Understanding HexSSPs

Friday, February 2, 2024 4:39 PM

We randomly choose values u_k, $k = 0, \dots, N-1$.
Then we form the vector $\vec{u} = [u_0, \dots, u_{N-1}]^T$.
We use $\vec{u}$ as the phases to construct a unitary complex vector,

$$\vec{U} = [e^{iu_0}, \dots, e^{iu_{N-1}}]^T = e^{i\vec{u}}$$

all elements have modulus 1.

This vector $\vec{U}$ is a hypervector, $\vec{U} \in \mathbb{C}^N$.

If we treat $\vec{U}$ as a vector of Fourier coefficients, then we get "pseudo-spatial" vectors. If $\vec{U}$ is conjugate symmetric, then the vectors are real-valued. This is the default in Nengo. (see the HRR page)

However, we will stick with the complex vectors of FHRR.
Define

$$\theta(x) = \vec{U}^x \quad \text{or} \quad \theta_{\vec{u}}(x) \quad \text{or} \quad \theta(x;\vec{u})$$

For Nengo users, $\phi(x) = \mathrm{IDFT}(\theta(x))$

You can increment x by Δx using

$$\theta(x) \odot \theta(\Delta x) = e^{i\vec{u}x} \odot e^{i\vec{u}\Delta x} = e^{i\vec{u}(x+\Delta x)} = \theta(x + \Delta x)$$

or $\theta(x;\vec{u}) \odot \theta(\Delta x;\vec{u})$

Hadamard (element-wise) product: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \odot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \end{bmatrix}$

What happens when we assess the similarity between two SSPs?

$$\theta(x) \cdot \overline{\theta(y)}$$

← complex conjugate

> Note: Parseval's theorem gives us that
> $\phi(x) \cdot \overline{\phi(y)} = \theta(x) \cdot \overline{\theta(y)}$ with a few caveats that we'll ignore here
> Thanks Parseval!!

Let's look closely at similarity.

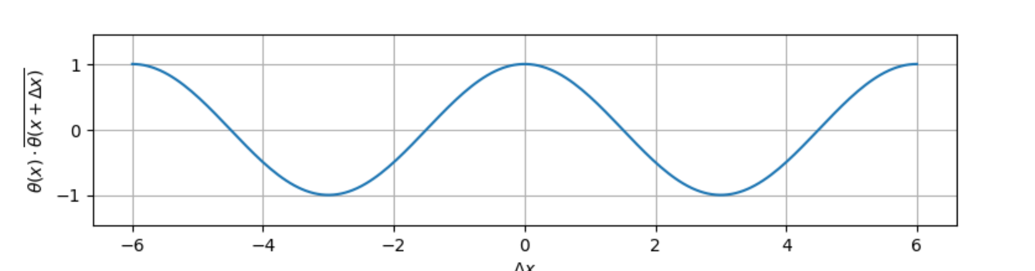
$$\theta(x) \cdot \overline{\theta(y)} = \frac{1}{N} \sum_{n=0}^{N-1} e^{iu_n x} e^{-iu_n y} = \frac{1}{N} \sum_{n=0}^{N-1} e^{iu_n (x-y)}$$

For now, let's think of y as fixed.
Then, the similarity $\theta(x) \cdot \overline{\theta(y)}$ is a function of x.
It is a sum of N Fourier basis functions (FBFs), with frequencies u_n, $n = 0, \dots, N-1$.

Let's look at some examples.

Here is a simple case, $\vec{u} = [\frac{\pi}{3}]$.
Then $\theta(x;\vec{u}) = [e^{i\frac{\pi}{3}x}]$.

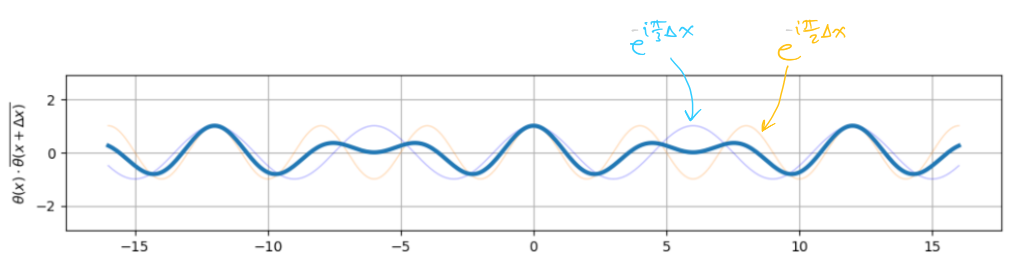
$$\theta(x, [\tfrac{\pi}{3}]) \cdot \overline{\theta(x+\Delta x, [\tfrac{\pi}{3}])} = e^{i\frac{\pi}{3}(x-(x+\Delta x))} = e^{-i\frac{\pi}{3}\Delta x}$$



Repeats when $\Delta x \frac{\pi}{3} = 2\pi k$ for $k \in \mathbb{Z}$ $\Rightarrow \Delta x = 6k$
∴ period is 6.

It gets interesting when we have more than 1 element in $\vec{u}$.
Eg. $\vec{u} = [\frac{\pi}{2}, \frac{\pi}{3}]^T$

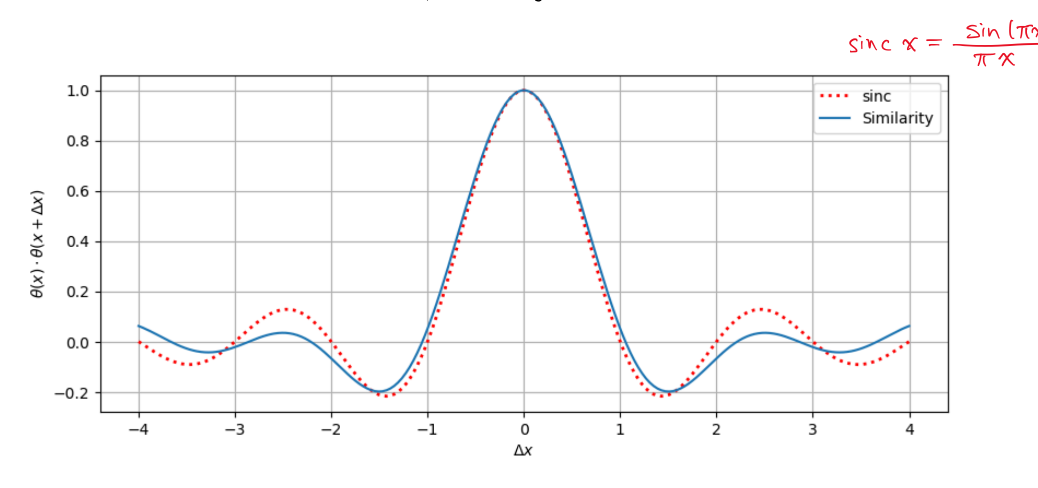
$$\theta(x) \cdot \overline{\theta(x+\Delta x)} = e^{i\frac{\pi}{2}\Delta x} + e^{i\frac{\pi}{3}\Delta x}$$



This is an interference pattern between two Fourier basis functions (FBFs), with freq. $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

Let's include LOTS of elements in $\vec{u}$, and observe the interference pattern between all those FBFs...
Let $\vec{u} \sim \mathrm{Uniform}(-\pi, \pi)^{500}$ (i.e. 500 elements)

$\mathrm{sinc}\, x = \frac{\sin(\pi x)}{\pi x}$



2 Dimensions

Question:
What if we are encoding vectors, like $(x, y) \in \mathbb{R}^2$?

Answer:
We use 2 sets of phases, one for each coord, and bind the resulting vectors together.

Consider $\vec{u} \in [-\pi, \pi]^N$ and $\vec{v} \in [-\pi, \pi]^N$.
Then $\vec{X} = e^{i\vec{u}}$ and $\vec{Y} = e^{i\vec{v}}$ are coord vectors.

We can encode x using $\theta(x;\vec{u}) = \vec{X}^x$
We can encode y using $\theta(y;\vec{v}) = \vec{Y}^y$

Then we encode (x, y) by binding those two

$$\theta(x;\vec{u}) \odot \theta(y;\vec{v}) = \vec{X}^x \odot \vec{Y}^y$$
$$= [e^{iu_0 x}, \dots, e^{iu_{N-1}x}]^T \odot [e^{iv_0 y}, \dots, e^{iv_{N-1}y}]^T$$
$$= [e^{iu_0 x} e^{iv_0 y}, \dots, e^{iu_{N-1}x} e^{iv_{N-1}y}]^T$$
$$= [e^{i(u_0, v_0)\cdot(x,y)}, \dots, e^{i(u_{N-1}, v_{N-1})\cdot(x,y)}]^T$$
$$= e^{i \underbrace{\begin{bmatrix} u_0 & v_0 \\ \vdots & \vdots \\ u_{N-1} & v_{N-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}}} = e^{iA\vec{x}}$$

To save time and paper, let's define

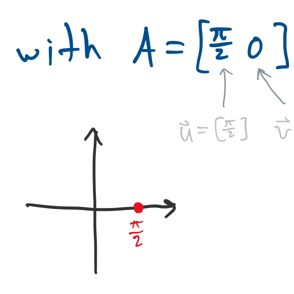
$$\theta(\vec{x}; (\vec{u}, \vec{v})) = \theta(x;\vec{u}) \odot \theta(y;\vec{v})$$

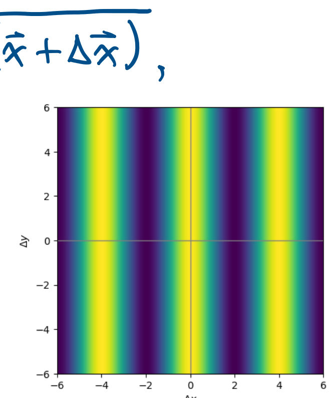
or simply $\theta(\vec{x})$ for short, where $\theta(\vec{x}) = e^{iA\vec{x}}$.

Let's investigate similarity between vector encodings in 2D.

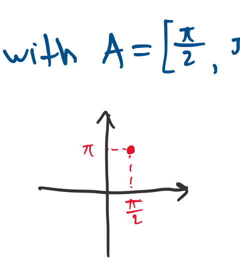
Consider $\theta(\vec{x}) \cdot \overline{\theta(\vec{x} + \Delta\vec{x})}$,

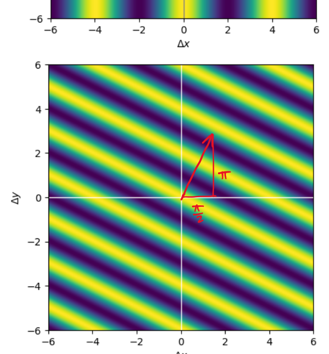
with $A = [\frac{\pi}{2}\ 0]$.
$\vec{u} = [\frac{\pi}{2}]$ $\vec{v} = [0]$





with $A = [\frac{\pi}{2}, \pi]$

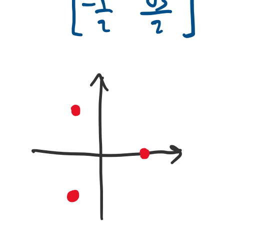


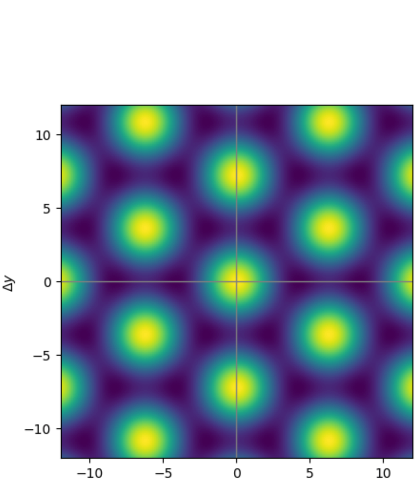


Each row of A corresponds to a FBF (or red dot).
When A has multiple rows, the interference pattern is the sum of those FBFs.

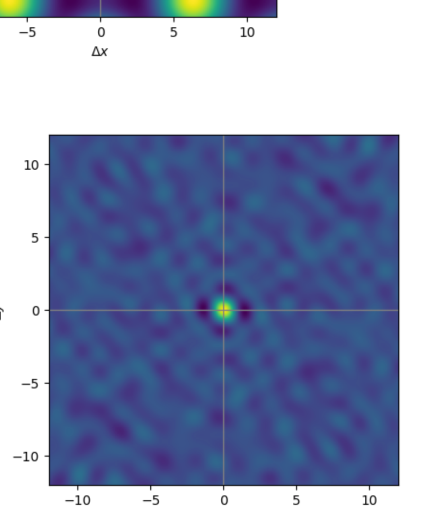
Example:

$$A = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$





Or with random A
$A \in \mathrm{Uniform}([-\pi, \pi]^2)^{200}$



Random SSPs

If we select phases from $\mathcal{U}[-\pi, \pi]^2$, can we find simplexes that create grid cells?
Yes!