Diagonalization and decidability problems

Textbook: Chapter 4

$$
\begin{array}{ll}
s_1 & = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
s_2 & = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\
s_3 & = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\
s_4 & = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots \\
s_5 & = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots \\
s_6 & = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots \\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\ddots
\end{array}
$$

$$
s = 1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1\ldots
$$

Sizing infinite sets

- The size of a finite set is the number of elements in it
- What about infinite sets?
- We can show that a set A is the same size as another, B, by making a **1-to-1 mapping** from each element of A to B
- This works even for infinite sets!

Examples

Ex: An infinite amount of \$100 bills and an infinite amount of \$1 bills **are worth the same amount.**

Ex: Are there more integers ($\mathbb{Z}$) or even integers ($2\mathbb{Z}$)? It would seem that, since only half of all integers are even, that there would be more of them. However, **this isn't true!**

Let $f : \mathbb{Z} \to 2\mathbb{Z}$ be the function $f(k) = 2k$. This is a **bidirectional mapping**, so the sets it maps to and from must be the same size!

Different infinity sizes

Ex: Is the set of all natural numbers $\{0, 1, 2, 3, \ldots\}$ the same size as the set of all real numbers (EG $\pi, 4, \frac{1}{2}, e, \ldots$)? They are both infinite, so one would assume so.

As it turns out, **no**! Some infinities are larger than others.

Cantor's diagonalization proof for $\mathbb{N}$ and $\mathbb{R}$

Thm: There are more real numbers between 0 and 1 than natural numbers 0 to ∞.

Pf: We will try to construct an arbitrary-order table where the first column is an increasing integer index starting at 0 and the second column is a list of all the real numbers from 0 to 1. If there are the same number, all reals will be contained in this list.

Index	Real number
0	$c_0 = 0.12345678\ldots$
1	$c_1 = 0.31415926\ldots$
2	$c_2 = 0.27182818\ldots$
3	$c_3 = 0.00000000\ldots$
4	$c_4 = 0.99999999\ldots$
$\vdots$	$\vdots$

Cantor pt. 2

Now we construct a new number c'. The i-th decimal digit of c' will be one more than the i-th digit of c_i (or 0 if it was 9). By definition, c' is not in our list (it is different from every element in at least one digit)!

$$
\begin{aligned}
&c_0 = 0.\mathbf{1}2345\ldots \\
&c_1 = 0.3\mathbf{1}415\ldots \\
&c_2 = 0.27\mathbf{1}82\ldots \\
&c_3 = 0.000\mathbf{0}0\ldots \\
&c_4 = 0.9999\mathbf{9}\ldots \\
&\vdots \\
&c' = 0.\mathbf{22210}\ldots
\end{aligned}
$$

Since we already used every natural number for indices, **there must be more real numbers than natural ones.** End of proof.

Note: We used the range $[0, 1]$, but this holds for any $[a, b] : a, b \in \mathbb{R}$! There are more real numbers between any two real numbers than there are natural numbers.

Infinity sizes

Def: An infinite set is said to be **countable** iff there exists a correspondence between it and the natural numbers.

Def: An infinite set is said to be **uncountable** iff there exists a correspondence between it and the real numbers.

Corollary: Any uncountable set is larger than any countable set. Any countable set is larger than any finite set.

Proving unrecognizable languages exist

Thm: There are more languages on any Σ than there are Turing machines.

Pf: We will prove the set of all Turing machines is countably infinite, while the set of all languages on Σ is uncountably infinite. Therefore, there are undecidable languages on every alphabet.

Encoding TMs in finite binary strings

Def: An **encoding** onto alphabet Σ of a TM M is some finite string $\langle M \rangle \in \Sigma^*$ which can be used by a UTM to simulate M.

This is intentionally vague! Just convince yourself that such a string exists and is finite for all valid TMs.

Def: A binary encoding of a TM encodes it onto alphabet $\{0, 1\}$.

Language characteristic strings

- A language on an alphabet Σ is some (possibly infinite) subset of Σ^*
- Without loss of generality, let us assume $\Sigma = \{0, 1\}$ (the smallest useful alphabet)
- Then Σ^* is $\{\epsilon, 0, 1, 01, 10, 11, \ldots\}$
- Let σ_i be the i-th element of Σ^* (0-indexed)
- **Note:** Every σ_i is of finite length!

Def: The characteristic string c_L of language L on alphabet Σ is an **infinite** binary string $c = c_0 c_1 c_2 \cdots c_i \cdots$ where $c_i = 1$ iff $\sigma_i \in L$.

- Ex: $c_\emptyset = 0000000\cdots$
- Ex: $c_{\Sigma^*} = 1111111\cdots$
- Ex: $c_{\{\epsilon\}} = 1000000\cdots$
- Ex: $c_{\{1,11\}} = 0010001\cdots$

Diagonalizing the TMs

Create a table such that:

- Every TM M_i's binary encoding a_i is in one column
- The characteristic string c_i of $\mathcal{L}(M_i)$ (the set all strings *recognized* by M_i) is in the other
- By convention, let any invalid TM encoding have a characteristic string of all 0 (follows from UTM def.)

TM Binary encoding	Characteristic string
$a_0 = 00000$	$c_0 = 01010101\ldots$
$a_1 = 111111111$	$c_1 = 11111111\ldots$
$a_2 = 0$	$c_2 = 00000000\ldots$
$a_3 = 1111\ldots1111$	$c_3 = 10101111\ldots$
$\vdots$	$\vdots$

Every TM is listed in the left column.

Diagonalization pt. 2

Let c' be an infinite binary characteristic string. Let the i-th bit of c' be the negation of the i-th bit of c_i.

TM Binary encoding	Characteristic string
$a_0 = 0000\ldots0000$	$c_0 = \mathbf{0}101\ldots$
$a_1 = 111111111$	$c_1 = 1\mathbf{1}11\ldots$
$a_2 = 0$	$c_2 = 00\mathbf{0}0\ldots$
$a_3 = 1111\ldots1111$	$c_3 = 101\mathbf{0}\ldots$
$\vdots$	$\vdots$

$c' = \mathbf{1011}\ldots$

The language of c' is not recognized by the list! Since the list has every TM, no TM recognizes c': It is said to be **unrecognizable**.

Therefore, some languages are not recognizable.

Proving the halting problem is recognizable

Def: The halting problem / entscheidungsproblem. Given some TM M and input w, does M ever halt? We define the language A_{TM} (A for "accept") to be:

$$A_{TM} = \{\langle M, w\rangle : M \texttt{ is a TM and accepts } w\}$$

Thm: The halting problem (A_{TM}) is *recognizable* (a positive answer can be given, but not always in finite time).

Pf: By construction. Let A be a UTM taking input $\langle M, w\rangle$ and accepting only if $M(w)$ halts. Being a UTM, it can simulate $M(w)$. If $M(w)$ ever halts, A accepts. By definition, A recognizes (but doesn't decide) A_{TM}. End of proof.

Proving the halting problem is undecidable

Thm: A_{TM} is undecidable.

Pf: By contradiction. Assume A_{TM} is decidable. Then there exists a TM H **deciding** A_{TM}. We will derive a contradiction similar to Russell's paradox.

$$H(\langle M, w \rangle) = \begin{cases} \textit{accept} & \texttt{if } M \texttt{ accepts } w \\ \textit{reject} & \texttt{if } M \texttt{ rejects or loops on } w \end{cases}$$

Let D be a new TM using H as a subroutine (this is legal). Since H is a decider, it always completes in finite time. We will design D to be a decider as well.

Halting problem undecidability pt. 2

$D =$ "On input $\langle M \rangle$, where M is a TM:

1. Run H on input $\langle M, \langle M \rangle \rangle$ (determine whether M halts when given its own description)
2. If H accepted, **reject**. If H rejected, **accept**."

This means that $D(\langle M \rangle)$ accepts if $M(\langle M \rangle)$ rejects or loops and rejects if $M(\langle M \rangle)$ accepts.

What happens if we run $D(\langle D \rangle)$?

Halting problem undecidability pt. 3

$$D(\langle D \rangle) = \begin{cases} \textit{accept} & \texttt{if } D \texttt{ rejects or loops on } \langle D \rangle \\ \textit{reject} & \texttt{if } D \texttt{ accepts } \langle D \rangle \end{cases}$$

- If $D(\langle D \rangle)$ accepts:
  - By definition, this means it must reject
  - **Contradiction!**
- If $D(\langle D \rangle)$ rejects:
  - By definition, this means it must accept
  - **Contradiction!**

Therefore, $D(\langle D \rangle)$ accepts if and only if it does not: A contradiction. The assumption that A_{TM} is decidable must be false. End of proof.

Proven by Church and Turing in 1936.

Co-Turing-recognizability

Def: A language L is said to be **co-Turing-recognizable** if $\overline{L}$ is Turing-recognizable.

Thm: A language is decidable iff it is Turing-recognizable and co-Turing-recognizable. (pf excluded)

Corollary: A language is undecidable iff it or its complement are not Turing-recognizable.

Note: All languages which are Turing *decidable* are trivially co-Turing-decidable and vice versa.

A Turing-unrecognizable language

Thm: $\overline{A_{TM}}$ is not Turing-recognizable.

Pf: Since A_{TM} is undecidable, either it or its complement must be Turing-unrecognizable. We already proved A_{TM} is Turing-recognizable. Therefore, $\overline{A_{TM}}$ must be Turing-unrecognizable. End of proof.

Next time: Reducibility