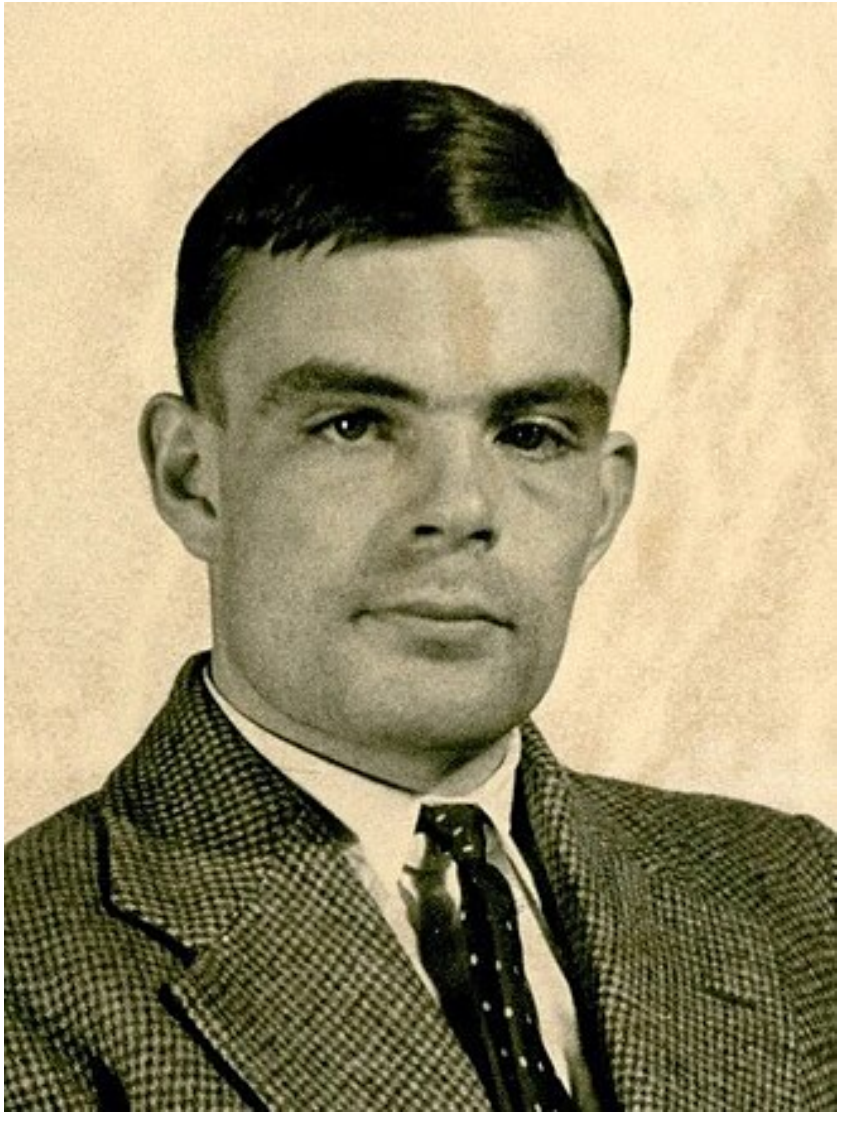
# The Church-Turing thesis and Turing machines



Textbook: Chapter 3

# History

Via Wikipedia

- Alonzo Church was Alan Turing's PhD advisor
  - Church formalized mathematics via lambda calculus
  - Turing then proved lambda calculus was equivalent to Turing machines, effectively founding computer science
- Turing then worked as a codebreaker during WW2 at Bletchley Park in England, being a large reason why Germany didn't win
- He was later persecuted by the English government (that he kept from being overthrown by Nazis) for being gay
  - They chemically castrated and fired him
- He died of cyanide poisoning at 41
  - Speculated to be a suicide
- The English government finally pardoned him in **2009**, 55 years after he died

# Turing machines (informal)

- DFA have finite memory, PDA have restricted infinite memory: What if we have unrestricted infinite memory?
- Equivalently, what if a DFA could move around freely and write on an infinite input "tape"?
- A **Turing machine (TM)** satisfies these conditions
- They have an infinite read/write tape controlled by a finite set of states and transitions
- They can move left or right
- Some states are special and immediately halt operation with either a positive (accept) or negative (reject) result

| Criteria | DFA/NFA | TM |
|---|---|---|
| Tape usage | Read-only | Read/write |
| Movement | Leftwards | Bidirectional |
| Input size | Finite | Infinite to the right |
| Halting | At input end | Only when entering special states |

# Church-Turing thesis

- *Hypothesizes* (does not prove or disprove) that no models are more powerful than Turing machines
  - "Anything that can be computed can be computed on a Turing machine"
- This holds for all models of computation that we've come up with so far

# Turing machines (formal)

**Def:** Turing machines. A TM is a 7-tuple

$$(Q, \Sigma, \Gamma, \delta, q_0, q_{\texttt{accept}}, q_{\texttt{reject}})$$

where $Q, \Sigma, \Gamma$ are all finite sets and:

1. $Q$ is the state set
2. $\Sigma$ is the input alphabet **not containing the blank symbol** $\sqcup$
3. $\Gamma$ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$
4. $\delta : (Q \times \Gamma) \to (Q \times \Gamma \times \{L, R\})$ is the transition function
5. $q_0 \in Q$ is the start state
6. $q_{\texttt{accept}} \in Q$ is the accept state
7. $q_{\texttt{reject}} \in Q$ is the reject state, where $q_{\texttt{reject}} \neq q_{\texttt{accept}}$

Where $L, R$ mean to move the R/W head left and right one cell respectively.

# TM configurations

- An automaton's **configuration** is a representation of all the information which can change
- A DFA configuration is an element of $Q$, while a PDA needs that and the contents of its stack

**Def:** A TM configuration contains the current state, the current tape contents, and the current head location. By convention, we write $uqv$ for state $q$ with tape contents $uv$ where the R/W head is on the first cell of $v$.

**Ex:** The TM configuration $1011q_70111$ means that the tape contents are 1011 to the left of the R/W head, 0 under it, and 111 to its right. The machine is in state $q_7$.

# TM computation

- A TM receives its input on the leftmost cells of the tape
- All cells after the input contain the empty symbol $\sqcup$
- A TM **cannot go past the left end of the tape**. If it tries to, it stays at the same cell
- We say configuration $C_1$ **yields** $C_2$ iff the TM can legally go from $C_1$ to $C_2$ in a single move
- $\delta(q_i, b) = (q_j, c, L)$:
  - If in state $q_i$ with $b$ under the R/W head, move to state $q_j$, write $c$, and move one cell to the left
  - If already on left edge of tape, do all that without moving
- $\delta(q_i, b) = (q_j, c, R)$:
  - If in state $q_i$ with $b$ under the R/W head, move to state $q_j$, write $c$, and move one cell to the right

# Configurations

**Def:** The **start configuration** of $M$ on $w$ is $q_0 w$. A **rejecting configuration** is any of the form $uq_{\texttt{reject}}v\$$, and an **accepting configuration** is of the form $uq_{\texttt{accept}}v\$$. Rejecting and accepting configurations are **halting configurations** and never yield anything.

**Def:** Computation. A TM $M$ **accepts** a string $w$ iff a series of configurations $C_1, C_2, \ldots, C_k$ exists such that:

1. $C_1 = q_0 w$ (the start configuration of $M$ on $w$)
2. $C_i$ yields $C_{i+1}$ for all $1 \leq i < k$
3. $C_k$ is an accepting configuration

Similar definitions hold for rejection and halting. The **language** of $M$, denoted $\mathcal{L}(M)$ is the set of all strings accepted by $M$.

# Universal TMs

**Def:** A universal Turing machine (UTM) $U$ is a TM which takes in a 2-tuple $(M, w)$ (where $M$ is some encoding of a TM and $w$ is a string on $M$'s input alphabet) and simulates the operation of $M$ on $w$. $U$ is such that:

1. $U(M, w)$ halts iff $M(w)$ halts
2. $U(M, w)$ accepts iff $M(w)$ accepts
3. $U(M, w)$ rejects when $M(w)$ rejects or (by convention) when $M$ is not an encoding of a valid TM

# Turing completeness and equivalence

Via Wikipedia

- By definition, a UTM is a TM which can simulate any TM
- Thus, if a system $Q$ can simulate a UTM, it can simulate every TM

**Def:** A system which can simulate a UTM is said to be **Turing complete** (or computationally universal).

**Def:** A system which can simulate *and be simulated by* a UTM is said to be **Turing equivalent.**

**Corollary:** Simulating any Turing complete system is sufficient to be Turing complete.

# Turing completeness and equivalence cont.

**Corollary:** All Turing complete programming languages are computationally equivalent: They only differ in speed and the "coolness factor".

**Corollary:** The Church-Turing thesis can be restated "all Turing complete systems are Turing equivalent."

**Note:** In most cases the restrictions of finite memory are ignored. If this is done, all known physically-constructable systems are Turing equivalent. If not, none are even Turing complete.

# Turing recognizability

**Def:** A language $L$ is **Turing-recognizable** iff some TM $M$ recognizes it ($\mathcal{L}(M) = L$).

- This handles *accept* states, but what about reject states?
- **What if the machine enters an infinite loop?**

# Turing decidability

**Def:** A **decider** on language $L$ is a TM which never enters an infinite loop: It only ever accepts or rejects. This TM is said to **decide** $L$.

**Def:** A language is **Turing-decidable** or simply **decidable** iff some Turing machine **decides** it.

**Corollary:** All Turing-decidable languages are Turing-recognizable.

# Next time: The diagonalization proof

### Unintentional Turing completeness [ edit ]

Some software and video games are Turing-complete by accident, i.e. not by design.

Software:

- Microsoft Excel[17]

Games:

- *Dwarf Fortress*[18]
- *Cities: Skylines*[19]
- *Opus Magnum*[20]
- *Minecraft*[21]
- *Magic: The Gathering*[22][23]
- Infinite-grid Minesweeper[24]

Social media:

- *Habbo Hotel*[25]

Computational languages:

- C++ templates[26]
- printf format string[27]
- TypeScript's type system[28]
- x86 assembly's MOV instruction[29][30][31]

Biology:

- Chemical reaction networks[32][33][34][35] and enzyme-based DNA computers[36] have been shown to be Turing-equivalent