Week 8 Time complexity of Algorithms

Example **Algorithms LARGEST 1**

1) Initially, place the # in register X, in a register called max.
2) For $i = 2, 3, \cdots, n$, do the following: compare the number in register X_i with the number in register max. If the # in X_i is larger than the # in max, move the number in register X_i to register max; otherwise, do nothing.
3) Finally, the # in register max is the largest of the n number in registers $X_1, X_2, X_3, \ldots, X_n$.

if i > max
swap;
i++;

at end.

how many comparisons? $\boxed{n-1}$

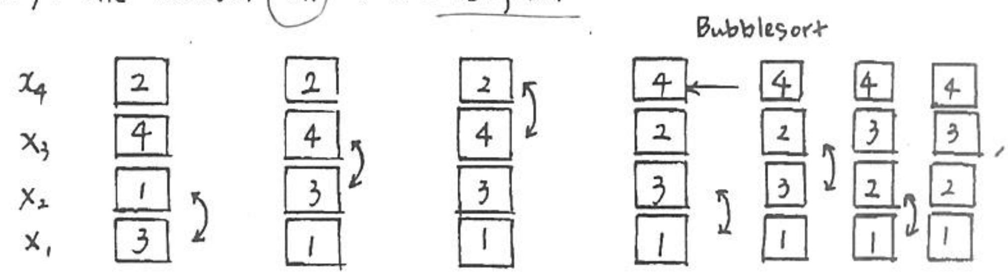
$\approx n$ (because $n \gg 1$)

Example **Algorithms LARGEST 2**

1) Do the following: for $i = 1, 2, \ldots n-1$, Compare the numbers X_i and X_{i+1}. place the larger one in X_{i+1}. and smaller one in X_i.
2) Finally, the number X_n is the largest.

$x_i > x_{i+1}$
swap;
i++;

i.e.



Example **Algorithms Bubblesort**

1) Do the following: for $i = n, n-1, \cdots, 3, 2$: Use Algorithm LARGEST 2 to place in register X_i the largest of $X_1, X_2, \cdots, X_i$
2) Finally, $X_1, X_2, \cdots X_n$ are in ascending order.

$$(n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n-1)}{2} \approx \frac{n^2}{2} \rightarrow n^2$$

Example **Shortest Paths Algorithm**

Let $G(V, E)$ be a weighted graph, where W is a function from E to the set of positive real #s.

How to find some shortest paths from vertex a to vertex z? L = length

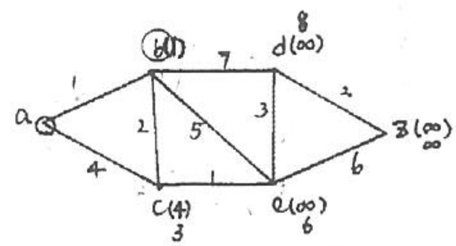
1) Initially, let $P = \{a\}$. $T = V - \{a\}$. For every vertex $t \in T$, let
$L(t) = w(a, t)$

2) Select the vertex in T that has the smallest index with respect to P.
Let x denote this vertex.

3) If $x = z$, stop; if not, $P' = P \cup \{x\}$. $T' = T - \{x\}$. For every vertex $t \in T'$.
compute its index respect to P' as

$$\boxed{L'(t) = \min\{L(t),\ L(t) + w(x, t)\}}$$

4) Repeat step 2 and 3 using P' as P and T' as T.



	b	c	d	e	z
Choose a	1.	4,	∞	∞	∞
Choose b		3.	8	6	∞
Choose c			8	4	∞
Choose e			7		10
Choose d					9

abc~~d~~edz

	a	b	c	d	e	z
a	0	1	4	∞	∞	∞
b	×	0	3	8	6	∞
c	×	×	0	8	4	∞
d	×	×	×	0	4	~~8~~ 9
e	×	×	×	7	0	~~10~~ 10
z						

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Week 11

$$ax + b = 0 \qquad \boxed{x = -\frac{b}{a}}$$

$$ax^2 + bx + c = 0$$

$$x^2 + 2 \cdot \frac{b}{2a} x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{(2a)^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \boxed{\frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

$$\underline{2}x^3 + 4x^2 + 3x + \underline{8} = 0$$

$8: \; 1, 2, 4, 8$

$2: \; 1, 2$

$$\frac{\pm 8}{1}, \frac{\pm 8}{2}, \frac{\pm 4}{1}, \frac{\pm 4}{2}, \frac{\pm 2}{1}, \frac{\pm 2}{2}, \frac{\pm 1}{1}, \frac{\pm 1}{2}$$

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

rational zeros $\qquad x = \frac{p}{q}$ ← factor of e, ← factor of a

$$1 + 2 + 3 + \dots + 10 = \sum_{k=1}^{10} k$$

$$1^2 + 2^2 + 3^2 + \dots + 10^2 = \sum_{k=1}^{10} k^2$$

<u>Recurrence Relations</u>

<u>Definition:</u> For a numeric function $(a_0, a_1, a_2, \dots, a_r, \dots)$ an <u>equation relating a_r for any r, to one or more of the a_i's $(i<r)$</u> is called <u>recurrence relation</u> or a <u>difference equation</u>. a_3 related to a_1, a_2, a_4 related to $a_1, a_2, a_3 \dots$

The bounding conditions are starting values from which we can carry out <u>a step-by-step computation</u> to determine the other a_i's.

Ex: $\begin{cases} a_r = 3a_{r-1} \\ a_0 = 1 \end{cases}$ bounding condition $\Rightarrow a_r = ?$ $\qquad a_r = 3^r$, [redacted] $r = 0, 1, 2, \dots$

Ex: $\begin{cases} a_r = a_{r-1} + a_{r-2} \\ a_0 = 1, \; a_1 = 1 \end{cases}$ $\quad a_r = ?$ $\qquad \boxed{a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{r+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{r+1}}$

$(1, 1, 2, 3, 5, 8, 13, 21, \dots)$

Linear Recurrence Relations with constant coefficients.

$c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = f(r)$

c_i's are constant. $c_0 \neq 0$, $c_k \neq 0$. k^{th} order

$a_r - 3a_{r-1} = 0$. $r = 1, 2, \cdots$ $\Rightarrow k = 1 \Rightarrow (a_0 = 1)$ 1 initial condition

$a_r - a_{r-1} - a_{r-2} = 0$ $\Rightarrow k = 2 \Rightarrow (a_0 = 1, a_1 = 1)$ 2 initial condition

Ex. $3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$

$a_3 = 3$. $a_4 = 6$

if a_3, a_4 are unique, do we know how to compute a_i ?

Solution: $a_5 = \frac{1}{3}(5 \cdot 6 - 2 \cdot 3 + 5^2 + 5) = 18$. the only value for a_5

$a_6 = \frac{119}{3}$. $a_2 = 9$ $(r=4)$. $a_1 = 25$. $a_0 = \frac{107}{2}$ $\Rightarrow$ there is only one value.

Ex: $3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$

$a_3 = 3$, $a_{20} = 5$

$\Rightarrow$ We don't know whether we have the unique solution.

Thm. The problem $\boxed{c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = f(r)}$

$a_{m-k} = b_k$, $a_{m-k-1} = b_{k-1}$. $a_{m-1} = b_1$

where c_i's and b_i's are all constants and $c_0 \neq 0$, $c_k \neq 0$

completely determines a numeric function a.

b_1 a_{m-1}

b_2 a_{m-2}

b_3 a_{m-3}

$\vdots$

b_{k-1} a_{m-k-1}

b_k a_{m-k}

Ex: $\begin{cases} a_r + a_{r-1} + a_{r-2} = 4 \\ a_r = 2 \end{cases}$

$\rightarrow$ 2 0 2 | 2 0 2 ⋯ possible solutions

$\rightarrow$ 2 2 0 | 2 2 0 ⋯

2 5 −3 | 2 5 −3 ⋯

2 −3 5

Ex. $\begin{cases} a_r + a_{r-1} + a_{r-2} = 4 \\ a_0 = 2, \quad a_1 = 2, \quad a_2 = 2 \end{cases}$

$a_r = ?$ no solution

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Homogeneous Solutions

Given $\boxed{c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = f(r)}$

Let $(a^{(k)} = a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \cdots, a_r^{(k)})$ denote the solution to

$$\boxed{c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = 0}$$

We call $a^{(k)}$ a homogeneous solution.

For a solution $a^{(P)} = a_0^{(P)}, a_1^{(P)}, \cdots, a_r^{(P)}$ that satisfies

$$\boxed{c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = f(r).}$$

We call $a^{(P)}$ a particular solution

NOTES:

$h+h \rightarrow h$ 1. The sum of two homogeneous solutions is a homogeneous solution.

$P+P \neq P$ 2. Normally, the sum of two particular solutions is not a particular solution.

However,

$h+P \rightarrow$ total sol^n 3. The sum of a homogeneous solution and a particular solution is a (total) solution to $c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = f(r)$

Ex: $2a^{(h)} = a^h + a^h$ is a homogeneous solution.

$m \cdot a^{(h)}$ — is a homogeneous solution

$m a^{(h)} + n a^{(h)}$ — is a homogeneous solution.

linear closeness of the operation.

Finding a homogeneous solution $a^{(h)}$

$$\boxed{c_0 x^k + c_1 x^{k-1} + \cdots + c_k = 0}$$

Case 1: the characteristic equation of k^{th} degree has k roots / zeros.

(Special case of case 2) $\alpha_1, \alpha_2, \cdots, \alpha_k$

the homogeneous solution $a_r^{(h)} = \sum_{i=1}^{k} A_i \alpha_i^r$

Case 2: α_1 has multiplexity m_1

α_2 has multiplexity m_2

α_n has multiplexity m_n

$\sqrt{}\ \boxed{m_1 + m_2 + \cdots + m_n = K} \Rightarrow$ # roots

$\alpha_1 : (A_1^{(1)} r^{m_1-1} + A_2^{(1)} r^{m_1-2} + \cdots + A_{m_1-1}^{(1)} r + A_{m_1}^{(1)}) \alpha_1^r$

$\alpha_2 : (A_1^{(2)} r^{m_2-1} + A_2^{(2)} r^{m_2-2} + \cdots + A_{m_2-1}^{(2)} r + A_{m_2}^{(2)}) \alpha_2^r \quad \cdots \quad \alpha_n$

$$\alpha_r^{(h)} = a_r^{(h)} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_j^{(i)} r^{m_i - j} \right) \alpha_i^r$$

1. form char. eq^n
2. find roots
3. form sol^n eq^n with const
4. find const. with ~~sol^n~~ initial cond^n
5. final a_r

Ex: Solve $\begin{cases} a_r = a_{r-1} + a_{r-2} \\ a_0 = 1, \quad a_1 = 1 \end{cases}$

Solution: $a_r - a_{r-1} - a_{r-2} = 0$

its characteristic equation is: $\alpha^2 - \alpha - 1 = 0$

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2}$$

two roots

2 constants $\Rightarrow$ $\therefore \ a_r^{(h)} = A\left(\frac{1+\sqrt{5}}{2}\right)^r + B\left(\frac{1-\sqrt{5}}{2}\right)^r$ where A. B are constant.

Since $f(r) = 0$. $\quad a_r = a_r^{(h)}$

$$a_r = A\left(\frac{1+\sqrt{5}}{2}\right)^r + B\left(\frac{1-\sqrt{5}}{2}\right)^r \quad r = 0, 1, 2, \ldots \quad A, B \text{ are Constant.}$$

$$\begin{cases} a_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = 1 \\ a_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 \end{cases} \Rightarrow \begin{cases} A + B = 1 \\ A - B = \frac{1}{\sqrt{5}} \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \cdot \frac{5+\sqrt{5}}{5} \\ B = \frac{1}{2} \cdot \frac{5-\sqrt{5}}{5} \end{cases}$$

$$\Rightarrow \underline{a_r = \frac{1}{2} \cdot \frac{5+\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^r + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^r} \quad r = 0, 1, 2 \cdots$$

Check the solution with initial condition.

Ex. Find the general form of the homogeneous solution to

$$a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$$

Solution: the characteristic equation

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0 \rightarrow (\alpha - 2)^3$$

$$-8 + 24 - 24 + 8 = 0$$

$8: 1, 2, 4, 8$

$1: 1$

$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm\frac{8}{1}$

We can check that $\alpha = -2$ is a zero

$\alpha = -2$

$\therefore \alpha + 2 = 0$

$$\begin{array}{r} \alpha^2 + 4\alpha + 4 \\ \alpha + 2 \overline{\big) \alpha^3 + 6\alpha^2 + 12\alpha + 8} \\ \underline{\alpha^3 + 2\alpha^2 \qquad\qquad} \\ 4\alpha^2 + 12\alpha \qquad \\ \underline{4\alpha^2 + 8\alpha \qquad} \\ 4\alpha + 8 \\ \underline{4\alpha + 8} \\ 0 \end{array}$$

$\alpha_1 = -2$ is a zero

$\alpha_2 = -2$

$\alpha_3 = -2$

$\therefore \alpha = -2$ is the zero of multiplexity 3.

$\therefore a_r^{(h)} = (Ar^2 + Br + C)(-2)^r. \quad r = 0, 1, 2, \ldots$ where A. B. C are constant.

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Week 12 $\quad a = (a_1, a_2, a_3, \cdots a_r, \cdots)$

$$\begin{cases} C_0 a_r + C_1 a_{r-1} + \dots + C_k a_{r-k} = f(r) \\ a_0 = b_0,\ a_1 = b_1,\ \cdots,\ a_{k-1} = b_{k-1} \end{cases}$$

$$\boxed{a_r = a_r^{(h)} + a_r^{(p)}}$$

$$C_0 \alpha^k + C_1 \alpha^{k-1} + \dots + C_k = 0$$

Case 1: k different zeros. $\alpha_1, \alpha_2, \cdots, \alpha_k$

$$\boxed{a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r + \cdots + A_k \alpha_k^r}$$

Case 2: $\alpha_1 \cdots m_1 \qquad \underline{m_1 + m_2 + \dots + m_n = k}$

$\alpha_2 \cdots m_2$

$\vdots$

$\alpha_n \cdots m_n \qquad \alpha_1 : \underline{\left(\text{i.e. } A_1 r^{m_1-1} + A_2 r^{m_1-2} + \dots + A_{m_1}\right) \alpha_1^r}$ — m_1-1 degree polynomial

zero degree polynomial — constant

one degree polynomial — linear

Example: $\quad a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$

$\rightarrow 1 \cdot \alpha^3 + 6\alpha^2 + 12\alpha + \underline{8} = 0$

$\alpha = \frac{p}{q}$ $\quad$ p must be a factor of 8; q must be a factor of 1.

8: 1, 2, 4, 8

α could be $\pm 1, \pm 2, \pm 4, \pm 8$;

α can not be positive $\rightarrow -1, -2, -4, -8$.

$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = (\alpha + 2)(\qquad) = 0$

$$\begin{array}{r|l} & \alpha^2 + 4\alpha + 4 \\ \hline \alpha + 2 & \alpha^3 + 6\alpha^2 + 12\alpha + 8 \\ & \alpha^3 + 2\alpha^2 \\ \hline & \quad 4\alpha^2 + 12\alpha + 8 \\ & \quad 4\alpha^2 + 8\alpha \\ \hline & \qquad 4\alpha + 8 \\ & \qquad 4\alpha + 8 \\ \hline & \qquad\quad 0 \end{array}$$

$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = (\alpha + 2)^3$

$\alpha = -2$, multiplicity $= 3$

$a_r^{(h)} = (\underbrace{Ar^2 + Br + C}_{\text{degree 2}})(-2)^r$, where A, B, C are unknown constants.

Example: Find the homogenous solution to

$$4a_r - 20a_{r-1} + 17a_{r-2} - 4a_{r-3} = 0$$

Solution: the char. polyn. is

$$4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$$

$4 : 1, 2, 4,$

$\pm\frac{1}{1} \pm \frac{2}{1} \pm \frac{4}{1}$

$\pm\frac{1}{2} \pm \frac{1}{4}$

$\alpha_i = \frac{p}{q}$ p — factor of -4; q — factor of 4

$\pm 1, \pm 2, \pm 4, \pm\frac{1}{2}, \pm\frac{1}{4}.$

α cannot be negative $\rightarrow 1, 2, 4, \frac{1}{2}, \frac{1}{4}.$

$$4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = (2\alpha - 1)(2\alpha^2 - 9\alpha + 4) = 0$$

$$\begin{array}{r} 2\alpha^2 - 9\alpha + 4 \\ 2\alpha - 1 \overline{\big) 4\alpha^3 - 20\alpha^2 + 17\alpha - 4} \\ \underline{4\alpha^3 - 2\alpha^2 \quad\quad\quad\quad} \\ -18\alpha^2 + 17\alpha - 4 \\ \underline{-18\alpha^2 + 9\alpha \quad\quad} \\ 8\alpha - 4 \\ \underline{8\alpha - 4} \\ 0 \end{array}$$

$\therefore \alpha_2 = 4$

$\alpha_3 = \frac{1}{2}$

$\alpha_1 = \frac{1}{2}$

$\alpha_2 = 4$

$\alpha_3 = \frac{1}{2}$

$$a_r^{(h)} = (Ar + B)\left(\tfrac{1}{2}\right)^r + C4^r$$

where A, B, C are unknown constants.

Case 1: $f(r) = F_1 r^t + F_2 r^{t-1} + \cdots F_t r + F_{t+1}$

Then $a_r^{(p)} = P_1 r^t + P_2 r^{t-1} + \cdots + P_t r + P_{t+1}$

Example: Find a particular solution to $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$.

Solution:

$$a_r^{(p)} = P_1 r^2 + P_2 r + P_3$$

where P_1, P_2, P_3 are constants

$a_r = a_r^{(p)} + a_r^{(h)}$

$$(P_1 r^2 + P_2 r + P_3) + 5(P_1(r-1)^2 + P_2(r-1) + P_3) + 6(P_1(r-2)^2 + P_2(r-2) + P_3) = 3r^2$$

$$12P_1 r^2 - (34P_1 - 12P_2)r + (29P_1 - 17P_2 + 12P_3) = 3r^2$$

$$\therefore \begin{cases} 12P_1 = 3 \\ 34P_1 - 12P_2 = 0 \\ 29P_1 - 17P_2 + 12P_3 = 0 \end{cases} \Rightarrow P_1 = \frac{1}{4},\ P_2 = \frac{17}{24},\ P_3 = \frac{115}{288}$$

$\therefore$ a particular solution is $a_r^{p} = \frac{1}{4}r^2 + \frac{17}{24}r + \frac{115}{288}$. $r = 0, 1, 2, \cdots$

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Week 13 Case 2: $f(r) = (F_1 r^t + F_2 r^{t-1} + \dots + F_t r + F_{t+1})\beta^r$

and β is a characteristic zero of multiplicity m, then

$$a_r^{(p)} = r^m (P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1})\beta^r$$

when $\beta = 1$. $a_r^{(p)} = r^m (P_1 r^t + P_2 r^{t-1} + \dots + P_t r + P_{t+1})$

Example: Find a particular solution to

$$a_r - 2a_{r-1} = 3 \cdot 2^r$$

Solution: Since $\beta = 2$ is a zero of the char. polynomial with multiplicity $= 1$

$\therefore$ a particular solution is $a_r^{(p)} = r \cdot A \cdot 2^r$ for some unknown constant A.

$\alpha - 2 = 0$

$\alpha = 2$

$\therefore \quad A \cdot r \cdot \cancel{2^r} - \cancel{2} A(r-1)\cancel{2^{r-1}} = 3 \cdot \cancel{2^r}$

$\Rightarrow \quad A = 3$

that's $a_r^{(p)} = 3r \cdot 2^r$ is a particular solution.

Example: For $a_r = a_{r-1} + 7$ find a particular solution.

Solution: $a_r - a_{r-1} = 7 \cdot 1^r$

Since $\beta = 1$ is a zero, therefore a particular solution is

$a_r^{(p)} = r \cdot A \cdot 1^r = Ar$ for some unknown constant A.

and $Ar - A(r-1) = 7$

$\therefore \quad A = 7$

that's $a_r^{(p)} = 7r$ is a particular solution.

Example: Solve $\begin{cases} a_r = a_{r-1} + 7 \\ a_0 = 2 \end{cases}$

Solution: $a_r = a_r^{(h)} + a_r^{(p)}$

as $\beta = 1$ is a zero. $a_r^{(p)} = Ar$ for some unknown constant A.

$\therefore Ar - A(r-1) = 7 \Rightarrow A = 7$

that's $a_r^{(p)} = 7r$

and since $\beta = 1$ is a zero. $a_r^{(h)} = B \cdot 1^r$ for some unknown constant B.

$\therefore \quad a_r = B + 7r$ is the general format of the solution

$\alpha - 1 = 7$

$\alpha = 8$

as $a_0 = 2$, $B + 7 \cdot 0 = 2 \Rightarrow B = 2$. ??

$a_r^{(h)} = (Ar)(8)^r$ $\therefore$ the solution is $a_r = 2 + 7r$

Generating Function

given $a = (a_0, a_1, a_2, \cdots, a_r, \cdots)$

we define $\boxed{A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots = \sum_{i=0}^{\infty} a_i z^i}$

as the generating function of a

From the equation. find $A(z)$, the generating function of a.

$A(z) \Rightarrow a_r$

Example. Let $a = (3^0, 3^1, 3^2, \cdots 3^r, \cdots)$. $a_r = 3^r$. $r = 0, 1, 2, \cdots$

$$A(z) = \sum_{i=0}^{\infty} 3^i \cdot z^i = \sum_{i=0}^{\infty} (3z)^i = \frac{1}{1-3z}$$

$$\boxed{\therefore\ A(z) = \frac{1}{1-3z}}$$

Some properties of generating function

(I) if $b = \alpha a$. where α is a constant, then

$\boxed{B(z) = \alpha A(z)}$

where $A(z)$ is the generating function of a, $B(z)$ is the generating function of b.

Example: Find the generating function of $a_r = 3^{r+2}$, $r = 0, 1, 2, \cdots$

Solution: $a_r = 3^{r+2} = 9 \cdot 3^r$

and $(3^0, 3^1, 3^2, \cdots 3^r, \cdots)$'s generating function is $\frac{1}{1-3z}$

$\therefore$ the generating function of $a_r = 3^{r+2}$, $r = 0, 1, 2, \cdots$ is

$$\boxed{A(z) = \frac{9}{1-3z}}$$

$$\begin{array}{r} 3z \\ 1-2z \overline{\smash{)}\,-6z^2+3z+2} \\ \underline{-6z^2+3z} \\ 2 \end{array}$$

(II) if $\boxed{c = a+b}$ then $\boxed{C(z) = A(z) + B(z)}$

if $\boxed{c = \alpha a + \beta b}$ then $\boxed{C(z) = \alpha A(z) + \beta B(z)}$

Example: Given $A(z) = \frac{2+3z-6z^2}{1-2z}$. what's a_r $(r \geq 0)$?

Solution:

Since $A(z) = \frac{2+3z-6z^2}{1-2z} = 3z + \frac{2}{1-2z}$

and $3z$ is the generating function of $(0, 3, 0, 0, \cdots)$

$\frac{2}{1-2z}$ is the generating function of $2 \cdot 2^r$. $r = 0, 1, 2, \cdots$ $(2, 4, 8 \cdots)$

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$$\therefore \quad a_r = \begin{cases} 2 & r=0 \\ 7 & r=1 \\ 2\cdot 2^r & r \geq 2 \end{cases} \qquad 2\cdot 2^r$$

$$A(z) = 1 \quad \Rightarrow \quad a = (1, 0, 0, 0, \cdots 0, \cdots)$$

$$A(z) = z \quad \Rightarrow \quad a = (0, 1, 0, 0, \cdots 0, \cdots)$$

$$A(z) = z^2 \quad \Rightarrow \quad a = (0, 0, 1, 0, \cdots 0, \cdots)$$

Example: Given $A(z) = \dfrac{3-8z}{1-5z+6z^2}$. compute a_r $(r \geq 0)$

Solution: $A(z) = \dfrac{3-8z}{1-5z+6z^2} = \dfrac{2}{1-2z} + \dfrac{1}{1-3z}$

$$\therefore \quad a_r = 2\cdot 2^r + 3^r \qquad r = 0, 1, 2, \cdots$$

$6z^2 - 5z + 1 = 0$

$3z(2z-1)$
$-1(2z-1)$

$\dfrac{3-8z}{(2z-1)(3z-1)}$

$\dfrac{A}{(2z-1)} + \dfrac{B}{(3z-1)} = \dfrac{3-8z}{(\ )(\ )}$

<u>(III)</u> if $\boxed{b_r = \alpha^r \cdot a_r}$ then $\boxed{B(z) = A(\alpha z)}$
where α is a constant.

Example: $a_r = 1, \quad r \geq 0$

$$A(z) = \sum_{i=1}^{\infty} z^i = \frac{1}{1-z}$$

$$b_r = \alpha^r \cdot 1, \quad r \geq 0$$

$$B(z) = A(\alpha z) = \frac{1}{1-\alpha z}$$

<u>(IV)</u> If $\boxed{c_r = a_r b_r, \qquad C(z) \neq A(z)\, B(z)}$

Example: Let $a_r = 2^r, \quad b_r = 3^r$

then $c_r = a_r \cdot b_r = 2^r \cdot 3^r = 6^r$

But $A(z) = \dfrac{1}{1-2z}$. $\quad B(z) = \dfrac{1}{1-3z}$

$C(z) = \dfrac{1}{1-6z} \neq A(z)\cdot B(z)$

<u>(V)</u> Let $c = a * b$. which is defined as

$$c_r = a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0 = \sum_{\substack{i,j=0,\cdots,r \\ i+j=r}} a_i b_j$$

then $C(z) = A(z)\cdot B(z)$

$\frac{1}{(1-z)^2}$

Example: Let $A(z)$ be the generating function of a_r $(r \geq 0)$. What's the generating function of

$$b_r = a_0 + a_1 + \cdots + a_r \quad (r \geq 0)?$$

Solution:

$$b_r = a_0 \cdot 1 + a_1 \cdot 1 + \cdots + a_r \cdot 1$$

$$B(z) = A(z) \cdot \frac{1}{1-z} = \frac{A(z)}{1-z}$$

Example: If $A(z) = \frac{1}{(1-z)^2}$ is the generating function of a_r $(r \geq 0)$ what's a_r $(r \geq 0)$?

Solution: $A(z) = \frac{1}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{1}{1-z}$

and $\frac{1}{1-z}$ is the generarionq function of $a_r = 1$, $r \geq 0$.

$(1-z)$ $(1-z)$ $1-z$

$\therefore A(z) = \frac{1}{(1-z)^2}$ is the generating function of

$$a_r = 1 + 1 + \cdots + 1 = r + 1 \quad , \quad r \geq 0$$

$\frac{1}{(1-z)^2(1-z)}$ $3^2 + \frac{1}{\ }$

Example: $A(z) = \frac{1}{(1-2z)^2}$

Solution: Let $B(z) = \frac{1}{(1-z)^2}$. then $A(z) = B(2z)$

$$\Rightarrow a_r = 2^r (r+1)$$

$2^2 - 4z + z^2$

Example: $A(z) = \frac{1}{(2-z)^2}$

Solution Let $B(z) = \frac{1}{(1-z)^2}$, then $A(z) = \frac{1}{4} \cdot B(\frac{1}{2}z)$

$$\Rightarrow a_r = \frac{1}{4} \cdot \left(\frac{1}{2}\right)^r (r+1)$$

$\longrightarrow$ Three types of $A(z)$:

$$\left.\begin{array}{ll} 1. & \text{polynomial of } z \qquad z, z^2, \cdots \\ 2. & \frac{A}{B - Cz} \\ 3. & \frac{A}{(B-Cz)^2} \end{array}\right\} \text{we know } a_r.$$

Example: Solve $\begin{cases} a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r \\ a_0 = 1, \quad a_1 = 1 \end{cases}$ $A \cdot r \cdot 2^r + (Dr + E)$

Solution: Method 1

The characteristic equation is $\alpha^2 - 5\alpha + 6 = 0$

$$\alpha_1 = 2, \quad \alpha_2 = 3$$

$$\therefore a_r^{(h)} = A \cdot 2^r + B \cdot 3^r \quad , \quad r \geq 0. \quad A, B \text{ are constants.}$$

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and $\alpha = 2$ is a zero

$\therefore \quad a_r^{(p)} = r\cdot C\cdot 2^r + (Dr+E)$, where C, D, E are unknown constants.

$$Cr\cdot 2^r - 5\cdot C(r-1)2^{r-1} + 6C(r-2)2^{r-2} = 2^r \quad \Rightarrow \quad C=-2$$

$$Dr + E - 5[D(r-1)+E] + 6[D(r-2)+E] = r \quad \Rightarrow \quad D=\frac{1}{2}\ ,\ E=\frac{7}{4}$$

$$\therefore \quad a_r^{(p)} = -2r\cdot 2^r + \frac{r}{2} + \frac{7}{4}$$

that's $a_r = A\cdot 2^r + B\cdot 3^r - 2r\cdot 2^r + \frac{r}{2} + \frac{7}{4}$

$$\begin{cases} a_0 = A + B - 0 + 0 + \frac{7}{4} = 1 \\ a_1 = A\cdot 2 + B\cdot 3 - 2\cdot 2 + \frac{1}{2} + \frac{7}{4} = 1 \end{cases} \Rightarrow \begin{cases} A = -5 \\ B = \frac{17}{4} \end{cases}$$

$\therefore$ the Solution is

$$a_r = -5\cdot 2^r + \frac{17}{4}\cdot 3^r - 2r\cdot 2^r + \frac{r}{2} + \frac{7}{4} \quad . \quad r \geq 0$$

Method 2. $\quad \sum_{r=2}^{\infty} (z^r\cdot(a_r - 5a_{r-1} + 6a_{r-2})) = \sum_{r=2}^{\infty} (2^r + r) z^r$

$$(A(z) - a_0 - a_1 z) - 5(A(z) - a_0)z + 6(A(z))\cdot z^2 = \frac{4z^2}{1-2z} + \frac{z - z(1-z)^2}{(1-z)^2}$$

$$\left[\begin{array}{l} \sum_{r=2}^{\infty} 2^r z^r = \sum_{r=0}^{\infty} 2^r z^r - 1 - 2z = \frac{1}{1-2z} - 1 - 2z \\ \sum_{r=2}^{\infty} r z^r = \sum_{r=2}^{\infty} (r+1) z^r - \sum_{r=2}^{\infty} z^r = \left(\frac{1}{(1-z)^2} - 1 - 2z\right) - \left(\frac{1}{1-z} - 1 - z\right) \end{array}\right]$$

$$A(z) = \frac{1 - 8z + 27z^2 - 35z^3 + 14z^4}{(1-z)^2(1-2z)^2(1-3z)}$$

$$= \frac{17/4}{1-3z} + \frac{1/2}{(1-z)^2} - \frac{2}{(1-2z)^2} + \frac{5/4}{1-z} - \frac{3}{1-2z}$$

$$\therefore \quad a_r = \frac{17}{4}3^r + \frac{1}{2}(r+1) - 2(r+1)2^r + \frac{5}{4}\cdot 1 - 3\cdot 2^r$$

$$= -5\cdot 2^r + \frac{17}{4}\cdot 3^r - 2r\cdot 2^r + \frac{r}{2} + \frac{7}{4}. \qquad r \geq 0$$

$\frac{-4 \pm \sqrt{16-16}}{2}$

$z^2 + 4z + 4 = 0$

$z^2 + 2z + 2z + 2 = 0$

(-2)

$z = -2$

$-2z = 0$

$(z+2)^2 = 0$

$\delta(n)$

$u(n)$

$$\sum_{r=2}^{\infty} (2^r + r)\cdot z^r$$

$$= \sum_{r=2}^{\infty} 2^r\cdot z^r = \frac{1}{1-2z} - 1 - 2z$$

lim

$$\sum_{r=2}^{\infty} (r+1) z^r - \sum_{r=2}^{\infty} z^r = \frac{1}{(1-z)^2} - 1 - 2z - \frac{1-1-z}{1-z}$$

$\frac{1}{z^3} =$

Week 14 Example: Solve $\begin{cases} a_r = 3a_{r-1} + 2b_{r-1} & r \geq 1 \\ b_r = a_{r-1} + b_{r-1} & r \geq 1 \\ a_0 = 1, \quad b_0 = 0 \end{cases}$

Solution: $\sum_{r=1}^{\infty} a_r z^r = \sum_{r=1}^{\infty} (3a_{r-1} z^r + 2b_{r-1} z^r)$

$$\Rightarrow A(z) - 1 = 3zA(z) + 2zB(z) \qquad a_0 = 1$$

$$\sum_{r=1}^{\infty} b_r z^r = \sum_{r=1}^{\infty} (a_{r-1} z^r + b_{r-1} z^r)$$

P25 (6.1) initial condition

$$\Rightarrow B(z) = zA(z) + zB(z) \qquad b_0 = 0$$

$$A(z) = \frac{1-z}{1-4z+z^2} \qquad B(z) = \frac{z}{1-4z+z^2}$$

$$A(z) = \frac{1-z}{1-4z+z^2} = \frac{(3-\sqrt{3})/6}{1-(2-\sqrt{3})z} + \frac{(3+\sqrt{3})/6}{1-(2+\sqrt{3})z}$$

$$B(z) = \frac{z}{1-4z+z^2} = \frac{\sqrt{3}/6}{1-(2+\sqrt{3})z} - \frac{\sqrt{3}/6}{1-(2-\sqrt{3})z}$$

$$a_r = \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^r + \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^r \qquad r \geq 0$$

$$b_r = \frac{\sqrt{3}}{6}(2+\sqrt{3})^r - \frac{\sqrt{3}}{6}(2-\sqrt{3})^r \qquad r \geq 0$$

$\sum_{r=0}^{\infty} a_r z^r$

$\sum_{r=1}^{\infty} a_{r-1} z^r$

$a_0 z^0$

$1, a_1, 2, \ldots$

$\sum_{r=0}^{\infty} a_{r-1} z^{r-1} = A(z)$

$z A(z)$

Example: solve $ra_r + ra_{r-1} - a_{r-1} = 2^r$, $r \geq 1$, given that $a_0 = 273$.

Solution: $ra_r + (r-1)a_{r-1} = 2^r$

Let $b_r = ra_r$

then $b_r + b_{r-1} = 2^r \qquad r \geq 1$

$$\sum_{r=1}^{\infty} b_r z^r + \sum_{r=1}^{\infty} b_{r-1} z^r = \sum_{r=1}^{\infty} 2^r z^r$$

$$\Rightarrow B(z) - b_0 + zB(z) = \frac{1}{1-2z} - 1 \qquad b_0 = 0$$

$$(1+z)B(z) = \frac{2z}{1-2z}$$

$$B(z) = \frac{2z}{(1-2z)(1+z)} = \frac{\frac{2}{3}}{1-2z} + \frac{-\frac{2}{3}}{1+z}$$

$$b_r = \frac{2}{3}2^r + \left(-\frac{2}{3}\right)(-1)^r$$

$$a_r = \frac{1}{r}\left[\frac{2}{3}2^r + \left(-\frac{2}{3}\right)(-1)^r\right] \qquad r \geq 1.$$

$$a_0 = 273$$

(20)

Week 15 Iterative Solutions for Non-linear Equations

Given $f(x)=0$, we try to look at some elementary methods / iterative methods for finding a solution within some designed error.

Example: $f(x) = x^3 - x - 1$, find the zeros.

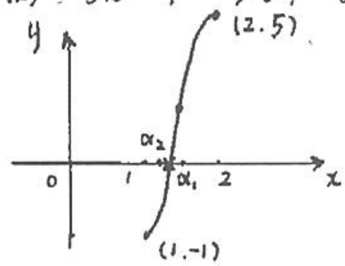
$f(1) = -1 < 0$

$f(2) = 5 > 0$

$f'(x) = 3x^2 - 1 > 0$, when $x \in [1, 2]$

$\Downarrow$

there must be a zero between 1 and 2



Bisection Method: $\alpha_1 = \frac{1}{2}(1+2) = 1.5$

$f(\alpha_1) = 0.875 > 0$ zero between 1 and 1.5

$\alpha_2 = \frac{1}{2}(1+1.5) = 1.25$.

$f(\alpha_2) = -0.276 < 0$ zero between 1.25 and 1.5

$\alpha_3 = \frac{1}{2}(1.5 + 1.25) = 1.375$. ...

$|\alpha_i - x_0| \le \varepsilon_0$, $|f(\alpha_i)| \le \varepsilon_0$?

Algorithm: (Bisection Method)

Given a function $f(x)$ continuous on the interval $[a_0, b_0]$; $f(a_0)\cdot f(b_0) < 0$.

and a designed error ε_0: $|m - x_0| \le \varepsilon_0$ $\quad |\alpha_i - x_0| \le (\frac{1}{2})^i$

For $n = 0, 1, 2, \dots$ until $|f(m)| \le \varepsilon_0$ do: $\quad i \to \infty$, ε_0

Set $m = \frac{1}{2}(a_n + b_n)$

if $f(a_n)\cdot f(\overset{m}{\underline{b_n}}) < 0$. set $a_{n+1} = a_n$. $b_{n+1} = m$;

otherwise. Set $a_{n+1} = m$, $b_{n+1} = b_n$.

Finally, m would be the solution with absolute error $\le \varepsilon_0$.

Algorithm: (False - Position) $\quad f(1) = -1$, $f(2) = 5 \Rightarrow$ zero is closer to 1

Given a function $f(x)$ continuous on the interval $[a_0, b_0]$. $f(a_0)\cdot f(b_0) < 0$. and a designed error ε_0:

$$\alpha = \frac{|f(2)|\cdot 1 + |f(1)|\cdot 2}{|f(2)| + |f(1)|} = 1.167$$

For $n = 0, 1, 2, \dots$ until the designed error ε_0. do:

take $w = \dfrac{|f(b_n)|\, a_n - |f(a_n)|\, b_n}{|f(b_n)| - |f(a_n)|}$

if $f(a_n) \cdot f(w) < 0$, set $a_{n+1} = a_n$. $b_{n+1} = w$;

otherwise. set $a_{n+1} = w$, $b_{n+1} = b_n$.

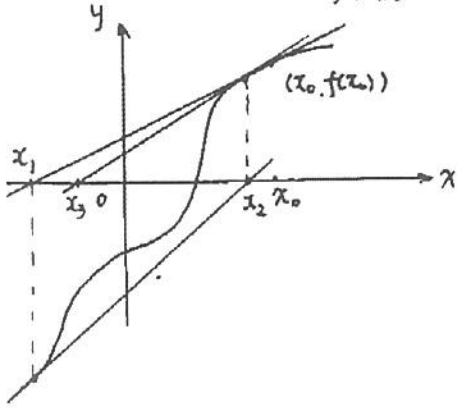
Finally, w will be the solution within absolute error ε_0.

Algorithm: (Newton's)

Given $f(x)$ continuous differentiable, and a point x_0, a designed error $\varepsilon_0 > 0$

For $n = 0, 1, 2, \dots$, until designed error, do:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



little by little, move closer to 'zero'

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{-1}{2}$$

Example: $f(x) = x^3 - x - 1$

$x_0 = 1$

$\vdots$

$x_4 = 1.324718$

$f(x_4) = 9.24 \times 10^{-7}$

$3x^2 - 1$

$\frac{1}{8} - \frac{1}{2} - 1$

$\frac{1-4-8}{8} =$

$3 \times \frac{1}{4} - 1$

$x_1 = \frac{1}{2}$

$x_2 = \frac{1}{2} - \frac{-11/8}{-1/4}$

$= \frac{1}{2} - \frac{11}{2}$

Fixed-Point iteration $\rightarrow$ Newton's Method

Let $g(x) = x - \dfrac{f(x)}{f'(x)}$

and $x_{n+1} = g(x_n)$

2

(1) if $x_0, x_1, x_2, \cdots, x_n, x_{n+1} \longrightarrow$ the zero of $f(x)$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(x_n)$$

$$\underset{\text{zero of } f(x)}{\underline{\alpha}} = g(\lim_{n\to\infty} x_n) = g(\alpha)$$

<u>Definition:</u> (iteration function)

$x = g(x)$

If any solution of $x = g(x)$, i.e. <u>any fixed point of $g(x)$</u>, is a solution of $f(x) = 0$, then we call $g(x)$ an <u>iteration function</u> of $f(x) = 0$

Example: Let $f(x) = x^2 - x - 2$, then all these functions are iteration function

(a) $g(x) = x^2 - 2$ fixed point: $x = g(x) = x^2 - 2$

(b) $g(x) = \sqrt{x+2}$

(c) $g(x) = 1 + \frac{2}{x}$

(d) $g(x) = x - \frac{x^2 - x - 2}{m}$, for any constant m.

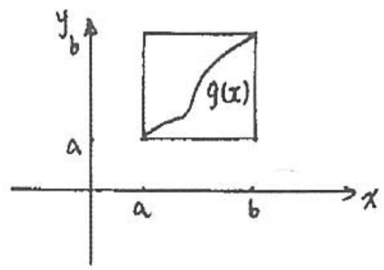
Algorithm: (Fixed-Point iteration)

Given an iteration function $g(x)$, and a starting point x_0, for $n = 0, 1, 2, \cdots$, <u>until satisfied</u>, do: $x_{n+1} = g(x_n)$

Example: if $g(x) = \sqrt{x}$, how can we iterate?

<u>Assumption 1:</u> There is an interval $I = [a, b]$, such that for all $x \in I$, $g(x)$ is defined and $g(x) \in I$.

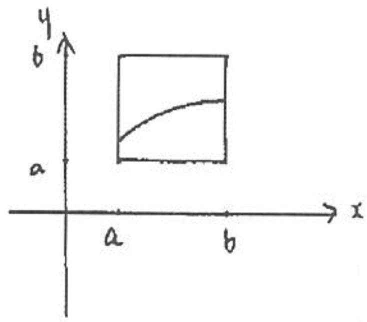
that's, the function $g(x)$ maps I into itself.



<u>Assumption 2:</u> The iteration function is differentiable on $I = [a, b]$, Further, there exists a non-negative constant $K < 1$, such that

$$|g'(x)| < K, \quad \text{for all } x \in I.$$

differential controllable (mild)



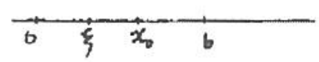
$g'(x)$ is small

<u>Theorem</u>: Let $g(x)$ be an iteration function satisfying Assumption 1 and 2. then $g(x)$ has exactly one fixed point ξ in I, and starting with any point in I, the sequence $x_1, x_2, \cdots x_n, \cdots$, generated by fixed-point iteration algorithm converges to ξ.

Further, let $e_n = \xi - x_n$, $n = 0, 1, 2, \cdots$, we can show that

$$|e_n| \le K^n |e_0| \le K^n (b-a). \quad 0 < K < 1$$

$\varepsilon_0 > 0$. $\quad |e_n| \le \varepsilon_0$



$$|e_n| \le K^n |e_0| \le K^n (b-a) \le \varepsilon_0$$

Example: Let $f(x) = x^2 - x - 2$.

choose $g(x) = \sqrt{2+x}$

$$g'(x) = \frac{1}{2 \cdot \sqrt{2+x}}$$

now $x > 0$ implies $g(x) > 0$

and $0 < g'(x) \le \underbrace{\frac{1}{\sqrt{8}}}_{K} < 1$

and if $x \le 7$. then $g(x) \le 3$

therefore. $\quad I = [0, 7]$

therefore. for any $x_0 \in [0, 7]$. sequence $x_1, x_2, \cdots$ converges to ξ.

Take $x_0 = 0$. then

$x_1 = \sqrt{2} = 1.41421$

$x_2 = \sqrt{3.41421} \doteq 1.84775$, $\qquad |e_2| \le \left(\frac{1}{\sqrt{8}}\right)^2 \cdot 2 = 0.25$