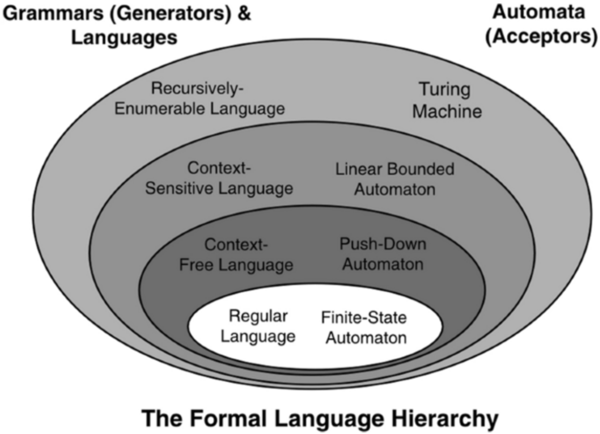
Finite-State Acceptors



Textbook: 1.1 and 1.2

Definitions

- An **automaton** (plural automata) is a self-moving machine. In this context, it just means "small, simple, simulated machine".
- An **acceptor** is an automaton which takes in a string and either accepts or rejects it through a series of state transitions. **Note:** Many texts fail to distinguish between *automata* and *acceptors*. Acceptors are a proper subset of automata.
- A **finite-state** automaton has a finite number of states that it can transition through.
- A **deterministic** automaton has exactly one set of state transitions which follow from a given string. A nondeterministic one, on the other hand, has multiple series of state transitions which may follow.

Kleene operators

- For an alphabet Σ, the **Kleene star** is an operator creating the set of all strings of zero or more characters
- If Σ^i is the set of all strings of length i over Σ, then $\Sigma^* = \bigcup_{i=0}^{\infty} \Sigma^i$
- Ex:
$$\{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, \ldots\}$$
- The **Kleene plus** gives the set of all strings of *one* or more characters over an alphabet. It is less common
$$\Sigma^* = \Sigma^+ \cup \{\epsilon\}$$

DFAs

- A **deterministic finite-state acceptor (DFA)** is an automaton built to accept strings using a finite number of states where each string corresponds to exactly one series of state transitions. Formally:

Def: Finite-state acceptor. A finite acceptor is a 5-tuple $A = (Q, \Sigma, \delta, q_0, F)$ where:

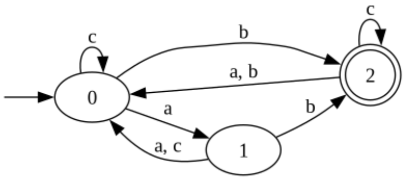
1) Q is a finite set called the **states**
2) Σ is a finite set of symbols called the **alphabet**
3) $\delta : (Q \times \Sigma) \to Q$ is the **transition function**
4) $q_0 \in Q$ is the **start state**
5) $F \subseteq Q$ is the set of acceptance states

Note: A DFA must define all transitions! Every state must have a specified target state for every character in the alphabet.

DFA operation

Consider the DFA A with states $Q = \{q_0, q_1, q_2\}$, starting state q_0, alphabet $\Sigma = \{a, b, c\}$, and acceptance state set $F = \{q_2\}$. The following graph demonstrates δ.

Note: An arrow from nothing means "initial state". In these diagrams, a double circle means "accepting state".



This will accept any string of the forms abc^i or bc^i, where i is any nonnegative integer.

Formal definition of computation

We say that a DFA A **accepts** string $w = w_1 w_2 \dots w_n \in \Sigma^*$ if and only if there exists some series of states $r_0 r_1 r_2 \dots r_n \in Q^+$ such that the following all hold:

1) $r_0 = q_0$
2) $\delta(r_i, w_{i+1}) = r_{i_1}$ for all $i = 0, \dots, n_1$
3) $r_n \in F$

Def: The "language of" operator. For an automaton A over alphabet Σ, the language of A is given by the operator $\mathcal{L}(A) = \{w \in \Sigma^* : A \texttt{ accepts } w\}$.

Ex: If A recognizes all nonnegative even integers, then $\mathcal{L}(A) = \{0, 2, 4, 6, 8, 10, \dots\}$

NFAs

- A **nondeterministic finite-state acceptor (NFA)** is a DFA that can follow multiple series of state transitions
- The NFA accepts iff any of its branches do
- The only formal difference is in the transition function: δ **can output any number of output states**

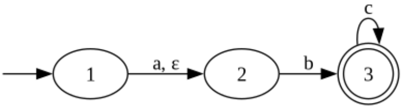
Def: Nondeterministic finite-state acceptor (NFA). An NFA is a 5-tuple $A = (Q, \Sigma, \delta, q_0, F)$ where:

1) Q is a finite set called the states
2) Σ is a finite set of symbols called the alphabet
3) $\delta : (Q \times \{\Sigma \cup \epsilon\}) \to \mathcal{P}(Q)$ is the **nondeterministic transition function**
4) $q_0 \in Q$ is the start state
5) $F \subseteq Q$ is the set of acceptance states

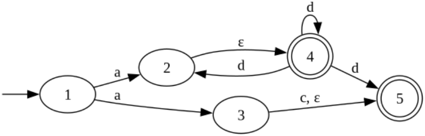
Note: If an NFA can leave transitions undefined, unlike a DFA. If it comes to a fork in the road, it duplicates and goes down both paths.

NFA operation

This NFA accepts the same language as our previous DFA:



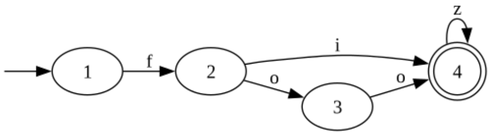
This NFA has more “weird” branches:



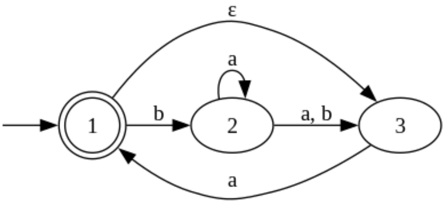
Practice problems

Do these problems on your own, then we will do them on the board.

1) Does the following DFA accept `fizz`? `foo`? `buzz`?



2) Describe the set of all strings accepted by the this NFA.



3) Construct an NFA **and** a DFA to recognize the set of all strings consisting of repetitions of `abc`.

Automaton equivalence

Def: Automaton equivalence. Automata A and B are equivalent if and only if they recognize the same language: That is,
$(A = B) \iff (\mathcal{L}(A) = \mathcal{L}(B))$.

Ex: Let A be such that

$$\mathcal{L}(A) = \{w : w \texttt{ is an even nonnegative integer}\}$$

and B be such that

$$\mathcal{L}(B) = \{w : w \texttt{ is a non-empty string of numbers ending in 0,}$$

Since any nonnegative even integer is a non-empty string of numbers and ends in 0, 2, 4, 6, or 8, all items recognized by A are recognized by B. Since all non-empty strings of numbers ending in 0, 2, 4, 6, or 8 are nonnegative even integers, all items recognized by B are recognized by A. Therefore $\mathcal{L}(A) = \mathcal{L}(B)$ and $A = B$.

DFA-NFA equivalence

- Counterintuitively, **NFA and DFA are equivalent**: Every DFA has an equivalent NFA and vice versa
- To simulate an NFA, you can keep track of all the states currently held by any branch
- To advance to the next step, let the new states be the union of all valid transitions from existing states
- It turns out the set of all possible combinations of NFA states is also a valid DFA state set! We will use this in our proof

DFA-NFA equivalence proof pt. 1

We will first prove that all DFA are NFA, then prove that all NFA have equivalent DFA.

Thm: All DFA have equivalent NFA. **Pf:** The set of all DFA is a subset of the set of all NFA. Therefore, all DFA are trivially NFA. End of proof.

Thm: All NFA have equivalent DFA. **Pf:** By construction. DFA have $\delta_{\texttt{DFA}} : (Q \times \Sigma) \to Q$, while NFA have $\delta_{\texttt{NFA}} : (Q \times \{\Sigma \cup \epsilon\}) \to \mathcal{P}(Q)$. We will show that all $\delta_{\texttt{NFA}}$ have an equivalent modification in the form of $\delta_{\texttt{DFA}}$, proving that all NFA are equivalent to DFA. Let A be an arbitrary NFA $= (Q, \Sigma, \delta, q_0, F)$.

1) Q vs $\mathcal{P}(Q)$. Since Q is a finite set, $\mathcal{P}(Q)$ is a finite set of size $2^{|Q|}$. Any finite set can be used as a set of DFA states, so $\mathcal{P}(Q)$ works. **Let** $Q' = \mathcal{P}(Q)$.

DFA-NFA equivalence proof pt. 2

2) ϵ transitions. We must eliminate any nonconsumptive transitions in order to be a valid DFA. Let $E : Q \to \mathcal{P}(Q)$ be the ϵ-**closure** function mapping a state to the set all all states reachable by ϵ transitions from it. Note that $E : Q \to Q'$. **Let** $\delta'(q, \sigma) = E(\delta_{\mathtt{NFA}}(E(q), \sigma))$. Note that $\delta' : (Q' \times \Sigma) \to Q'$.

3) Starting states. The starting state may have outgoing ϵ transitions which must be accounted for. Thus, **let** $q_0' = E(q_0)$.

4) Accepting states. An NFA's F is a subset of the set of its states ($F \subset Q$). However, it is not a subset of Q' yet: Thus, we must adjust it. **Let** $F' = \{R \in Q' : R \texttt{ contains an accept state from } F\}$.

Now, for an arbitrary NFA $A = (Q, \Sigma, \delta, q_0, F)$, let $A' = (Q', \Sigma, \delta', q_0', F')$. By the above rules, A' is a valid DFA equivalent to A. End of proof.

Operation closure

Def: Closure under an operation. A set is **closed** under an operation iff that operation maps back to the set. We will study closure under unary and binary relations, but the concept generalizes to k-ary relations.

Ex: Integers. Since, for all integers a, b the results $a + b$ and ab are integers, the set of all integers is closed under addition and multiplication. Since there exist integers b, c such that b/c is not an integer, the set of all integers is *not* closed under division.

Ex: Derived closure. Since -1 is an integer, then a, b being arbitrary integers implies that $a - b = a + (-1 * b)$ is also an integer. Since addition and multiplication were previously shown to be closed, we can now say subtraction is as well.

The regular operations

- Just like in numeric math we have standard operations like $+$ and $\times$, in the study of regular languages we have the following **"regular"** operations

Def: The regular operations. The following three operations are known as the regular operations.

- **Union**: $A \cup B = \{x : (x \in A) \vee (x \in B)\}$
- **Concatenation**: $A \circ B = \{xy : (x \in A) \wedge (y \in B)\}$
- **Kleene ("KLAY-nee") Star**: $A^* = \{x_1 x_2 \ldots x_k : k \geq 0, \forall i [x_i \in A]\}$

Note: Kleene plus. If a set A is closed under concatenation and Kleene star, then $A^+ = A \circ A^*$ is also closed.

State machine closure under regular operations

Thm: Finite-state-acceptor recognizable languages are closed under the regular operations: Union, concatenation, and Kleene star.

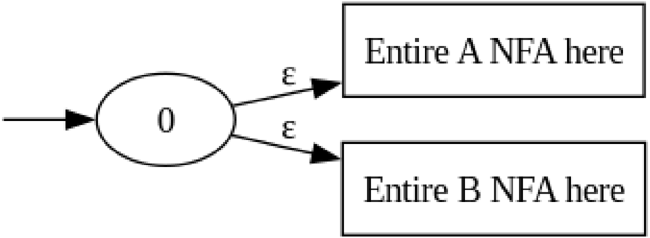
Pf: By construction in 3 parts. This can be done using only DFA (as in book), but for simplicity and without loss of generality we will instead use NFA here.

1) Union. Given NFA A, B, we want to construct some NFA representing $A \cup B$.
2) Concatenation. Given NFA A, B, we want to construct some NFA representing $A \circ B$.
3) Kleene star. Given NFA A, we want to construct some NFA representing A^*.

This proof will rely on graph representations over formal definitions.

1: Closure under union

Let Q_A and Q_B be the state sets Q from A and B, respectively. They are relabeled this way to avoid collisions. Let a_0 and b_0 be the initial states of their corresponding NFAs. To create an NFA for $A \cup B$, we just need to add a new initial state q_0' with outgoing ϵ transitions to a_0 and b_0. After this, our new state set Q', accepting state set F', and δ' are just trivial unions of A and B's.

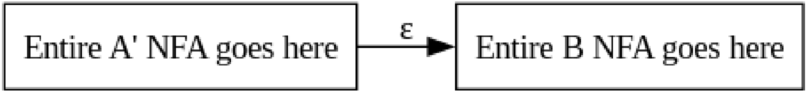


$$\delta'(q_0, \epsilon) = \{a_0, b_0\}$$

This NFA accepts a string matching A or B, therefore NFA are

2: Closure under concatenation

Let us define A' as a modification of A where all accepting states are no longer accepting and have outgoing ϵ transitions to the initial state of B. Let all states of A and B be relabelled so they do not conflict. This is represented in the diagram below.



Our new NFA has the initial state of A, the state transition map of A' merged with B, the state set of relabelled states from both, and the final states of B.

This NFA accepts a string composed of a substring matching A followed by a substring matching B, therefore NFA are closed under concatenation.

3: Closure under Kleene star

Let us define a new non-conflicting initial state q_0. All states of A will be relabelled to avoid a collision with q_0. Let A' be the modification of A such that all accepting states are no longer accepting and have an outgoing ϵ transition to q_0. q_0, in turn, has an outgoing ϵ transition to the old starting node of A. The below diagram demonstrates this.

This NFA accepts zero or more instances of a substring matching A, therefore NFA are closed under Kleene star.

End of proof

Since NFA are closed under union, concatenation, and Kleene star, DFA are too. Thus, finite-state machines are closed under the regular operations. End of proof.

Next up: Regular languages and expressions

"The syntactic component of a grammar must specify, for each sentence, a deep structure that determines its semantic interpretation and a surface structure that determines its phonetic interpretation." - **Noam Chomsky**

"Give Orange Me Give Eat Orange Me Eat Orange Give Me Eat Orange Give Me You" - **Nim Chimpsky**