①

Mrunmai

ESE 554 Computational Models for Computer Engineers

Week 1 Sets

Defin: A set is a collection of objects ; objects must be well-defined.

$S = \{1, 2, 3, 4\}$

$a \in S$ $\qquad a \notin S$

Some examples:

$S = \{x \mid x$ is a positive real #.$\}$

$S = \{x \mid x = a+b$, where $a \in \{5, 10, 25\}$, $b \in \{100, 200, 500\}\}$

$N = \{0, 1, 2, 3, \dots\}$ natural numbers

$P = \{1, 2, 3, 4, 5 \dots\}$ positive integer

$Z = \{0, 1, -1, 2, -2 \dots\}$ integers

$R = \{x \mid x$ is a real number$\}$ real numbers

$Q = \{x \mid x = \frac{m}{n}$, where $m, n \in Z$, and $n \neq 0\}$ rational numbers

ϕ – empty set . $\phi = \{\ \}$

$P(A) = \{B \mid B \subseteq A\}$. — the power of set A

Defin: A set S is said to be a subset of set T if any element $x \in S$, $x \in T$.

$S \rightarrow T$ Subset if $x \in S$ && $x \in T$

$\phi \in P(A)$ $\qquad \phi \subseteq P(A)$

$A \in P(A)$ $\qquad A \notin P(A)$

Defin: $\boxed{S = T \Leftrightarrow S \subseteq T \text{ and } T \subseteq S}$

$P \subset N \subset Z \subset Q \subset R$ "$\subset$" real subset

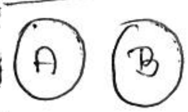
Set Operation:

A . B

Union >> $A \cup B = \{x \mid x \in A$ or $x \in B\}$ $\qquad A \cup B = B \cup A$

Intersection $A \cap B = \{x \mid x \in A$ and $x \in B\}$ $\qquad A \cap B = B \cap A$

$\boxed{\text{A and B are disjoint iff } A \cap B = \phi}$ $\qquad$ iff – if and only if

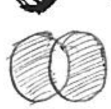


(Venn Diagram)

$A - B = \{x \mid x \in A$ and $x \notin B\}$ difference

$(A \setminus B)$

→ $A \oplus B = \{x \mid x \in A$ or $x \in B$ but not both$\}$ ⊛ Symmetric difference



$= \boxed{(A \cup B) - (A \cap B)} = \boxed{(A-B) \cup (B-A)}$

$\bar{A} = A^c = U - A$ where U is the universal set

- commutative laws

$A \cup B = B \cup A$. $A \cap B = B \cap A$

- associate laws

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

- distributive laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- identity laws

$A \cup \phi = A$

$A \cup U = U$

$A \cap \phi = \phi$

$A \cap U = A$

- De Morgan laws

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{\overline{A}} = A$

$\overline{A} \cup A = U$

$\overline{A} \cap A = \emptyset$

(*)

$$\boxed{\begin{aligned} (A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c \\ (A^c)^c &= A \\ A^c \cup A &= U \end{aligned}} \quad \boxed{A^c \cap A = \phi}$$

Counting and Principle of Inclusion and Exclusion

<u>Defin</u>: Give a finite set S. the cardinality of S is ?

$|S|$ = the # of elements of S ⟶ Size of S.

$S = \{1, 2, 3\}$, $|S| = 3$ # unique elements.

elements repeat ⟵ $S = \{1, 1, 2, 3\} = \{1, 2, 3\}$

<u>multi-set</u>

Theorem of inclusion exclusion

$|A \cup B| = \underbrace{|A| + |B|}_{\text{inclusion}} - \underbrace{|A \cap B|}_{\text{exclusion}}$ *

if $A \cap B = \phi$ (A) (B)

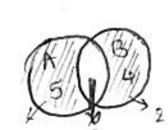
$\Rightarrow |A \cup B| = |A| + |B|$, by definition

Proof: $\because |A \cup B| = |A| + |B - A|$. as $A \cup B = A \cup (B - A)$. and $A \cap (B - A) = \phi$

$|B| = |B - A| + |A \cap B|$. as $B = (B - A) \cup (A \cap B)$. and $(B - A) \cap (A \cap B) = \phi$

$\therefore |B - A| = |B| - |A \cap B|$

$\boxed{\Rightarrow |A \cup B| = |A| + |B| - |A \cap B|}$



$\underset{7}{|A \cup B|} = \underbrace{\underset{5}{|A|} + \underset{4}{|B|}}_{9} - \underset{2}{|A \cap B|}$

②

Also.

$$|A^c \cap B^c| = |(A \cup B)^c|$$
$$= |U| - |A \cup B|$$

$$\boxed{|A^c \cap B^c| = |U| - |A| - |B| + |A \cap B|}$$

$$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|}$$

Example:

1-300
Not ÷ 3, 5, 7
÷ 3 Not ÷ 5, 7

Among the integers 1 - 300, how many of them are not divisible by 3, nor by 5, nor by 7? How many of them are divisible by 3, but not by 5 nor 7?

Solu: let A = the subset of all integers 1-300 that are divisible by 3.

$|A| = 100$

1-300

let B = " " by 5.

$|B| = 60$

let C = " " by 7.

$|C| = 42$

$U = \{1, 2, \cdots, 299, 300\}$. $|U| = 300$

$$|A^c \cap B^c \cap C^c| = |(A \cup B \cup C)^c| = |U| - |A \cup B \cup C|$$

Not ÷ 3, 5, 7

$$= 300 - (100 + 60 + 42 - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|)$$

? 15, 35, 21

$$= 300 - (202 - (20 + 14 + 8) + 2)$$
$$= 138$$

20

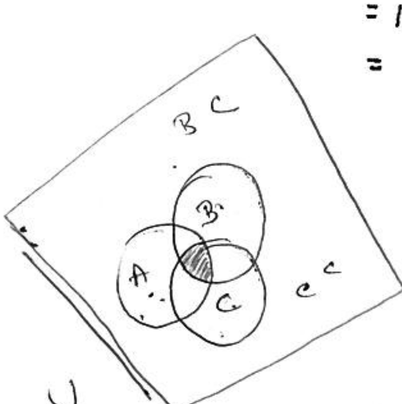
$$|A \cap B^c \cap C^c| = |(A \cap B^c) \cap (A \cap C^c)|$$ (take A as the universal set)

8

$$= |A| - (|B \text{ in } A \cup C \text{ in } A|) \quad ??$$

14

$$= |A| - (|(B \cap A) \cup (C \cap A)|)$$
$$= 100 - \underline{(20 + 14 - 2)}$$
$$= 68$$



$|A \cup B| = |A| + |B| - |A \cap B|$

~~$(A \cap B)^C$~~

A → U
$A \cap B^C$

Week 2

Example:

a) In how many ways can we make up the pattern

$$\begin{array}{ccccc} & & X_{E1} & & \\ X_6 & X_7 & X_{E2} & X_8 & X_9 \\ & & X_3 & & \\ & & X_4 & & \\ & & X_5 & & \end{array}$$

with 0s and 1s?

b) How many of these patterns are not symmetric with respect to the vertical axis?

a) $2 \times 2 \times \cdots \times 2 = 2^9$

b) $2^9 -$ the # of symmetric patterns

$= 2^9 - 2^7$

Combinatories — the math of choices

A die — 6 ways

Two dice — 21 ways $6 + 5 + 4 + 3 + 2 + 1 = 21$

Rule of Product and Rule of Sum

E1 — n outcomes

E2 — m outcomes

EK

Rule of Product — There are $n \times m$ possible outcomes when both of E1 and E2 take place

Rule of Sum — There are $n+m$ possible outcomes when exactly one of E1 and E2 takes place.

E1 or E2

Example: Let $S = \{\overset{E1}{1}, \overset{E2}{2}, 3, 4, \ldots \overset{E_{10}}{10}\}$ E1 - take 1 or not (2 outcomes)

How many subsets does p(s) have?   All the 10 events take place.

$\boxed{|p(s)| = 2^{10}}$ power set → Rule of Product

Permutation:

Defin: A permutation of n items taken r at a time is defined as select r items from among the n distinct items and arrange them in the order selected.

Example: $S = \{1, 2, 3\}$.

then 12, 21, 13, 31, 23, 32 are all the permutations taken two at a time.

Permutation Rules:

The # of permutations of n distinct items taken r at a time is $\boxed{0! = 1}$

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

★ $\boxed{P(n, r) = \frac{n!}{(n-r)!}}$

Example: For a 25.000 students university. what is the minimal length of passwords you need? Assume all single entry is from $\{0, 1, 2, \ldots, 9\}$ and there are no repeated numbers in the password!

Solution: it's a permutation problem. It's to find the smallest r such that

$P(10, r) \geq 25000$. $r = 5$

$$P(n,r) = \frac{n!}{(n-r)!}$$

Combinations:

Defin: A subset of r items from a set of n items is called a combination of n items taken r at a time. no order

Combination Rule:

the # of combinations of n distinct items taken r at a time is

$$\boxed{C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}}$$

$C(n,r)$ — binomial coefficients.

$$(x+y)^n$$
$$= C(n,0)\,x^n + C(n,1)\,x^{n-1}y + C(n,2)\,x^{n-2}y^2 + \dots + C(n,n)\,y^n$$

Example: Let's say ESE Dept has 20 professors and you are going to pick 3 of them to be your committee members. How many ways you can pick them?

Solution: $C(20,3) = 1140$

Repetitions:

Example: How many distinct arrangements of the letters of the word "bed"? "bee"?

Solution: "bed" — $P(3,3) = 3\times2\times1 = 6$

"bee" — $\frac{3!}{2!} = 3$ ÷ common!

Permutations & Repetition Thm

For a multi-set of n objects which has r_1 repeated objects of first kind, r_2 of the second kind, ..., r_k of k^{th} kind, $r_1 + r_2 + \dots + r_k = n$. then, the # of distinct permutations of n objects is

$$\boxed{\frac{n!}{[r_1!][r_2!]\cdots[r_k!]}}$$

→ multiplication of repeated objects of each kind.

Example: What's the # of different messages that can be represented by the sequence of three dashes ("-"), two dots (".") and a comma (",")?

Solution: $\frac{6!}{3!\,2!\,1!} = 60$

Partitions

Ordered Partition Theorem (I)

The # of ways to partition a set of n distinct objects into r distinct categories such that n_1 are in the first category, n_2 in the second category, ..., n_k in the k^{th} category, where $n_1+n_2+\dots+n_k=n$, is

Same as permutations with repeatations

$$\boxed{\frac{n!}{(n_1)!\,(n_2)!\cdots(n_k)!}}$$

Example: Let's say you have 9 projects to be assigned to John, Lisa and Mark. John will be assigned 4 projects, Lisa 3 and Mark 2. How many ways are there for the assignments?

Solution:
$$\frac{9!}{4!\,3!\,2!} = 1260$$

Ordered Partition Thm (II)

The # of ways to divide up n indistinguishable objects into r distinct categories is

$$C(n+r-1,\, n)$$

Example:

4 dice: 4 indistinguishable objects, 6 distinct categories
$C(4+6-1,\ 4)$

2 dice: $C(2+6-1,\ 2) = 21$

#4 Relations and Functions

Defin: Let S, T be sets. The Cartesian product of S and T (in this order) is defined as

$$S \times T = \{(a,b) \mid a \in S,\ b \in T\}$$

Defin: Let S, T be sets.

(a) a binary relation R from S to T is a subset of the Cartesian product $S \times T$

(b) a binary relation on a set S is a subset of $S \times S = S^2$

$$a R b \iff (a,b) \in R$$

$$a \not R b \iff (a,b) \notin R$$

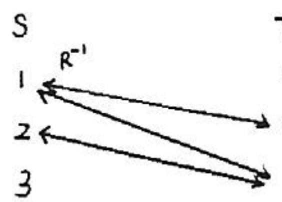
Example: The identity relation on S is $\boxed{R = \{(a,a) \mid a \in S\}}$

Example: Let $S = T = \{1,2,3\}$ $\quad R = \{(1,2), (1,3), (2,3)\}$

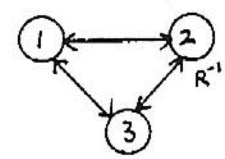
i.e. $1R2,\ 1R3,\ 2R3$

Representations.

identity relation on $S = \{(a,a) \mid a \in S\}$



arrow diagram



directed graph

$$M_R = \begin{array}{c|c} {}_{S\downarrow}\ ^{T\rightarrow} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$M_{R^{-1}} = M_R^T$

matrix (boolean matrix)

Defin: Let R be a relation from S to T. Then the inverse of R, written as R^{-1}, is the set of the ordered pairs

$$\boxed{R^{-1} = \{(b,a) \mid (a,b) \in R\}}$$

Defin: Let α be a relation from A to B, β be a relation from B to C. The composition of α with β, denoted by $\alpha \circ \beta$ or $\alpha\beta$, is defined as

$$\boxed{\alpha\beta = \{(a,c) \mid \text{there is a } b \in B, \text{ such that } (a,b) \in \alpha,\ (b,c) \in \beta\}}$$

Notes:

(1) $\alpha\beta \neq \beta\alpha$

(2) $M_{\alpha\beta} = M_\beta \cdot M_\alpha$

(3) $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$

(4) $M_{(\alpha\beta)^{-1}} = M_{\alpha^{-1}} \cdot M_{\beta^{-1}} = (M_\alpha)^T \cdot (M_\beta)^T$

Thm: Let α, β, γ be binary relations such that $(\alpha\beta)\gamma$ is well-defined. Then

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

and

$$M_\gamma \cdot (M_\beta \cdot M_\alpha) = (M_\gamma \cdot M_\beta) \cdot M_\alpha$$

Equivalence Relations and partitions

Defin: Let R denote a binary relation on a set A

MR.

(a) R is reflexive if $\forall\, a \in A \Rightarrow a R a$ diagonal = 1

* (b) R is symmetric if $\forall\, a \in A, b \in A, a R b \Rightarrow b R a$ $M_a = M_a^T$

(c) R is transitive if $\forall\, a, b, c \in A, a R b, b R c \Rightarrow a R c$.

* (d) R is anti-symmetric if $\forall\, a, b \in A, a R b$ and $b R a \Rightarrow a = b$

$R \leftrightarrow$ "|"

Example: $\mathbb{Z}$ (all the integers)

the relation $a|b$ iff b divides a.

"not symmetric", for example $2|4$ but $\cancel{4|2}$

"not anti-symmetric", for example $2|(-2)$, $(-2)|2$, but $2 \neq -2$.

Example: The relation "$\leq$" (is less than or equal to) on the set of real numbers $\mathbb{R}$

(1) reflexive yes

(2) symmetric no

(3) transitive yes

(4) anti-symmetric yes

R = mod 3

$a \bmod 3\ b \Leftrightarrow a-b=3n$

Example: Let $\mathbb{Z}$ be the set of integers. $R = \{(a,b) \mid a-b = 3n$ for some integers n and $a, b \in \mathbb{Z}\}$

(a) reflexive yes (b) symmetric yes

(c) transitive yes (d) anti-symmetric no

$\mathbb{Z} = [0] \cup [1] \cup [2]$

Def'n: If R is a binary relation on a set A, then R is an equivalence relation if R is reflexive, symmetric and transitive.

Note: Let R be an equivalence relation on A, then for $x \in A$,

$$[x] = \{ y \in A \mid xRy \} \subseteq A$$

Proof: $\forall\ y \in [x]$, $[y] = [x]$

Def'n: A partition of a set A is a set of non-empty sets $\{A_1, A_2 \cdots A_k\}$ such that $\boxed{A_i \subseteq A,\ i = 1, 2, \dots k, \text{ and } \bigcup_{i=1} A_i = A}$, $\boxed{A_i \cap A_j = \emptyset, \text{ if } i \neq j,}$ $i, j = 1, 2, \cdots, k$.

Thm: Let R be an equivalence relation on a set A, then $\boxed{A = \bigcup_{x \in A} [x] \text{ is a partition of set } A.}$

$$R$$

$$M_R = \begin{pmatrix} & & \\ & \cdot & \\ & & \end{pmatrix}_{n \times n}$$

(1) R is reflexive $\Leftrightarrow$ diagonal$(M_R) = 1$ $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

(2) R is symmetric $\Leftrightarrow$ $M_R^T = M_R$

(3) R is transitive $\Leftrightarrow$

Thm. Let R be a relation on a set of S. Then R is transitive iff $R^2 \subseteq R$ iff "$M_R * M_R \leq M_R$", where M_R is the matrix of R.

Let's define the boolean operation $\vee$ and $\wedge$ as

$m \vee n = \max\{m, n\}$, $m \wedge n = \min\{m, n\}$

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

For $M_1, M_2 \in A = \{M \mid M \text{ is a } n \times n \text{ matrix and } M[i,j] = 0 \text{ or } 1, \text{ for all } i, j = 1, 2, \dots n\}$ define "$\leq$" to be the relation $M_1 \leq M_2$, iff $M_1[i,j] \leq M_2[i,j]$, for $i, j = 1, 2, \dots, n$

$$M_1 * M_2[i,k] = \bigvee_{j=1}^{n} (M_1[i,j] \wedge M_2[j,k])$$

??

$R \downarrow M_R$

① diag$(M_R) = 1$

② $M_R^T = M_R$

③ $M_R * M_R \leq M_R$

⑥

<u>equivalence</u>
reflexive
transitive
summetric

<u>Partial order</u>
relflexive
transitive
Anti-symmetric

POSET

R
- reflexive $\Leftrightarrow$ $Diag(M_R) = 1$
- Symmetric $\Leftrightarrow$ $M_R^T = M_R$
- transitive $\Leftrightarrow$ $M_R * M_R \leq M_R$
- anti-symmetric $\Leftrightarrow$ $M_R \wedge M_R^T \leq I$

Partial Order Relations

An equivalence relation $\longrightarrow$ partition

Defin: A relation R on a set A is a partial order relation if R is

(1) reflexive

(2) transitive

(3) anti-symmetric

Read

(A, R) is a partially ordered set, or simply say poset

(A, R) is a poset.

- $R = $ "$\leq$". $(A, \leq)$ is a poset.
- $a \leq b$. "a preceeds b"

"a is less than or equal to b".

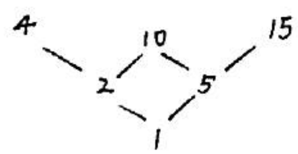
Example: $(\mathbb{Z}, \leq)$ is a poset. Yes

Example: $A = \{1, 2, 4, 5, 10, 15\}$. $\{1, 2, 4\} \subseteq A$

$\leq \; = a \mid b$

$(A, \leq)$ is a poset.

$4 \nleq 10$, $5 \leq 10$, $5 \leq 15$, $4 \nleq 15$.



$M_R * M_R \leq M_R$

Thm. Let R be a relation on a (finite) set S. M_R is its matrix. Then R is anti-symmetric iff $M_R \wedge M_R^T \leq I$, where I is the identity matrix. M_R^T is the transpose of M_R.

$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{bmatrix}$

For M_1, M_2, $\forall i, j$.

$$(M_1 \wedge M_2)[i,j] = M_1[i,j] \wedge M_2[i,j]$$

Example. $M_R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

(1) reflexive no. $diag(M_R) \neq 1$

(2) symmetric no. $M_R^T \neq M_R$

(3) transitive $M_R * M_R \leq M_R$? ↘

1 0 0 $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ [

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$ yes

$M_R * M_R$ 矩阵乘法. 出现大于等于1的数字 即改为1

Order

Defin: Let $(A, \leq)$ be a poset

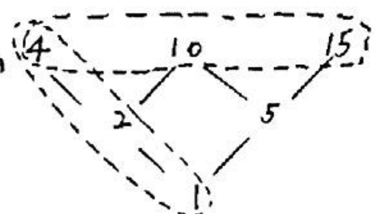
(a) an element $x \in A$ is maximal. if $x \leq y \Rightarrow y = x$ i.e. no element y is proceeded by x

(b) an element $x \in A$ is minimal if $y \leq x \Rightarrow y = x$ i.e. no element y proceeds x

Example: $A = \{1, 2, 3, 4, 5\}$. $R =$ " $\leq$ "

1 − the only minimal 5 − the only maximal

Defin: A subset of A is a <u>chain</u> if every two elements in the subset are related. The # of elements in a chain is called the length of the chain. Similarly, a subset of A is an <u>anti-chain</u> if no two distinct elements in the subset are related.

← chain  anti-chain $R = a | b$

P121. Textbook

$A = \{ M \mid M$ is a $n \times n$ matrix and $M[i,j] = 0$ or 1 for all $i,j \}$

$M_1 * M_2 [i,k] = \sum_{j=1}^{n} (M[i,j] \wedge M[j,k])$

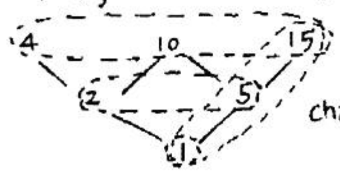
$M_1 \le M_2$ iff $M_1[i,j] \le M_2[i,j]$ for all i, j.
$\Downarrow$
if and only if

Week 5 totally ordered set
$A = \{1, 2, 3, 4, 5\}$. $R = \le$

Thm (Dilworth's)

If $(A, \le)$ is a finite poset and the size of the largest chains in A is n, then A can be partitioned into n disjoint anti-chains.

$A = \{1, 2, 4, 5, 10, 15\}$ $R = a \mid b$



$A = \{1\} \cup \{2, 5\} \cup \{4, 10, 15\}$

Corollary: Let $(A, \le)$ be a finite poset, consisting of $mn+1$ elements. Then either there is an anti-chain consisting of $m+1$ elements or there is a chain consisting of $n+1$ elements in A.

Example? a totally ordered set $(A, \le)$, only one chain.

Functions

Defin: A binary relation R from A to B is called a function if $\forall a \in A$, there is a unique element $b \in B$ such that $(a, b) \in R$.

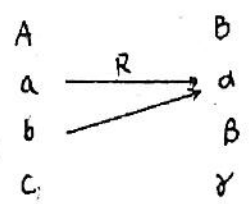
$f = A \to B$

— "image of a in A" $\in B$

— range of A = {all images} $\subseteq B$

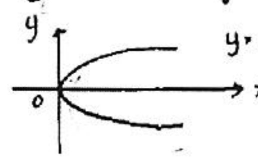
Example: $f(x) = x^2$
for all $x \in R$. $f: R \to R$

Example:



R is not an function
because c does not have an image.

Example:



$y = \pm\sqrt{x}$, $x \ge 0$.

not a function.

example: encryption and decryption

encrytion $F(x) = 3x + 5 \pmod{26}$

"Leavingtoday" ⇒ "110400 .... 24" ⇒ $F(11)F(04)F(00)\cdots$

⇒ 12170516 .... 25 ⇒ MRFQ ---- FZ

$F^{-1} = 9x + 7 \pmod{26}$ ⇒ 121705 ... 25 ⇒ $F^{-1}(12)F^{-1}(17)\cdots F^{-1}(25)$

⇒ 115/26

⇒ 11 04 ... ⇒ LEAVIGTODAY

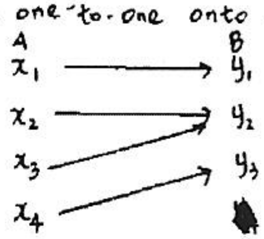
<u>Defin:</u> Let $f: A \to B$ is a function.

(1) "f maps A onto B" if the range of f is B.

(2) "f maps <u>one-to-one</u> into B" if there are 2 elements in A, the outcomes will be different

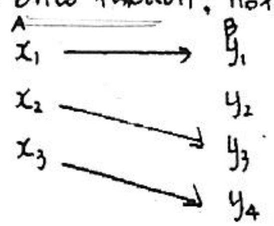
(3) "f is a one-to-one onto function" if a) one to one relation & b) full relation from A to B

Example.



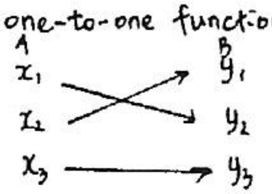
It's an onto fuction, not a one-to-one onto function.

Example



It's a one-to-one function, not an onto function.

Example.



It's a one-to-one onto function. ⇒ the size of A = the size of B

<u>Defin:</u> If $f: A \to B$ is a one-to-one onto function, then we can define the inverse of f as

$$f^{-1} = \{(b, a) \in B \times A \mid (a, b) \in A \times B\}$$

or

$$f^{-1} : B \to A$$

<u>Thm (Pigeonhole Principle):</u> Let $f: A \to B$ be a function. A, B are both finite sets.
If $|A| > |B|$, then there exists $a_1, a_2 \in A$.
Such that $f(a_1) = f(a_2)$

Example: Let A be a 10-element subset of $\{1, 2, \ldots, 50\}$. Show that A possesses two different 4-element subsets, the sum of whose elements are equal.

i.e. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\{4, 5, 6, 7\}$, $\{1, 2, 9, 10\}$, ...

Notes 1. $f: A \to B$. both A and B are finite $|B| = n$

if f is k to 1, the $|A| \le kn$

2. $\lceil \frac{m}{n} \rceil = \lfloor \frac{m-1}{n} \rfloor + 1$

$\lceil \frac{m}{n} \rceil < \frac{m}{n} + 1$

<u>Proof:</u> $f: S \to T$

$|S| > |T| \Rightarrow \exists\, a_1, a_2 \in S,\ a_1 \neq a_2$

$f(a_1) = f(a_2)$

S = all 4-element subsets of A

f = sum of the elements of the subset.

$T = \{\overset{1+2+3+4}{10}, 11, 12, \ldots \overset{47+48+49+50}{194}\}$

$|T| = 185$, $|S| = C(10,4) = 210$

$\therefore$ By Pigeonhole Principle, there are two different elements $S_1, S_2 \in S$.

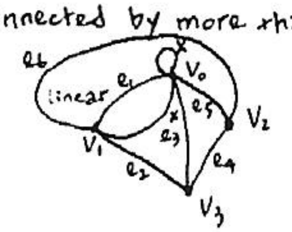
Such that $f(S_1) = f(S_2)$, which says their element sums are equal.

Graph Theory.

<u>Defin:</u> A graph $G(V,E)$ is defined to be a set V of vertices and another set E of edges such that each edge is incident with exactly two (possibly identical) vertices.

<u>Notes:</u> (a) if E is a binary relation on V. $G(V,E)$ is a directed graph; if E is defined by multisets of two elements from V, we call $G(V,E)$ an undirected graph.

(b) a graph is also called a multigraph; a linear graph is a multigraph such that every edge has two distinct endpoints and no vertices are connected by more than one edge.



$V_1 e_2 V_3 e_4 V_2$: from V_1 to V_2

$V_1 e_6 V_2$

$\vdots$

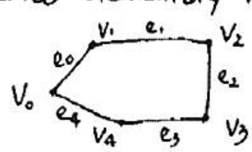
<u>Defin:</u> a path in a graph $G(V,E)$ is an alternating sequence of its vertices and edges of the form

$V_0 e_1 V_1 e_2 V_2 \cdots e_n V_n$ this path connects V_0 to V_n

<u>Defin:</u> a connected graph: every two vertices are connected.

an unconnected graph: We can find a pair of vertices that are not connected.

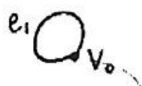
<u>Defin:</u> a path is called elementary if all of its vertices are distinct.



path: $\underline{V_0} e_0 V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 \underline{V_0}$ exception

Def'n: a path is called simple if all of its edges are distinct.

a path is a circuit if $V_0 = V_n$.



$V_0 e_1 V_0$: elementary, simple & a circuit

An Eulerian path is a path that transverses each edge once and only once.

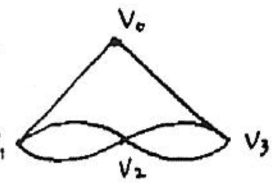
An Eulerian circuit is a circuit that transverses each edge once and only once.

$\deg(V_0) = 2$

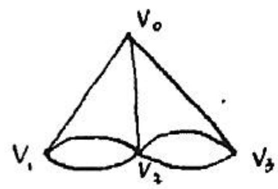
$\deg(V_1) = 3$

$\deg(V_2) = 4$

$\deg(V_3) = 3$



G_1 ✓



G_2 X

an Eulerian path?

$\deg(V_0) = 3$

$\deg(V_1) = 3$

$\deg(V_2) = 5$

$\deg(V_3) = 3$

Def'n: The degree of a vertex is the # of edges incident with it.

$\deg(V)$

Thm (Euler's). An undirected graph passes an ~~Euler~~ Eulerian path iff it is connected and has either zero or two vertices of odd degree.

Corollary: An undirected graph passes an Eulerian circuit iff it's connected and its vertices are all of even degree.

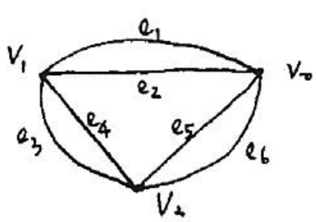
an algorithm: (Find an Eulerian circuit)

Step 1: Initial vertex: pick any vertex V_0 in G.

Step 2: Initial circuit: construct a circuit C

$V_0 e_1 V_1 \cdots V_0$ as: Find an e_1 adjacent to V_0 and to a second vertex V_1, find a new edge e_2 adjacent to V_1 and so on.

Step 3: Repeat Step 1 and 2 until C contains every edge



Step 1: V_1

Step 2: $V_1 e_4 V_2 e_3 V_1$

Step 3: $V_2 e_3 V_1 e_4 V_2 e_6 V_0 e_1 V_1 e_2 V_0 e_5 V_2$

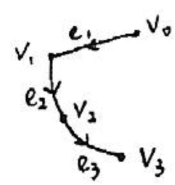
9

Week 6 directed graphs

Defn: In a directed graph, the incoming degree of a vertex is the # of edges that are incident into it, and the outgoing degree of a vertex is the # of edges that are incident from it.

Thm (Euler's): A directed graph has an Eulerian circuit iff it's connected and $deg_{in}(V) = deg_{out}(V)$ for all vertices.

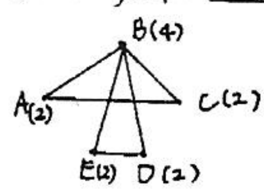
A directed graph has an Eulerian path iff it's connected and $deg_{in}(V) = deg_{out}(V)$ for all vertices except 2 vertices for which $deg_{in}(V) = deg_{out}(V) + 1$ for one vertex and $deg_{in}(V) = deg_{out}(V) - 1$ for the other vertex.



an Eulerian path / circuit $\longrightarrow$ a simple path.

Hamiltonian paths / circuit

Defn: A Hamilton path/circuit is a path/circuit that passes through each of the vertices of the graph exactly once.



$n = 5$

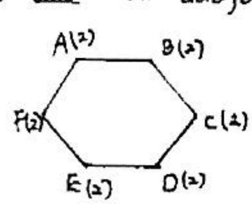
$n-1 = 4$

CABDE

Thm: Let G be a connected linear graph with ($n \geq 3$) vertices.

a) G has a Hamiltonian path if the sum of the degrees of each pair of the vertices is at least $n-1$.

b) G has a Hamilton circuit if the sum of the degrees of each pair of the [redacted] non-adajact vertices is at least n.



Thm (Tree ~~Classification~~ Characterization) Let G be a graph with n vertices and m edges. Then the following statements are equivalent.

1. G is a tree
2. G is connected and $m = n-1$
3. G does not have a cycle and $m = n-1$.

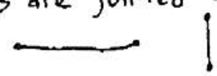
<u>Classification of graphs</u>

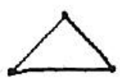
<u>Defin:</u> a <u>cycle</u> is a circuit in which all vertices and edges are distinct.

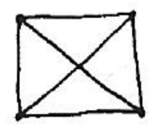
a <u>tree</u> is a connected (undirected) graph with no cycles.

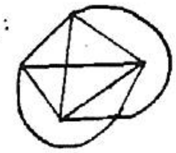
a collection of disjoint trees is called a <u>forest.</u>

<u>Defin:</u> a completed graph on n vertices, denoted by K_n, is such that each pair of distinct vertices are jointed by an edge.

K_2: 

K_3: 

K_4: 

K_5: 

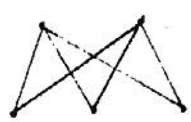
<u>Defin:</u> A bi-partite graph is one that the vertex set can be partitioned into 2 disjoint sets $A = \{A_1, A_2, \cdots, A_m\}$ and $B = \{B_1, B_2, \cdots B_n\}$ such that no vertices in A or B are adjacent.

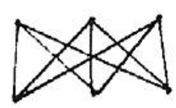
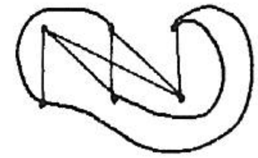
A complete bi-partite $K_{m,n}$ is a bi-partite graph in which each vertex in A is joined to every verrex in B.

$K_{m,n}$ $\begin{cases} A \\ B \end{cases}$ 

$K_{1,2}$: 

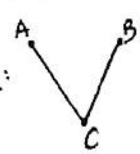
$K_{2,2}$: 

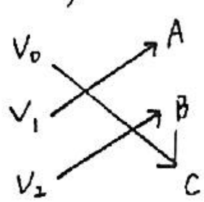
$K_{2,3}$: 

$K_{3,3}$:  

midterm < <u>Defin: (Isomorphism).</u> Given two graphs G1 and G2 with vertex sets $A = \{A_1, A_2, \cdots A_n\}$ and $B = \{B_1, B_2, \dots, B_n\}$ respectively. An isomorphism π between G1 and G2 is a one-to-one onto mapping. $\pi: A \to B$. such that if $\pi(A_i) = B_i$, $\pi(A_j) = B_j$. then A_i is adjacent to A_j <u>iff</u> B_i is adjacent to B_j.

G_1: 

G_2: 

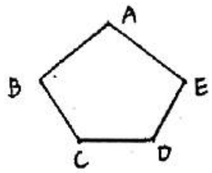
π: 

$G_1 \overset{\pi}{\underset{\pi}{\cong}} G_2$ $\quad$ G_1 is isomophic to G_2

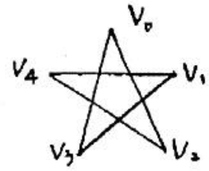
10

A rooted tree:
- an vertice $deg_{in} = 0$

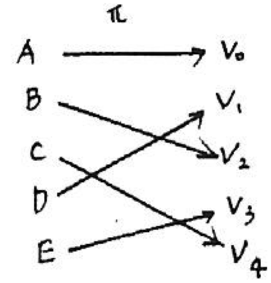
Planar - no intercections



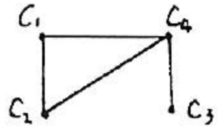
G_1



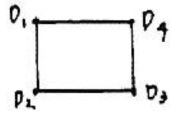
G_2



$G_1 \cong G_2$



G_1



G_2

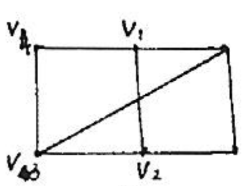
$G_1 \ncong G_2$

<u>Thm (Isomorphism Preservation)</u> : If 2 graphs G1 and G2 are isomorphis, then G1 and G2 have:

1) the same # of vertices and edges
2) the same degree of corresponding vertices.
3) the same # of connected components.

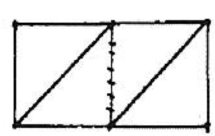
and

4) G1 has an Eulerian path (circuit) iff G2 has an Eulerian path (circuit).
5) G1 has a Hamiltonian path (circuit) iff G2 has the same.
6) if $\pi(A_1) = B_1$, $\pi(A_2) = B_2$. and there is a path (circuit) of length k from A_1 to A_2. then there is the same from B_1 to B_2.



G_1

$\ncong$

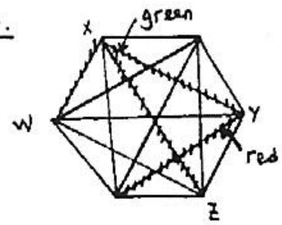


G_2

$V_1V_2V_3V_4V_1 \longrightarrow G_2$ doesn't have such a cycle.

Week 7 Example Show that among the group of six people. there are either three who are mutual friends or three who are strangers to each other.

Proof.



$K_6 \Rightarrow$ green K_3 (friends) or red K_3 (strangers)

Six Vertices, Pick x, then there will be five edges.

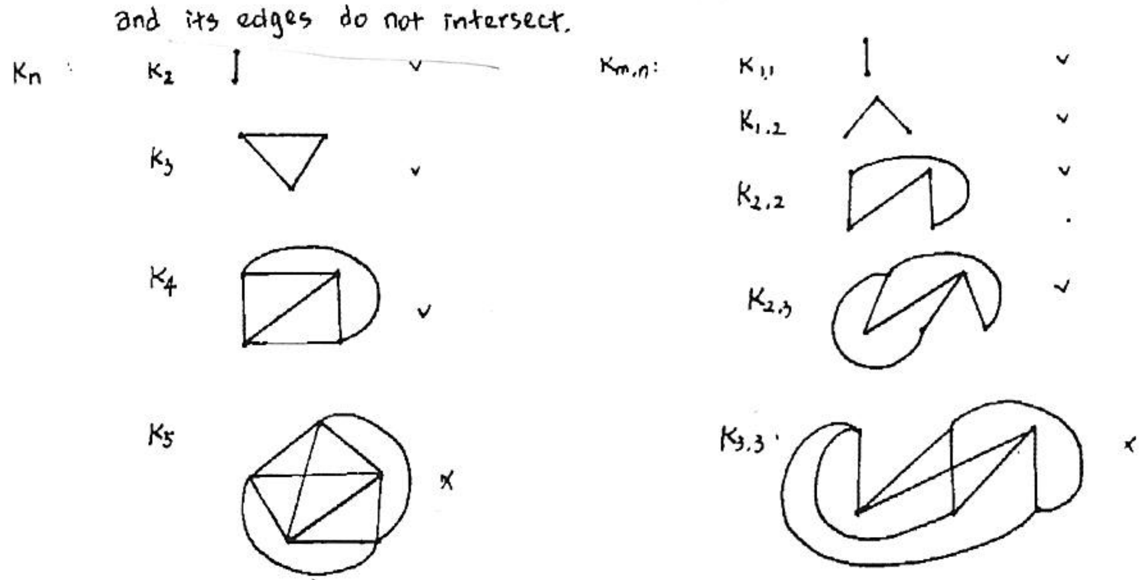
At least 3 of the 5 edges are of the same color. (i.e. green)

If Y, Z, W are all green $\Longrightarrow$ done.

If not, at least one of them is green. & Choose the other two green edges $\Rightarrow$ done.

Planar Graphs and Euler's Formular.

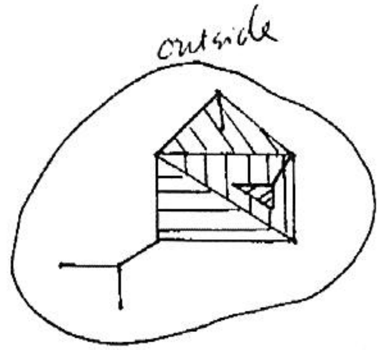
Defin: A multigraph is called "Planar" if it can be (re)drawn in a plane and its edges do not intersect.



K_5 is not planar, $K_{3,3}$ is not planer.

Defin: a region of a planar graph is an area of the plane that is bounded by edges and is not further divided into sub-areas.

(2)



$r = 5$ regions

$v = 12$

$e = 15$

<u>Thm (Euler's Formula)</u> : If a connected planar graph has v vertices, e edges and r regions, then

$$v - e + r = 2$$

$4 - 6 + 4 = 2$

Example: Show that $K_{3,3}$ is not planar.

<u>Proof</u>: For $K_{3,3}$, $v = 6$, $\underline{e = 9}$ (3×3)

<u>if $K_{3,3}$ is planar</u>, then according to Euler's formular,

$r = e - v + 2 = 9 - 6 + 2 = 5$

Since no 3 vertices of $K_{3,3}$ form a triangle, each region would have 4 or more edges at the boundary.

$\therefore$ the total # of edges making up the regions $\geq 4 \times 5 = 20$ where some edges may be counted twice because they may be boundaring 2 regions.

Hence, the total # of edges of $K_{3,3} \geq \frac{20}{2} = \underline{10}$, that's $e \geq 10$.

This contradicts $e = 9$.

$\therefore$ $K_{3,3}$ is not planar.

<u>Thm</u>: A connected planar graph with v vertices, e edges $(e \geq 2)$ and r regions <u>without loop or multiple edges</u> satisfies

$$\boxed{3v - 6 \geq e \geq \frac{3}{2} r}$$

Example: Show that K_5 is not planar.

<u>Proof</u>: For K_5, $v = 5$, $e = 10$ (C_5^2)

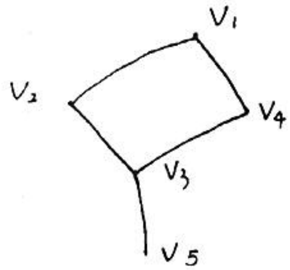
but $3v - 6 = 3 \times 5 - 6 = 9 \ngeq e = 10$

$\therefore$ K_5 is not planar.

Thm (Kuratowski) : a graph is planar iff it does not contain any Subgraph that is isomorphic, within vertices of degree 2. to $K_{3,3}$ or K_5.

Defin : Two graphs G1 and G2 are isomorphic to within vertices of degree 2. if they are isomorphic or if they can be transformed into isomorphic graphs by repeated insertions and/or removals of vertices of degree 2.

Connectedness



Matrix

$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \quad 5 \times 5$$

Thm: If A is the adjacent matrix of a graph. then the # of paths from vertex V_i to vertex V_j of length K is given by the ij-th entry of A^K.

Thm: If A is the adjacent matrix of a graph of n vertices, then

i) G is connected iff $\underbrace{I + A + A^2 + \dots + A^{n-1}}_{\text{should be symmetric}}$ has only true entries. = all entries are non-ze

ii) G is connected iff $(I + A)^{n-1}$ has only true entries.