

THE GAUSS–KRUEGER PROJECTION: Karney-Krueger equations

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ABSTRACT

The Gauss-Krueger projection has two forms. One has the Karney-Krueger equations capable of micrometre accuracy anywhere within 30° of a central meridian of longitude. The other has equations limited to millimetre accuracy within 6° of a central meridian. These latter equations are complicated but are widely used. The former equations are simple, easily adapted to computers, but not in wide use. This paper gives a complete development of the Karney-Krueger equations.

INTRODUCTION

The Gauss-Krueger projection is a conformal mapping of a reference ellipsoid of the earth onto a plane where the equator and central meridian λ_0 remain as straight lines and with constant scale factor on the central meridian. All other meridians λ and parallels ϕ are complex curves (Figure 5). The projection is one of a family of Transverse Mercator (TM) projections and the spherical form was originally developed by Johann Heinrich Lambert (1728-1777) and sometimes called the Gauss-Lambert projection acknowledging the contribution of Carl Friedrich Gauss (1777–1855) to the development of the TM projection. Snyder (1993) and Lee (1976) have excellent summaries of the history paraphrased below.

Gauss (c.1822) developed the ellipsoidal TM as an example of his investigations in conformal mapping using complex algebra and used it for the survey of Hannover in the 1820's. This projection had constant scale along the central meridian and was known as Gauss' Hannover projection. Also (c.1843) Gauss developed a 'double projection' combining a conformal mapping of the ellipsoid onto a sphere followed by a mapping from the sphere to the plane using the spherical TM formula. This projection was adapted by Oskar Schreiber and used for the Prussian Land Survey of 1876-1923. It is also called the Gauss-Schreiber projection and scale along the central meridian is not constant. Gauss left few details of his original developments and Schreiber (1866, 1897) published analyses of Gauss' methods, and Louis Krueger (1912) re-evaluated both Gauss' and Schreiber's work, hence the name Gauss-Krueger as a synonym for the TM projection.

We show a derivation of the Karney-Krueger equations for the TM projection that give micrometre accuracy anywhere within 30° of a central meridian and the appellation 'Karney-Krueger' distinguishes these equations from others and also acknowledges the work of one of the authors (Karney 2011) who provides a complete analysis of the accuracy of Krueger's series with the addition of iterative formula for the inverse transformation. At the heart of these equations are Krueger's two key series linking conformal latitude ϕ' and rectifying latitude μ and we note our extensive use of the computer algebra systems MAPLE and Maxima in showing these series to high orders of n ; unlike Krueger who only had patience. Without these computer tools the potential of his series could not be realized.

Krueger also gave other equations recognisable as Thomas's or Redfearn's equations (Thomas 1952, Redfearn 1948) that are in wide use. But they are complicated and unnecessarily inaccurate. We

outline the development of these equations but do not give them explicitly, as we do not wish to promote their use. We also show that using these equations can lead to large errors in some circumstances.

This paper supports the work of Engsager & Poder (2007) who use Krueger's series in their algorithms for a highly accurate TM projection but for reasons of space provided no derivation of the formulae.

SOME PRELIMINARIES

The Gauss-Krueger (or TM) projection is a mapping of a reference ellipsoid onto a plane and definition of the ellipsoid and associated constants are given. We define and give equations for isometric latitude ψ , meridian distance M , quadrant length Q , rectifying radius A , rectifying latitude μ and conformal latitude ϕ' . These basic 'elements' are required for our development of the two key series linking ϕ' and μ .

The ellipsoid

The ellipsoid is a surface of revolution created by rotating an ellipse (whose semi-axes lengths are a and b and $a > b$) about its minor axis and is the mathematical surface that idealizes the irregular shape of the earth. It has the following geometrical constants:

$$\text{flattening} \quad f = \frac{a-b}{a} \quad (1)$$

$$\text{eccentricity} \quad \varepsilon = \sqrt{\frac{a^2 - b^2}{a^2}} \quad (2)$$

$$\text{2nd eccentricity} \quad \varepsilon' = \sqrt{\frac{a^2 - b^2}{b^2}} \quad (3)$$

$$\text{3rd flattening} \quad n = \frac{a-b}{a+b} \quad (4)$$

$$\text{polar radius} \quad c = \frac{a^2}{b} \quad (5)$$

The constants are inter-related

$$\frac{b}{a} = 1 - f = \sqrt{1 - \varepsilon^2} = \frac{1}{\sqrt{1 + \varepsilon'^2}} = \frac{1-n}{1+n} = \frac{a}{c} \quad (6)$$

And since $0 < n < 1$ an absolutely convergent series for ε^2 can be obtained from (6)

$$\varepsilon^2 = \frac{4n}{(1+n)^2} = 4n - 8n^2 + 12n^3 - 16n^4 + 20n^5 - \dots \quad (7)$$

Radii of curvature ρ (meridian plane) and ν (prime vertical plane) at latitude ϕ are

$$\rho = \frac{a(1 - \varepsilon^2)}{(1 - \varepsilon^2 \sin^2 \phi)^{3/2}} = \frac{a(1 - \varepsilon^2)}{W^3} = \frac{c}{V^3} \quad \text{and} \quad \nu = \frac{a}{(1 - \varepsilon^2 \sin^2 \phi)^{1/2}} = \frac{a}{W} = \frac{c}{V} \quad (8)$$

where V and W are defined as

$$W^2 = 1 - \varepsilon^2 \sin^2 \phi \quad \text{and} \quad V^2 = 1 + \varepsilon'^2 \cos^2 \phi \quad (9)$$

Isometric latitude ψ

ψ is a variable angular measure along a meridian defined by considering the diagonal ds of the differential rectangle on the ellipsoid (Deakin & Hunter 2010b)

$$\begin{aligned} ds^2 &= (\rho d\phi)^2 + (\nu \cos \phi d\lambda)^2 \\ &= \nu^2 \cos^2 \phi \left\{ \left(\frac{\rho d\phi}{\nu \cos \phi} \right)^2 + (d\lambda)^2 \right\} \\ &= \nu^2 \cos^2 \phi \left\{ (d\psi)^2 + (d\lambda)^2 \right\} \end{aligned} \quad (10)$$

ψ is defined by the relationship

$$d\psi = \frac{\rho}{\nu \cos \phi} d\phi \quad (11)$$

Integration gives

$$\psi = \ln \left\{ \tan \left(\frac{1}{4} \pi + \frac{1}{2} \phi \right) \left(\frac{1 - \varepsilon \sin \phi}{1 + \varepsilon \sin \phi} \right)^{\frac{1}{2} \varepsilon} \right\} \quad (12)$$

Note: for a spherical surface of radius R ; $\rho = \nu = R$, $\varepsilon = 0$ and

$$\psi = \ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} \phi \right) \quad (13)$$

Meridian distance M

M is defined as the arc of the meridian ellipse from the equator to latitude ϕ

$$M = \int_0^\phi \rho d\phi = \int_0^\phi \frac{a(1 - \varepsilon^2)}{W^3} d\phi = \int_0^\phi \frac{c}{V^3} d\phi \quad (14)$$

This elliptic integral cannot be expressed in terms of elementary functions; instead, the integrand is expanded by using the binomial series and the integral evaluated by term-by-term integration. The usual series formula for M is a function of ϕ and powers of ε^2 ; but the German geodesist F.R. Helmert (1880) gave a series for M as a function of ϕ and powers of n requiring fewer terms for the same accuracy. Using Helmert's method (Deakin & Hunter 2010a) M can be written as

$$M = \frac{a}{1+n} \left\{ c_0 \phi + c_2 \sin 2\phi + c_4 \sin 4\phi + c_6 \sin 6\phi + c_8 \sin 8\phi + c_{10} \sin 10\phi + c_{12} \sin 12\phi \right. \\ \left. + c_{14} \sin 14\phi + c_{16} \sin 16\phi + \dots \right\} \quad (15)$$

where the coefficients $\{c_n\}$ are to order n^8 as follows

$$\begin{aligned}
c_0 &= 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \frac{1}{256}n^6 + \frac{25}{16384}n^8 \dots, & c_2 &= -\frac{3}{2}n + \frac{3}{16}n^3 + \frac{3}{128}n^5 + \frac{15}{2048}n^7 + \dots, \\
c_4 &= \frac{15}{16}n^2 - \frac{15}{64}n^4 - \frac{75}{2048}n^6 - \frac{105}{8192}n^8 - \dots, & c_6 &= -\frac{35}{48}n^3 + \frac{175}{768}n^5 + \frac{245}{6144}n^7 + \dots, \\
c_8 &= \frac{315}{512}n^4 - \frac{441}{2048}n^6 - \frac{1323}{32768}n^8 - \dots, & c_{10} &= -\frac{693}{1280}n^5 + \frac{2079}{10240}n^7 + \dots, \\
c_{12} &= \frac{1001}{2048}n^6 - \frac{1573}{8192}n^8 - \dots, & c_{14} &= -\frac{6435}{14336}n^7 + \dots, \\
c_{16} &= \frac{109395}{262144}n^8 - \dots
\end{aligned} \tag{16}$$

[This is Krueger's equation for X shown in §5, p.12, extended to order n^8]

Quadrant length Q

Q is the length of the meridian arc from the equator to the pole and is obtained from (15) by setting $\phi = \frac{1}{2}\pi$, noting that $\sin 2\phi, \sin 4\phi, \dots$ all equal zero, giving

$$Q = \frac{a\pi}{2(1+n)} c_0 \tag{17}$$

[This is Krueger's equation for \mathfrak{M} shown in §5, p.12.]

Rectifying radius A

Dividing Q by $\frac{1}{2}\pi$ gives the rectifying radius A of a circle having the same circumference as the meridian ellipse, and to order n^8

$$A = \frac{a}{1+n} \left\{ 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \frac{1}{256}n^6 + \frac{25}{16384}n^8 + \dots \right\} \tag{18}$$

Rectifying latitude μ

μ is defined in the following way (Adams 1921):

“If a sphere is determined such that the length of a great circle upon it is equal in length to a meridian upon the earth, we may calculate the latitudes upon this sphere such that the arcs of the meridian upon it are equal to the corresponding arcs of the meridian upon the earth.”

If μ denotes this latitude on the sphere of radius R then M is given by

$$M = R\mu \tag{19}$$

and since $\mu = \frac{1}{2}\pi$ when $M = Q$ then $R = A$ and μ is defined as

$$\mu = \frac{M}{A} \tag{20}$$

An expression for μ as a function of ϕ is obtained by dividing (18) into (15) giving to order n^4

$$\mu = \phi + d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + d_8 \sin 8\phi + \dots \tag{21}$$

where the coefficients $\{d_n\}$ are

$$\begin{aligned}
d_2 &= -\frac{3}{2}n + \frac{9}{16}n^3 - \dots, & d_4 &= \frac{15}{16}n^2 - \frac{15}{32}n^4 + \dots, \\
d_6 &= -\frac{35}{48}n^3 + \dots, & d_8 &= \frac{315}{512}n^4 - \dots
\end{aligned} \tag{22}$$

[This is Krueger's eq. (6), §5, p.12.]

An expression for ϕ as a function of μ is obtained by reversion of a series using Lagrange's theorem (Bromwich 1991), and to order n^4

$$\phi = \mu + D_2 \sin 2\mu + D_4 \sin 4\mu + D_6 \sin 6\mu + D_8 \sin 8\mu + \dots \tag{23}$$

where the coefficients $\{D_n\}$ are

$$\begin{aligned}
D_2 &= \frac{3}{2}n - \frac{27}{32}n^3 + \dots, & D_4 &= \frac{21}{16}n^2 - \frac{55}{32}n^4 + \dots, \\
D_6 &= \frac{151}{48}n^3 - \dots, & D_8 &= \frac{1097}{512}n^4 - \dots
\end{aligned} \tag{24}$$

[This is Krueger's eq. (7), §5, p.13.]

Conformal latitude ϕ'

Suppose we have a conformal mapping of the ellipsoid to a sphere having curvilinear coordinates ϕ', λ . Adams (1921) shows that the conformal latitude ϕ' is defined by the function

$$\tan\left(\frac{1}{4}\pi + \frac{1}{2}\phi'\right) = \tan\left(\frac{1}{4}\pi + \frac{1}{2}\phi\right) \left(\frac{1 - \varepsilon \sin \phi}{1 + \varepsilon \sin \phi}\right)^{\frac{1}{2}\varepsilon} \tag{25}$$

Series involving conformal latitude and rectifying latitude

Two key series are developed in this section; (i) a series for conformal latitude ϕ' as a function of the rectifying latitude μ , and (ii) a series for μ as a function of ϕ' . The method of development is not the same as employed by Krueger, but does give insight into his labour as he had only pencil, paper and perseverance. We have the benefit of computer algebra systems.

A series for ϕ' as a function of latitude ϕ can be developed using a method given by Yang *et al.* (2000) where ϕ' can be solved from equation (25) and expressed as

$$\phi' = 2F(\phi, \varepsilon) - \frac{1}{2}\pi \tag{26}$$

with

$$F(\phi, \varepsilon) = \tan^{-1} \left\{ \tan\left(\frac{1}{4}\pi + \frac{1}{2}\phi\right) \left(\frac{1 - \varepsilon \sin \phi}{1 + \varepsilon \sin \phi}\right)^{\frac{1}{2}\varepsilon} \right\}$$

Now since $0 < \varepsilon < 1$, $F(\phi, \varepsilon)$ can be expanded into a power series of ε about $\varepsilon = 0$

$$F(\phi, \varepsilon) = F(\phi, \varepsilon)|_{\varepsilon=0} + \varepsilon \left[\frac{\partial}{\partial \varepsilon} F(\phi, \varepsilon) \right]_{\varepsilon=0} + \frac{\varepsilon^2}{2!} \left[\frac{\partial^2}{\partial \varepsilon^2} F(\phi, \varepsilon) \right]_{\varepsilon=0} + \frac{\varepsilon^3}{3!} \left[\frac{\partial^3}{\partial \varepsilon^3} F(\phi, \varepsilon) \right]_{\varepsilon=0} + \dots \tag{27}$$

All odd-order partial derivatives evaluated at $\varepsilon = 0$ are zero and the even-order partial derivatives evaluated at $\varepsilon = 0$ are

$$\begin{aligned}
\left[\frac{\partial^2}{\partial \varepsilon^2} F(\phi, \varepsilon) \right]_{\varepsilon=0} &= -\frac{1}{2} \sin 2\phi \\
\left[\frac{\partial^4}{\partial \varepsilon^4} F(\phi, \varepsilon) \right]_{\varepsilon=0} &= -\frac{5}{2} \sin 2\phi + \frac{5}{4} \sin 4\phi \\
\left[\frac{\partial^6}{\partial \varepsilon^6} F(\phi, \varepsilon) \right]_{\varepsilon=0} &= -\frac{135}{4} \sin 2\phi + \frac{63}{2} \sin 4\phi - \frac{39}{4} \sin 6\phi \\
\left[\frac{\partial^8}{\partial \varepsilon^8} F(\phi, \varepsilon) \right]_{\varepsilon=0} &= -\frac{1967}{2} \sin 2\phi + \frac{4879}{4} \sin 4\phi - \frac{1383}{2} \sin 6\phi + \frac{1237}{8} \sin 8\phi \\
&\dots
\end{aligned}$$

[The last of these derivatives is incorrectly shown in Yang *et al.* (2000, p.80).]

Substituting these, with $F(\phi, \varepsilon)|_{\varepsilon=0} = \frac{1}{4}\pi + \frac{1}{2}\phi$ into equation (27), and re-arranging into (26) gives

$$\begin{aligned}
\phi' = \phi + &\left(-\frac{1}{2}\varepsilon^2 - \frac{5}{24}\varepsilon^4 - \frac{3}{32}\varepsilon^6 - \frac{281}{5760}\varepsilon^8 - \dots \right) \sin 2\phi + \left(\frac{5}{48}\varepsilon^4 + \frac{7}{80}\varepsilon^6 + \frac{697}{11520}\varepsilon^8 + \dots \right) \sin 4\phi \\
&+ \left(-\frac{13}{480}\varepsilon^6 - \frac{461}{13440}\varepsilon^8 - \dots \right) \sin 6\phi + \left(\frac{1237}{161280}\varepsilon^8 + \dots \right) \sin 8\phi + \dots
\end{aligned} \quad (28)$$

[Note that (28) is incorrectly shown in Yang *et al.* (2000, eq. (3.5.8), p.80) due to the error noted previously.]

Extending the process to higher even-powers of ε and higher even-multiples of ϕ ; and then using the series (7) to replace $\varepsilon^2, \varepsilon^4, \varepsilon^6, \dots$ with powers of n gives a series for ϕ' as a function of ϕ to order n^4 as

$$\phi' = \phi + g_2 \sin 2\phi + g_4 \sin 4\phi + g_6 \sin 6\phi + g_8 \sin 8\phi + \dots \quad (29)$$

where the coefficients $\{g_n\}$ are

$$\begin{aligned}
g_2 &= -2n + \frac{2}{3}n^2 + \frac{4}{3}n^3 - \frac{82}{45}n^4 + \dots, & g_4 &= \frac{5}{3}n^2 - \frac{16}{15}n^3 - \frac{13}{9}n^4 + \dots, \\
g_6 &= -\frac{26}{15}n^3 + \frac{34}{21}n^4 + \dots, & g_8 &= \frac{1237}{630}n^4 - \dots
\end{aligned} \quad (30)$$

[This is Krueger's eq. (8), §5, p.14.]

A series for ϕ' as a function of μ can be obtained using Taylor's theorem (Krueger 1912, p.14) where

$$\sin 2\phi = \sin 2\mu + (2\phi - 2\mu) \cos 2\mu - \frac{(2\phi - 2\mu)^2}{2!} \sin 2\mu - \frac{(2\phi - 2\mu)^3}{3!} \cos 2\mu + \frac{(2\phi - 2\mu)^4}{4!} \sin 2\mu + \dots$$

Replacing 2ϕ and 2μ with higher even multiples of ϕ and μ , and with (23), expressions for $\sin 2\phi, \sin 4\phi, \sin 6\phi$, etc. as functions of n and $\sin 2\mu, \sin 4\mu, \sin 6\mu$, etc. can be developed.

Substituting these expressions into (29) and simplifying gives a series for ϕ' to order n^8

$$\begin{aligned}
\phi' = \mu + \beta_2 \sin 2\mu + \beta_4 \sin 4\mu + \beta_6 \sin 6\mu + \beta_8 \sin 8\mu + \beta_{10} \sin 10\mu + \beta_{12} \sin 12\mu \\
+ \beta_{14} \sin 14\mu + \beta_{16} \sin 16\mu + \dots
\end{aligned} \quad (31)$$

where the coefficients $\{\beta_n\}$ are

$$\begin{aligned}
\beta_2 &= -\frac{1}{2}n + \frac{2}{3}n^2 - \frac{37}{96}n^3 + \frac{1}{360}n^4 + \frac{81}{512}n^5 - \frac{96199}{604800}n^6 + \frac{5406467}{38707200}n^7 - \frac{7944359}{67737600}n^8 + \dots \\
\beta_4 &= -\frac{1}{48}n^2 - \frac{1}{15}n^3 + \frac{437}{1440}n^4 - \frac{46}{105}n^5 + \frac{1118711}{3870720}n^6 - \frac{51841}{1209600}n^7 - \frac{24749483}{348364800}n^8 + \dots \\
\beta_6 &= -\frac{17}{480}n^3 + \frac{37}{840}n^4 + \frac{209}{4480}n^5 - \frac{5569}{90720}n^6 - \frac{9261899}{58060800}n^7 + \frac{6457463}{17740800}n^8 - \dots \\
\beta_8 &= -\frac{4397}{161280}n^4 + \frac{11}{504}n^5 + \frac{830251}{7257600}n^6 - \frac{466511}{2494800}n^7 - \frac{324154477}{7664025600}n^8 + \dots \\
\beta_{10} &= -\frac{4583}{161280}n^5 + \frac{108847}{3991680}n^6 + \frac{8005831}{63866880}n^7 - \frac{22894433}{124540416}n^8 - \dots \\
\beta_{12} &= -\frac{20648693}{638668800}n^6 + \frac{16363163}{518918400}n^7 + \frac{2204645983}{12915302400}n^8 - \dots \\
\beta_{14} &= -\frac{219941297}{5535129600}n^7 + \frac{497323811}{12454041600}n^8 + \dots \\
\beta_{16} &= -\frac{191773887257}{3719607091200}n^8 + \dots
\end{aligned} \tag{32}$$

[This is Krueger's eq. (10), §5, p.14 extended to order n^8]

μ is obtained by reversion of the series (31) using Lagrange's theorem (Bromwich 1991).

$$\begin{aligned}
\mu &= \phi' + \alpha_2 \sin 2\phi' + \alpha_4 \sin 4\phi' + \alpha_6 \sin 6\phi' + \alpha_8 \sin 8\phi' + \alpha_{10} \sin 10\phi' + \alpha_{12} \sin 12\phi' \\
&\quad + \alpha_{14} \sin 14\phi' + \alpha_{16} \sin 16\phi' + \dots
\end{aligned} \tag{33}$$

where the coefficients $\{\alpha_n\}$ are

$$\begin{aligned}
\alpha_2 &= \frac{1}{2}n - \frac{2}{3}n^2 + \frac{5}{16}n^3 + \frac{41}{180}n^4 - \frac{127}{288}n^5 + \frac{7891}{37800}n^6 + \frac{72161}{387072}n^7 - \frac{18975107}{50803200}n^8 + \dots \\
\alpha_4 &= \frac{13}{48}n^2 - \frac{3}{5}n^3 + \frac{557}{1440}n^4 + \frac{281}{630}n^5 - \frac{1983433}{1935360}n^6 + \frac{13769}{28800}n^7 + \frac{148003883}{174182400}n^8 - \dots \\
\alpha_6 &= \frac{61}{240}n^3 - \frac{103}{140}n^4 + \frac{15061}{26880}n^5 + \frac{167603}{181440}n^6 - \frac{67102379}{29030400}n^7 + \frac{79682431}{79833600}n^8 + \dots \\
\alpha_8 &= \frac{49561}{161280}n^4 - \frac{179}{168}n^5 + \frac{6601661}{7257600}n^6 + \frac{97445}{49896}n^7 - \frac{40176129013}{7664025600}n^8 + \dots \\
\alpha_{10} &= \frac{34729}{80640}n^5 - \frac{3418889}{1995840}n^6 + \frac{14644087}{9123840}n^7 + \frac{2605413599}{622702080}n^8 + \dots \\
\alpha_{12} &= \frac{212378941}{319334400}n^6 - \frac{30705481}{10378368}n^7 + \frac{175214326799}{58118860800}n^8 + \dots \\
\alpha_{14} &= \frac{1522256789}{1383782400}n^7 - \frac{16759934899}{3113510400}n^8 + \dots \\
\alpha_{16} &= \frac{1424729850961}{743921418240}n^8 - \dots
\end{aligned} \tag{34}$$

[This is Krueger's eq. (11), §5, p.14 extended to order n^8]

THE GAUSS-KRUEGER PROJECTION

The Gauss-Krueger (or TM) projection is a triple-mapping in two parts (Bugayevskiy & Snyder 1995). The first part is a conformal mapping of the ellipsoid to the conformal sphere of radius a followed by a conformal mapping of this sphere to the plane using the spherical TM projection equations with spherical latitude ϕ replaced by conformal latitude ϕ' . This two-step process is also known as the Gauss-Schreiber projection (Snyder 1993) and the scale along the central meridian is not constant. The second part is the conformal mapping from the Gauss-Schreiber to the Gauss-Krueger projection where the scale factor along the central meridian is made constant.

To understand this process we first discuss the spherical Mercator and Transverse Mercator (TM) projections. We then give the equations for the Gauss-Schreiber projection and show that the scale factor along the central meridian of this projection is not constant. Finally, using complex functions and principles of conformal mapping developed by Gauss, we show the conformal mapping from the Gauss-Schreiber projection to the Gauss-Krueger projection.

Having established the 'forward' mapping $\phi, \lambda \rightarrow X, Y$ we show how the 'inverse' mapping $X, Y \rightarrow \phi, \lambda$ from the plane to the ellipsoid is achieved.

In addition to the equations for the forward and inverse mappings we derive equations for scale factor m and grid convergence γ .

Mercator projection of the sphere

The Mercator projection of the sphere is a conformal projection with the well known equations (Lauf 1983)

$$X = R(\lambda - \lambda_0) = R\omega \quad \text{and} \quad Y = R \ln \tan\left(\frac{1}{4}\pi + \frac{1}{2}\phi\right) = \frac{R}{2} \ln\left(\frac{1 + \sin\phi}{1 - \sin\phi}\right) \quad (35)$$

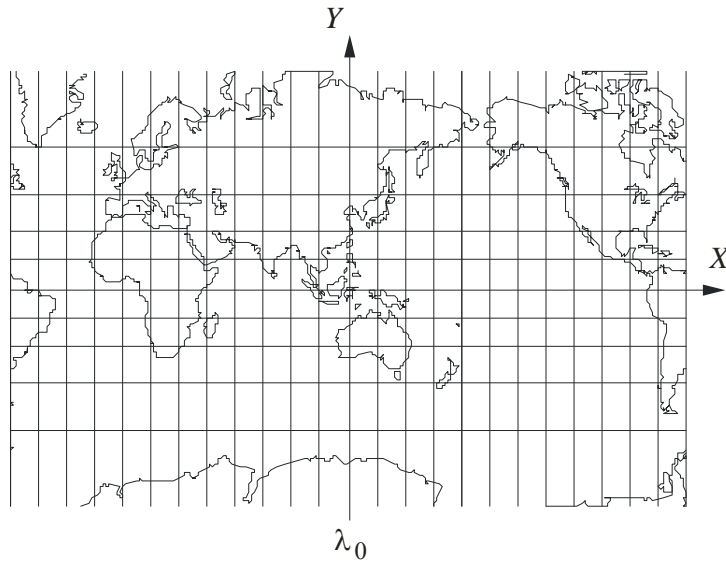


Figure 2 Mercator projection
graticule interval 15° , central meridian $\lambda_0 = 120^\circ \text{E}$

TM projection of the sphere (Gauss-Lambert projection)

The equations for the Gauss-Lambert projection can be derived by considering Figure 3 that shows P having curvilinear coordinates ϕ, λ that are angular quantities measured along great circles (meridian and equator).

Now consider the great circle NBS (the oblique equator) with a pole A that lies on the equator and the great circle through APC making an angle θ with the equator.

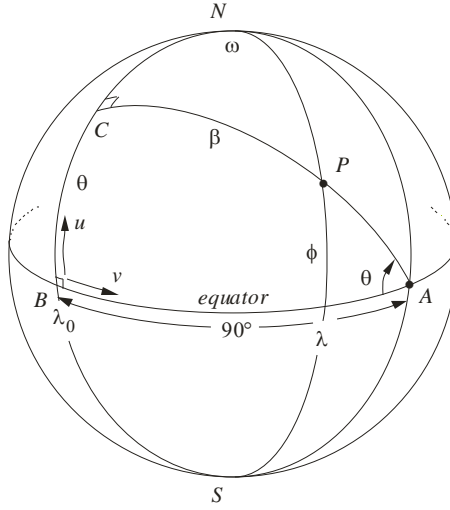


Figure 3 Oblique pole A on equator

β and θ are oblique latitude and oblique longitude respectively and equations linking β, θ and ϕ, λ can be obtained from the right-angled spherical triangle CNP having sides $\beta, \frac{1}{2}\pi - \theta, \frac{1}{2}\pi - \phi$ and an angle at N of $\omega = \lambda - \lambda_0$

$$\sin \beta = \cos \phi \sin \omega \quad (36)$$

$$\tan \theta = \frac{\tan \phi}{\cos \omega} \quad (37)$$

Squaring both sides of (36) and using the trigonometric identity $\sin^2 x + \cos^2 x = 1$ gives

$$\tan \beta = \frac{\sin \omega}{\sqrt{\tan^2 \phi + \cos^2 \omega}} \quad (38)$$

Replacing ϕ with β and ω with θ in (35); then using (36), (37) and the identity

$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ for $-1 < x < 1$ leads to the equations for the Gauss-Lambert projection (Lauf 1983, Snyder 1987)

$$\begin{aligned} u &= R \theta = R \tan^{-1} \left(\frac{\tan \phi}{\cos \omega} \right) \\ v &= \frac{R}{2} \ln \left(\frac{1 + \sin \beta}{1 - \sin \beta} \right) = \frac{R}{2} \ln \left(\frac{1 + \cos \phi \sin \omega}{1 - \cos \phi \sin \omega} \right) = R \tanh^{-1} (\cos \phi \sin \omega) \end{aligned} \quad (39)$$

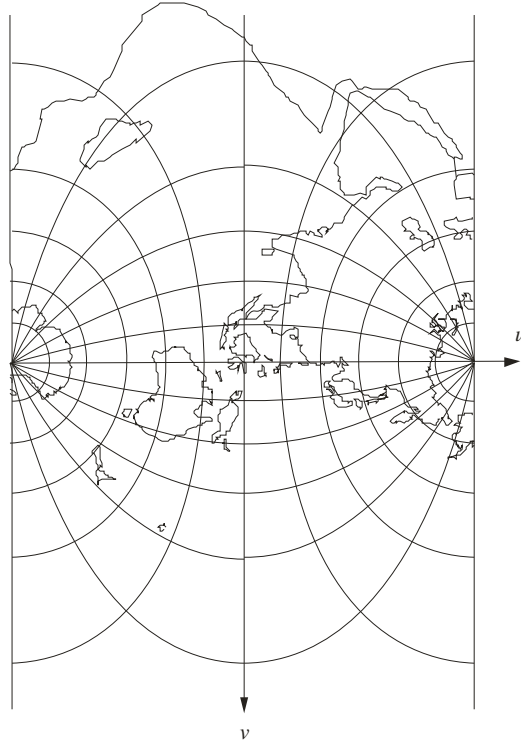


Figure 4 Gauss-Lambert projection
graticule interval 15° , central meridian $\lambda_0 = 120^\circ \text{E}$

Gauss-Lambert scale factor

The Gauss-Lambert projection is conformal and hence the scale factor m is the same in all directions around any point (Lauf 1983) and

$$m = \frac{1}{\sqrt{1 - \cos^2 \phi \sin^2 \omega}} = 1 + \frac{1}{2} \cos^2 \phi \sin^2 \omega + \frac{3}{8} \cos^4 \phi \sin^4 \omega + \dots \quad (40)$$

Along the central meridian $\omega = 0$ and the central meridian scale factor $m_0 = 1$

Gauss-Lambert grid convergence

The grid convergence γ is the angle between the meridian and the grid-line parallel to the u -axis and is defined as

$$\tan \gamma = \left| \frac{dv}{du} \right| \quad (41)$$

Equations (39) show that $u = u(\phi, \lambda)$ and $v = v(\phi, \lambda)$ thus the total differentials du and dv are

$$du = \frac{\partial u}{\partial \phi} d\phi + \frac{\partial u}{\partial \omega} d\omega \quad \text{and} \quad dv = \frac{\partial v}{\partial \phi} d\phi + \frac{\partial v}{\partial \omega} d\omega \quad (42)$$

But along a meridian ω is constant and $d\omega = 0$ so the grid convergence is obtained from

$$\tan \gamma = \left| \frac{\partial v}{\partial \phi} / \frac{\partial u}{\partial \phi} \right| \quad (43)$$

Substituting the partial derivatives of (39) gives (Lauf 1983)

$$\gamma = \tan^{-1} (\sin \phi \tan \omega) \quad (44)$$

TM projection of the conformal sphere (Gauss-Schreiber projection)

The equations for the TM projection of the conformal sphere of radius a are simply obtained by replacing spherical latitude ϕ with conformal latitude ϕ' in (39) to give

$$u = a \tan^{-1} \left(\frac{\tan \phi'}{\cos \omega} \right) \quad \text{and} \quad v = a \tanh^{-1} (\cos \phi' \sin \omega) \quad (45)$$

[These are Krueger's eq's (36), §8, p. 20]

Alternatively, replacing ϕ with β and ω with θ in (35) then using (37), (38) and the identity $\ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} x \right) = \sinh^{-1} \tan x$; and finally ϕ with ϕ' gives (Karney 2011)

$$u = a \tan^{-1} \left(\frac{\tan \phi'}{\cos \omega} \right) \quad \text{and} \quad v = a \sinh^{-1} \left(\frac{\sin \omega}{\sqrt{\tan^2 \phi' + \cos^2 \omega}} \right) \quad (46)$$

$\tan \phi'$ can be evaluated as follows. Using the identities $\ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} x \right) = \sinh^{-1} \tan x$ and $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ for $-1 < x < 1$ we write

$$\ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} \phi' \right) = \sinh^{-1} \tan \phi' \quad (47)$$

And (25) can be written as

$$\ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} \phi' \right) = \ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} \phi \right) - \frac{1}{2} \varepsilon \ln \left(\frac{1 + \varepsilon \sin \phi}{1 - \varepsilon \sin \phi} \right) = \sinh^{-1} \tan \phi - \varepsilon \tanh^{-1} (\varepsilon \sin \phi) \quad (48)$$

Equating (47) and (48) gives

$$\sinh^{-1} \tan \phi' = \sinh^{-1} \tan \phi - \varepsilon \tanh^{-1} (\varepsilon \sin \phi) \quad (49)$$

With the substitution

$$\sigma = \sinh \left\{ \varepsilon \tanh^{-1} \left(\frac{\varepsilon \tan \phi}{\sqrt{1 + \tan^2 \phi}} \right) \right\} \quad (50)$$

equation (49) can be rearranged as (Karney 2011)

$$\tan \phi' = \tan \phi \sqrt{1 + \sigma^2} - \sigma \sqrt{1 + \tan^2 \phi} \quad (51)$$

Gauss-Schreiber scale factor

The scale factor m is defined by

$$m^2 = \frac{(dS)^2}{(ds)^2} \quad (52)$$

Where dS is a differential distance on the projection plane and ds is the differential distance on the ellipsoid, and from (10) noting that $d\omega = d\lambda$

$$(ds)^2 = \rho^2 (d\phi)^2 + v^2 \cos^2 \phi (d\omega)^2 \quad (53)$$

For the projection plane $u = u(\phi, \omega)$ and $v = v(\phi, \omega)$

$$(dS)^2 = (du)^2 + (dv)^2 \quad (54)$$

and the total differentials are

$$du = \frac{\partial u}{\partial \phi} d\phi + \frac{\partial u}{\partial \omega} d\omega \quad \text{and} \quad dv = \frac{\partial v}{\partial \phi} d\phi + \frac{\partial v}{\partial \omega} d\omega \quad (55)$$

Since the projection is conformal, scale is the same in all directions around any point. It is sufficient then to choose any one direction, say along a meridian where ω is constant and $d\omega = 0$.

Hence

$$m^2 = \frac{1}{\rho^2} \left\{ \left(\frac{\partial u}{\partial \phi} \right)^2 + \left(\frac{\partial v}{\partial \phi} \right)^2 \right\} \quad (56)$$

The partial derivatives are evaluated using the chain rule for differentiation and (12), (25) and (45)

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial \phi'} \frac{\partial \phi'}{\partial \psi} \frac{\partial \psi}{\partial \phi} \quad \text{and} \quad \frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial \phi'} \frac{\partial \phi'}{\partial \psi} \frac{\partial \psi}{\partial \phi} \quad (57)$$

with

$$\begin{aligned} \frac{\partial \psi}{\partial \phi} &= \frac{\varepsilon^2 - 1}{(1 - \varepsilon^2 \sin^2 \phi) \cos \phi} & \frac{\partial \phi'}{\partial \psi} &= \cos \phi' \\ \frac{\partial u}{\partial \phi'} &= \frac{a \cos \omega}{1 - \cos^2 \phi' \sin^2 \omega} & \frac{\partial v}{\partial \phi'} &= \frac{-a \sin \phi' \sin \omega}{1 - \cos^2 \phi' \sin^2 \omega} \end{aligned} \quad (58)$$

Substituting (58) into (57) and then into (56) and simplifying gives the scale factor as

$$m = \frac{\sqrt{1 + \tan^2 \phi} \sqrt{1 - \varepsilon^2 \sin^2 \phi}}{\sqrt{\tan^2 \phi' + \cos^2 \omega}} \quad (59)$$

Along the central meridian of the projection $\omega = 0$ and the central meridian scale factor is

$$m_0 = \frac{\cos \phi'}{\cos \phi} \sqrt{1 - \varepsilon^2 \sin^2 \phi} = \frac{\cos \phi'}{\cos \phi} \left\{ 1 - \frac{1}{2} \varepsilon^2 \sin^2 \phi - \frac{1}{8} \varepsilon^4 \sin^4 \phi - \dots \right\} \quad (60)$$

m_0 is not constant and varies slightly from unity, but a final conformal mapping from the u, v Gauss-Schreiber plane to the X, Y Gauss-Krueger plane can be made that will have $m_0 = \text{constant}$.

Gauss-Schreiber grid convergence

The grid convergence is defined by (43) with the partial derivatives evaluated using the chain rule for differentiation [see (57)] giving

$$\tan \gamma = \left| \frac{\partial v}{\partial \phi} / \frac{\partial u}{\partial \phi} \right| = \left| \frac{\partial v}{\partial \phi'} / \frac{\partial u}{\partial \phi'} \right| \quad (61)$$

And using (58)

$$\gamma = \tan^{-1} (\sin \phi' \tan \omega) = \tan^{-1} \left(\frac{\tan \phi' \tan \omega}{\sqrt{1 + \tan^2 \phi'}} \right) \quad (62)$$

Conformal mapping from the Gauss-Schreiber to the Gauss-Krueger projection

Using the theory of conformal mapping and complex functions developed by Gauss suppose that the mapping from the u, v Gauss-Schreiber plane (Figure 4) to the X, Y Gauss-Krueger plane (Figure 5) is given by

$$\frac{1}{A}(Y + iX) = f(u + iv) \quad (63)$$

where A is the rectifying radius.

Let the complex function $f(u + iv)$ be

$$f(u + iv) = \frac{u}{a} + i\frac{v}{a} + \sum_{r=1}^{\infty} \kappa_{2r} \sin\left(2r\left(\frac{u}{a}\right) + i2r\left(\frac{v}{a}\right)\right) \quad (64)$$

where a is the radius of the conformal sphere and κ_{2r} are as yet, unknown coefficients.

Expanding the complex trigonometric function in (64) gives

$$f(u + iv) = \frac{u}{a} + i\frac{v}{a} + \sum_{r=1}^{\infty} \kappa_{2r} \left(\sin 2r\left(\frac{u}{a}\right) \cosh 2r\left(\frac{v}{a}\right) + i \cos 2r\left(\frac{u}{a}\right) \sinh 2r\left(\frac{v}{a}\right) \right) \quad (65)$$

and equating real and imaginary parts gives

$$\frac{Y}{A} = \frac{u}{a} + \sum_{r=1}^{\infty} \kappa_{2r} \sin 2r\left(\frac{u}{a}\right) \cosh 2r\left(\frac{v}{a}\right) \quad \text{and} \quad \frac{X}{A} = \frac{v}{a} + \sum_{r=1}^{\infty} \kappa_{2r} \cos 2r\left(\frac{u}{a}\right) \sinh 2r\left(\frac{v}{a}\right) \quad (66)$$

Now, along the central meridian $v = 0$ and $\cosh 2v = \cosh 4v = \dots = 1$ and Y/A becomes

$$\frac{Y}{A} = \frac{u}{a} + \kappa_2 \sin 2\left(\frac{u}{a}\right) + \kappa_4 \sin 4\left(\frac{u}{a}\right) + \kappa_6 \sin 6\left(\frac{u}{a}\right) \dots \quad (67)$$

Furthermore, along the central meridian u/a is an angular quantity identical to the conformal latitude ϕ' and (67) becomes

$$\frac{Y}{A} = \phi' + \kappa_2 \sin 2\phi' + \kappa_4 \sin 4\phi' + \kappa_6 \sin 6\phi' \dots \quad (68)$$

Now, if the central meridian scale factor is unity then the Y coordinate is the meridian distance M , and $Y/A = M/A = \mu$ is the rectifying latitude and (68) becomes

$$\mu = \phi' + \kappa_2 \sin 2\phi' + \kappa_4 \sin 4\phi' + \kappa_6 \sin 6\phi' \dots \quad (69)$$

This equation is identical in form to (33) and we may conclude that the coefficients $\{\kappa_{2r}\}$ are equal to the coefficients $\{\alpha_{2r}\}$ in (33); and the Gauss-Krueger projection is given by

$$\begin{aligned} X &= A \left\{ \frac{v}{a} + \sum_{r=1}^{\infty} \alpha_{2r} \cos 2r\left(\frac{u}{a}\right) \sinh 2r\left(\frac{v}{a}\right) \right\} \\ Y &= A \left\{ \frac{u}{a} + \sum_{r=1}^{\infty} \alpha_{2r} \sin 2r\left(\frac{u}{a}\right) \cosh 2r\left(\frac{v}{a}\right) \right\} \end{aligned} \quad (70)$$

[These are Krueger's equations (42), §8, p. 21.]

A is given by (18), u/a and v/a are given by (46) and we use coefficients α_{2r} up to $r = 8$ given by (34).

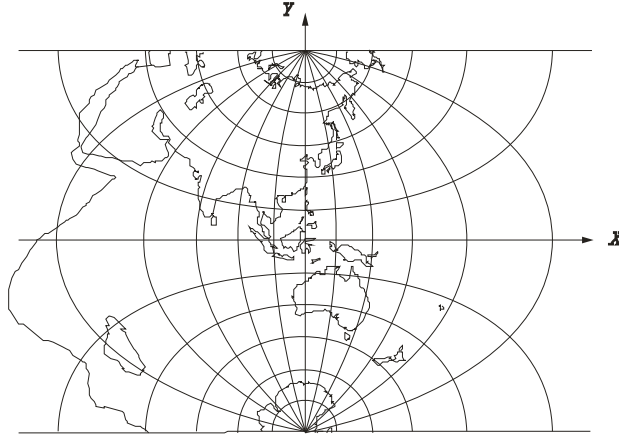


Figure 5 Gauss-Krueger projection

graticule interval 15° , central meridian $\lambda_0 = 120^\circ \text{E}$

[Note: the graticules of Figures 4 and 5 are for different projections but are indistinguishable at the printed scales and for the longitude extent shown. If the two mappings were scaled so that the distances from the equator to the pole were identical, there would be some obvious differences between the graticules at large distances from the central meridian. One of the authors (Karney 2011, Fig. 1) has examples of these differences.]

Finally, X and Y are scaled and shifted to give E (east) and N (north) coordinates related to a false origin

$$E = m_0 X + E_0 \quad \text{and} \quad N = m_0 Y + N_0 \quad (71)$$

m_0 is the central meridian scale factor and the quantities E_0, N_0 are offsets that make the E, N coordinates positive in the area of interest. The origin of X, Y coordinates is the true origin at the intersection of the equator and the central meridian. The origin of E, N coordinates is known as the false origin and it is located at $X = -E_0, Y = -N_0$.

Gauss-Krueger scale factor

The Gauss-Krueger scale factor can be derived in a similar way to the scale factor for the Gauss-Schreiber projection and we have

$$(ds)^2 = \rho^2 (d\phi)^2 + \nu^2 \cos^2 \phi (d\omega)^2 \quad \text{and} \quad (dS)^2 = (dX)^2 + (dY)^2$$

where $X = X(u, v)$, $Y = Y(u, v)$ and the total differentials dX and dY are

$$dX = \frac{\partial X}{\partial u} du + \frac{\partial X}{\partial v} dv \quad \text{and} \quad dY = \frac{\partial Y}{\partial u} du + \frac{\partial Y}{\partial v} dv \quad (72)$$

du and dv are given by (55) and substituting these into (72) gives

$$\begin{aligned} dX &= \frac{\partial X}{\partial u} \left\{ \frac{\partial u}{\partial \phi} d\phi + \frac{\partial u}{\partial \omega} d\omega \right\} + \frac{\partial X}{\partial v} \left\{ \frac{\partial v}{\partial \phi} d\phi + \frac{\partial v}{\partial \omega} d\omega \right\} \\ dY &= \frac{\partial Y}{\partial u} \left\{ \frac{\partial u}{\partial \phi} d\phi + \frac{\partial u}{\partial \omega} d\omega \right\} + \frac{\partial Y}{\partial v} \left\{ \frac{\partial v}{\partial \phi} d\phi + \frac{\partial v}{\partial \omega} d\omega \right\} \end{aligned}$$

Choosing to evaluate the scale along a meridian where ω is constant and $d\omega = 0$ gives

$$dX = \left(\frac{\partial X}{\partial u} \frac{\partial u}{\partial \phi} + \frac{\partial X}{\partial v} \frac{\partial v}{\partial \phi} \right) d\phi \quad \text{and} \quad dY = \left(\frac{\partial Y}{\partial u} \frac{\partial u}{\partial \phi} + \frac{\partial Y}{\partial v} \frac{\partial v}{\partial \phi} \right) d\phi \quad (73)$$

and

$$m^2 = \frac{(dS)^2}{(ds)^2} = \frac{(dX)^2 + (dY)^2}{\rho^2 (d\phi)^2} \quad (74)$$

Differentiating (70) gives

$$\frac{\partial X}{\partial u} = \frac{A}{a} q, \quad \frac{\partial X}{\partial v} = \frac{A}{a} p, \quad \frac{\partial Y}{\partial u} = \frac{\partial X}{\partial v}, \quad \frac{\partial Y}{\partial v} = -\frac{\partial X}{\partial u} \quad (75)$$

where

$$q = -\sum_{r=1}^{\infty} 2r\alpha_{2r} \sin 2r\left(\frac{u}{a}\right) \sinh 2r\left(\frac{v}{a}\right) \quad \text{and} \quad p = 1 + \sum_{r=1}^{\infty} 2r\alpha_{2r} \cos 2r\left(\frac{u}{a}\right) \cosh 2r\left(\frac{v}{a}\right) \quad (76)$$

Substituting (75) into (73) and then into (74) and simplifying gives

$$m^2 = \left(\frac{A}{a}\right)^2 (q^2 + p^2) \left\{ \frac{1}{\rho^2} \left(\left(\frac{\partial u}{\partial \phi} \right)^2 + \left(\frac{\partial v}{\partial \phi} \right)^2 \right) \right\} \quad (77)$$

The term in braces $\{ \}$ is the square of the Gauss-Schreiber scale factor [see (56)] and so, using (59), we may write the scale factor for the Gauss-Krueger projection as

$$m = m_0 \left(\frac{A}{a} \right) \sqrt{q^2 + p^2} \left\{ \frac{\sqrt{1 + \tan^2 \phi} \sqrt{1 - \varepsilon^2 \sin^2 \phi}}{\sqrt{\tan^2 \phi' + \cos^2 \omega}} \right\} \quad (78)$$

q and p are found from (76), $\tan \phi'$ from (51) and A from (18).

Gauss-Krueger grid convergence

The grid convergence is defined by

$$\tan \gamma = \left| \frac{dX}{dY} \right| \quad (79)$$

Using (73) and (75) we may write (79) as

$$\tan \gamma = \left| \frac{q \frac{\partial u}{\partial \phi} + p \frac{\partial v}{\partial \phi}}{p \frac{\partial u}{\partial \phi} - q \frac{\partial v}{\partial \phi}} \right| = \left| \frac{\frac{q}{p} + \left(\frac{\partial v}{\partial \phi} / \frac{\partial u}{\partial \phi} \right)}{1 - \frac{q}{p} \left(\frac{\partial v}{\partial \phi} / \frac{\partial u}{\partial \phi} \right)} \right| \quad (80)$$

Let $\gamma = \gamma_1 + \gamma_2$, then using a trigonometric addition formula write

$$\tan \gamma = \tan(\gamma_1 + \gamma_2) = \frac{\tan \gamma_1 + \tan \gamma_2}{1 - \tan \gamma_1 \tan \gamma_2} \quad (81)$$

Noting the similarity between (80) and (81) we may define

$$\tan \gamma_1 = \left| \frac{q}{p} \right| \quad \text{and} \quad \tan \gamma_2 = \left| \frac{\partial v}{\partial \phi} / \frac{\partial u}{\partial \phi} \right| \quad (82)$$

and γ_2 is the Gauss-Schreiber grid convergence [see (61) and (62)]. So the grid convergence on the Gauss-Krueger projection is

$$\gamma = \tan^{-1}\left(\frac{q}{p}\right) + \tan^{-1}\left(\frac{\tan \phi' \tan \omega}{\sqrt{1 + \tan^2 \phi'}}\right) \quad (83)$$

Conformal mapping from the Gauss-Krueger plane to the ellipsoid

The conformal mapping from the Gauss-Krueger plane to the ellipsoid is achieved in three steps:

- (i) A conformal mapping from the Gauss-Krueger to the Gauss-Schreiber plane giving u, v coordinates, then
- (ii) Solving for $\tan \phi'$ and $\tan \omega$ given the u, v Gauss-Schreiber coordinates from which $\lambda = \lambda_0 \pm \omega$, and finally
- (iii) Solving for $\tan \phi$ by Newton-Raphson iteration and then obtaining ϕ .

The development of the equations for these steps is set out below.

Gauss-Schreiber coordinates from Gauss-Krueger coordinates

Suppose that the mapping from the X, Y Gauss-Krueger plane to the u, v Gauss-Schreiber plane is given by the complex function

$$\frac{1}{a}(u + iv) = F(Y + iX) \quad (84)$$

If E, N are given and E_0, N_0 and m_0 are known, then from (71)

$$X = \frac{E - E_0}{m_0} \quad \text{and} \quad Y = \frac{N - N_0}{m_0} \quad (85)$$

Let the complex function $F(Y + iX)$ be

$$F(Y + iX) = \frac{Y}{A} + i \frac{X}{A} + \sum_{r=1}^{\infty} K_{2r} \sin \left(2r \left(\frac{Y}{A} \right) + i 2r \left(\frac{X}{A} \right) \right) \quad (86)$$

where K_{2r} are as yet unknown coefficients.

Expanding the complex trigonometric function in (86) and equating real and imaginary parts gives

$$\frac{u}{a} = \frac{Y}{A} + \sum_{r=1}^{\infty} K_{2r} \sin 2r \left(\frac{Y}{A} \right) \cosh 2r \left(\frac{X}{A} \right) \quad \text{and} \quad \frac{v}{a} = \frac{X}{A} + \sum_{r=1}^{\infty} K_{2r} \cos 2r \left(\frac{Y}{A} \right) \sinh 2r \left(\frac{X}{A} \right) \quad (87)$$

Along the central meridian $Y/A = M/A = \mu$ and $X = 0$ [and $\cosh(0) = 1$]. Also, u/a is an angular quantity that is identical to ϕ' and we can write the first of (87) as

$$\phi' = \mu + K_2 \sin 2\mu + K_4 \sin 4\mu + K_6 \sin 6\mu + \dots \quad (88)$$

This equation is identical in form to (31) and we may conclude that the coefficients $\{K_{2r}\}$ are equal to the coefficients $\{\beta_{2r}\}$ in (31) and the ratios u/a and v/a are given by

$$\frac{u}{a} = \frac{Y}{A} + \sum_{r=1}^{\infty} \beta_{2r} \sin 2r \left(\frac{Y}{A} \right) \cosh 2r \left(\frac{X}{A} \right) \quad \text{and} \quad \frac{v}{a} = \frac{X}{A} + \sum_{r=1}^{\infty} \beta_{2r} \cos 2r \left(\frac{Y}{A} \right) \sinh 2r \left(\frac{X}{A} \right) \quad (89)$$

where A is given by (18) and we use coefficients β_{2r} up to $r = 8$ given by (32).

Conformal latitude and longitude difference from Gauss-Schreiber coordinates

Equations (46) can be re-arranged and solved for $\tan \phi'$ and $\tan \omega$ as functions of the ratios u/a and v/a giving

$$\tan \phi' = \frac{\sin\left(\frac{u}{a}\right)}{\sqrt{\sinh^2\left(\frac{v}{a}\right) + \cos^2\left(\frac{u}{a}\right)}} \quad \text{and} \quad \tan \omega = \sinh\left(\frac{v}{a}\right) / \cos\left(\frac{u}{a}\right) \quad (90)$$

To evaluate $\tan \phi$ after obtaining $\tan \phi'$ from (90), consider (50) and (51) with $t = \tan \phi$ and $t' = \tan \phi'$

$$t' = t\sqrt{1+\sigma^2} - \sigma\sqrt{1+t^2} \quad (91)$$

and

$$\sigma = \sinh\left\{\varepsilon \tanh^{-1}\left(\frac{\varepsilon t}{\sqrt{1+t^2}}\right)\right\} \quad (92)$$

t can be evaluated using the Newton-Raphson method for the real roots of the equation $f(t) = 0$ given in the form of an iterative equation

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)} \quad (93)$$

where t_n denotes the n^{th} iterate and $f(t)$ is given by

$$f(t) = t\sqrt{1+\sigma^2} - \sigma\sqrt{1+t^2} - t' \quad (94)$$

The derivative $f'(t) = \frac{d}{dt}\{f(t)\}$ is given by

$$f'(t) = \left(\sqrt{1+\sigma^2}\sqrt{1+t^2} - \sigma t\right) \frac{(1-\varepsilon^2)\sqrt{1+t^2}}{1+(1-\varepsilon^2)t^2} \quad (95)$$

where $t' = \tan \phi'$ is fixed.

An initial value for t_1 can be taken as $t_1 = t' = \tan \phi'$ and the functions $f(t_1)$ and $f'(t_1)$ evaluated from (92), (94) and (95). t_2 is now computed from (93) and this process repeated to obtain t_3, t_4, \dots . This iterative process can be concluded when the difference between t_{n+1} and t_n reaches an acceptably small value, and then the latitude is given by $\phi = \tan^{-1} t_{n+1}$.

This concludes the development of the Gauss-Krueger projection.

ACCURACY OF THE TRANSFORMATIONS

One of the authors (Karney, 2011) has compared Krueger's series to order n^8 (set out above) with an exact transverse Mercator projection defined by Lee (1976) and shows that errors in positions computed from these series are less than 5 nanometres anywhere within a distance of 4200 km of the central meridian (equivalent to $\omega \approx 37.7^\circ$ at the equator). So we can conclude that Krueger's series (to order n^8) are easily capable of micrometre precision within 30° of a central meridian.

THE 'OTHER' GAUSS-KRUEGER PROJECTION

Krueger (1912) develops the mapping equations that we have shown above in the first part of his manuscript followed by examples of the forward and inverse transformations. He then develops and explains an alternative approach: direct transformations from the ellipsoid to the plane and from the plane to the ellipsoid.

This alternative approach is outlined in the Appendix and for the forward transformation [see (98)] the equations involve functions containing powers of the longitude difference $\omega^2, \omega^3, \dots$ and

derivatives $\frac{dM}{d\psi}, \frac{d^2M}{d\psi^2}, \frac{d^3M}{d\psi^3}, \dots$ For the inverse transformation [see (103) and (105)] the

equations involve powers of the X coordinate X^2, X^3, X^4, \dots and derivatives $\frac{d\phi_1}{d\psi_1}, \frac{d^2\phi_1}{d\psi_1^2}, \frac{d^3\phi_1}{d\psi_1^3}, \dots$

and $\frac{d\psi_1}{dY}, \frac{d^2\psi_1}{dY^2}, \frac{d^3\psi_1}{dY^3}, \dots$ For both transformations, the higher order derivatives become

excessively complicated and are not generally known (or approximated) beyond the eighth derivative.

Redfearn (1948) and Thomas (1952) derive identical formulae, extending (slightly) Krueger's equations, and updating the notation and formulation. These are regarded as the standard for transformations between the ellipsoid and the TM projection. For example, GeoTrans (2010) uses Thomas' equations and Geoscience Australia define Redfearn's equations as the method of transformation between the Geocentric Datum of Australia (ellipsoid) and Map Grid Australia (transverse Mercator) [GDAV2.3].

The apparent attractions of these formulae are:

- (i) their wide-spread use and adoption by government mapping authorities, and
- (ii) there are no hyperbolic functions.

The weakness of these formulae are:

- (a) they are only accurate within relatively small bands of longitude difference about the central meridian (mm accuracy for $\omega < 6^\circ$) and
- (b) at large longitude differences ($\omega > 30^\circ$) they can give wildly inaccurate results.

The inaccuracies in Redfearn's (and Thomas's) equations are most evident in the inverse transformation $X, Y \rightarrow \phi, \omega$. Table 1 shows a series of points each having latitude $\phi = 75^\circ$ but with increasing longitude differences ω from a central meridian. X, Y coordinates are computed using Krueger's series and can be regarded as exact (at mm accuracy) and the column headed Redfearn ϕ, ω are the values obtained from Redfearn's equations for the inverse transformation. The error is the distance on the ellipsoid between the given ϕ, ω in the first column and the Redfearn ϕ, ω in the third column.

The values in the table have been computed for the GRS80 ellipsoid ($a = 6378137$ m, $f = 1/298.257222101$) with $m_0 = 1$

point	Gauss-Krueger	Redfearn	error
$\phi 75^\circ$ $\omega 6^\circ$	X 173137.521 Y 8335703.234	$\phi 75^\circ 00' 00.0000''$ $\omega 5^\circ 59' 59.9999''$	0.001
$\phi 75^\circ$ $\omega 10^\circ$	X 287748.837 Y 8351262.809	$\phi 75^\circ 00' 00.0000''$ $\omega 9^\circ 59' 59.9966''$	0.027
$\phi 75^\circ$ $\omega 15^\circ$	X 429237.683 Y 8381563.943	$\phi 75^\circ 00' 00.0023''$ $\omega 14^\circ 59' 59.8608''$	1.120
$\phi 75^\circ$ $\omega 20^\circ$	X 567859.299 Y 8423785.611	$\phi 75^\circ 00' 00.0472''$ $\omega 19^\circ 59' 57.9044''$	16.888
$\phi 75^\circ$ $\omega 30^\circ$	X 832650.961 Y 8543094.338	$\phi 75^\circ 00' 03.8591''$ $\omega 29^\circ 58' 03.5194''$	942.737
$\phi 75^\circ$ $\omega 35^\circ$	X 956892.903 Y 8619555.491	$\phi 75^\circ 00' 23.0237''$ $\omega 34^\circ 49' 57.6840''$	4.9 km

Table 1

This problem is highlighted when considering Greenland (Figure 6), which is an ideal 'shape' for a transverse Mercator projection, having a small east-west extent (approx. 1650 km) and large north-south extent (approx. 2600 km).

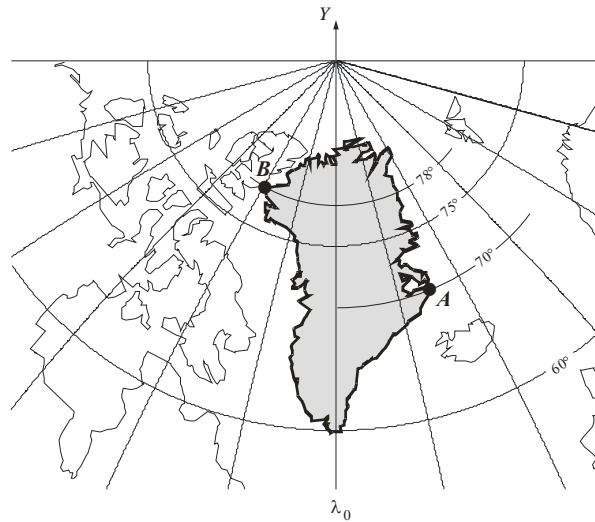


Figure 6 Gauss-Krueger projection of Greenland
graticule interval 15° , central meridian $\lambda_0 = 45^\circ \text{W}$

A and B represent two extremes if a central meridian is chosen as $\lambda_0 = 45^\circ \text{W}$. A is a point furthest from the central meridian (approx. 850 km); and B would have the greatest west longitude.

Table 2 shows the errors at A and B for the GRS80 ellipsoid with $m_0 = 1$ for the inverse transformation using Redfearn's equations.

point	Gauss-Krueger	Redfearn	error
A $\phi 70^\circ$ $\omega 22.5^\circ$	X 842115.901 Y 7926858.314	$\phi 75^\circ 00' 00.2049''$ $\omega 22^\circ 29' 53.9695''$	64.282
B $\phi 78^\circ$ $\omega -30^\circ$	X -667590.239 Y 8837145.459	$\phi 78^\circ 00' 03.1880''$ $\omega -29^\circ 57' 59.2860''$	784.799

Table 2

CONCLUSION

We have provided a derivation of the Karney-Krueger equations for the Gauss-Krueger projection that allow micrometre accuracy in the forward and inverse mappings between the ellipsoid and plane. And we have provided some commentary on the 'other' Gauss-Krueger equations in wide use in the geospatial community. These other equations offer only limited accuracy and should be abandoned in favour of the equations (and methods) we give.

Our work is not original; indeed some of these equations were developed by Krueger almost a century ago. But with the aid of computer algebra systems we have extended Krueger's original series – as others have done (Engsager & Poder 2007) – so that they are capable of very high accuracy at large distances from a central meridian. This makes the Transverse Mercator (TM) projection a much more useful projection for the geospatial community.

We also hope that this paper may be useful to mapping organisations wishing to 'upgrade' transformation software that use formulae given by Redfearn (1948) or Thomas (1952) – they are unnecessarily inaccurate.

APPENDIX

Conformal mapping and complex functions

A theory due to Gauss states that a conformal mapping from the ψ, ω datum surface to the X, Y projection surface can be represented by the complex expression

$$Y + iX = f(\psi + i\omega) \quad (96)$$

Providing that ψ and ω are isometric parameters and the complex function $f(\psi + i\omega)$ is analytic. $i = \sqrt{-1}$ and the left-hand side of (96) is a complex function consisting of a real and imaginary part. The right-hand-side of (96) is another complex function of real and imaginary parameters ψ and ω respectively.

The alternative approach to developing a transverse Mercator projection of the ellipsoid is to expand (96) as a power series (Lauf 1983)

$$Y + iX = f(\psi + i\omega) = f(\psi) + i\omega f^{(1)}(\psi) + \frac{(i\omega)^2}{2!} f^{(2)}(\psi) + \frac{(i\omega)^3}{3!} f^{(3)}(\psi) + \dots \quad (97)$$

where $f^{(1)}(\psi)$, $f^{(2)}(\psi)$, etc. are first, second and higher order derivatives of the function $f(\psi)$ and equating real and imaginary parts (noting that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc. and $f(\psi) = M$) gives

$$\begin{aligned} X &= \omega \frac{dM}{d\psi} - \frac{\omega^3}{3!} \frac{d^3 M}{d\psi^3} + \frac{\omega^5}{5!} \frac{d^5 M}{d\psi^5} - \frac{\omega^7}{7!} \frac{d^7 M}{d\psi^7} + \dots \\ Y &= M - \frac{\omega^2}{2!} \frac{d^2 M}{d\psi^2} + \frac{\omega^4}{4!} \frac{d^4 M}{d\psi^4} - \frac{\omega^6}{6!} \frac{d^6 M}{d\psi^6} + \dots \end{aligned} \quad (98)$$

In this alternative approach, the inverse transformation from the plane to the ellipsoid is represented by another complex expression

$$\psi + i\omega = F(Y + iX) \quad (99)$$

And similarly to before, $F(Y + iX)$ can be expanded as a power series giving

$$\psi + i\omega = F(Y) + iXF^{(1)}(Y) + \frac{(iX)^2}{2!} F^{(2)}(Y) + \frac{(iX)^3}{3!} F^{(3)}(Y) + \dots \quad (100)$$

When $X = 0$, $\omega = 0$; but when $X \neq 0$ the point $P(\phi, \omega)$ becomes $P_1(\phi_1, 0)$, a point on the central meridian having 'foot-point' latitude ϕ_1 . Now ψ_1 is the isometric latitude for ϕ_1 and we have $F(Y) = \psi_1$

Substituting (100) into (99) and equating real and imaginary parts gives

$$\begin{aligned}\psi &= \psi_1 - \frac{X^2}{2!} \frac{d^2\psi_1}{dY^2} + \frac{X^4}{4!} \frac{d^4\psi_1}{dY^4} - \frac{X^6}{6!} \frac{d^6\psi_1}{dY^6} + \dots \\ \omega &= X \frac{d\psi_1}{dY} - \frac{X^3}{3!} \frac{d^3\psi_1}{dY^3} + \frac{X^5}{5!} \frac{d^5\psi_1}{dY^5} - \frac{X^7}{7!} \frac{d^7\psi_1}{dY^7} + \dots\end{aligned}\quad (101)$$

The first of (101) gives ψ in terms of ψ_1 but we require ϕ in terms of ϕ_1 . Write the first of (101) as

$$\psi = \psi_1 + \delta\psi \quad (102)$$

where

$$\delta\psi = -\frac{X^2}{2!} \frac{d^2\psi_1}{dY^2} + \frac{X^4}{4!} \frac{d^4\psi_1}{dY^4} - \frac{X^6}{6!} \frac{d^6\psi_1}{dY^6} + \dots \quad (103)$$

And $\phi = g(\psi) = g(\psi_1 + \delta\psi)$ can be expanded as another power series

$$\phi = g(\psi_1) + \delta\psi g^{(1)}(\psi_1) + \frac{(\delta\psi)^2}{2!} g^{(2)}(\psi_1) + \frac{(\delta\psi)^3}{3!} g^{(3)}(\psi_1) + \dots \quad (104)$$

Noting that $g(\psi_1) = \phi_1$ we may write the transformation as

$$\begin{aligned}\phi &= \phi_1 + \delta\psi \frac{d\phi_1}{d\psi_1} + \frac{(\delta\psi)^2}{2!} \frac{d^2\phi_1}{d\psi_1^2} + \frac{(\delta\psi)^4}{4!} \frac{d^4\phi_1}{d\psi_1^4} + \dots \\ \omega &= X \frac{d\psi_1}{dY} - \frac{X^3}{3!} \frac{d^3\psi_1}{dY^3} + \frac{X^5}{5!} \frac{d^5\psi_1}{dY^5} - \frac{X^7}{7!} \frac{d^7\psi_1}{dY^7} + \dots\end{aligned}\quad (105)$$

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