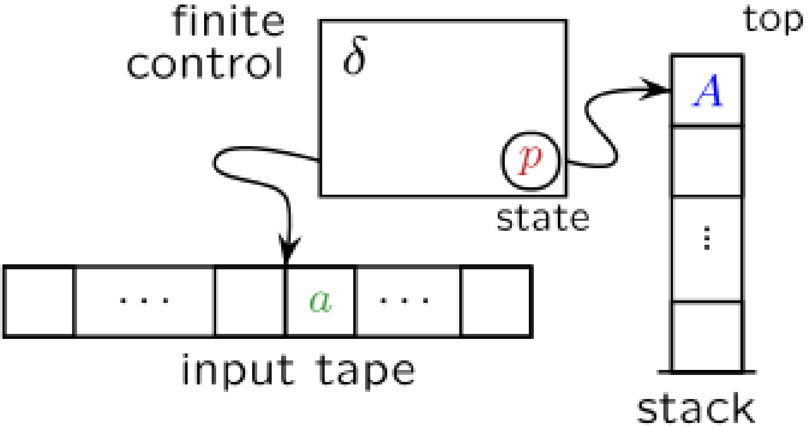
PDA, the Chomsky hierarchy, and the pumping lemma for context-free languages



Textbook: 2.2 and 2.3

PushDown Automata (PDA)

- We have been looking at *finite* state machines
- What if we need to remember an **infinite** amount information?
- Would allow us to match the set of strings containing matching parenthesis
- We can give out state machine an additional memory store in the form of an infinitely large **stack**
- Allow it to push or pop from that stack
- **Not** random access: Last in, first out

nPDA

- Unlike DFA/NFA, **dPDA and nPDA are not equivalent in power**
- Unlike other models, **when we say PDA we mean nPDA not dPDA**
- We will be ignoring deterministic PDA because they are not equivalent to context-free grammars: nondeterministic PDA are

Formal definition

Def: nPDA. A **nondeterministic pushdown automaton (nPDA)** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, F are all finite sets and:

1. Q is the set of states
2. Σ is the **input alphabet**
3. Γ is the **stack alphabet**
4. $\delta : (Q \times \Sigma_\epsilon \times \Gamma_\epsilon) \to \mathcal{P}(Q \times \Gamma_\epsilon)$ is the **nondeterministic transition function**
5. $q_0 \in Q$ is the start state
6. $F \subseteq Q$ is the set of acceptance states

We let \$ be the special symbol indicating that the stack is empty.

Interpreting an nPDA transition

For some mapping

$$\delta(q, \sigma, \gamma) = (q', \gamma')$$

- The current state is q
- The transition is taken when the input is σ (or ϵ for nonconsumptive transitions)
- γ is the item to pop off the stack (or ϵ if we don't want to)
- q' is the new state
- γ' is the item to push to the top of the stack (or ϵ if we don't want to)

Note: ϵ means "don't pop from the stack", $ means "empty stack". We disallow the removal of $.

Equivalence with context-free grammars

Thm: A language is context free iff some nPDA recognizes it. This proof is 7 pages of the textbook, so we will only go over the broad strokes.

Lemma 1: A language being context free implies some nPDA recognizes it. (pg 115)

- By construction
- We show how to use an nPDA to determine if the CFG derives any input string

Lemma 2: Any language recognized by a nPDA is context free. (pg 119)

- By construction
- We show how to create grammar rules from an nPDA in CFG form

The Chomsky hierarchy

- The **Chomsky hierarchy** states the power of different models of computation
- Each entry is a linguistic class and corresponds to a type of automata

In increasing power:

1. Finite-state automata / regular languages
2. Pushdown automata / context-free grammars
3. Linear bounded automata / context-sensitive grammars
4. Turing machines / unrestricted grammars

Chomsky hierarchy grammar rules

Grammar	Languages	Recognizing Automaton	Production rules (constraints)[a]	Examples[5][6]
Type-3	Regular	Finite-state automaton	$A \to \mathrm{a}$ $A \to \mathrm{a}B$ (right regular) or $A \to \mathrm{a}$ $A \to B\mathrm{a}$ (left regular)	$L = \{a^n \vert n > 0\}$
Type-2	Context-free	Non-deterministic pushdown automaton	$A \to \alpha$	$L = \{a^n b^n \vert n > 0\}$
Type-1	Context-sensitive	Linear-bounded non-deterministic Turing machine	$\alpha A \beta \to \alpha \gamma \beta$	$L = \{a^n b^n c^n \vert n > 0\}$
Type-0	Recursively enumerable	Turing machine	$\gamma \to \alpha$ (γ non-empty)	$L = \{w \vert w$ describes a terminating Turing machine$\}$

Via Wikipedia

The pumping lemma for context-free languages

- Just like we have a pumping lemma to prove that a given language is nonregular, we have another to prove a given language is not context-free
- Very similar to the pumping lemma for regular languages

Def: Pumping lemma for context-free languages. If A is a context-free language, then there is a number p (the pumping length) where, if s is any string $\in A$ such that $|s| \geq p$, then s may be divided into five pieces $s = uvxyz$ such that:

1. For each $i \geq 0, uv^ixy^iz \in A$
2. $|vy| > 0$
3. $|vxy| \leq p$

CFG Pumping lemma proof

Pf: By construction. We will derive a minimal bound for the pumping length p, then show that any string longer than this must have a derivation where some variable R derives itself. This implies that the lemma holds, ending the proof.

Lemma 1: String sizes. Let b be the maximum number of variables on the RHS of any rule in our CFG. Then at most b leaves are within 1 step from the starting variable, b^2 within 2, and b^h within h. Thus, any parse tree of height h produces a string of size at most b^h. Conversely, if a generates string is at least $b^h + 1$ long, each of its parse trees must be at least $h + 1$ high.

If $|V|$ is the number of variables in our CFG, we let $p = b^{|V|+1}$. Thus, any string longer than the pumping length must have parse trees of height at least $|V| + 1$, since $b^{|V|+1} \geq b^{|V|} + 1$.

CFG Pumping lemma proof pt. 2

Since the height of the parse tree is the number of variable replacements, there must be $|V| + 1$ replacements of $|V|$ variables. Thus, there must be a variable which is repeated. This variable derives another instance of itself. Thus, we can replace its first occurrence with its final occurrence or infinitely pump the in-between region while remaining in the language.

Conditions 2 and 3 are proven in the book (pg 125).

CFG Pumping lemma proof pt. 3

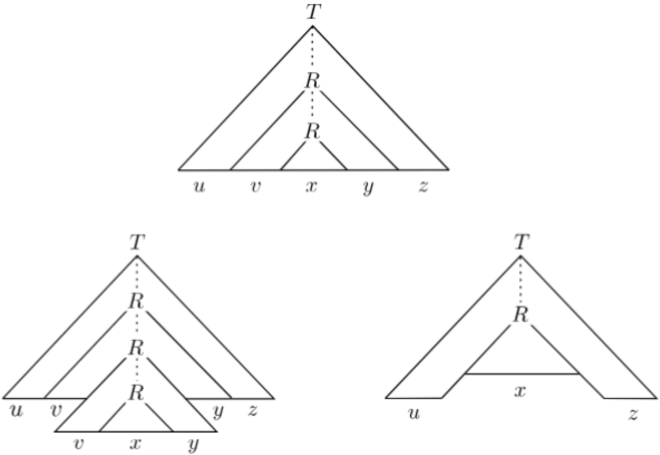


FIGURE 2.35
Surgery on parse trees

Non-context-free languages

Practice: Let $A = \{a^n b^n c^n : n \geq 0\}$. Prove that A is not context free.

- See pg 126 of textbook

Next up: **Part 2: Computability theory**, the Church-Turing thesis, and Turing machines

An assignment on part 1 of the textbook should come soon