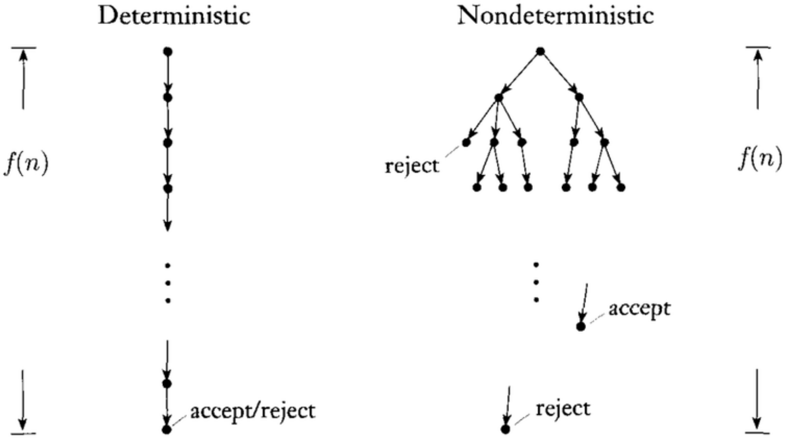
# Nondeterministic Turing machines

Textbook: Chapter 3.2



# Idea

- We turned DFA into NFA by modifying the form of $\delta$
- **What if we do this to a TM?**

Recall: A TM's transition function is of the form:

$$\delta : (Q \times \Gamma) \to (Q \times \Gamma \times \{L, R\})$$

For state set $Q$, tape alphabet $\Gamma$. A nondeterministic version would look like:

$$\delta : (Q \times \Gamma) \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

# Definition of NTMs

**Def:** Nondeterministic Turing machines (NTMs). An NTM is a 6-tuple

$$(Q, \Sigma, \Gamma, \delta, q_0, q_{\texttt{accept}})$$

where $Q, \Sigma, \Gamma$ are all finite sets and:

1. $Q$ is the state set
2. $\Sigma$ is the input alphabet, $\sqcup \notin \Sigma$
3. $\Gamma$ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$
4. $\delta$ is the nondeterministic transition function

$$\delta : (Q \times \Gamma) \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

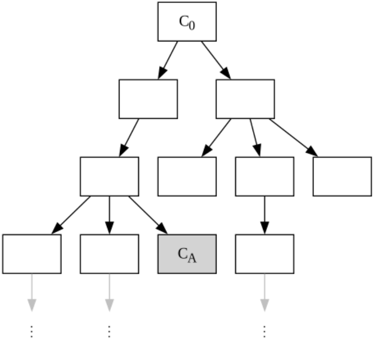
5. $q_0 \in Q$ is the start state
6. $q_{\texttt{accept}} \in Q$ is the accept state

**An NTM accepts iff any branch of its nondeterminism does**

# Equivalence with (D)TM

**Thm:** TM and NTM are computationally equivalent.

**Pf:** By construction. If we let an NTM configuration be a node in a tree from which zero or more may follow:




We want to find any $C_A$ in the tree following $C_0$. How?
**Breadth-First search!**

# Notes

- Note: We cannot use depth-first search, since any given branch may never terminate (DFS doesn't work on infinite trees)
- Note: This algorithm will always find the shortest accepting computation history if one exists

Convince yourself that queues are legal data structures to use within a TM (recall TMs and algorithms are equivalent)

# Breadth-first search on NTM configuration trees

Let TM $R$ take input $\langle M \rangle, w$, where $w$ is the input to NTM $M$:

1. Create an empty queue of configurations
2. Push the entrance configuration $q_0 w$ to the queue
3. For as long as the queue is nonempty:
   3.1 Pop a configuration $c$ from the queue
   3.2 If $c$ contains a halting state, halt
   3.3 Push each configuration that follows from $c$ to the queue

Note that $R$ accepts iff $M$ does on some branch, $R$ rejects iff $M$ does on some branch, and $R$ loops iff $M$ does on all branches. $R$ is a legal TM emulating an arbitrary NTM, so TM and NTM are equivalent. End of proof.

# Complexity

**Thm:** A TM can simulate $i$ steps of an NTM in $O(2^i)$ time and space.

- If we let $b$ be the max number of configurations output by $\delta$, then there are at most $b^i$ nodes after $i$ steps
- Therefore, the TM will have to simulate a total of up to $\sum_{j=0}^{i} b^j = b^{i+1} - 1$ nodes! **Very bad!**

**Computational equivalence does not imply polynomiality!**

# Corollaries

**Corollary:** A system is Turing-recognizable iff some NTM recognizes it, Turing-decidable iff some NTM decides it, Turing-complete iff it simulates all NTM, and Turing-equivalent iff it is equivalent to an NTM.

- A language is decided by an NTM iff it halts on all branches of its determinism

# Next up: Advanced computability theory