \frac{f(x)}{g(x)} \right < \infty$
$f(n) \in o(g(n))$	Asymptotically negligible	$\lim_{x \rightarrow \infty} \left \frac{f(x)}{g(x)} \right = 0$
$f(n) \in \Omega(g(n))$	Asymptotic lower bound	$\lim_{x \rightarrow \infty} \left \frac{f(x)}{g(x)} \right > 0$
$f(n) \in \omega(g(n))$	Asymptotically dominant	$\lim_{x \rightarrow \infty} \left \frac{f(x)}{g(x)} \right = \infty$
$f(n) \in \Theta(g(n))$	Asymptotically tight bound	$0 < \lim_{x \rightarrow \infty} \left \frac{f(x)}{g(x)} \right < \infty$