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Coordinate conversions and transformations including formulas

EPSG
GEODETIC PARAMETER DATASET
https://epsg.org

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Introduction

The EPSG Geodetic Parameter Dataset, abbreviated to the EPSG Dataset, is a repository of parameters required to:

- define a *coordinate reference system* (CRS) which ensures that coordinates describe position unambiguously;
- define transformations and conversions that allow coordinates to be changed from one CRS to another CRS. Transformations and conversions are collectively called *coordinate operations*.

The EPSG Dataset is maintained by IOGP. It conforms to ISO 19111:2019 – *Referencing by coordinates*. It is distributed in three ways:

i) the **EPSG Registry**, in full the *EPSG Geodetic Parameter Registry*, a web-based delivery platform with a graphic user interface (GUI) and an application programming interface (API). From this repository descriptions of CRSs and transformations may be output in well-known text (WKT) conformant to ISO 19162 or as GML documents

ii) the **EPSG Database**, in full *the EPSG Geodetic Parameter Database*, an MS Access relational database with content derived from the online registry

iii) in a relational data model as **SQL scripts** which enable a user to create an Oracle, MySQL, PostgreSQL, or other relational database and populate that database with the EPSG Dataset

The terms of use of the EPSG Dataset are available here.

IOGP Reports 373-07-01 through 373-07-06 form a multi-part document for users of the EPSG Dataset.

- IOGP Report 373-07-01 (EPSG Guidance Note 7-1) – *Understanding the EPSG Dataset*, sets out detailed information about the Dataset content and its maintenance.

- IOGP Report 373-07-02 (EPSG Guidance Note 7-2) – *Coordinate conversions and transformations including formulas*, (this document), provides a detailed explanation of formulas necessary for executing coordinate conversions and transformations using the coordinate operation methods supported in the EPSG Dataset. Geodetic parameters in the EPSG Dataset are consistent with these formulas.

- IOGP Report 373-07-03 (EPSG Guidance Note 7-3) – *EPSG Registry API user guide*, is primarily intended to assist computer application developers who wish to use the RESTful API of the EPSG Registry to query and retrieve entities and attributes from the Dataset.

- IOGP Report 373-07-04 (EPSG Guidance Note 7-4) – *EPSG Database and SQL Script user guide*, provides guidance for users of the EPSG Database which may be obtained from the EPSG Registry's *Download Dataset* page.

- IOGP Report 373-07-05 (EPSG Guidance Note 7-5) – *EPSG null and copy transformations to WGS 84*, explains aspects of the policies for the inclusion in the EPSG Dataset of datum ensembles and so-called null coordinate transformations to WGS 84, including limitations in the description of WGS 84 through code EPSG:4326.

- IOGP Report 373-07-06 (EPSG Guidance Note 7-6) – *EPSG Dataset - Policies and procedures for data management*, documents strategy for populating the EPSG Dataset.

The complete texts may be found on the EPSG Registry web site under Support Documentation.

This Part 2 of the multi-part Guidance Note is primarily intended to assist computer application developers in using the coordinate operation methods supported by the EPSG Dataset. It may also be useful to other users of the data.

1 Implementation notes

1.1 Ellipsoid parameters

In the formulas in this Guidance Note, the basic ellipsoidal parameters are represented by symbols and derived as follows:

Primary ellipsoid parameters

Parameter Name	Symbol	Description
semi-major axis	a	Length of the semi-major axis of the reference ellipsoid, the radius of the equator.
semi-minor axis	b	Length of the semi-minor axis of the reference ellipsoid, the distance along the ellipsoid axis between equator and pole.
inverse flattening	$1/f$	$= a/(a - b)$. Dimensionless.

Derived ellipsoid parameters

Parameter Name	Symbol	Description
flattening	f	$= 1 / (1/f)$
eccentricity	e	$= \sqrt{(2f - f^2)}$
eccentricity squared	e^2	$= (2f - f^2)$
second eccentricity	e'	$= \sqrt{[e^2 /(1 - e^2)]}$
radius of curvature in the meridian	ρ	radius of curvature of the reference ellipsoid in the plane of the meridian at latitude φ, where $\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{3/2}$
radius of curvature in the prime vertical	ν	radius of curvature of the reference ellipsoid perpendicular to the meridian at latitude φ, where $\nu = a /(1 - e^2\sin^2\varphi)^{1/2}$
radius of authalic sphere	R_A	radius of sphere having same surface area as the ellipsoid. $R_A = a [(1 - \{(1 - e^2) / (2 e)\} \{\ln[(1 - e) / (1 + e)]\}) 0.5]^{1/2}$
radius of conformal sphere	R_C	$= \sqrt{(\rho\ \nu)} = [a\ \sqrt{(1 - e^2)} / (1 - e^2 \sin^2\varphi)]$ This is a function of latitude and therefore not constant. When used for spherical map projections the use of φ_O (or φ_1 as relevant to method) for φ is suggested, except if the map projection is equal area when R_A (see above) should be used.

Ellipsoid units

In the EPSG Dataset the primary ellipsoid parameters are given in the linear unit defined in the information source. Usually, but not always, this is metres. When applied in the map projection formulas given in Section §3 below, these first need to be converted to the same unit as the map projection before use in the formulas. The relevant ellipsoid is obtained through the reference frame (datum) element of the projected coordinate reference system. The map projection linear unit is that of the coordinate system element of the projected coordinate reference system and is also defined in the false easting and northing of the map projection.

1.2 Longitude 'wrap-around'

The formulas in this Guidance Note assume longitudes are described using the range -π to +π radians ($-180 \le \lambda \le +180$ degrees). If the area of interest crosses the 180° meridian and an alternative longitude range convention is being used, longitudes need to be converted to fall into this $-180 \le \lambda \le +180$ degrees range. This may be achieved by applying the following:

If $(\lambda - \lambda_O) \le -180°$ then $\lambda = \lambda + 360°$. This may be required when $\lambda_O > 0°$.
If $(\lambda - \lambda_O) \ge 180°$ then $\lambda = \lambda - 360°$. This may be required when $\lambda_O < 0°$.

In the formulas in this Guidance Note the symbol λ_C or λ_F may be used rather than λ_O, but the same principle applies.

1.3 Arctangent function

Unless there is a note to the contrary, and if the expression for which the arctangent is sought has a numerator over a denominator, the formulas are arranged so that the atan2 function should be used. For atan(y/x), the arctangent is normally returned as an angle between $-\pi/2$ and $+\pi/2$ (between -90° and +90°), whereas the atan2(y,x) output accounts for the quadrant resulting in output between $-\pi$ and $+\pi$ (-180° and +180°).

Readers should note that using the atan2 function the order of the arguments for atan(y/x) varies across different programming languages and tools. This Guidance Note uses the convention atan2(y,x) for atan(y/x).

Conditions not resolved by the atan2 function, but requiring adjustment for almost any programme, are as follows:

1) If x and y are both zero, the arctangent is indeterminate, but may normally be given an arbitrary value of 0, and
2) If x or y is infinite, the arctangent is $\pm\pi/2$ (±90°), the sign depending on other conditions. In any case, the final longitude should be adjusted, if necessary, so that it is an angle between $-\pi$ (or -180°) and $+\pi$ (or +180°). This is done by adding or subtracting multiples of 360° (or 2π) as required (see also longitude 'wrap-around' in Section §1.2 above).

1.4 Angular units

All angles are assumed to be in radians unless otherwise stated.

1.5 Offsets

Several transformation methods which utilise offsets in coordinate values are recognized. The offset methods may be in n-dimensions. These include longitude rotations, geographic CRS ellipsoidal coordinate offsets, Cartesian grid offsets and vertical offsets.

Mathematically, if the origin of a one-dimensional coordinate system is shifted along the positive axis and placed at a point with ordinate A, then the transformation formula is:

$$X_{new} = X_{old} - A$$

However it is common practice in coordinate reference system transformations to apply the shift as an addition, with the sign of the shift parameter value having been suitably reversed to compensate for the practice. This practice has been adopted for the EPSG Dataset. Hence transformations allow calculation of coordinates in the target system by adding a correction parameter to the coordinate values of the point in the source system:

$$X_t = X_s + A$$

where X_s and X_t are the values of the coordinates in the source and target coordinate systems and A is the value of the transformation parameter to transform source coordinate reference system coordinate to target coordinate reference system coordinate.

Offset methods are reversible. For the reverse transformation, the offset parameter value is applied with its sign reversed.

1.6 Coordinate transformation direction

Note that it is very important to ensure that the signs of the parameter values used in the transformations are correct in respect of the transformation being executed. Preferably one should always express transformations in terms of "From".........."To"........... thus avoiding the ambiguity and confusion which may result from interpreting a hyphen or a dash symbol as a minus sign.

1.7 Axis order and axis units associated with affine and polynomial transformations

When considering the relationship of the coordinate system axes of the source and target CRSs there are two categories of formulas:

- formulas where an intrinsic unambiguous relationship exists, e.g. all map projections, longitude rotation, etc.
- formulas where no intrinsic relationship exists, in particular affine and polynomial transformations.

Formulas in the first category are insensitive to axis order and units in input and output coordinates, because software is expected to apply axis swapping and unit conversions automatically as dictated by the base or source CRS and target CRS definitions. For example, in longitude rotations it is obvious from the method formula that the longitude offset is applied to the longitude in the latitude/longitude coordinate tuple. The Transverse Mercator formulas are applied in the map projection conversions within projected CRSs such as ETRS89 / UTM zone 30N (EPSG:25830) with coordinate axis order easting, northing and also in ETRS89 / UTM zone 30N (N-E) (EPSG:3042) with coordinate axis order northing, easting. It is obvious from the method formula that latitude is converted to northing and longitude to easting regardless of the axis order. Similarly in projected CRSs such as NAD83(2011) / Texas South (EPSG:6585) with axes in metres and NAD83(2011) / Texas South (ftUS) (EPSG:6586) with axes in US survey feet, the output coordinate units are dictated by the parameter values, not the method formulas. Most coordinate operation methods and coordinate operations using them fall into this category.

For formulas in the second category axis order and unit matters because (X,Y) means "first and second coordinate" whatever they are, not necessarily "easting, northing" or "northing, easting". Software should not reorder coordinates or convert units before applying the formula. Implementations should apply the affine coefficients (Section §4.7) or polynomial terms (§4.6) without making any assumptions about the physical orientation of coordinate system axes in the source and target CRSs. In these methods any unit conversion factor is embedded in the coefficients. For example, in the affine parametric transformation from Jamaica 1875 / Jamaica (Old Grid) to JAD69 / Jamaica National Grid (EPSG:10087) used as the example in Section §4.7.1, the foot to metre conversion is embedded within the A1 and B2 coefficients. A different unit conversion such as metre to foot in the reverse calculation requires different coefficient values. Similarly, any change in axis order in the source or target CRS requires corresponding change in the operation parameter values in order to be reflected in the output. This can be deduced by examining the differences between the seismic bin grid method matrices described in Section §4.8.

1.8 Examples

Formulas in this document are accompanied by examples in which values of intermediate parameters are given. These intermediate values are not given to full precision; they are not intended to be used as a starting or ending point of a calculation.

2 Coordinate conversions and transformations - Basic concepts

Coordinate conversions and coordinate transformations change coordinate values in one coordinate reference system to coordinate values in another coordinate reference system. A **coordinate system** is a set of mathematical rules for specifying how coordinates are to be assigned to points. It includes the definition of the coordinate axes, the units to be used and the geometry of the axes. A coordinate system is an abstract concept, unrelated to the Earth. A coordinate system is related to the Earth through a **datum**. The combination of coordinate system and datum is a **coordinate reference system** (CRS). If coordinates are referenced to a different datum, their values change. Colloquially the term coordinate system has historically been used to mean coordinate reference system.

Coordinates may be changed from one coordinate reference system to another through the application of a **coordinate operation**. Two types of coordinate operation may be distinguished:

- **Coordinate conversion**, where no change of datum is involved and the parameters are chosen and thus error free.
- **Coordinate transformation**, where the target CRS is based on a different datum to the source CRS. Transformation parameters are empirically determined and thus subject to measurement errors.

Certain coordinate operation methods do not readily fit the ISO 19111 classification of being either a coordinate conversion (no change of datum involved) or a coordinate transformation. These methods (for example polynomials and affine transformations) change coordinates directly from one coordinate reference system to another and may be applied with or without change of datum, depending upon whether the source and target coordinate reference systems are based on the same or different datums. EPSG follows the general mathematical usage of these methods and classifies them as transformations. In practice, most usage of these methods does in fact include a change of datum.

Coordinates may also change within a coordinate reference system due to plate motion or other tectonic activity. **Plate motion operation**s describe this change of coordinate value with time. It is emphasized that this change is within a coordinate reference system, in contrast with coordinate conversions and coordinate transformations, which are between coordinate reference systems. However, for convenience of using the same data model, point motion operations are modelled as if they were a type of coordinate operation.

Geographic coordinates (latitude and longitude) are calculated on a model of the Earth. The definition of the model is part of a geodetic datum definition.

A projected coordinate reference system is the result of the application of a **map projection** to a geographic coordinate reference system. A map projection is a type of coordinate conversion. It uses an identified method with specific formulas and a set of parameters specific to that coordinate conversion method. Map projection formulas are given in Section §3 of this Guidance Note.

Some transformation methods operate directly between geographic coordinates. Others are between geocentric coordinates (3-dimensional Cartesian coordinates where the coordinate system origin is at the centre of the earth). Section §4 of this Guidance Note covers conversions and transformations between geographic coordinate reference systems, both directly and indirectly through geocentric systems. This last part also describes transformations of vertical coordinates.

Coordinate handling software may execute more complicated operations, concatenating a number of steps linking together geographic, projected and/or engineering coordinates referenced to different datums. Other than as mentioned above, these concatenated operations are beyond the scope of this document.

3 Map projections

3.1 Map projection methods

Setting aside the large number of map projection methods which may be employed for atlas maps, equally small-scale illustrative exploration maps, and wall maps of the world or continental areas, the EPSG Dataset provides reference parameter values for orthomorphic or conformal map projections which are used for medium or large scale topographic or exploration mapping. Here accurate positions are important and sometimes users may wish to scale accurate positions, distances or areas from the maps.

Small scale maps normally assume a spherical model of the Earth and the inaccuracies inherent in this assumption are of no consequence at the usual scale of these maps. For medium- and large-scale sheet maps, and for coordinates held digitally to a high accuracy, it is essential that due regard is paid to the actual shape of the Earth. Such coordinate reference systems are therefore invariably based on an ellipsoid and its derived map projections. The EPSG Dataset and this supporting conversion documentation usually considers only map projection methods for the ellipsoid. Projection formulas for the sphere are simpler but the spherical figure is inadequate to represent positional data with great accuracy at large map scales for the real Earth. Projections of the sphere are only suitable for illustrative maps at scale of 1:1 million or less where precise positional definition is not critical.

Though not exhaustive, the following list of named map projection methods are those which are most frequently encountered for medium and large-scale mapping, some of them much less frequently than others since they are designed to serve only one particular country. They are grouped according to their possession of similar construction properties. Except where indicated all are conformal.

Lambert Conical Conformal	Conical
with one standard parallel	
with two standard parallels	
Mercator	Cylindrical
with one standard parallel	
with two standard parallels	
Cassini-Soldner (note: not conformal)	Transverse Cylindrical
Transverse Mercator Group	Transverse Cylindrical
Transverse Mercator (including south oriented version)	
Universal Transverse Mercator	
Gauss-Kruger	
Gauss-Boaga	
Oblique Mercator Group	Oblique Cylindrical
Hotine Oblique Mercator	
Laborde Oblique Mercator	
Stereographic	Azimuthal
Polar	
Oblique and equatorial	

The map projection formulas which follow are largely adapted from Snyder (1987). As well as providing an extensive overview of most map projections in current general use, and the formulas for their construction for both the spherical and ellipsoidal earth, this excellent publication provides computational hints and details of the accuracies attainable by the formulas. It is strongly recommended that all those who have to deal with map projections for medium- and large-scale mapping should follow its guidance.

There are a number of different formulas available in the literature for map projections other than those quoted by Snyder. Some are closed formulas; others, for ease of calculation, may depend on series expansions and their precision will generally depend on the number of terms used for computation. Generally, those formulas which follow in this Section will provide results which are accurate to within a few centimetres, which is normally adequate for exploration mapping purposes. Coordinate expression and computations for engineering operations are usually performed in grid terms.

3.1.1 **Reversibility**

Different formulas are required for forward and reverse map projection conversions: the forward formula cannot be used for the reverse conversion. However both forward and reverse formulas are explicitly given in the sections below as parts of a single conversion method. As such, map projection methods are described in the EPSG Dataset as being reversible. Forward and reverse formulas for each conversion method use the map projection parameters appropriate to that method with parameter values unchanged.

3.2 Map projection parameters

A map projection grid is related to the geographical graticule of an ellipsoid through the definition of a coordinate conversion method and a set of parameters appropriate to that method. Different conversion methods may require different parameters. Any one coordinate conversion method may take several different sets of associated parameter values, each set related to a particular map projection zone applying to a particular country or area of the world. Before setting out the formulas involving these parameters, which enable the coordinate conversions for the projection methods listed above, it is as well to understand the nature of the parameters.

The plane of the map and the ellipsoid surface may be assumed to have one particular point in common. This point is referred to as the **natural origin**. It is the point from which the values of both the geographic coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively it may be considered as the point which, in the absence of application of false coordinates, has grid coordinates of (0,0). For example, for projected coordinate reference systems using the Cassini-Soldner or Transverse Mercator methods, the natural origin is at the intersection of a chosen parallel and a chosen meridian. The chosen parallel will frequently, but not necessarily, be the equator. The chosen meridian will usually be central to the mapped area. For the Stereographic projection, the origin is at the centre of the projection where the plane of the map is imagined to be tangential to the ellipsoid.

Since the natural origin may be at or near the centre of the projection and under normal coordinate circumstances would thus give rise to negative coordinates over parts of the map, this origin is usually given false coordinates which are large enough to avoid this inconvenience. Hence each natural origin will normally have **False Easting, FE**, and **False Northing, FN** values**.** For example, the false easting for the origins of all Universal Transverse Mercator zones is 500000m. As the UTM origin lies on the equator, areas north of the equator do not need and are not given a false northing but for mapping southern hemisphere areas the equator origin is given a false northing of 10,000,000m, thus ensuring that no point in the southern hemisphere will take a negative northing coordinate. Figure 8 illustrates the UTM arrangements.

These arrangements suggest that if there are false easting and false northing for the real or natural origin, there is also a **Grid Origin** which has coordinates (0,0). In general, this point is of no consequence though its geographic position may be computed if needed. For example, for the WGS 84 / UTM zone 31N coordinate reference system which has a natural origin at 0°N, 3°E where false easting is 500000m E (and false northing is 0.0m N), the grid origin is at 0°N, 1°29'19.478"W. Sometimes, however, rather than base the easting and northing coordinate reference system on the natural origin by giving it **FE** and **FN** values, it may be convenient to select a **False Origin** at a specific meridian/parallel intersection and attribute the false coordinates **Easting at False Origin, E_F** and **Northing at False Origin, N_F** to this. The related easting and northing of the natural origin may then be computed, if required.

Longitudes are most commonly expressed relative to the **Prime Meridian** of Greenwich but some countries, particularly in former times, have preferred to relate their longitudes to a prime meridian through their national astronomical observatory, usually sited in or near their capital city, for example, Paris for France, Bogota for Colombia. The meridian of the projection zone origin is known as the **Longitude of Origin**. For certain projection types it is often termed the **Central Meridian** or abbreviated as **CM** and provides the direction of the northing axis of the projected coordinate reference system**.**

Because of the steadily increasing distortion in the scale of the map with increasing distance from the origin, central meridian or other line on which the scale is the nominal scale of the projection, it is usual to limit the extent of a projection to within a few degrees of latitude or longitude of this point or line. Thus, for example, a UTM or other Transverse Mercator projection zone will normally extend only 2 or 3 degrees from the central meridian. Beyond this area another **zone** of the projection, with a new origin and central meridian, needs to be used or created. The UTM system has a specified 60 numbered zones, each 6 degrees wide, covering the ellipsoid between the 84 degree North and 80 degree South latitude parallels. Other Transverse Mercator projection zones may be constructed with different central meridians, and different origins chosen to suit the countries or states for which they are used. A number of these are included in the EPSG Dataset. Similarly a Lambert Conic Conformal zone distorts most rapidly in the north-south direction and may, as in Texas, be divided into latitudinal bands.

In order to further limit the scale distortion within the coverage of the zone or projection area, some projections introduce a **scale factor** at the origin (on the central meridian for Transverse Mercator projections), which has the effect of reducing the nominal scale of the map here and making it have the nominal scale some distance away. For example, in the case of the UTM and some other Transverse Mercator projections a scale factor of slightly less than unity is introduced on the central meridian thus making it unity on two roughly north-south lines either side of the central one, and reducing its departure from unity beyond these. The scale factor is a required parameter whether or not it is unity and is usually symbolised as $\mathbf{k_O}$.

Thus, for projections in the Transverse Mercator group the parameters which are required to completely and unambiguously define the projection method are:

Latitude of natural origin (φ_O)
Longitude of natural origin (the central meridian) (λ_O)
Scale factor at natural origin (on the central meridian) (k_O)
False easting (FE)
False northing (FN)

This set of parameters may also used by other projection methods. However, for some of these other methods, including the Lambert Conic Conformal, the projection may be defined using alternative parameters. It is possible to replace the latitude of natural origin and scale factor by instead giving the latitude of the two parallels along which scale is true, the 'standard parallels'. This results in the following set of parameters, which we call variant B:

Latitude of first standard parallel (φ_1)
Latitude of second standard parallel (φ_2)
Longitude of natural origin (the central meridian) (λ_O)
False easting (FE)
False northing (FN)

For the Lambert Conic Conformal projection on the ellipsoid, the single parallel at which scale is a minimum – the latitude of natural origin – is slightly poleward of midway between the two standard parallels. It is usual to choose values which are round numbers for the two standard parallels, which results in the latitude of natural origin having an irregular value. It is then more convenient to define the false grid coordinates at some other point, itself having round number values. This results in the following set of parameters (variant C):

Latitude of first standard parallel (φ_1)
Latitude of second standard parallel (φ_2)
Latitude of false origin (φ_F)
Longitude of false origin (λ_F)
False easting (E_F)
False northing (N_F)

The limiting cases of the conic aspect are a cylinder tangential at the equator and a plane tangential at a pole, but the conic formulas should not be used for these cases as some trigonometrical functions break down. Instead, different formulas or methods exist. For the Lambert Conic Conformal these are the Mercator and Polar Stereographic methods respectively. For the Mercator projection the two standard parallels have the same latitude in opposite hemispheres. It is therefore only necessary to have one of them as a defining parameter. Similarly, in the other limiting case, the Polar Stereographic projection, there is only one standard parallel. In these projection methods, for variant B the set of parameters reduces to:

Latitude of standard parallel (φ_1)
Longitude of natural origin (the central meridian) (λ_O)
False easting (FE)
False northing (FN)

with a similar reduction for variant C.

Variants A and B are alternative ways of defining one projection. Regardless of whether variant A or variant B is used for the projection definition, the conversion of geographic coordinates will produce the same grid coordinates. However, in the case of variant C, the conversion of the same geographic coordinates will produce different grid coordinates to those of variants A and B, except in the special and unusual case when the latitude of the false origin is chosen to exactly coincide with the natural origin.

It is EPSG Dataset policy to document the parameters and their values for these variants as they are defined by the information source. To accomplish this, the EPSG Dataset recognises the variants as different methods, each with their own name and formulas:

General Method	Variant A	Variant B	Variant C
Lambert Conic Conformal	Lambert Conic Conformal (1SP)	*(not used)*	Lambert Conic Conformal (2SP)
Mercator	Mercator (variant A) alias: Mercator (1SP)	Mercator (variant B)	Mercator (variant C)
Polar Stereographic	Polar Stereographic (variant A)	Polar Stereographic (variant B)	Polar Stereographic (variant C)

This concept is continued with some other projection methods in which variants are found, but in these the relationship between the alternative sets of parameters may be less straightforward:

General Method	Variant A	Variant B	Variant C
Oblique Mercator	Hotine Oblique Mercator (variant A) aliases: Hotine Oblique Mercator, Rectified Skew Orthomorphic	Hotine Oblique Mercator (variant B) aliases: Oblique Mercator, Rectified Skew Orthomorphic	Laborde Oblique Mercator alias: Laborde Madagascar

For the Oblique Mercator it is also possible to define the azimuth of the initial line through the latitude and longitude of two widely spaced points along that line. This approach is not included in the EPSG Dataset.

Tables 1 and 2 give the definitions of projection parameters used in the projection formulas which follow. For Ellipsoid parameters see the general Implementation Notes in Section §1.1.

Table 1 – Parameters used in map projection conversions.

Parameter Name	Symbol	Description
Angle from Rectified to Skew Grid	γ_C	The angle at the natural origin of an oblique projection through which the natural coordinate reference system is rotated to make the projection north axis parallel with true north.
Azimuth of initial line	α_C	The azimuthal direction (north zero, east of north being positive) of the great circle which is the centre line of an oblique projection. The azimuth is given at the projection center.
Central meridian		See Longitude of natural origin.
Easting at false origin	E_F	The easting value assigned to the false origin.
Easting at projection centre	E_C	The easting value assigned to the projection centre.
False easting	FE	The value assigned to the abscissa (east or west) axis of the projection grid at the natural origin.
False northing	FN	The value assigned to the ordinate (north or south) axis of the projection grid at the natural origin.
False origin		A specific parallel/meridian intersection other than the natural origin to which the grid coordinates E_F and N_F, are assigned.
Grid origin		The point which has coordinates (0,0). It is offset from the natural origin by the false easting and false northing. In some projection methods, it may alternatively be offset from the false origin by Easting at false origin and Northing at false origin. In general, grid origin is of no consequence.
Initial line		The line on the surface of the model of the Earth which forms the axis for the grid of an oblique projection.
Initial longitude	λ_I	The longitude of the western limit of the first zone of a Transverse Mercator zoned grid system.
Latitude of 1st standard parallel	φ_1	For a conic projection with two standard parallels, this is the latitude of one of the parallels at which the cone intersects with the ellipsoid. It is normally but not necessarily that nearest to the equator. Scale is true along this parallel.
Latitude of 2nd standard parallel	φ_2	For a conic projection with two standard parallels, this is the latitude of one of the parallels at which the cone intersects with the ellipsoid. It is normally but not necessarily that nearest to the pole. Scale is true along this parallel.

Parameter Name	**Symbol**	**Description**
Latitude of false origin	φ_F	The latitude of the point which is not the natural origin and at which grid coordinate values easting at false origin and northing at false origin are defined.
Latitude of natural origin	φ_O	The latitude of the point from which the values of both the geographic coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively, it may be considered as the latitude of the point which in the absence of application of false coordinates has grid coordinates of (0,0).
Latitude of projection centre	φ_C	For an oblique projection, this is the latitude of the point at which the azimuth of the initial line is defined.
Latitude of pseudo standard parallel	φ_P	Latitude of the parallel on which the conic or cylindrical projection is based. This latitude is not geographic, but is defined on the conformal sphere AFTER its rotation to obtain the oblique aspect of the projection.
Latitude of standard parallel	φ_F	For polar aspect azimuthal projections, the parallel on which the scale factor is defined to be unity.
Longitude of false origin	λ_F	The longitude of the point which is not the natural origin and at which grid coordinate values easting at false origin and northing at false origin are defined.
Longitude of natural origin	λ_O	The longitude of the point from which the values of both the geographic coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively, it may be considered as the longitude of the point which in the absence of application of false coordinates has grid coordinates of (0,0). Sometimes known as "central meridian (CM)".
Longitude of origin	λ_O	For polar aspect azimuthal projections, the meridian along which the northing axis increments and also across which parallels of latitude increment towards the north pole.
Longitude of projection centre	λ_C	For an oblique projection, this is the longitude of the point at which the azimuth of the initial line is defined.
Natural origin		The point from which the values of both the geographic coordinates on the ellipsoid and the grid coordinates on the projection are deemed to increment or decrement for computational purposes. Alternatively, it may be considered as the point which in the absence of application of false coordinates has grid coordinates of (0,0). For example, for projected coordinate reference systems using the Transverse Mercator method, the natural origin is at the intersection of a chosen parallel and a chosen central meridian.
Northing at false origin	N_F	The northing value assigned to the false origin.
Northing at projection centre	N_C	The northing value assigned to the projection centre.
Origin		See natural origin, false origin and grid origin.
Projection centre		On an oblique cylindrical or conical projection, the point at which the direction of the cylinder or cone and false coordinates are defined.
Scale factor at natural origin	k_O	The factor by which the map grid is reduced or enlarged during the projection process, defined by its value at the natural origin.
Scale factor on initial line	k_C	The factor by which an oblique projection's map grid is reduced or enlarged during the projection process, defined by its value along the centre line of the cylinder or cone.

Parameter Name	**Symbol**	**Description**
Scale factor on pseudo standard parallel	k_P	The factor by which the map grid is reduced or enlarged during the projection process, defined by its value at the pseudo-standard parallel.
Zone width	W	The longitude width of a zone of a Transverse Mercator zoned grid system.

Table 2 – Summary of Coordinate Operation Parameters required for selected Map Projection methods.

Coordinate Operation Parameter Name	**Coordinate Operation Method**									
	Mercator (variant A)	Mercator (variant B)	Cassini-Soldner	Transverse Mercator	Hotine Oblique Mercator (variant A)	Hotine Oblique Mercator (variant B)	Lambert Conic Conformal (1 SP variant A)	Lambert Conic Conformal (1 SP variant B)	Lambert Conic Conformal (2 SP)	Oblique Stereo-graphic
Latitude of natural origin	1 (=equator)		1	1			1	1		1
Longitude of natural origin	2	2	2	2			2			2
Scale factor at natural origin	3			3			3	2		3
False easting	4	3	3	4	6		4			4
False northing	5	4	4	5	7		5			5
Latitude of false origin								3	1	
Longitude of false origin								4	2	
Latitude of 1st standard parallel		1							3	
Latitude of 2nd standard parallel									4	
Easting at false origin								5	5	
Northing at false origin								6	6	
Latitude of projection centre					1	1				
Longitude of projection centre					2	2				
Scale factor on initial line					3	3				
Azimuth of initial line					4	4				
Angle from Rectified to Skewed grid					5	5				
Easting at projection centre						6				
Northing at projection centre						7				

3.3 Map Projections supporting projected 3D coordinate reference systems

3D map projections support 3-dimensional projected coordinate reference systems. These are standard horizontal 2D map projection grids with a third, vertical, axis of ellipsoidal height. Latitude, longitude and ellipsoidal height are converted to and from easting, northing and ellipsoidal height. The conversion of latitude and longitude to and from easting and northing follows the standard 2D map projection as described in the following sections; the ellipsoidal height (h) is carried across unchanged.

Because the tangential map plane is coincident with the ellipsoid only at the projection origin and elsewhere is above the ellipsoid surface, the ellipsoid height differs from a height measured from the map plane. For a latitude or longitude offset from the projection origin by 3°, the difference exceeds 8km. In practice, the height above the projection plane is not used.

The ellipsoid curvature also means that the height is not exactly perpendicular to the map grid plane, so the coordinate system is not Cartesian but instead is affine. However, in practice it is treated as a Cartesian coordinate system with orthogonal axes.

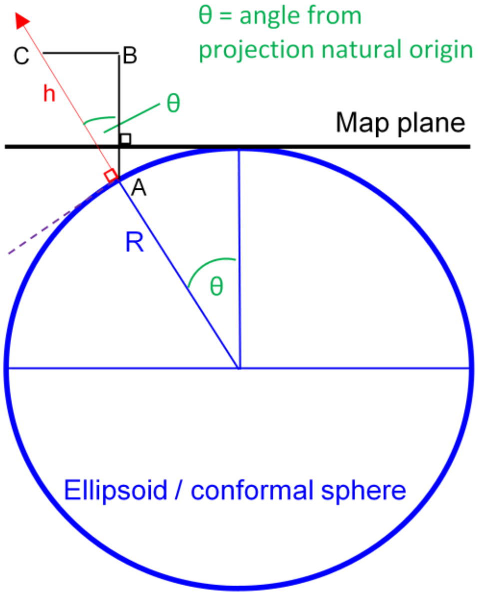


Figure 1 – Orthogonality of ellipsoid height and map plane.

A small error in relative height is introduced by assuming that ellipsoid height is perpendicular to the map plane. Referring to Figure 1, the difference between ellipsoidal height (AC) and the distance perpendicular to the map plane (AB) is proportional to the height difference. It is under 1 centimetre per 100m height difference for a point's latitude or longitude offset from the projection origin by less than 0.5 degrees, but 3 degrees from the origin it is over 1 decimetre per 100m height difference - see Table 3.

Table 3 – Error in relative height introduced through assumption of a Cartesian 3D coordinate system.

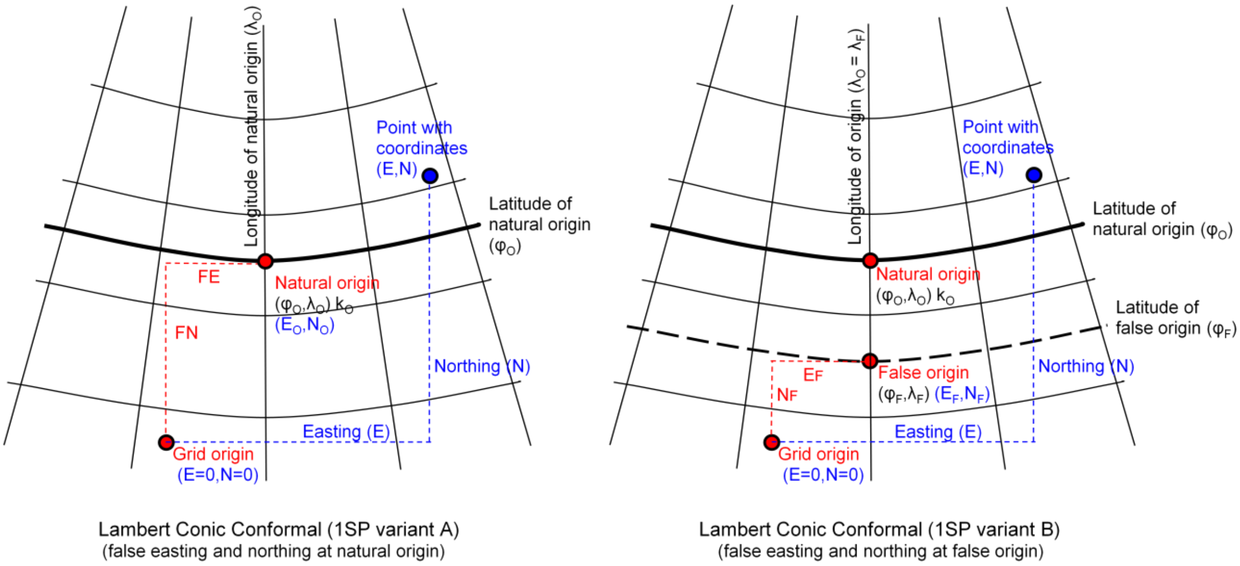
Angular distance from projection origin (θ deg)	0°	0.1°	0.2°	0.3°	0.4°	0.5°	1°	2°	3°
Height difference AC-AB per 100m of height	0.000m	0.000m	0.001m	0.001m	0.002m	0.004m	0.015m	0.061m	0.137m

3.4 Conic Map Projections

3.4.1 Lambert Conic Conformal

The Lambert Conic Conformal may often be adopted for territories with limited latitudinal extent but wide longitudinal width. But if the latitudinal extent is also large there may still be a need to use two or more zones if the scale distortion at the extremities of the one zone becomes too large to be tolerable.

Conical projections with one standard parallel are normally considered to maintain the nominal map scale along the parallel of latitude which is the line of contact between the imagined cone and the ellipsoid. For a one standard parallel Lambert the natural origin of the projected coordinate reference system is the intersection of the standard parallel with the longitude of origin (central meridian) (see Figure 2a below). To maintain the conformal property the spacing of the parallels is variable and increases with increasing distance from the standard parallel, while the meridians are all straight lines radiating from a point on the prolongation of the ellipsoid's minor axis. False grid coordinates FE and FN may be assigned to the natural origin to ensure that coordinates remain positive to the south and west of the origin.



Figures 2a (left) and 2b (right) – Lambert Conic Conformal projection (1SP).

A variation of these arrangements is for false coordinates E_F and N_F to be assigned to some other point on the longitude of origin (central meridian), the false origin, as shown in Figure 2b.

Although a one standard parallel Lambert is normally considered to have unity scale factor on the standard parallel, a scale factor of slightly less than unity is sometimes introduced on this parallel. This is a regular feature of the mapping of some former French territories and has the effect of making the scale factor unity on two other parallels either side of the standard parallel. The projection thus, strictly speaking, becomes a Lambert Conic Conformal projection with **two** standard parallels. From the one standard parallel and its scale factor it is possible to derive the equivalent two standard parallels and then treat the projection as a two standard parallel Lambert conical conformal, but this procedure is seldom adopted since the two parallels obtained in this way will generally not have integer values of degrees or degrees minutes and seconds. It is instead usually preferred to select two specific parallels with latitude in 'round numbers' on which the scale factor is to be unity, as for several State Plane Coordinate systems in the United States (see Figure 3).

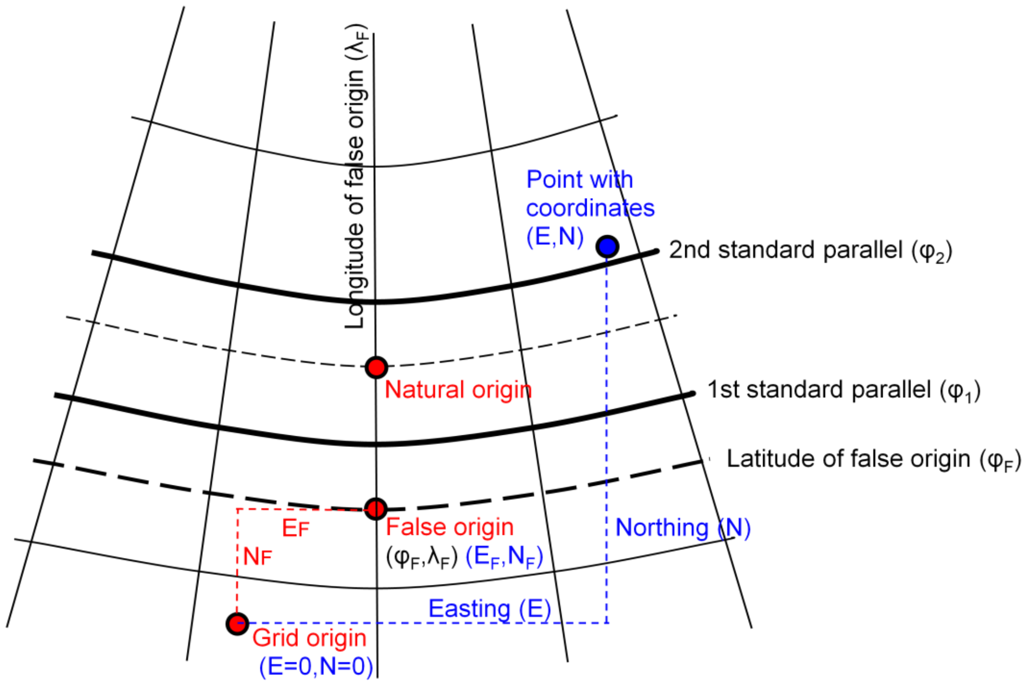


Figure 3 – Lambert Conic Conformal projection (2SP).

The choice of the two standard parallels will usually be made according to the latitudinal extent of the area which it is wished to map, the parallels usually being chosen so that they each lie a proportion inboard of the north and south margins of the mapped area. Various schemes and formulas have been developed to make this selection such that the maximum scale distortion within the mapped area is minimised, but regardless of which two standard parallels are adopted, the formulas are the same.

EPSG recognises three variants of the Lambert Conic Conformal. Parameters for these are shown in Table 4.

Table 4 – Coordinate Operation Parameters for Lambert Conic Conformal methods.

	Method		
	1 SP		2 SP
	Variant A	Variant B	
EPSG Dataset coordinate operation method code:	9801	1102	9802
Parameter			
Latitude of natural origin (φ_O)	x	x	
Longitude of natural origin (λ_O)	x		
Scale factor at natural origin (k_O)	x	x	
False easting (easting at natural origin) (FE)	x		
False northing (northing at natural origin) (FN)	x		
Latitude of false origin (φ_F)		x	x
Longitude of false origin (λ_F) (note 1)		x	x
Latitude of 1st standard parallel			x
Latitude of 2nd standard parallel			x
Easting at false origin (E_F)		x	x
Northing at false origin (N_F)		x	x

Note: In both of the LCC 1SP variant B and LCC 2SP methods, the false origin is on the same meridian as that passing through the natural origin, i.e., $\lambda_F = \lambda_O$.

These variants are similar to the Mercator and Polar Stereographic variants. The Mercator and Polar Stereographic methods may be considered logically as end-member cases of the Lambert Conic Conformal, where the apex of the cone is at infinity or a pole respectively, but the Lambert formulas should not be used for these cases as some trigonometrical functions break down.

3.4.1.1 Lambert Conic Conformal (2SP)

(EPSG Dataset coordinate operation method code 9802)

To derive the projected Easting and Northing coordinates of a point with geographic coordinates (φ,λ), the formulas for the Lambert Conic Conformal **two standard parallel case** are:

Easting, $E = E_F + r \sin\theta$
Northing, $N = N_F + r_F - r \cos\theta$

where $m = \cos\varphi/(1 - e^2\sin^2\varphi)^{0.5}$ for m_1, φ_1, and m_2, φ_2 where φ_1 and φ_2 are the latitudes of the two standard parallels
$t = \tan(\pi/4 - \varphi/2)/[(1 - e \sin\varphi)/(1 + e \sin\varphi)]^{e/2}$ for t_1, t_2, t_F and t using φ_1, φ_2, φ_F and φ respectively
$n = (\ln m_1 - \ln m_2)/(\ln t_1 - \ln t_2)$
$F = m_1/(n t_1^{\,n})$
$r = a F t^n$ for r_F and r, using t_F and t respectively, where r_F is the radius of the parallel of latitude of the false origin
$\theta = n(\lambda - \lambda_F)$
As with other conics, a negative n and r result for projections centered in the Southern Hemisphere.

The reverse formulas to derive the latitude and longitude of a point from its Easting and Northing values are:
$\varphi = \pi/2 - 2\operatorname{atan}\{t'[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{e/2}\}$
$\lambda = \theta'/n + \lambda_F$
where
n, F, and r_F are derived as for the forward calculation;
$r' = \pm\{(E - E_F)^2 + [r_F - (N - N_F)]^2\}^{0.5}$, taking the sign of n
$t' = (r'/(aF))^{1/n}$
If n is positive
$\theta' = \operatorname{atan2}\{(E - E_F), [r_F - (N - N_F)]\}$
but, if n is negative, the signs of both arguments of the atan2 function must be reversed
$\theta' = \operatorname{atan2}\{-(E - E_F), -[r_F - (N - N_F)]\}$

Note that the formula for φ requires iteration. First calculate t' and then a trial value for φ using $\varphi = \pi/2 - 2\operatorname{atan}(t')$. Then use the full equation for φ substituting the trial value into the right hand side of the equation to derive a new value for φ. Iterate the process until φ does not change significantly. The solution should quickly converge, in 3 or 4 iterations.

Example 1 (Northern Hemisphere)
For Projected Coordinate Reference System: NAD27 / Texas South Central (EPSG CRS code 32040)

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.4 metres	b = 6356583.8 metres
	from which:	a = 20925832.164 US survey feet	
		1/f =294.9786982 e = 0.08227185	e^2 =0.00676866

Latitude of false origin	φ_F	27°50'00"N	
Longitude of false origin	λ_F	99°00'00"W	
Latitude of 1st standard parallel	φ_1	28°23'00"N	
Latitude of 2nd standard parallel	φ_2	30°17'00"N	
Easting at false origin	E_F	2000000.00	US survey feet
Northing at false origin	N_F	0.00	US survey feet

Forward calculation for:

Latitude φ = 28°30'00.00"N Longitude λ = 96°00'00.00"W

first gives:

φ_F	=	0.48578331 rad	λ_F	=	-1.72787596 rad
φ_1	=	0.49538262 rad	φ_2	=	0.52854388 rad
m_1	=	0.88046050	m_2	=	0.86428642
t_1	=	0.59823957	t_2	=	0.57602212
t_F	=	0.60475101	n	=	0.48991263
F	=	2.31154807	r_F	=	37807441.20

then:

φ	=	0.49741884 rad	λ	=	-1.67551608 rad
t	=	0.59686306	r	=	37565039.86
θ	=	0.02565177			

Then Easting E = 2963503.91 US survey feet
Northing N = 254759.80 US survey feet

Reverse calculation for same easting and northing first gives:

θ' = 0.025651765
t' = 0.59686306
r' = 37565039.86

Then Latitude φ = 28°30'00.000"N
Longitude λ = 96°00'00.000"W

Example 2 (Southern Hemisphere)

For Projected Coordinate Reference System: AGD66 / Vicgrid66 (EPSG CRS code 3110)

Parameters:

Ellipsoid: Australian National Spheroid a = 6378160.0 metres 1/f = 298.25
from which: e = 0.081820179996 e^2 = 0.00669454185

Latitude of false origin	φ_F	37°00'00.000"S	
Longitude of false origin	λ_F	145°00'00.000"E	
Latitude of 1st standard parallel	φ_1	36°00'00.000"S	
Latitude of 2nd standard parallel	φ_2	38°00'00.000"S	
Easting at false origin	E_F	2500000.000	metres
Northing at false origin	N_F	4500000.000	metres

Forward calculation for:

Latitude φ = 37°45'00.000"S Longitude λ = 144°45'00.000"E

first gives:

φ_F	=	-0.64577182 rad	λ_F	=	2.530727415 rad
φ_1	=	-0.62831853 rad	φ_2	=	-0.66322512 rad
m_1	=	0.809954211	m_2	=	0.789012446
t_1	=	1.954896966	t_2	=	2.0418636298
t_F	=	1.997618776	n	=	-0.6018461050
F	=	-2.0145905003	r_F	=	-8472661.322

then:

φ	=	-0.658861793 rad	λ	=	2.5438173848 rad
t	=	2.030654821	r	=	-8389432.783
θ	=	0.002626049			

Then Easting E = 2477968.963 metres
Northing N = 4416742.535 metres

Reverse calculation for same easting and northing first gives:

(Since n is negative, we reverse the signs of both ATAN2 arguments to calculate θ')

θ' = 0.002626049
t' = 2.030654821
r' = -8389432.783

Then Latitude φ = 37°45'00.000"S
Longitude λ = 144°45'00.000"E

Further examples of input and output may be found in test procedure 5103 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.4.1.2 <u>Lambert Conic Conformal (1SP)</u> [Also known as Lambert Conic Conformal (1SP variant A)] (EPSG Dataset coordinate operation method code 9801)

For the Lambert Conic Conformal single standard parallel variant A, the formulas for the Lambert Conic Conformal (2SP) variant can be used with minor modifications:

$E = FE + r \sin\theta$

$N = FN + r_O - r \cos\theta$, using the natural origin rather than the false origin.

where

$m_O = \cos\varphi_O/(1 - e^2\sin^2\varphi_O)^{0.5}$ where φ_O is the latitude of natural origin

$t_O = \tan(\pi/4 - \varphi_O/2)/[(1 - e\sin\varphi_O)/(1 + e\sin\varphi_O)]^{e/2}$

$t = \tan(\pi/4 - \varphi/2)/[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{e/2}$

$n = \sin\varphi_O$

$F = m_O/(n t_O^{\,n})$

$r_O = a F t_O^{\,n} k_O$

$r = a F t^n k_O$

$\theta = n(\lambda - \lambda_O)$

As with other conics, a negative n and r result for projections centered in the Southern Hemisphere.

The reverse formulas to derive the latitude and longitude of a point from its Easting and Northing values are:

$\varphi = \pi/2 - 2\operatorname{atan}\{t'[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{e/2}\}$

$\lambda = \theta'/n + \lambda_O$

where

n, F, and r_O are derived as for the forward calculation;

$r' = \pm\{(E - FE)^2 + [r_O - (N - FN)]^2\}^{0.5}$, taking the sign of n

$t' = (r'/(a\,k_O\,F))^{1/n}$

If n is positive,

$\theta' = \operatorname{atan2}\{(E - FE), [r_O - (N - FN)]\}$

but if n is negative the signs of both arguments of the atan2 function must be reversed

$\theta' = \operatorname{atan2}\{-(E - FE), -[r_O - (N - FN)]\}$

Note that the formula for φ requires iteration. First calculate t' and then a trial value for φ using $\varphi = \pi/2 - 2\operatorname{atan}(t')$. Then use the full equation for φ substituting the trial value into the right hand side of the equation. Thus derive a new value for φ. Iterate the process until φ does not change significantly. The solution should quickly converge, in 3 or 4 iterations.

Example

For Projected Coordinate Reference System: JAD69 / Jamaica National Grid (EPSG CRS code 24200)

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.4 metres	b = 6356583.8 metres	
	from which:	1/f = 294.9786982	e = 0.08227185	e^2 = 0.00676866

Latitude of natural origin	φ_O	18°00'00"N
Longitude of natural origin	λ_O	77°00'00"W
Scale factor at natural origin	k_O	1.000000
False easting	FE	250000.00 metres
False northing	FN	150000.00 metres

Forward calculation for:

Latitude φ = 17°55'55.80"N Longitude λ = 76°56'37.26"W

first gives:

φ_O	= 0.31415927 rad	λ_O	= -1.34390352 rad				
m_O	= 0.95136402	t_O	= 0.72806411				
n	= 0.30901699	F	= 3.39591092	r_O	= 19636447.86		

then:

φ	=	0.31297535 rad	λ	=	-1.34292061 rad
t	=	0.728965259	r	=	19643955.26
θ	=	0.00030374			

Then	Easting	E	=	255966.58 metres
	Northing	N	=	142493.51 metres

Reverse calculation for the same easting and northing first gives

θ'	=	0.000303736
t'	=	0.728965259
r'	=	19643955.26

Then	Latitude	φ	=	17°55'55.80"N
	Longitude	λ	=	76°56'37.26"W

Further examples of input and output may be found in test procedure 5102 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.4.1.3 Lambert Conic Conformal (1SP variant B)

(EPSG Dataset coordinate operation method code 1102)

For the Lambert Conic Conformal single standard parallel variant B, as with variant A, the formulas for the Lambert Conic Conformal (2SP) variant can be used with minor modifications:

Easting, $E = E_F + r \sin\theta$

Northing, $N = N_F + r_F - r \cos\theta$

where

$m_O = \cos\varphi_O/(1 - e^2\sin^2\varphi_O)^{0.5}$ where φ_O is the latitude of natural origin

$t_O = \tan(\pi/4 - \varphi_O/2)/[(1 - e \sin\varphi_O)/(1 + e \sin\varphi_O)]^{e/2}$

$t_F = \tan(\pi/4 - \varphi_F/2)/[(1 - e \sin\varphi_F)/(1 + e \sin\varphi_F)]^{e/2}$

$t = \tan(\pi/4 - \varphi/2)/[(1 - e \sin\varphi)/(1 + e \sin\varphi)]^{e/2}$

$n = \sin \varphi_O$

$F = m_O/(n t_O^{\,n})$

$r_F = a F t_F^{\,n} k_O$

$r = a F t^n k_O$

$\lambda_O = \lambda_F$

$\theta = n(\lambda - \lambda_O)$

As with other conics, a negative n and r result for projections centered in the Southern Hemisphere.

The reverse formulas to derive the latitude and longitude of a point from its Easting and Northing values are:

$\varphi = \pi/2 - 2\mathrm{atan}\{t'[(1 - e \sin\varphi)/(1 + e \sin\varphi)]^{e/2}\}$

$\lambda = \theta'/n + \lambda_O$

where

n, F, r_F and λ_O are derived as for the forward calculation;

$r' = \pm\{(E - E_F)^2 + [r_F - (N - N_F)]^2\}^{0.5}$, taking the sign of n

$t' = (r'/(a k_O F))^{1/n}$

If n is positive,

$\theta' = \mathrm{atan2}\{(E - E_F), [r_F - (N - N_F)]\}$

but if n is negative the signs of both arguments of the atan2 function must be reversed

$\theta' = \mathrm{atan2}\{-(E - E_F), -[r_F - (N - N_F)]\}$

Note that the formula for φ requires iteration. First calculate t' and then a trial value for φ using $\varphi = \pi/2 - 2\operatorname{atan}(t')$. Then use the full equation for φ substituting the trial value into the right hand side of the equation to derive a new value for φ. Iterate the process until φ does not change significantly. The solution should quickly converge, in 3 or 4 iterations.

Example
For Projected Coordinate Reference System: Lyon-Turin Ferroviaire 2004 (EPSG CRS code 9549)
(note: area of applicability of CRS is extended for this example.)

Parameters:

Ellipsoid:	GRS 1980	a = 6378137.0 metres	1/f = 298.257222101
	from which:	e = 0.081819191	e^2 = 0.006694380

Latitude of natural origin	φ_O	44°22'45"N	
Scale factor at natural origin	k_O	1.000000	
Latitude of false origin	φ_F	45°11'00"N	
Longitude of false origin	λ_F	6°49'00"E	
Easting at false origin	E_F	150000.00	metres
Northing at false origin	N_F	50000.00	metres

Forward calculation for:

Latitude φ = 47°00'00.00"N Longitude λ = 7°00'00.00"E

first gives :

φ_O	=	0.774562578 rad	m_O	=	0.715900163
φ_F	=	0.788597934 rad	λ_F	=	0.118973277 rad
t_O	=	0.422551185	t_F	=	0.414305398
n	=	0.699403505	F	=	1.869760448
r_F	=	6439208.575			

then:

φ = 0.820304748 rad λ = 0.122173048 rad
t = 0.395846092 r = 6237180.887 θ = 0.002237931

Then Easting E = 163958.366 metres
Northing N = 252043.307 metres

Reverse calculation for the same easting and northing first gives
r' = 6237180.887 t' = 0.395846092 θ' = 0.002237931

Then Latitude φ = 47°00'00.000"N
Longitude λ = 7°00'00.000"E

3.4.1.4 Lambert Conic Conformal (1SP West Orientated)
(EPSG Dataset coordinate operation method code 9826)

In older mapping of Denmark and Greenland, the Lambert Conic Conformal (1SP variant A) is used with axes positive north and **west**. To derive the projected Westing and Northing coordinates of a point with geographic coordinates (φ, λ), the formulas are as for the standard Lambert Conic Conformal (1SP) variant A above (EPSG Dataset coordinate operation method code 9801) except for:

$$W = FE - r \sin\theta$$

In this formula, the term FE retains its definition, i.e. in the Lambert Conic Conformal (West Orientated) method it increases the Westing value at the natural origin. In this method it is effectively false westing (FW).

The reverse formulas to derive the latitude and longitude of a point from its Westing and Northing values are as for the standard Lambert Conic Conformal (1SP variant A) except for:

$\theta' = \text{atan2}[(FE - W) , \{r_O - (N - FN)\}]$
$r' = \pm[(FE - W)^2 + \{r_O - (N - FN)\}^2]^{0.5}$, taking the sign of n

3.4.1.5 Lambert Conic Conformal (2SP Belgium)

(EPSG Dataset coordinate operation method code 9803)

In 1972, in order to retain approximately the same grid coordinates after a change of geodetic datum, a modified form of the two standard parallel case was introduced in Belgium. In 2000 this modification was replaced through use of the regular Lambert Conic Conformal (2SP) method with appropriately modified parameter values.

For the Lambert Conic Conformal (2SP Belgium) methodthe formulas for the regular Lambert Conic Conformal (2SP) variant given above are used except for:

Easting, $E = E_F + r \sin (\theta - a)$

Northing, $N = N_F + r_F - r \cos (\theta - a)$

and for the reverse formulas

$\lambda = [(\theta' + a)/n] + \lambda_F$

where a = 29.2985 arc-seconds.

Example

For Projected Coordinate Reference System: BD72 / Belge Lambert 72 (EPSG CRS code 31300)

Parameters:

Ellipsoid:	International 1924	a = 6378388 metres	1/f = 297.0
	from which:	e = 0.08199189	e^2 = 0.006722670

Latitude of false origin	φ_F	90°00'00"N	
Longitude of false origin	λ_F	4°21'24.983"E	
Latitude of 1st standard parallel	φ_1	49°50'00"N	
Latitude of 2nd standard parallel	φ_2	51°10'00"N	
Easting at false origin	E_F	150000.01	metres
Northing at false origin	N_F	5400088.44	metres

Forward calculation for:

Latitude φ = 50°40'46.461"N Longitude λ = 5°48'26.533"E

first gives:

φ_F	=	1.57079633 rad	λ_F	=	0.07604294 rad
φ_1	=	0.86975574 rad	φ_2	=	0.89302680 rad
m_1	=	0.64628304	m_2	=	0.62834001
t_1	=	0.36750382	t_2	=	0.35433583
t_F	=	0.00	n	=	0.77164219
F	=	1.81329763	r_F	=	0.00

then:

φ	=	0.88452540 rad	λ	=	0.10135773 rad
t	=	0.35913403	r	=	5248041.03
θ	=	0.01953396	a	=	0.00014204

Then	Easting	E	=	251763.20 metres
	Northing	N	=	153034.13 metres

Reverse calculation for same easting and northing first gives:

θ'	=	0.01939192
t'	=	0.35913403
r'	=	5248041.03

Then	Latitude	φ	=	50°40'46.461"N
	Longitude	λ	=	5°48'26.533"E

3.4.1.6 Lambert Conic Conformal (2SP Michigan)

(EPSG Dataset coordinate operation method code 1051)

In 1964, the US state of Michigan redefined its State Plane CS27 zones to better reflect the geography of the state, changing them from being Transverse Mercator zones orientated north-south to being Lambert Conic Conformal zones orientated east-west. In making this change, it was decided to scale the ellipsoid to the average height of the state, 800 feet above sea level, in order to eliminate the need for height reduction in engineering surveys (in the later CS83 definitions this modification was eliminated). In the 1964 modification the formulas for the regular Lambert Conic Conformal (2SP) variant are used except for:

in the forward formulas

$r = a\,K\,F\,t^n$ for r_F and r

and in the reverse formulas

$t' = (r'/(a\,K\,F))^{1/n}$ where K = is the ellipsoid scaling factor.

Example

For Projected Coordinate Reference System: NAD27 / Michigan Central (EPSG CRS code 6201)

Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.4 metres		b = 6356583.8 metres
	from which:	a = 20925832.164 US survey feet		
		1/f = 294.9786982	e = 0.08227185	e^2 = 0.00676866

Latitude of false origin	φ_F	43°19'00"N	
Longitude of false origin	λ_F	84°20'00"W	
Latitude of 1st standard parallel	φ_1	44°11'00"N	
Latitude of 2nd standard parallel	φ_2	45°42'00"N	
Easting at false origin	E_F	2000000.00	US survey feet
Northing at false origin	N_F	0.00	US survey feet
Ellipsoid scaling factor	K	1.0000382	

Forward calculation for:

Latitude φ = 43°45'00.000"N Longitude λ = 83°10'00.000"W

first gives:

φ_F	=	0.756018454 rad	λ_F	=	-1.471894336 rad
φ_1	=	0.771144641 rad	φ_2	=	0.797615468 rad
m_1	=	0.718295175	m_2	=	0.699629151
t_1	=	0.424588396	t_2	=	0.409053868
t_F	=	0.433541026	n	=	0.706407410
F	=	1.862317735	r_F	=	21594768.40

then:

φ	=	0.763581548 rad	λ	=	-1.451532161 rad			
t	=	0.429057680	r	=	21436775.51	θ	=	0.014383991

Then Easting X = 2308335.75 ftUS, Northing Y = 160210.48 ftUS.

Reverse calculation for same easting and northing first gives:

θ' = 0.014383991
t' = 0.429057680
r' = 21436775.51

Then Latitude φ = 43°45'00.000"N
Longitude λ = 83°10'00.000"W

3.4.1.7 Lambert Conic Near-Conformal

(EPSG Dataset coordinate operation method code 9817)

The Lambert Conformal Conic with one standard parallel formulas, as published by the Army Map Service, are still in use in several countries. The AMS uses series expansion formulas for ease of computation, as was normal before the electronic computer made such approximate methods unnecessary. Where the expansion series have been carried to enough terms, the results are the same to the centimetre level as through the Lambert Conic Conformal (1SP) formulas above. However, in some countries, the expansion formulas were truncated to the third order and the map projection is not fully conformal. The full formulas are used in Libya but from 1915 for France, Morocco, Algeria, Tunisia, and Syria the truncated formulas were used. In 1943 in Algeria and Tunisia, from 1948 in France, from 1953 in Morocco, and from 1973 in Syria, the truncated formulas were replaced with the full formulas.

To compute the Lambert Conic Near-Conformal, the following formulas are used. First compute constants for the projection:

$n = f / (2 - f)$
$A = 1 / (6 \rho_O \nu_O)$ where ρ_O and ν_O are computed as in the Implementation Notes in this document.
$A' = a [1 - n + 5 (n^2 - n^3) / 4 + 81 (n^4 - n^5) / 64] \pi /180$
$B' = 3 a [n - n^2 + 7 (n^3 - n^4) / 8 + 55 n^5 / 64] / 2$
$C' = 15 a [n^2 - n^3 + 3 (n^4 - n^5) / 4] / 16$
$D' = 35 a [n^3 - n^4 + 11 n^5 / 16] / 48$
$E' = 315 a [n^4 - n^5] / 512$
$r_O = k_O \nu_O / \tan \varphi_O$
$s_O = A' \varphi_O - B' \sin 2\varphi_O + C' \sin 4\varphi_O - D' \sin 6\varphi_O + E' \sin 8\varphi_O$
where in the first term φ_O is in degrees, in the other terms φ_O is in radians.

Then, for the computation of easting and northing from latitude and longitude:

$s = A' \varphi - B' \sin 2\varphi + C' \sin 4\varphi - D' \sin 6\varphi + E' \sin 8\varphi$
where in the first term φ is in degrees, in the other terms φ is in radians.
$M = [s - s_O]$
$M = k_O (m + Am^3)$ (see footnote[1])
$r = r_O - M$
$\theta = (\lambda - \lambda_O) \sin \varphi_O$

and

$E = FE + r \sin\theta$
$N = FN + M + r \sin\theta \tan (\theta / 2)$

[1] This is the term that is truncated to the third order. To be equivalent to the Lambert Conic Conformal (1SP) (variant A) it would be $M = k_O (m + Am^3 + Bm^4 + Cm^5 + Dm^6)$. B, C and D are not detailed here.

The reverse formulas for φ and λ from E and N are:

$$\theta' = \text{atan2}\,\{(E - FE)\,,\,[r_O - (N - FN)]\}$$
$$r' = \pm\{(E - FE)^2 + [r_O - (N - FN)]^2\}^{0.5}, \text{ taking the sign of } \varphi_O$$
$$M' = r_O - r'$$

If an exact solution is required, it is necessary to solve for m and φ using iteration of the two equations Firstly:

$$m' = m' - [M' - k_O\, m' - k_O\, A\, (m')^3] / [- k_O - 3\, k_O\, A\, (m')^2]$$

using M' for m' in the first iteration. This will usually converge (to within 1mm) in a single iteration.
Then

$$\varphi' = \varphi' + \{m' + s_O - [A'\, \varphi'\, (180/\pi) - B' \sin 2\varphi' + C' \sin 4\varphi' - D' \sin 6\varphi' + E' \sin 8\varphi']\}/A'\, (\pi/180)$$

first using $\varphi' = \varphi_O + m'/A'\,(\pi/180)$.

However, the following non-iterative solution is accurate to better than 0.001" (3mm) within 5 degrees latitude of the projection origin and should suffice for most purposes:

$$m' = M' - [M' - k_O\, M' - k_O\, A\, (M')^3] / [- k_O - 3\, k_O\, A\, (M')^2]$$
$$\varphi' = \varphi_O + m'/A'\, (\pi/180)$$
$$s' = A'\, \varphi' - B' \sin 2\varphi' + C' \sin 4\varphi' - D' \sin 6\varphi' + E' \sin 8\varphi'$$

where in the first term φ' is in degrees, in the other terms φ' is in radians.

$$Ds' = A'(180/\pi) - 2B' \cos 2\varphi' + 4C' \cos 4\varphi' - 6D' \cos 6\varphi' + 8E' \cos 8\varphi'$$
$$\varphi = \varphi' - [(m' + s_O - s') / (-ds')] \text{ radians}$$

Then, after solution of φ (using either of the two methods above):

$$\lambda = \lambda_O + \theta' / \sin \varphi_O \quad \text{where } \lambda_O \text{ and } \lambda \text{ are in radians}$$

Example

For Projected Coordinate Reference System: Deir ez Zor / Levant Zone (EPSG CRS code 22700)

Parameters:

Ellipsoid:	Clarke 1880 (IGN)	a = 6378249.2 metres	b = 6356515.0 metres
	from which:	1/f = 293.4660213	n = 0.001706682563

Latitude of natural origin	φ_O	34°39'00"N
Longitude of natural origin	λ_O	37°21'00"E
Scale factor at natural origin	k_O	0.99962560
False easting	FE	300000.00 metres
False northing	FN	300000.00 metres

Forward calculation for:

Latitude φ = 37°31'17.625"N Longitude λ = 34°08'11.291"E

first gives:

φ_O	=	0.604756586 rad	λ_O	=	0.651880476 rad
A	=	4.1067494 E-15	A'	=	111131.8633
B'	=	16300.64407	C'	=	17.38751
D'	=	0.02308	E'	=	0.000033
s_O	=	3835482.233	r_O	=	9235264.405

then:

φ	=	0.654874806 rad	λ	=	0.595793792 rad
s	=	4154101.458	m	=	318619.225
M	=	318632.72	r	=	8916631.685
θ	=	-0.03188875			

Then Easting E = 15707.96 metres (c.f. E = 15708.00 using full formulas)
Northing N = 623165.96 metres (c.f. N = 623167.20 using full formulas)

Reverse calculation for same easting and northing first gives:

θ' = -0.031888749
r' = 8916631.685
M' = 318632.717

Using the non-iterative solution:

m' = 318619.222
φ' = 0.654795830
s' = 4153599.259
ds' = 6358907.456

Then Latitude φ = 0.654874806 rad = 37°31'17.625"N
Longitude λ = 0.595793792 rad = 34°08'11.291"E

3.4.2 <u>**Krovak**</u>

The normal case of the Lambert Conic Conformal is for the axis of the cone to be coincident with the minor axis of the ellipsoid, that is the axis of the cone is normal to the ellipsoid at a geographic pole. For the Oblique Conic Conformal, the axis of the cone is normal to the ellipsoid at a defined location and its extension cuts the ellipsoid minor axis at a defined angle. The map projection method is similar in principle to the Oblique Mercator. It is used in the Czech Republic and Slovakia under the name ‘Krovak’ projection, where, like the Laborde Oblique Mercator cylindrical projection in Madagascar, the rotation to north is made in spherical rather than plane coordinates. The geographic coordinates on the ellipsoid are first reduced to conformal coordinates on the conformal (Gaussian) sphere. These spherical coordinates are then rotated to north and the rotated spherical coordinates then projected onto the oblique cone and converted to grid coordinates. The pseudo standard parallel is defined on the conformal sphere after its rotation. It is then the parallel on this sphere at which the map projection is true to scale; on the ellipsoid it maps as a complex curve. A scale factor may be applied to the map projection to increase the useful area of coverage.

Four variants of the classical Krovak method developed in 1922 are recognized:

i) For the Czech Republic and Slovakia the original 1922 method by **Krovak** results in coordinates which increment to the south and west.
ii) A 21st century modification to this, **Krovak (North Orientated)**, causes the coordinate system axes to increment to the east and north, making the system more readily incorporated into GIS systems.
iii) A second modification to the original method for distortions in the surveying network, **Krovak Modified**.
iv) A third modification, **Krovak Modified (North Orientated)**, applies both of the first and second modifications, that is it compensates for surveying network distortions and also results in coordinates for the Czech Republic and Slovakia incrementing to the east and north.

3.4.2.1 Krovak

(EPSG Dataset coordinate operation method code 9819)

The defining parameters for the Krovak method are:

φ_C = latitude of projection centre, the latitude of the point used as reference for the calculation of the Gaussian conformal sphere
λ_O = longitude of origin
α_C = rotation in plane of meridian of origin of the conformal coordinates
= co-latitude of the cone axis at its point of intersection with the conformal sphere
φ_P = latitude of pseudo standard parallel
k_P = scale factor on pseudo standard parallel
FE = False Easting
FN = False Northing

For the projection devised by Krovak in 1922 for Czechoslovakia the axes are positive south (X) and west (Y). The symbols FE and FN represent the westing and southing coordinates at the grid natural origin. This is at the intersection of the cone axis with the surface of the conformal sphere.

From these, the following constants for the projection may be calculated:

$$A = a\,(1 - e^2)^{0.5} / [\,1 - e^2 \sin^2(\varphi_C)\,]$$
$$B = \{1 + [e^2 \cos^4\varphi_C / (1 - e^2)]\}^{0.5}$$
$$\gamma_O = \operatorname{asin}[\sin(\varphi_C) / B]$$
$$t_O = \tan(\pi/4 + \gamma_O/2)\,.\,[\,(1 + e \sin(\varphi_C)) / (1 - e \sin(\varphi_C))\,]^{e.B/2} / [\tan(\pi/4 + \varphi_C/2)]^B$$
$$n = \sin(\varphi_P)$$
$$r_O = k_P\, A / \tan(\varphi_P)$$

To derive the projected Southing and Westing coordinates of a point with geographic coordinates (φ,λ) the formulas for the Krovak are:

$$U = 2\,(\operatorname{atan}\{\,t_O \tan^B(\varphi/2 + \pi/4) / [(1 + e\sin(\varphi)) / (1 - e\sin(\varphi))]^{e.B/2}\} - \pi/4)$$
$V = B\,(\lambda_O - \lambda)$ where λ_O and λ must both be referenced to the same prime meridian,
$$T = \operatorname{asin}[\,\cos(\alpha_C)\sin(U) + \sin(\alpha_C)\cos(U)\cos(V)\,]$$
$$D = \operatorname{asin}[\,\cos(U)\sin(V) / \cos(T)\,]$$
$$\theta = n\,D$$
$$r = r_O \tan^n(\pi/4 + \varphi_P/2) / \tan^n(T/2 + \pi/4)$$
$$Xp = r\cos\theta$$
$$Yp = r\sin\theta$$

Then

Southing X = Xp + FN
Westing Y = Yp + FE

Note also that the formula for D is satisfactory for the normal use of the projection within the pseudo-longitude range on the conformal sphere of ±90 degrees from the central line of the projection. Should there be a need to exceed this range (which is not necessary for application in the Czech and Slovak Republics) then for the calculation of D use:

$$\sin(D1) = \cos(U)\sin(V) / \cos(T)$$
$$\cos(D1) = \{[\cos(\alpha_C)\ \sin(T) - \sin(U)] / [\sin(\alpha_C)\ \cos(T)]\}$$
$$D = \operatorname{atan2}(\sin(D1), \cos(D1))$$

Reverse

The reverse formulas to derive the latitude and longitude of a point from its Southing (X) and Westing (Y) values are:

$Xp' = \text{Southing} - FN$
$Yp' = \text{Westing} - FE$
$r' = [(Xp')^2 + (Yp')^2]^{0.5}$
$\theta' = \text{atan2}(Yp', Xp')$
$D' = \theta' / \sin(\varphi_P)$
$T' = 2\{\text{atan}[(r_O / r')^{1/n} \tan(\pi/4 + \varphi_P/2)] - \pi/4\}$
$U' = \text{asin}[\cos(\alpha_C)\sin(T') - \sin(\alpha_C)\cos(T')\cos(D')]$
$V' = \text{asin}[\cos(T')\sin(D') / \cos(U')]$

Latitude φ is found by iteration using U' as the value for φ_{j-1} in the first iteration

$$\varphi_j = 2 (\text{atan} \{ t_O^{-1/B} \tan^{1/B} (U'/2 + \pi/4) [(1 + e \sin(\varphi_{j-1}))/(1 - e \sin(\varphi_{j-1}))]^{e/2} \} - \pi/4)$$

Three iterations will usually suffice to reach sub-millimetre precision. Then:

$\lambda = \lambda_O - V' / B$ where λ is referenced to the same prime meridian as λ_O.

Example

For Geographic CRS *S-JTSK* and Projected CRS *S-JTSK (Ferro) / Krovak*

Parameters:

Ellipsoid: Bessel 1841 a = 6377397.155metres 1/f = 299.1528128
from which: e = 0.081696831 e^2 = 0.006674372

Latitude of projection centre	φ_C	49°30'00"N	
Longitude of origin	λ_O	42°30'00"E of Ferro	
Co-latitude of cone axis	α_C	30°17'17.30311"	
Latitude of pseudo standard parallel	φ_P	78°30'00"N	
Scale factor on pseudo standard parallel	k_P	0.9999	
False Easting	FE	0.00	metres
False Northing	FN	0.00	metres

Projection constants:

φ_C	=	0.863937979 rad	λ_O	=	0.741764932 rad
α_C	=	0.528627763 rad	φ_P	=	1.370083463 rad
A	=	6380703.611	B	=	1.000597498
γ_O	=	0.863239103	t_O	=	1.003419164
n	=	0.979924705	r_O	=	1298039.005

Forward calculation for:

Latitude	φ	=	50°12'32.442"N	=	0.876312568 rad
Longitude	λ_G	=	16°50'59.179"E of Greenwich		

Firstly, because the projection definition includes longitudes referenced to the Ferro meridian, the Greenwich longitude of the point needs to be transformed to be referenced to the Ferro meridian using the Longitude Rotation method (EPSG method code 9601, see Implementation Notes on Offsets in Section §1.5)[2].

Longitude of point when referenced to Greenwich meridian	λ_G	=	16°50'59.179"E of Greenwich
Longitude of Ferro		=	17°40'00" west of Greenwich

and then

Longitude of point when referenced to Ferro meridian	λ	=	34°30'59.179"E of Ferro
		=	0.602425500 rad

Then the forward calculation first gives:

U	=	0.875596949
V	=	0.139422687
T	=	1.386275049
D	=	0.506554623
θ	=	0.496385389
r	=	1194731.014
Xp	=	1050538.643
Yp	=	568990.997

and	Southing	X	=	1050538.64 metres
	Westing	Y	=	568991.00 metres

For the Krovak grid in Czechia and Slovakia southing (X) values are greater than 900000m and westing (Y) values are less than 900000m.

Reverse calculation for the same Southing and Westing gives

Xp'	=	1050538.643
Yp'	=	568990.997
r'	=	1194731.014
θ'	=	0.496385389
D'	=	0.506554623
T'	=	1.386275049
U'	=	0.875596949
V'	=	0.139422687

Then by iteration

φ_1 = 0.876310601 rad φ_2 = 0.876312560 rad φ_3 = 0.876312566 rad

and

Latitude	φ	=	0.876312566 rad	=	50°12'32.442"N
Longitude of point when referenced to Ferro meridian	λ	=	0.602425500 rad	=	34°30'59.179"E of Ferro

Then using the Longitude Rotation method (EPSG method code 9601)

Longitude of Ferro = 17°40'00" west of Greenwich

[2] An alternative approach would be to define the projection with longitude of origin value of 24º50'E referenced to the Greenwich meridian. This approach would eliminate the need to transform the longitude of every point from Greenwich to Ferro before conversion, which arguably is more practical. It is not documented here because the published projection definition is referenced to Ferro.

Then

Longitude of point when referenced to Greenwich meridian	λ_G	=	16°50'59.179"E of Greenwich

3.4.2.2 Krovak (North Orientated)

(EPSG Dataset coordinate operation method code 1041)

Having axes positive south and west, as is the case with the projection defined by Krovak in 1922, causes problems in some GIS systems. To resolve this, the Czech and Slovak authorities modified the Krovak projection method to have a coordinate system with axes incrementing to the east and north.

To derive the projected Easting and Northing coordinates of a point with geographic coordinates (φ,λ), the formulas are the same as those for the Krovak (south-orientated) method (EPSG method code 9819) described in the previous section as far as the calculation of Southing and Westing. Then

$$\text{Easting}_{\text{Krovak (North Orientated)}} \quad X = -\text{Westing}_{\text{Krovak}}$$
$$\text{Northing}_{\text{Krovak (North Orientated)}} \quad Y = -\text{Southing}_{\text{Krovak}}$$

Note that in addition to changing the sign of the coordinate values, the axes abbreviations X and Y are transposed from south to east and from west to north respectively. Note also that for this north-orientated variant, although the grid coordinates increment to the east and north their numerical values are negative in the Czech Republic and Slovakia area. For the north-orientated grid easting (X) values are greater than –900000m i.e. between –900000m and –400000m) and northing (Y) values are less than –900000m (between –900000m and –1400000m).

For the reverse conversion to derive the latitude and longitude of a point from its Easting (X) and Northing (Y) values, first:

$$\text{Southing}_{\text{Krovak}} = -\text{Northing}_{\text{Krovak (North Orientated)}}$$
$$\text{Westing}_{\text{Krovak}} = -\text{Easting}_{\text{Krovak (North Orientated)}}$$

after which latitude and longitude are derived as described for the south-orientated Krovak method in the previous section.

Example

For Projected Coordinate Reference System: S-JTSK (Ferro) / Krovak East North (EPSG CRS code 5221)

Forward calculation for the same point as in the south-orientated case in Section §3.4.2.1 above:

Latitude	φ	=	50°12'32.442"N
Longitude	λ_G	=	34°30'59.179"E of Ferro

proceeds as in the south-orientated case as far as the calculation of Southing and Westing:

U	=	0.875596949
V	=	0.139422687
T	=	1.386275049
D	=	0.506554623
θ	=	0.496385389
r	=	1194731.014
Xp	=	1050538.643
Yp	=	568990.997

Southing = 1050538.643
Westing = 568990.997

Then

Easting X = –568991.00 metres
Northing Y = –1050538.64 metres

Reverse calculation for the same Easting and Northing (–568990.995, –1050538.634) first gives

Southing = 1050538.643
Westing = 568990.997

then proceeds as in the calculation for the (south-orientated) Krovak method to give

Latitude φ = 50°12'32.442"N
Longitude λ = 34°30'59.179"E of Ferro

3.4.2.3 Krovak Modified

(EPSG Dataset coordinate operation method code 1042)

After the initial introduction of the S-JTSK (Ferro) / Krovak projected CRS, further geodetic measurements were observed and calculations made in the projected CRS resulted in changed grid coordinate values. The relationship through the Krovak projection between the original S-JTSK geographic CRS coordinates and the new projected CRS coordinates was no longer exact. In 2009, Czech authorities introduced a modification which modelled the survey network distortions.

To derive the projected Easting and Northing coordinates of a point with geographic coordinates (φ,λ) the formulas for the Krovak Modified method are the same as those for the original (unmodified, south-orientated) Krovak method as far as the calculation of Xp and Yp.

Then:

$$Xr = Xp - X_O$$
$$Yr = Yp - Y_O$$

and

$$dX = C_1 + C_3.Xr - C_4.Yr - 2.C_6.Xr.Yr + C_5.(Xr^2 - Yr^2) + C_7.Xr.(Xr^2 - 3.Yr^2) - C_8.Yr.(3.Xr^2 - Yr^2) + 4.C_9.Xr.Yr.(Xr^2 - Yr^2) + C_{10}.(Xr^4 + Yr^4 - 6.Xr^2.Yr^2)$$

$$dY = C_2 + C_3.Yr + C_4.Xr + 2.C_5.Xr.Yr + C_6.(Xr^2 - Yr^2) + C_8.Xr.(Xr^2 - 3.Yr^2) + C_7.Yr.(3.Xr^2 - Yr^2) - 4.C_{10}.Xr.Yr.(Xr^2 - Yr^2) + C_9.(Xr^4 + Yr^4 - 6.Xr^2.Yr^2)$$

where X_O, Y_O and C_1 through C_{10} are additional defining parameters for the modified method.

Finally:

Southing $X = Xp - dX + FN$
Westing $Y = Yp - dY + FE$

For the reverse conversion to derive the latitude and longitude of a point from its Southing (X) and Westing (Y) values:

$$Xr' = (\text{Southing} - FN) - X_O$$
$$Yr' = (\text{Westing} - FE) - Y_O$$

$$dX' = C_1 + C_3.Xr' - C_4.Yr' - 2.C_6.Xr'.Yr' + C_5.(Xr'^2 - Yr'^2) + C_7.Xr'.(Xr'^2 - 3.Yr'^2) - C_8.Yr'.(3.Xr'^2 - Yr'^2) + 4.C_9.Xr'.Yr'.(Xr'^2 - Yr'^2) + C_{10}.(Xr'^4 + Yr'^4 - 6.Xr'^2.Yr'^2)$$

$$dY' = C_2 + C_3.Yr' + C_4.Xr' + 2.C_5.Xr'.Yr' + C_6.(Xr'^2 – Yr'^2) + C_8.Xr'.(Xr'^2 – 3.Yr'^2) + C_7.Yr'.(3.Xr'^2 – Yr'^2) – 4.C_{10}.Xr'.Yr'.(Xr'^2 – Yr'^2) + C_9.(Xr'^4 + Yr'^4 – 6.Xr'^2.Yr'^2)$$

$$Xp' = (\text{Southing} – FN) + dX'$$
$$Yp' = (\text{Westing} – FE) + dY'$$

After which latitude and longitude are derived from Xp' and Yp' as described for the classic Krovak method (EPSG method code 9819) described above.

It may be noted that the calculation of dX and dY in the forward formulas (and dX' and dY' in the reverse fomulas) is a reversible polynomial of degree 4 – see Section §4.6. The coefficients C_1 through C_{10} can be mapped to the coefficients A_0 through A_{14} and B_0 through B_{14} as:

A_0	=	C_1	A_7	=	$-3.C_8$	B_0	=	C_2	B_7	=	$3.C_7$
A_1	=	C_3	A_8	=	$-3.C_7$	B_1	=	C_4	B_8	=	$-3.C_8$
A_2	=	$-C_4$	A_9	=	C_8	B_2	=	C_3	B_9	=	$-C_7$
A_3	=	C_5	A_{10}	=	C_{10}	B_3	=	C_6	B_{10}	=	C_9
A_4	=	$-2.C_6$	A_{11}	=	$4.C_9$	B_4	=	$2.C_5$	B_{11}	=	$-4.C_{10}$
A_5	=	$-C_5$	A_{12}	=	$-6.C_{10}$	B_5	=	$-C_6$	B_{12}	=	$-6.C_9$
A_6	=	C_7	A_{13}	=	$-4.C_9$	B_6	=	C_8	B_{13}	=	$4.C_{10}$
			A_{14}	=	C_{10}				B_{14}	=	C_9

and the scaling parameter m is unity. The polynomial coefficient values have been empirically derived so in principle this is a transformation which induces inaccuracy into the output coordinates. However, the polynomial is decreed to be exact and for practical purposes the Krovak Modified method is treated as a map projection. The embedding of this polynomial means that the Krovak Modified method is not truly conformal.

Example

For Projected Coordinate Reference System: S-JTSK/05 (Ferro) / Modified Krovak (EPSG CRS code 5224)

Parameters:

Ellipsoid:	Bessel 1841	a = 6377397.155 metres	1/f = 299.1528128
	from which:	e = 0.081696831	e^2 = 0.006674372

Latitude of projection centre	φ_C	49°30'00"N	= 0.863937979 rad
Longitude of origin	λ_O	42°30'00"E of Ferro	
Co-latitude of cone axis	α_C	30°17'17.30311"	= 0.528627763 rad.
Latitude of pseudo standard parallel	φ_P	78°30'00"N	= 1.370083463 rad
Scale factor on pseudo standard parallel	k_P	0.9999	
False Easting	FE	5000000.00	metres
False Northing	FN	5000000.00	metres
Ordinate 1 of evaluation point	X_O	1089000.00	metres
Ordinate 2 of evaluation point	Y_O	654000.00	metres
C1		2.946529277E-02	
C2		2.515965696E-02	
C3		1.193845912E-07	
C4		-4.668270147E-07	
C5		9.233980362E-12	
C6		1.523735715E-12	
C7		1.696780024E-18	
C8		4.408314235E-18	
C9		-8.331083518E-24	
C10		-3.689471323E-24	

Forward calculation for the same point as in the Krovak (south-orientated) method example:

Latitude	φ	=	50°12'32.442"N
Longitude	λ_G	=	34°30'59.179"E of Ferro

proceeds as in the original Krovak method as far as the calculation of Xp and Yp. Then:

Xr	=	-38461.369
Yr	=	-85009.005
dX	=	-0.077
dY	=	0.088

and	Southing	X	=	6050538.71 metres
	Westing	Y	=	5568990.91 metres

For the reverse calculation of these coordinates:

Xr'	=	-38461.292	Yr'	=	-85009.093
dX'	=	-0.077	dY'	=	0.088
Xp'	=	1050538.631	Yp'	=	568990.995
r'	=	1194731.002	θ'	=	0.496385393
D'	=	0.506554627	T'	=	1.386275051
U'	=	0.875596951	V'	=	0.139422687

Then by iteration

φ_1 = 0.876310603 rad φ_2 = 0.876312562 rad φ_3 = 0.876312568 rad

Latitude	φ	=	0.876312568 rad	=	50°12'32.442"N
Longitude	λ	=	0.602425500 rad	=	34°30'59.179"E of Ferro

3.4.2.4 Krovak Modified (North Orientated)

(EPSG Dataset coordinate operation method code 1043)

A further modification of the Krovak method is to apply the axis orientation change as described to the Krovak Modified method described in the previous section. The calculations for Southing and Westing is as for the Krovak Modified method (EPSG method code 1042), but then:

$$\text{Easting}_{\text{Krovak Modified (North Orientated)}} \quad X = -\text{Westing}_{\text{Krovak Modified}}$$
$$\text{Northing}_{\text{Krovak Modified (North Orientated)}} \quad Y = -\text{Southing}_{\text{Krovak Modified}}$$

As with the Krovak (North Orientated) projection, the axis abbreviations X and Y are transposed from south to east and from west to north respectively.

For the reverse calculation of easting and northing to latitude and longitude,

$$\text{Southing}_{\text{Krovak Modified}} = -\text{Northing}_{\text{Krovak Modified (North Orientated)}}$$
$$\text{Westing}_{\text{Krovak Modified}} = -\text{Easting}_{\text{Krovak Modified (North Orientated)}}$$

and the calculation then follows that of the Krovak Modified method.

Example

For Projected Coordinate Reference System: S-JTSK/05 (Ferro) / Modified Krovak East North

Forward calculation for the same point as in the Krovak method in Section §3.4.2.1 above:

Latitude φ = 50°12'32.442"N
Longitude λ_G = 34°30'59.179"E of Ferro

proceeds as in the example for the Krovak Modified method described in the previous section as far as the calculation of Southing and Westing.

Then Easting X = –5568990.91 metres
Northing Y = –6050538.71 metres

Reverse calculation for the same Easting and Northing first gives

Southing = 6050538.71
Westing = 5568990.91

and then continues as in the calculation for the Krovak Modified method to give

Latitude φ = 50°12'32.442"N
Longitude λ_G = 34°30'59.179"E of Ferro

3.4.3 **Albers Equal Area**

(EPSG Dataset coordinate operation method code 9822)

The formulas to convert geodetic latitude and longitude (φ, λ) to Easting (E) and Northing (N) are:

$$\text{Easting (E)} = E_F + r \sin\theta$$
$$\text{Northing (N)} = N_F + r_F - r \cos\theta$$

where

$$\theta = n (\lambda - \lambda_F)$$
$$r = [a (C - n\,\alpha)^{0.5}] / n$$
$$r_F = [a (C - n\,\alpha_F)^{0.5}] / n$$

and

$$C = m_1^2 + (n\,\alpha_1)$$
$$n = (m_1^2 - m_2^2) / (\alpha_2 - \alpha_1)$$
$$m_1 = \cos\varphi_1 / (1 - e^2\sin^2\varphi_1)^{0.5}$$
$$m_2 = \cos\varphi_2 / (1 - e^2\sin^2\varphi_2)^{0.5}$$
$$\alpha = (1 - e^2) \{[\sin\varphi / (1 - e^2\sin^2\varphi)] - [1/(2e)] \ln [(1 - e\sin\varphi) / (1 + e\sin\varphi)]\}$$
$$\alpha_F = (1 - e^2) \{[\sin\varphi_F / (1 - e^2\sin^2\varphi_F)] - [1/(2e)] \ln [(1 - e\sin\varphi_F) / (1 + e\sin\varphi_F)]\}$$
$$\alpha_1 = (1 - e^2) \{[\sin\varphi_1 / (1 - e^2\sin^2\varphi_1)] - [1/(2e)] \ln [(1 - e\sin\varphi_1) / (1 + e\sin\varphi_1)]\}$$
$$\alpha_2 = (1 - e^2) \{[\sin\varphi_2 / (1 - e^2\sin^2\varphi_2)] - [1/(2e)] \ln [(1 - e\sin\varphi_2) / (1 + e\sin\varphi_2)]\}$$

The reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing values are:

$$\varphi = \beta' + [(e^2/3 + 31e^4/180 + 517e^6/5040) \sin 2\beta'] + [(23e^4/360 + 251e^6/3780) \sin 4\beta'] + [(761e^6/45360) \sin 6\beta']$$

$$\lambda = \lambda_F + (\theta / n)$$

where

C, n, and r_F are as in the forward equations, ß' is the authalic latitude, and:

$$\beta' = \operatorname{asin}(\alpha' / \{1 - [(1 - e^2) / 2e] \ln [(1 - e) / (1 + e)]\})$$
$$\alpha' = [C - (\rho'^2 n^2 / a^2)] / n$$
$$r' = \{(E - E_F)^2 + [r_F - (N - N_F)]^2\}^{0.5}$$

If n is positive,

$$\theta = \text{atan2}\ \{(E - E_F)\ ,\ [r_F - (N - N_F)]\}$$

but if n is negative, the signs of both arguments of the atan2 function must be reversed:

$$\theta = \text{atan2}\ \{-(E - E_F),\ -[r_F - (N - N_F)]\}$$

Example 1 (Northern Hemisphere)

For Projected Coordinate Reference System: NAD83 / Great Lakes Albers (EPSG CRS code 3174)

Parameters:

Ellipsoid:	GRS 1980	a = 6378137.0 metres	1/f = 298.257222101
	from which:	e = 0.081819191	e^2 = 0.00669438

Latitude of false origin	φ_F	45°34'08.3172"N	
Longitude of false origin	λ_F	84°27'21.4380"W	
Latitude of 1st standard parallel	φ_1	42°07'21.9864"N	
Latitude of 2nd standard parallel	φ_2	49°00'54.6480"N	
Easting at false origin	E_F	1000000.000	metre
Northing at false origin	N_F	1000000.000	metre

Forward calculation for:

Latitude	φ	=	42°45'00.000"N
Longitude	λ	=	78°45'00.000"W

first gives:

φ_F	=	0.48578331 rad	λ_F	=	-1.72787596 rad
φ_1	=	0.49538262 rad	φ_2	=	0.52854388 rad
m_1	=	0.7428286888	m_2	=	0.6571136237
n	=	0.7128137342	C	=	1.5035043911
r_F	=	6263350.4332	α_F	=	1.4218650778
α_1	=	1.3351453325	α_2	=	1.5034868510

then:

φ	=	0.746128255 rad	λ	=	-1.374446786 rad
α	=	1.351293970	r	=	6577011.9827
θ	=	0.0709874815			

Then	Easting	E	=	1466493.492 metre
	Northing	N	=	702903.006 metre

Reverse calculation for same easting and northing first gives:

θ'	=	0.0709874815	α'	=	1.3512939695
r'	=	6577011.983	β'	=	0.7438962839

Then	Latitude	φ	=	42°45'00.000"N
	Longitude	λ	=	78°45'00.000"W

Example 2 (Southern Hemisphere)

Parameters:

Ellipsoid: GRS 1967 Modified		a = 6378160.0 metres	1/f = 298.25
	from which:	e = 0.081820180	e^2 = 0.006694542

Latitude of false origin	φ_F	32°00'00.000"S	
Longitude of false origin	λ_F	60°00'00.000"W	
Latitude of 1st standard parallel	φ_1	5°00'00.000"S	
Latitude of 2nd standard parallel	φ_2	42°00'00.000"S	
Easting at false origin	E_F	0.000	metre
Northing at false origin	N_F	0.000	metre

Forward calculation for:

Latitude φ = 18°30'02.016"S   Longitude λ = 46°00'01.538"W

first gives :

φ_F	=	0.48578331 rad	λ_F	=	-1.72787596 rad
φ_1	=	0.49538262 rad	φ_2	=	0.52854388 rad
m_1	=	0.9962200286	m_2	=	0.7442610814
n	=	-0.378429434	C	=	1.0579795608
r_F	=	-13683051.84	α_F	=	-1.054065016
α_1	=	-0.173150420	α_2	=	-1.331965641

then:

φ	=	-1.0471975512 rad	λ	=	-0.802858912 rad
α	=	-0.630662757	r	=	-15255863.89
θ	=	-0.092464933			

Then Easting E = 1408623.196 metre
Northing N = 1507641.482 metre

Reverse calculation for same easting and northing first gives:

(Since n is negative, we reverse the signs of both atan2 arguments to calculate θ')

θ'	=	-0.092464933
α'	=	-0.630662757
r'	=	15255863.888
β'	=	-0.321550075

Then Latitude φ = 18°30'02.016"S
Longitude λ = 46°00'01.538"W

Further examples of input and output may be found in test procedure 5109 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.4.4 **Bonne**

(EPSG Dataset coordinate operation method code 9827)

The Bonne projection was frequently adopted for 18th and 19th century mapping, but being equal area rather than conformal its use for topographic mapping was replaced during the 20th century by conformal map projection methods.

The formulas to convert geodetic latitude and longitude (φ, λ) to Easting and Northing are:

$$E = FE + (r \sin \theta)$$
$$N = FN + (a\, m_O / \sin \varphi_O - r \cos \theta)$$

where

$$m = \cos\varphi / (1 - e^2\sin^2\varphi)^{0.5}$$

with φ in radians and m_O for φ_O, the latitude of the origin, derived in the same way.

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 -....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024+....)\sin2\varphi$$
$$+ (15e^4/256 + 45e^6/1024 +.....)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$$

with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

$$r = a\ m_O / \sin \varphi_O + M_O - M$$
$$\theta = a\ m (\lambda - \lambda_O) / \rho \qquad \text{with } \lambda \text{ and } \lambda_O \text{ in radians}$$

For the reverse calculation:

$$X = E - FE$$
$$Y = N - FN$$
$$r = \pm [X^2 + (a\ .\ m_O / \sin \varphi_O - Y)^2]^{0.5} \text{ taking the sign of } \varphi_O$$
$$M = a\ m_O / \sin \varphi_O + M_O - r$$
$$\mu = M / [a (1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)]$$
$$e_1 = [1 - (1 - e^2)^{0.5}] / [1 + (1 - e^2)^{0.5}]$$
$$\varphi = \mu + (3e_1/2 - 27e_1^3/32 +.....)\sin2\mu + (21e_1^2/16 - 55e_1^4/32 + ...)\sin4\mu$$
$$+ (151e_1^3/96 +.....)\sin6\mu + (1097e_1^4/512 - ....)\sin8\mu + ...$$
$$m = \cos \varphi / (1 - e^2 \sin^2 \varphi)^{0.5}$$

If φ_O is not negative

$$\lambda = \lambda_O + r\{\operatorname{atan2}[X , (a\ .\ m_O / \sin \varphi_O - Y)]\} / a\ .\ m$$

but if φ_O is negative the signs of both arguments of the atan2 function must be reversed

$$\lambda = \lambda_O + r\{\operatorname{atan2}[- X , - (a\ .\ m_O / \sin \varphi_O - Y)]\} / a\ .\ m$$

In either case, if $\varphi = \pm 90°$, $m = 0$ and the equation for λ is indeterminate, so use $\lambda = \lambda_O$.

3.4.4.1 Bonne (South Orientated)

(EPSG Dataset coordinate operation method code 9828)

A special case of the Bonne method with coordinate system axes positive south and west has been used for older mapping in Portugal. The formulas are as for the general case above except:

$$W = FE - (r\ \sin \theta)$$
$$S = FN - (a\ m_O / \sin \varphi_O - r\ \cos \theta)$$

In these formulas, the terms FE and FN retain their definition, i.e. in the Bonne (South Orientated) method they increase the Westing and Southing value at the natural origin. In this method they are effectively false westing (FW) and false southing (FS) respectively.

For the reverse formulas, those for the standard Bonne method above apply, with the exception that:

$$X = FE - W$$
$$Y = FN - S$$

3.4.5 Equidistant Conic

(EPSG Dataset coordinate operation method code 1119)

The equidistant conic projection is neither conformal nor equal area, but a compromise between the two. It is most frequently used for very small scale maps in atlases based on a sphere. EPSG carries only the ellipsoidal development in which the meridional arc distance between the parallels of latitude reflects that on the ellipsoid.

The formulas to convert geodetic latitude and longitude (φ, λ) to easting (E) and northing (N) are:

Easting, $E = E_F + r \sin\theta$
Northing, $N = N_F + r_F - r \cos\theta$

where
$$M = a\,[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)\ \varphi$$
$$- (3e^2/8 + 3e^4/32 + 45e^6/1024 + \ldots)\ \sin 2\varphi$$
$$+ (15e^4/256 + 45e^6/1024 + \ldots)\ \sin 4\varphi$$
$$- (35e^6/3072 + \ldots)\ \sin 6\varphi + \ldots]$$
with φ in radians, and with M_F, M_1 and M_2 for φ_F (latitude of false origin), φ_1 (latitude of 1st standard parallel) and φ_2 (latitude of 2nd standard parallel) derived in the same way;

$m = \cos\varphi / (1 - e^2 \sin^2\varphi)^{0.5}$ with φ in radians, and m_1 and m_2 for φ_1 and φ_2 derived in the same way;
$n = a\,(m_1 - m_2) / (M_2 - M_1)$
$G = (m_1 / n) + (M_1 / a)$
$r_F = a\,G - M_F$

$r = a\,G - M$
$\theta = n\,(\lambda - \lambda_F)$

For the reverse calculation:

G and r_F are constants for the projection calculated as for the forward calculation above. Then:

$r' = \pm\ \{(E - E_F)^2 + [r_F - (N - N_F)]^2\}^{0.5}$, taking the sign of n

If n is positive (northern hemisphere case of the projection),
$$\theta' = \operatorname{atan2}\{(E - E_F)\,,\ [r_F - (N - N_F)]\}$$
but if n is negative (southern hemisphere case of the projection), the signs of both arguments of the atan2 function must be reversed:
$$\theta' = \operatorname{atan2}\{-(E - E_F),\ -[r_F - (N - N_F)]\}$$

$M = a\,G - r'$
$\mu = M / [a\,(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)]$

$$\text{Latitude, } \varphi = \mu + (3e_1/2 - 27e_1^3/32 + \ldots)\ .\ \sin 2\mu$$
$$+ (21e_1^2/16 - 55e_1^4/32 + \ldots)\ .\ \sin 4\mu$$
$$+ (151e_1^3/96 - \ldots)\ .\ \sin 6\mu$$
$$+ (1097e_1^4/512 - \ldots)\ .\ \sin 8\mu \quad \text{with } \mu \text{ in radians}$$

where $e_1 = [1 - (1 - e^2)^{0.5}] / [1 + (1 - e^2)^{0.5}]$

Longitude, $\lambda = \lambda_F + (\theta' / n)$

Example (Northern Hemisphere)
Parameters:

Ellipsoid:	Clarke 1866	a = 6378206.4 metres		b = 6356583.8 metres
	from which:	1/f =294.9786982	e = 0.08227185	e^2 =0.00676866
		e_1 = 0.001697916		

Latitude of false origin	φ_F	23°00'00"N
Longitude of false origin	λ_F	96°00'00"W
Latitude of 1st standard parallel	φ_1	29°30'00"N
Latitude of 2nd standard parallel	φ_2	45°30'00"N
Easting at false origin	E_F	0.00 metres
Northing at false origin	N_F	0.00 metres

Forward calculation for:
Latitude φ = 35°00'00.00"N Longitude λ = 75°00'00.00"W

first gives :

φ_F	=	0.401425728 rad	λ_F	=	-1.675516082 rad
φ_1	=	0.514872129 rad	φ_2	=	0.794124810 rad
M_1	=	3264511.20 m	m_1	=	0.871070821
M_2	=	5040295.01 m	m_2	=	0.702119143
M_F	=	2544389.75 m	n	=	0.606835507
G	=	1.947254290	r_F	=	9875600.03 m

Then :

φ	=	0.610865238 rad	λ	=	-1.308996939 rad
M	=	3874395.26 m	m	=	0.820065623
r	=	8545594.52 m	θ	=	0.222416830 rad

then Easting E = 1885051.86 m
Northing N = 1540507.64 m

Reverse calculation for same easting and northing gives:

r'	=	8545594.52 m	θ'	=	0.222416830 rad
M	=	3874395.26	μ	=	0.608473702
φ	=	0.610865238 rad	λ	=	-1.308996939 rad

Then Latitude φ = 35°00'00.000"N
Longitude λ = 75°00'00.000"W

3.4.6 **American Polyconic**
(EPSG Dataset coordinate operation method code 9818)

This projection was popular before the advent of modern computers due to its ease of mechanical construction, particularly in the United States. It is neither conformal nor equal area, and is distortion-free only along the longitude of origin. A modified form of the polyconic projection was adopted in 1909 for the International Map of the World series of 1/1,000,000 scale topographic maps. A general study of the polyconic family of projections by Oscar Adams of the US Geological Survey was published in 1919 (and reprinted in 1934).

The formulas to derive the projected Easting and Northing coordinates are:

If $\varphi = 0$:

Easting, $E = FE + a(\lambda - \lambda_O)$
Northing, $N = FN - M_O$

If φ is not zero:

Easting, $E = FE + \nu \cot\varphi \sin L$

Northing, $N = FN + M - Mo + \nu \cot\varphi(1 - \cos L)$

where $L = (\lambda - \lambda_O)\sin\varphi$
$\nu = a / (1 - e^2\sin^2\varphi)^{0.5}$
$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + ...)\sin 2\varphi$
$+ (15e^4/256 + 45e^6/1024 + ...)\sin 4\varphi - (35e^6/3072 + ...)\sin 6\varphi + ...]$
with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

The reverse formulas to convert Easting and Northing projected coordinates to latitude and longitude require iteration. This iteration will not converge if $(\lambda - \lambda_O)$>90º but the projection should not be used in that range.

First M_O is calculated using the formula for M given in the forward case. Then:

If $M_O = (N-FN)$ then:

$\varphi = 0$
$\lambda = \lambda_O + (E-FE)/a$

If M_O does not equal (N-FN) then:

$A = [M_O + (N-FN)] / a$
$B = A^2 + \{[(E-FE)^2] / a^2\}$
$C = (1 - e^2 \sin^2 \varphi_n)^{0.5} \tan \varphi_n$ initially using $\varphi_n = A$ for the first iteration
M is found
$H = Mn$
$J = H / a$
$\varphi'' = \varphi' - [A(C J +1) - J - 0.5C(J^2 + B)] /$
$\{e^2 \sin 2\varphi' (J^2 + B - 2A J) / 4C + (A - J) (C H - [2 / \sin 2\varphi')] - H\}$

Then after solution of φ

$\lambda = \lambda_O + \{\text{asin}[(E-FE) C / a]\} / \sin\varphi$

Example

Examples of input and output may be found in test procedure 5107 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.5 Cylindrical Map Projections

3.5.1 Mercator

(EPSG Dataset coordinate operation method codes 9804, 9805 and 1044)

The Mercator map projection is a special limiting case of the Lambert Conic Conformal map projection with the equator as the single standard parallel. All other parallels of latitude are straight lines and the meridians are also straight lines at right angles to the equator, equally spaced. It is the basis for the transverse and oblique forms of the projection. It is little used for land mapping purposes but is in almost universal use for navigation charts. As well as being conformal, it has the particular property that straight lines drawn on it are lines of constant bearing. Thus navigators may derive their course from the angle the straight course line makes with the meridians.

Three variants of the Mercator projection are recognised, differentiated by the parameters used in the projection definition:

- Variant A, also known as Mercator (1SP). The projection is defined with the equator as the single standard parallel, with scale factor on the equator also defined. False grid coordinates are applied at the natural origin of the projection, the intersection of the equator and the longitude of origin. See figure 4 below. In the few cases in which the Mercator projection is used for terrestrial applications or land mapping, such as in Indonesia prior to the introduction of the Universal Transverse Mercator, this is the normal method of definition.

- Variant B, defined through the latitude of two parallels equidistant either side of the equator upon which the grid scale is true. False grid coordinates are applied at the natural origin of the projection, the intersection of the equator and the longitude of origin. See Figure 4 below. Variants A and B are alternative ways of defining the same projection – coordinates of a point will have the same values regardless of whether the definition is variant A or variant B.

- Variant C, defined through the latitude of two parallels equidistant either side of the equator upon which the grid scale is true, as in variant (B). However in variant C false grid coordinates are applied at a point other than the natural origin of the projection, called the false origin. See Figure 5 below. If the values of the standard parallels and false grid coordinates are the same, coordinates for variant C will differ from those for variant B.

Variants A and B are special cases of the more general variant C. The defining parameters for the three Mercator variants are summarised in Table 5.

Table 5 – Coordinate Operation Parameters for Mercator methods.

	Method		
	1 SP	2 SPs	
	Variant A	Variant B	Variant C
EPSG Dataset coordinate operation method code:	9804	9805	1108
Parameter			
Latitude of natural origin (φ_O)	(x) (note 1)		
Longitude of natural origin (λ_O)	x	x	
Scale at natural origin (k_O)	x		
Latitude of 1st standard parallel (φ_1)		x (note 2)	x (note 2)
False easting (easting at natural origin) (FE)	x	x	
False northing (northing at natural origin) (FN)	x	x	
Latitude of false origin (φ_F)			x
Longitude of false origin (λ_F)			x
Easting at false origin (E_F)			x
Northing at false origin (N_F)			x

Notes:
1. For the Mercator variant A (1SP) method, in the EPSG Dataset, to be fully transparent about the location of the projection origin, the parameter *latitude of natural origin* (φ_O) is included in the defining parameters of the map projection method and map projections. It must have a value of zero because, by definition, the location of the natural origin for this method is on the equator. However, this parameter is not used in the conversion formulas.

2. For each of variants B and C (the 2SP cases), there is a pair of standard parallels with equal latitude value but different signs. Only one needs to be defined.

Several equations are common between the variants.

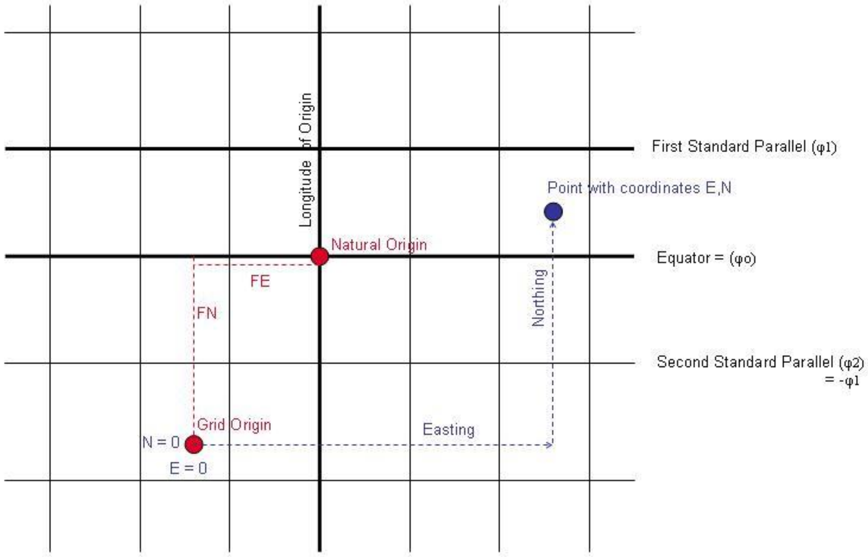


Figure 4 – Mercator variants A and B.

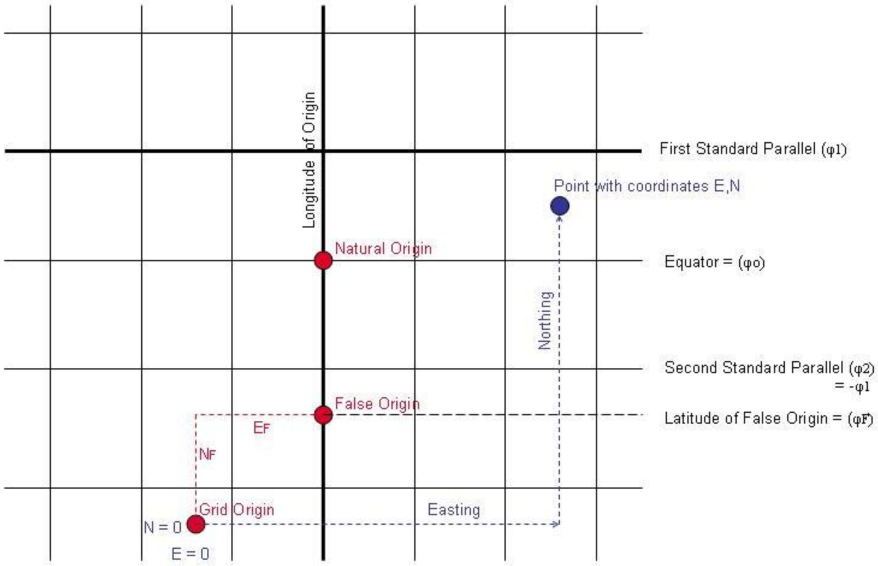


Figure 5 – Mercator variant C.

The formulas to derive projected Easting and Northing coordinates are:

For variants B and C, the two standard parallel cases either with false coordinates defined at the natural origin of the projection (variant B) or with the false origin (variant C), first calculate k_O, the scale factor at the equator or natural origin, from

$$k_O = \cos\varphi_1 / (1 - e^2 \sin^2\varphi_1)^{0.5}$$

where φ_1 is the absolute value of the first standard parallel (i.e., positive).

Then, for both of variants A and B,

$$E = FE + a\, k_O\, (\lambda - \lambda_O)$$

$$N = FN + a\, k_O \ln\{\tan(\pi/4 + \varphi/2)[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{(e/2)}\}$$

where symbols are as listed above and logarithms are natural.

But for variant C only:

$$M = a\, k_O \ln\{\tan(\pi/4 + \varphi_F/2)[(1 - e\sin\varphi_F)/(1 + e\sin\varphi_F)]^{(e/2)}\}$$

$$E = E_F + a\, k_O\, (\lambda - \lambda_F)$$

$$N = (N_F - M) + a\, k_O \ln\{\tan(\pi/4 + \varphi/2)[(1 - e\sin\varphi)/(1 + e\sin\varphi)]^{(e/2)}\}$$

The reverse formulas to derive latitude and longitude from E and N values are:

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi)$$
$$+ (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi)$$
$$+ (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

where for both variants A and B:

$\chi = \pi/2 - 2 \text{ atan } (t)$
$t = B^{(FN-N)/(a. kO)}$
B = base of the natural logarithm, 2.7182818…
and for the 2 SP variant B, k_O is calculated as for the forward transformation above.

Then:
$\lambda = [(E - FE)/(a\ k_O)] + \lambda_O$

For variant C:
$t = B^{(NF - M - N)/(a. kO)}$
where, B is base of the natural logarithm and M and k_O are calculated as for the forward transformation.

Then
$\lambda = [(E - E_F)/(a\ k_O)] + \lambda_F$

Examples
<u>1. Mercator (variant A)</u>
Also known as Mercator (1SP)
(EPSG Dataset coordinate operation method code 9804)

For Projected Coordinate Reference System: Makassar / NEIEZ (EPSG CRS code 30020)

Parameters:

Ellipsoid:	Bessel 1841	a = 6377397.155metres	1/f = 299.1528128
	from which:	e = 0.081696831	

Latitude of natural origin	φ_O	0°00'00"N	=	0.0 rad
Longitude of natural origin	λ_O	110°00'00"E	=	1.91986218 rad
Scale factor at natural origin	k_O	0.997		
False easting	FE	3900000.00	metres	
False northing	FN	900000.00	metres	

Forward calculation for:

Latitude	φ	=	3°00'00.00"S	=	-0.05235988 rad
Longitude	λ	=	120°00'00.00"E	=	2.09439510 rad

gives	Easting	E	=	5009726.58 metres
	Northing	N	=	569150.82 metres

Reverse calculation for same easting and northing first gives:

t	=	1.0534121
χ	=	-0.0520110

Then	Latitude	φ	=	3°00'00.000"S
	Longitude	λ	=	120°00'00.000"E

Further examples of input and output may be found in test procedure 5111 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, <u>https://gigs.iogp.org/</u>.

2. Mercator variant B
(EPSG Dataset coordinate operation method code 9805)

For Projected Coordinate Reference System: Pulkovo 1942 / Caspian Sea Mercator (EPSG CRS code 3388)

Parameters:

Ellipsoid:	Krassowsky 1940	a = 6378245.0 metres	1/f = 298.3
	from which:	e = 0.08181333	e^2 = 0.00669342

Latitude of 1st standard parallel	φ_1	42°00'00"N	=	0.73303829 rad
Longitude of natural origin	λ_O	51°00'00"E	=	0.89011792 rad
False easting	FE	0.00	metres	
False northing	FN	0.00	metres	

then scale factor at natural origin k_O (at latitude of natural origin at 0°N) = 0.744260894.

Forward calculation for:

Latitude	φ	=	53°00'00.00"N	=	0.9250245 rad
Longitude	λ	=	53°00'00.00"E	=	0.9250245 rad

gives	Northing	N	=	5171848.07 metres
	Easting	E	=	165704.29 metres

Reverse calculation for same northing and easting first gives:

t = 0.336391288
χ = 0.921795958

Then	Latitude	φ	=	53°00'00.000"N
	Longitude	λ	=	53°00'00.000"E

Further examples of input and output may be found in test procedure 5112 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3. Mercator variant C
(EPSG Dataset coordinate operation method code 1108)

Parameters:

Ellipsoid:	Krassowsky 1940	a = 6378245.0 metres	1/f = 298.3
	from which:	e = 0.08181333	e^2 = 0.00669342

Latitude of 1st standard parallel	φ_1	42°00'00"N	=	0.73303829 rad
Latitude of false origin	φ_F	42°00'00"N	=	0.73303829 rad
Longitude of false origin	λ_F	51°00'00"E	=	0.89011792 rad
Easting at false origin	E_F	0.00	metres	
Northing at false origin	N_F	0.00	metres	

First calculate projection constants:

k_O = 0.744260894 M = 3819897.85

Forward calculation for:

Latitude	φ	=	53°00'00.00"N	=	0.9250245 rad
Longitude	λ	=	53°00'00.00"E	=	0.9250245 rad

gives	Easting	E	=	165704.29 metres
	Northing	N	=	1351950.22 metres

(Note the significant difference in northings from the variant B example).

Reverse calculation for same easting and northing first gives:

$t = 0.336391288$
$\chi = 0.921795958$

Then	Latitude	φ	=	53°00'00.000"N
	Longitude	λ	=	53°00'00.000"E

3.5.1.1 Mercator (Spherical)

(EPSG Dataset coordinate operation method code 1026)

The formulas to derive projected Easting and Northing coordinates from latitude φ and longitude λ using a sphere rather than ellipsoid as model of the earth are:

$$E = FE + R\,(\lambda - \lambda_O)$$
$$N = FN + R \ln[\tan(\pi/4 + \varphi/2)]$$

where λ_O is the longitude of natural origin and FE and FN are false easting and false nothing.

R is the radius of the sphere. It will normally be given as one of the CRS parameters. However, if the datum for the CRS uses an ellipsoid rather than a sphere, R should be calculated as the radius of the conformal sphere at the latitude of natural origin $\varphi_O = 0$ using the formula for R_C given in the Implementation Notes in Section §1.1 of this document.

If latitude $\varphi = 90^o$, N is infinite. The above formula for N will fail near to the pole, and should not be used poleward of 88^o.

The reverse formulas to derive latitude and longitude on the sphere from E and N values are:

$$D = -(N - FN) / R = (FN - N) / R$$
$$\varphi = \pi/2 - 2 \operatorname{atan}(\mathbf{e}^D) \text{ where } \mathbf{e}\text{=base of natural logarithms, 2.7182818…}$$
$$\lambda = [(E - FE)/R] + \lambda_O$$

Note that for the Mercator (Spherical) method, in the EPSG Dataset, to be fully transparent about the location of the projection origin, the parameter *latitude of natural origin* (φ_O) is included in the defining parameters of the map projection method. It must have a value of zero because, by definition, the location of the natural origin for this method is on the equator. However, this parameter is not used in the conversion formulas.

Example

Parameters:

Sphere: R = 6371007.0 metres

Latitude of natural origin	φ_O	0°00'00.000"N	=	0.0 rad
Longitude of natural origin	λ_O	0°00'00.000"E	=	0.0 rad
False easting	FE	0.00	metres	
False northing	FN	0.00	metres	

Forward calculation for:

Latitude	φ	=	24°22'54.433"N	=	0.425542460 rad
Longitude	λ	=	100°20'00.000"W	=	–1.751147016 rad

whence

$E = –11\ 156\ 569.90$ m
$N = 2\ 796\ 869.94$ m

Reverse calculation for the same point (–11 156 569.90 m E, 2 796 869.94m N) first gives:

$D = –0.438999665$

Then	Latitude	φ	=	0.425542460 rad	=	24°22'54.433"N
	Longitude	λ	=	–1.751147016 rad	=	100°20'00.000"W

3.5.1.2 Popular Visualisation Pseudo-Mercator ("Web Mercator")

(EPSG Dataset coordinate operation method code 1024)

This method is used by some popular web mapping and visualisation applications. Strictly speaking the inclusion of 'Mercator' in the method name is misleading: it is *not* a Mercator projection. It is a different map projection and uses its own distinct formula: it is a separate method. Unlike either the spherical or ellipsoidal Mercator projection methods, this method is not conformal: scale factor varies as a function of azimuth, which creates angular distortion. Despite angular distortion, there is no convergence in the meridian, so the graticule has a similar appearance to the graticule of a Mercator projection, but the graticules of the two projections do not overlay. See IOGP Report 373-23 - *Geomatics Guidance Note 23 – Web Mercator* for more information.

The formulas to derive projected easting and northing coordinates from ellipsoidal latitude φ and longitude λ are:

$$E = FE + a\,(\lambda - \lambda_O)$$
$$N = FN + a \ln[\tan(\pi/4 + \varphi/2)]$$

where a is the ellipsoid semi-major axis, λ_O is the longitude of natural origin, FE and FN are false easting and false nothing, and logarithms are natural.

If latitude $\varphi = 90$º, N is infinite. The above formula for N will fail near to the pole, and should not be used poleward of 88º. Web applications often truncate the projection at approximately ±85.06°, a value which coincides with the boundary of tiles constructed on the map grid.

The reverse formulas to derive latitude and longitude on the ellipsoid from E and N values are:

$$D = – (N – FN) / a = (FN – N) / a$$
$$\varphi = \pi/2 - 2\,\mathrm{atan}(\mathbf{e}^D) \text{ where } \mathbf{e}\text{=base of natural logarithms, 2.7182818…}$$
$$\lambda = [(E - FE)/a] + \lambda_O$$

Note that in these formulas the parameter *latitude of natural origin* (φ_O) is not used. However, for completeness in CRS labelling the EPSG Dataset includes this parameter, which must have a value of zero.

Distortion

q_α is the scale factor at a given azimuth α. It is a function of the radius of curvature of the ellipsoid at that azimuth.

$$Q_\alpha = a / (R' \cos\varphi)$$

where

$$R' = \rho\,\nu / (\nu \cos^2\alpha + \rho \sin^2\alpha)$$
$$\rho = a\,(1 – e^2) / (1 – e^2 \sin^2\varphi)^{3/2}$$
$$\nu = a / (1 – e^2 \sin^2\varphi)^{1/2}$$

ρ and ν are the radii of curvature of the ellipsoid at latitude φ in the plane of the meridian and perpendicular to the meridian respectively;

When the azimuth is 0º, 180º, 90º or 270º the scale factors in the meridian (h) and on the parallel (k) are:

$$h = q_0 = q_{180} = a / (\rho \cos \varphi)$$
$$k = q_{90} = q_{270} = a / (\nu \cos \varphi)$$

h and k are not equal, which demonstrates the non-conformality of the Pseudo-Mercator method.

Maximum angular distortion ω is a function of latitude and is found from:

$$\omega = 2 \text{ asin}\{[\text{ABS}(h – k)] / (h + k)\}$$

Example

For Projected Coordinate Reference System: WGS 84 / Pseudo-Mercator (EPSG CRS code 3857)

Parameters:

Ellipsoid: WGS 84 a = 6378137.0 metres 1/f = 298.257223563

Latitude of natural origin	φ_O	0°00'00.000"N	=	0.0 rad
Longitude of natural origin	λ_O	0°00'00.000"E	=	0.0 rad
False easting	FE	0.00	metres	
False northing	FN	0.00	metres	

Forward calculation for the same coordinate values as used for the Mercator (Spherical) example:

Latitude	φ	=	24°22'54.433"N	=	0.425542460 rad
Longitude	λ	=	100°20'00.000"W	=	–1.751147016 rad

whence

E = –11 169 055.58 m N = 2 800 000.00 m

(in comparison, the spherical Mercator coordinates are E = –11 156 569.90m, N = 2 796 869.9m)

and h = 1.1034264
k = 1.0972914
ω = 0°19'10.01"

Reverse calculation for a point 10km north on the grid (–11 169 055.58 m E, 2 810 000.00m N) first gives:

D = –0.44056752

Then	Latitude	φ	=	0.426970023 rad	=	24°27'48.889"N
	Longitude	λ	=	–1.751147016 rad	=	100°20'00.000"W

In comparision, the same WGS 84 ellipsoidal coordinates, when converted to the WGS 84 / World Mercator projected coordinate reference system (EPSG CRS code 3395) using the ellipsoidal Mercator (1SP) variant A method described above, results in a grid distance between the two points of 9944.4m, a scale difference of ~0.5%.

Table 6 – Comparison of Pseudo-Mercator (Web Mercator) and Mercator coordinates.

WGS 84		WGS 84 / Pseudo-Mercator (EPSG CRS code 3857)		WGS 84 / World Mercator (EPSG CRS code 3395)	
Latitude	Longitude	Easting	Northing	Easting	Northing
24°27'48.889"N	100°20'00.000"W	–11169055.58m	2810000.00m	–11169055.58m	2792311.49m
24°22'54.433"N	100°20'00.000"W	–11169055.58m	2800000.00m	–11169055.58m	2782367.06m
		0.00m	10000.00m	0.00m	9944.43m

3.5.2 Cassini-Soldner

(EPSG Dataset coordinate operation method code 9806)

The Cassini-Soldner projection is the ellipsoidal version of the Cassini projection for the sphere. It is not conformal but, as it is relatively simple to construct, it was extensively used in the last century and is still useful for mapping areas with limited longitudinal extent. It has now largely been replaced by the conformal Transverse Mercator which it resembles. Like this, it has a straight central meridian along which the scale is true, all other meridians and parallels are curved, and the scale distortion increases rapidly with increasing distance from the central meridian.

The formulas to derive projected Easting and Northing coordinates are:

$$\text{Easting, } E = FE + \nu[A - TA^3/6 - (8 - T + 8C)TA^5/120]$$

$$\text{Northing, } N = FN + X$$

where $X = M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 6C)A^4/24]$

$A = (\lambda - \lambda_O)\cos\varphi$

$T = \tan^2\varphi$

$C = e^2 \cos^2\varphi/(1 - e^2)$

$\nu = a /(1 - e^2\sin^2\varphi)^{0.5}$

and M, the distance along the meridian from equator to latitude φ, is given by

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 -....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 +....)\sin2\varphi$$
$$+ (15e^4/256 + 45e^6/1024 +.....)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$$

with φ in radians.

M_O is the value of M calculated for the latitude of the natural origin φ_O. This may not necessarily be chosen as the equator.

The reverse formulas to compute latitude and longitude from Easting and Northing are:

$$\varphi = \varphi_1 - (\nu_1\tan\varphi_1/\rho_1)[D^2/2 - (1 + 3T_1)D^4/24]$$
$$\lambda = \lambda_O + [D - T_1D^3/3 + (1 + 3T_1)T_1D^5/15]/\cos\varphi_1$$

where

$\nu_1 = a /(1 - e^2\sin^2\varphi_1)^{0.5}$

$\rho_1 = a(1 - e^2)/(1 - e^2\sin^2\varphi_1)^{1.5}$

φ_1 is the latitude of the point on the central meridian which has the same Northing as the point whose coordinates are sought, and is found from:

$$\varphi_1 = \mu_1 + (3e_1/2 - 27e_1^3/32 +.....)\sin2\mu_1 + (21e_1^2/16 - 55e_1^4/32 + ...)\sin4\mu_1$$
$$+ (151e_1^3/96 +.....)\sin6\mu_1 + (1097e_1^4/512 - ....)\sin8\mu_1 + ...$$

where

$e_1 = [1 - (1 - e^2)^{0.5}]/[1 + (1 - e^2)^{0.5}]$

$\mu_1 = M_1/[a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)]$

$M_1 = M_O + (N - FN)$

$T_1 = \tan^2\varphi_1$

$D = (E - FE)/\nu_1$

Example
For Projected Coordinate Reference System: Trinidad 1903 / Trinidad Grid (EPSG CRS code 30200)

Parameters:

Ellipsoid:	Clarke 1858	a = 20926348 ftCla	b = 20855233 ftCla
	from which:	a = 31706587.879 Clarke's links	
		1/f = 294.2606764	e^2 = 0.006785146

Latitude of natural origin	φ_O	10°26'30"N	
Longitude of natural origin	λ_O	61°20'00"W	
False easting	FE	430000.00	Clarke's links (lkCla)
False northing	FN	325000.00	Clarke's links (lkCla)

Forward calculation for:
Latitude φ = 10°00'00.00"N Longitude λ = 62°00'00.00"W

first gives :

φ_O	=	0.182241463 rad	λ_O	=	-1.070468608 rad
M_O	=	5739691.117			

then

φ	=	0.174532925 rad	λ	=	-1.082104136 rad
A	=	-0.011458759	C	=	0.006625504
T	=	0.031091204	M	=	5496860.238
ν	=	31709831.916			

Then	Easting	E	=	66644.94 Clarke's links
	Northing	N	=	82536.22 Clarke's links

Reverse calculation for same easting and northing first gives:

e_1	=	0.001702066	D	=	-0.011458751
T_1	=	0.031095442	M_1	=	5497227.336
ν_1	=	31709832.345	μ_1	=	0.173673055
φ_1	=	0.174544579	ρ_1	=	31501122.396

Then	Latitude	φ	=	10°00'00.000"N
	Longitude	λ	=	62°00'00.000"W

Further examples of input and output may be found in test procedure 5108 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.5.2.1 Hyperbolic Cassini-Soldner

(EPSG Dataset coordinate operation method code 9833)

The grid for the island of Vanua Levu, Fiji, uses a modified form of the standard Cassini-Soldner projection known as the Hyperbolic Cassini-Soldner.

Easting is calculated as for the standard Cassini-Soldner above. The standard Cassini-Soldner formula to derive projected Northing is modified to:

$$\text{Northing, } N = FN + X - (X^3/6\rho\nu)$$

where $\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{1.5}$ and X and ν are as in the standard Cassini-Soldner above.

For the reverse calculation of latitude and longitude from easting and northing, the standard Cassini-Soldner formulas given in the previous section need to be modified to account for the hyperbolic correction factor ($X^3/6\rho\nu$). Specifically for the Fiji Vanua Levu grid, the following may be used. The standard Cassini-Soldner formulas given in the previous section are used except that the equation for M_1 is modified to

$$M_1 = M_O + (N - FN) + q$$

where

$$\varphi_1' = \varphi_O + (N - FN)/315320$$
$$\rho_1' = a(1 - e^2)/(1 - e^2\sin^2\varphi_1')^{1.5}$$
$$\nu_1' = a/(1 - e^2\sin^2\varphi_1')^{0.5}$$
$$q' = (N - FN)^3 / 6\,\rho_1'\,\nu_1'$$
$$q = (N - FN + q')^3 / 6\,\rho_1'\,\nu_1'$$

Example

For Projected Coordinate Reference System: Vanua Levu 1915 / Vanua Levu Grid (EPSG CRS code 3139)

Parameters:

Ellipsoid:	Clarke 1880		a = 20926202 ft	b =20854895 ft
		From which:	a = 317063.667 chains	
			1/f = 293.4663077	e^2 = 0.006803481

Latitude of natural origin	φ_O	16°15'00"S	
Longitude of natural origin	λ_O	179°20'00"E	
False easting	FE	12513.318	chains
False northing	FN	16628.885	chains

Forward calculation for:

Latitude	φ	=	16°50'29.2435"S
Longitude	λ	=	179°59'39.6115"E

first gives :

φ_O	=	-0.283616003 rad	λ_O	=	3.129957125 rad
M_O	=	-89336.59			

then :

φ	=	-0.293938867 rad	λ	=	3.141493807 rad
A	=	0.011041875	C	=	0.006275088
T	=	0.091631819	M	=	-92590.02
ν	=	317154.24	ρ	=	315176.48
X	=	-3259.28			

Then	Easting	E	=	16015.2890 chains
	Northing	N	=	13369.6601 chains

Reverse calculation for same easting and northing first gives:

φ_1'	=	0.293952249	q'	=	-0.058
ν_1'	=	317154.25	q	=	-0.058
ρ_1'		315176.50			
e_1	=	0.001706681	D	=	0.011041854
T_1	=	0.091644092	M_1	=	-92595.87
ν_1	=	317154.25	μ_1	=	-0.292540098
ρ_1	=	315176.51	φ_1	=	-0.293957437

Then	Latitude	φ	=	16°50'29.2435"S
	Longitude	λ	=	179°59'39.6115"E

3.5.3 Transverse Mercator

3.5.3.1 General Case

(EPSG Dataset coordinate operation method code 9807)

The Transverse Mercator projection in its various forms is the basis for the most widely used projected coordinate reference system for world topographical and offshore mapping. All versions have the same basic characteristics and formulas. The differences which distinguish the different forms of the projection which are applied in different countries arise from variations in the choice of values for the coordinate conversion parameters, namely the latitude of the natural origin, the longitude of the natural origin (central meridian), the scale factor at the natural origin (on the central meridian), and the values of False Easting and False Northing, which embody the units of measurement, given to the origin. Additionally there are variations in the width of the longitudinal zones for the projections used in different territories.

Table 7 indicates the variations in the coordinate conversion parameters which distinguish the different forms of the Transverse Mercator projection and are used in the EPSG Dataset Transverse Mercator map projection operations.

Table 7 – Transverse Mercator variants.

Coordinate Operation Method Name	Areas used	Latitude of natural origin	Longitude of natural origin	Scale Factor on central meridian	Zone width	False Easting	False Northing
Transverse Mercator	Various, world wide	Various	Various	Various	Usually less than 6°	Various	Various
Gauss-Kruger	Former USSR, Yugoslavia, Germany, S. America, China	Usually 0°	Various, according to area of cover	Usually 1.000000	Often 6° or 3°, sometimes less than 3°	Various, but often 500000m prefixed by zone number	Various
Gauss Boaga	Italy	Various	Various	0.9996	6°	Various	0m
Transverse Mercator (south oriented)	Southern Africa	0°	2° intervals E of 11°E	1.000000	2°	0m	0m
UTM North hemisphere	World wide equator to 84°N	Always 0°	6° intervals E of 177°W	Always 0.9996	Always 6°	500000m	0m
UTM South hemisphere	World wide north of 80° S to equator	Always 0°	6° intervals E of 177°W	Always 0.9996	Always 6°	500000m	10000000 m

In the EPSG Dataset, the Transverse Mercator, Gauss-Kruger and Gauss-Boaga coordinate conversion methods are considered to be the same under the name “Transverse Mercator”. The most familiar and commonly used Transverse Mercator in the energy industry is the Universal Transverse Mercator (UTM), a global system of 120 TM zones where many of the defining parameters have the same value in each zone.

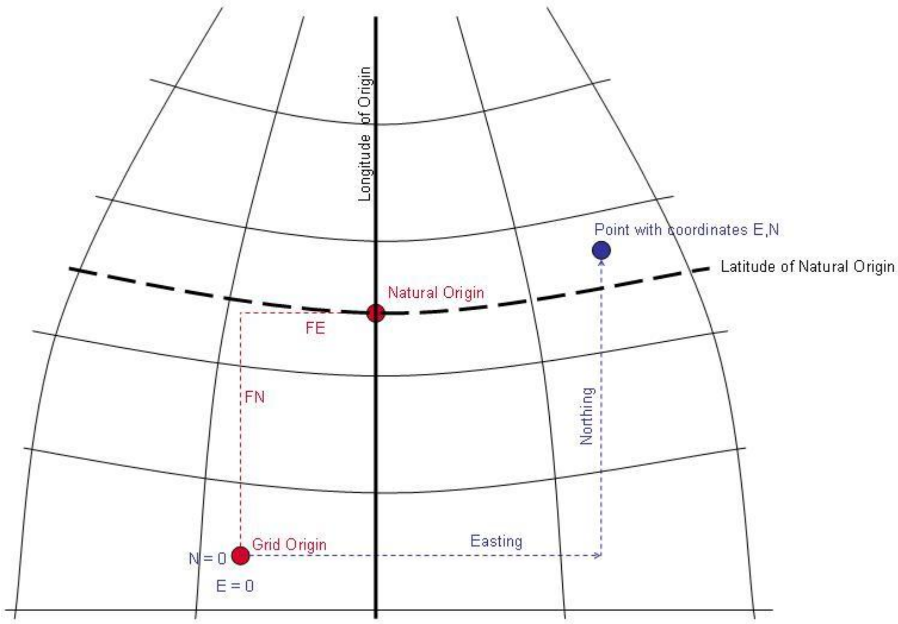


Figure 6 – Transverse Mercator in the northern hemisphere.

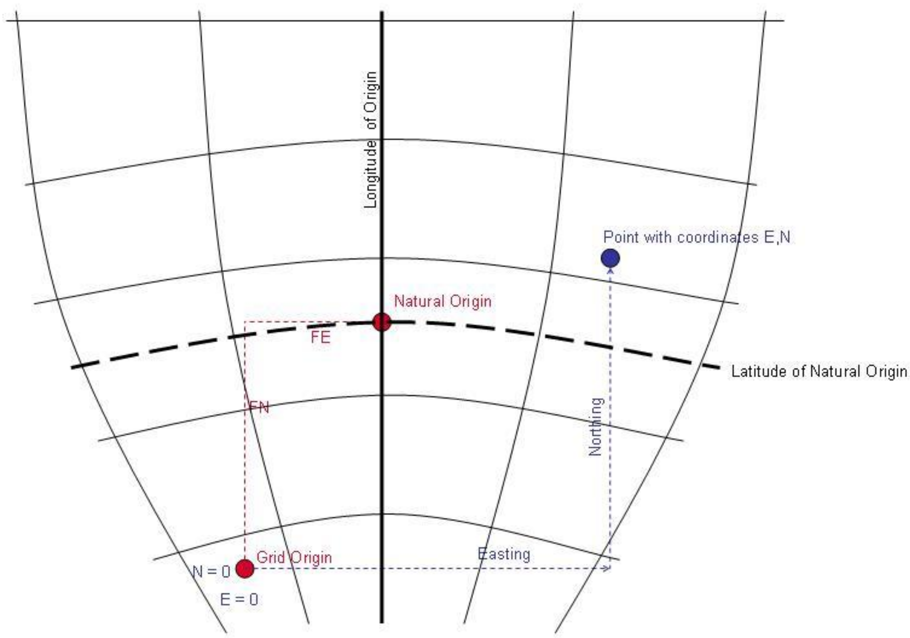


Figure 7 – Transverse Mercator in the southern hemisphere.

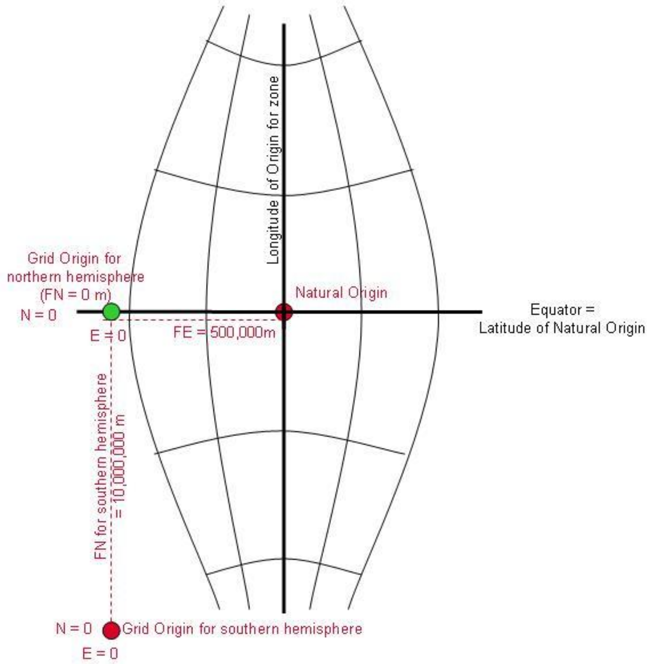


Figure 8 – One zone of the Universal Transverse Mercator system.

From April 2010, the EPSG Dataset supports two distinct formulas for the Transverse Mercator projection (EPSG Dataset coordinate operation method code 9807):

- Our original formulas which are based on those by Snyder (1987), published in US Geological Survey Professional Paper No. 1395[3] (referred to below as 'USGS formulas')
- Formulas which are based on those of Krüger and published in Finland by the public administration information management board (JUHTA (2008))[4], (referred to below as 'JHS formulas')

Within ±4º of longitude east or west of the projection longitude of origin, the results from these two formulas can be considered the same. Within this longitude range, differences in results are never more than 3mm (forward conversion) and 0.0005" (reverse conversion), with these differences greatest at the outer limits of this ±4º longitude band at high latitudes. Both USGS and JHS formulas are satisfactory within ±4º east or west of the longitude of origin, the area for which the characteristics of the projection method are best suited. However, the JHS formulas are more robust over a wider area – to ±40º of the longitude of origin – and have better forward and reverse round trip closure. For these reasons, the JHS formulas are now recommended.

Warning: more than ±4º of longitude east or west of the projection longitude of origin, the USGS and JHS formulas produce different results. Their use should not be mixed, else location integrity will be compromized.

[3] Snyder, John P., "Map Projections - A Working Manual". *United States Geological Survey*, Professional Paper No.1395, 1987.
[4] JUHTA, "ETRS89-related map projections, grid coordinates and map sheet division", *Public Administration Information Management Board*, Specification JHS 154, Appendix 1, "Projection formulas". June 2008.

It must be recognized that although the JHS formulas allow more accurate coordinate conversion over this wide band, the increasing distortions (in distance, area, and angle which are inherent in the Transverse Mercator projection method well away from the longitude of origin) cannot be avoided. These distortions make the method quite unsuitable in these outer areas for many purposes, for example, engineering and large-scale mapping.

JHS formulas:

For the calculation of easting and northing from latitude and longitude, first calculate_constants for the projection:

$n = f / (2-f)$
$B = [a/(1+n)] (1 + n^2/4 + n^4/64)$

$h_1 = n/2 – (2/3)n^2 + (5/16)n^3 + (41/180)n^4$
$h_2 = (13/48)n^2 – (3/5)n^3 + (557/1440)n^4$
$h_3 = (61/240)n^3 – (103/140)n^4$
$h_4 = (49561/161280)n^4$

Then the meridional arc distance from equator to the projection origin (M_O) is computed from:

If $\varphi_O = 0$ then $M_O = 0$
else if $\varphi_O \equiv 90°N \equiv \pi/2$ radians then $M_O = B (\pi/2)$
else if $\varphi_O \equiv 90°S \equiv -\pi/2$ radians then $M_O = B (-\pi/2)$
else

$Q_O = \operatorname{asinh}(\tan\varphi_O) – [e \operatorname{atanh}(e \sin\varphi_O)]$

$\beta_O = \operatorname{atan}(\sinh Q_O)$
$\xi_{O0} = \operatorname{asin} (\sin \beta_O)$

Note: The previous two steps are taken from the generic calculation flow given below for latitude φ, but here for φ_O may be simplified to $\xi_{O0} = \beta_O = \operatorname{atan}(\sinh Q_O)$.

$\xi_{O1} = h_1 \sin(2\xi_{O0})$
$\xi_{O2} = h_2 \sin(4\xi_{O0})$
$\xi_{O3} = h_3 \sin(6\xi_{O0})$
$\xi_{O4} = h_4 \sin(8\xi_{O0})$
$\xi_O = \xi_{O0}+ \xi_{O1}+ \xi_{O2}+ \xi_{O3}+ \xi_{O4}$
$M_O = B\, \xi_O$

end

Note: if the projection grid origin is very close to but not exactly at the pole (within 2" or 50m), in the equation above for Q_O, the tangent function is unstable and may fail. Rather than using Qo as above, M_O may instead be calculated (as in the USGS formula below) from:

$$M_O = a[(1 – e^2/4 – 3e^4/64 – 5e^6/256 –....)\varphi_O – (3e^2/8 + 3e^4/32 + 45e^6/1024+....)\sin2\varphi_O + (15e^4/256 + 45e^6/1024 +.....)\sin4\varphi_O – (35e^6/3072 + ...)\sin6\varphi_O + ...]$$

with φ_O in radians.

Then

$Q = \operatorname{asinh}(\tan\varphi) – [e \operatorname{atanh}(e \sin\varphi)]$
$\beta = \operatorname{atan}(\sinh Q)$
$\eta_0 = \operatorname{atanh} [\cos \beta \sin(\lambda – \lambda_O)]$
$\xi_0 = \operatorname{asin} (\sin \beta \cosh \eta_0)$

$\xi_1 = h_1 \sin(2\xi_0) \cosh(2\eta_0)$ $\qquad$ $\eta_1 = h_1 \cos(2\xi_0) \sinh(2\eta_0)$

$\xi_2 = h_2 \sin(4\xi_0) \cosh(4\eta_0)$ $\eta_2 = h_2 \cos(4\xi_0) \sinh(4\eta_0)$
$\xi_3 = h_3 \sin(6\xi_0) \cosh(6\eta_0)$ $\eta_3 = h_3 \cos(6\xi_0) \sinh(6\eta_0)$
$\xi_4 = h_4 \sin(8\xi_0) \cosh(8\eta_0)$ $\eta_4 = h_4 \cos(8\xi_0) \sinh(8\eta_0)$
$\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3 + \xi_4$ $\eta = \eta_0 + \eta_1 + \eta_2 + \eta_3 + \eta_4$

and

Easting, $E = FE + k_O B \eta$
Northing, $N = FN + k_O (B \xi - Mo)$

For the reverse formulas to convert Easting and Northing projected coordinates to latitude and longitude, first calculate constants of the projection where n is as for the forward conversion, as are B and M_O:

$h_1' = n/2 - (2/3)n^2 + (37/96)n^3 - (1/360)n^4$
$h_2' = (1/48)n^2 + (1/15)n^3 - (437/1440)n^4$
$h_3' = (17/480)n^3 - (37/840)n^4$
$h_4' = (4397/161280)n^4$

Then

$\eta' = (E - FE) / (B k_O)$
$\xi' = [(N - FN) + k_O M_O] / (B k_O)$

$\xi_1' = h_1' \sin(2\xi') \cosh(2\eta')$ $\eta_1' = h_1' \cos(2\xi') \sinh(2\eta')$
$\xi_2' = h_2' \sin(4\xi') \cosh(4\eta')$ $\eta_2' = h_2' \cos(4\xi') \sinh(4\eta')$
$\xi_3' = h_3' \sin(6\xi') \cosh(6\eta')$ $\eta_3' = h_3' \cos(6\xi') \sinh(6\eta')$
$\xi_4' = h_4' \sin(8\xi') \cosh(8\eta')$ $\eta_4' = h_4' \cos(8\xi') \sinh(8\eta')$
$\xi_0' = \xi' - (\xi_1' + \xi_2' + \xi_3' + \xi_4')$ $\eta_0' = \eta' - (\eta_1' + \eta_2' + \eta_3' + \eta_4')$

$\beta' = \text{asin}(\sin \xi_0' / \cosh \eta_0')$
$Q' = \text{asinh}(\tan \beta')$
$Q'' = Q' + [e \,\text{atanh}(e \tanh Q')]$
$= Q' + [e \,\text{atanh}(e \tanh Q'')]$ which should be iterated until the change in Q" is insignificant

Then

$\varphi = \text{atan}(\sinh Q'')$
$\lambda = \lambda_O + \text{asin}(\tanh(\eta_0') / \cos \beta')$

<u>USGS formulas</u> (presented here only for backward compatibility within 4° of longitude of origin):

Easting, $E = FE + k_O\nu[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^2)A^5/120]$

Northing, $N = FN + k_O\{M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 9C + 4C^2)A^4/24 + (61 - 58T + T^2 + 600C - 330e'^2)A^6/720]\}$

where $T = \tan^2\varphi$
$C = e^2 \cos^2\varphi / (1 - e^2)$
$A = (\lambda - \lambda_O)\cos\varphi$, with λ and λ_O in radians
$\nu = a / (1 - e^2\sin^2\varphi)^{0.5}$
$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + ...)\sin2\varphi + (15e^4/256 + 45e^6/1024 + ...)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$
with φ in radians and M_O for φ_O, the latitude of the origin, derived in the same way.

The reverse formulas to convert Easting and Northing projected coordinates to latitude and longitude are:

$$\varphi = \varphi_1 - (\nu_1 \tan\varphi_1/\rho_1)[D^2/2 - (5 + 3T_1 + 10C_1 - 4C_1^2 - 9e'^2)D^4/24 + (61 + 90T_1 + 298C_1 + 45T_1^2 - 252e'^2 - 3C_1^2)D^6/720]$$

$$\lambda = \lambda_O + [D - (1 + 2T_1 + C_1)D^3/6 + (5 - 2C_1 + 28T_1 - 3C_1^2 + 8e'^2 + 24T_1^2)D^5/120] / \cos\varphi_1$$

where

$\nu_1 = a / (1 - e^2\sin^2\varphi_1)^{0.5}$

$\rho_1 = a(1 - e^2) / (1 - e^2\sin^2\varphi_1)^{1.5}$

φ_1 may be found as for the Cassini projection from:

$$\varphi_1 = \mu_1 + (3e_1/2 - 27e_1^3/32 + ...)\sin 2\mu_1 + (21e_1^2/16 - 55e_1^4/32 + ...)\sin 4\mu_1 + (151e_1^3/96 + ...)\sin 6\mu_1 + (1097e_1^4/512 - ....)\sin 8\mu_1 + ...$$

and where

$e_1 = [1 - (1 - e^2)^{0.5}] / [1 + (1 - e^2)^{0.5}]$

$\mu_1 = M_1/[a(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)]$

$M_1 = M_O + (N - FN) / k_O$

$T_1 = \tan^2\varphi_1$

$C_1 = e'^2\cos^2\varphi_1$

$e'^2 = e^2 / (1 - e^2)$

$D = (E - FE) / (\nu_1 k_O)$

For areas south of the equator, the value of latitude φ will be negative and the formulas to compute the E and N given above will automatically result in the correct values. Note that the false northings of the origin, if the equator, will need to be large to avoid negative northings and for the UTM projection is in fact 10,000,000m. Alternatively, as in the case of Argentina's Transverse Mercator (Gauss-Kruger) zones, the origin is at the south pole with a northings of zero. However, each zone central meridian takes a false easting of 500000m prefixed by an identifying zone number. This ensures that instead of points in different zones having the same eastings, every point in the country, irrespective of its projection zone, will have a unique set of projected system coordinates. Strict application of the above formulas, with south latitudes negative, will result in the derivation of the correct Eastings and Northings.

Similarly, in applying the reverse formulas to determine a latitude south of the equator, a negative sign for φ results from a negative φ_1 which in turn results from a negative M_1.

Examples

For Projected Coordinate Reference System OSGB36 / British National Grid (EPSG CRS code 27700)

Parameters:

Ellipsoid:	Airy 1830	a = 6377563.396 metres	1/f = 299.3249646
	from which:	e^2 = 0.00667054	e'^2 = 0.00671534

Latitude of natural origin	φ_O	49°00'00"N	
Longitude of natural origin	λ_O	2°00'00"W	
Scale factor at natural origin	k_O	0.9996012717	
False easting	FE	400000.00	metres
False northing	FN	-100000.00	metres

<u>JHS formulas</u>:

Forward calculation for:

Latitude	φ	=	50°30'00.00"N
Longitude	λ	=	00°30'00.00"E

Constants of the projection:

φ_O	=	0.85521133 rad	λ_O	=	-0.03490659 rad
n	=	0.00167322	B	=	6366914.609
h_1	=	0.0008347452	h_2	=	0.0000007554

h_3	=	1.18487E-09	h_4	=	2.40864E-12
Q_O	=	0.9787671618			
ξ_{O0}	=	0.8518980373	ξ_{O1}	=	0.0008273732
ξ_{O2}	=	-0.0000001986	ξ_{O3}	=	-1.0918E-09
ξ_{O4}	=	1.2218E-12	M_O	=	5429228.602
φ	=	0.88139127 rad	λ	=	0.00872665 rad
Q	=	1.0191767215	β	=	0.8781064142
η_0	=	0.0278629616	ξ_0	=	0.8785743280
η_1	=	-0.0000086229	ξ_1	=	0.0008215669
η_2	=	-0.0000000786	ξ_2	=	-0.0000002768
η_3	=	1.05551E-10	ξ_3	=	-1.01855E-09
η_4	=	3.97791E-13	ξ_4	=	1.67447E-12
η	=	0.0278542603	ξ	=	0.8793956171

Then Easting E = 577274.99 metres
Northing N = 69740.50 metres

Reverse calculation for same easting and northing first gives:

h_1'	=	0.0008347455	h_2'	=	0.0000000586
h_3'	=	1.65563E-10	h_4'	=	2.13692E-13

Then

ξ'	=	0.87939562	η'	=	0.0278542603
ξ_1'	=	0.0008213109	η_1'	=	-0.0000086953
ξ_2'	=	-0.0000000217	η_2'	=	-0.0000000061
ξ_3'	=	-1.41881E-10	η_3'	=	1.486E-11
ξ_4'	=	1.49609E-13	η_4'	=	3.50657E-14
ξ_0'	=	0.8785743280	η_0'	=	0.0278629616
β'	=	0.8781064142	Q'	=	1.0191767215
Q" 1st iteration	=	1.0243166838	Q" 2nd iteration	=	1.0243306667
Q" 3rd iteration	=	1.0243307046	Q" 4th iteration	=	1.0243307047

Then Latitude φ = 50°30'00.000"N
Longitude λ = 00°30'00.000"E

<u>USGS formulas</u>:
Forward calculation for:
Latitude φ = 50°30'00.00"N
Longitude λ = 00°30'00.00"E

first gives:

φ_O	=	0.85521133 rad	λ_O	=	-0.03490659 rad
M_O	=	5429228.60			

then:

φ	=	0.88139127 rad	λ	=	0.00872665 rad
A	=	0.02775415	C	=	0.00271699
T	=	1.47160434	M	=	5596050.46
ν	=	6390266.03			

Then Easting E = 577274.99 metres

Northing N = 69740.50 metres

Reverse calculation for same easting and northing first gives:

e_1	=	0.00167322	μ_1	=	0.87939562
M_1	=	5599036.80	ν_1	=	6390275.88
φ_1	=	0.88185987	D	=	0.02775243
ρ_1	=	6372980.21	C_1	=	0.00271391
T_1	=	1.47441726			

Then Latitude φ = 50°30'00.000"N
Longitude λ = 00°30'00.000"E

Table 8 – Comparison of results from JHS and USGS formulas.

Forward conversion, JHS minus USGS

	Latitude = 0°		Latitude = 25°		Latitude = 50°		Latitude = 75°	
	dE (mm)	dN (mm)	dE (mm)	dN (mm)	dE (mm)	dN (mm)	dE (mm)	dN (mm)
$\lambda = \lambda_O + 1°$	0	0	0	0	0	0	0	-1
$\lambda = \lambda_O + 2°$	0	0	0	0	0	0	0	-1
$\lambda = \lambda_O + 3°$	0	0	0	0	0	0	0	-1
$\lambda = \lambda_O + 4°$	1	0	0	0	0	0	0	-1
$\lambda = \lambda_O + 5°$	3	0	-1	0	-1	0	0	-1
$\lambda = \lambda_O + 10°$	404	0	-117	23	-75	-10	12	0
$\lambda = \lambda_O + 20°$	52772	0	-15803	5730	-9383	-2453	1584	161

Reverse conversion, JHS minus USGS

	Latitude = 0°		Latitude = 25°		Latitude = 50°		Latitude = 75°	
	dN (")	dE (")	dN (")	dE (")	dN (")	dE (")	dN (")	dE (")
$\lambda = \lambda_O + 1°$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\lambda = \lambda_O + 2°$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\lambda = \lambda_O + 3°$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\lambda = \lambda_O + 4°$	0.0000	0.0000	0.0000	-0.0001	0.0000	-0.0001	0.0000	-0.0002
$\lambda = \lambda_O + 5°$	0.0000	-0.0001	0.0000	-0.0002	0.0000	-0.0006	0.0000	-0.0011
$\lambda = \lambda_O + 10°$	0.0000	-0.0013	0.0024	-0.0322	0.0035	-0.0857	0.0018	-0.1408
$\lambda = \lambda_O + 20°$	0.0000	-1.7750	0.6895	-4.7213	0.0600	-12.7781	0.5258	-20.9142

Further examples of input and output may be found in test procedure 5101 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.5.3.2 Transverse Mercator Zoned Grid System
(EPSG Dataset coordinate operation method code 9824)

When the growth in distortion away from the projection origin is of concern, a projected coordinate reference system cannot be used far from its origin. A means of creating a grid system over a large area, but also limiting distortion, is to have several grid zones with most defining parameters being made common. Coordinates throughout the system are repeated in each zone. To make coordinates unambiguous, the easting is prefixed by the relevant zone number. This procedure was adopted by German mapping in the 1930s through the Gauss-Kruger systems and later by American military mapping through the Universal Transverse Mercator (or UTM) grid system. Note: subsequent civilian adoption of the systems usually ignores the zone prefix to easting. Where this is the case, the formulas below do not apply; use the standard TM formulas separately for each zone.

The parameter Longitude of natural origin (λ_O) is changed from being a defining parameter to a derived parameter, replaced by two other defining parameters, the Initial Longitude (λ_I) (the western limit of zone 1) and the Zone Width (W). Each of the remaining four Transverse Mercator defining parameters – Latitude of natural origin, Scale Factor at natural origin, False Easting, and False Northing – have the same parameter values in every zone.

The standard Transverse Mercator formulas above are modified as follows:

Zone number, Z, = $[\text{INT}((\lambda - \lambda i) / W) \bmod(360 / W)] + 1$ with λ, λ_I and W in degrees.

In the formula above, mod = (a - floor(a / b) b) where 'floor' is defined as rounding toward negative infinity.
Z is in the range $1 \le Z \le (360/W)$ and for UTM $1 \le Z \le 60$.

Then,

$$\lambda o = \lambda_I + (Z\ W) - (W/2)$$

For the forward calculation,

$$\text{Easting, } E = Z*10^6 + FE + k_O.\nu\ [A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^{\,2})A^5/120]$$

and in the reverse calculation for longitude,

$$D = (E - [FE + Z*10^6])/(\nu_1\ k_O)$$

3.5.3.3 **Transverse Mercator (South Orientated)**
(EPSG Dataset coordinate operation method code 9808)

For the mapping of parts of southern Africa a south oriented Transverse Mercator map projection method is used. Here the coordinate axes are called Westings and Southings and increment to the West and South from the origin respectively. See Figure 9 for a diagrammatic illustration (and compare this with Figures 6 and 7).

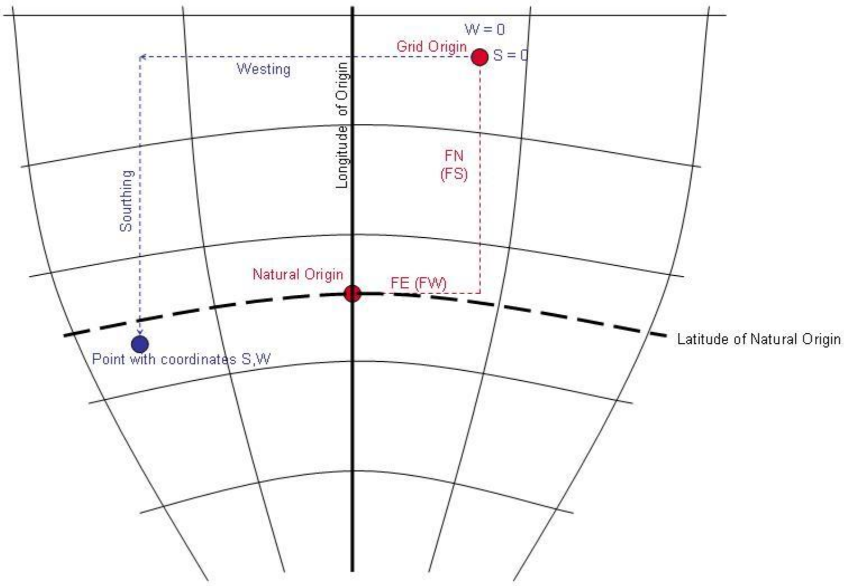


Figure 9 – Transverse Mercator (South Orientated) in the southern hemisphere.

The general case of the Transverse Mercator formulas given above need to be modified to cope with this arrangement with

$$\text{Westing, } W = FE - k_O\,\nu[A + (1 - T + C)A^3/6 + (5 - 18T + T^2 + 72C - 58e'^{\,2})A^5/120]$$

$$\text{Southing, } S = FN - k_O\{M - M_O + \nu\tan\varphi[A^2/2 + (5 - T + 9C + 4C^2)A^4/24 + (61 - 58T + T^2 + 600C - 330e'^2)A^6/720]\}$$

In these formulas, the terms FE and FN retain their definition, i.e., in the Transverse Mercator (South Orientated) method they increase the Westing and Southing value at the natural origin. In this method, they are effectively false westing (FW) and false southing (FS) respectively.

For the reverse formulas, those for the general case of the Transverse Mercator given above apply, with the exception that:

$$M_1 = M_O - (S - FN)/k_O$$

and $D = -(W - FE)/(\nu_1\, k_O)$, with $\nu_1 = \nu$ for φ_1

Example
For Projected Coordinate Reference System Hartebeesthoek94 / Lo29 (EPSG CRS code 2053)

Parameters:

Ellipsoid:	WGS 84	a = 6378137 metres	1/f = 298.257223563
	from which:	$e^2 = 0.006694380$	$e'^2 = 0.006739497$

Latitude of natural origin	φ_O	0°00'00"N
Longitude of natural origin	λ_O	29°00'00"E
Scale factor at natural origin	k_O	1.0
False westing	FW	0.00
False southing	FS	0.00

Forward calculation for:

Latitude	φ	=	25°43'55.302"S
Longitude	λ	=	28°16'57.479"E

first gives:

φ_O	=	0.0 rad	λ_O	=	0.506145483 rad
A	=	-0.11278823	C	=	0.005469120
M_O	=	0.00			

then:

φ	=	-0.449108618 rad	λ	=	0.493625066 rad
A	=	-0.11278823	C	=	0.005469120
T	=	0.232281740	M	=	-2847147.08
ν	=	6382165.01			

Then	Westing	Y	=	71984.49 metres
	Southing	X	=	2847342.74 metres

Reverse calculation for same easting and northing first gives:

e_1	=	0.001679220	μ_1	=	-0.447171650
M_1	=	-2847342.74	ν_1	=	6382165.53
φ_1	=	-0.449139443	D	=	-0.011279007
ρ_1	=	6347451.58	C_1	=	0.005468958
T_1	=	0.232318356			

Then	Latitude	φ	=	25°43'55.302"S
	Longitude	λ	=	28°16'57.479"E

Further examples of input and output may be found in test procedure 5113 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.5.4 **Oblique Mercator**

It has been noted that the Transverse Mercator map projection method is employed for the topographical mapping of longitudinal bands of territories, limiting the amount of scale distortion by limiting the extent of the projection either side of the central meridian. Sometimes the shape, general trend, and extent of some countries makes it preferable to apply a single zone of the same kind of projection but with its central line aligned with the trend of the territory concerned rather than with a meridian. So, instead of a meridian forming this true scale central line for one of the various forms of Transverse Mercator, or the equator forming the line for the Mercator, a line with a particular azimuth traversing the territory is chosen and the same principles of construction are applied to derive what is now an Oblique Mercator. Such a single zone projection suits areas which have a large extent in one direction but limited extent in the perpendicular direction and whose trend is oblique to the bisecting meridian, such as East and West Malaysia and the Alaskan panhandle. The ellipsoidal development of the Oblique Mercator was originally applied at the beginning of the 20th century to the mapping of Switzerland by Rosenmund (1903)[5], updated by Bolliger (1967)[6] and SwissTopo (2008)[7]. Further developments were made by Laborde (1928)[8] for the mapping of Madagascar and Reunion, and by Hotine (1946)[9] for the mapping of Malaysia, for which he introduced the name Rectified Skew Orthomorphic.

These approaches differ slightly from each other in the use of intermediate surfaces. Hotine projected the ellipsoid conformally onto a sphere of constant total curvature, called the 'aposphere', before projection onto the plane and then rotation of the grid to north. Like Hotine, Laborde used a triple projection technique to map the ellipsoid to the plane but, in the Laborde method, the rotation to north is made on the intermediate conformal sphere rather than in the projection plane. These different approaches give similar results, but beyond a few hundred kilometres from the projection centre the results from the Hotine development diverge rapidly from those given by Laborde's formulas, particularly in the direction along the initial line.

EPSG therefore documents both the Hotine and Laborde Oblique Mercator methods. For the Hotine method, we use Snyder's work[10] which modifed Hotine's hyperbolic functions to use exponential functions. For the Hotine method we recognize two variants differentiated only by the point at which false grid coordinates are defined. Referring to Figure 10 below, if the false grid coordinates are defined at the intersection of the initial line and the aposphere (the equator on one of the intermediate surfaces inherent in the method), that is at the natural origin of the coordinate system, the map projection method is known as variant A. If the false grid coordinates are defined at the projection centre the projection method is known as variant B. The defining parameters for these methods and variants are summarized in Table 9. Snyder describes a further variation of the Hotine method where the initial line is defined by two points through which it passes. This approach is not currently included in the EPSG Dataset.

Table 9 – Coordinate Operation Parameters for Oblique Mercator methods.

Method:	Hotine variant A	Hotine variant B	Laborde
EPSG Dataset coordinate operation method code:	9812	9815	9813
Parameter			
Latitude of the projection centre (φ_C)	x	x	x
Longitude of the projection centre (λ_C)	x	x	x
Azimuth of the initial line (α_C)	x	x	x
Angle from the rectified grid to the skew (oblique) grid (γ_C)	x	x	
Scale factor on the initial line (k_C)	x	x	x
False Easting (easting at the natural origin) (FE)	x		
False Northing (northing at the natural origin) (FN)	x		
Easting at the projection centre (E_C)		x	x
Northing at the projection centre (N_C)		x	x

[5] Rosenmund, M. " Die Änderung des Projektionssystems der schweizerischen Landesvermessung", 1903.
[6] Bollinger, J., "Die projecktionen der schweizerischen Plan und Kartenwerke", 1967
[7] Federal Office of Topography (SwissTopo), "Formulas and constants for the calculation of the Swiss conformal cylindrical projection and for the transformation between coordinate systems", 2008.
[8] Laborde, J.,"La nouvelle projection du Service Geographique de Madagascar", 1928.
[9] Hotine, M., Series of Articles in *Empire Survey Review* numbers 62-66, 1946 and 1947.
[10] Snyder, John P., "Map Projections - A Working Manual". Professional Paper No.1395, United States Geological Survey, 1987.

Note: if the Hotine (variant B) method is used as an approximation to the Laborde method, the additional parameter required by that method, the angle from the rectified grid to the skew (oblique) grid γ_C, takes the same value as the azimuth of the initial line passing through the projection centre, i.e. $\gamma_C = \alpha_C$.

3.5.4.1 Hotine Oblique Mercator

(EPSG datset coordinate operation method codes 9812 and 9815)

The Hotine Oblique Mercator coordinate system is defined by:

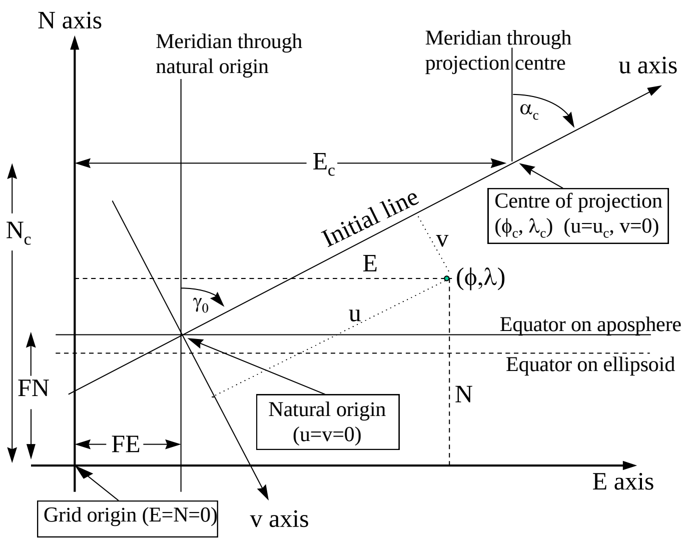


Figure 10 – Hotine Oblique Mercator Projection.

The initial line central to the map area of given azimuth α_C passes through a defined centre of the projection (φ_C, λ_C) . The point where the projection of this line cuts the equator on the aposphere is the origin of the (u , v) coordinate system. The u axis is along the initial line and the v axis is perpendicular to (90° clockwise from) this line.

In applying the formulas for both variants of the Hotine Oblique Mercator projection, the first set of coordinates computed are referred to the (u, v) coordinate axes defined with respect to the initial line. These coordinates are then 'rectified' to the usual Easting and Northing by applying an orthogonal conversion. Hence, the alternative name as the Rectified Skew Orthomorphic. The angle from rectified to skewed grid may be defined such that grid north coincides with true north at the natural origin of the projection, that is where the initial line of the projection intersects the equator on the aposphere. In some circumstances, particularly where the projection is used in non-equatorial areas such as the Alaskan panhandle, the angle from rectified to skewed grid is defined to be identical to the azimuth of the initial line at the projection centre; this results in grid and true north coinciding at the projection centre rather than at the natural origin.

To ensure that all coordinates in the map area have positive grid values, false coordinates are applied. These may be given values (E_C, N_C) if applied at the projection centre [variant B] or be applied as false easting (FE) and false northing (FN) at the natural origin [variant A].

The formulas can be used for the following cases:

Alaska State Plane Zone 1
Hungary EOV
East and West Malaysia Rectified Skew Orthomorphic grids
Swiss Cylindrical projection

The Swiss and Hungarian systems are a special case of variant B where the azimuth of the line through the projection centre is 90 degrees. The variant B formulas may also be used as an approximation to the Laborde Oblique Mercator method and has been used in this way for the Madagascar grid (see following section).

The following constants for the map projection may be calculated for both variants of the Hotine Oblique Mercator from the defining parameters:

$B = \{1 + [e^2 \cos^4\varphi_C / (1 - e^2)]\}^{0.5}$
$A = a\, B\, k_C (1 - e^2)^{0.5} / (1 - e^2 \sin^2 \varphi_C)$
$t_O = \tan(\pi/4 - \varphi_C / 2) / [(1 - e \sin \varphi_C) / (1 + e \sin \varphi_C)]^{e/2}$
$D = B (1 - e^2)^{0.5} / [\cos \varphi_C (1 - e^2 \sin^2 \varphi_C)^{0.5}]$
To avoid problems with computation of F, if D < 1 make $D^2 = 1$
$F = D + (D^2 - 1)^{0.5} . \mathrm{SIGN}(\varphi_C)$
$H = F\, t_O^{B}$
$G = (F - 1/F) / 2$
$\gamma_O = \mathrm{asin}[\sin (\alpha_C) / D]$
$\lambda_O = \lambda_C - [\mathrm{asin}(G \tan\gamma_O)] / B$

Then for variant B only (EPSG method code 9815), two further constants for the map projection, the (u_C , v_C) coordinates for the centre point (φ_C , λ_C), are calculated from:
$v_C = 0$

In general
$u_C = (A / B) \mathrm{atan2}[(D^2 - 1)^{0.5} , \cos (\alpha_C)] * \mathrm{SIGN}(\varphi_C)$

but for the special cases where α_c = 90 degrees (e.g., Hungary, Switzerland) then
$u_C = A (\lambda_C - \lambda_O)$

Forward case: to compute (E,N) from a given (φ,λ). For both variants of the Hotine Oblique Mercator method:

$t = \tan(\pi / 4 - \varphi / 2) / [(1 - e \sin \varphi) / (1 + e \sin \varphi)]^{e/2}$
$Q = H / t^B$
$S = (Q - 1/Q) / 2$
$T = (Q + 1/Q) / 2$
$V = \sin(B (\lambda - \lambda_O))$
$U = (- V \cos(\gamma_O) + S \sin(\gamma_O)) / T$
$v = A \ln[(1 - U) / (1 + U)] / (2 B)$

Then either (a) for variant A, (where the FE and FN values have been specified with respect to the natural origin of the (u , v) axes, EPSG method code 9812):
$u = A\, \mathrm{atan2}\{(S \cos\gamma_O + V \sin\gamma_O) , \cos[B (\lambda - \lambda_O)]\} / B$
and the rectified skew coordinates are then derived from:
$E = v \cos(\gamma_C) + u \sin(\gamma_C) + FE$
$N = u \cos(\gamma_C) - v \sin(\gamma_C) + FN$

or (b) for variant B (where the false easting and northing values (E_C , N_C) have been specified with respect to the centre of the projection (φ_C , λ_C) (EPSG method code 9815):
In general:
$u = (A\, \mathrm{atan2}\{(S \cos\gamma_O + V \sin\gamma_O) , \cos[B (\lambda - \lambda_O)]\} / B) - [\mathrm{ABS}(u_C) * \mathrm{SIGN}(\varphi_C)]$
but in the special case where α_C = 90 degrees:
if $\lambda = \lambda_C$, u = 0
else $u = (A\, \mathrm{atan2}\{(S \cos\gamma_O + V \sin\gamma_O) , \cos[B (\lambda - \lambda_O)]\} / B) - [\mathrm{ABS}(u_C) * \mathrm{SIGN}(\varphi_C) * \mathrm{SIGN}(\lambda_C - \lambda)]$

and in all cases of this variant B the rectified skew coordinates are then derived from:

$E = v \cos(\gamma_C) + u \sin(\gamma_C) + E_C$

$N = u \cos(\gamma_C) - v \sin(\gamma_C) + N_C$

Reverse

To compute (φ,λ) from a given (E,N) :

For variant A:

$v' = (E - FE) \cos(\gamma_C) - (N - FN) \sin(\gamma_C)$

$u' = (N - FN) \cos(\gamma_C) + (E - FE) \sin(\gamma_C)$

or for variant B:

$v' = (E - E_C) \cos(\gamma_C) - (N - N_C) \sin(\gamma_C)$

$u' = (N - N_C) \cos(\gamma_C) + (E - E_C) \sin(\gamma_C) + [ABS(u_C) * SIGN(\varphi_C)]$

then for both variants:

$Q' = \mathbf{e}^{-(B v' / A)}$ where $\mathbf{e}$ is the base of natural logarithms.

$S' = (Q' - 1 / Q') / 2$

$T' = (Q' + 1 / Q') / 2$

$V' = \sin (B u' / A)$

$U' = (V' \cos(\gamma_O) + S' \sin(\gamma_O)) / T'$

$t' = \{H / [(1 + U') / (1 - U')]^{0.5}\}^{1 / B}$

$\chi = \pi / 2 - 2 \operatorname{atan}(t')$

$$\varphi = \chi + \sin(2\chi) (e^2 / 2 + 5 e^4 / 24 + e^6 / 12 + 13 e^8 / 360) + \sin(4\chi) (7 e^4 /48 + 29 e^6 / 240 + 811 e^8 / 11520) + \sin(6\chi) (7 e^6 / 120 + 81 e^8 / 1120) + \sin(8\chi) (4279 e^8 / 161280)$$

$\lambda = \lambda_O - \operatorname{atan2} [(S' \cos\gamma_O - V' \sin\gamma_O) , \cos(B u' / A)] / B$

Examples

For Projected Coordinate Reference System Timbalai 1948 / R.S.O. Borneo (m) (EPSG CRS code 29873)

Using variant B: (EPSG Dataset coordinate operation method code 9815):

Parameters:

Ellipsoid: Everest 1830 (1967 Definition) a = 6377298.556 metres 1/f = 300.8017
from which: e = 0.081472981 e^2 = 0.006637847

Latitude of projection centre	φ_C	4°00'00"N	
Longitude of projection centre	λ_C	115°00'00"E	
Azimuth of initial line	α_C	53°18'56.9537"	
Angle from Rectified to Skew Grid	γ_C	53°07'48.3685"	
Scale factor on initial line	k_C	0.99984	
Easting at projection centre	E_C	590476.87	metres
Northings at projection centre	N_C	442857.65	metres

Constants for the map projection:

φ_C	=	0.069813170 rad	λ_C	=	2.007128640 rad
α_C	=	0.930536611 rad	γ_C	=	0.927295218 rad
B	=	1.003303209	F	=	1.072121256
A	=	6376278.686	H	=	1.000002991
t_O	=	0.932946976	γ_O	=	0.927295218

D	=	1.002425787	λ_O	=	1.914373469
D^2	=	1.004857458			
u_c	=	738096.09	v_c	=	0.00

Forward calculation for:

Latitude	φ	=	5°23'14.1129"N
Longitude	λ	=	115°48'19.8196"E

first gives:

φ	=	0.094025313 rad	λ	=	2.021187362 rad
t	=	0.910700729	Q	=	1.098398182
S	=	0.093990763	T	=	1.004407419
V	=	0.106961709	U	=	0.010967247
v	=	-69702.787	u	=	163238.163

Then	Easting	E	=	679245.73 metres
	Northing	N	=	596562.78 metres

Reverse calculation for same easting and northing first gives:

v'	=	-69702.787	u'	=	901334.257
Q'	=	1.011028053			
S'	=	0.010967907	T'	=	1.000060146
V'	=	0.141349378	U'	=	0.093578324
t'	=	0.910700729	χ	=	0.093404829

Then	Latitude	φ	=	5°23'14.113"N
	Longitude	λ	=	115°48'19.820"E

If the same projection is defined using variant A then:

False easting	FE	=	0.0 metres
False northing	FN	=	0.0 metres
Then	u	=	901334.257

and all other values are as for variant B.

Further examples of input and output may be found in test procedures 5105 and 5106 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.5.4.2 Laborde Oblique Mercator
(EPSG Dataset coordinate operation method code 9813)

The defining parameters for the Laborde Oblique Mercator projection method are:

φ_C = latitude of the projection centre
λ_C = longitude of the projection centre
α_C = azimuth (true) of the initial line passing through the projection centre
k_C = scale factor on the initial line of the projection
E_C = Easting at the projection centre
N_C = Northing at the projection centre

All angular units should be converted to radians prior to use and all longitudes reduced to the Paris Meridian using the Paris lLongitude of 2.5969213 grads (converted from 2°20'14.025"E) east of Greenwich.

The following constants for the map projection may be calculated from these defining parameters:

$B = \{1+[e^2 \cos^4\varphi_C]/(1- e^2)\}^{0.5}$
$\varphi_s = \text{asin}[\sin\varphi_C / B]$
$R = a\ k_C \{(1-e^2)^{0.5} / [1-e^2 \sin^2\varphi_C]\}$
$C = \ln[\tan(\pi/4+\varphi_s /2)] - B. \ln\{\tan(\pi/4+\varphi_c /2) ([1 - e \sin\varphi_c]/[1+e \sin\varphi_c])^{(e/2)}\}$

Forward case to compute (E,N) from a given (φ,λ):
$L = B.(\lambda-\lambda_C)$
$q = C + B . \ln\{\tan(\pi/4+\varphi/2) ([1-e \sin\varphi] / [1+e \sin\varphi])^{(e/2)}\}$
$P = 2.\text{atan}(\mathbf{e}^{q}) - \pi/2$ where **e** is the base of natural logarithms.
$U = \cos P.\cos L.\cos\varphi_s + \sin P.\sin\varphi_s$
$V = \cos P.\cos L.\sin\varphi_s - \sin P.\cos\varphi_s$
$W = \cos P.\sin L$
$d = (U^2+V^2)^{0.5}$
if d <> 0 then L' = 2.atan(V/(U+d)) and P' = atan(W/d)
if d = 0 then L' = 0 and P' = SIGN(W). π/2
$H = -L' + i.\ln(\tan(\pi/4+P'/2))$ where $i^2 = -1$
$G = (1 - \cos(2.\alpha_C) + i.\sin(2.\alpha_C))/12$
$E = E_C + R . \text{IMAGINARY}(H+G.H^3)$
$N = N_C + R . \text{REAL}(H + G.H^3)$

Reverse
To compute (φ,λ) from a given (E,N):
$G = (1-\cos(2.\alpha_C) + i.\sin(2.\alpha_C))/12$ where $i^2 = -1$

To solve for Latitude and Longitude, an iterative solution is required, where the first two elements are
$H_0 = (N-N_C)/R + i.(E-E_C)/R$ ie k = 0
$H_1 = H_0/(H_0 + G.H_0^3)$, i.e. k = 1,
and in subsequent reiterations, k increments by 1
$H_{k+1} = (H_0+2.G.H_k^3)/(3.G.H_k^2+1)$
Re-iterate until $\text{ABSOLUTE}(\text{REAL}([H_0\text{-}H_k\text{-}G.H_k^3])) < 1E\text{-}11$

$L' = -1.\text{REAL}(H_k)$
$P' = 2.\text{atan}(\mathbf{e}^{\text{IMAGINARY}(Hk)}) - \pi/2$ where **e** is the base of natural logarithms.
$U' = \cos P'.\cos L'.\cos\varphi_s + \cos P'.\sin L'.\sin\varphi_s$
$V' = \sin P'$
$W' = \cos P'.\cos L'.\sin\varphi_s - \cos P'.\sin L'.\cos\varphi_s$
$d = (U'^2+ V'^2)^{0.5}$
if d <> 0 then L = 2 atan[V'/(U'+d)] and P = atan(W'/d)
if d = 0 then L = 0 and P = SIGN(W') . π/2
$\lambda = \lambda_C + (L/B)$

$q' = \{\ln[\tan(\pi/4+P/2)] - C\}/B$
The final solution for latitude requires a second re-iterative process, where the first element is
$\varphi'_0 = 2.\text{atan}(\mathbf{e}^{q'}) - \pi/2$ where **e** is the base of natural logarithms.
And the subsequent elements are
$\varphi'_k = 2.\text{atan}\{(\{1+e.\sin[\varphi'_{k-1}]\} / \{1-e.\sin[\varphi'_{k-1}]\})^{(e/2)}. \ \mathbf{e}^{q'}\} - \pi/2$ for $k = 1 \rightarrow$
Iterate until $\text{ABSOLUTE}(\varphi'_k\text{-} \varphi'_{k-1}) < 1E\text{-}11$
$\varphi = \varphi'_k$

Example
For Projected Coordinate Reference System Tananarive (Paris) / Laborde Grid (EPSG CRS code 29701)

Parameters:

Ellipsoid:	International 1924	a = 6378388 metres	1/f = 297
	from which:	e = 0.081991890	e^2 = 0.006722670

Latitude of projection centre	φ_C	21 grads S	
Longitude of projection centre	λ_C	49 grads E of Paris =51.5969213 grads E of Greenwich	
Azimuth of initial line	α_C	21 grads	
Scale factor on initial line	k_C	0.9995	
Easting at projection centre	E_C	400000	metres
Northings at projection centre	N_C	800000	metres

Constants for the map projection:

φ_C	=	-0.329867229 rad	λ_C	=	0.810482544 rad
α_C	=	0.329867229 rad			
B	=	1.002707541	φ_s	=	-0.328942879
R	=	6358218.319	C	=	-0.0002973474

Forward calculation for:

Latitude	φ	=	16°11'23.280"S		
			17.9886666667 grads S	=	-0.282565315 rad
Longitude	λ	=	44°27'27.260"E of Greenwich		
			46.800381173 grads E of Paris	=	0.735138668 rad

first gives :

L	=	-0.034645081	q	=	-0.285595283	P	=	-0.281790207
U	=	0.998343010	V	=	-0.046948995	W	=	-0.033271994
d	=	0.999446334	L'	=	-0.046992297	P'	=	-0.033278135
H	=	0.046992297 - 0.033284279i	G	=	0.017487082 + 0.051075588i			

Then	Easting	E	=	188333.848 metres
	Northing	N	=	1098841.091 metres

Reverse calculation for same easting and northing first gives:

G	=	0.017487082 + 0.051075588i	H_0	=	0.047000760 - 0.033290167i	H_1	=	0.999820949 - 0.000001503i
H_k	=	0.046992297 - 0.033284279i	L'	=	-0.046992297	P'	=	-0.033278136
U'	=	0.959982752	V'	=	-0.033271994	W'	=	-0.278075693
d	=	0.960559165	L	=	-0.3464508142	P	=	-0.281790207
q'	=	-0.284527565	φ'_0	=	-0.280764449			

Then	Latitude	φ	=	-0.282565315 rad	=	17.9886666667 grads S
					=	16°11'23.280"S
	Longitude	λ	=	0.735138668 rad	=	46.8003811733 grads East of Paris
					=	44°27'27.260"E of Greenwich

Table 10 – The Hotine Oblique Mercator (variant B) method as an approximation of the Laborde formulas.

		Using Laborde formula		Using Hotine variant B			
Latitude	Greenwich Longitude	Northing X	Easting Y	Northing X	Easting Y	dX (m)	dY (m)
18°54'S	47°30'E	799665.521	511921.054	799665.520	511921.054	0.00	0.00
16°12'S	44°24'E	1097651.447	182184.982	1097651.426	182184.985	0.02	0.00
25°40'S	45°18'E	50636.222	285294.334	50636.850	285294.788	0.63	0.45
12°00'S	49°12'E	1561109.146	701354.056	1561109.350	701352.935	0.20	1.12

Within 450 kilometres of the projection origin near Antananarivo, Laborde's formulas can be approximated to better than 2cm by the Hotine method, which is satisfactory for most purposes. But at 600 kilometres from the origin along the initial line, the Hotine method approximates Laborde's formulas to no better than 1 metre.

3.5.5 Equidistant Cylindrical

(EPSG Dataset coordinate operation method code 1028)

The characteristics of the Equidistant Cylindrical projection are that the scale is true along two standard parallels equidistant from the equator (or at the equator if that is the standard parallel) and along the meridians. The formulas usually given for this method employ spherical equations with a mean radius of curvature sphere calculated from the latitude of standard parallel. This is a compromise, often satisfactory for the low resolution purposes to which it is put. However, in the spherical implementation the distance is not true along the meridians nor along the standard parallel(s). Spherical formulas are given in the following section.

The ellipsoidal forward equations to convert latitude and longitude to easting and northing are

$$E = FE + \nu_1 \cos\varphi_1 (\lambda - \lambda_O)$$
$$N = FN + M$$

where

$\nu_1 = a /(1 - e^2\sin^2\varphi_1)^{0.5}$ (see the Implementation Notes in Section §1.1)

and

$$M = a(1-e^2)\int_0^{\varphi}(1-e^2\sin^2\varphi)^{-3/2}\,d\varphi \qquad (1)$$

or

$$M = a[\int_0^{\varphi}(1-e^2\sin^2\varphi)^{1/2}\,d\varphi - e^2\sin\varphi\cos\varphi/(1-e^2\sin^2\varphi)^{1/2}] \qquad (2)$$

The first calculation (1) of M above contains an elliptic integral of the third kind. The alternative calculation (2) of M contains an elliptic integral of the second kind. If software supports the functions for these integrals, then the functions can be used directly. Otherwise, the value of M can be computed through a series equation. The following series equation is adequate for any ellipsoid with a flattening of 1/290 or less, which covers all Earth-based ellipsoids of record.

$$M = a[(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \tfrac{5}{256}e^6 - \tfrac{175}{16384}e^8 - \tfrac{441}{65536}e^{10} - \tfrac{4851}{1048576}e^{12} - \tfrac{14157}{4194304}e^{14})\varphi$$
$$+ (-\tfrac{3}{8}e^2 - \tfrac{3}{32}e^4 - \tfrac{45}{1024}e^6 - \tfrac{105}{4096}e^8 - \tfrac{2205}{131072}e^{10} - \tfrac{6237}{524288}e^{12} - \tfrac{297297}{33554432}e^{14})\sin 2\varphi$$
$$+ (\tfrac{15}{256}e^4 + \tfrac{45}{1024}e^6 + \tfrac{525}{16384}e^8 + \tfrac{1575}{65536}e^{10} + \tfrac{155925}{8388608}e^{12} + \tfrac{495495}{33554432}e^{14})\sin 4\varphi$$
$$+ (-\tfrac{35}{3072}e^6 - \tfrac{175}{12288}e^8 - \tfrac{3675}{262144}e^{10} - \tfrac{13475}{1048576}e^{12} - \tfrac{385385}{33554432}e^{14})\sin 6\varphi$$
$$+ (\tfrac{315}{131072}e^8 + \tfrac{2205}{524288}e^{10} + \tfrac{43659}{8388608}e^{12} + \tfrac{189189}{33554432}e^{14})\sin 8\varphi$$
$$+ (-\tfrac{693}{1310720}e^{10} - \tfrac{6237}{5242880}e^{12} - \tfrac{297297}{167772160}e^{14})\sin 10\varphi$$
$$+ (\tfrac{1001}{8388608}e^{12} + \tfrac{11011}{33554432}e^{14})\sin 12\varphi$$
$$+ (-\tfrac{6435}{234881024}e^{14})\sin 14\varphi]$$

The inverse equations are

$$\lambda = \lambda_0 + X/(\nu_1 \cos\varphi_1) = \lambda_0 + X(1 - e^2 \sin^2\varphi_1)^{1/2}/(a\cos\varphi_1)$$
$$\varphi = \mu + (\tfrac{3}{2}n - \tfrac{27}{32}n^3 + \tfrac{269}{512}n^5 - \tfrac{6607}{24576}n^7)\sin 2\mu$$
$$+ (\tfrac{21}{16}n^2 - \tfrac{55}{32}n^4 + \tfrac{6759}{4096}n^6)\sin 4\mu$$
$$+ (\tfrac{151}{96}n^3 - \tfrac{417}{128}n^5 + \tfrac{87963}{20480}n^7)\sin 6\mu$$
$$+ (\tfrac{1097}{512}n^4 - \tfrac{15543}{2560}n^6)\sin 8\mu$$
$$+ (\tfrac{8011}{2560}n^5 - \tfrac{69119}{6144}n^7)\sin 10\mu$$
$$+ (\tfrac{293393}{61440}n^6)\sin 12\mu$$
$$+ (\tfrac{6845701}{860160}n^7)\sin 14\mu$$

where

X = E - FE

Y = N - FN

$$\mu = Y/[a(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \tfrac{5}{256}e^6 - \tfrac{175}{16384}e^8 - \tfrac{441}{65536}e^{10} - \tfrac{4851}{1048576}e^{12} - \tfrac{14157}{4194304}e^{14})]$$
$$n = \frac{1-(1-e^2)^{1/2}}{1+(1-e^2)^{1/2}}$$

Example

For Projected Coordinate Reference System: WGS 84 / World Equidistant Cylindrical (EPSG:4087)

Parameters:

Ellipsoid:	WGS 1984	a = 6378137.0 metres	1/f =298.257223563
	from which:	e = 0.081819191	

Latitude of first standard parallel	φ_1	=	0°00'00.000"N	=	0.0 rad
Longitude of natural origin	λ_O	=	0°00'00.000"E	=	0.0 rad
False easting	FE	=	0.00	metres	
False northing	FN	=	0.00	metres	

Forward calculation for:

Latitude	φ	=	55°00'00.000"N	=	0.959931086 rad
Longitude	λ	=	10°00'00.000"E	=	0.174532925 rad

First gives

Radius of curvature in prime vertical at φ_1	ν_1 =	6378137.0
Radius of curvature of parallel at φ_1	$\nu_1 \cos \varphi_1$ =	6378137.0
Meridional arc distance from equator to φ	M =	6097230.3131

whence

E = 1113194.91 m
N = 6097230.31 m

Reverse calculation for the same Easting and Northing (1113194.91 E, 6097230.31 N) first gives:

Rectifying latitude (radians)	μ =	0.9575624671
Second flattening	n =	0.001679220386

Then	Latitude	φ	=	55°00'00.000"N
	Longitude	λ	=	10°00'00.000"E

3.5.5.1 Equidistant Cylindrical (Spherical)

(EPSG Dataset coordinate operation method code 1029)

This method has one of the simplest formulas available. If the latitude of natural origin (φ_1) is at the equator the method is also known as Plate Carrée. It is not used for rigorous topographic mapping because its distortion characteristics are unsuitable. Formulas are included to distinguish this map projection method from an approach sometimes mistakenly called by the same name and used for simple computer display of geographic coordinates – see Pseudo Plate Carrée in Section §3.5.5.2 below.

For the forward calculation of the Equidistant Cylindrical method:

$$E = FE + R\,(\lambda - \lambda_O)\cos(\varphi_1)$$
$$N = FN + R\,\varphi$$

where φ_1, λ_O, φ and λ are expressed in radians.

R is the radius of the sphere and will normally be one of the CRS parameters. If the figure of the earth used is an ellipsoid rather than a sphere then R should be calculated as the radius of the conformal sphere at the projection origin at latitude φ_1 using the formula for R_C given in the Implementation Notes in Section §1.1. Note however that if applying spherical formula to ellipsoidal coordinates, the equidistant projection properties are not preserved.

For the reverse calculation:

$$\varphi = (N - FN) / R$$
$$\lambda = \lambda_O + \{[E - FE] / [R\cos(\varphi_1)]\}$$

where R is as for the forward method.

3.5.5.2 Pseudo Plate Carrée
(EPSG Dataset coordinate operation method code 9825)

This is not a map projection in the true sense as the coordinate system units are angular (for example, decimal degrees) and therefore of variable linear scale. It is used only for depiction of graticule (latitude/longitude) coordinates on a computer display. The origin is at latitude (φ) = 0, longitude (λ) = 0. See above for the formulas for the proper Plate Carrée map projection method.

For the forward calculation:

$$X = \lambda$$
$$Y = \varphi$$

For the reverse calculation:

$$\varphi = Y$$
$$\lambda = X$$

3.5.6 Lambert Cylindrical Equal Area
(EPSG Dataset coordinate operation method code 9835)

See Snyder (1987) for formulas and example.

3.5.6.1 Lambert Cylindrical Equal Area (Spherical)
(EPSG Dataset coordinate operation method code 9834)

For the forward calculation, for the normal aspect of the projection in which φ_1 is the latitude of the standard parallel:

$$E = FE + R\,(\lambda - \lambda_O)\cos(\varphi_1)$$
$$N = FN + R\sin(\varphi) / \cos(\varphi_1)$$

where φ_1, φ and λ are expressed in radians

R is the radius of the sphere and will normally be one of the CRS parameters. If the model of the Earth used is an ellipsoid rather than a sphere then R should be calculated as the radius of the authalic sphere using the formula for R_A given in the Implementation Notes in Section §1.1.

For the reverse calculation:

$$\varphi = \mathrm{asin}\{[(N - FN) / R] \cos(\varphi_1)\}$$
$$\lambda = \lambda_O + \{[E - FE] / [R\cos(\varphi_1)]\}$$

where R is as for the forward method.

For formulas for oblique and polar aspects and examples see Snyder (1987).

3.6 Azimuthal Map Projections

3.6.1 Stereographic

The Stereographic projection may be imagined to be a projection of the Earth's surface onto a plane in contact with the Earth at a single tangent point from a projection point at the opposite end of the diameter through that tangent point.

This projection is best known in its polar form and is frequently used for mapping polar areas where it complements the Universal Transverse Mercator used for lower latitudes. Its spherical form has also been widely used by the US Geological Survey for planetary mapping and the mapping at small scale of continental hydrocarbon provinces. In its transverse or oblique ellipsoidal forms, it is useful for mapping limited areas centred on the point where the plane of the projection is regarded as tangential to the ellipsoid, e.g., the Netherlands. The tangent point is the origin of the projected coordinate reference system and the meridian through it is regarded as the central meridian. In order to reduce the scale error at the extremities of the projection area, it is usual to introduce a scale factor at the origin of less than unity; the point scale factor then will be equal to one on a near circle centred at the origin and some distance from it.

The coordinate conversion from geographic to projected coordinates is executed via the distance and azimuth of the point from the centre point or origin. For a sphere, the formulas are relatively simple. For the ellipsoid, the parameters defining the conformal sphere at the tangent point as origin are first derived. The conformal latitudes and longitudes are substituted for the geodetic latitudes and longitudes of the spherical formulas for the origin and the point.

An alternative approach is given by Snyder (1987), where, instead of defining a single conformal sphere at the origin point, the conformal latitude at each point on the ellipsoid is computed. The conformal longitude is then always equivalent to the geodetic longitude. This approach is a valid alternative to that given here, but gives slightly different results away from the origin point. The USGS formula is therefore considered by IOGP to be a different coordinate operation method to that described here. In the US, the approach followed, as documented in the following Section, is sometimes called a double stereographic projection. However, most projections of the ellipsoid involve a sphere as an intermediate surface and 'double' is not unique to the stereographic projection.

3.6.1.1 Oblique and Equatorial Stereographic

(EPSG Dataset coordinate operation method code 9809)

Given the geodetic origin of the projection at the tangent point (φ_O, λ_O), the parameters defining the conformal sphere are:

$$R = (\rho_O \ \nu_O)^{0.5}$$
$$n = \{1+[(e^2 \cos^4\varphi_O) / (1 - e^2)]\}^{0.5}$$
$$c = (n + \sin\varphi_O) (1 - \sin\chi_{OO}) / [(n - \sin\varphi_O) (1 + \sin\chi_{OO}]$$

where: $\sin \chi_{OO} = (w_1 - 1) / (w_1 + 1)$
$w_1 = [S_1 (S_2)^e]^n$
$S_1 = (1 + \sin\varphi_O)/(1 - \sin\varphi_O)$
$S_2 = (1 - e \sin\varphi_O)/(1 + e \sin\varphi_O)$

The conformal latitude of the origin (χ_O) is then computed from:
$$\chi_O = \sin^{-1} [(w_2 - 1)/(w_2 + 1)]$$

where $w_2 = c\ w_1$

For any point with geodetic latitude (φ) the equivalent conformal latitude (χ) is then computed from:

$$\chi = \sin^{-1} [(w - 1)/(w + 1)]$$

where $w = c\, [S_a\, (S_b)^{e}]^n$
$S_a = (1 + \sin\varphi) / (1 - \sin\varphi)$
$S_b = (1 - e \sin\varphi) / (1 + e \sin\varphi)$

Then

$\delta\Lambda = (\Lambda - \Lambda_O) = n(\lambda - \lambda_O)$ where Λ is the conformal longitude at geodetic longitude λ
$B = 1+\sin\chi \sin\chi_O + \cos\chi \cos\chi_O \cos(\delta\Lambda)$

and

$$E = FE + 2\, R\, k_O \cos\chi \sin(\delta\Lambda) / B$$
$$N = FN + 2\, R\, k_O\, [\sin\chi \cos\chi_O - \cos\chi \sin\chi_O \cos(\delta\Lambda)] / B$$

The reverse formulas to compute the geodetic coordinates from the grid coordinates involve computing the conformal values, then the isometric latitude, and finally the geodetic values.

The parameters defining the conformal sphere and conformal latitude at the origin are computed as above. Then for any point with Stereographic grid coordinates (E,N):

$$\chi = \chi_O + 2 \tan^{-1}\{[(N - FN) - (E - FE) \tan (j/2)] / (2\, Rk_O)\}$$

where $g = 2\, Rk_O \tan (\pi/4 - \chi_O/ 2)$
$h = 4\, Rk_O \tan \chi_O + g$
$i = \text{atan2} \{(E - FE) , [h + (N - FN)]\}$
$j = \text{atan2} \{(E - FE) , [g - (N - FN)]\} - i$

Then isometric latitude ψ is calculated from:

$$\psi = 0.5 \ln \{(1 + \sin\chi) / [c\, (1 - \sin\chi)]\} / n$$

The first approximation for geodetic latitude is found from:

$$\varphi_1 = 2 \text{ atan } (\mathbf{e}^{\Psi}) - \pi / 2$$

where

e=base of natural logarithms
ψ = isometric latitude at $\varphi_1 = \ln\{\tan(\varphi_1/2 +\pi/4)\ \ [(1 - e \sin\varphi_1)/(1 + e \sin\varphi_1)]^{(e/2)}\}$

Then iterate

$$\psi_i= \ln\{\tan(\varphi_i/2 +\pi/4)\ \ [(1 - e \sin\varphi_i)/(1 + e \sin\varphi_i)]^{(e/2)}\}$$

and

$$\varphi_{i+1} = \varphi_i - (\psi_i - \psi) \cos \varphi_i\, (1 - e^2 \sin^2\varphi_i) / (1 - e^2)$$

until the change in φ is sufficiently small.

Geodetic longitude $\lambda = (\delta\Lambda) / n + \lambda_O$ where $\delta\Lambda= (j + 2\, i)$, so $\lambda = (j + 2\, i) / n + \lambda_O$

For stereographic projections centred on points in the southern hemisphere, the signs of E, N, λ_O and λ must be reversed in the equations above.

If the projection is the equatorial case, φ_O and χ_O will be zero degrees and the formulas may be simplified as a result, but the above formulas remain valid.

For the polar case, φ_O and χ_O will be 90 degrees and the formulas become indeterminate. See below for formulas for the polar case.

Example
For Projected Coordinate Reference System: Amersfoort / RD New (EPSG CRS code 28992)

Parameters:

Ellipsoid: Bessel 1841 a = 6377397.155 metres 1/f = 299.1528128
from which: e = 0.081696831

Latitude of natural origin	φ_O	52°09'22.178"N	
Longitude of natural origin	λ_O	5°23'15.500"E	
Scale factor at natural origin	k_O	0.9999079	
False easting	FE	155000.00	metres
False northing	FN	463000.00	metres

Forward calculation for the point:

Latitude φ = 53°N Longitude λ = 6°E

first gives the conformal sphere constants:

φ_O = 0.910296727 rad λ_O = 0.094032038 rad
ρ_O = 6374588.710 ν_O = 6390710.613 R = 6382644.571
S_1 = 8.509582274 S_2 = 0.878790173 w_1 = 8.428769183
sin χ_{OO} = 0.787883237 n = 1.000475857 c = 1.007576465

then w_2 = 8.492629457

and the conformal coordinates of the map projection origin are:

χ_O = 0.909684757 $\Lambda_O = \lambda_O$ = 0.094032038 rad

For the point (φ,λ)

φ = 0.925024504 rad λ = 0.104719755 rad
S_a = 8.932237807 S_b = 0.877500613 w = 8.913578649
χ = 0.924394997 $\delta\Lambda$ = 0.010692803 B = 1.999870665

Then

E = 196105.283 m N = 557057.739 m

Reverse calculation for the same Easting and Northing (196105.283E, 557057.739N) first gives:

g = 4379954.188 h = 37197327.960 i = 0.001102272 j = 0.008488259

Then χ = 0.924394997 and ψ = 1.089495505 and φ_1 = 0.921804179

Iteration for ψ and φ:
ψ_1 = 1.084170545 φ_2 = 0.925031393
ψ_2 = 1.089506925 φ_3 = 0.925024504
ψ_3 = 1.089495505 φ_4 = 0.925024504 rad = 53°00'00.000"N

Also $\delta\Lambda$ = 0.010692803 whence λ = 0.104719755 rad = 6°00'00.000"E

Further examples of input and output may be found in test procedure 5104 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

3.6.1.2 Polar Stereographic

For the polar sterographic projection, three variants are recognized, differentiated by their defining parameters. In the basic variant (**variant A**), the latitude of origin is either the north pole or the south pole, at which is defined a scale factor at the natural origin, the meridian along which the northing axis increments and along which intersecting parallels increment *towards the north pole* (the longitude of origin), and false grid coordinates. In **variant B**, instead of the scale factor at the pole being defined, the (non-polar) latitude at which the scale is unity – the standard parallel – is defined. In **variant C**, the latitude of a standard parallel along which the scale is unity is defined; the intersection of this parallel with the longitude of origin is the false origin, at which grid coordinate values are defined.

Table 11 – Coordinate Operation Parameters for Polar Stereographic methods.

	Method		
Parameter	Variant A	Variant B	Variant C
Latitude of natural origin (φ_O)	x (note 1)	(note 2)	(note 2)
Latitude of standard parallel (φ_F)		x	x
Longitude of origin (λ_O)	x	x	x
Scale at natural origin (k_O)	x		
False easting (easting at natural origin = pole) (FE)	x	x	
False northing (northing at natural origin = pole) (FN)	x	x	
Easting at false origin (E_F)			x
Northing at false origin (N_F)			x

In all three variants, the formulas for the **south pole case** are straightforward but some require modification for the **north pole case** to allow the longitude of origin going towards (as opposed to away from) the natural origin and for the anticlockwise increase in longitude value when viewed from the north pole (see Ffigure 11). Several equations are common between the variants and cases.

Notes:
1. In variant A, the parameter *Latitude of natural origin* is used only to identify which hemisphere case is required. The only valid entries are ±90° or equivalent in alternative angle units.
2. For variants B and C, whilst it is mathematically possible for the standard parallel to be in the opposite hemisphere to the pole at which is the projection natural origin, such an arrangement would be unsatisfactory from a cartographic perspective as the rate of change of scale would be excessive in the area of interest. The EPSG Dataset therefore excludes the hemisphere of pole as a defining parameter for these variants. In the formulas that follow, for these variants B and C, the hemisphere of pole is taken to be that of the hemisphere of the standard parallel.

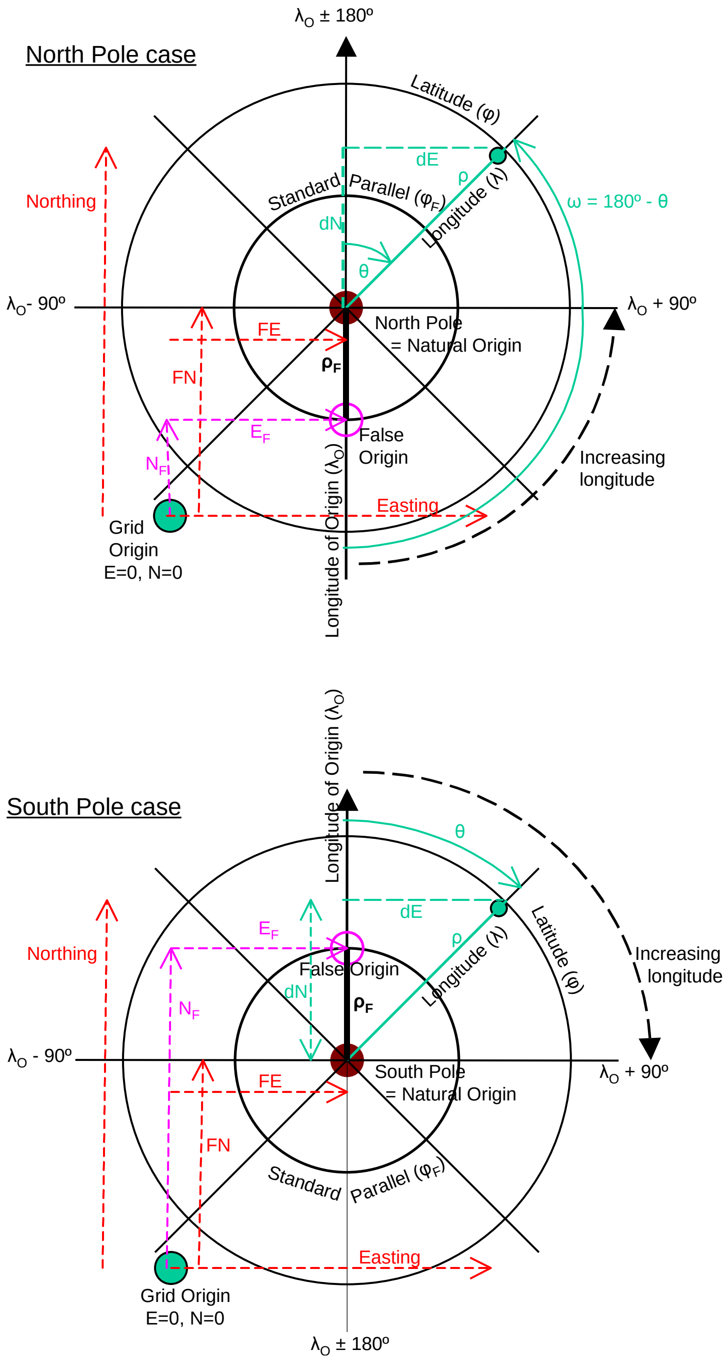


Figure 11 – Polar Stereographic Projection.

Polar Stereographic (Variant A) (EPSG Dataset coordinate operation method code 9810).

For the forward conversion from latitude and longitude, for the south pole case

$$dE = \rho \sin(\theta)$$

and

$$dN = \rho \cos(\theta)$$

where $\theta = (\lambda - \lambda_O)$

Then

$$E = dE + FE = FE + \rho \sin(\lambda - \lambda_O)$$
$$N = dN + FN = FN + \rho \cos(\lambda - \lambda_O)$$

where

$t = \tan(\pi/4 + \varphi/2) / \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$\rho = 2\, a\, k_O\, t / \{[(1+e)^{(1+e)}\, (1-e)^{(1-e)}]^{0.5}\}$

For the north pole case,

$dE = \rho \sin(\theta) = \rho \sin(\omega)$

$dN = \rho \cos(\theta) = -\rho \cos(\omega)$

where, as shown in Figure 11, ω = longitude λ measured anticlockwise in the projection plane.

ρ and E are found as for the south pole case but

$t = \tan(\pi/4 - \varphi/2) \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$N = FN - \rho \cos(\lambda - \lambda_O)$

For the reverse conversion from easting and northing to latitude and longitude,

$$\varphi = \chi + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi) + (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi) + (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)$$

where $\rho' = [(E - FE)^2 + (N - FN)^2]^{0.5}$

$t' = \rho' \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\} / 2\, a\, k_O$

and for the south pole case

$\chi = 2 \operatorname{atan}(t') - \pi/2$

but for the north pole case

$\chi = \pi/2 - 2 \operatorname{atan}(t')$

Then for both north and south cases if E = FE, $\lambda = \lambda_O$

else for the south pole case

$\lambda = \lambda_O + \operatorname{atan2}[(E - FE), (N - FN)]$

and for the north pole case

$\lambda = \lambda_O + \operatorname{atan2}[(E - FE), (FN - N)]$

Example

For Projected Coordinate Reference System: WGS 84 / UPS North (EPSG CRS code 5041)

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.257223563
	from which:	e = 0.081819191	

Latitude of natural origin	φ_O	90°00'00.000"N	=	1.570796327 rad
Longitude of natural origin	λ_O	0°00'00.000"E	=	0.0 rad
Scale factor at natural origin	k_O	0.994		
False easting	FE	2000000.00	metres	
False northing	FN	2000000.00	metres	

Forward calculation for:

Latitude	φ	=	73°N
Longitude	λ	=	44°E

φ = 1.274090354 rad λ = 0.767944871 rad

t = 0.150412808 ρ = 1900814.564

whence

E = 3320416.75 m

N = 632668.43 m

Reverse calculation for the same Easting and Northing (3320416.75 E, 632668.43 N) first gives:

ρ' = 1900814.566
t' = 0.150412808
χ = 1.2722090

Then	Latitude	φ	=	73°00'00.000"N
	Longitude	λ	=	44°00'00.000"E

Polar Stereographic (Variant B) (EPSG Dataset coordinate operation method code 9829).

For the forward conversion from latitude and longitude:
for the south pole case

$t_F = \tan(\pi/4 + \varphi_F/2) / \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$
$m_F = \cos\varphi_F / (1 - e^2 \sin^2\varphi_F)^{0.5}$
$k_O = m_F \{[(1+e)^{(1+e)} (1-e)^{(1-e)}]^{0.5}\} / (2\ t_F)$

then t, ρ, E, and N are found as in the south pole case of variant A.

For the north pole case, m_F and k_O are found as for the south pole case above, but

$t_F = \tan(\pi/4 - \varphi_F/2)\ \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

Then t, ρ, E, and N are found as in variant A.

For the reverse conversion from easting and northing to latitude and longitude, first k_O is found from m_F and t_F as in the forward conversion above, then φ and λ are found as for variant A.

Example

For Projected Coordinate Reference System: WGS 84 / Australian Antarctic Polar Stereographic

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.257223563
	from which:	e = 0.081819191	

Latitude of standard parallel	φ_F	71°00'00.000"S	=	-1.239183769 rad
Longitude of origin	λ_O	70°00'00.000"E	=	1.221730476 rad
False easting	FE	6000000.00	metres	
False northing	FN	6000000.00	metres	

Forward calculation for:

Latitude φ = 75°S   Longitude λ = 120°E

φ = -1.308996939 rad	λ = 2.094395102 rad	
t_F = 0.168407325	m_F = 0.326546781	k_O = 0.97276901
t = 0.132508348	ρ = 1638783.238	

whence

E = 7255380.79 m
N = 7053389.56 m

Reverse calculation for the same Easting and Northing (7255380.79 E, 7053389.56 N) first gives:

t_F = 0.168407325 m_F = 0.326546781   and k_O = 0.97276901

then ρ' = 1638783.236 t' = 0.132508347 χ = -1.3073146

Then: Latitude φ = 75°00'00.000"S   Longitude λ = 120°00'00.000"E

Polar Stereographic (Variant C) (EPSG Dataset coordinate operation method code 9830).

For the forward conversion from latitude and longitude, for the south pole case

$E = E_F + \rho \sin(\lambda - \lambda_O)$

$N = N_F - \rho_F + \rho \cos(\lambda - \lambda_O)$

where

m_F is found as in variant B = $\cos\varphi_F \,/\, (1 - e^2 \sin^2\varphi_F)^{0.5}$

t_F is found as in variant B = $\tan(\pi/4 + \varphi_F/2) \,/\, \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

t is found as in variants A and B = $\tan(\pi/4 + \varphi/2) \,/\, \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$\rho_F = a\, m_F$

$\rho = \rho_F\, t / t_F$

For the north pole case, m_F, ρ_F, ρ, and E are found as for the south pole case but

t_F is found as in variant B = $\tan(\pi/4 - \varphi_F/2)\, \{[(1 + e \sin\varphi_F) / (1 - e \sin\varphi_F)]^{(e/2)}\}$

t is found as in variants A and B = $\tan(\pi/4 - \varphi/2)\, \{[(1 + e \sin\varphi) / (1 - e \sin\varphi)]^{(e/2)}\}$

$N = N_F + \rho_F - \rho \cos(\lambda - \lambda_O)$

For the reverse conversion from easting and northing to latitude and longitude,

$$\begin{aligned}\varphi = \chi \quad & + (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360) \sin(2\chi) \\ & + (7e^4/48 + 29e^6/240 + 811e^8/11520) \sin(4\chi) \\ & + (7e^6/120 + 81e^8/1120) \sin(6\chi) + (4279e^8/161280) \sin(8\chi)\end{aligned}$$

(as for variants A and B)

where for the south pole case

$\rho' = [(E-E_F)^2 + (N - N_F + \rho_F)^2]^{0.5}$

$t' = \rho'\, t_F / \rho_F$

$\chi = 2\,\text{atan}(t') - \pi/2$

and where m_F and t_F are as for the forward conversion.

For the reverse conversion north pole case, m_F, t_F, and ρ_F are found as for the north pole case of the forward conversion, and

$\rho' = [(E-E_F)^2 + (N - N_F - \rho_F)^2]^{0.5}$

t' is found as for the south pole case of the reverse conversion = $\rho'\, t_F / \rho_F$

$\chi = \pi/2 - 2\,\text{atan}(t')$

Then for both north and south pole cases

if $E = E_F$, $\lambda = \lambda_O$

else for the south pole case

$\lambda = \lambda_O + \text{atan2}[(E - E_F), (N - N_F + \rho_F)]$

and for the north pole case

$\lambda = \lambda_O + \text{atan2}[(E - E_F), (N_F + \rho_F - N)]$

Example

For Projected Coordinate Reference System: Petrels 1972 / Terre Adelie Polar Stereographic (EPSG:2985)

Parameters:

Ellipsoid:	International 1924	a = 6378388.0 metres	1/f = 297.0
	from which:	e = 0.081991890	

Latitude of false origin	φ_F	67°00'00.000"S
Longitude of origin	λ_O	140°00'00.000"E
Easting at false origin	E_F	300000.00
Northing at false origin	N_F	200000.00

Forward calculation for:

Latitude	φ	=	66°36'18.820"S
Longitude	λ	=	140°04'17.040"E

φ_F = -1.169370599 rad	λ_O = 2.443460953 rad	
φ = -1.162480524 rad	λ = 2.444707118 rad	
m_F = 0.391848769	ρ_F = 2499363.488	t_F = 0.204717630
t = 0.208326304	ρ = 2543421.183	

whence

E = 303169.52 m
N = 244055.72 m

Reverse calculation for the same Easting and Northing (303169.522 E, 244055.721 N) first gives:

ρ' = 2543421.183
t' = 0.208326304
χ = -1.1600190

Then	Latitude	φ	=	66°36'18.820"S
	Longitude	λ	=	140°04'17.040"E

3.6.2 **Lambert Azimuthal Equal Area**

(EPSG Dataset coordinate operation method code 9820)

Oblique aspect

To derive the projected coordinates of a point, geodetic latitude (φ) is converted to authalic latitude (ß). The formulas to convert geodetic latitude and longitude (φ, λ) to Easting and Northing are:

$$\text{Easting, } E = FE + \{(B\,D)\ [\cos ß\ \sin(\lambda - \lambda_O)]\}$$

$$\text{Northing, } N = FN + (B / D)\ \{(\cos ß_O\ \sin ß) - [\sin ß_O\ \cos ß\ \cos(\lambda - \lambda_O)]\}$$

where

$$B = R_q\ (2 / \{1 + \sin ß_O\ \sin ß + [\cos ß_O\ \cos ß\ \cos(\lambda - \lambda_O)]\})^{0.5}$$

$$D = a\,[\cos \varphi_O / (1 - e^2 \sin^2 \varphi_O)^{0.5}] / (R_q\ \cos ß_O)$$

$$R_q = a\,(q_P / 2)^{0.5}$$

$$ß = \operatorname{asin}(q / q_P)$$

$$ß_O = \operatorname{asin}(q_O / q_P)$$

$$q = (1 - e^2)\,([\sin \varphi / (1 - e^2\sin^2\varphi)] - \{[1/(2e)]\ \ln [(1 - e \sin \varphi) / (1 + e \sin \varphi)]\})$$

$$q_O = (1 - e^2)\,([\sin \varphi_O / (1 - e^2\sin^2\varphi_O)] - \{[1/(2e)]\ \ln [(1 - e \sin \varphi_O) / (1 + e \sin \varphi_O)]\})$$

$$q_P = (1 - e^2)\,([\sin \varphi_P / (1 - e^2\sin^2\varphi_P)] - \{[1/(2e)]\ \ln [(1 - e \sin \varphi_P) / (1 + e \sin \varphi_P)]\})$$

where $\varphi_P = \pi/2$ radians, thus

$$q_P = (1 - e^2)\,([1 / (1 - e^2)] - \{[1/(2e)]\ \ln [(1 - e) / (1 + e)]\})$$

The reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing values are:

$$\varphi = ß' + [(e^2/3 + 31e^4/180 + 517e^6/5040) \sin 2ß'] + [(23e^4/360 + 251e^6/3780) \sin 4ß'] + [(761e^6/45360) \sin 6ß']$$

$$\lambda = \lambda_O + \text{atan2}\{(E–FE) \sin C , [D\ \rho \cos ß_O \cos C – D^2 (N–FN) \sin ß_O \sin C]\}$$

where

$$ß' = \text{asin}\{(\cos C \sin ß_O) + [(D\ (N–FN) \sin C \cos ß_O) / \rho]\}$$

$$C = 2 \text{ asin}(\rho / 2 R_q)$$

$$\rho = \{[(E–FE)/D]^2 + [D\ (N –FN)]^2\}^{0.5}$$

and D, R_q, and $ß_O$ are as in the forward equations.

Example

For Projected Coordinate Reference System: ETRS89-extended / ETRS-LAEA (EPSG CRS code 3035)

Parameters:

Ellipsoid:	GRS 1980	a = 6378137.0 metres	1/f = 298.257222101
	from which:	e = 0.081819191	

Latitude of natural origin	φ_O	52°00'00.000"N
Longitude of natural origin	λ_O	10°00'00.000"E
False easting	FE	4321000.00 metres
False northing	FN	3210000.00 metres

Forward calculation for:

Latitude	φ	=	50°00'00.000"N
Longitude	λ	=	5°00'00.000"E

First gives

φ_O =	0.907571211 rad	λ_O =	0.174532925 rad
φ =	0.872664626 rad	λ =	0.087266463 rad
q_P =	1.995531087	q_O =	1.569825704
q =	1.525832247	R_q =	6371007.181
$ß_O$ =	0.905397517	ß =	0.870458708
D =	1.000425395	B =	6374393.455

whence

E = 3962799.45 m
N = 2999718.85 m

Reverse calculation for the same Easting and Northing (3962799.45 E, 2999718.85 N) first gives:

ρ =	415276.208
C =	0.065193736
ß' =	0.870458708

Then	Latitude	φ	=	50°00'00.000"N
	Longitude	λ	=	5°00'00.000"E

Further examples of input and output may be found in test procedure 5110 of IOGP's *Geospatial Integrity in Geoscience Software (GIGS)* Test Dataset, https://gigs.iogp.org/.

Polar aspect

For the polar aspect of the Lambert Azimuthal Equal Area projection, some of the above equations are indeterminate. Instead, for the forward case from latitude and longitude (φ, λ) to Easting (E) and Northing (N):

For the north polar case:

$$\text{Easting, } E = FE + [\rho \sin(\lambda - \lambda_O)]$$
$$\text{Northing, } N = FN - [\rho \cos(\lambda - \lambda_O)]$$

where

$$\rho = a\,(q_P - q)^{0.5}$$

and q_P and q are found as for the general case above.

For the south polar case:

$$\text{Easting, } E = FE + [\rho \sin(\lambda - \lambda_O)]$$
$$\text{Northing, } N = FN + [\rho \cos(\lambda - \lambda_O)]$$

where

$$\rho = a\,(q_P + q)^{0.5}$$

and q_P and q are found as for the general case above.

For the reverse formulas to derive the geodetic latitude and longitude of a point from its Easting and Northing:

$$\varphi = ß' + [(e^2/3 + 31e^4/180 + 517e^6/5040) \sin 2ß'] + [(23e^4/360 + 251e^6/3780) \sin 4ß'] + [(761e^6/45360) \sin 6ß']$$

as for the oblique case, but where

$$ß' = \pm\text{asin}\, [1 - \rho^2 / (a^2\{1 - [(1 - e^2)/2e] \ln[(1-e)/(1+e)]\})], \text{ taking the sign of } \varphi_O$$

and $\rho = \{[(E - FE)]^2 + [(N - FN)]^2\}^{0.5}$

Then

$$\lambda = \lambda_O + \text{atan2}\, [(E - FE) , (N - FN)] \text{ for the south pole case}$$

and

$$\lambda = \lambda_O + \text{atan2}\, [(E - FE) , - (N - FN)] = \lambda_O + \text{atan2}\, [(E - FE) , (FN - N)] \text{ for the north pole case.}$$

3.6.2.1 Lambert Azimuthal Equal Area (Spherical)

(EPSG Dataset coordinate operation method code 1027)

The US National Atlas uses the spherical form of the oblique case, so exceptionally IOGP includes this method in the EPSG Dataset. For formulas and example see Snyder (1987).

R is the radius of the sphere and will normally be one of the CRS parameters. If the figure of the Earth used is an ellipsoid rather than a sphere then R should be calculated as the radius of the authalic sphere using the formula for R_A given in the Implementation Notes given in Section §1.1. Note however that if applying spherical formula to ellipsoidal coordinates, the authalic projection properties are not preserved.

3.6.3 Azimuthal Equidistant

3.6.3.1 Modified Azimuthal Equidistant

(EPSG Dataset coordinate operation method code 9832)

For various islands in Micronesia, the US National Geodetic Survey has developed formulas for the oblique form of the ellipsoidal projection which calculates distance from the origin along a normal section rather than the geodesic. For the distances over which these projections are used (under 800km) this modification introduces no significant error.

First calculate a constant for the projection:

$$\nu_O = a /(1 - e^2 \sin^2\varphi_O)^{1/2}$$

Then the forward conversion from latitude and longitude is given by:

$$\nu = a /(1 - e^2 \sin^2\varphi)^{1/2}$$
$$\psi = \text{atan}\ [(1 - e^2) \tan \varphi + e^2\ \nu_O \sin \varphi_O\ / (\nu \cos \varphi)]$$
$$\alpha = \text{atan2}\ \{\sin (\lambda - \lambda_O)\ , [\cos \varphi_O \tan \psi - \sin \varphi_O \cos (\lambda - \lambda_O)]\}$$
$$G = e \sin \varphi_O /(1 - e^2)^{1/2}$$
$$H = e \cos \varphi_O \cos \alpha /(1 - e^2)^{1/2}$$

Then

$$\text{if } (\sin \alpha) = 0,\ \ s = \text{asin}\ (\cos \varphi_O \sin \psi - \sin \varphi_O \cos \psi) * \text{SIGN}(\cos \alpha)$$
$$\text{if } (\sin \alpha) \neq 0,\ \ s = \text{asin}\ [\sin (\lambda - \lambda_O) \cos \psi / \sin \alpha]$$

and in either case

$$c = \nu_O\ s\ \{[1 - s^2\ H^2\ (1 - H^2) /6] + [(s^3/8)GH(1\text{-}2H^2)] + (s^4/120)[H^2(4\text{-}7H^2) - 3G^2(1\text{-}7H^2)] - [(s^5/48)GH]\}$$

Then

$$E = FE + c \sin \alpha$$
$$N = FN + c \cos \alpha$$

For the reverse conversion from easting and northing to latitude and longitude:

$$c' = [(E - FE)^2 + (N - FN)^2]^{0.5}$$
$$\alpha' = \text{atan2}\ [(E - FE)\ , (N - FN)]$$
$$A = - e^2 \cos^2 \varphi_O \cos^2 \alpha' / (1 - e^2)$$
$$B = 3e^2\ (1\text{-}A) \sin \varphi_O\ \cos \varphi_O \cos \alpha' / (1 - e^2)$$
$$D = c' / \nu_O$$
$$J = D - [A\ (1 + A)\ D^3 / 6] - [B\ (1 + 3A)\ D^4 / 24]$$
$$K = 1 - (A\ J^2 / 2) - (B\ J^3 / 6)$$
$$\Psi' = \text{asin}\ (\sin \varphi_O \cos J + \cos \varphi_O \sin J \cos \alpha')$$

Then

$$\varphi = \text{atan}\ [(1 - e^2\ K \sin \varphi_O / \sin \Psi') \tan \Psi' / (1 - e^2)]$$
$$\lambda = \lambda_O + \text{asin}\ (\sin \alpha' \sin J / \cos \Psi')$$

Example
For Projected Coordinate Reference System: Guam 1963 / Yap Islands (EPSG CRS code 3295)

Parameters:

Ellipsoid: Clarke 1866 a = 6378206.4 metres b = 6356583.8 metres
from which: 1/f = 294.9786982 e = 0.08227185 e^2 = 0.00676866

Latitude of natural origin	φ_O	9°32'48.15"N
Longitude of natural origin	λ_O	138°10'07.48"E
False easting	FE	40000.00 metres
False northing	FN	60000.00 metres

Forward calculation for:

Latitude	φ	=	9°35'47.493"N
Longitude	λ	=	138°11'34.908"E

first gives

φ_O =	0.166621493 rad	λ_O =	2.411499514 rad
φ =	0.167490973 rad	λ =	2.411923377 rad
ν_O =	6378800.24	G =	0.013691332
ν =	6378806.40	H =	0.073281276
ψ =	0.167485249	s =	0.000959566
α =	0.450640866	c =	6120.88

whence

E	=	42665.90
N	=	65509.82

Reverse calculation for the same Easting and Northing (42665.90 m E, 65509.82 m N) first gives:

c' =	6120.88	D =	0.000959566
α' =	0.450640866	J =	0.000959566
A =	-0.005370145	K =	1.000000002
B =	0.003026119	Ψ' =	0.167485249

whence

φ	=	0.167490973 rad	=	9°35'47.493"N
λ	=	2.411923377 rad	=	138°11'34.908"E

3.6.3.2 Guam Projection
(EPSG Dataset coordinate operation method code 9831)

The Guam projection is a simplified form of the oblique case of the azimuthal equidistant projection. For the Guam projection, the forward conversion from latitude and longitude is given by:

$$x = a (\lambda - \lambda_O) \cos\varphi / [(1 - e^2 \sin^2\varphi)^{(1/2)}]$$
$$E = FE + x$$
$$N = FN + M - M_O + \{x^2 \tan\varphi \, [(1 - e^2 \sin^2\varphi)^{(1/2)}] / (2a)\}$$

where

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - ....)\varphi - (3e^2/8 + 3e^4/32 + 45e^6/1024+....)\sin2\varphi$$
$$+ (15e^4/256 + 45e^6/1024 +.....)\sin4\varphi - (35e^6/3072 + ...)\sin6\varphi + ...]$$

with φ in radians and M_O for φ_O, the latitude of the natural origin, derived in the same way.

The reverse conversion from easting and northing to latitude and longitude requires iteration of three equations. The Guam projection uses three iterations, which is satisfactory over the small area of application. First M_O for the latitude of the origin φ_O is derived as for the forward conversion. Then:

$e_1 = [1 – (1 – e^2)^{0.5}] / [1 + (1 – e^2)^{0.5}]$

and

$M' = M_O + (N – FN) – \{(E – FE)^2 \tan \varphi_O [(1 – e^2 \sin^2\varphi_O)^{(1/2)}] / (2a)\}$

$\mu' = M' / a(1 – e^2/4 – 3e^4/64 – 5e^6/256 –....)$

$\varphi' = \mu' + (3e_1/2 – 27e_1^3/32)\sin(2\mu') + (21e_1^2/16 – 55e_1^4/32)\sin(4\mu') + (151e_1^3/96)\sin(6\mu') + (1097e_1^4/512)\sin(8\mu')$

$M'' = M_O + (N – FN) – \{(E – FE)^2 \tan \varphi' [(1 – e^2 \sin^2\varphi')^{(1/2)}] / (2a)\}$

$\mu'' = M'' / a(1 – e^2/4 – 3e^4/64 – 5e^6/256 –....)$

$\varphi'' = \mu'' + (3e_1/2 – 27e_1^3/32)\sin(2\mu'') + (21e_1^2/16 – 55e_1^4/32)\sin(4\mu'') + (151e_1^3/96)\sin(6\mu'') + (1097e_1^4/512)\sin(8\mu'')$

$M''' = M_O + (N – FN) – \{(E – FE)^2 \tan \varphi'' [(1 – e^2 \sin^2\varphi'')^{(1/2)}] / (2a)\}$

$\mu''' = M''' / a(1 – e^2/4 – 3e^4/64 – 5e^6/256 –....)$

$\varphi''' = \mu''' + (3e_1/2 – 27e_1^3/32)\sin(2\mu''') + (21e_1^2/16 – 55e_1^4/32)\sin(4\mu''') + (151e_1^3/96)\sin(6\mu''') + (1097e_1^4/512)\sin(8\mu''')$

Then

$\lambda = \lambda_O + \{(E – FE) [(1 – e^2 \sin^2\varphi''')^{(1/2)}] / (a \cos \varphi''')\}$

Example

For Projected Coordinate Reference System: Guam 1963 / Guam SPCS
(EPSG CRS code 3993)

Parameters:

Ellipsoid: Clarke 1866 a = 6378206.4 metres b = 6356583.8 metres
from which: 1/f = 294.9786982 e = 0.08227185 e^2 = 0.00676866

Latitude of natural origin	φ_O	13°28'20.87887"N
Longitude of natural origin	λ_O	144°44'55.50254"E
False easting	FE	50000.00 metres
False northing	FN	50000.00 metres

Forward calculation for:

Latitude φ = 13°20'20.53846"N Longitude λ = 144°38'07.19265"E

first gives

φ_O =	0.235138896 rad	λ_O =	2.526342288 rad
M_O =	1489888.76 m		
φ =	0.232810140 rad	λ =	2.524362746 rad
x =	– 12287.52 m	M =	1475127.96 m

whence

E = 37,712.48 m N = 35,242.00 m

Reverse calculation for the same Easting and Northing (37,712.48 m E, 35,242.00 m N) first gives:

M_O = 1489888.76 m e_1 = 0.001697916

and

	M (metres)	µ (radians)	φ (radians)		
First iteration:	1475127.93	0.231668814	0.232810136		
Second iteration:	1475127.96	0.231668819	0.232810140		
Third iteration:	1475127.96	0.231668819	0.232810140	=	13°20'20.538"N

Then

λ = 2.524362746 rad = 144°38'07.193"E

3.6.3.3 Azimuthal Equidistant

(EPSG Dataset coordinate operation method code 1125)

The Equi-7 grid developed at the Technical University of Vienna (TU Wien) to support continental-scale satellite imagery georeferencing is constructed on a series of azimuthal equidistant projections with rigorous calculations for the direct and inverse geodetic problems[11] as developed by Karney (2013)[12]. The equations have been implemented in C in the PROJ (1995)[13] library. The projection is neither conformal nor equal area.

For the forward conversion of latitude and longitude to easting and northing:

$$E = FE + s_{12} \sin \alpha_{12}$$
$$N = FN + s_{12} \cos \alpha_{12}$$

where s_{12} is the distance along the geodesic from projection origin φ_O,λ_O to point φ,λ and α_{12} is its true azimuth, s_{12} and α_{12} being computed using Karney's formula for the inverse geodetic problem.

Reverse conversion of easting and northing to latitude and longitude:

$$\alpha_{12} = \mathrm{atan2}(E-FE, N-FN)$$
$$s_{12} = [(E-FE)^2 + (N-FN)^2]^{(1/2)}$$

Then φ,λ are computed from $\varphi_O,\lambda_O,$ s_{12} and α_{12} using Karney's formula for the direct geodetic problem.

Example

For the WGS 84 / Equi-7 Europe zone:

Latitude of natural origin (φ_O)	= 53.0°N
Longitude of natural origin (λ_O)	= 24.0°E
False easting	= 5,837,287.820 m
False northing	= 2,121,415.696 m

Then for point latitude φ = 63.0°N, longitude λ = 44.0°E,
Easting E = 6,840,895.297 m, Northing N = 3,382,726.731 m.

[11] Forward geodetic problem: given coordinates φ_1,λ_1, geodesic distance s_{12} and true azimuth α_{12}, find coordinates φ_2,λ_2. Inverse geodetic problem: given coordinates of two points φ_1,λ_1, and φ_2,λ_2, find the geodesic distance s_{12} and true azimuths α_{12} and α_{21}.
[12] Karney, C. F. F. "Algorithms for geodesics." *Journal of Geodesy* 87(1), 2013. p. 43-55.
[13] OSGeo/PROJ source code. https://github.com/OSGeo/PROJ/blob/master/src/projections/aeqd.cpp (Accessed 4th January 2024).

3.6.4 Perspectives

3.6.4.1 Introduction

Geophysical and reservoir interpretation and visualization systems now work in a 3D "cube" offering continuous, scalable, viewing and mapping in a single Cartesian 3D coordinate system. Subsurface mapping historically has been undertaken in pseudo-3D coordinate reference systems consisting of a vertical component together with an independent horizontal component which had to be changed to maintain cartographic correctness over large areas. Map projections are inherently distorted. Typically, distances and areas measured on the map-grid only approximate their true values. Over small areas near the projection origin, the distortions can be managed to be within acceptable limits. But it is impossible to map large areas without significant distortion. This creates problems when there is a requirement to map areas beyond the limits of a map zone, typically overcome by moving to another zone.

The motivation here is to offer a method of overcoming these limitations by describing geodetically well-defined CRSs that can be implemented in 3D within a visualization environment and can be scaled (from reservoirs to regions) without distortion. There are three components:

- the use of geodetically rigorous 3D geocentric and topocentric coordinates, the relationship of which to geographic coordinates is described in Section §4.1
- perspective realizations of topocentric coordinates in 2D
- an ellipsoidal development of the orthographic projection; this 2D representation contains the quantifiable mapping distortions inherent in this projection method.

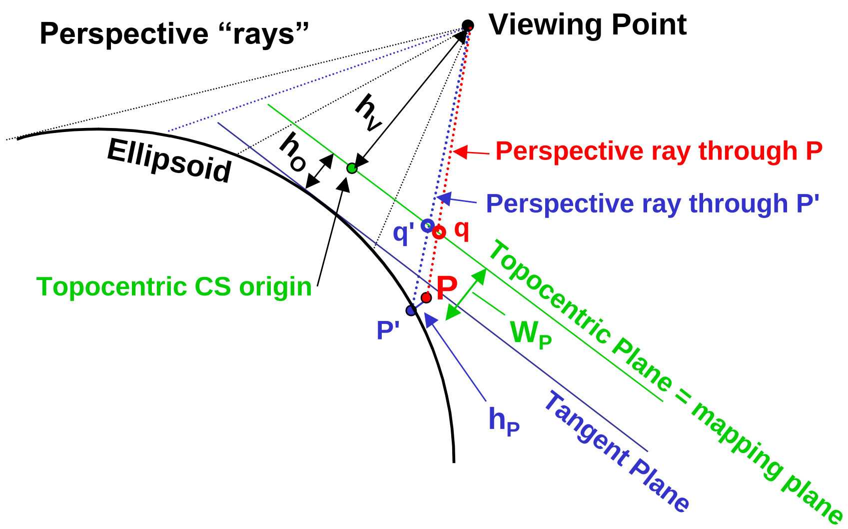


Figure 12 – Vertical perspective.

Classical map projections map 2D latitude and longitude onto the plane. With reference to Figure 12 above, point P at a height h_P above the ellipsoid is first reduced to the ellipsoid surface at P', and P' is then mapped onto the mapping plane at q'. The height of the point is not material.

In contrast, perspectives map points on, above or below the surface of the ellipsoid onto the mapping plane; point P is mapped onto the mapping plane at q. The height of a point above or depth below the surface of the ellipsoid will change the horizontal coordinates at which the point maps. Perspectives are a view of the Earth from space without regard to cartographic properties such as conformality or equality of area.

Perspectives can be classified as vertical or tilted. Consider a point anywhere on the ellipsoid, a plane tangent to the ellipsoid at that point, and a perpendicular to the ellipsoid (and the tangent plane) at that point. Vertical perspectives are the view of the Earth from a point on the perpendicular through a mapping plane which is either the tangent plane or a plane parallel to the tangent plane. Tilted perspectives are the view from a point which is not on the perpendicular. Tilted perspectives are not considered further in this Guidance Note.

In addition to vertical and tilted, perspectives can be classified as positive or negative. Perspectives with a positive viewing height h_V are the view of the Earth from above, as from a satellite or from another celestial body (and as shown in Figure 12). Perspectives with a negative viewing height h_V are the "view" of the Earth from below, which is mathematically but not physically possible. The mapping equations, however, are identical; only the sign of one term (the viewing height, h_V) differs. The viewing point cannot be on the mapping plane.

In this development, vertical perspectives are based upon topocentric coordinates that are valid for an ellipsoidal Earth. The introduction of an intermediate topocentric coordinate system simplifies the mathematical exposition of vertical perspectives. In such a topocentric Cartesian coordinate system, two of the three axes represent the horizontal plane. A change of perspective (zooming in and out) is achieved by moving the viewing point along the perpendicular. The mapping plane is the plane parallel to the tangent plane which passes through the topocentric origin (rather than the tangent plane itself). In the special case of the topocentric origin being on the ellipsoid surface, the mapping plane will be the tangent plane.

3.6.4.2 Vertical Perspective

(EPSG Dataset coordinate operation method code 9838)

This general case deals with a viewing point at a finite height above the origin. If the viewing point is at infinity ($h_V = \infty$), the formulas for the orthographic case given in Section §3.6.4.3 below should be used.

The forward equations for the Vertical Perspective to convert geographic 3D coordinates (φ, λ, h) to Easting (E) and Northing (N) begin with the methods of Section §4.1.3 to convert the geographic coordinates to topocentric coordinates U, V, W. The perspective projection origin is coincident with the topographic origin and has coordinates (φ_O, λ_O, h_O). As in Section §4.1.3:

$$U = (\nu + h) \cos\varphi \sin(\lambda - \lambda_O)$$
$$V = (\nu + h) [\sin\varphi \cos\varphi_O - \cos\varphi \sin\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \cos\varphi_O$$
$$W = (\nu + h) [\sin\varphi \sin\varphi_O + \cos\varphi \cos\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \sin\varphi_O - (\nu_O + h_O)$$

Then, given the height h_V of the perspective viewing point above the origin, the perspective coordinates (E, N) are calculated from topocentric coordinates (U, V, W) as:

$$E = U\, h_V / (h_V - W)$$
$$N = V\, h_V / (h_V - W)$$

The reverse calculation from E,N to U,V,W and φ,λ,h is indeterminate.

Example

For Projected Coordinate Reference System: WGS 84 / Vertical Perspective example

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.257223563
	then	e = 0.081819191	e^2 = 0.006694380

Topographic origin latitude	φ_O	=	55°00'00.000"N
Topographic origin longitude	λ_O	=	5°00'00.000"E
Topographic origin ellipsoidal height	h_O	=	200 metres
Height of viewpoint	h_V	=	5900000 meters

Forward calculation for:

Latitude	φ	=	53°48'33.82"N
Longitude	λ	=	2°07'46.38"E
Ellipsoidal height	h	=	73 metres

φ_O = 0.95993109 rad	λ_O = 0.08726646 rad
φ = 0.939151101 rad	λ = 0.037167659 rad
ν_O = 6392 510.73 m	ν = 6392 088.02 m

U = –189 013.869 m
V = –128 642.040 m
W = – 4 220.171 m

Then:

E = –188 878.767 m
N = –128 550.090 m

3.6.4.3 Vertical Perspective (orthographic case)

(EPSG Dataset coordinate operation method code 9839)

The orthographic vertical perspective is a special case of the vertical perspective with the viewing point at infinity ($h_V = \infty$). Therefore, all projection "rays" are parallel to one another and all are perpendicular to the tangent plane. Since the rays are parallel, coordinates in the tangent-plane are the same in any other parallel mapping plane, i.e., are consistent for any value of h_O, which therefore becomes irrelevant to the forward formulas.

The orthographic vertical perspective forward conversion from 3D geographic coordinates latitude, longitude, and ellipsoidal height (φ, λ, h) to Easting (E) and Northing (N) is given by:

$$E = U = \text{limit}\ (U\ h_V / (h_V - W),\ h_V \to \infty)$$
$$N = V = \text{limit}\ (V\ h_V / (h_V - W),\ h_V \to \infty)$$

where, as in Sections §4.1.3 and §3.6.4.2:

$$U = (\nu + h) \cos\varphi \sin(\lambda - \lambda_O)$$
$$V = (\nu + h) [\sin\varphi \cos\varphi_O - \cos\varphi \sin\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \cos\varphi_O$$

The reverse calculation from E,N to U,V,W and φ,λ,h is indeterminate.

Example

For Projected Coordinate Reference System: WGS 84 / Vertical Perspective (Orthographic case) example

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.257223563
	from which:	e = 0.081819191	e^2 = 0.006694380

Topographic origin latitude	φ_O	=	55°00'00.000"N		
Topographic origin longitude	λ_O	=	5°00'00.000"E		
Topographic origin ellipsoidal height	h_O	=	200 metres		

Forward calculation for:

Latitude	φ	=	53°48'33.82"N	=	0.939151101 rad
Longitude	λ	=	2°07'46.38"E	=	0.037167659 rad
Ellipsoidal height	h	=	73 metres		

The projection origin and example point are the same as those used in the general case of the Vertical Perspective in the previous section. Note that the ellipsoidal height at the point to be converted (h) is 73 metres. The ellipsoidal height at the topocentric center (h_O) is not used in any of the equations for the numerical examples that follow. But h_O will be used for the reverse case if W is known (for which a numerical example can be found in Section §4.1.3).

φ_O = 0.95993109 rad λ_O = 0.08726646 rad
φ = 0.939151101 rad λ = 0.037167659 rad
ν_O = 6392 510.727 m ν = 6392 088.017 m

Then:

E = –189013.869 m
N = –128642.040 m

3.6.5 **Orthographic**

(EPSG Dataset coordinate operation method code 9840)

Most cartographic texts which describe the orthographic projection do so using a spherical development. This section describes an ellipsoidal development. This allows the projected coordinates to be consistent with those for the vertical perspectives described in the previous section. If the projection origin is at the topocentric origin, the ellipsoidal Orthographic projection is a special case of the orthographic vertical perspective in which the ellipsoid height of all mapped points is zero (h = 0). The projection is neither conformal nor equal-area, but near the point of tangency there is no significant distortion. Within 90km of the origin, the scale change is less than 1 part in 10,000.

The Orthographic projection forward conversion from 2D geographic coordinates latitude and longitude (φ, λ) and the origin on the ellipsoid (φ_O, λ_O) is given by:

$$E = FE + \nu \cos \varphi \sin (\lambda - \lambda_O)$$
$$N = FN + \nu [\sin \varphi \cos \varphi_O - \cos \varphi \sin \varphi_O \cos (\lambda - \lambda_O)] + e^2 (\nu_O \sin \varphi_O - \nu \sin \varphi) \cos \varphi_O$$

where

ν is the prime vertical radius of curvature at latitude φ; $\nu = a /(1 - e^2\sin^2\varphi)^{1/2}$,
ν_O is the prime vertical radius of curvature at latitude of origin φ_O; $\nu_O = a /(1 - e^2\sin^2\varphi_O)^{1/2}$,
e is the eccentricity of the ellipsoid and $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$
a and b are the ellipsoidal semi-major and semi-minor axes,
1/f is the inverse flattening, and
the latitude and longitude of the projection origin are φ_O and λ_O.

These formulas are similar to those for the orthographic case of the vertical perspective except that, for the Orthographic projection given here, h = 0 and the term (ν + h) reduces to ν. The projection origin is at the topocentric system origin φ_O, λ_O with false origin coordinates FE and FN.

For the reverse formulas for latitude and longitude corresponding to a given Easting (E) and Northing (N), iteration is required as the prime vertical radius (ν) is a function of latitude.

Begin by seeding the iteration with the center of projection (or some better guess):

$\varphi = \varphi_O$
$\lambda = \lambda_O$

Enter the iteration here with the (next) best estimates of φ and λ. Then solve for the radii of curvature in the prime vertical (ν) and meridian (ρ):

$\nu = a / (1 - e^2 \sin^2\varphi)^{0.5}$
$\rho = a (1 - e^2) / (1 - e^2 \sin^2\varphi)^{1.5}$

Compute test values of E and N (E' and N') using the forward equations:

$E' = FE + \nu \cos\varphi \sin(\lambda - \lambda_O)$
$N' = FN + \nu [\sin\varphi \cos\varphi_O - \cos\varphi \sin\varphi_O \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin\varphi_O - \nu \sin\varphi) \cos\varphi_O$

Partially differentiate the forward equations to solve for the elements of the Jacobian matrix:

$J_{11} = \partial E/\partial\varphi = - \rho \sin\varphi \sin(\lambda - \lambda_O)$
$J_{12} = \partial E/\partial\lambda = \nu \cos\varphi \cos(\lambda - \lambda_O)$
$J_{21} = \partial N/\partial\varphi = \rho (\cos\varphi \cos\varphi_O + \sin\varphi \sin\varphi_O \cos(\lambda - \lambda_O))$
$J_{22} = \partial N/\partial\lambda = \nu \sin\varphi_O \cos\varphi \sin(\lambda - \lambda_O)$

Solve for the determinant of the Jacobian:

$D = J_{11} J_{22} - J_{12} J_{21}$

Solve the northerly and easterly differences this iteration:

$\Delta E = E - E'$
$\Delta N = N - N'$

Adjust the latitude and longitude for the next iteration by inverting the Jacobian and multiplying by the differences:

$\varphi = \varphi + (J_{22} \Delta E - J_{12} \Delta N) / D$
$\lambda = \lambda + (-J_{21} \Delta E + J_{11} \Delta N) / D$

Return to the entry point with new estimates of latitude and longitude and iterate until the change in φ and λ is not significant.

Example

For Projected Coordinate Reference System: WGS 84 / Orthographic projection example

Parameters:

Ellipsoid:	WGS 84	a = 6378137.0 metres	1/f = 298.2572236
	from which:	e = 0.081819191	e^2 = 0.006694380

Latitude of natural origin	φ_O	55°00'00.000"N
Longitude of natural origin	λ_O	5°00'00.000"E
False easting	FE	0 metres
False northing	FN	0 metres

Forward calculation for:

Latitude	φ	=	53°48'33.82"N
Longitude	λ	=	2°07'46.38"E

Note that ellipsoidal heights at the topocentric center (h_O) and at the point to be converted (h) may be the same as in the Vertical Perspective examples in the previous section. Neither height is used in the formulas that follow.

φ_O = 0.95993109 rad λ_O = 0.08726646 rad
φ = 0.939151101 rad λ = 0.037167659 rad
ν_O = 6392 510.727 m ν = 6392 088.017 m

Then:

Easting, E = –189 011.711 m
Northing, N = –128 640.567 m

Reverse calculation for these E, N coordinates into latitude (φ) and longitude (λ) is iterative. The following values are constant for every iteration.

ν_O = 6392 510.73 m φ_O = 0.95993109 rad λ_O = 0.08726646 rad

The following values change during 4 iterations to convergence:

	1	2	3	4
Latitude	0.95993109	0.9397628327	0.9391516179	0.9391511016
Longitude	0.08726646	0.0357167858	0.0371688977	0.0371676590
ν	6392510.727	6392100.544	6392088.028	6392088.017
ρ	6378368.440	6377140.690	6377103.229	6377103.198
E'	0	–194318.490	–189006.908	–189011.711
N'	0	–124515.840	–128637.469	–128640.567
J_{11}	0	265312.746	257728.730	257734.999
J_{12}	3666593.522	3766198.868	3769619.566	3769621.986
J_{21}	6378368.440	6370240.831	6370437.125	6370436.766
J_{22}	0	–159176.388	–154825.395	–154829.329
D	–2338688440386	–24033825331760	–24054027385585	–24054043431047
ΔN	–128640.567	–4124.727	–3.098	0
ΔE	–189011.711	5306.779	–4.803	0
Latitude	0.9397628327	0.9391516179	0.9391511016	0.9391511016
Longitude	0.0357167858	0.0371688977	0.0371676590	0.0371676590

which results in:

Latitude	φ	=	0.939151102 rad	=	53°48'33.82"N
Longitude	λ	=	0.037167659 rad	=	2°07'46.38"E

3.6.6 **Local Orthographic**

(EPSG Dataset coordinate operation method code 1130)

This is an ellipsoidal implementation which is equivalent to the ellipsoidal Orthographic projection described in previous Section §3.6.5 but includes two additional defining parameters applied at the projection origin, a scale factor and an azimuth, used to align the grid axes with local infrastructure. It is an infinite perspective vertical projection of the ellipsoid centered at a given latitude and longitude. The projection is neither conformal nor equal-area, but near the point of tangency or the line of secancy there is no significant distortion. Within 90km of the origin, the scale change is less than 1 part in 10,000. The method is also referred to as a local vertical projection or a local secant rectangular system.

The forward conversion from 2D geographic coordinates latitude and longitude (φ, λ) begins by:

$$X_p = \nu \cos(\varphi) \sin(\lambda - \lambda_C)$$
$$Y_p = -\sin(\varphi_C)\ [\nu \cos(\varphi) \cos(\lambda - \lambda_C) - \nu_C \cos(\varphi_C)] + \cos(\varphi_C)\ [\nu\ (1 - e^2) \sin(\varphi) - \nu_C\ (1 - e^2) \sin(\varphi_C)]$$

where

φ_C, λ_C are the latitude and longitude of the projection centre on the ellipsoid,
e is the eccentricity of the ellipsoid and $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$,
ν is the radius of curvature in the prime vertical at latitude φ; $\nu = a / (1 - e^2\sin^2\varphi)^{1/2}$,
ν_C is the radius of curvature in the prime vertical at the latitude of the projection centre φ_C;
$\nu_C = a / (1 - e^2\sin^2\varphi_C)^{1/2}$.

Then applying the azimuth (α_C, positive clockwise from true North), scale factor (k_C), and false easting (E_C) and false northing (N_C) at the projection centre:

$$E = E_C + k_C [\cos(\alpha_C) X_p - \sin(\alpha_C) Y_p]$$
$$N = N_C + k_C [\sin(\alpha_C) X_p + \cos(\alpha_C) Y_p]$$

Reverse

For the reverse formulas for latitude and longitude corresponding to a given Easting (E) and Northing (N), first remove the false easting, false northing, rotation, and scale factor:

$$X_p = \cos(\alpha_C) (E - FE) / k_C + \sin(\alpha_C) (N - FN) / k_C$$
$$Y_p = -\sin(\alpha_C) (E - FE) / k_C + \cos(\alpha_C) (N - FN) / k_C$$

To simplify the calculation of the latitude and longitude values, some interim quantities are calculated:

$$B = 1 - e^2 \cos^2(\varphi_C)$$
$$C = Y_p - \nu_C \sin(\varphi_C) \cos(\varphi_C) + \nu_C (1 - e^2) \cos(\varphi_C) \sin(\varphi_C)$$
$$D = \{(1 - e^2) [(a^2 - Xp^2) (1 - e^2 \cos^2(\varphi_C)) - C^2]\}^{1/2}$$
$$X_g = [-C \sin(\varphi_C) + D \cos(\varphi_C)] / B$$
$$Y_g = X_p$$
$$Z_g = [C \cos(\varphi_C) (1 - e^2) + D \sin(\varphi_C)] / B$$

Then:

$$\varphi = \mathrm{atan2}(Z_g, (1 - e^2) [X_g^2 + Y_g^2]^{1/2})$$
$$\lambda = \lambda_C + \mathrm{atan2}(Y_g, X_g)$$

Example

For Projected Coordinate Reference System: NAD83(2011) / San Francisco SFO B18.

Parameters:

Ellipsoid:	GRS80	a = 6378137.0 metres	1/f = 298.257222101
	from which:	e = 0.081819191	e^2 = 0.006694380

Latitude of projection centre	φ_C	37°37'44.289"N	= 0.656749406 rad
Longitude of projection centre	λ_C	122°23'38.190"W	= -2.136177266 rad
Azimuth at projection centre	α_C	27°47'34"	= 0.485075480 rad
Scale factor at projection centre	k_C	0.9999968	
Easting at projection centre	E_C	0 m	
Northing at projection centre	N_C	0 m	

Forward calculation for:

Latitude	φ	=	37°37'33.877"N	= 0.656698927 rad
Longitude	λ	=	122°23'04.700"W	= -2.136014902 rad

ν_C = 6386110.046 m ν = 6386109.000 m
X_p = 821.217 m Y_p = -320.965 m

Then:

Easting, E = 876.136 m
Northing, N = 98.974 m

For the reverse calculation for these E, N coordinates into latitude (φ) and longitude (λ):

X_p = 821.217 m	Y_p = -320.965 m	
B = 0.995801057	C = -20992.833	D = 6343357.666
X_g = 5057874.044	Yg = 821.217	Z_g = 3872656.155

which results in:

Latitude	φ	=	0.656698927 rad	=	37°37'09'33.877"N
Longitude	λ	=	-2.136014902 rad	=	122°23'04.700"W

3.7 Miscellaneous Map Projections

3.7.1 New Zealand Map Grid
(EPSG Dataset coordinate operation method code 9811)

This projection system typifies the development in the design and formulation of map projections where, by more complex mathematics yielding formulas readily handled by modern computers, it is possible to maintain the conformal property and minimise scale distortion over the total extent of a country area regardless of shape. Thus, both North and South Islands of New Zealand, previously treated not very satisfactorily in two zones of a Transverse Mercator projection, can now be projected as one zone of what resembles most closely a curved version Oblique Mercator but which, instead of being based on a minimum scale factor straight central line, has a central line which is a complex curve roughly following the trend of both North and South Islands. The projected coordinate reference system achieves this by a form of double projection where a conformal projection of the ellipsoid is first made to say an oblique Stereographic projection and then the Cauchy-Riemann equations are invoked in order to further project the rectangular coordinates on this to a modification in which lines of constant scale can be made to follow other than the normal great or small circles of Central meridians or standard parallels. The mathematical treatment of the New Zealand Map Grid is described in Stirling (1973)[14].

3.7.2 Tunisia Mining Grid
(EPSG Dataset coordinate operation method code 9816)

This grid is used as the basis for mineral leasing in Tunisia. Lease areas are approximately 2 km x 2 km or 400 hectares. The southwest corner of these blocks is defined through a six-figure grid reference where the first three digits are an easting in kilometres and the last three digits are a northing. The origin of the grid is Djebel Kebar at 38.81973gN, 7.83445gE (of Paris) at which the grid reference is 270582. The latitudes and longitudes for block corners at 2 km intervals are tabulated in a mining decree dated 1 January 1953. From this tabulation in which geographic coordinates are given to 5 decimal places it can be seen that:

a) the minimum easting is 94 km, on which the longitude is 5.68989 grads east of Paris
b) the maximum easting is 490 km, on which the longitude is 10.51515 grads east of Paris
c) each 2 km grid easting interval equals 0.02437 grads
d) the minimum northing is 40 km, on which the latitude is 33.39 grads
e) the maximum northing is 860 km, on which the latitude is 41.6039 grads
f) between 40 km N and 360 km N, each 2 km grid northing interval equals 0.02004 grads
g) between 360 km N and 860 km N, each 2 km grid northing interval equals 0.02003 grads

From the above the latitude of N=360 can be determined to be 36.5964gN.

This grid could be considered to be two equidistant cylindrical projection zones, north and south of the 360 km northing line. However, this would require the introduction of two spheres of unique dimensions. IOGP has therefore implemented the Tunisia mining grid as a coordinate conversion method in its own right. Formulas are:

Grads from Paris

$$\varphi = \varphi_O + [A\,(N - N_F)]$$
$$= 36.5964 + [A\,(N - 360)]$$

where φ and φ_O are in grads, N and N_F are in kilometres and A = 0.010015 if N > N_F = 360, else A = 0.01002.

[14] Stirling, I.F., "New Zealand Map Grid". New Zealand *Department of Lands and Survey*, Technical Circular 1973/32. 1973.

$\lambda = \lambda_O + [0.012185 (E - E_F)]$
$= 7.83445 + [0.012185 (E - 270)]$
where φ and φ_O are in grads relative to the Paris meridian and E and E_F are in kilometres.

The reverse formulas are:

$E (km) = E_F + [(\lambda - \lambda_O) / 0.012185]$
$= 270 + [(\lambda - 7.83445) / 0.012185]$.

$N (km) = N_F + [(\varphi - \varphi_O) / A]$
$= 360 + [(\varphi - 36.5964) / A]$
where $A = 0.010015$ if $\varphi > \varphi_O = 36.5964$, else $A = 0.01002$.

Degrees from Greenwich

Modern practice in Tunisia is to quote latitude and longitude in degrees with longitudes referenced to the Greenwich meridian. The formulas required in addition to the above are:

φ_d (degrees) = $(0.9\ \varphi_g)$ where φ_g is in grads.
$\lambda_{Greenwich}$ (degrees) = $[0.9 (\lambda_{Paris} + 2.5969213)]$ where λ_{Paris} is in grads and 2.5969213g is converted from the conventional value of 2°20'14.025"E.

φ_g (grads) = $(\varphi_d / 0.9)$ where φ_d is in decimal degrees.
λ_{Paris} (grads) = $[(\lambda_{Greenwich} / 0.9) - 2.5969213)]$ where $\lambda_{Greenwich}$ is in decimal degrees.

Example

Defining parameters are:
φ_O = 36.5964gN
λ_O = 7.83445gE of Paris
E_F = 270 km
N_F = 360 km

For grid location 302598, first derive the easting (302 km) and northing (598 km) from the grid reference.
Then:
Latitude φ = 36.5964 + [A (598 – 360)]. As N > 360, A = 0.010015.
= 38.97997 grads North = 35.08197 degrees North.

Longitude λ = 7.83445 + [0.012185 (302 – 270)]
= 8.22437 grads East of Paris = 9.73916 degrees East of Greenwich.

3.7.3 **Colombia Urban**
(EPSG Dataset coordinate operation method code 1052)

The capital cites of each department in Colombia use an urban projection for large scale topographic mapping of the urban areas. It is based on a plane through the origin at an average height for the area, eliminating the need for corrections to engineering survey observations.

The forward conversion from latitude and longitude to easting and northing is:

$E = FE + A\,\nu \cos\varphi\,(\lambda - \lambda_O)$
$N = FN + G\,\rho_O\,[(\varphi - \varphi_O) + \{B\,(\lambda - \lambda_O)^2\,\nu^2 \cos^2\varphi\}]$

where

φ_O, λ_O and h_O are the latitude, longitude, and height of the projection origin,

ρ is the radius of curvature of the ellipsoid in the plane of the meridian at latitude φ, ρ_O at latitude φ_O, and ρ_M at mid-latitude $(\varphi_O + \varphi)/2$; $\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{3/2}$ for φ, φ_O and φ_M

ν is the radius of curvature of the ellipsoid perpendicular to the meridian at latitude φ with ν_O at latitude φ_O; $\nu = a\,/(1 - e^2\sin^2\varphi)^{0.5}$ for φ and φ_O

e is the eccentricity of the ellipsoid and $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$
a is the ellipsoidal semi-major and 1/f is the inverse flattening

$A = 1 + h_O/\nu_O$
$B = \tan\varphi_O / 2\rho_O\nu_O$
$G = 1 + h_O/\rho_M$

Reverse
The reverse conversion from easting and northing to latitude and longitude is:

$\varphi = \varphi_O + [(N - FN) / D] - B\,[(E - FE) / C]^2$
$\lambda = \lambda_O + (E - FE) / C\,\nu \cos(\varphi)$

where
B is as in the forward conversion,
$C = 1 + h_O/a$
$D = \rho_O\,(1 + h_O/a(1-e^2))$

Example
For Projected Coordinate Reference System: MAGNA-SIRGAS / Bogota urban grid (EPSG:6247)

Parameters:

Ellipsoid:	GRS 1980	a = 6378137.0 metres	1/f = 298.257222101
	from which:	$e^2 = 0.006694380$	

Latitude of natural origin	φ_O	4°40'49.75"N
Longitude of natural origin	λ_O	74°08'47.73"W
False easting	FE	92334.879 metres
False northing	FN	109320.965 metres
Height of projection origin	h_O	2550.000 metres

Forward calculation for:
Latitude φ = 4°48'00.000"N Longitude λ = 74°15'00.000"W
First calculate constants for the projection:

$\varphi_O = 0.081689893$ rad	$\lambda_O = -1.294102154$ rad
$\rho_O = 6335862.944$	$\nu_O = 6378279.154$
A = 1.000399794	B = 1.012970E-15
C = 1.000399803	D = 6338413.114

Then:

$\varphi = 0.083775804$ rad	$\lambda = -1.295906970$ rad
$\nu = 6378286.489$	G = 1.000402470

and

$E = 80859.033m \qquad N = 122543.174m$

Reverse calculation for the same Easting and Northing (80859.033E, 122543.174N) first gives:

	Latitude	φ	=	4°48'00.000"N
then		ν	=	6378286.489
and	Longitude	λ	=	74°15'00.000"W

3.7.4 Equal Earth

(EPSG Dataset coordinate operation method code 1078)

The Equal Earth projection is an equal-area pseudo-cylindrical projection. It is appropriate for mapping global phenomena or for any other thematic world map that requires areas at their correct relative sizes. Its key features are its resemblance to the Robinson projection and continents with a visually pleasing appearance similar to those found on a globe.

Forward

To derive the projected coordinates of a point, geodetic latitude (φ) is converted to authalic latitude (ß). The formulas to convert geodetic latitude and longitude (φ, λ) to Easting (E) and Northing (N) are:

Easting, $E = FE + R_q\, 2\, (\lambda - \lambda_O) \cos(\theta) / \{\sqrt{3}\, [1.340264 - 0.243318\, \theta^2 + \theta^6\, (0.006251 + 0.034164\, \theta^2)]\}$

Northing, $N = FN + R_q\, \theta\, [1.340264 - 0.081106\, \theta^2 + \theta^6\, (0.000893 + 0.003796\, \theta^2)]$

where

$\theta = \text{asin}\, [\sin(ß)\, \sqrt{3} / 2]$

$R_q = a\, (q_P / 2)^{0.5}$

$ß = \text{asin}(q / q_P)$

$q = (1 - e^2)\, (\{\sin(\varphi) / [1 - e^2 \sin^2(\varphi)]\} - 1/(2e) \ln\{[1 - e \sin(\varphi)] / [1 + e \sin(\varphi)]\})$

$q_P = (1 - e^2)\, (\{\sin(\varphi_P) / [1 - e^2 \sin^2(\varphi_P)]\} - 1/(2e) \ln\{[1 - e \sin(\varphi_P)] / [1 + e \sin(\varphi_P)]\})$

where $\varphi_P = \pi/2$ radians, thus

$q_P = (1 - e^2)\, ([1 / (1 - e^2)] - \{[1/(2e)]\, \ln [(1 - e) / (1 + e)]\})$

Reverse

The reverse conversion from easting and northing to latitude and longitude requires iteration of the northing (N) equation to obtain θ. $\Theta_0 = (N–FN) / R_q$ is used as the first trial θ. A correction $\Delta\theta_n$ is calculated and subtracted from the trial θ_n value to obtain the next trial θ_{n+1}. N is the suffix number of iteration.

$\Theta_0 = (N–FN) / R_q$

$\Delta\theta_0 = \{ \theta_0\, [1.340264 - 0.081106\, \theta_0^2 + \theta_0^6\, (0.000893 + 0.003796\, \theta_0^2)] - [N–FN] / R_q \} / \{1.340264 - 0.243318\, \theta_0^2 + \theta_0^6\, [0.006251 + 0.034164\, \theta_0^2]\}$

$\theta_1 = \theta_0 - \Delta\theta_0$

$\Delta\theta_1 = \{\theta_1\, [1.340264 - 0.081106\, \theta_1^2 + \theta_1^6\, (0.000893 + 0.003796\, \theta_1^2)] - [N–FN] / R_q \} / \{1.340264 - 0.243318\, \theta_1^2 + \theta_1^6\, [0.006251 + 0.034164\, \theta_1^2]\}$

$\theta_2 = \theta_1 - \Delta\theta_1$

etc.

The calculation is repeated until $\Delta\theta_n$ is less than a predetermined convergence value. Then, using the final θ_{n+1} as θ, the geodetic latitude and longitude of a point are determined as follows:

$$\varphi = ß + \{[(e^2/3 + 31e^4/180 + 517e^6/5040) \sin(2ß)] + [(23e^4/360 + 251e^6/3780) \sin(4ß)] + [(761e^6/45360) \sin(6ß)]\}$$
$$\lambda = \lambda_O + \sqrt{3}\,(E–FE)\,\{1.340264 – 0.243318\,\theta^2 + \theta^6\,(0.006251 + 0.034164\,\theta^2)\} / \{2\,R_q \cos(\theta)\}$$

where $ß = \text{asin}\{2 \sin(\theta) / \sqrt{3}\}$ and R_q is defined as in the forward equations.

Sphere
For the spherical form of the projection, $ß = \varphi$ and $R_q = R$, where R is the radius of the sphere.

Example
For Projected Coordinate Reference System: WGS 84 / Equal Earth Americas (EPSG CRS code 8858)

Parameters:

Ellipsoid:	WGS 1984	a = 6378137.0 metres	1/f =298.257223563
	from which:	e = 0.08181919084262	

Longitude of natural origin	λ_O	90°00'00.00"W	= -1.5707963268 rad
False easting	FE	=	0.000 m
False northing	FN	=	0.000 m

Forward calculation for:

Latitude	φ	=	34°03'27.169" N	=	0.5944163293 rad
Longitude	λ	=	117°11'48.349" W	=	-2.0454693977 rad

First gives

$ß = 0.5923399644$ $\quad R_q = 6371007.181$
$\theta = 0.5046548375$

whence

E = -2390749.042 m
N = 4242849.758 m

Reverse calculation for the same Easting and Northing (-2390749.042 E, 4242849.758 N) starts with an iteration to obtain θ:

$(N–FN) / R_q = 0.665962169$

Step 1:	$\theta_0 = 0.665962169$	$\Delta\theta_0 = 0.164312383$
Step 2:	$\theta_1 = 0.501649786$	$\Delta\theta_1 = -0.003004201$
Step 3:	$\theta_2 = 0.504653987$	$\Delta\theta_2 = -8.512742e-07$
Step 4:	$\theta_3 = 0.504654838$	$\Delta\theta_3 = -6.859963e-14$

$\theta = \theta_4 = 0.504654838$

This gives:

$ß = 0.592339965$

Then	Latitude	φ	=	34°03'27.169" N
	Longitude	λ	=	117°11'48.349" W

4 Formulas for Coordinate Operations other than Map Projections

4.1 Conversions between geodetic Coordinate Reference Systems

4.1.1 Geographic/Geocentric conversions

(EPSG datset coordinate operation method code 9602)

Latitude, **φ,** and Longitude, **λ**, and ellipsoidal height, **h**, in terms of a 3-dimensional geographic coordinate reference system may be expressed in terms of a geocentric (Earth centred) Cartesian coordinate reference system X, Y, Z with the Z axis corresponding with the Earth's rotation axis positive northwards, the X axis through the intersection of the prime meridian and equator, and the Y axis through the intersection of the equator with longitude 90°E. The geographic and geocentric systems are based on the same geodetic datum.

Geocentric coordinate reference systems are conventionally taken to be defined with the X axis through the intersection of the Greenwich meridian and equator. This requires that the equivalent geographic coordinate reference system be based on the Greenwich meridian. In application of the formulas below, geographic coordinate reference systems based on a non-Greenwich prime meridian should first be transformed to their Greenwich equivalent. Geocentric coordinates X, Y, and Z take their units from the units for the ellipsoid axes (a and b). As it is conventional for X, Y, and Z to be in metres, if the ellipsoid axis dimensions are given in another linear unit, they should first be converted to metres.

If the ellipsoidal semi major axis is **a,** semi minor axis **b**, and inverse flattening **1/f**, then

$$X = (\nu + h) \cos\varphi \cos\lambda$$
$$Y = (\nu + h) \cos\varphi \sin\lambda$$
$$Z = [(1 - e^2)\,\nu + h] \sin\varphi$$

where ν is the prime vertical radius of curvature at latitude φ, where $\nu = a / (1 - e^2 \sin^2\varphi)^{0.5}$,
φ and λ are respectively the latitude and longitude (related to the prime meridian) of the point,
h is height above the ellipsoid, (see note below), and
e is the eccentricity of the ellipsoid where $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$

(Note that h *is the height above the ellipsoid. This is the height value that is delivered by GPS satellite observations but is not the gravity-related height value which is normally used for national mapping and levelling operations. The gravity-related height (*H*) is usually the height above mean sea level or an alternative level reference for the country. If one starts with a gravity-related height* H*, it will be necessary to convert it to an ellipsoid height* (h) *before using the above transformation formulas (*see Section §4.11.1)*. For the WGS 84 ellipsoid the difference between ellipsoid and mean sea level can vary between values of -100m in the Sri Lanka area to +80m in the North Atlantic.)*

For the reverse conversion, Cartesian coordinates in the geocentric coordinate reference system may be converted to geographic coordinates in terms of the geographic 3D coordinate reference system by:

$$\varphi = \text{atan2}\,[(Z + e^2 \nu \sin\varphi)\,,\,(X^2 + Y^2)^{0.5}] \text{ by iteration}$$
$$\lambda = \text{atan2}\,(Y\,,\,X)$$
$$h = X \sec\lambda \sec\varphi - \nu$$

where λ is relative to the Greenwich prime meridian.

To avoid iteration for φ it may alternatively be found from:

$$\varphi = \text{atan2}[(Z + \varepsilon\, b \sin^3 q)\,,\,(p - e^2\, a \cos^3 q)]$$

where

$$\varepsilon = e^2 / (1 - e^2)$$
$$b = a(1 - f)$$
$$p = (X^2 + Y^2)^{0.5}$$
$$q = \text{atan2}[(Z\, a)\,,\,(p\, b)]$$

Then h may more conveniently be found from

$$h = (p / \cos \varphi) - \nu$$

Example

Consider a North Sea point with position derived by GPS satellite in the WGS 84 coordinate reference system. The WGS 84 ellipsoid parameters are:

a = 6378 137.000m
1/f = 298.257223563

from which

e^2 = 0.006694380
ε = 0.006739497
b = 6356 752.314 m

Using the reverse direction direct formulas above, the conversion of WGS 84 geocentric coordinates of

X = 3771 793.968 m
Y = 140 253.342 m
Z = 5124 304.349 m

is:

p = 3774400.712
q = 0.937549875
φ = 0.939151102 rad
ν = 6392088.017

Then WGS 84 geographic 3D coordinates are:

latitude φ = 53°48'33.820"N
longitude λ = 2°07'46.380"E
and ellipsoidal height h = 73.0m

Further examples of input and output may be found in test procedure 5201 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.1.2 Geocentric/topocentric conversions

(EPSG Dataset coordinate operation method code 9836)

A topocentric coordinate system is a 3-D Cartesian system having mutually perpendicular axes U, V, W with an origin on or near the surface of the Earth. The U-axis is locally east, the V-axis is locally north, and the W-axis is up forming a right-handed coordinate system. It is applied in two particular settings:

(i) the height axis W is chosen to be along the direction of gravity at the topocentric origin. The other two axes are then in the horizontal plane. A special case of this, often applied in engineering applications, is when the topocentric origin is on the vertical datum surface; then topocentric height W approximates to gravity-related height H.

(ii) the topocentric height axis W is chosen to be the direction through the topocentric origin and perpendicular to the surface of the ellipsoid. The other two topocentric axes (U and V) are in the "topocentric plane", a plane parallel to the tangent to the ellipsoid surface at the topocentric origin and passing through the topocentric origin (see Figures 13 and 14 below). The coordinates defining the topocentric origin will usually be expressed in ellipsoidal terms as latitude φ_O, longitude λ_O and ellipsoidal height h_O but may alternatively be expressed as geocentric Cartesian coordinates X_O, Y_O, Z_O. In this context, the geocentric coordinates of the topocentric origin should not be confused with those of the geocentric origin where X=Y=Z=0.

A special case of this is when the topocentric origin is chosen to be exactly on the ellipsoid surface and h_O = 0. Then the topocentric U and V axes are in the ellipsoid tangent plane and at (and only at) the topocentric origin topocentric height W = ellipsoidal height h.

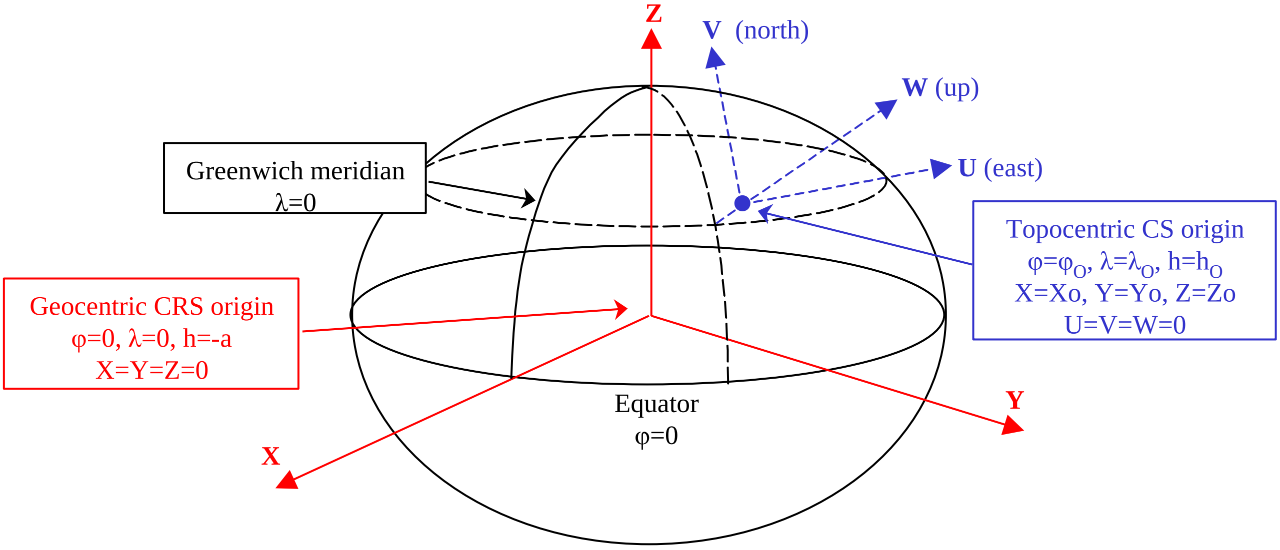


Figure 13 – Topocentric and geocentric systems.

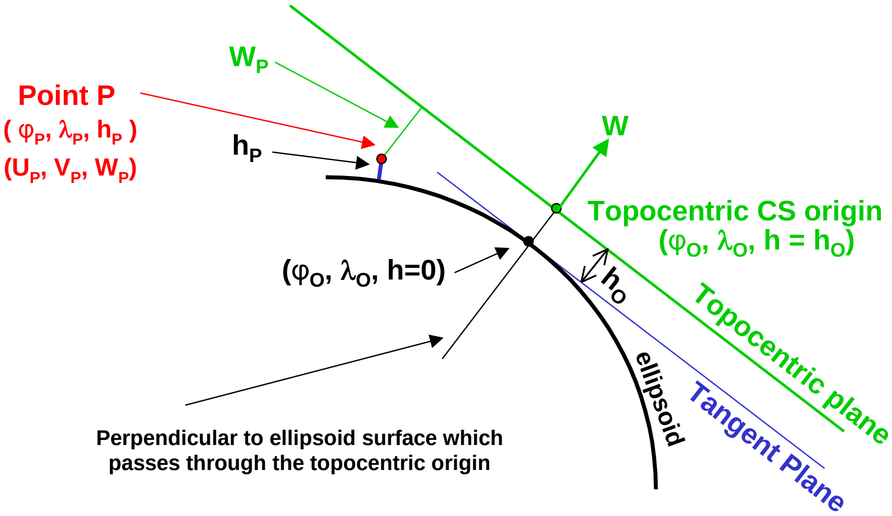


Figure 14 – Topocentric and ellipsoidal heights.

In this and the following section, we are concerned with the second of the two settings for topocentric coordinate systems where the system is associated with the ellipsoid and a particular geodetic datum. The application of such topocentric coordinates includes scalable mapping and visualization systems as described in Section §3.6.4. The following section covers the conversion between ellipsoidal coordinates and topocentric coordinates. The remainder of this section describes how geocentric coordinates X, Y, Z may be converted into topocentric coordinates U, V, W given the geocentric coordinates of the topocentric CS origin (X_O, Y_O, Z_O).

First, it is necessary to derive ellipsoidal values φ_O, λ_O of the topocentric origin from their geocentric values X_O, Y_O, Z_O through the reverse formulas given in Section §4.1.1. (The value h_O for the ellipsoidal height of the topocentric origin is not required in what follows.)

Then topocentric coordinates [U, V, W] are computed as follows:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \boldsymbol{R} \left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \begin{pmatrix} X_O \\ Y_O \\ Z_O \end{pmatrix} \right)$$

where,

$$\boldsymbol{R} = \begin{pmatrix} -\sin\lambda_O & \cos\lambda_O & 0 \\ -\sin\varphi_O\cos\lambda_O & -\sin\varphi_O\sin\lambda_O & \cos\varphi_O \\ \cos\varphi_O\cos\lambda_O & \cos\varphi_O\sin\lambda_O & \sin\varphi_O \end{pmatrix}$$

Or, expressed as scalar equations:

$$U = -(X-X_O)\sin\lambda_O + (Y-Y_O)\cos\lambda_O$$
$$V = -(X-X_O)\sin\varphi_O\cos\lambda_O - (Y-Y_O)\sin\varphi_O\sin\lambda_O + (Z-Z_O)\cos\varphi_O$$
$$W = (X-X_O)\cos\varphi_O\cos\lambda_O + (Y-Y_O)\cos\varphi_O\sin\lambda_O + (Z-Z_O)\sin\varphi_O$$

The reverse formulas to calculate geocentric coordinates from topocentric coordinates are:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boldsymbol{R}^{-1} \begin{pmatrix} U \\ V \\ W \end{pmatrix} + \begin{pmatrix} X_O \\ Y_O \\ Z_O \end{pmatrix}$$

where,

$$\boldsymbol{R}^{-1} = \boldsymbol{R}^{T} = \begin{pmatrix} -\sin\lambda_0 & -\sin\varphi_0\cos\lambda_0 & \cos\varphi_0\cos\lambda_0 \\ \cos\lambda_0 & -\sin\varphi_0\sin\lambda_0 & \cos\varphi_0\sin\lambda_0 \\ 0 & \cos\varphi_0 & \sin\varphi_0 \end{pmatrix}$$

and, as for the forward case, φ_O and λ_O are calculated through the formulas in Section §4.1.1.

Or, expressed as scalar equations:

$X = X_O - U \sin \lambda_O - V \sin \varphi_O \cos \lambda_O + W \cos \varphi_O \cos \lambda_O$
$Y = Y_O + U \cos \lambda_o - V \sin \varphi_O \sin \lambda_O + W \cos \varphi_O \sin \lambda_O$
$Z = Z_O + V \cos \varphi_O + W \sin \varphi_O$

Example

For Geocentric CRS	=	WGS 84 (EPSG CRS code 4978)
and		
Topocentric origin Xo	=	3652 755.3058 m
Topocentric origin Yo	=	319 574.6799 m
Topocentric origin Zo	=	5201 547.3536 m

Ellipsoid parameters: a = 6378137.0 metres 1/f = 298.257223563

First calculate additional ellipsoid parameters:

e^2 = 0.006694380 ε = 0.006739497 b = 6356752.314

Next, derive φ_O, λ_O from Xo,Yo,Zo by the formulas given in Section §4.1.1:

p	=	3666708.2376	
q	=	0.9583523313	
φ_O	=	0.9599310885	rad
λ_O	=	0.0872664625	rad

Forward calculation for point with geocentric coordinates:

X = 3771 793.968 m Y = 140 253.342 m Z = 5124 304.349 m

gives topocentric coordinates

U = –189 013.869 m V = –128 642.040 m W = – 4 220.171 m

The reverse calculation contains no intermediate terms other than those solved for above and is a trivial reversal of the forward.

4.1.3 <u>**Geographic/topocentric conversions**</u>

(EPSG Dataset coordinate operation method code 9837)

Topocentric coordinates may be derived from geographic coordinates indirectly by concatenating the geographic/geocentric conversion described in Section §4.1.1 above with the geocentric/topocentric conversion described in Section §4.1.2 above. Alternatively the conversion may be made directly:

To convert latitude φ, longitude λ and ellipsoidal height h into topocentric coordinates U,V,W:

$U = (\nu + h) \cos(\varphi) \sin(\lambda - \lambda_O)$
$V = (\nu + h) [\sin(\varphi) \cos(\varphi_O) - \cos(\varphi) \sin(\varphi_O) \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin(\varphi_O) - \nu \sin(\varphi)) \cos(\varphi_O)$
$W = (\nu + h) [\sin(\varphi) \sin(\varphi_O) + \cos(\varphi) \cos(\varphi_O) \cos(\lambda - \lambda_O)] + e^2 (\nu_O \sin(\varphi_O) - \nu \sin(\varphi)) \sin(\varphi_O) - (\nu_O + h_O)$

where φ_O, λ_O, h_O are the ellipsoidal coordinates of the topocentric origin

and ν is the radius of curvature in the prime vertical at latitude φ, where $\nu = a / (1 - e^2 \sin^2(\varphi))^{0.5}$
ν_O is the radius of curvature in the prime vertical at latitude φ_O, where $\nu_O = a / (1 - e^2 \sin^2(\varphi_O))^{0.5}$
e is the eccentricity of the ellipsoid where $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$

The reverse formulas to convert topocentric coordinates (U, V, W) into latitude, longitude, and ellipsoidal height (φ, λ, h) first draws on the reverse case of Section §4.1.2 to derive geocentric coordinates X, Y, Z and then on the reverse case in Section §4.1.1 to derive latitude, longitude and height.

First

$X = X_O - U \sin(\lambda_O) - V \sin(\varphi_O) \cos(\lambda_O) + W \cos(\varphi_O) \cos(\lambda_O)$
$Y = Y_O + U \cos(\lambda_O) - V \sin(\varphi_O) \sin(\lambda_O) + W \cos(\varphi_O) \sin(\lambda_O)$
$Z = Z_O + V \cos(\varphi_O) + W \sin(\varphi_O)$

where

$X_O = (\nu_O + h_O) \cos(\varphi_O) \cos(\lambda_O)$
$Y_O = (\nu_O + h_O) \cos(\varphi_O) \sin(\lambda_O)$
$Z_O = [(1 - e^2)\, \nu_O + h_O] \sin(\varphi_O)$
φ_O, λ_O, h_O are the ellipsoidal coordinates of the topocentric origin,
ν_O is the radius of curvature in the prime vertical at latitude φ_O, $\nu_O = a / (1 - e^2 \sin^2(\varphi_O))^{0.5}$, and
e is the eccentricity of the ellipsoid where $e^2 = (a^2 - b^2)/a^2 = 2f - f^2$.

Then

$\varphi = \text{atan2}[(Z + \varepsilon\, b \sin^3(q)) , (p - e^2\, a \cos3(q))]$
$\lambda = \text{atan2}(Y , X)$

where

$\varepsilon = e^2 / (1 - e^2)$
$b = a(1 - f)$
$p = (X^2 + Y^2)^{0.5}$
$q = \text{atan2}[(Z\, a) , (p\, b)]$
λ is relative to the Greenwich prime meridian.

And

$h = (p / \cos(\varphi)) - \nu$

where

ν is the radius of curvature in the prime vertical at latitude φ, where $\nu = a / (1 - e^2 \sin^2(\varphi))^{0.5}$

Example

For Geographic 3D CRS = WGS 84 (EPSG CRS code 4979)
and

Topocentric origin latitude	φ_O	55°00'00.000"N
Topocentric origin longitude	λ_O	5°00'00.000"E
Topocentric origin ellipsoidal height	h_O	200 metres

Ellipsoid parameters: a = 6378137.0 metres 1/f = 298.257223563

First calculate additional ellipsoid parameter e^2 and radius of curvature ν_O at the topocentric origin:

e^2	=	0.006694380	ν_O	=	6392510.727
then					
φ_O	=	0.95993109 rad	λ_O	=	0.08726646 rad

Forward calculation for:

Latitude	φ	=	53°48'33.82"N	=	0.93915110 rad
Longitude	λ	=	2°07'46.38"E	=	0.03716765 rad
Height	h	=	73.0 metres		

ν = 6392088.017

then

U	=	–189 013.869 m
V	=	–128 642.040 m
W	=	– 4 220.171 m

Reverse calculation for:

U	=	–189 013.869 m
V	=	–128 642.040 m
W	=	– 4 220.171 m

First calculate additional ellipsoid parameter e^2 and radius of curvature ν_O at the topocentric origin:

e^2 = 0.006694380 ν_O = 6392510.727

then the following intermediate terms:

X_O	=	3652 755.306	ε	=	0.0067394967
Y_O	=	319 574.680	b	=	6356 752.314
Z_O	=	5201 547.353	p	=	3774 400.712
			q	=	0.937549875
X	=	3771 793.968	φ	=	0.9391511015 rad
Y	=	140 253.342	ν	=	6392 088.017
Z	=	5124 304.349	λ	=	0.03716765908 rad

for a final result of:

Latitude	φ	=	53°48'33.820"N
Longitude	λ	=	2°07'46.380"E
Height	h	=	73.0 metres

4.1.4 **Geographic 3D to 2D conversions**

(EPSG Dataset coordinate operation method code 9659)

The forward case is trivial. A geographic 2D CRS is derived from a 3-dimensional geographic coordinate reference system with coordinates geodetic latitude, geodetic longitude and ellipsoidal height by the simple expedient of dropping the height. The change is described by the following matrix operation:

$$\begin{pmatrix} \text{Derived CRS 1st coordinate} \\ \text{Derived CRS 2nd coordinate} \\ \text{Derived CRS 3rd coordinate} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \text{Base CRS 1st coordinate} \\ \text{Base CRS 2nd coordinate} \\ \text{Base CRS 3rd coordinate} \end{pmatrix}$$

The reverse conversion, from 2D to 3D, is indeterminate. It is, however, a requirement when a geographic 2D coordinate reference system is to be transformed using a geocentric method which is 3-dimensional (see Section §4.4.1). In practice, an artificial ellipsoidal height is created and appended to the geographic 2D coordinate reference system to create a geographic 3D coordinate reference system referenced to the same geodetic datum. The assumed ellipsoidal height is usually either set to the gravity-related height of a position in a compound coordinate reference system, or set to zero. As long as the height chosen is within a few kilometres of sea level, no error will be induced into the horizontal position resulting from the later geocentric transformation; the vertical coordinate will, however, be meaningless.

Example

A location in the ETRS89 Geographic 3D coordinate reference system

	latitude φ_s	=	53°48'33.82"N
	longitude λ_s	=	2°07'46.38"E
and	ellipsoidal height h_s	=	73.0m

is converted to the ETRS89 Geographic 2D coordinate reference system as

latitude φ_s	=	53°48'33.82"N
longitude λ_s	=	2°07'46.38"E

For the reverse conversion of the same point in the ETRS89 Geographic 2D coordinate reference system with horizontal coordinates of

latitude φ_s	=	53°48'33.82"N
longitude λ_s	=	2°07'46.38"E

an arbitrary value is given to the ellipsoidal height resulting in coordinates in the ETRS89 Geographic 3D coordinate reference system of

	latitude φ_s	=	53°48'33.82"N
	longitude λ_s	=	2°07'46.38"E
and	ellipsoidal height h_s	=	0.0m

4.1.5 Horizontal axis order reversal

a) Horizontal axis reversal – Geographic 3D CRS
(EPSG Dataset coordinate operation method code 9844)

A 3-dimensional geographic coordinate reference system comprising geodetic latitude, geodetic longitude, and ellipsoidal height is converted to one comprising geodetic longitude, geodetic latitude, and ellipsoidal height. This is a parameter-less conversion. The axis order change operates on coordinates of a point whose order change is described by the following matrix operation:

$$\begin{pmatrix} \text{Derived CRS 1st coordinate} \\ \text{Derived CRS 2nd coordinate} \\ \text{Derived CRS 3rd coordinate} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Base CRS 1st coordinate} \\ \text{Base CRS 2nd coordinate} \\ \text{Base CRS 3rd coordinate} \end{pmatrix}$$

b) Horizontal axis reversal – 2D CRS
(EPSG Dataset coordinate operation method code 9843)

This is a parameter-less conversion. It is used when GIS systems require ordinate before abscissa, for example:

- a 2-dimensional geographic coordinate reference system comprising geodetic latitude and geodetic longitude is converted to one comprising geodetic longitude and geodetic latitude
- a projected coordinate reference system comprising of northing and easting is converted to one comprising of easting and northing

The axis order change operates on coordinates of a point whose order change is described by the following matrix operation:

$$\begin{pmatrix} \text{Derived CRS 1st coordinate} \\ \text{Derived CRS 2nd coordinate} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \text{Base CRS 1st coordinate} \\ \text{Base CRS 2nd coordinate} \end{pmatrix}$$

4.2 Point Motion Operations

Classical coordinate reference systems are 'static', that is the spatial coordinates of a point do not change with time. Modern global geodetic reference frames have an Earth-centred Cartesian 3D coordinate system that co-rotates with the Earth as a whole. Points on the crust of the Earth are subject to tectonic motion and deformation and their coordinates may change with time, either because the point on the tectonic plate moves (slowly) through the coordinate space due to secular tectonic plate motion, or as a consequence of deformation caused by, e.g., an earthquake or post-glacial isostatic adjustment. Such systems are called 'dynamic'. Their coordinates are qualified by their *coordinate epoch*. By convention, coordinate epoch is given in decimal years referenced to the Gregorian calendar. Coordinates of a point at coordinate epoch t_1 may be calculated at some other coordinate epoch t_2 by one of the methods below.

It is stressed that the change of coordinates due to point motion is due entirely to the motion of the point and that there is *no change of coordinate reference system*. The coordinate change due to point motion happens *within* a coordinate reference system.

The velocities necessary for these coordinate operations may come from the station solutions of the reference frame definition (datum parameters), or from a plate motion velocity grid. The velocity vector **V** may be resolved into geocentric Cartesian components V_X, V_Y, V_Z, or alternatively given in linear units resolved into north, east and up components V_N, V_E, V_U which are applied to ellipsoidal coordinates in a geographic 3D CRS.

This motion compensation may then be concatenated with a 'static' transformation to form a time-specific transformation (refer to Section §4.3.6).

4.2.1 Point motion (geocentric Cartesian)

(EPSG Dataset coordinate operation method code 1064)

The geocentric Cartesian 3D coordinates of a point at coordinate epoch t_1 may be calculated at any other coordinate epoch t_2 from:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{t_2} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{t_1} + (t_2 - t_1) \begin{pmatrix} V_X \\ V_Y \\ V_Z \end{pmatrix}$$

where **V** describes the linear velocities of the Cartesian coordinates, with the velocity (regardless of whether positive or negative) implicitly taken to be positive towards the future.

Reverse
For the reverse operation the same formula is used. The calculation will be exact only if the velocity vector **V** used in the reverse calculation is the same as that used in the forward calculation. However, because the velocity field changes spatially, the velocity vector for the reverse operation in principle will usually differ from that for the forward operation, and if there are discontinuities in the velocity field the operation may not be reversible. As such the operation method is described as non-reversible. However, when implemented through a velocity grid as long as there are no discontinuities in the velocity field a reverse solution by iteration may be possible - see Section §4.2.3.

Example
Given a point with geocentric Cartesian coordinates at coordinate epoch 2005.00 and having linear velocities:

X	=	2845 456.0813 m	V_X	=	–0.0212 m/yr
Y	=	2160 954.2453 m	V_Y	=	+0.0124 m/yr
Z	=	5265 993.2296 m	V_Z	=	+0.0072 m/yr
t_1	=	2005.00			

whose coordinates are required at coordinate epoch 2010.0.

$$X_{(t=2010.0)} = 2845\,456.0813 + (2010.00 - 2005.00) * (-0.0212)$$
$$Y_{(t=2010.0)} = 2160\,954.2453 + (2010.00 - 2005.00) * (+0.0124)$$
$$Z_{(t=2010.0)} = 5265\,993.2296 + (2010.00 - 2005.00) * (+0.0072)$$

Then

X	=	2845 455.9753 m
Y	=	2160 954.3073 m
Z	=	5265 993.2656 m
t_2	=	2010.00

4.2.2 Point motion (ellipsoidal)

(EPSG Dataset coordinate operation method code 1067)

The ellipsoidal coordinates of a point at coordinate epoch t_1 may be calculated at any other coordinate epoch t_2 from:

$$\begin{pmatrix} \varphi \\ \lambda \\ h \end{pmatrix}_{t_2} = \begin{pmatrix} \varphi \\ \lambda \\ h \end{pmatrix}_{t_1} + (t_2 - t_1) \begin{pmatrix} V_\varphi \\ V_\lambda \\ V_h \end{pmatrix}$$

where **V** describes the velocities of the ellipsoidal coordinates, positive towards the future. These are usually given as a rate in linear units (for instance millimetres per year) resolved into north, east and up components V_N, V_E, and V_U. The north and east components are converted into latitude and longitude components by:

$$V\varphi = V_N / (\rho + h)$$
$$V\lambda = V_E / [(\nu + h) \cos \varphi]$$

where

ρ is radius of curvature of the CRS's ellipsoid in the plane of the meridian at latitude φ

$\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{3/2}$

ν is radius of curvature of the ellipsoid perpendicular to the meridian at latitude φ

$\nu = a /(1 - e^2\sin^2\varphi)^{1/2}$

The up component V_U is in the geometric domain so $Vh = V_U$.

Reverse

For the reverse operation, the same formula is used. The calculation will be exact only if the velocity vector **V** used in the reverse calculation is the same as that used in the forward calculation. However, because the velocity field changes spatially, the velocity vector for the reverse operation in principle will usually differ from that for the forward operation, and if there are discontinuities in the velocity field the operation may not be reversible. As such the operation method is described as non-reversible. However, when implemented through a velocity grid as long as there are no discontinuities in the velocity field a reverse solution by iteration may be possible - see Section §4.2.3.

Example

Given a point with ellipsoidal coordinates at coordinate epoch 2017.55 and having linear velocities:

φ	=	51.0°N	V_N	=	+15.12 mm/yr
λ	=	141.0°W	V_E	=	–2.86 mm/yr
h	=	1000 m	V_U	=	+1.10 mm/yr
t	=	2017.55 (years)			

whose coordinates are required at coordinate epoch 1997.00. The point is referenced to a CRS which uses the GRS 1980 ellipsoid for which a = 6378137.0m and 1/f = 298.257222101 from which e^2 = 0.006694380.

Then (after conversion to appropriate units):

$V\varphi$ = 2.37174E-09 rad/yr
$V\lambda$ = -7.10972E-10 rad/yr
Vh = 0.0011 m / yr

and

φ = 0.890117919 + 2.37174E-09 (1997.00-2017.55) = 0.890117870 radians
λ = 0.890117919 -7.10972E-10 (1997.00-2017.55) = -2.460914231 radians
h = 1000.0 + 0.0011 (1997.00-2017.55) = 999.977m

φ = 50°59'59.98995"N, λ = 140°59'59.99699"W, h = 999.977m
t = 1997.00

4.2.3 Point motion by velocity grid interpolation

The velocities of points in some geodetic coordinate reference systems are available through gridded data sets (sometimes called velocity grids). First, the velocities at a point are obtained through interpolation within the grid; bilinear interpolation is the most usual grid interpolation mechanism, although the providers of some gridded data sets may suggest an alternative interpolation method. Then, the interpolated velocities are applied as for the point motion (geocentric Cartesian) or point motion (ellipsoidal) methods in the previous Sections §4.2.1 and §4.2.2.

For the purposes of interpolation within the grid, horizontal coordinates of the point are required. The providers of gridded data sets define the horizontal CRS upon which the grid has been constructed and which is to be used for the grid interpolation; this is the geographical 2D subset of the geodetic CRS to which coordinates are referenced.

The EPSG Dataset differentiates methods by the domain in which the velocities are given (geocentric Cartesian or ellipsoidal) and the format of the gridded data file. The grid file format is given in documentation available from the information source. Examples include *NTv2_Vel* which is used by Natural Resources Canada for point motion in NAD83(CSRS) in the ellipsoidal domain and the Indonesian deformation model (INADEFORM) in the geocentricCartesian domain.

Reverse
The reversibility of point motion operations by velocity grid interpolation is complex and a detailed description is beyond the scope of this Guidance Note. For the reverse operation, the same formula is used. As for the single point methods described in the previous two sections, theoretically the calculation cannot be reversed exactly. However, in practice, reversal is possible using iteration if certain conditions are met. As such, the EPSG operation method is described as reversible, but it is left to implementations to evaluate whether this is possible with any particular data set.

Because the velocity field changes spatially, the velocity vector for the reverse transformation will differ from that for the forward transformation. The displacement vector at the location output from the forward calculation cannot be used directly because the domain of the grid is in the *source* CRS and epoch. However, in certain circumstances, an iterative reverse may be possible using the velocity at the target coordinates from the forward calculation as an approximation for that at the (forward) source location, then iterating until sufficiently small differences have been obtained. The criteria when an iterative reverse is possible are met when the velocity vectors at neighbouring nodes are of similar magnitude and direction (that is, the curvature of the grid at the point of interest is small) and the time period between source and target epochs is sufficiently small to not change this. When there are discontinuities in the velocity field causing large variations in vector magnitude or direction, the iterative solution may diverge rather than converge.

4.2.4 Velocity grid used for transformation

In Canada the velocity grid described in the previous section as a point motion operation within a CRS is also used as a transformation between the different realizations of NAD83(CSRS) (see section Section §4.4.6 below). It is also used as a transformation between CGVD2013 vertical CRS snapshots (see Section §4.10.3 below).

4.3 Transformations between Geocentric Coordinate Reference Systems

4.3.1 Overview

The methods in this section operate in the geocentric Cartesian coordinate domain. However they are most frequently used as the middle part of a transformation of coordinates from one geographic coordinate reference system into another forming a concatenated operation of:

[(geographic to geocentric) + (geocentric to geocentric) + (geocentric to geographic)]

in terms of coordinate reference systems or:

[(ellipoidal to Cartesian) + (Cartesian to Cartesian) + (Cartesian to ellipoidal)]

in terms of their coordinates.

The formulas given in the remainder of this section are for the transformation in the geocentric Cartesian coordinate domain forming the middle step. See Section §4.4.1 below for a fuller description of these concatenated operations, and EPSG Dataset Guidance Note 7 part 1 (IOGP Report 373-07-01 - *Using the EPSG geodetic parameter dataset*) for a general discussion of implicit concatenated operations created by application software.

The coordinate transformation methods in this section consist of a family of three-dimensional similarity transformations which progresses to a more extensive modelling of the CRS relationships:

- 3 translations model the offset of the source and target geocentric Cartesian coordinate reference system origins near the centre of the Earth. They assume that the two systems are orientated such that the axes are exactly parallel and have exactly the same scale. These conditions are rarely encountered in reality, but the modelling has been widely adopted in part because of its simplicity and in part because there may be insufficient data to develop a more rigorous transformation.

- 7-parameter Helmert transformations add three rotations and a scale change to the three translations. These better represent geodetic reality, especially over large areas. A variation used for modelling secular tectonic plate motion is to have the values of the seven parameters change with time.

- 10-parameter transformations add the three Cartesian coordinates of a non-geocentric point about which the rotations of the Helmert transformation are applied. These reduce correlations between the seven Helmert parameters when the transformation derivation is poorly conditioned, for example derived over small surface areas (subtending less than 30° at the centre of the Earth).

In implementation, the 3-parameter geocentric translation method can be treated as a special case of the 7-parameter Helmert transformation which in turn is a special case of the 10-parameter method, and they are documented below in sequence from 10-parameter to 3 -parameter. However, issues connected with reversibility described below should be noted.

In the formulas in this Section §5.3.1, the parameters used in geocentric Cartesian coordinate transformation methods are represented by symbols as given in Table 12:

Table 12 – Coordinate Operation Parameters used in geocentric coordinate transformations.

a) Parameters used in geocentric transformations (including Helmert transformations)		
Parameter Name	Symbol	Description
	X_S Y_S Z_S	Coordinates in the source coordinate reference system.
	X_T Y_T Z_T	Coordinates in the target coordinate reference system.
	X_P Y_P Z_P	Coordinates of a pivot point in the source coordinate reference system.
X-axis translation	tX	Translation vector, to be added to the point's position vector in the source CRS in order to transform from source CRS to target CRS; also: the coordinates of the origin of source CRS in the target CRS.
Y-axis translation	tY	
Z-axis translation	tZ	
X-axis rotation	rX	Rotations to be applied about the axes of the source CRS. There are two opposing conventions for defining the positive rotation direction, described below.
Y-axis rotation	rY	
Z-axis rotation	rZ	
Scale factor for source CRS axes Scale difference	M dS	Multiplication factor to be applied to coordinates in the source CRS to obtain the correct scale of the target CRS. M = (1 + dS). Scale factor for source CRS axes minus one. dS = (M – 1). If a distance of 100 km in the source coordinate reference system translates into a distance of 100.0001 km in the target coordinate reference system, the scale difference is 1 ppm (the ratio being 1.000001). When the scale difference dS is expressed in parts per million (ppm), $M = (1 + dS*10^{-6})$. When the scale difference dS is expressed in parts per billion (ppb), $M = (1 + dS*10^{-9})$.
b) Additional parameters used only in time-dependent Helmert transformations		
Parameter Name	Symbol	Description
Rate of change of X-axis translation	δtX	First derivative of X-axis translation with respect to time.
Rate of change of Y-axis translation	δtY	First derivative of Y-axis translation with respect to time.
Rate of change of Z-axis translation	δtZ	First derivative of Z-axis translation with respect to time.
Rate of change of X-axis rotation	δrX	First derivative of X-axis rotation with respect to time.
Rate of change of Y-axis rotation	δrY	First derivative of Y-axis rotation with respect to time.
Rate of change of Z-axis rotation	δrZ	First derivative of Z-axis rotation with respect to time.
Rate of change of scale difference	δdS	First derivative of scale difference with respect to time.
Parameter reference epoch	t_0	The reference epoch for the parameter values of a time-dependent transformation.
c) Additional parameter used only in time-specific Helmert transformations		
Transformation reference epoch	t_{trans}	The reference epoch for a time-specific transformation.

R is a rotation matrix combining the three rotations about the X, Y, and Z axes respectively. There are three issues with this, discussed in the following Sections §4.3.1.1, §4.3.1.2, and §4.3.1.1:

i) the sign convention used
ii) the order in which the rotations are applied
iii) the magnitude of the rotations

4.3.1.1 Coordinate Frame and Position Vector rotation conventions

Two opposing conventions are in use for the rotations: they may be applied to the position vector in the source CRS or to the coordinate frame of the source CRS.

a) The **Coordinate Frame** rotation convention (CF)

Rotations are applied to the coordinate frame (the set of coordinate axes). The sign convention is such that a positive rotation of the frame about an axis when viewed from the origin of the Cartesian coordinate system in the positive direction of that axis is defined as a *clockwise rotation of the coordinate frame*. A positive rotation in the source CRS about its Z-axis only will result in a smaller longitude value for the point in the target CRS.

$$\boldsymbol{R_Z} = \begin{pmatrix} \cos(rZ) & \sin(rZ) & 0 \\ -\sin(rZ) & \cos(rZ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{R_Y} = \begin{pmatrix} \cos(rY) & 0 & -\sin(rY) \\ 0 & 1 & 0 \\ \sin(rY) & 0 & \cos(rY) \end{pmatrix} \quad \boldsymbol{R_X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(rX) & \sin(rX) \\ 0 & -\sin(rX) & \cos(rX) \end{pmatrix}$$

b) The **Position Vector** rotation convention (PV)

Rotations are applied to the point's vector. The sign convention is such that a positive rotation about an axis is defined as a *clockwise rotation of the position vector* when viewed from the origin of the Cartesian coordinate system in the positive direction of that axis, e.g., a positive rotation about the source system Z-axis only will result in a larger longitude value for the point in the target system.

$$\boldsymbol{R_Z} = \begin{pmatrix} \cos(rZ) & -\sin(rZ) & 0 \\ \sin(rZ) & \cos(rZ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{R_Y} = \begin{pmatrix} \cos(rY) & 0 & \sin(rY) \\ 0 & 1 & 0 \\ -\sin(rY) & 0 & \cos(rY) \end{pmatrix} \quad \boldsymbol{R_X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(rX) & -\sin(rY) \\ 0 & \sin(rX) & \cos(rX) \end{pmatrix}$$

This is the opposite of the Coordinate Frame convention in (a). The same rotation that is defined as positive in the Coordinate Frame convention is negative in the Position Vector convention, and vice versa: $r_{PV} = -r_{CF}$. So that the correct parameter values may be associated with the coordinate transformation method (algorithm) implemented in software it is crucial that the convention underlying the definition of the rotation parameters is clearly understood and is communicated when exchanging transformation parameter values.

Note: in some literature, such as that from the International Earth Rotation Service (IERS), this rotation convention is described as an *anticlockwise* (*counter-clockwise*) *rotation of the coordinate frame*.

4.3.1.2 Sequence of rotations

In general, with three rotations about the X, Y, and Z axes respectively, the sequence in which the rotations are applied is significant. Applying them in a different sequence will result in change in output. Although the difference will be sub-millimetre if the rotations are under 10 arc-seconds, the error grows quadratically and for rotations of one degree the difference in result is hundreds of metres. By convention, the rotations are first about the X-axis, then the Y-axis, then the Z-axis. For the Coordinate Frame rotation convention this leads to the following compounded rotation matrix:

$$\boldsymbol{R}_{\mathbf{CF}} = \boldsymbol{R_Z} * \boldsymbol{R_Y} * \boldsymbol{R_X}$$

$$= \begin{pmatrix} \cos(rY)\cos(rZ) & \cos(rX)\sin(rZ) + \sin(rX)\sin(rY)\cos(rZ) & \sin(rX)\sin(rZ) - \cos(rX)\sin(rY)\cos(rZ) \\ -\cos(rY)\sin(rZ) & \cos(rX)\cos(rZ) - \sin(rX)\sin(rY)\sin(rZ) & \sin(rX)\cos(rZ) + \cos(rX)\sin(rY)\sin(rZ) \\ \sin(rY) & -\sin(rX)\cos(rY) & \cos(rX)\cos(rY) \end{pmatrix}$$

4.3.1.3 Magnitude of rotations - small angle approximation

When the rotation angles are very small, cos(r) ≈ 1, sin(r) ≈ r in radians and the product of multiple small angles can be considered to be insignificant and ignored. The full (compounded) matrix formula can be simplified using the small angle approximation as:

for the Coordinate Frame rotation convention: $$\boldsymbol{R}_{\mathbf{CF}} = \begin{pmatrix} 1 & rZ & -rY \\ -rZ & 1 & rX \\ rY & -rX & 1 \end{pmatrix}$$

and for the Position Vector transformation convention: $$\boldsymbol{R}_{\mathbf{PV}} = \begin{pmatrix} 1 & -rZ & rY \\ rZ & 1 & -rX \\ -rY & rX & 1 \end{pmatrix}$$

For this small angle approximation to be better than 3mm at the Earth's surface, the rotations must be under 10 arc-seconds. If the rotations are less than 3 arc-seconds, the approximation is valid to better than 0.3 mm at the Earth's surface.

For larger angles, an error may occur when using the approximation, but only when the transformation parameter values have been derived with an algorithm that includes the full rotation matrix and the application of those values uses the approximation, or vice versa. If the parameter values for a coordinate transformation are applied with the same algorithm as those used in their derivation, whether that be the full matrix or the small angle approximation, no error occurs.

In geodetic practice most transformations have small rotations. The formulas using the small angle approximation then produce results insignificantly different to the rigorous 'full matrix' formulas. In the EPSG Dataset, these similarity (shape-preserving) transformations between geocentric CRSs are generally associated with formulas using the small angle approximation. Implemetations may use these transformations with either the 'full matrix' or the small angle approximation formulas, regardless of through which they have been derived. When transformations have large rotations, in the EPSG Dataset they are associated with methods using the 'full matrix' formulas; these must be implemented using 'full matrix' formulas and should not be implemented using small angle approximation formulas.

Note: When using the small angle approximation, for the reverse calculation the same formulas as for the forward calculation are used but with the signs of the parameter values changed. However, when using the full matrix formulas the reverse uses different formulas from the forward calculation, and in this the signs of the parameter values are not changed.

For transformations in the geocentric Cartesian coordinate domain, the section in which they are detailed below is given in Table 13, together with their EPSG method code. The method code for when they are embedded in a transformation in the ellipsoidal coordinate domain as discussed in Section §4.4.1.2 is given in parentheses.

Table 13 – Similarity transformation methods between Geocentric Coordinate Reference Systems.

Method	**Small angle approximation**		**Full rotation matrix**	
	Section	**EPSG codes**	**Section**	**EPSG codes**
Molodensky-Badekas 10-parameter (PV)	§**Error! Reference source not found.**	1061 (1063)		
Molodensky-Badekas 10-parameter (CF)	§4.3.2.2	1034 (9636)		
Helmert 7-parameter (Position Vector)	§4.3.3.1	1033 (9606)		
Helmert 7-parameter (Coordinate Frame)	§4.3.3.2	1032 (9607)	§4.3.3.3	1132 (1133)
Geocentric translations (3-parameter)	§4.3.4	1031 (9603)		

Time-dependent Helmert (PositionVector)	§4.3.5.1	1053 (1054)		
Time-dependent Helmert (Coordinate Frame)	§4.3.5.2	1056 (1057)		
Time-specific Helmert (PositionVector)	§4.3.6	1065		
Time-specific Helmert (Coordinate Frame)	§4.3.6	1066		

4.3.2 Molodensky-Badekas 10-parameter transformations

Several different formulas are documented in the literature. Each may be encountered using the Coordinate Frame or the Position Vector rotation convention.

Variant A: $\boldsymbol{V_T} = (1 + \mathrm{dS})\, \boldsymbol{R_A}\, (\boldsymbol{V_S} - \boldsymbol{V_P}) + \boldsymbol{V_P} + \boldsymbol{T}$

Variant B: $\boldsymbol{V_T} = \boldsymbol{R_B}\, (\boldsymbol{V_S} - \boldsymbol{V_P}) + \mathrm{dS}\, (\boldsymbol{V_S} - \boldsymbol{V_P}) + \boldsymbol{V_S} + \boldsymbol{T}$

Variant C: $\boldsymbol{V_T} = \boldsymbol{R_C}\, (\boldsymbol{V_S} - \boldsymbol{V_P}) + \boldsymbol{V_S} + \boldsymbol{T}$

where

$$\boldsymbol{V_T} = \begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} \quad \boldsymbol{V_S} = \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} \quad \boldsymbol{V_P} = \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix} \quad \boldsymbol{T} = \begin{pmatrix} tX \\ tY \\ tZ \end{pmatrix}$$

and, using the Coordinate Frame rotation convention,

$$\boldsymbol{R_A} = \begin{pmatrix} 1 & rZ & -rY \\ -rZ & 1 & rX \\ rY & -rX & 1 \end{pmatrix} \quad \boldsymbol{R_B} = \begin{pmatrix} 0 & rZ & -rY \\ -rZ & 0 & rX \\ rY & -rX & 0 \end{pmatrix} \quad \boldsymbol{R_C} = \begin{pmatrix} dS & rZ & -rY \\ -rZ & dS & rX \\ rY & -rX & dS \end{pmatrix}$$

The parameters in the matrices are defined in Table 12 in Section §4.3.1 above.

Variants B and C can be shown to be approximations to variant A: for rotation angles of under 10 arc-seconds, the approximation is good to better than 0.1mm. In implementation, the 7-parameter Helmert transformation is a special case of variant A, but not of variants B or C. For both of these reasons, only variant A is documented in the EPSG dataset. These formulas are commonly referred to by the name Molodensky-Badekas, but the formulation is not exactly due to either Molodensky or Badekas.

Reverse

Variant A is: $\boldsymbol{V_T} = (1 + \mathrm{dS})\, \boldsymbol{R_A}\, (\boldsymbol{V_S} - \boldsymbol{V_P}) + \boldsymbol{V_P} + \boldsymbol{T}$

the exact reverse of which is: $\boldsymbol{V_S} = [1/(1 + \mathrm{dS})]\, \boldsymbol{R_A}^{-1}\, (\boldsymbol{V_T} - \boldsymbol{V_P}) + \boldsymbol{V_P} - [1/(1 + \mathrm{dS})]\, \boldsymbol{R_A}^{-1}\, \boldsymbol{T}$

This may be approximated by: $\boldsymbol{V_S} = (1 - \mathrm{dS})\, \boldsymbol{R_A}^{-1}\, (\boldsymbol{V_T} - \boldsymbol{V_P}) + \boldsymbol{V_P} - \boldsymbol{T}$

This approximation uses the same formula as in the forward calculation but with the signs of the translation, rotation and scale parameters changed. (The signs of the coordinates of the pivot point are unchanged). The approximation is valid only when the transformation parameter values are small compared to the magnitude of the geocentric coordinates. Under this condition, the methods are considered to be reversible for practical purposes. The misclosure in a round trip calculation depends on the magnitude of the translation, rotation, and scale parameter values. The two approximation errors are:

$$\boldsymbol{\Delta_1} = [1/(1 + \mathrm{dS})]\, \boldsymbol{R_A}^{-1}\, \boldsymbol{T} - \boldsymbol{T}$$
$$\boldsymbol{\Delta_2} = [\mathrm{dS}^2/(1 + \mathrm{dS})]\, \boldsymbol{R_A}^{-1}\, (\boldsymbol{V_T} - \boldsymbol{V_P})$$

In practical circumstances, these two approximation errors when combined do not exceed 2cm and are usually much less.

Note: The EPSG Dataset documents reversibility using the approximation, that is using the same formula as for the forward calculation but with the signs of the parameter values changed. The alternative of using the exact reverse formula is treated as a different method in which the reverse uses a different formula from for the forward calculation and in this the signs of the parameter values are not changed.

4.3.2.1 Molodensky-Badekas transformation (Position Vector geocentric domain)

(EPSG Dataset coordinate operation method code 1061)

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M \begin{pmatrix} 1 & -rZ & +rY \\ +rZ & 1 & -rX \\ -rY & +rX & 1 \end{pmatrix} \begin{pmatrix} X_S & - & X_P \\ Y_S & - & Y_P \\ Z_S & - & Z_P \end{pmatrix} + \begin{pmatrix} X_P & + & tX \\ Y_P & + & tY \\ Z_P & + & tZ \end{pmatrix}$$

where M = 1 + dS.

4.3.2.2 Molodensky-Badekas transformation (Coordinate Frame geocentric domain)

(EPSG Dataset coordinate operation method code 1034)

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M \begin{pmatrix} 1 & +rZ & -rY \\ -rZ & 1 & +rX \\ +rY & -rX & 1 \end{pmatrix} \begin{pmatrix} X_S & - & X_P \\ Y_S & - & Y_P \\ Z_S & - & Z_P \end{pmatrix} + \begin{pmatrix} X_P & + & tX \\ Y_P & + & tY \\ Z_P & + & tZ \end{pmatrix}$$

where M = 1 + dS.

For an example see Section §4.4.1.2 below, where the method is embedded within a transformation in the geographic 2D CRS domain. Further examples of input and output may be found in test procedure 5205 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.3.3 Helmert 7-parameter transformations

If the coordinates of pivot point P in the Molodensky-Badekas transformation methods are constrained to be zero, i.e., $X_P = Y_P = Z_p = 0$, then the formulas in the previous section simplify to a 7-parameter Helmert transformation, expressed in matrix form in what is sometimes known as the "Bursa-Wolf" formula.

4.3.3.1 Position Vector transformation (geocentric domain)

(EPSG Dataset coordinate operation method code 1033)

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M \begin{pmatrix} 1 & -rZ & +rY \\ +rZ & 1 & -rX \\ -rY & +rX & 1 \end{pmatrix} \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} tX \\ tY \\ tZ \end{pmatrix}$$

where M = 1 + dS.

Example

Transformation from WGS 72 to WGS 84 (EPSG Dataset transformation code 1238). Transformation parameter values:

tX	=	0.000 m		
tY	=	0.000 m		
tZ	=	+4.5 m		
rX	=	0.000 sec		
rY	=	0.000 sec		
rZ	=	+0.554 sec	=	0.000002685868 radians
dS	=	+0.219 ppm		from which M = 1.000000219

Input point coordinate reference system: WGS 72 (geocentric Cartesian coordinates):

X_S = 3 657 660.66 m
Y_S = 255 768.55 m
Z_S = 5 201 382.11 m

Application of the 7 parameter Position Vector transformation results in:

X_T = 3 657 660.78 m
Y_T = 255 778.43 m
Z_T = 5 201 387.75 m

on the WGS 84 geocentric coordinate reference system.

Further examples of input and output may be found in test procedure 5203 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.3.3.2 Coordinate Frame rotation (geocentric domain)

(EPSG Dataset coordinate operation method code 1032)

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M \begin{pmatrix} 1 & +rZ & -rY \\ -rZ & 1 & +rX \\ +rY & -rX & 1 \end{pmatrix} \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} tX \\ tY \\ tZ \end{pmatrix}$$

where M = 1 + dS.

The same example as for the Position Vector transformation can be calculated: however, the following transformation parameter values have to be applied to achieve the same input and output in terms of coordinate values:

Transformation parameters Coordinate Frame rotation convention:

tX	=	0.000 m		
tY	=	0.000 m		
tZ	=	+4.5 m		
rX	=	–0.000 sec		
rY	=	–0.000 sec		
rZ	=	–0.554 sec	=	–0.000002685868 radians
dS	=	+0.219 ppm		from which M = 1.000000219

Note that the only difference compared to the Position Vector transformation parameters is that the sign of te rotations is changed.

Reverse

Using vector notation the forward formula is

$$\boldsymbol{V_T} = (1 + dS)\, \boldsymbol{R}\, \boldsymbol{V_S} + \boldsymbol{T}$$

the exact reverse of which is

$$\boldsymbol{V_S} = [1/(1 + dS)]\, \boldsymbol{R}^{-1} (\boldsymbol{V_T} - \boldsymbol{T})$$

This may be approximated by

$$\boldsymbol{V_S} = (1 - dS)\, \boldsymbol{R}^{-1}\, \boldsymbol{V_T} - \boldsymbol{T}$$

This approximation uses the same formula as in the forward calculation but with the signs of the translation, rotation, and scale parameters changed. The approximation is valid only when the transformation parameter values are small compared to the magnitude of the geocentric coordinates. Under this condition, the methods are considered to be reversible for practical purposes. The misclosure in a round-trip calculation depends on the magnitude of the translation, rotation, and scale parameter values. The two approximation errors are:

$$\boldsymbol{\Delta_1} = [1/(1 + dS)]\, \boldsymbol{R}^{-1}\, \boldsymbol{T} - \boldsymbol{T}$$

$$\boldsymbol{\Delta_2} = [dS^2/(1 + dS)]\, \boldsymbol{R}^{-1}\, \boldsymbol{V_T}$$

In practical circumstances, these two approximation errors when combined do not exceed 2cm and are usually much less.

Further examples of input and output may be found in test procedure 5204 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.3.3.3 Coordinate Frame rotation full matrix (geocentric domain)

(EPSG Dataset coordinate operation method code 1132)

In some circumstances a Helmert transformation is used, but the rotation parameter values are large enough to make the small angle rotation approximation invalid. Then the full rotation matrix in Section §4.3.1.2 above must be used.

$$\boldsymbol{V_T} = (1 + dS)\, \boldsymbol{R}\, \boldsymbol{V_S} + \boldsymbol{T}$$

where $\boldsymbol{R}$ is the full (compounded) matrix:

$$\boldsymbol{R} = \begin{pmatrix} \cos(rY)\cos(rZ) & \cos(rX)\sin(rZ) + \sin(rX)\sin(rY)\cos(rZ) & \sin(rX)\sin(rZ) - \cos(rX)\sin(rY)\cos(rZ) \\ -\cos(rY)\sin(rZ) & \cos(rX)\cos(rZ) - \sin(rX)\sin(rY)\sin(rZ) & \sin(rX)\cos(rZ) + \cos(rX)\sin(rY)\sin(rZ) \\ \sin(rY) & -\sin(rX)\cos(rY) & \cos(rX)\cos(rY) \end{pmatrix}$$

Reverse

With large translation values, significant error may arise in applying the reversibility assumption described in Section §4.3.3.2 immediately above. For a reverse transformation, using the same parameter values, a different formula must be used.

$$\boldsymbol{V_S} = [1/(1 + dS)]\, \boldsymbol{R}^{-1} (\boldsymbol{V_T} - \boldsymbol{T})$$

Unlike the Coordinate Frame method using small angle approximations (Section §4.3.3.2 immediately above), in this reverse formula the signs of the parameter values are not reversed.

Example
Transformation from Saba to Saba 2020 (EPSG Dataset transformation code 10646).
Transformation parameter values:

tX	=	1138.7432 m		
tY	=	–2064.4761 m		
tZ	=	110.7016 m		
rX	=	–214.615206 sec	=	–0.001040484 rad
rY	=	479.360036 sec	=	0.002324003 rad
rZ	=	–164.703951 sec	=	–0.000798507 rad
dS	=	–402.32073 ppm		from which M = –0.000402321

$$\boldsymbol{R_Z} = \begin{pmatrix} 0.9999997 & -0.0007985 & 0 \\ 0.0007985 & 0.9999997 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{R_Y} = \begin{pmatrix} 0.9999973 & 0 & -0.0023240 \\ 0 & 1 & 0 \\ 0.0023240 & 0 & 0.9999973 \end{pmatrix} \quad \boldsymbol{R_X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.9999995 & -0.0010405 \\ 0 & 0.0010405 & 0.9999995 \end{pmatrix}$$

$$\boldsymbol{R}_{\mathbf{CF}} = \boldsymbol{R_Z} * \boldsymbol{R_Y} * \boldsymbol{R_X} = \begin{pmatrix} 0.999996981 & -0.000800925 & -0.002323168 \\ 0.000798505 & 0.999999138 & -0.001042339 \\ 0.002324001 & 0.001040481 & 0.999996758 \end{pmatrix}$$

Input point coordinate reference system: Saba (geocentric Cartesian coordinates):

X_{Saba} = 2738467.761 m
Y_{Saba} = –5429579.811 m
Z_{Saba} = 1919164.067 m

Application of the 7 parameter Coordinate Frame full matrix rotation results in:

$X_{Saba2020}$ = 2738386.686 m
$Y_{Saba2020}$ = –5429268.990 m
$Z_{Saba2020}$ = 1919210.970 m

on the Saba 2020 geocentric coordinate reference system.

With the large rotation values in these transformation parameters, application of the small angle approximation formula (method code 1032) leads to significant error in output coordinates. For comparison:

		full matrix	small angle approximation	difference
$X_{Saba2020}$	=	2738386.686 m	2738380.228 m	6.458 m
$Y_{Saba2020}$	=	–5429268.990 m	–5429270.104 m	1.113 m
$Z_{Saba2020}$	=	1919210.970 m	1919217.179 m	–6.208 m

Reverse
The reverse transformation from Saba 2020 coordinates of:

$X_{Saba2020}$ = 2738386.686 m
$Y_{Saba2020}$ = –5429268.990 m
$Z_{Saba2020}$ = 1919210.970 m

gives coordinates referenced to Saba of:

X_{Saba} = 2738467.761 m
Y_{Saba} = –5429579.811 m
Z_{Saba} = 1919164.067 m

With the large translation values in these transformation parameters, application of the approximate reverse formula in the reverse direction leads to significant error in output coordinates. For comparison:

	source XYZ	round trip full matrix	using reversibility approximation	difference
X	2738467.761 m	2738467.761 m	2738466.381 m	1.38 m
Y	–5429579.811 m	–5429579.811 m	–5429580.558 m	0.75 m
Z	1919164.067 m	1919164.067 m	1919163.307 m	–0.76 m

4.3.4 **Geocentric translations (3-parameter)**

(EPSG Dataset coordinate operation method code 1031)

If we assume that the axes of the ellipsoids are parallel, that the prime meridian is Greenwich, and that there is no scale difference between the source and target coordinate reference system, then geocentric coordinate reference systems may be related to each other through three translations (colloquially known as shifts) **tX, tY, tZ** in the sense from source geocentric coordinate reference system to target geocentric coordinate reference system. They may then be applied as

$$X_T = X_S + tX$$
$$Y_T = Y_S + tY$$
$$Z_T = Z_S + tZ$$

This is a specific case of the Helmert 7-parameter transformation methods in Section §4.3.3 above in which rX = rY = rZ = 0 and dS = 0 so M = 1.

Reverse

The geocentric translations method is exactly reversible: the same formula is used as for the forward transformation but in the reverse case the signs of the three translations are changed.

Example

Consider a North Sea point with coordinates derived by GPS satellite in the WGS 84 geocentric coordinate reference system, with coordinates of:

$$X_S = 3771\,793.97 \text{ m}$$
$$Y_S = 140\,253.34 \text{ m}$$
$$Z_S = 5124\,304.35 \text{ m}$$

whose coordinates are required in terms of the ED50 coordinate reference system which takes the International 1924 ellipsoid. The three parameter geocentric translations method's parameter values from WGS 84 to ED50 for this North Sea area are given as tX = +84.87m, tY = +96.49m, tZ = +116.95m.

Applying the quoted geocentric translations to these, we obtain new geocentric values now related to ED50:

$$X_T = 3771\,793.97 + 84.87 = 3771\,878.84 \text{ m}$$
$$Y_T = 140\,253.34 + 96.49 = 140\,349.83 \text{ m}$$
$$Z_T = 5124\,304.35 + 116.95 = 5124\,421.30 \text{ m}$$

Further examples of input and output may be found in test procedure 5211 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.3.4.1 **Geocentric translations by grid interpolation**

In France the national mapping agency (IGN) has promulgated a transformation between the old geographic 2D coordinate reference system NTF and the modern 3-dimensional system RGF93 which uses geocentric translations interpolated from a grid file. The method has also been used in New Caledonia and Saudi Arabia. This is a multi-step operation which first requires interpolation of a gridded dataset in the geographic 2D domain to obtain the geocentric translations, then the transformation between geographic coordinates is made using the Geocentric translations (geog 2D domain) method. This is described in detail in Section §4.4.1.3 below.

4.3.5 Time-dependent Helmert 7-parameter transformations

To account for secular tectonic plate motion (but not local deformation), the rate of change of each of the seven parameters may be included in a time-dependent variation of the Helmert 7-parameter transformation. This leads to a 15-parameter transformation: the three translations, three rotations, and the scale change, the rates for these seven parameters, and the parameter reference epoch for these parameters[15]. For a time-dependent transformation, first an adjustment to the parameter values is calculated, and then the adjusted values used for the Helmert coordinate transformation. See the example below for the process to be followed when applying a time-dependent transformation.

4.3.5.1 Time-dependent Position Vector transformation (geocentric domain)

(EPSG Dataset coordinate operation method code 1053)

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M' \begin{pmatrix} 1 & -rZ' & +rY' \\ +rZ' & 1 & -rX' \\ -rY' & +rX' & 1 \end{pmatrix} \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} tX' \\ tY' \\ tZ' \end{pmatrix}$$

where tX', tY', tZ', rX', rY', rZ', and M' are time-adjusted parameter values:

$$\begin{aligned}
tX' &= tX + \delta tX\,(t - t_0) \\
tY' &= tY + \delta tY\,(t - t_0) \\
tZ' &= tZ + \delta tZ\,(t - t_0) \\
rX' &= rX + \delta rX\,(t - t_0) \\
rY' &= rY + \delta rY\,(t - t_0) \\
rZ' &= rZ + \delta rZ\,(t - t_0) \\
dS' &= dS + \delta dS\,(t - t_0) \\
M' &= 1 + dS'
\end{aligned}$$

and t is the coordinate epoch of the dynamic coordinates and t_0 the parameter reference epoch for the transformation parameters.

Reverse

For the reverse transformation the signs of all parameters need to be reversed except for that of the parameter reference epoch t_0.

4.3.5.2 Time-dependent Coordinate Frame rotation (geocentric domain)

(EPSG Dataset coordinate operation method code 1056)

$$\begin{pmatrix} X_T \\ Y_T \\ Z_T \end{pmatrix} = M' \begin{pmatrix} 1 & +rZ' & -rY' \\ -rZ' & 1 & +rX' \\ +rY' & -rX' & 1 \end{pmatrix} \begin{pmatrix} X_S \\ Y_S \\ Z_S \end{pmatrix} + \begin{pmatrix} tX' \\ tY' \\ tZ' \end{pmatrix}$$

where tX', tY', tZ', rX', rY', rZ', and M' are as in the Time-dependent Position Vector method in the previous Section.

Example

Transformation (EPSG code 6276) from ITRF2008 to GDA94 at coordinate epoch 2013.90.

Source ITRF2008 geocentric Cartesian coordinates at coordinate epoch 2013.90 are:

[15] Sometimes referred to as a 14-parameter transformation, but this ignores the parameter reference epoch which defines when the non-time-dependent values apply.

X_S = –3789470.710 m
Y_S = 4841770.404 m
Z_S = –1690893.952 m
t = 2013.90 years

Transformation parameter values:

tX	=	–84.68 mm	δtX	=	+1.42 mm/yr
tY	=	–19.42 mm	δtY	=	+1.34 mm/yr
tZ	=	+32.01 mm	δtZ	=	+0.90 mm/yr
rX	=	–0.4254 msec	δrX	=	+1.5461 msec/yr
rY	=	+2.2578 msec	δrY	=	+1.1820 msec/yr
rZ	=	+2.4015 msec	δrZ	=	+1.1551 msec/yr
dS	=	+0.00971 ppm	δdS	=	+0.000109 ppm/yr
t_0	=	1994.00			

First, apply the correction due to rate of change to each of the 7 transformation parameters for the period (t-t_0), taking care to convert the translations to the same units as the source CRS (in this case metres) and the rotations to radians:

tX' = –84.68 + [(+1.42 (2013.90 – 1994.00)] = –56.42 mm = –0.056 m
tY' = –19.42 + [(+1.34 (2013.90 – 1994.00)] = +7.25 mm = +0.007 m
tZ' = +32.01 + [(+0.90 (2013.90 – 1994.00)] = +49.92 mm = +0.050 m
rX' = –0.4254 + [(+1.5461 (2013.90 – 1994.00)] = +30.3420 msec = +1.471021E-07 rad
rY' = +2.2578 + [(+1.1820 (2013.90 – 1994.00)] = +25.7796 msec = +1.249830E-07 rad
rZ' = +2.4015 + [(+1.5551 (2013.90 – 1994.00)] = +25.3880 msec = +1.230844E-07 rad
dS' = +0.00971 + [+0.000109 (2013.90 – 1994.00)] = +0.01188 ppm
from which M' = 1.00000001188

Using these seven time-adjusted parameter values, application of the 7 parameter Coordinate Frame rotation formula to the given (source) ITRF2008 coordinates results in GDA94 (target) geocentric coordinates of:

X_T = –3789470.004 m
Y_T = 4841770.686 m
Z_T = –1690895.108 m

(In practice, these coordinates often would be further converted to geographic 3D or compound geographic 2D + gravity-related height values. See Sections §4.4.1 and §4.11.2 respectively).

For the reverse transformation from GDA94 coordinates to ITRF08 coordinates at coordinate epoch 2013.90, the signs of all parameters need to be reversed except for the parameter reference epoch. Then:

tX' = +84.68 + [(–1.42 (2013.90 –1994.00)] = +56.42 mm = +0.056 m

and similar for the other six parameters. Hence tY' = –0.007 m, tZ' = –0.050 m, rX' = –1.471021E-07 rad, rY' = –1.249830E-07 rad, rZ' = –1.230844E-07 rad and dS' = –0.01188 ppm,
from which M' = 0.99999998812.

Using these time-adjusted parameters values, applying the 7 parameter Coordinate Frame rotation formula to the GDA94 coordinates of

X_S = –3789470.004 m
Y_S = 4841770.686 m
Z_S = –1690895.108 m

results in coordinates referenced to ITRF08 at coordinate epoch 2013.90 of:

X_T = –3789470.710 m
Y_T = 4841770.404 m
Z_T = –1690893.952 m

4.3.6 Time-specific Helmert 7-parameter transformations

The time-specific coordinate transformation is an alternative approach to the time-dependent coordinate transformation. In this method, the transformation parameter values are declared for a specified *transformation reference epoch* (an additional parameter for the transformation) and coordinates are adjusted to this epoch before the transformation is applied. This requires that the velocities for points whose coordinates are to be transformed are available. The time-specific transformation may then be applied as part of a concatenated operation in conjunction with one, or two, point motion operations (see Section §4.2) in two or three steps:

i) change the source CRS Cartesian coordinates from their dataset coordinate epoch to the transformation reference epoch at which the time-specific transformation is valid using a point motion operation

ii) apply the Helmert 7-parameter coordinate transformation

iii) change the target CRS Cartesian coordinates from the transformation reference epoch at which the time-specific transformation is valid to any other desired coordinate epoch using a point motion operation

The transformation reference epoch of the Helmert 7-parameter transformation is used as the target epoch (t_2) in step (i) and as the source epoch (t_1) in step (iii). It is not used in step (ii).

The time-specific Helmert transformation may use either the Position Vector or Coordinate Frame rotation convention:

- Time-specific Position Vector transform (geocen)
  (EPSG Dataset coordinate operation method code 1065)
- Time-specific Coordinate Frame rotation (geocen)
  (EPSG Dataset coordinate operation method code 1066)

Example
Initial coordinates are referenced to the ITRF2008 CRS (EPSG code 5332) at coordinate epoch 2005.00. (Note: this coordinate epoch happens to be the frame reference epoch for ITRF2008 but in other circumstances the coordinate set could be referenced to a coordinate epoch different from the CRS's frame reference epoch).

Coordinates are required to be referenced to the PZ-90.11 CRS (EPSG code 7679) at coordinate epoch 2013.90.

Input point geocentric Cartesian coordinates and linear velocities:

X_S	=	2845 456.0813 m	V_X	=	–0.0212 m/yr
Y_S	=	2160 954.2453 m	V_Y	=	+0.0124 m/yr
Z_S	=	5265 993.2296 m	V_Z	=	+0.0072 m/yr
t	=	2005.0			

These are referenced to ITRF2008 at coordinate epoch 2005.00

Step 1: convert the coordinates in the source CRS from the dataset coordinate epoch (2005.00) to the epoch of validity of the time-specific transformation (2010.00) using the Point Motion (geocentric Cartesian) method described in Section §4.2.1. The formulas are repeated here for convenience:

$$\begin{pmatrix} Xs_{(t2)} \\ Ys_{(t2)} \\ Zs_{(t2)} \end{pmatrix} = \begin{pmatrix} Xs_{(t1)} \\ Ys_{(t1)} \\ Zs_{(t1)} \end{pmatrix} + (t_2 - t_1) \begin{pmatrix} V_X \\ V_Y \\ V_Z \end{pmatrix}$$

$$X_{S(t=2010.0)} = 2845\,456.0813 + (-0.0212) * (2010.0 - 2005.0) = 2845\,455.9753 \text{ m}$$
$$Y_{S(t=2010.0)} = 2160\,954.2453 + (+0.0124) * (2010.0 - 2005.0) = 2160\,954.3073 \text{ m}$$
$$Z_{S(t=2010.0)} = 5265\,993.2296 + (+0.0072) * (2010.0 - 2005.0) = 5265\,993.2656 \text{ m}$$

Step 2: transform the coordinates in the source CRS at coordinate epoch 2010.00 to the target CRS at coordinate epoch 2010.00 using the Coordinate Frame rotation (geocentric domain) method described in Section §4.3.3. The time-specific transformation from PZ-90.11 to ITRF2008 (EPSG Dataset transformation code 7960) parameter values are:

tX	=	–0.003 m	
tY	=	–0.001 m	
tZ	=	0.000 m	
rX	=	0.019 msec	
rY	=	–0.042 msec	
rZ	=	0.002 msec	
dS	=	0.000 ppm	from which M = 1.0
t	=	2010.0	

This uses the Coordinate Frame rotation convention described in Section §4.3.3.

Because the transformation is required in the reverse direction to that defined, the relevant parameters need to have their signs reversed. Transformation parameter values (from ITRF2008 to PZ-90.11) then are:

tX	=	0.003 m		
tY	=	0.001 m		
tZ	=	0.000 m		
rX	=	–0.019 msec	=	– 9.21145994108117E-11 radians
rY	=	0.042 msec	=	2.03621746066005E-10 radians
rZ	=	–0.002 msec	=	– 9.69627362219071E-12 radians
dS	=	0.000 ppm		M = 1.0
t	=	2010.0		

Then:

$$X_{T(t=2010.0)} = 2845\,455.9772$$
$$Y_{T(t=2010.0)} = 2160\,954.3078$$
$$Z_{T(t=2010.0)} = 5265\,993.2652$$

Step 3: convert the coordinates in the target CRS from epoch 2010.00 to the required coordinate epoch 2013.90 using the Point Motion (geocentric Cartesian) method:

$$X_{T(t=2013.9)} = 2845\,455.9772 + [-0.0212 * (2013.9 - 2010.0)] = 2845\,455.8945 \text{ m}$$
$$Y_{T(t=2013.9)} = 2160\,954.3078 + [+0.0124 * (2013.9 - 2010.0)] = 2160\,954.3562 \text{ m}$$
$$Z_{T(t=2013.9)} = 5265\,993.2652 + [+0.0072 * (2013.9 - 2010.0)] = 5265\,993.2945 \text{ m}$$

4.4 Transformations between Geographic Coordinate Reference Systems

4.4.1 Transformations using geocentric methods

4.4.1.1 Geographic 3D domain

Transformation of coordinates from one geographic coordinate reference system into another is often carried out as a concatenation of the following operations:

(geographic to geocentric) + (geocentric to geocentric) + (geocentric to geographic)

See Section §4.4 of EPSG Dataset Guidance Note 7 part 1 (IOGP Report 373-07-01 - *Using the EPSG geodetic parameter dataset*) for a fuller description of the concatenation technique.

The middle step of the concatenated transformation, from geocentric to geocentric, may be through any of the methods described in Section §4.3 above: 10-parameter Molodensky-Badekas transformations, 7-parameter Helmert or Bursa-Wolf transformations (static or dynamic (time-dependent)), or 3-parameter geocentric translations. The geographic 3D to/from geocentric steps of the concatenated transformation are described in Section §4.1.1 above. When involving geographic 2D coordinates, the techniques described in the Geographic 3D to 2D conversions Section §4.1.4 above are also used as additional steps at each end of the concatenation.

The concatenated transformations are:

a) Molodensky-Badekas (CF geog3D domain), EPSG method code 1039

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 3D to Geocentric	9602	§4.1.1
2	Molodensky-Badekas (CF geocentric domain)	1034	§4.3.2.2
3	Geocentric to Geographic 3D	9602	§4.1.1

b) Molodensky-Badekas (PV geog3D domain), EPSG method code 1062

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 3D to Geocentric	9602	§4.1.1
2	Molodensky-Badekas (PV geocentric domain)	1061	§**Error! Reference source not found.**
3	Geocentric to Geographic 3D	9602	§4.1.1

c) Position Vector transformation (geog3D domain), EPSG method code 1037

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 3D to Geocentric	9602	§4.1.1
2	Position Vector transformation (geocentric domain)	1033	§4.3.3.1
3	Geocentric to Geographic 3D	9602	§4.1.1

d) Coordinate Frame rotation (geog3D domain), EPSG method code 1038

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 3D to Geocentric	9602	§4.1.1
2	Coordinate Frame rotation (geocentric domain)	1032	§4.3.3.2
3	Geocentric to Geographic 3D	9602	§4.1.1

e) Time-dependent Position Vector transformation (geog3D domain), EPSG method code 1055

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 3D to Geocentric	9602	§4.1.1
2	Time-dependent Position Vector transformation (geocentric domain)	1053	§4.3.5.1
3	Geocentric to Geographic 3D	9602	§4.1.1

f) Time-dependent Coordinate Frame rotation (geog3D domain), EPSG method code 1058

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 3D to Geocentric	9602	§4.1.1
2	Time-dependent Coordinate Frame rotation (geocentric domain)	1056	§4.3.5.2
3	Geocentric to Geographic 3D	9602	§4.1.1

g) Geocentric Translations (geog3D domain), EPSG method code 1035

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 3D to Geocentric	9602	§4.1.1
2	Geocentric Translations (geocentric domain)	1031	§4.3.4
3	Geocentric to Geographic 3D	9602	§4.1.1

4.4.1.2 Geographic 2D domain

The concatenated transformations for source and target coordinates referenced to CRSs in the geographic 2D domain are:

a) Molodensky-Badekas (CF geog2D domain), EPSG method code 9636

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Molodensky-Badekas (CF geocentric domain)	1034	§4.3.2.2
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

See below for an example using this concatenated method.

b) Molodensky-Badekas (PV geog2D domain), EPSG method code 1063

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Molodensky-Badekas (PV geocentric domain)	1061	§**Error! Reference source not found.**
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

c) Position Vector transformation (geog2D domain), EPSG method code 9606

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Position Vector transformation (geocentric domain)	1033	§4.3.3.1
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

d) Coordinate Frame rotation (geog2D domain), EPSG method code 9607

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Coordinate Frame rotation (geocentric domain)	1032	§4.3.3.2
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

e) Coordinate Frame rotation (geog2D domain full matrix), EPSG method code 1133

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Coordinate Frame rotation full matrix (geocentric domain)	1132	§4.3.3.3
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

f) Time-dependent Position Vector transformation (geog2D domain), EPSG method code 1054

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Time-dependent Position Vector transformation (geocentric domain)	1053	§4.3.5.1
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

g) Time-dependent Coordinate Frame rotation (geog2D domain), EPSG method code 1057

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Time-dependent Coordinate Frame rotation (geocentric domain)	1056	§4.3.5.2
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

h) Geocentric Translations (geog2D domain), EPSG method code 9603

Step #	Step Method Name	EPSG Method Code	Section
1	Geographic 2D to Geographic 3D	9659	§4.1.4
2	Geographic 3D to Geocentric	9602	§4.1.1
3	Geocentric Translations (geocentric domain)	1031	§4.3.4
4	Geocentric to Geographic 3D	9602	§4.1.1
5	Geographic 3D to Geographic 2D	9659	§4.1.4

Example
(using the Molodensky-Badekas (CF geog2D domain) method as the middle step [see (a) above])

Transformation from La Canoa to REGVEN between geographic 2D coordinate reference systems (EPSG Dataset transformation code 1771).

The ten Molodensky-Badekas (CF) transformation parameter values for this transformation are:

tX	=	-270.933 m		
tY	=	+115.599 m		
tZ	=	-360.226 m		
rX	=	-5.266 sec	=	-0.000025530288 radians
rY	=	-1.238 sec	=	-0.000006001993 radians
rZ	=	+2.381 sec	=	+0.000011543414 radians
dS	=	-5.109 ppm		
Ordinate 1 of evaluation point	=	2464351.59 m		
Ordinate 2 of evaluation point	=	-5783466.61 m		
Ordinate 3 of evaluation point	=	974809.81 m		

M = 1 + dS = 0.999994891

Ellipsoid Parameters for the source and target coordinate reference systems are:

CRS name	Ellipsoid name	Semi-major axis (a)	Inverse flattening (1/f)
La Canoa	International 1924	6378388.0 metres	1/f = 297.0
REGVEN	WGS 84	6378137.0 metres	1/f = 298.257223563

Input point coordinate system: La Canoa (geographic 2D)

Latitude φ_S = 9°35'00.386"N
Longitude λ_S = 66°04'48.091"W

Step 1: Using the technique described in Section §4.1.4 above, this is taken to be geographic 3D with an assumed ellipsoidal height h_S = 201.46 m

Step 2: Using the geographic (3D) to geocentric conversion method given in Section §4.1.1, these three coordinates convert to Cartesian geocentric coordinates:

X_S = 2 550 408.96 m
Y_S = -5 749 912.26 m
Z_S = 1 054 891.11 m

Step 3: Application of the Molodensky-Badekas (CF geocentric domain) transformation (Section §4.3.2) results in:

X_T = 2 550 138.46 m
Y_T = -5 749 799.87 m
Z_T = 1 054 530.82 m

on the REGVEN geocentric coordinate reference system (CRS code 4962)

Step 4: Using the reverse formulas for the geographic/geocentric conversion method given in Section §4.1.1 on the REGVEN geographic 3D coordinate reference system (CRS code 4963) this converts into:

Latitude φ_T = 9°34'49.001"N
Longitude λ_T = 66°04'54.705"W
Ellipsoidal height h_T = 180.51 m

Step 5: Because the source coordinates were 2D, using the geographic 3D to 2D method (Section §4.1.4) the target system ellipsoidal height is dropped and the results treated as a geographic 2D coordinate reference system (CRS code 4189):

Latitude φ_T	=	9°34'49.001"N
Longitude λ_T	=	66°04'54.705"W

Further examples of input and output for the Geocentric Translation (geographic domain) methods 9603 and 1035 may be found in test procedures 5212 and 5213 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.4.1.3 Geocentric translations by grid interpolation (France)

(EPSG Dataset coordinate operation method code 1087)

In France, the national mapping agency (IGN) has promulgated a transformation between the historic classical geographic 2D coordinate reference system NTF and the modern 3-dimensional system RGF93 which uses geocentric translations interpolated from a grid file. The method has also been used in New Caledonia and Saudi Arabia. It is described in IGN (1997) document NTG-88[16]. This is a multi-step operation which first requires interpolation of a gridded dataset in the geographic 2D domain to obtain the geocentric translations, then the transformation is made between geographic coordinates using the Geocentric translations (geographic 2D domain) method (code 9603), which itself has five steps, as described in the Section §4.4.1.2(h) above. The CRS in which the grid has been constructed (the interpolation CRS) is in the modern system. The geocentric offsets within the file are additive in the direction from historic CRS to modern CRS (in France, from NTF to RGF93). This means that, unusually, the interpolation CRS is the target CRS of the forward transformation.

In summary:

- Within the grid file the sense of the parameter values is from old CRS to modern CRS (in France *from* NTF *to* RGF93). The offsets (see Section §1.5) are additive when the old CRS (NTF) is the source CRS and the modern CRS (in France, RGF93) is the target CRS.
- The interpolation CRS to which the grid file nodes are referenced is the modern geographic 2D CRS (in France, RGF93).
- Unusually, the interpolation CRS is the target CRS in the forward transformation.

For the forward transformation, old CRS to modern CRS (in France, NTF to RGF93), because the interpolation CRS is not the source CRS, in principle iteration is required for interpolation within the grid. The source CRS coordinates are used for the initial bilinear interpolation of the grid to obtain preliminary geocentric translation offsets. These are applied to calculate provisional target CRS coordinates:

$$X'_t = X_s + tX$$
$$Y'_t = Y_s + tY$$
$$Z'_t = Z_s + tZ$$

This process is iterated until the changes in target coordinates are insignificant.

However, if a suitable 'standard transformation' is available, an approximation to an accuracy of 1cm may be made. This avoids iteration. The steps for the forward transformation then are:

- Transform coordinates in the source CRS to coordinates in the interpolation CRS by using the standard transformation.

[16] IGN, "Grille de paramètres de transformation de coordonnées GR3DF97A". *Institut Geografique National, Service de Géodésie et Nivellement*, NTG_88. Paris. 1997.

- Using these coordinates, interpolate within the grid file to obtain the three geocentric translations (tX, tY, tZ) applicable at the point.
- Using the five steps of the Geocentric translation (geog 2D domain) method (Section §4.4.1), transform coordinates of the point in the source CRS to the target CRS. In the middle step:

  Xt = Xs + tX
  Yt = Ys + tY
  Zt = Zs + tZ

Reverse
For the reverse transformation from modern CRS to old CRS (in France, RGF93 to NTF), the steps are:

- Using the modern geographic coordinates (in France, RGF93), interpolate within the grid file to obtain the three geocentric translations (tX, tY, tZ) applicable at the point.
- Transform RGF93 coordinates to NTF coordinates, taking into account the sense of the geocentric translation parameter values. In the middle step:

  Xt' = Xs' + (–tX)
  Yt' = Ys' + (–tY)
  Zt' = Zs' + (–tZ)

Example
Forward transformation from NTF to RGF93 for a point in France with NTF coordinates of:

latitude φ_{NTF} = 48°50'40.2441"N,
longitude λ_{NTF} = 2°25'32.4187"E (of Greenwich).

First, using NTF to ETRS89 (1) (EPSG coordinate transformation code 1651) which uses the Geocentric translations (geog2D domain) method, code 9603, with parameter values of

tX = –168 m
tY = –60 m
tZ = +320 m

gives approximate RGF93 coordinates of

latitude $\varphi_{RGF93'}$ = 48°50'39.9967"N = 48.84444352°
longitude $\lambda_{RGF93'}$ = 2°25'29.8273"E = 2.42495203°

Then, using these approximate RGF93 coordinates, bilinear interpolation of the relevant four grid node values

48.9°N, 2.4°E	tX =	–168.275	48.9°N, 2.5°E	tX =	–168.253
	tY =	–58.606		tY =	–58.554
	tZ =	320.189		tZ =	320.165
48.8°N, 2.4°E	tX =	–168.252	48.8°N, 2.5°E	tX =	–168.204
	tY =	–58.630		tY =	–58.594
	tZ =	320.170		tZ =	320.125

gives interpolated tX = –168.253 m, tY = –58.609 m and tX = 320.170 m.

Then, using the Geocentric translations (geog2D domain) method, NTF coordinates

latitude φ_{NTF} = 48°50'40.2441"N,
longitude λ_{NTF} = 2°25'32.4187"E (of Greenwich)

and an assumed ellipsoid height of 0.000m
convert to geocentric Cartesian values

X_{NTF} = 4201905.725 m
Y_{NTF} = 177998.072 m
Z_{NTF} = 4778904.260 m

Applying the interpolated geocentric translations as an additive correction:

X_{RGF93} = 4201905.725 + –168.253 = 4201737.472 m
Y_{RGF93} = 177998.072 + –58.606 = 177939.463 m
Z_{RGF93} = 4778904.260 + 320.189 = 4779224.430 m

then

latitude φ_{RGF93} = 48°50'40.0050"N
longitude λ_{RGF93} = 2°25'29.8960"E

For the reverse calculation from RGF93 to NTF of the same point:

latitude φ_{RGF93} = 48°50'40.0050"N
longitude λ_{RGF93} = 2°25'29.8960"E

first use these RGF93 coordinates to interpolate within the grid to give geocentric translations of

tX = –168.253 m, tY = –58.609 m and tX = 320.170 m.

Then, using the Geocentric translations (geog2D domain) method, the RGF93 geographical coordinates (with assumed ellipsoidal height of 0.000m) are converted to geocentric Cartesian coordinates and, in the middle step, the interpolated geocentric translations are applied with reverse sign:

X_{NTF} = 4201737.472 – –168.253 = 4201905.725 m
Y_{NTF} = 177939.463 – –58.606 = 177998.072 m
Z_{NTF} = 4779224.430 – 320.189 = 4778904.260 m

then

latitude φ_{NTF} = 48°50'40.2441"N,
longitude λ_{NTF} = 2°25'32.4187"E (of Greenwich).

4.4.2 Abridged Molodensky transformation

(EPSG Dataset coordinate operation method code 9605)

As an alternative to the computation of the new latitude, longitude, and ellipsoid height by concatenation of three operations (geographic 3D to geocentric + geocentric to geocentric + geocentric to geographic 3D), the changes in these coordinates may be derived directly as geographic coordinate offsets through formulas derived by Molodensky (EPSG Dataset coordinate operation method code 9604, not detailed in this Guidance Note). Abridged versions of these formulas, which are satisfactory for most practical purposes, are as follows:

$$\varphi_t = \varphi_s + d\varphi$$
$$\lambda_t = \lambda_s + d\lambda$$
$$h_t = h_s + dh$$

where

$$d\varphi'' = (-\,tX \sin\varphi_s \cos\lambda_s - tY \sin\varphi_s \sin\lambda_s + tZ \cos\varphi_s + [a_s\, df + f_s\, da\,] \sin 2\varphi_s) / (\rho_s \sin 1'')$$
$$d\lambda'' = (-\,tX \sin\lambda_s + tY \cos\lambda_s) / (\nu_s \cos\varphi_s \sin 1'')$$
$$dh = tX \cos\varphi_s \cos\lambda_s + tY \cos\varphi_s \sin\lambda_s + tZ \sin\varphi_s + (a_s\, df + f_s\, da) \sin^2\varphi_s - da$$

and where tX, tY, and tZ are the geocentric translation parameters, ρ_s and ν_s are the meridian and prime vertical radii of curvature at the given latitude φ_s on the first ellipsoid, da is the difference in the semi-major axes of the target and source ellipsoids and df is the difference in the flattening of the two ellipsoids:

$\rho_s = a_s (1 - e_s^{\ 2}) / (1 - e_s^{\ 2}\sin^2\varphi_s)^{3/2}$
$\nu_s = a_s / (1 - e_s^{\ 2}\sin^2\varphi_s)^{1/2}$
$da = a_t - a_s$
$df = f_t - f_s = 1/(1/f_t) - 1/(1/f_s)$.

The formulas for $d\varphi$ and $d\lambda$ indicate changes in φ and λ in arc-seconds.

Example

For a North Sea point with coordinates derived by GPS satellite in the WGS 84 geographic coordinate reference system, with coordinates of:

latitude φ_s = 53°48'33.82"N,
longitude λ_s = 2°07'46.38"E,
and ellipsoidal height h_s = 73.0m,

whose coordinates are required in terms of the ED50 geographic coordinate reference system which takes the International 1924 ellipsoid.

The three geocentric translations parameter values *from* WGS 84 *to* ED50 for this North Sea area are given as tX = +84.87m, tY = +96.49m, tZ = +116.95m.
Ellipsoid Parameters are:

WGS 84	a = 6378137.0 metres	1/f = 298.257223563
International 1924	a = 6378388.0 metres	1/f = 297.0

Then

da = 6378388 – 6378137 = 251
df = 0.003367003 – 0.003352811 = 1.41927E-05

whence

$d\varphi$	=	(-68.448-2.894+69.056+87.079)/30.917	=	2.743"
$d\lambda$	=	(-3.154+96.423)/18.299	=	5.097"
dh	=	50.079+2.117+94.385+59.510-251.000	=	– 44.909 m

ED50 values on the International 1924 ellipsoid are then:

latitude φ_t = 53°48'36.563"N
longitude λ_t = 2°07'51.477"E
and ellipsoidal height h_t = 28.091 m

Because ED50 is a geographic 2D coordinate reference system the height is dropped to give:

latitude φ_t = 53°48'36.56"N
longitude λ_t = 2°07'51.48"E

For comparison, better values computed through the concatenation of the three operations (geographic to geocentric + geocentric to geocentric + geocentric to geographic) are:

latitude φ_t = 53°48'36.565"N
longitude λ_t = 2°07'51.477"E
and ellipsoidal height h_t = 28.02 m

Further examples of input and output may be found in test procedure 5213 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.4.3 Longitude rotation

(EPSG Dataset coordinate operation method code 9601)

A longitude rotation allows calculation of longitude in the target geographic coordinate reference system by adding a correction parameter A to the longitude value of the point in the source system:

$$\lambda_2 = \lambda_1 + A_{1>2}$$

where

λ_2 = longitude value in the target geographic coordinate reference system.
λ_1 = longitude value in the source geographic coordinate reference system.
$A_{1>2}$ is the longitude rotation, an offset to be applied for the transformation from CRS 1 to CRS 2. Its value for the forward calculation is the value of the origin of the source CRS 1 in the target CRS 2.

Note: The sign convention used here is that which is applied in geodesy; this differs from that in general mathematics - see Implementation Notes - Offsets (Section §1.5).

For the **reverse** transformation from CRS 2 to CRS 1 the same formula is used but with the sign of the offset $A_{1>2}$ reversed:

$$\lambda_1 = \lambda_2 + (-A_{1>2})$$

Example

For coordinate transformation: MGI (Ferro) to MGI (1), code 3895:

Transformation Parameter: Longitude offset $A_{1>2}$ = –17°40'

Consider a point having a longitude referenced to the Ferro prime meridian of 30°E. Its value referenced to the Greenwich prime meridian is

$$\lambda_{Greenwich} = 30° + (-17°40') = 12°20'E.$$

For the reverse calculation to transform the longitude referenced to the Greenwich meridian of 12°20'E to longitude referenced to the Greenwich meridian:

$$\lambda_{Ferro} = 12°20'E - (-17°40') = 30°E.$$

Examples of input and output for the Longitude Rotation method may be found in test procedure 5208 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.4.4 Geographic Offsets

(EPSG Dataset coordinate operation method codes 9619 and 9660)

This transformation method is between two geographic coordinate reference systems, and is normally used only for purposes where low accuracy can be tolerated. It is generally used for transformations in two dimensions, latitude and longitude, where:

$$\varphi_t = \varphi_s + d\varphi$$
$$\lambda_t = \lambda_s + d\lambda$$

(EPSG Dataset coordinate operation method code 9619).

In very rare circumstances, a transformation in three dimensions, additionally including ellipsoidal height, may be encountered:

$$\varphi_t = \varphi_s + d\varphi$$
$$\lambda_t = \lambda_s + d\lambda$$
$$h_t = h_s + dh$$

(EPSG coordinate operation method code 9660)

This should not be confused with the Geographic2D with Height Offsets method used in Japan, where the height difference is between the ellipsoidal height component of a 3D geographic coordinate reference system and a gravity-related height system. This is discussed in Section §4.11.3.

Reverse
For the reverse transformation, the same formula is used but with the sign of the offset reversed.

Example
A position with coordinates of 38°08'36.565"N, 23°48'’16.235"E referenced to the old Greek geographic 2D coordinate reference system (EPSG CRS code 4120) is to be transformed to the newer GGRS87 system (EPSG CRS code 4121). Transformation parameters from Greek to GGRS87 are:

$d\varphi$ = -5.86"
$d\lambda$ = +0.28"

Then	φ_{GGRS87}	=	38°08'36.565"N	+	(-5.86")	=	38°08'30.705"N
and	λ_{GGRS87}	=	23°48'16.235"E	+	0.28"	=	23°48'16.515"E

For the reverse transformation for the same point,

φ_{GREEK}	=	38°08'30.705"N	+	5.86"	=	38°08'36.565"N
λ_{GREEK}	=	23°48'16.515"E	+	(-0.28")	=	23°48'’16.235"E

4.4.5 Geographic Offset by Interpolation of Gridded Offset Data

The relationship between some geographic coordinate reference systems is available through gridded data sets of latitude and longitude (and, in the 3D case, ellipsoidal height) offsets. This family of methods includes:

- *NADCON* and *NADCON 5* (EPSG Dataset coordinate operation method codes 9613, 1074 and 1075) which are used by the US National Geodetic Survey for transformation between US systems
- *NTv1 and its successor NTv2* (EPSG Dataset coordinate operation method codes 9614 and 9615) which originated in the national mapping agency of Canada and was subsequently adopted in Australia, New Zealand, and then several other countries

The offsets at a point are derived by interpolation within the gridded data. In some methods, separate grid files are given for latitude and longitude (and, in the 3D case, ellipsoidal height) offsets whilst in other methods the offsets for both latitude and longitude (and, in the 3D case, ellipsoidal height) are given within a single grid file. The EPSG Dataset differentiates methods by the format of the gridded data file(s). The grid file format is given in documentation available from the information source. Although the authors of some data sets suggest a particular interpolation method within the grid(s), generally the density of grid nodes should be such that any reasonable grid interpolation method will give the same offset value. Bilinear interpolation is the most usual grid interpolation mechanism. The interpolated value of the offset A is then added to the source CRS coordinate value to give the coordinates in the target CRS, see Section §4.4.3.

Reverse
The coordinate reference system for the coordinates of the grid nodes will be given either in the file itself or in accompanying documentation. This will normally be the source coordinate reference system for the forward transformation. Then, in forward transformations, the offset is obtained through straightforward interpolation of the grid file. But for the reverse transformation the first grid interpolation entry will be the value of the point in the second coordinate reference system, the offsets are interpolated and applied *with sign reversed*, and the result used in further iterations of interpolation and application of offset until the difference between results from successive iterations is insignificant. An iterative reverse is only possible when the curvature of the grid is small, i.e., when source coordinates are approximately equal to target coordinates, with only small displacement vectors between them.

Example
Examples for the NADCON and NTv2 methods may be found in test procedures 5206 and 5207 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.4.6 Geographic Offset by Interpolation of Gridded Velocity Data

(EPSG Dataset coordinate operation method code 1114)

In Canada, each version of NAD83(CSRS) is defined by a transformation from successive realizations of ITRF. The NAD83(CSRS) versions represent a single conventional reference system. They differ only through the inclusion of additional data and any changes in the ITRF adjustment technique, which for many practical purposes are not significant. However, in practice, deformation between the frame epochs, mostly due to post-glacial isostatic adjustment, does lead to differences that may be significant for some purposes. The Canadian velocity grid in NTv2_Vel format is used not only as a point motion operation within one of the realizations (see Section §4.2.3) but also as a transformation between NAD83(CSRS) versions at different epochs. It is considered to be reversible.

First, the velocities are interpolated for the required point from within the velocity grid where **V** describes the velocities of the ellipsoidal coordinates, positive towards the future. These are usually given as a rate in linear units (say millimetres per year) resolved into north, east, and up components V_N, V_E, and V_U. NRCan uses biquadratic interpolation. Then the north and east components are converted into latitude and longitude components by:

$$V\varphi = V_N / (\rho + h)$$
$$V\lambda = V_E / [(\nu + h) \cos \varphi]$$

where

ρ is the radius of curvature of the CRS's ellipsoid in the plane of the meridian at latitude φ
$$\rho = a(1 - e^2)/(1 - e^2\sin^2\varphi)^{3/2}$$
ν is the radius of curvature of the ellipsoid perpendicular to the meridian at latitude φ
$$\nu = a /(1 - e^2\sin^2\varphi)^{1/2}$$

The up component V_U is in the geometric domain so $Vh = V_U$.

Then (as in Section §4.2.2) the ellipsoidal coordinates of a point at coordinate epoch t_1 may be calculated at any other coordinate epoch t_2 from:

$$\begin{pmatrix} \varphi \\ \lambda \\ h \end{pmatrix}_{t_2} = \begin{pmatrix} \varphi \\ \lambda \\ h \end{pmatrix}_{t_1} + (t_2 - t_1) \begin{pmatrix} V_\varphi \\ V_\lambda \\ V_h \end{pmatrix}$$

In general application of the method, t_1 and and t_2 would be expected to be user-input parameter values. For transformations between NAD83(CSRS)v2 and later realizations using EPSG Dataset coordinate operation method code 1114, t_1 and and t_2 are given in the Datum.Anchor_Epoch attribute of the source CRS and target CRS respectively.

Reverse
In principle, a velocity grid is not reversible where it contains discontinuities. NRCan considers that the Canada v6 and v7 velocity grids are reversible because, for most of Canada, the changes in the velocity vector over short distances are small and the grid spacing is relatively large.

For the reverse operation, the same formula is used but the sign of $t_2 - t_1$ is reversed. The velocity grid is interpolated at the new location coordinates. Iteration using the output coordinates may be required until there is no significant change in the interpolated velocities, although in practice this may not be necessary.

Example

Given a point with NAD83(CSRS)v6 ellipsoidal coordinates at coordinate epoch 2010.00 of:

$\varphi_{NAD83(CSRS)v6}$ = 49°53'09.2927"N, $\lambda_{NAD83(CSRS)v6}$ = 99°54'41.0572"W, $h_{NAD83(CSRS)v6}$ = 373.795m

to be transformed to NAD83(CSRS)v3 epoch 1997.00.

Grid file: Canadian velocity grid v7.0

First, t_1 and t_2 may be obtained through the datum.anchor_epoch attribute values for the datums of the source and target CRSs respectively.

t_1 (for NAD83(CSRS)v6) = 2010.00

t_2 (for NAD83(CSRS)v3) = 1997.00

(Should this epoch data not be available this way, it should be user-input).

Then the velocities V_N, V_E, and V_U are interpolated from within the velocity grid at location (φ, λ):

V_N = –1.00 mm/yr
V_E = 2.46 mm/yr
V_U = –1.85 mm/yr

Then (after conversion to appropriate units):

$V\varphi$ = –1.5690696E-10 rad/yr
$V\lambda$ = 5.9740384E-10 rad/yr
Vh = –0.00185 m / yr

and

$\varphi_{NAD83(CSRS)v3}$ = 0.870673461 –1.5690696E-10 (1997.00–2010.00) = 0.870673463 radians
$\lambda_{NAD83(CSRS)v3}$ = –1.743782974 + 5.9740384E-10 (1997.00–2010.00) = –1.743782982 radians
$h_{NAD83(CSRS)v3}$ = 373.795 + –0.00185 (1997.00–2010.00) = 373.819 m

φ = 49°53'09.2931"N, λ = 99°54'41.0588"W, h = 373.819m
t = 1997.00

For the reverse transformation from coordinate epoch 1997.00 to coordinate epoch 2010.00, first the velocity grid is interpolated at t=1997 coordinates of

φ' = 49°53'09.2931"N, λ' = 99°54'41.0588"W, h' = 373.819m

V_N = –1.00 mm/yr
V_E = 2.46 mm/yr
V_U = –1.85 mm/yr

from which

φ = 49°53'09.2927"N, λ = 99°54'41.0572"W, h = 373.795m
t = 2010.00

In this example, the interpolated velocities are as for the forward transformation and no further iteration is required.

4.5 Transformations between Projected Coordinate Reference Systems

4.5.1 Cartesian Grid Offsets

(EPSG Dataset coordinate operation method code 9656)

A Cartesian grid offset allows calculation of coordinates in the target projected coordinate reference system by adding correction parameters to the coordinate values of the point in the source system:

$$E_2 = E_1 + \delta E_{1>2}$$
$$N_2 = N_1 + \delta N_{1>2}$$

where

E_2 and N_2 = values in the target coordinate reference system
E_1 and N_1 = values in the source vertical coordinate reference system
$\delta E_{1>2}$ and $\delta N_{1>2}$ are the offsets to be applied for the transformation from CRS 1 to CRS 2. Their value for the forward calculation is the value of the origin of the source CRS 1 in the target CRS 2.

Note: The sign convention used here is that which is applied in geodesy; this differs from that in general mathematics - see Implementation Notes - Offsets (Section §1.5).

For the **reverse** transformation from CRS 2 to CRS 1, the same formula is used but with the sign of the offsets reversed:

$$E_1 = E_2 + (-\delta E_{1>2})$$
$$N_1 = N_2 + -\delta N_{1>2})$$

The method may be applied between any two 2D CRSs having Cartesian coordinates.

4.5.2 Cartesian Grid Offsets from Form Function

(EPSG Dataset coordinate operation method code 1036)

In the German state of Schleswig-Holstein, the Cartesian grid offsets to be applied are determined through interpolation within an irregular grid of points at which coordinates in both source and target coordinate reference systems are given. The interpolation uses a finite element method form function procedure described in papers by Joachim Boljen (2003)[17].

4.5.3 Ordnance Survey Transformation

(EPSG Dataset coordinate operation method code 9633)

In Great Britain the relationship between the ETRS89 geographic 2D CRS and the OSGB36 / National Grid projected CRS uses a two-step pipeline through an unpublished intermediate projected CRS (ETRS89 / National Grid). First, the ETRS89 geographic coordinates φ,λ are converted to the intermediate grid coordinates E',N' using the Transverse Mercator map projection method described in Section §3.5.3.1. Then, grid offsets δE and δN modelling the transformation between the ETRS89 reference frame and the OSGB36 datum are derived by bilinear interpolation within the OSTN gridded data.

National Grid easting $E = E' + \delta E$
National Grid northing $N = N' + \delta N$

[17] Boljen, J., "Bezugssystemumstellung DHDN90 ↔ ETRS89 in Schleswig-Holstein". In *Zeitschrift für Vermessungswesen* (ZfV, the Journal of the German Association of Surveying (DVW)), 128. 2003. p. 244-250 and 129. 2004. p. 258-260.

4.6 Polynomial transformations

Note: In the sub-sections that follow, the general mathematical symbols X and Y, representing the axes of a coordinate reference system, must not be confused with the specific axis abbreviations or axis order in particular coordinate reference systems. To properly apply these mathematical operations, the axis order for a given source and target coordinate reference system must be explicitly known: see Implementation Notes in Section §1.7.

4.6.1 General case

Polynomial transformations between two coordinate reference systems are typically applied in cases where one or both coordinate reference systems exhibits lack of homogeneity in orientation and scale. The small distortions are then approximated by polynomial functions in latitude and longitude or in easting and northing. Depending on the degree of variability in the distortions, approximation may be carried out using polynomials of degree 2, 3, or higher. In the case of transformations between two projected coordinate reference systems, the additional distortions resulting from the application of two map projections and a datum transformation can be included in a single polynomial approximation function.

Polynomial approximation functions themselves are subject to variations, as different approximation characteristics may be achieved by different polynomial functions. The simplest of all polynomials is the general polynomial function. In order to avoid problems of numerical instability, this type of polynomial should be used after reducing the coordinate values in both the source and the target coordinate reference system to 'manageable' numbers, between –10 and +10 at most. This is achieved by working with offsets relative to a central evaluation point, scaled to the desired number range by applying a scaling factor to the coordinate offsets.

Hence, an evaluation point is chosen in the source coordinate reference system (X_{S0}, Y_{S0}) and in the target coordinate reference system (X_{T0}, Y_{T0}). Often, these two sets of coordinates do not refer to the same physical point but two points are chosen that have the same coordinate values in both the source and the target coordinate reference system. (When the two points have identical coordinates, these parameters are conveniently eliminated from the formulas, but the general case where the coordinates differ is given here).

The selection of an evaluation point in each of the two coordinate reference systems allows the point coordinates in both to be reduced as follows:

$$X_S - X_{S0}$$
$$Y_S - Y_{S0}$$

and

$$X_T - X_{T0}$$
$$Y_T - Y_{T0}$$

These coordinate differences are expressed in their own unit of measure, which may not be the same as that of the corresponding coordinate reference system.[18]

A further reduction step is usually necessary to bring these coordinate differences into the desired numerical range by applying a scaling factor to the coordinate differences in order to reduce them to a value range that may be applied to the polynomial formulas below without introducing numerical precision errors:

$$U = m_S (X_S - X_{S0})$$
$$V = m_S (Y_S - Y_{S0})$$

where:

[18] If the source and/or the target coordinate reference system are geographic, the coordinates themselves may be expressed in sexagesimal degrees (degrees, minutes, seconds), which cannot be directly processed by a mathematical formula.

X_S , Y_S are coordinates in the source coordinate reference system,
X_{S0} , Y_{S0} are coordinates of the evaluation point in the source coordinate reference system,
m_S is the scaling factor applied to the coordinate differences in the source coordinate reference system.

The normalized coordinates U and V of the point whose coordinates are to be transformed are used as input to the polynomial transformation formula. In order to control the numerical range of the polynomial coefficients A_n and B_n, the output coordinate differences dX and dY are multiplied by a scaling factor, m_T.

$$
\begin{aligned}
m_T.dX \;=\; & A_0 + A_1U + A_2V + A_3U^2 + A_4UV + A_5V^2 && \text{(to degree 2)}\\
+\; & A_6U^3 + A_7U^2V + A_8UV^2 + A_9V^3 && \text{(degree 3 terms)}\\
+\; & A_{10}U^4 + A_{11}U^3V + A_{12}U^2V^2 + A_{13}UV^3 + A_{14}V^4 && \text{(degree 4 terms)}\\
+\; & A_{15}U^5 + A_{16}U^4V + A_{17}U^3V^2 + A_{18}U^2V^3 + A_{19}UV^4 + A_{20}V^5 && \text{(degree 5 terms)}\\
+\; & A_{21}U^6 + A_{22}U^5V + A_{23}U^4V^2 + A_{24}U^3V^3 + A_{25}U^2V^4 + A_{26}UV^5 + A_{27}V^6 && \text{(degree 6 terms)}\\
+\; & \ldots + A_{104}V^{13} && \text{(degree 13 terms)}
\end{aligned}
$$

$$
\begin{aligned}
m_T.dY \;=\; & B_0 + B_1U + B_2V + B_3U^2 + B_4UV + B_5V^2 && \text{(to degree 2)}\\
+\; & B_6U^3 + B_7U^2V + B_8UV^2 + B_9V^3 && \text{(degree 3 terms)}\\
+\; & B_{10}U^4 + B_{11}U^3V + B_{12}U^2V^2 + B_{13}UV^3 + B_{14}V^4 && \text{(degree 4 terms)}\\
+\; & B_{15}U^5 + B_{16}U^4V + B_{17}U^3V^2 + B_{18}U^2V^3 + B_{19}UV^4 + B_{20}V^5 && \text{(degree 5 terms)}\\
+\; & B_{21}U^6 + B_{22}U^5V + B_{23}U^4V^2 + B_{24}U^3V^3 + B_{25}U^2V^4 + B_{26}UV^5 + B_{27}V^6 && \text{(degree 6 terms)}\\
+\; & \ldots + B_{104}V^{13} && \text{(degree 13 terms)}
\end{aligned}
$$

from which dX and dY are evaluated. These will be in the units of the target coordinate reference system.

In the EPSG Dataset, the polynomial coefficients are given as parameters with names of the form *Aumvn* and *Bumvn*, where m is the power to which U is raised and n is the power to which V is raised. For example, polynomial coefficient A_{17} is represented by the coordinate operation parameter named Au3v2.
Note: for brevity, polynomial coefficients with a value of zero are omitted from the EPSG Dataset.

The relationship between the two coordinate reference systems can now be written as follows:

$$(X_T - X_{TO}) = (X_S - X_{SO}) + dX$$
$$(Y_T - Y_{TO}) = (Y_S - Y_{SO}) + dY$$

or

$$X_T = X_S - X_{SO} + X_{TO} + dX$$
$$Y_T = Y_S - Y_{SO} + Y_{TO} + dY$$

where:

X_T , Y_T are coordinates in the target coordinate reference system,
X_S , Y_S are coordinates in the source coordinate reference system,
X_{SO} , Y_{SO} are coordinates of the evaluation point in the source coordinate reference system,
X_{TO} , Y_{TO} are coordinates of the evaluation point in the target coordinate reference system,
dX, dY are derived through the scaled polynomial formulas.

Other (arguably better) approximating polynomials are described in mathematical textbooks such as Phillips and Taylor[19].

[19] Phillips, G.M. and Taylor, P.J. "Theory and applications of numerical analysis". *Academic Press*. 1973.

Example
For General polynomial of degree 6 (EPSG Dataset coordinate operation method code 9648)
Coordinate transformation TM75 to ETRS89 (1)

Ordinate 1 of evaluation point X_O in source CRS:	$X_{SO} = \varphi_{SO}$ = 53°30'00.000"N = +53.5 degrees
Ordinate 2 of evaluation point Y_O in source CRS:	$Y_{SO} = \lambda_{SO}$ = 7°42'00.000"W = – 7.7 degrees
Ordinate 1 of evaluation point X_O in target CRS :	$X_{TO} = \varphi_{TO}$ = 53°30'00.000"N = +53.5 degrees
Ordinate 2 of evaluation point Y_O in target CRS :	$Y_{SO} = \lambda_{TO}$ = 7°42'00.000"W = – 7.7 degrees
Scaling factor for source CRS coordinate differences:	$m_S = 0.1$
Scaling factor for target CRS coordinate differences:	$m_T = 3600$

Polynomial coefficients (see EPSG Dataset transformation code 1041 for complete set of values):

$A_0 = 0.763$	$A_1 = -4.487$	….	$A_{24} = -265.898$	...	$A_{27} = 0$
$B_0 = -2.810$	$B_1 = -0.341$	….	$B_{24} = -853.950$	...	$B_{27} = 0$

Note: for brevity, polynomial coefficients with a value of zero are omitted from the EPSG Dataset.

Forward calculation for:

Latitude $\varphi_{TM75} = X_s =$ 55°00'00"N = + 55.000 degrees
Longitude $\lambda_{TM75} = Y_s =$ 6°30'00"W = – 6.500 degrees

$X_S - X_{SO} = \varphi_{TM75} - \varphi_{S0} = 55.0 - 53.5 = 1.5$ degrees
$Y_S - Y_{SO} = \lambda_{TM75} - \lambda_{S0} = -6.5 - (-7.7) = 1.2$ degrees

$U = m_S (X_S - X_{S0}) = m_S (\varphi_{TM75} - \varphi_{S0}) = 0.1*(1.5) = 0.15$
$V = m_S (Y_S - Y_{S0}) = m_S (\lambda_{TM75} - \lambda_{S0}) = 0.1*(1.2) = 0.12$

$$dX = (A_0 + A_1U + ... + A_5V^2 + ... + A_{24}U^3V^3) / m_T$$
$$= [0.763 + (-4.487 * 0.15) + ... + (0.183 * 0.12^2) + ... + (-265.898 * 0.15^3 * 0.12^3)] / 3600$$

$$dY = (B_0 + B_1U + ... + B_{24}U^3V^3) / m_T$$
$$= [-2.81 + (-0.341 * 0.15) + ... + (-853.95 * 0.15^3 * 0.12^3)] / 3600$$

	to degree 2	degree 3	degree 4	degree 5	degree 6	Sum / m_T
dX =	0.1029127	-0.002185407	0.0064009440	0.0014247770	-0.0015507171	0.0000297229
dY =	-3.3955340	0.022364019	-0.0230149836	-0.0156886729	-0.0049802364	-0.0009491261

Then Latitude $\varphi_{ETRS89} = X_T = X_S + dX = 55.0 + 0.00002972$ degrees = 55°00'00.107"N
Longitude $\lambda_{ETRS89} = Y_T = Y_S + dY = -6.5 - 0.00094913$ degrees = 6°30'03.417"W

4.6.1.1 Polynomial reversibility

Approximation polynomials are in a strict mathematical sense ***not reversible***, i.e., the same polynomial coefficients cannot be used to execute the reverse transformation.

In principle, two options are available to execute the reverse transformation:

1. By applying a similar polynomial transformation with a different set of polynomial coefficients for the reverse polynomial transformation. This would result in a separate forward and reverse transformation being stored in the EPSG Dataset.

2. By applying the polynomial transformation with the same coefficients but with their signs reversed and then iterate to an acceptable solution, the number of iteration steps being dependent on the desired accuracy. (Note that only the signs of the polynomial coefficients should be reversed and not the coordinates of the evaluation points or the scaling factors). The iteration procedure is usually described by the information source of the polynomial transformation.

However, under certain conditions, described below, a satisfactory solution for the reverse transformation may be obtained using the forward coefficient values in a single step, rather than multiple step iteration. If such a solution is possible, in the EPSG Dataset the polynomial coordinate transformation method is classified as a ***reversible polynomial of degree n***.

A (general) polynomial transformation is reversible only when the following conditions are met:

1. The coordinates of source and target evaluation point are (numerically) the same.
2. The unit of measure of the coordinate differences in source and target coordinate reference system are the same.
3. The scaling factors applied to source and target coordinate differences are the same.
4. The spatial variation of the differences between the coordinate reference systems around any given location is sufficiently small.

Clarification on conditions for polynomial reversibility:

Conditions 1 and 2: In the reverse transformation the roles of the source and target coordinate reference systems are reversed. Consequently, the coordinates of the evaluation point in the *source* coordinate reference system become those in the *target* coordinate reference system in the reverse transformation. Usage of the same transformation parameters for the reverse transformation will therefore only be valid if the evaluation point coordinates are numerically the same in source and target coordinate reference system and in the same units. That is, $X_{S0} = X_{T0} = X_0$ and $Y_{S0} = Y_{T0} = Y_0$.

Condition 3: The same holds for the scaling factors of the source and target coordinate differences and for the units of measure of the coordinate differences. That is, $m_S = m_T = m$.

Condition 4: If conditions 1, 2, and 3 are all satisfied, it then may be possible to use the forward polynomial algorithm with the forward parameters for the reverse transformation. This is the case if the spatial variations in dX and dY around any given location are sufficiently constant. The signs of the polynomial coefficients are then reversed but the scaling factor and the evaluation point coordinates retain their signs. If these spatial variations in dX and dY are too large, for the reverse transformation iteration would be necessary. It is therefore not the algorithm that determines whether a single step solution is sufficient or whether iteration is required, but the desired accuracy combined with the degree of spatial variability of dX and dY.

An example of a reversible polynomial transformation is ED50 to ED87 (1) for the North Sea. The suitability of this transformation to be described by a reversible polynomial can easily be explained. In the first place, both source and target coordinate reference systems are of type geographic 2D. The typical difference in coordinate values between ED50 and ED87 is in the order of 2 metres ($\approx 10^{-6}$ degrees) in the area of application. The polynomial functions are evaluated about central points with coordinates of 55°N, 0°E in both coordinate reference systems. The reduced coordinate differences (*in degrees*) are used as input parameters to the polynomial functions. The output coordinate differences are corrections to the input coordinate offsets of about *10^{-6} degrees*. This difference of several orders of magnitude between input and output values is the property that makes a polynomial function reversible in practice (although not in a formal mathematical sense).

The error made by the polynomial approximation formulas in calculating the reverse correction is of the same order of magnitude as the ratio of output versus input:

$$\frac{\textit{output error}}{\textit{input error}} \approx \frac{\textit{output value}}{\textit{input value}} \quad (\approx 10^{-6})$$

As long as the input values (the coordinate offsets from the evaluation point) are orders of magnitude larger than the output (the corrections), and provided the coefficients are used with changed signs, the polynomial transformation may be considered to be reversible.

Hence the EPSG Dataset acknowledges two classes of general polynomial functions, reversible and non-reversible, as distinguished by whether the coefficients may be used in both forward and reverse transformations, i.e., are reversible. The EPSG Dataset does not describe the iterative solution as a separate algorithm. The iterative solution for the reverse transformation, when applicable, is deemed to be implied by the (forward) algorithm.

Example
For Reversible polynomial of degree 4 (EPSG Dataset coordinate operation method code 9651)
Coordinate transformation ED50 to ED87 (1)

Ordinate 1 of evaluation point: $X_O = \varphi_O$ = 55°00'00.000"N = +55 degrees
Ordinate 2 of evaluation point: $Y_O = \lambda_O$ = 0°00'00.000"E = +0 degrees

Scaling factor for coordinate differences: m = 1.0

Parameters:
A_0 = – 5.56098E-06 A_1 = – 1.55391E-06 ... A_{14} = – 4.01383E-09
B_0 = + 1.48944E-05 B_2 = + 2.68191E-05 ... B_{14} = + 7.62236E-09

Forward calculation for:
Latitude $\varphi_{ED50} = X_s$ = 52°30'30"N = + 52.508333333 degrees
Longitude $\lambda_{ED50} = Y_s$ = 2°E = + 2.0 degrees

$U = m (X_S – X_0) = m (\varphi_{ED50} – \varphi_0)$ = 1.0 (52.508333333 – 55.0) = – 2.491666667 degrees
$V = m (Y_S – Y_0) = m (\lambda_{ED50} – \lambda_0)$ = 1.0 (2.0 – 0.0) = 2.0 degrees

$dX = (A_0 + A_1U + ... + A_{14}V^4) / k_{CD}$
= [– 5.56098E–06 + (– 1.55391E-06 * – 2.491666667) + ... + (– 4.01383E-09 * 2.0^4)]/1.0
= – 3.12958E–06 degrees

$dY = (B_0 + B_1U + ... + B_{14}V^4) / k_{CD}$
= [+1.48944E–05 + (2.68191E-05 * – 2.491666667) + ... + (7.62236E-09 * 2.0^4)]/1.0
= +9.80126E–06 degrees

Then: Latitude $\varphi_{ED87} = X_T = X_S + dX$ = 52.508333333 – 3.12958E–06 degrees = 52°30'29.9887"N
Longitude $\lambda_{ED87} = Y_T = Y_S + dY$ = 2°00'00.0353"E

Reverse
For coordinate transformation ED50 to ED87 (1).
The polynomial to degree 4 transformation for the ED50 to ED87 (1) coordinate transformation is reversible. The same formulas may be applied for the reverse calculation, but coefficients A_0 through A_{14} and B_0 through B_{14} are applied with reversal of their signs. Sign reversal is not applied to the coordinates of the evaluation point or scaling factor for coordinate differences. Thus:

Ordinate 1 of evaluation point: $X_O = \varphi_O$ = 55°00'00.000"N = +55 degrees
Ordinate 2 of evaluation point: $Y_O = \lambda_O$ = 0°00'00.000"E = +0 degrees
Scaling factor for coordinate differences: m = 1.0

A_0 = +5.56098E-06 A_1 = +1.55391E-06 ... A_{14} = +4.01383E-09
B_0 = –1.48944E-05 B_1 = –2.68191E-05 ... B_{14} = –7.62236E-09

Reverse calculation for:
Latitude $\varphi_{ED87} = X_S$ = 52°30'29.9887"N = +52.5083301944 degrees
Longitude $\lambda_{ED87} = Y_S$ = 2°00'00.0353"E = +2.0000098055 degrees

U = 1.0 (52.5083301944 – 55.0) = – 2.4916698056 degrees
V = 1.0 (2.0000098055 – 0.0) = 2.0000098055 degrees

$dX = (A_0 + A_1U + ... + A_{14}V^4)/k$
= [+5.56098E-06 + (1.55391E– 06 * – 2.491666667) + ...
... + (4.01383E-09 * 2.0000098055^4)]/1.0
= +3.12957E–06 degrees

$dY = (B_0 + B_1.U + ... + B_{14}.V^4)/k$
= [– 1.48944E-05 + (-2.68191E-05 * -2.491666667) + ...
... + (– 7.62236E-09 * 2.0000098055^4)]/1.0
= – 9.80124E–06 degrees

Then: Latitude $\varphi_{ED50} = X_T = X_S + dX$ = 52.5083301944 + 3.12957E–06 degrees = 52°30'30.000"N
Longitude $\lambda_{ED50} = Y_T = Y_S + dY$ = = 2°00'00.000"E

4.6.2 Polynomial transformation with complex numbers

The relationship between two projected coordinate reference systems may be approximated more elegantly by a single polynomial regression formula written in terms of complex numbers. The advantage is that the dependence between the 'A' and 'B' coefficients (for U and V) is taken into account in the formula, resulting in fewer coefficients for the same order polynomial. A polynomial to degree 3 in complex numbers is used in Belgium. A polynomial to degree 4 in complex numbers is used in the Netherlands for transforming coordinates referenced to the Amersfoort / RD system to and from ED50 / UTM.

$M_T (dX + i\,dY) = (A_1 + i\,A_2) (U + i\,V) + (A_3 + i\,A_4) (U + i\,V)^2$ (to degree 2)
$+ (A_5 + i\,A_6) (U + i\,V)^3$ (additional degree 3 terms)
$+ (A_7 + i\,A_8) (U + i\,V)^4$ (additional degree 4 terms)

where $U = m_S (X_S – X_{S0})$
$V = m_S (Y_S – Y_{S0})$
and m_S, m_T are the scaling factors for the coordinate differences in the source and target coordinate reference systems.

The polynomial to degree 4 can alternatively be expressed in matrix form as

$$\begin{pmatrix} m_T.dX \\ \\ m_T.dY \end{pmatrix} = \begin{pmatrix} +A_1 & -A_2 & +A_3 & -A_4 & +A_5 & -A_6 & +A_7 & -A_8 \\ \\ +A_2 & +A_1 & +A_4 & +A_3 & +A_6 & +A_5 & +A_8 & +A_7 \end{pmatrix} \begin{pmatrix} U \\ V \\ U^2–V^2 \\ 2UV \\ U^3–3UV^2 \\ 3U^2V–V^3 \\ U^4–6U^2V^2 + V^4 \\ 4U^3V–4UV^3 \end{pmatrix}$$

Then, as for the general polynomial case above
$X_T = X_S – X_{SO} + X_{TO} + dX$
$Y_T = Y_S – Y_{SO} + Y_{TO} + dY$

where, as above,

X_T , Y_T are coordinates in the target coordinate system,
X_S , Y_S are coordinates in the source coordinate system,
X_{SO} , Y_{SO} are coordinates of the evaluation point in the source coordinate reference system,
X_{TO} , Y_{TO} are coordinates of the evaluation point in the target coordinate reference system.

Note that the zero order coefficients of the general polynomial, A_0 and B_0, have apparently disappeared. In reality, they are absorbed by the different coordinates of the source and of the target evaluation point, which in this case, are numerically very different because of the use of two different projected coordinate systems for source and target.

The transformation parameter values (the coefficients) are not reversible. For the reverse transformation, a different set of parameter values are required, used within the same formulas as the forward direction.

Example
Complex polynomial of degree 4 (EPSG Dataset coordinate operation method code 9653)
Coordinate transformation: Amersfoort / RD New to ED50 / UTM zone 31N (1) (code 1044):

Coordinate transformation parameter name	Formula symbol	Parameter value	Unit
ordinate 1 of the evaluation point in the source CRS	X_{SO}	155,000.000	metre
ordinate 2 of the evaluation point in the source CRS	Y_{SO}	463,000.000	metre
ordinate 1 of the evaluation point in the target CRS	X_{TO}	663,395.607	metre
ordinate 2 of the evaluation point in the target CRS	Y_{TO}	5,781,194.380	metre
scaling factor for source CRS coordinate differences	m_S	10^{-5}	
scaling factor for target CRS coordinate differences	m_T	1.0	
A1	A_1	–51.681	coefficient
A2	A_2	+3,290.525	coefficient
A3	A_3	+20.172	coefficient
A4	A_4	+1.133	coefficient
A5	A_5	+2.075	coefficient
A6	A_6	+0.251	coefficient
A7	A_7	+0.075	coefficient
A8	A_8	–0.012	coefficient

For input point:

Easting, $X_{AMERSFOORT/RD} = X_S = 200{,}000.00$ metres
Northing, $Y_{AMERSFOORT/RD} = Y_S = 500{,}000.00$ metres

$U = m_S (X_S - X_{S0}) = (200{,}000 - 155{,}000)\ 10^{-5} = 0.45$
$V = m_S (Y_S - Y_{S0}) = (500{,}000 - 463{,}000)\ 10^{-5} = 0.37$

$dX = (-1{,}240.050) / 1.0$
$dY = (1{,}468.748) / 1.0$

Then: Easting, $E_{ED50/UTM31} = X_T = X_S - X_{SO} + X_{TO} + dX$
$= 200{,}000 - 155{,}000 + 663{,}395.607 + (-1{,}240.050)$
$= 707{,}155.557$ metres

Northing, $N_{ED50/UTM31N} = Y_T = Y_S - Y_{S0} + Y_{T0} + dY$
$= 500{,}000 - 463{,}000 + 5{,}781{,}194.380 + 1{,}468.748$
$= 5{,}819{,}663.128$ metres

This method is not reversible. For the reverse, a transformation with different parameter values but using the same formula is required.
Coordinate transformation: ED50 / UTM zone 31N to Amersfoort / RD New (1) (code 6303):

Coordinate transformation parameter name	Formula symbol	Parameter value	Unit
ordinate 1 of the evaluation point in the source CRS	X_{SO}	663,395.607	metre
ordinate 2 of the evaluation point in the source CRS	Y_{SO}	5,781,194.380	metre
ordinate 1 of the evaluation point in the target CRS	X_{TO}	155,000.000	metre
ordinate 2 of the evaluation point in the target CRS	Y_{TO}	463,000.000	metre
scaling factor for source CRS coordinate differences	m_S	10^{-5}	
scaling factor for target CRS coordinate differences	m_T	1.0	
A1	A_1	–56.619	coefficient
A2	A_2	–3,290.362	coefficient
A3	A_3	–20.184	coefficient
A4	A_4	+0.861	coefficient
A5	A_5	–2.082	coefficient
A6	A_6	+0.023	coefficient
A7	A_7	–0.070	coefficient
A8	A_8	+0.025	coefficient

For input point:

Easting, $E_{ED50/UTM31} = X_S$ = 707,155.557 metres
Northing, $N_{ED50/UTM31} = Y_S$ = 5,819,663.128 metres

$U = m_S (X_S – X_{S0}) = (707,155.557 – 663,395.607)\ 10^{-5} = 0.437599497$
$V = m_S (Y_S – Y_{S0}) = (5,819,663.128 – 5,781,194.380)\ 10^{-5} = 0.384687482$

$dX = (1240.0499) / 1.0$
$dY = (–1,468.7484) / 1.0$

Then: Easting, $X_{AMERSFOORT/RD} = X_T = X_S – X_{SO} + X_{TO} + dX$
= 707,155.557 – 663,395.607 + 155,000.000 + 1,240.050
= 200,000.000 metres

Northing, $Y_{AMERSFOORT/RD} = Y_T = Y_S – Y_{S0} + Y_{T0} + dY$
= 5,819,663.128 – 5,781,194.380 + 463,000.000 + (–1,468.748)
= 500,000 .000 metres

4.6.3 **Polynomial transformation for Spain**

(EPSG Dataset coordinate operation method code 9617)

The original geographic coordinate reference system for the Spanish mainland was based on Madrid 1870 datum, Struve 1860 ellipsoid, with longitudes related to the Madrid meridian. Three second-order polynomial expressions have been empirically derived by El Servicio Geográfico del Ejército to transform geographic coordinates based on this system to equivalent values based on the European Datum of 1950 (ED50). The polynomial coefficients derived can be used to transform coordinates from the Madrid 1870 (Madrid) geographic coordinate reference system to the ED50 system. Three pairs of expressions have been derived: each pair is used to calculate the shift in latitude and longitude respectively for (i) a mean for all Spain, (ii) a better fit for the north of Spain, and (iii) a better fit for the south of Spain.

The polynomial expressions are:

$d\varphi$ (arc sec) = $A_0 + (A_1\ \varphi_s) + (A_2\ \lambda_s) + (A_3\ H_s)$
$d\lambda$ (arc sec) = $B_{00} + B_0 + (B_1\ \varphi_s) + (B_2\ \lambda_s) + (B_3\ H_s)$

where latitude φ_s and longitude λ_s are in decimal degrees referred to the Madrid 1870 (Madrid) geographic coordinate reference system and H_s is gravity-related height in metres. B_{00} is the longitude (in seconds) of the Madrid meridian measured from the Greenwich meridian; it is the value to be applied to a longitude relative to the Madrid meridian to transform it to a longitude relative to the Greenwich meridian.

The results of these expressions are applied through the formulas:

$\varphi_{ED50} = \varphi_{M1870(M)} + d\varphi$

and $\lambda_{ED50} = \lambda_{M1870(M)} + d\lambda$.

Example

Input point coordinate reference system: Madrid 1870 (Madrid) (geographic 2D)

Latitude φ_s = 42°38'52.77"N
= +42.647992 degrees

Longitude λ_s = 3°39'34.57"E of Madrid
= +3.659603 degrees from the Madrid meridian.

Gravity-related height H_s = 0 m

For the north zone transformation:

A_0 = 11.328779	B_{00} = -13276.58
A_1 = -0.1674	B_0 = 2.5079425
A_2 = -0.03852	B_1 = 0.08352
A_3 = 0.0000379	B_2 = -0.00864
	B_3 = -0.0000038

$d\varphi$ = +4.05 seconds

Then latitude φ_{ED50} = 42°38'52.77"N + 4.05"
= 42°38'56.82"N

$d\lambda$ = -13270.54 seconds = -3°41'10.54"

Then longitude λ_{ED50} = 3°39'34.57"E – 3°41'10.54"
= 0°01'35.97"W of Greenwich.

4.7 Miscellaneous Linear Coordinate Operations (Affine Transformations)

An affine 2D transformation is used for converting or transforming a coordinate reference system possibly with non-orthogonal axes and possibly different units along the two axes to an isometric coordinate reference system (i.e. a system of which the axes are orthogonal and have equal scale units, for example a projected CRS). The transformation therefore involves a change of origin, differential change of axis orientation and a differential scale change. The EPSG Dataset distinguishes five methods to implement this class of coordinate operation:

- the parametric representation,
- the geometric representation,
- a simplified case of the geometric representation known as the Similarity Transformation in which the degrees of freedom are constrained.
- two variations of the geometric representation for seismic bin grids.

Note: In the sub-sections that follow, the general mathematical symbols X and Y, representing the axes of a coordinate reference system, must not be confused with the specific axis abbreviations or axis order in particular coordinate reference systems. To properly apply these generic mathematical operations, the axis order for a given source and target coordinate reference system must be explicitly known: see Implementation Notes in Section §1.7.

4.7.1 Affine Parametric Transformation

(EPSG Dataset coordinate operation method code 9624)

Mathematical and survey literature usually provides the parametric representation of the affine transformation. The parametric algorithm is commonly used for rectification of digitized maps. It is often embedded in CAD software and Geographic Information Systems where it is frequently referred to as "rubber sheeting".

The formula in matrix form is as follows:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R}\,\boldsymbol{V_S}$$

where:

$$\boldsymbol{V_T} = \begin{pmatrix} X_T \\ Y_T \end{pmatrix} \qquad \boldsymbol{V_{TO}} = \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \qquad \boldsymbol{R} = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{V_S} = \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

or using algebraic coefficients:

$$X_T = A_0 + (A_1\,X_S) + (A_2\,Y_S)$$
$$Y_T = B_0 + (B_1\,X_S) + (B_2\,Y_S)$$

where

X_T , Y_T are the coordinates of a point P in the target coordinate reference system;
X_S , Y_S are the coordinates of P in the source coordinate reference system.

This form of describing an affine transformation is analogous to the general polynomial transformation formulas (Section §4.6.1 above). An affine transformation could be considered to be a first order general polynomial transformation but without the reduction to source and target evaluation points.

Reverse
The reverse operation is another affine parametric transformation using the same formulas but with different parameter values. The reverse parameter values, indicated by a prime ('), can be calculated from those of the forward operation as follows:

$D = (A_1 B_2) – (A_2 B_1)$
$A_0' = [(A_2 B_0) – (B_2 A_0)] / D$
$B_0' = [(B_1 A_0) – (A_1 B_0)] / D$
$A_1' = +B_2 / D$
$A_2' = – A_2 / D$
$B_1' = – B_1 / D$
$B_2' = +A_1 / D$

Then

$$X_S = A_0' + (A_1' X_T) + (A_2' Y_T)$$
$$Y_S = B_0' + (B_1' X_T) + (B_2' Y_T)$$

Or in matrix form:

$$\boldsymbol{V_S} = \boldsymbol{R}^{-1} (\boldsymbol{V_T} – \boldsymbol{V_{TO}})$$

Example
Coordinate transformation: Jamaica 1875 / Jamaica (Old Grid) to JAD69 / Jamaica National Grid:

Coordinate transformation parameter	**Parameter value**	**Unit**
A_0	82357.457	metre
A_1	0.304794369	coefficient
A_2	0.000015417425	coefficient
B_0	28091.324	metre
B_1	-0.000015417425	coefficient
B_2	0.304794369	coefficient

The units for the Jamaica 1875 / Jamaica (Old Grid) are feet, those for the JAD69 / Jamaica National Grid are metres. The foot-metre conversion factor is embedded within the transformation, together with the scale and orientation differences between the two coordinate reference systems.

For input point:
Easting, $E_{\text{Jamaica 1875 / Jamaica (Old Grid)}} = X_S$ = 553900.00 feet
Northing, $N_{\text{Jamaica 1875 / Jamaica (Old Grid)}} = Y_S$ = 482500.00 feet

Then: Easting, $E_{\text{JAD69 / Jamaica National Grid}} = X_T$ = 251190.497 metres
Northing, $N_{\text{JAD69 / Jamaica National Grid}} = X_T$ = 175146.067 metres

For the reverse calculation, first calculate derived parameter values:

Coordinate transformation parameter	Parameter value	Unit
D	0.092899608	coefficient
A_0'	-270201.960	Clarke's foot
A_1'	3.280900499	coefficient
A_2'	-0.000165958	coefficient
B_0'	-92178.507	Clarke's foot
B_1'	0.000165958	coefficient
B_2'	3.280900499	coefficient

Then for the same point

Easting, $E_{\text{JAD69 / Jamaica National Grid}}$ $= X_T$ = 251190.497 metres
Northing, $N_{\text{JAD69 / Jamaica National Grid}}$ $= X_T$ = 175146.067 metres

the reverse transformation using the derived parameter values gives

Easting, $E_{\text{Jamaica 1875 / Jamaica (Old Grid)}}$ $= X_S$ = 553900.000 feet
Northing, $N_{\text{Jamaica 1875 / Jamaica (Old Grid)}}$ $= Y_S$ = 482500.000 feet

4.7.2 Affine Geometric Transformation

(EPSG Dataset coordinate operation method code 9623)

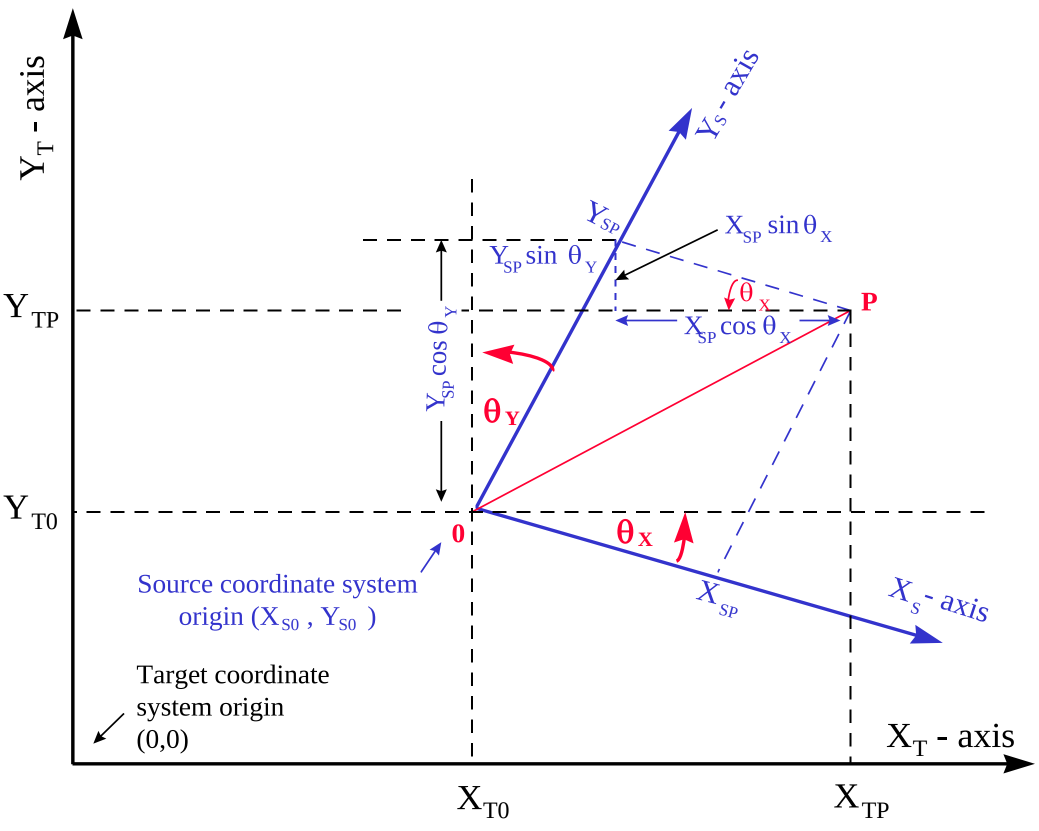


Figure 14 – Geometric representation of the affine coordinate transformation.
(Note that the scale parameters of the X_s and Y_s axes have been omitted to prevent cluttering of the figure).

From the diagram above it can be seen that:

$$X_{TP} = X_{TO} + Y_{SP} \sin \theta_Y + X_{SP} \cos \theta_X = X_{TO} + X_{SP} \cos \theta_X + Y_{SP} \sin \theta_Y$$
$$Y_{TP} = Y_{TO} + Y_{SP} \cos \theta_Y - X_{SP} \sin \theta_X = Y_{TO} - X_{SP} \sin \theta_X + Y_{SP} \cos \theta_Y$$

The scaling of both source and target coordinate reference systems adds some complexity to this formula. The operation will often be applied to transform an engineering coordinate reference system to a projected coordinate reference system. The orthogonal axes of the projected coordinate reference system have identical units. The engineering coordinate reference system may have different units of measure on its two axes: these have scale ratios of M_X and M_Y respective to the axes of the projected coordinate reference system.

The projected coordinate reference system is nominally defined to be in well-known units, e.g., metres. However, the distortion characteristics of the map projection only preserve true scale along certain defined lines or curves. Hence the projected coordinate reference system's unit of measure is, strictly speaking, only valid along those lines or curves. Everywhere else its scale is distorted by the map projection. For conformal map projections, the distortion at any point can be expressed by the point scale factor 'k' for that point. Note that this point scale factor 'k' should NOT be confused with the scale factor at the natural origin of the projection, denominated by 'k_0'. (For non-conformal map projections, the scale distortion at a point is bearing-dependent and is not described in this document).

It has developed as working practice to choose the origin of the source (engineering) coordinate reference system as the point in which to calculate this point scale factor 'k', although, for engineering coordinate reference systems with a large coverage area, a point in the middle of the area may be a better choice.

Adding the scaling between each pair of axes and dropping the suffix for point P, after rearranging the terms we have the geometric representation of the affine transformation:

$$X_T = X_{TO} + X_S\, k\, M_X \cos\theta_X + Y_S\, k\, M_Y \sin\theta_Y$$
$$Y_T = Y_{TO} - X_S\, k\, M_X \sin\theta_X + Y_S\, k\, M_Y \cos\theta_Y$$

where:

X_{TO}, Y_{TO} = the coordinates of the origin point of the source coordinate reference system, expressed in the target coordinate reference system;
M_X, M_Y = multiplication factor applied to coordinates in the source coordinate reference system to obtain the correct scale of the equivalent axis of the target coordinate reference system, for the first and second source and target axes pairs respectively;
k = point scale factor of the target coordinate reference system at a chosen reference point;
θ_X, θ_Y = the angles about which the source coordinate reference system axes X_S and Y_S must be rotated to coincide with the target coordinate reference system axes X_T and Y_T respectively (counter-clockwise being positive).

Alternatively, in matrix form:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R_1}\, k\, \boldsymbol{S_1}\, \boldsymbol{V_S}$$

where:

$$\boldsymbol{V_T} = \begin{pmatrix} X_T \\ Y_T \end{pmatrix} \qquad \boldsymbol{V_{TO}} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} \qquad \boldsymbol{V_S} = \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

and

$$\boldsymbol{R_1} = \begin{pmatrix} \cos\theta_X & \sin\theta_Y \\ -\sin\theta_X & \cos\theta_Y \end{pmatrix} \qquad \boldsymbol{S_1} = \begin{pmatrix} M_X & 0 \\ 0 & M_Y \end{pmatrix}$$

or

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + \begin{pmatrix} \cos\theta_X & \sin\theta_Y \\ -\sin\theta_X & \cos\theta_Y \end{pmatrix} k \begin{pmatrix} M_X & 0 \\ 0 & M_Y \end{pmatrix} \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

Comparing the algebraic representation with the parameters of the parametric form in Section §4.7.1 above it can be seen that the parametric and geometric forms of the affine coordinate transformation are related as follows:

$A_0 = X_{TO}$
$A_1 = k\, M_X \cos\theta_X$
$A_2 = k\, M_Y \sin\theta_Y$
$B_0 = Y_{TO}$
$B_1 = -k\, M_X \sin\theta_X$
$B_2 = k\, M_Y \cos\theta_Y$

Reverse
For the Affine Geometric Transformation, the reverse operation can be described by a different formula, as shown below, in which the same parameter values as the forward transformation may be used. In matrix form:

$$\boldsymbol{V_S} = (1/k)\, \boldsymbol{S_1}^{-1}\, \boldsymbol{R}_1^{-1}\, (\boldsymbol{V_T} - \boldsymbol{V_{TO}})$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = \frac{1}{k\,.Z} \begin{pmatrix} 1/M_X & 0 \\ 0 & 1/M_Y \end{pmatrix} \begin{pmatrix} \cos\theta_Y & -\sin\theta_Y \\ \sin\theta_X & \cos\theta_X \end{pmatrix} \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix}$$

where $Z = \cos(\theta_X - \theta_Y)$;

Algebraically:
$X_S = [(X_T - X_{TO}) \cos\theta_Y - (Y_T - Y_{TO}) \sin\theta_Y] / [k\, M_X \cos(\theta_X - \theta_Y)]$
$Y_S = [(X_T - X_{TO}) \sin\theta_X + (Y_T - Y_{TO}) \cos\theta_X] / [k\, M_Y \cos(\theta_X - \theta_Y)]$

<u>Orthogonal case</u>
If the source coordinate reference system happens to have orthogonal axes, that is both axes are rotated through the same angle to bring them into the direction of the orthogonal target coordinate reference system axes, i.e., $\theta_X = \theta_Y = \theta$, then the Affine Geometric Transformation can be simplified. In matrix form this is:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R_2}\, k\, \boldsymbol{S_1}\, \boldsymbol{V_S}$$

where $\boldsymbol{V_T}$, $\boldsymbol{V_{TO}}$, $\boldsymbol{S_1}$ and $\boldsymbol{V_S}$ are as in the general case but

$$\boldsymbol{R_2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Alternatively,

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} k \begin{pmatrix} M_X & 0 \\ 0 & M_Y \end{pmatrix} \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

Algebraically:
$X_T = X_{TO} + X_S\, k\, M_X \cos\theta + Y_S\, k\, M_Y \sin\theta$
$Y_T = Y_{TO} - X_S\, k\, M_X \sin\theta + Y_S\, k\, M_Y \cos\theta$

where:

X_{TO}, Y_{TO} = the coordinates of the origin point of the source coordinate reference system, expressed in the target coordinate reference system;

M_X , M_Y = multiplication factor applied to coordinates in the source coordinate reference system to obtain the correct scale of the equivalent axis in the target coordinate reference system, for the X axes and the Y axes respectively;
k = the point scale factor of the target coordinate reference system at a chosen reference point;
θ = the angle through which the source coordinate reference system axes must be rotated to coincide with the target coordinate reference system axes (counter-clockwise is positive). Alternatively, the bearing (clockwise positive) of the source coordinate reference system Y_S-axis measured relative to target coordinate reference system north.

The reverse formulas of the general case can also be simplified by replacing θ_X and θ_Y with θ. In matrix form:

$$\boldsymbol{V_S} = (1/k)\ \boldsymbol{S_1}^{-1}\ \boldsymbol{R_2}^{-1}\ (\boldsymbol{V_T} - \boldsymbol{V_{TO}})$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1/M_X & 0 \\ 0 & 1/M_Y \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix}$$

Algebraically:

$$X_S = [(X_T - X_{TO}) \cos\theta \;-\; (Y_T - Y_{TO}) \sin\theta\,] / [k\,M_X\,]$$
$$Y_S = [(X_T - X_{TO}) \sin\theta \;+\; (Y_T - Y_{TO}) \cos\theta] / [k\,M_Y\,]$$

In the EPSG Dataset, this orthogonal case has been deprecated. The formulas for the general case should be used, inserting θ for both θ_X and θ_Y. The orthogonal case has been documented here to show the progression through increasing constraints on the degrees of freedom between the general case and the Similarity Transformation.

4.7.3 <u>**Similarity Transformation**</u>

(EPSG Dataset coordinate operation method code 9621)

If the source coordinate reference system has orthogonal axes and has axes of the same scale, that is both axes are scaled by the same factor to bring them into the scale of the target coordinate reference system axes (i.e., $M_X = M_Y = M$), then the orthogonal case of the Affine Geometric Transformation can be simplified further to a Similarity Transformation.

From the diagram in Figure 15 below the Similarity Transformation in algebraic form is:

$$X_{TP} = X_{TO} + Y_{SP}\,M \sin\theta \;+ X_{SP}\,M \cos\theta$$
$$Y_{TP} = Y_{TO} + Y_{SP}\,M \cos\theta \;-\; X_{SP}\,M \sin\theta$$

Dropping the suffix for point P and rearranging the terms

$$X_T = X_{TO} \;+ X_S\,M \cos\theta \;+ Y_S\,M \sin\theta$$
$$Y_T = Y_{TO} \;-\; X_S\,M \sin\theta \;+ Y_S\,M \cos\theta$$

where:
X_{TO} , Y_{TO} = the coordinates of the origin point of the source coordinate reference system expressed in the target coordinate reference system;
M = multiplication factor applied to coordinates in the source coordinate reference system to obtain the correct scale of the target coordinate reference system;
θ = the angle about which the axes of the source coordinate reference system need to be rotated to coincide with the axes of the target coordinate reference system, counter-clockwise being positive. Alternatively, the bearing of the source coordinate reference system Y_S-axis measured relative to target coordinate reference system north.

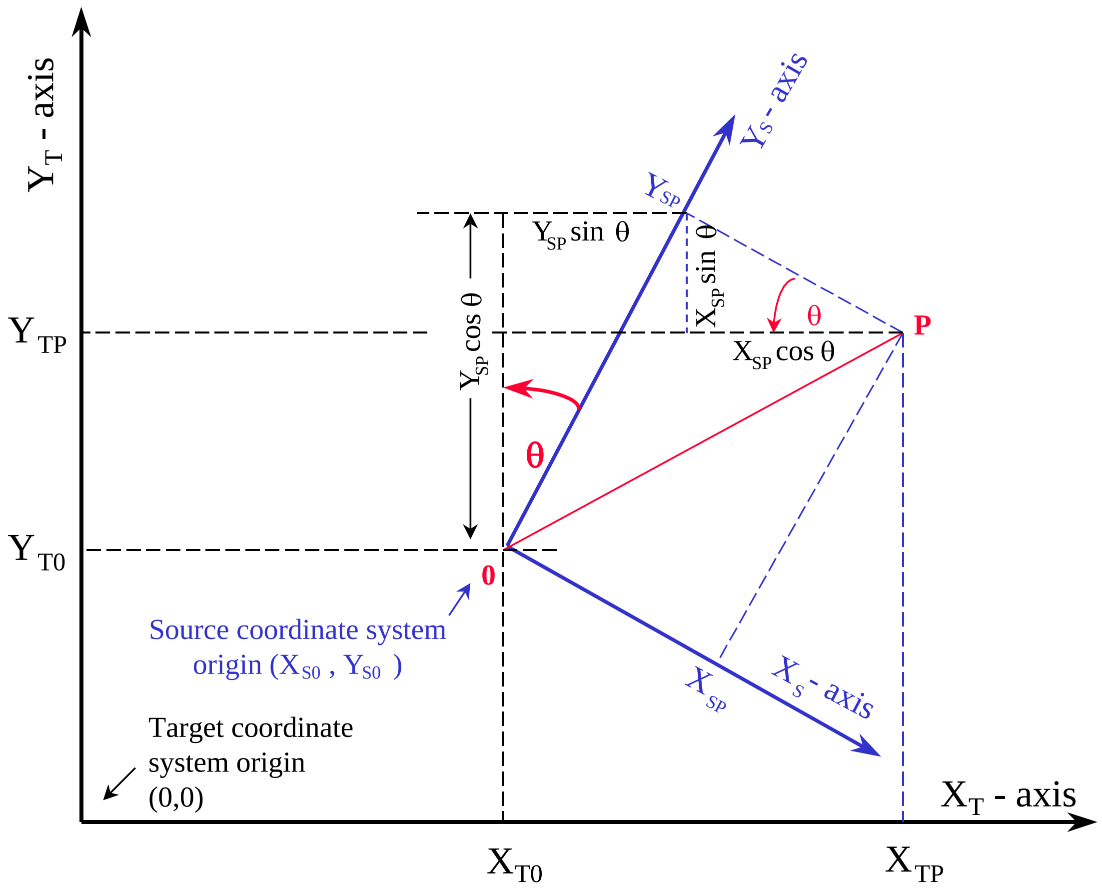


Figure 15 – Similarity Transformation.

The Similarity Transformation can also be described as a special case of the Affine Parametric Transformation where coefficients $A_1 = B_2$ and $A_2 = - B_1$.

In matrix form:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \mathrm{M}\, \boldsymbol{R_2}\, \boldsymbol{V_S}$$

where $\boldsymbol{V_T}$, $\boldsymbol{V_{TO}}$, $\boldsymbol{R_2}$ and $\boldsymbol{V_S}$ are as in the Affine Orthogonal Geometric Transformation method, or

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + \mathrm{M} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_S \\ Y_S \end{pmatrix}$$

Reverse

The reverse formula for the Similarity Transformation, in matrix form, is:

$$\boldsymbol{V_S} = (1/\mathrm{M})\, \boldsymbol{R_2}^{-1} (\boldsymbol{V_T} - \boldsymbol{V_{TO}})$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix}$$

Algebraically:

$$X_S = [(X_T - X_{TO}) \cos\theta \; - \; (Y_T - Y_{TO}) \sin\theta \,] / M$$
$$Y_S = [(X_T - X_{TO}) \sin\theta \; + \; (Y_T - Y_{TO}) \cos\theta] / M$$

An alternative approach for the reverse operation is to use the same formula as for the forward computation but with different parameter values:

$$X_T = X_{TO}' + X_S M' \cos\theta' + Y_S M' \sin\theta'$$
$$Y_T = Y_{TO}' - X_S M' \sin\theta' + Y_S M' \cos\theta'$$

The reverse parameter values, indicated by a prime ('), can be calculated from those of the forward operation as follows:

$$X_{TO}' = (Y_{TO} \sin\theta - X_{TO} \cos\theta) / M$$
$$Y_{TO}' = -(Y_{TO} \cos\theta + X_{TO} \sin\theta) / M$$
$$M' = 1/M$$
$$\theta' = -\theta$$

Example
ED50 / UTM zone 31N to ETRS89 / UTM zone 31N

Parameters of the Similarity Transformation:

Ordinate 1 of evaluation point in target CRS:	X_{TO}	=	-129.549 metres
Ordinate 2 of evaluation point in target CRS:	Y_{TO}	=	-208.185 metres
Scale factor for source CRS axes:	M	=	1.00000155
Rotation angle of source CRS axes:	θ	=	1.56504" = 0.000007588 rad

Forward computation of ED50 / UTM zone 31N (source) coordinates 300000m E, 4500000m N:

$$E_{ETRS89} = -129.549 + 300000.465 + 34.144$$
$$= 299905.060 \text{ m E}$$

$$N_{ETRS89} = -208.185 - 2.276 + 4500006.977$$
$$= 4499796.515 \text{m N}$$

Reverse computation of ETRS89 / UTM zone 31N coordinates 299905.060m E, 4499796.515m N:

$$E_{ED50} = (300034.609 - 34.144) / 1.00000155$$
$$= 300000.000 \text{m E}$$

$$N_{ED50} = (2.276 + 4500004.700) / 1.00000155$$
$$= 4500000.000 \text{m N}$$

Alternative reverse computation:

First calculate new parameter values:

$$X_{TO}' = 129.5472 \text{ m}$$
$$Y_{TO}' = 208.1857 \text{ m}$$
$$M' = 0.99999845$$
$$\theta' = -0.000007588 \text{ rad}$$

Then apply these values to forward formula:

$$E_{ED50} = 129.547 + 299904.595 + (–34.142) = 300000.000 \text{ m}$$
$$N_{ED50} = 208.186 – (–2.276) + 4499789.539 = 4500000.000 \text{ m}$$

When to use the Similarity Transformation

Similarity Transformations can be used when source and target coordinate reference systems

- each have orthogonal axes
- each have the same scale along both axes

and

- both have the same units of measure

for example, between engineering plant grids and projected coordinate reference systems.

Coordinate Operations between two coordinate reference systems where in either system either the scale along the axes differ, or the axes are not orthogonal, should be defined as an Affine Transformation in either the parametric or geometric form. But for seismic bin grids see the following section.

4.8 Seismic Bin Grid Transformations

The P6/98 seismic bin grid exchange format described a special case of the Affine Geometric Transformation. The method is also described in the SEG-Y revision 1 and subsequent revisions of the seismic data exchange format. The P6 mathematics are applied in two ways:

i) as a stand-alone transformation that includes a change of datum, used to geo-reference a previously ungeoreferenced bin grid (defined as a local (engineering) CRS) to a map grid.
ii) as the deriving conversion component of a bin grid described through the derived CRS construct. The bin grid inherits the geodetic datum of its base projected CRS.

For the P6 seismic bin grid case of the affine geometric transformation method (whether applied as a transformation or as a conversion):

- the base coordinate reference system is a map grid (projected CRS) which by definition has orthogonal axes with the same units;
- the bin grid axes are orthogonal;

and one or both of the following may apply:

- the origin of the bin grid may be assigned non-zero bin node coordinates;
- the bin node interval may increase in increments other than 1, i.e., Inc_{SX} and Inc_{SY} may not be unity.

4.8.1 P6 Right-handed Seismic Bin Grid Transformation

(EPSG Dataset coordinate operation method code 9666)

The P6/98 exchange format defined the bin grid axes to be named I and J, where the I-axis is rotated 90° clockwise from the J-axis (when viewed from above the plane containing the two axes). This is sometimes described as 'right-handed'.

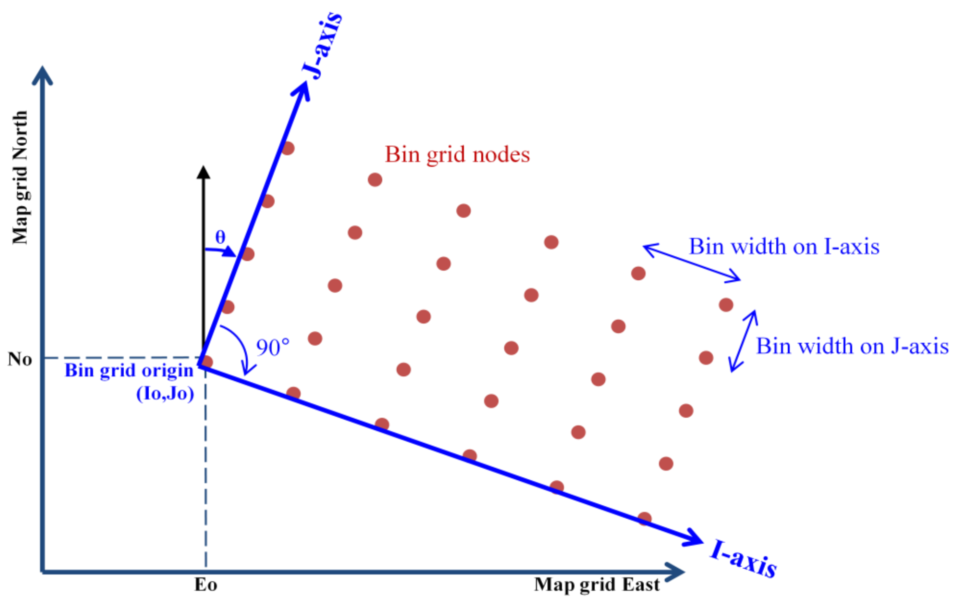


Figure 16 – Relationship of right-handed bin grid to map grid.

The defining parameters are:

P6 format term	**Equivalent affine geometric transformation term**[20]
Bin grid origin (Io)	Ordinate 1 of evaluation point in source CRS (X_{SO})
Bin grid origin (Jo)	Ordinate 2 of evaluation point in source CRS (Y_{SO})
Map grid easting of bin grid origin (Eo)	Ordinate 1 of evaluation point in target CRS (X_{TO})
Map grid northing of bin grid origin (No)	Ordinate 2 of evaluation point in target CRS (Y_{TO})
Scale factor of bin grid (SF)	Point scale factor (k)
Nominal bin width along I axis (I_bin_width)	Scale factor for source CRS first axis (M_X)
Nominal bin width along J axis (J_bin_width)	Scale factor for source CRS second axis (M_y)
Grid bearing of bin grid J axis (θ)	Rotation angle of source CRS axes (θ)
Bin node increment on I axis (I_bin_inc)	Bin node increment on I-axis
Bin node increment on J axis (J_bin_inc)	Bin node increment on J-axis

In this method the terms X_S, Y_S, M_X, and M_Y in the orthogonal case of the Affine Geometric Transformation formulas are replaced by $(X_S - X_{SO})$, $(Y_S - Y_{SO})$, (M_X / Inc_{SX}), and (M_Y / Inc_{SY}), respectively. Thus, the transformation from bin grid to map grid (source to target coordinate reference system in Affine Geometric Transformation formula nomenclature) is:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R_2}\, k\, \boldsymbol{S_2}\, \boldsymbol{V_2}$$

where, as in the orthogonal case of the Affine Geometric Transformation method:

$$\boldsymbol{V_T} = \begin{pmatrix} X_T \\ Y_T \end{pmatrix} \qquad \boldsymbol{V_{TO}} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{R_2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

but where

$$\boldsymbol{S_2} = \begin{pmatrix} M_X / Inc_{SX} & 0 \\ 0 & M_Y / Inc_{SY} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{V_2} = \begin{pmatrix} X_S - X_{SO} \\ Y_S - Y_{SO} \end{pmatrix}$$

That is,

$$\begin{pmatrix} X_T \\ Y_T \end{pmatrix} = \begin{pmatrix} X_{TO} \\ Y_{TO} \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \; k \; \begin{pmatrix} M_X / Inc_{SX} & 0 \\ 0 & M_Y / Inc_{SY} \end{pmatrix} \begin{pmatrix} X_S - X_{SO} \\ Y_S - Y_{SO} \end{pmatrix}$$

Algebraically:

$X_T = X_{TO} + [(X_S - X_{SO}) \cos\theta\ k\ M_X / Inc_{SX}] + [(Y_S - Y_{SO}) \sin\theta\ k\ M_Y / Inc_{SY}]$
$Y_T = Y_{TO} - [(X_S - X_{SO}) \sin\theta\ k\ M_X / Inc_{SX}] + [(Y_S - Y_{SO}) \cos\theta\ k\ M_Y / Inc_{SY}]$

[20] Note: In practice bin grids are either georeferenced to or (more usually) defined from an identified map grid (projected CRS). As such the example bin grids in the EPSG Dataset are described in the sense map grid to bin grid, with the map grid (projected CRS) as source CRS for transformations (codes 6918 and 6919) and as base CRS with deriving conversions (codes 32698 and 32699). However, to retain consistency with the syntax used in the description of the affine general transformations in the previous sections of this document, in the formulas in this section the bin grid is the affine source CRS and the map grid is the affine target CRS.

Using the symbol notation in the P6/98 document these expressions are:

$$\begin{pmatrix} E \\ N \end{pmatrix} = \begin{pmatrix} E_O \\ N_O \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \; SF \begin{pmatrix} \text{I\_bin\_width} / \text{I\_bin\_inc} & 0 \\ 0 & \text{J\_bin\_width} / \text{J\_bin\_inc} \end{pmatrix} \begin{pmatrix} I - I_O \\ J - J_O \end{pmatrix}$$

or

$$E = E_O + [(I - I_O) \cos\theta \; SF \; \text{I\_bin\_width} / \text{I\_bin\_inc}] + [(J - J_O) \sin\theta \; SF \; \text{J\_bin\_width} / \text{J\_bin\_inc}]$$

$$N = N_O - [(I - I_O) \sin\theta \; SF \; \text{I\_bin\_width} / \text{I\_bin\_inc}] + [(J - J_O) \cos\theta \; SF \; \text{J\_bin\_width} / \text{J\_bin\_inc}]$$

For the reverse transformation (map grid to bin grid):

$$\boldsymbol{V_S} = (1/k) \, \boldsymbol{S_2}^{-1} \boldsymbol{R_2}^{-1} (\boldsymbol{V_T} - \boldsymbol{V_{TO}}) + \boldsymbol{V_{SO}}$$

or

$$\begin{pmatrix} X_S \\ Y_S \end{pmatrix} = 1/k \begin{pmatrix} Inc_{SX} / M_X & 0 \\ 0 & Inc_{SY} / M_Y \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin & \cos\theta \end{pmatrix} \begin{pmatrix} X_T - X_{TO} \\ Y_T - Y_{TO} \end{pmatrix} + \begin{pmatrix} X_{SO} \\ Y_{SO} \end{pmatrix}$$

or algebraically:

$$X_S = \{[(X_T - X_{TO}) \cos\theta - (Y_T - Y_{TO}) \sin\theta] \, [Inc_{SX} / (k \, M_X)]\} + X_{SO}$$

$$Y_S = \{[(X_T - X_{TO}) \sin\theta + (Y_T - Y_{TO}) \cos\theta] \, [Inc_{SY} / (k \, M_Y)]\} + Y_{SO}$$

Using the symbol notation in the P6/98 document these reverse expressions are:

$$\begin{pmatrix} I \\ J \end{pmatrix} = 1/SF \begin{pmatrix} \text{I\_bin\_inc} / \text{I\_bin\_width} & 0 \\ 0 & \text{J\_bin\_inc} / \text{I\_bin\_width} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin & \cos\theta \end{pmatrix} \begin{pmatrix} E - E_O \\ N - N_O \end{pmatrix} + \begin{pmatrix} I_O \\ J_O \end{pmatrix}$$

or

$$I = I_O + \{[(E - E_O) \cos\theta - (N - N_O) \sin\theta] \, [\text{I\_bin\_inc} / (SF \; \text{I\_bin\_width})]\}$$

$$J = J_O + \{[(E - E_O) \sin\theta + (N - N_O) \cos\theta] \, [\text{J\_bin\_inc} / (SF \; \text{J\_bin\_width})]\}$$

Examples

Example 1: $k \neq 1$

This example is given in the P6/98 document[21]. The bin grid is based on projected CRS WGS 84 / UTM Zone 31N. The origin of the bin grid is defined to be at E = 456781.0, N = 5836723.0. The point scale factor at this point is 0.99984.

The bin width on the I-axis (X_S axis) is 25 metres, whilst the bin width on the J-axis (Y_S axis) is 12.5 metres. The origin of the grid has bin node values of 1,1.

In the map grid, the bearing of the bin grid I and J axes are 110° and 20°, respectively. Thus, the angle through which the bin grid axes need to be rotated to coincide with the map grid axes is +20 degrees.

The transformation parameter values are:

Parameter	EPSG Affine Transformation symbol	P6 symbol	Parameter value
Bin grid origin I	X_{SO}	Io	1
Bin grid origin J	Y_{SO}	Jo	1
Bin grid origin Easting	X_{TO}	Eo	456781.00 m
Bin grid origin Northing	Y_{TO}	No	5836723.00 m
Scale factor of bin grid	k	SF	0.99984
Bin Width on I-axis	M_X	I_bin_width	25 m
Bin Width on J-axis	M_Y	J_bin_width	12.5 m
Map grid bearing of bin grid J-axis	θ	θ	20 deg
Bin node increment on I-axis	Inc_{SX}	I_bin_inc	1
Bin node increment on J-axis	Inc_{SY}	J_bin_inc	1

Calculation of map grid coordinates for bin node with coordinates: I = 300, J = 247:

$$\text{Easting} = X_T = X_{TO} + [(X_S - X_{SO}) \cos\theta \; k \; M_X / Inc_{SX}] + [(Y_S - Y_{SO}) \sin\theta \; k \; M_Y / Inc_{SY}]$$
$$= 456781.000 + 7023.078 + 1051.544$$
$$= 464855.62 \text{ m.}$$

$$\text{Northing} = Y_T = Y_{TO} - [(X_S - X_{SO}) \sin\theta \; k \; M_X / Inc_{SX}] + [(Y_S - Y_{SO}) \cos\theta \; k \; M_Y / Inc_{SY}]$$
$$= 5836723.000 - 2556.192 + 2889.092$$
$$= 5837055.90 \text{ m.}$$

Calculation of bin node coordinates for this map grid location 464 855.62mE, 5 837 055.90mN:

$$\text{Bin grid I} = X_S = X_{SO} + \{[(X_T - X_{TO}) \cos\theta - (Y_T - Y_{TO}) \sin\theta] [Inc_{SX} / (k \; M_X)]\}$$
$$= 1 + (7473.804 * 0.040006401) \qquad = 300$$

$$\text{Bin grid J} = Y_S = Y_{SO} + \{[(X_T - X_{TO}) \sin\theta + (Y_T - Y_{TO}) \cos\theta] [Inc_{SY} / (k \; M_Y)]\}$$
$$= 1 + (3074.508 * 0.080012802) \qquad = 247$$

Example 2: $k = 1$

It is not uncommon to define the bin grid with the parameter "Scale factor of bin grid" (SF) set to 1.0. Using the above example, and with all other parameter values unchanged, then calculation of map grid coordinates for the bin node with coordinates: I = 300, J = 247 gives:

[21] U.K. Offshore Operators Association (Surveying and Positioning Committee). "*UKOOA Data Exchange Format P6/98, Definition of 3D seismic binning grids*". Revision 3, May 2000. https://www.iogp.org/wp-content/uploads/2016/12/P6.pdf (Accessed 22 July 2024).

Easting = 456781.000 + 7024.202 + 1051.712 = 464856.91 m.

Northing = 5836723.000 - 2556.601 + 2889.555 = 5837055.95 m.

Reverse calculation of bin node coordinates for this map grid location 464 856.91mE, 5 837 055.95mN:
Bin node I = 1 + (7475 * 0.04) = 300

Bin node J = 1 + (3075 * 0.08) = 247

Further examples of input and output may be found in test procedure 5209 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.8.2 P6 Left-handed Seismic Bin Grid Transformation

(EPSG Dataset coordinate operation method code 1049)

The P6/98 exchange format constrained the bin grid axes to be 'right-handed', i.e., by definition the bin grid I-axis orientation is 90° clockwise from the J-axis *when viewed from above the plane of the coordinate system*. The P6/11 format extends the P6/98 format to allow for a 'left-handed' bin grid in which the I-axis orientation is 90° counter-clockwise from the J-axis when viewed from above the plane of the coordinate system. (When using the P6/98 format, this can be accomplished through giving the *Bin node increment on I-axis* parameter a negative value).

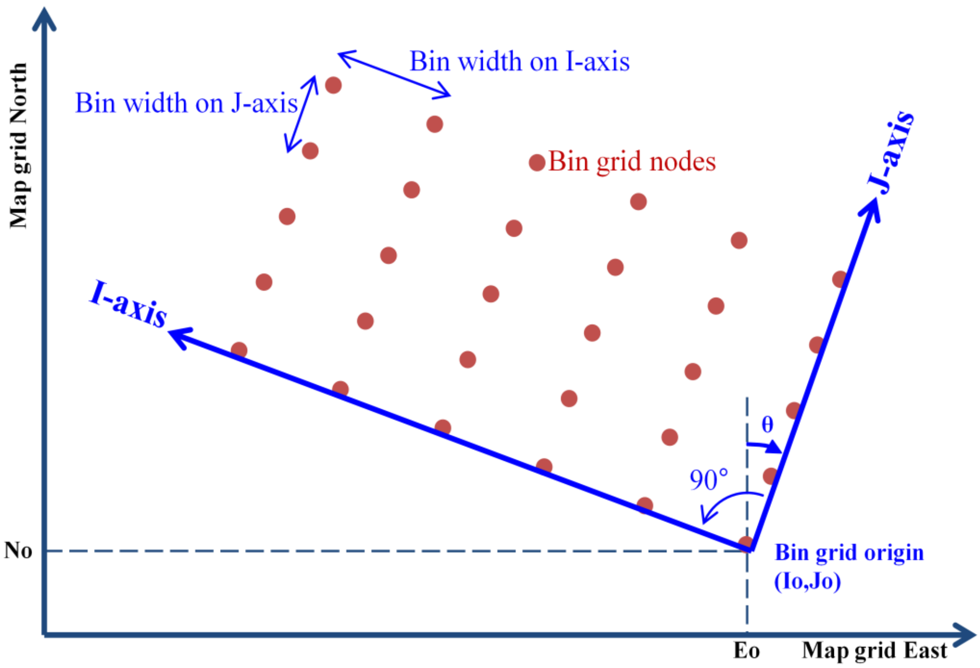


Figure 17 – Relationship of left-handed bin grid to map grid.

The transformation from bin grid to map grid is:

$$\boldsymbol{V_T} = \boldsymbol{V_{TO}} + \boldsymbol{R_2}\, k\, \boldsymbol{S_3}\, \boldsymbol{V_2}$$

where

$$\boldsymbol{S_3} = \begin{pmatrix} M_X / -Inc_{SX} & 0 \\ 0 & M_Y / Inc_{SY} \end{pmatrix}$$

and other matrices are as defined earlier,

or

$$\begin{pmatrix} E \\ N \end{pmatrix} = \begin{pmatrix} E_O \\ N_O \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad SF \quad \begin{pmatrix} \text{I\_bin\_width} / -\text{I\_bin\_inc} & 0 \\ 0 & \text{J\_bin\_width} / \text{J\_bin\_inc} \end{pmatrix} \begin{pmatrix} I - I_O \\ J - J_O \end{pmatrix}$$

or

$$E = E_O - [(I - I_O) \cos\theta \ SF \ \text{I\_bin\_width} / \text{I\_bin\_inc}] + [(J - J_O) \sin\theta \ SF \ \text{J\_bin\_width} / \text{J\_bin\_inc}]$$

$$N = N_O + [(I - I_O) \sin\theta \ SF \ \text{I\_bin\_width} / \text{I\_bin\_inc}] + [(J - J_O) \cos\theta \ SF \ \text{J\_bin\_width} / \text{J\_bin\_inc}]$$

For the transformation from map grid to bin grid:

$$\boldsymbol{V_S} = (1/k) \ \boldsymbol{S_3}^{-1} \ \boldsymbol{R_2}^{-1} \ (\boldsymbol{V_T} - \boldsymbol{V_{TO}}) + \boldsymbol{V_{SO}}$$

or

$$\begin{pmatrix} I \\ J \end{pmatrix} = 1/SF \quad \begin{pmatrix} -\text{I\_bin\_inc} / \text{I\_bin\_width} & 0 \\ 0 & \text{J\_bin\_inc} / \text{I\_bin\_width} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin & \cos\theta \end{pmatrix} \begin{pmatrix} E - E_O \\ N - N_O \end{pmatrix} + \begin{pmatrix} I_O \\ J_O \end{pmatrix}$$

or

$$I = I_O - \{[(E - E_O) \cos\theta - (N - N_O) \sin\theta] \ [\text{I\_bin\_inc} / (SF \ \text{I\_bin\_width})]\}$$
$$J = J_O + \{[(E - E_O) \sin\theta + (N - N_O) \cos\theta] \ [\text{J\_bin\_inc} / (SF \ \text{J\_bin\_width})]\}$$

Example

This example is given in the IOGP Report 483-6u - IOGP P6/11 Seismic Bin Grid Data Exchange Format – User Guide. The base map grid is a projected CRS NAD27 / BLM 16N (ftUS) upon which the origin of the bin grid is defined at E = 871200 ftUS, N = 10280160 ftUS. As the survey was acquired on the map grid, the bin grid scale factor at the bin grid origin is chosen to be 1.

The bin width on the I-axis (X_S axis) is 82.5 US survey feet, the bin width on the J-axis (Y_S axis) is 41.25 US survey feet. The origin of the bin grid has bin node values of I=5000, J=0.

In the map grid, the bearing of the bin grid I and J axes are 250° and 340° respectively. Thus the angle through which the bin grid axes need to be rotated to coincide with the map grid axes is +340 degrees.

The transformation parameter values are:

Parameter	EPSG Affine Transformation symbol	P6 symbol	Parameter value
Bin grid origin I	X_{SO}	Io	5000
Bin grid origin J	Y_{SO}	Jo	0
Bin grid origin Easting	X_{TO}	Eo	871200 ftUS
Bin grid origin Northing	Y_{TO}	No	10280160 ftUS
Scale factor of bin grid	k	SF	1
Bin Width on I-axis	M_X	I_bin_width	82.5 ftUS
Bin Width on J-axis	M_Y	J_bin_width	41.25 ftUS
Map grid bearing of bin grid J-axis	θ	θ	340 deg
Bin node increment on I-axis	Inc_{SX}	I_bin_inc	1
Bin node increment on J-axis	Inc_{SY}	J_bin_inc	1

Calculation of map grid coordinates of bin node with coordinates: I = 4700, J = 247:

Easting = $X_T = X_{TO} - [(X_S - X_{SO}) \cos\theta \; k \; M_X / Inc_{SX}] + [(Y_S - Y_{SO}) \sin\theta \; k \; M_Y / Inc_{SY}]$
= 871200.000 – (–23257.392 –3484.758) = 890 972.63 ftUS.

Northing = $Y_T = Y_{TO} + [(X_S - X_{SO}) \sin\theta \; k \; M_X / Inc_{SX}] + [(Y_S - Y_{SO}) \cos\theta \; k \; M_Y / Inc_{SY}]$
= 10280160.000 + (8464.999 + 9574.293) = 10 298 199.29 ftUS.

Calculation for bin node coordinates of the point with map grid 890 972.63 ftUS E, 10 298 199.29 ftUS N:

Bin node I = $X_S = X_{SO} - \{[(X_T - X_{TO}) \cos\theta - (Y_T - Y_{TO}) \sin\theta] [Inc_{SX} / (k \; M_X)]\}$
= 5000 - (24750 * 0.012121212) = 4700

Bin node J = $Y_S = Y_{SO} + \{[(X_T - X_{TO}) \sin\theta + (Y_T - Y_{TO}) \cos\theta] [Inc_{SY} / (k \; M_Y)]\}$
= 0 + (10188.75 * 0.024242424) = 247

4.9 Wellbore Local to Geodetic transformations

The P7/17 Wellbore Positional Data Exchange format describes methods to transform 3D ('cubic') coordinates of a wellbore to coordinates in geodetic space. Although considered 3D, the wellbore local coordinates (n,e,d) may (and usually will) have separate horizontal and vertical origin points, the well WRP (well reference point) and ZDP (zero depth point), respectively. In the EPSG data model these local coordinates are referenced to a compound CRS consisting of a local horizontal CRS and a local vertical CRS. The local coordinates themselves are the output of a computation such as minimum curvature using the borehole survey observables measured depth, inclination, and azimuth (MD, INC, AZ).

Local coordinates are unscaled and have the same unit of measure (metre or foot) on all three axes as MD. The local horizontal coordinates may be transformed to coordinates in either a geographic 2D or a projected CRS, and the local vertical coordinate may be transformed to coordinates in a geodetic vertical CRS. For more information on wellbore positioning see IOGP Report 483-7u - *P7/17 Wellbore Positioning Data Exchange User Guide*.

4.9.1 Local depth

The local depth *(d)* is given in a 1D vertical CRS with origin at the well Zero-Depth Point (ZDP) and with orientation "down".

Local depth is transformed to depth in the target CRS by applying the vertical offset method (coordinate operation method code 9616, Section §4.10.1):

$$D[i] = d[i] + D_{ZDP}$$

where:

D_{ZDP} is the depth of the ZDP point below the vertical reference surface (VRS) of the target vertical CRS, negative if the ZDP is above the VRS);
$d[i]$ is the local depth at survey station *i;*
$D[i]$ is the Depth below the VRS at survey station *i* (positive below VRS).

If the ZDP is referenced to a height vertical CRS, which often is the case, its height must first be converted to a depth using a Height Depth Reversal (EPSG coordinate operation method code 1068, Section §4.10.6). Combining this with the vertical offset above:

$$D[i] = d[i] – H_{ZDP}$$

Before application of these vertical offset formulas care must be taken that the local depths are converted to the same unit as used by the target CRS, if necessary through using a Change of Vertical Unit (EPSG method code 1069, Section §4.10.7).

4.9.2 Local horizontal coordinates

The local horizontal coordinates *(n,e)* are given in a 2D Cartesian CRS with its origin at the Well Reference Point (WRP) at known location. An example of such local CRS is EPSG code 8377.

The local north axis aligns either with true north or with grid north, according to local practice. To transform local coordinates to 'global' coordinates, the LMP method is used when the local north axis is aligned to true north and the GNL method when the local north axis is aligned to grid north. An advantage of the LMP method is that it eliminates the need to store the projected CRS grid convergence, which is a common source of error and confusion when managing wellbore survey data. The LMP method accounts for varying local grid scale factor and depth correction factor along the wellbore.

4.9.3 Wellbore local to geographic (LMP)

(EPSG Dataset coordinate operation method code 1076)

The LMP method is clearly described by Zinn (2005)[22]. This paper includes formulas and pseudo-code that are not reproduced here.
LMP requires that the input local horizontal coordinates are referenced to true north, and that depths (here referred to as *D*, but as *TVD* in the SPE paper) are referenced to the reference surface of the vertical CRS.

The output of LMP is a set of ellipsoidal coordinates in a geographic 2D CRS. Projected coordinates of the wellbore may be calculated from the geographic coordinates using the relevant map projection formulas provided in this Guidance Note.

Required inputs are:

Latitude of the Well Reference Point in the target 2D geographic CRS;
Longitude of the Well Reference Point in the target 2D geographic CRS;
dcf_flag : 0 for none; 2 to apply a variable depth correction factor along the path;
n[] : Array of length *num_points* containing local northing coordinates, oriented to true north;
e[] : Array of length *num_points* containing local easting coordinates, oriented to true east;
D[] : Array of length *num_points* containing depths below the Vertical Reference Surface at survey stations down the wellbore (referred to as *TVD* in the SPE paper).

In this Guidance Note (but not in the SPE paper) the method has an option for the application of the depth correction, to accommodate subsurface applications that do not correct for Earth curvature:

depth correction factor (*dcf*) is used to scale horizontal offsets according to the depth of the point in the wellbore: (i) no dcf correction is applied, or (ii) a dcf correction is applied at each point along the wellbore.

For option (ii), depths *D[]* are required as input but are not changed by the algorithm. The depths are used to compute a depth correction and therefore a depth of zero must be close to Mean Sea Level (strictly speaking to the reference ellipsoid, but that difference is insignificant and ignored for the computed depth correction).
For option (i), *D[]* is not required.

The following equations assume that the local coordinates *(n,e)* and depth *D* are in metres. If these use a different unit of measure, then as a first step they should be converted to metres.

When *dcf_flag* = 2, output arrays *LAT*, *LON* (in radians) are computed as:

for i=1:num_points

$$\rho = a\,(1 - e^2) / (1 - e^2 \sin^2(\mathrm{LAT}[i-1]))^{3/2}$$
$$\nu = a / (1 - e^2 \sin^2(\mathrm{LAT}[i-1]))^{1/2}$$
$$\mathrm{LAT}[i] = \mathrm{LAT}[i-1] + (n[i] - n[i-1]) / (\rho - D[i])$$
$$\mathrm{LON}[i] = \mathrm{LON}[i-1] + (e[i] - e[i-1]) / (\nu - D[i]) / \cos(\mathrm{LAT}[i-1])$$

where for i=1, for the first station: LAT[0] and LON[0] are the geographic coordinates of the WRP in radians; n[0] = 0, e[0] = 0;

and (per the Implementation Notes in this document):

a is the semi-major axis of the reference ellipsoid;
e is the eccentricity of the reference ellipsoid (not to be confused with local easting);
ρ is the radius of curvature in the meridian;
ν is the radius of curvature in the prime vertical.

[22] Zinn, N.D. "Accounting for Earth Curvature in Directional Drilling", *Society of Petroleum Engineers*. SPE-96813-MS. 2005. https://doi.org/10.2118/96813-MS (Accessed 22 July 2024).

Subtraction of D[i] from the radii of curvature in the above equations effectively accomplishes a depth correction as described in the SPE paper and may also be used in the GNL method. This correction should not be applied if the wellbore path is to be loaded into a subsurface application which is using a cubic model that does not apply a similar correction for its (seismic) data. When *dcf_flag* = 0, the equations above are modified to:

$$LAT[i] = LAT[i-1] + (n[i] - n[i-1]) / \rho$$
$$LON[i] = LON[i-1] + (e[i] - e[i-1]) / \{\nu \cos(LAT[i-1])\}$$

Reverse

To compute (n,e) from given (LAT,LON,D) arrays:

For i=1:num_points

$$\rho = a\,(1 - e^2) / (1 - e^2 \sin^2 (LAT[i-1]))^{3/2}$$
$$\nu = a / (1 - e^2 \sin^2 (LAT[i-1]))^{1/2}$$
$$n[i] = n[i-1] + (LAT[i] - LAT[i-1])\,(\rho - D[i])$$
$$e[i] = e[i-1] + (LON[i] - LON[i-1])\,(\nu - D[i]) \cos(LAT[i-1])$$

where for the first station i=0 variables are initialized as in the forward case.
Note that LAT[i-1], the latitude of the previous point, is used in the reverse formulas to compute the radii of curvature in order to be internally consistent in round trip calculations.

When *dcf_flag* = 0, the equations above are modified to:

$$n[i] = n[i-1] + (LAT[i] - LAT[i-1])\,\rho$$
$$e[i] = e[i-1] + (LON[i] - LON[i-1])\,\nu \cos(LAT[i-1])$$

Example

An example computation for the LMP method is provided below. This is not a realistic wellbore survey but demonstrates the computation steps. A straight path is modelled to a bottom hole at 5 km depth, approximately 14 km away from the surface location, using a constant inclination 70.54002°, azimuth 51.3178° from true north, and survey stations every 30.0167 metres.

Parameters:

Ellipsoid:	WGS 84	a = 6378137 metres	1/f = 298.257223563
	from which	e = 0.0818191908	e^2 = 0. 00669437999

Forward calculation for a well with surface coordinates (origin of the local coordinates)

Latitude of Well Reference Point	9.0364081° = 0.1577150739 rad
Longitude of Well Reference Point	-30.2713073° = -0.5283339813 rad
Depth Correction	dcf_flag = 2
Zero Depth Point height	ZDP_height = 25 m (not used in equations)

Local northing (n) and easting (e) given in Table 14 are the unscaled output of an algorithm such as minimum curvature as commonly applied in wellbore survey (described as cubical coordinates in the SPE paper). Axes n and e are aligned with true north and true east, respectively.

Table 14 – Computation for the LMP method.

Initialization: LAT[0] = 0.1577150739 rad; LON[0] = -0.5283339813 rad.							
I	**D (m)**	**n (m)**	**e (m)**	**ρ (m)**	**ν (m)**	**LAT (rad)**	**LON (rad)**
1	0.00	0.00	0.00	6,337,009.005	6,378,663.709	0.1577150739	-0.5283339813
2	10.00	15.00	24.00	6,337,009.005	6,378,663.709	0.1577174409	-0.5283301714
3	20.00	30.00	48.00	6,337,009.052	6,378,663.724	0.1577198080	-0.5283263616
4	30.00	45.00	72.00	6,337,009.099	6,378,663.740	0.1577221751	-0.5283225517
5	40.00	60.00	96.00	6,337,009.145	6,378,663.756	0.1577245421	-0.5283187419
...							
497	4960.00	7440.00	11904.00	6,337,032.231	6,378,671.502	0.1588895878	-0.5264433944
498	4970.00	7455.00	11928.00	6,337,032.278	6,378,671.517	0.1588919567	-0.5264395808
499	4980.00	7470.00	11952.00	6,337,032.325	6,378,671.533	0.1588943256	-0.5264357673
500	4990.00	7485.00	11976.00	6,337,032.373	6,378,671.549	0.1588966945	-0.5264319538
501	5000.00	7500.00	12000.00	6,337,032.420	6,378,671.565	0.1588990634	-0.5264281403

For convenience of interpretation, the results given in Table 14 are converted from radians to degrees, and to WGS 84 / UTM zone 25N, EPSG CRS code 32625:

Table 15 – LMP output (in radians) converted.

i	Latitude (deg)	Longitude (deg)	Easting (m)	Northing (m)
1	9.0364081	-30.2713073	800000.00	1000000.00
2	9.0365437	-30.2710890	800023.91	1000015.19
3	9.0366793	-30.2708707	800047.81	1000030.38
4	9.0368150	-30.2706524	800071.71	1000045.57
5	9.0369506	-30.2704341	800095.62	1000060.76
…				
497	9.1037028	-30.1629846	811860.25	1007539.67
498	9.1038385	-30.1627662	811884.17	1007554.88
499	9.1039742	-30.1625477	811908.08	1007570.09
500	9.1041100	-30.1623292	811932.00	1007585.30
501	9.1042457	-30.1621107	811955.92	1007600.52

The normal use for this transformation method is the forward case: from survey measurements to calculated path in local coordinates to geographic 2D CRS. Reverse calculations are not expected to be needed. For completeness the reverse calculations using the above example are given below.
Initialise for i=1:

LAT[0] = 0.1577150739 rad
LON[0] = -0.5283339813 rad
e[0] = 0
n[0] = 0

Then with given:

LAT[1] = 9.03640810° = 0.1577150739 rad
LON[1] = -30.27130730° = -0.5283339813 rad
D[1] = 0 m

yields:

ρ = 6,337,009.005 m
ν = 6,378,663.709 m
n[1] = 0 + 0.0 * (6337009.005 – 0) = 0.0
e[1] = 0 + 0.0 * (6378663.709 – 0) * cos(0.1577150739) = 0.0

Next, for i=2 with given:

LAT[2] = 9.03654372° = 0.1577174409 rad
LON[2] = -30.27108901° = -0.5283301714 rad
D[2] = 10 m

yields:

ρ = 6,337,009.005 m
ν = 6,378,663.709 m
n[2] = 0.0 + (0.1577174409 - 0.1577150739) * (6337009.005 - 10) = 15.00
e[2] = 0.0 + (-0.5283301714 - -0.5283339813) * (6378663.709 - 10)*cos(0.1577174409) = 24.00

etc.

4.9.4 Wellbore local to projected (GNL)

(EPSG Dataset coordinate operation method code 1077)

The GNL method requires as input (an array of) unscaled local coordinates with the local north axis aligned with grid north of the projected CRS.

If the input local coordinates are not aligned with grid north, but instead with magnetic north or true north, then they should be changed to grid north reference before input in the algorithm. In practice, this is often done by applying a constant rotation using the map grid convergence and/or magnetic declination at the surface location to all stations. It is also possible to compute these corrections along-hole, e.g., iteratively or using the planned trajectory. However, the LMP method is recommended whenever the inputs are aligned with true north or corrections which vary along-hole (convergence or scale) are desired.

The method has options for the application of two corrections:

i) **point scale factor** (*psf*) is used to scale horizontal offsets to the local scale for the projected CRS, resolved into east and north components (when a conformal map projection is used, psf_E = psf_N): (i) no psf correction is applied, (ii) the point scale factor at the well reference point is applied to the complete wellbore, and (iii) the point scale factor at each point along the wellbore is applied.

ii) **depth correction factor** (*dcf*) is used to scale horizontal offsets according to the depth of the point in the wellbore: (i) no dcf correction is applied, or (ii) a dcf correction is applied at each point along the wellbore.

The inputs are:

WRP_E : Easting of the Well Reference Point in the target projected CRS;
WRP_N : Northing of the Well Reference Point in the target projected CRS;
psf_flag : 0 for none; 1 for constant at first point; 2 to apply a variable point scale factor;
dcf_flag : 0 for none; 2 to apply a variable depth correction factor along the path;
n[] : Array of length *num_points* containing local northing coordinates, oriented to Grid North of the target projected CRS;
e[] : Array of length *num_points* containing local easting coordinates, oriented to Grid East of the target projected CRS;
D[] : Array of length *num_points* containing depths below the Vertical Reference Surface at survey stations down the wellbore.

Care must be taken to convert local easting and northing to the same unit as used by the target projected CRS before applying the following equations.

The method is specified for three of the six possible combinations of psf_flag and dcf_flag. Formulas can readily be extended. However, it is considered not sensible to apply a variable point scale factor without depth correction factor, nor to apply depth correction factor without scale factor. The LMP method is recommended when it is desired to apply variable corrections along the wellbore.

If *psf_flag=0* and *dcf_flag=0* then the target coordinates are computed using a simple 2D translation:

E = WRP_E + e
N = WRP_N + n

If *psf_flag=1* and *dcf_flag=0* then the target coordinates are computed using a scaled offset as:

E = WRP_E + e psf_E
N = WRP_N + n psf_N

If *psf_flag=2* and *dcf_flag=2* then an iterative method is needed to compute the variable point scale factor and depth correction factor along the wellbore. The LMP method is recommended in this situation, but if the GNL method is used then map convergence should also be calculated variable along-hole to reference the local coordinates to grid north.

Output arrays *E* and *N* of length *num_points* are computed as follows:

For i=1:num_points
    for psf_E and psf_N, *compute_psf*(E[i-1], N[i-1])
    dcf = R / (R – D[i])
    E[i] = E[i-1] + (e[i] – e[i-1]) psf_E dcf
    N[i] = N[i-1] + (n[i] – n[i-1]) psf_N dcf

where for i=1, for the first station: E[0] = WRP_E; N[0] = WRP_N; e[0] = 0; n[0] = 0, and:

compute_psf() is a function that returns the local point scale factors in easting and northing directions at the given location;

R = 6371000 m. Note: Using a constant rather than the ellipsoidal local radii of curvature for the depth correction factors in easting and northing directions at the given location results in an insignificantly small error (e.g., for a lateral of 10 km at 10 km depth the worst case difference is 8 centimetres: 6371/(6371-10)=1.001572 vs. 6340/(6340-10)=1.001580).

Reverse

To compute (n,e) from given (E,N):

For *psf_flag=0* and *dcf_flag=0*:

n = N – WRP_N
e = E – WRP_E

For *psf_flag=1* and *dcf_flag=0*:

n = (N – WRP_N / psf_N
e = (E – WRP_E) / psf_E

For *psf_flag=2* and *dcf_flag=2* (additionally requiring D as input):

For i=1:num_points
    for psf_E and psf_N, *compute_psf*(E[i-1], N[i-1])
    dcf = R / (R – D[i])
    n[i] = n[i-1] + (N[i] – N[i-1]) / (psf_N dcf_N)
    e[i] = e[i-1] + (E[i] – E[i-1]) / (psf_E dcf_E)

where for the first station i=0 is initialised as in the forward case.

Example

An example of the GNL method is provided below, similar to the example given for the LMP method. This is not a realistic wellbore survey but demonstrates the magnitude of differences in computed coordinates when different corrections are applied. A straight path is modelled to a bottom hole at 5 km depth, approximately 14 km away from the surface location, using a constant inclination 70.54002°, bearing 51.3178° from grid north, and survey stations every 30.0167 metres.

Forward calculation for a well with surface coordinates in WGS 84 / UTM zone 25N, EPSG CRS code 32625. Since this is a conformal projection, psf_E = psf_N.

Easting of Well Reference Point	WRP_E	= 800,000 m
Northing of Well Reference Point	WRP_N	= 1,000,000 m
Point scale factor at WRP	psf_E = psf_N	= 1.00071
Zero Depth Point height	ZDP_height	= 25 m (not used in equations)

Local axes n and e are aligned with grid north and grid east, respectively.

Table 16 – Computation for the GNL method.

Initialization: E[0] = 800,000 m; N[0] = 1,000,000 m.											
						psf_flag=0; dcf_flag=0		**psf_flag=1; dcf_flag=0**		**psf_flag=2; dcf_flag=2**	
i	**D (m)**	**n (m)**	**e (m)**	**psf_E = psf_N**	**dcf**	**E (m)**	**N (m)**	**E (m)**	**N (m)**	**E (m)**	**N (m)**
1	0.00	0.00	0.00	1.000714	*1.000000*	800000.00	1000000.00	800000.00	1000000.00	800000.00	1000000.00
2	10.00	15.00	24.00	1.000714	*1.000002*	800000.00	1000000.00	800000.00	1000000.00	800000.00	1000000.00
3	20.00	30.00	48.00	1.000714	*1.000003*	800024.00	1000015.00	800024.02	1000015.01	800024.02	1000015.01
4	30.00	45.00	72.00	1.000714	*1.000005*	800048.00	1000030.00	800048.03	1000030.02	800048.03	1000030.02
5	40.00	60.00	96.00	1.000715	*1.000006*	800072.00	1000045.00	800072.05	1000045.03	800072.05	1000045.03
…											
497	4960.00	7440.00	11904.00	1.000804	*1.000779*	811904.00	1007440.00	811912.50	1007445.31	811917.68	1007448.55
498	4970.00	7455.00	11928.00	1.000804	*1.000781*	811928.00	1007455.00	811936.52	1007460.32	811941.72	1007463.57
499	4980.00	7470.00	11952.00	1.000804	*1.000782*	811952.00	1007470.00	811960.53	1007475.33	811965.76	1007478.60
500	4990.00	7485.00	11976.00	1.000804	*1.000784*	811976.00	1007485.00	811984.55	1007490.34	811989.79	1007493.62
501	5000.00	7500.00	12000.00	1.000804	*1.000785*	812000.00	1007500.00	812008.57	1007505.35	812013.83	1007508.65

4.10 Transformations between Vertical Coordinate Reference Systems

The following coordinate operation methods apply only when the positive direction of the axis of the source and target CRS is the same and where the two vertical CRSs share the same unit of measure:

§4.10.1 Vertical Offset
§4.10.2 Vertical Offset by Interpolation of Gridded Data
§4.10.3 Vertical Offset by Interpolation of Gridded Velocity Data
§4.10.4 Vertical Offset and Slope
§4.10.5 Vertical change by geoid grid difference

If either the positive axis direction or unit of measure are different, then a concatenated operation including one or more conversions through intermediate vertical CRSs is required. The additional operation methods required for such conversions:

§4.10.6 Height Depth Reversal, and
§4.10.7 Change of Vertical Unit

are covered later in this section. Their application is discussed in IOGP Report 373-24 - *Geomatics Guidance Note 24 – Vertical data in oil and gas applications*.

4.10.1 Vertical Offset

(EPSG Dataset coordinate operation method code 9616)

A vertical offset allows calculation of coordinates in the target vertical coordinate reference system by adding a correction parameter A to the coordinate values of the point in the source system:

$$X_2 = X_1 + A_{1>2}$$

where

X_2 = value in the target vertical coordinate reference system.
X_1 = value in the source vertical coordinate reference system.
$A_{1>2}$ is the offset to be applied for the transformation from CRS 1 to CRS 2. Its value for the forward calculation is the value of the origin of the source CRS 1 in the target CRS 2.

Note: The sign convention used here is that which is applied in geodesy; this differs from that in general mathematics - see Implementation Notes - Offsets (Section §1.5).

For the **reverse** transformation from CRS 2 to CRS 1, the same formula is used but with the sign of the offset $A_{1>2}$ reversed:

$$X_1 = X_2 + (–A_{1>2})$$

Example

For coordinate transformation: Baltic 1977 height to Black Sea height (1), code 5447:

Transformation Parameter: Vertical Offset $A_{1>2}$ = 0.4m

Consider a point having a gravity-related height H_{Baltic} in the Baltic 1977 height system of 2.55m. Its value in the Black Sea height system is

$$H_{BlackSea} = 2.55m + 0.4m = 2.95m$$

For the reverse calculation to transform the Black Sea height of 2.95m to Baltic 1977 height:

$$H_{Baltic} = 2.95m + (–0.4m) = 2.55m.$$

Further examples of input and output may be found in test procedure 5210 of IOGP's *Geospatial Integrity in Geoscience Software* (GIGS) Test Dataset, https://gigs.iogp.org/.

4.10.2 Vertical Offset by Interpolation of Gridded Data

The relationship between some vertical coordinate reference systems is available through gridded data sets of vertical offsets (sometimes called height differences). The vertical offset at a point is first interpolated from within the grid of values. The interpolated offset is then applied as for the Vertical Offset method in the previous section.

For the purposes of interpolation within the grid, horizontal coordinates of the point are required. The providers of gridded data sets define the horizontal CRS upon which the grid has been constructed and which is to be used for the grid interpolation, known as the *Interpolation CRS*; this is usually a geographic 2D CRS.

The EPSG Dataset differentiates methods by the format of the gridded data file. The grid file format is given in documentation available from the information source. An example method is *Vertical Offset by Grid Interpolation (gtx)* (EPSG Dataset coordinate operation method code 1084) where the grid is provided in the US NGS .gtx file format) used in New Zealand where care is required in interpreting the direction of the transformation because of the file naming convention used: refer to the method formula and this document's Implementation Notes - Offsets (Section §1.5). But see Section §4.10.5 for the treatment of vertical CRS differences in Canada.

Although the providers of some gridded data sets suggest a particular interpolation method within the grid, the density of grid nodes should be such that any reasonable grid interpolation method will give the same offset value within an appropriately small tolerance. Bilinear interpolation is the most usual grid interpolation mechanism.

4.10.3 Vertical Offset by Interpolation of Gridded Velocity Data

(EPSG Dataset coordinate operation method code 1113)

In Canada, the realization of CGVD2013 is defined by application of a geoid model to the NAD83(CSRS) geographic 3D CRS. Although the vertical datum is static, deformation, mostly due to post-glacial isostatic adjustment, leads to changes in ellipsoidal height between the CSRS frame epochs that may be significant for some purposes. When the geoid model is applied to the ellipsoidal heights, any change in ellipsoidal height manifests itself as a change in CGVD2013 gravity-related height. The Canadian velocity grid is used not only as a point motion operation within one of the realizations but also as a transformation between NAD83(CSRS) realizations at different epochs (Section §4.4.6). The transformation between CGVD2013 snapshots at different epochs can be made through a concatenated operation involving the geoid model defining CGVD2013 and the application of the velocity grid. This concatenation simplifies to:

$$H_{(t2)} = H_{(t1)} + (t_2 – t_1)\, V_{U.}$$

where

V_U is the vertical velocity interpolated from the velocity grid, in which positive velocity values represent land uplift, i.e., CSRS ellipsoidal height and CGVD2013 height increasing with time;

t_1 and t_2 are the source (from) and target (to) coordinate epochs respectively, where t is positive towards the future.

In general application of the method, t_1 and and t_2 would be expected to be user-input parameter values. For transformations between CGVD2013 static snapshots [CGVD2013a(1997) height, CGVD2013a(2002) height, and CGVD2013a(2010) height] using EPSG Dataset coordinate operation method code 1113, t_1 and and t_2 are given in the Datum.Anchor_Epoch attribute of the source CRS and target CRS, respectively.

Reverse

In principle, interpolation of a velocity grid is not reversible when it crosses discontinuities. NRCan considers that the Canada v6 and v7 velocity grids are reversible because, for most of Canada, the changes in the velocity vector over short distances are small and the grid node spacing is relatively large. Any change in NAD83(CSRS) horizontal coordinates between the two epochs will be insignificant for the interpolation of the velocity grid and any realization of CSRS may be used for this. For the reverse calculation, the same formula is used but with the sign of the difference in epoch reversed, i.e., the source and target coordinate epochs are transposed and the sign of $(t_2 - t_1)$ changes.

Example

Given a point with NAD83(CSRS)v6 coordinates 49°53'09.293"N, 99°54'41.057"W and with gravity related height H = 396.737 m in CGVD2013(CGG2013a) epoch 2010 whose gravity-related height is required in CGVD2013(CGG2013a) epoch 1997:

First the velocity grid is interpolated at 49°53'09.293"N, 99°54'41.057"W for V_U. This is found to be -1.85 mm/year = –0.00185m/year.

Then, t_1 and t_2 may be obtained through the datum.anchor_epoch attribute values for the datums of the source and target CRSs respectively. (Should this epoch data not be available this way, it should be user-input).

Then $H_{1997} = H_{2010} + (1997.0 – 2010.0) * –0.00185)$
$= 396.737 + (–13.0 * –0.00185)$
$= 396.761$ m

For the reverse calculation, the same point with gravity related height H = 396.761 m in CGVD2013(CGG2013a) epoch 2010 whose gravity-related height is required in CGVD2013(CGG2013a) epoch 2010, first the velocity grid is interpolated as above using the same geographical coordinates:

$V_U = –0.00185$ m/year.

Then, t_1 and t_2 may be obtained through the datum.anchor_epoch attribute values for the datums of the source and target CRSs respectively.

Then $H_{2010} = H_{1997} + (2010.0 – 1997.0)\ V_U$:
$= 396.761 + (13.00 * –0.00185)$
$= 396.737$ m

4.10.4 Vertical Offset and Slope

(EPSG Dataset coordinate operation method code 1046)

In Europe, national vertical systems are related to the pan-European vertical system through three transformation parameters and the formula:

$$X_2 = X_1 + \{A_{1>2} + [I_{\varphi 1>2}\ \rho_O\ (\varphi – \varphi_O)] + [I_{\lambda 1>2}\ \nu_O\ (\lambda – \lambda_O) \cos\varphi]\}$$ where

X_2 = value in the target vertical coordinate reference system.
X_1 = value in the source vertical coordinate reference system.
$A_{1>2}$ is the offset to be applied for the transformation from CRS 1 to CRS 2. Its value is the value of the origin of the source CRS 1 in the target CRS 2.
$I_{\varphi 1>2}$ is the value in radians of the slope parameter in the latitude domain, i.e., in the plane of the meridian, derived at an evaluation point with coordinates of φ_O , λ_O. When I_φ is positive then to the north of the evaluation point latitude φ_O the source and target CRS surfaces converge.
$I_{\lambda 1>2}$ is the value in radians of the slope parameter in the longitude domain, i.e., perpendicular to the plane of the meridian. When I_λ is positive then to the east of the evaluation point longitude λ_O the CRS surfaces converge.
ρ_O is the radius of curvature of the meridian at latitude φ_O,
where $\rho_O = a(1 – e^2)/(1 – e^2\sin^2\varphi_O)^{3/2}$

ν_O is the radius of curvature on the prime vertical (i.e., perpendicular to the meridian) at latitude φ_O, where $\nu_O = a /(1 - e^2 \sin^2\varphi_O)^{1/2}$

φ , λ are the horizontal coordinates of the point in the ETRS89 coordinate reference system, in radians.

φ_O , λ_O are the coordinates of the evaluation point in the ETRS89 coordinate reference system, in radians.

The horizontal location of the point must always be given in ETRS89 terms. Care is required where compound coordinate reference systems are in use: if the horizontal coordinates of the point are known in the local CRS, they must first be transformed to ETRS89 values.

Reverse

Similarly to the Vertical Offset method described in previous sections above, the Vertical Offset and Slope method is reversible using a slightly different formula to the forward formula and in which the signs of the parameters A, I_φ and I_λ from the forward transformation are reversed in the reverse transformation:

$$X_1 = \{X_2 + -A_{1>2} + [-I_{\varphi 1>2}\ \rho_O (\varphi - \varphi_O)] + [-I_{\lambda 1>2}\ \nu_O (\lambda - \lambda_O) \cos\varphi]\}$$

Example

For coordinate transformation LN02 height to EVRF2000 height (1)

Transformation parameters:

Ordinate 1 of evaluation point:	φ_{S0} =	46°55'N	=	0.818850307 rad
Ordinate 2 of evaluation point:	λ_{S0} =	8°11'E (of Greenwich)	=	0.142826110 rad
Vertical Offset:	A =	–0.245 m		
Inclination in latitude:	I_φ =	–0.210"	=	-0.000001018 rad
Inclination in longitude:	I_λ =	–0.032"	=	-0.000000155 rad
Horizontal CRS code:			=	4258

Consider a point having a gravity-related height in the LN02 system (H_s) of 473.0m and with horizontal coordinates in the ETRS89 geographic coordinate reference system of:

Latitude φ_{ETRS89}	=	47°20'00.00"N	=	0.826122513 rad
Longitude λ_{ETRS89}	=	9°40'00.00"E	=	0.168715161 rad

ETRS89 uses the GRS1980 ellipsoid for which a = 6378137 m and 1/f = 298.257222101

Then

ρ_O = 6369526.88 m

I_φ term = –0.047 m

ν_O = 6389555.64 m

I_λ term = –0.017 m

whence EVRF2000 height $X_2 = H_{EVRF}$ = 473.0 +(–0.245) + (–0.047) + (–0.017)
= 472.69 m.

For the reverse transformation from EVRF2000 height of 472.69 m to LN02 height:

$X_1 = H_{LN02}$ = {472.69 + [–(–0.245)] + [–(–0.047)] + [–(–0.017)]}
= 473.00 m.

4.10.5 **Vertical change by geoid grid difference**
(EPSG Dataset coordinate operation method code 1126)

The relationship between CGVD28 and CGVD2013 gravity-related vertical coordinate reference systems in Canada is available through gridded data files of height differences. These have been derived as the difference between the CGVD28 hybrid geoid height and the CGVD2013 geoid height. These CGVD height difference files give height difference values that must be applied as a subtraction (for the forward transformation). In the EPSG Dataset, offsets are additive (for the forward transformation) (see Implementation Notes - Offsets Section §1.5). For that reason these transformations between CGVD28 and CVD2013 do not use the Vertical Offset by Interpolation of Gridded Data method (Section §4.10.2).

The vertical height difference at a point is first interpolated within the grid of values, in Canada using biquadratic interpolation. For many practical purposes bilinear interpolation will give results that are not significantly different. For interpolation within the height difference grid, horizontal positions in the Interpolation CRS are used.

The interpolated value is then applied as a correction parameter (A). The interpolated height difference is ***subtracted*** from the height in the source CRS:

$$H_2 = H_1 - A_{1>2}$$

where
H_2 = height value in the target vertical coordinate reference system (CRS 2).
H_1 = height value in the source vertical coordinate reference system (CRS 1).
$A_{1>2}$ is the height difference to be applied for the transformation from CRS 1 to CRS 2.

For the reverse transformation from CRS 2 to CRS 1, the same formula is used but with the sign of the height difference $A_{1>2}$ reversed:

$$H_1 = H_2 - (-A_{1>2}) = H_2 + A_{1>2}$$

Example
For a point at 49°53'09.293"N, 99°54'41.057"W referenced to NAD83(CSRS)v6 (CRS code 8252) with CGVD28 height of 397.140 metres, to find its CGVD2013(CGG2013a) epoch 2010 height:

First obtain the height differences at each of the surrounding grid nodes by interpolation of the HT2_2010_CGG2013a.byn grid file. Using biquadratic interpolation for 49°53'09.293"N, 99°54'41.057"W, the height difference = 0.382 m.

Then CGVD2013(CGG2013a) epoch 2010 height = 397.140 – 0.382 = 396.758 m.

For the reverse calculation, to find the CGVD28 height from CGVD2013(CGG2013a) epoch 2010 height of 396.758 m:

First interpolate the grid as for the forward transformation. Height difference = 0.382 m.

Then CGVD28 height = 396.758 + 0.382 = 397.140 m.

4.10.6 Height Depth Reversal

(EPSG Dataset coordinate operation method code 1068)

If there is a requirement to either (i) change a height in a source CRS to a depth in a target CRS referenced to the same vertical datum, or (ii) change a depth in the a source CRS to a height in a target CRS referenced to the same vertical datum, then the following generic formula can be used:

$$X_2 = -X_1$$

Examples

1. Source CRS = MSL height, in which a height = -300m.
   Then target CRS = MSL depth, and the depth = (-300 * -1) = 300m.

2. Source CRS = NAVD88 depth (ftUS), in which a depth = 500 ftUS.
   Then target CRS = NAVD88 height (ftUS), and the height = (500 * -1) = -500 ftUS.

4.10.7 Change of Vertical Unit

(EPSG Dataset coordinate operation method code 1069)

If there is a requirement to change the linear unit of a vertical axis, the following formulas can be used:

for the forward transformation:	$X_2 = (X_1 \ U_1) / U_2$
for the reverse transformation:	$X_1 = (X_2 \ U_2) / U_1$

where U_1 and U_2 are unit conversion ratios for the Unit of Measure of the coordinate system axis for CRS_1 and CRS_2 respectively. U = [(factor b) / (factor c)] from the EPSG Dataset Unit of Measure table, with respect to the SI base unit, for linear units the metre.

Then

for the forward transformation:	$X_2 = [X_1 \ (b_1 / c_1)] / (b_2 / c_2) = (X_1 \ b_1 \ c_2) / (b_2 \ c_1)$
for the reverse transformation:	$X_1 = [X_2 \ (b_2 / c_2)] / (b_1 / c_1) = (X_2 \ b_2 \ c_1) / (b_1 \ c_2)$

Example

Conversion of a height of 25 metres into US survey feet:

For the metre (EPSG UoM code 9001) factor b = 1 and factor c = 1, so U_1 has a value of 1.
For the US survey foot (EPSG UoM code 9003) factor b = 12 and factor c = 39.37, so U_2 has a value of approximately 0.304800609601219 (one US survey foot ≈ 0.304800609601219 metres).

Then H_{ftUS} = (25 * (1 / 1)) / (12 / 39.37) = 82.02 ftUS.

For the reverse conversion of 82.02 ftUS into metres:
U_1 and U_2 are as for the forward case, with values of 1/1 and 12/39.37 respectively.

Then H_m = (82.02 * (12 / 39.37) / (1 / 1)) = 25.00 m.

4.11 Geoid and Height Correction Models

4.11.1 Geographic3D to GravityRelatedHeight

Although superficially involving a change of dimension from three to one, this transformation method is actually one-dimensional. The transformation applies an offset to the ellipsoidal height component of a geographic 3D coordinate reference system with the result being a gravity-related height in a vertical coordinate reference system. However the ellipsoidal height component of a geographic 3D coordinate reference system cannot exist without the horizontal components, i.e. it cannot exist as a one-dimensional coordinate reference system.

Geodetic science distinguishes between geoid-ellipsoid separation models ('geoid height models') and height correction (sometimes referred to as hybrid) models. Geoid-ellipsoid separation models give the height of the geoid above the ellipsoid surface. It is positive when the geoid is above the ellipsoid. Height correction models give height difference between the ellipsoid surface and a previously-adopted national vertical CRS. The mathematics of the application of these models is identical and for the purposes of the EPSG Dataset they are considered to be one method. Height correction models to tidal surfaces (hydroid models) use slightly different mathematics and are described in Section §4.12below.

The correction value ζ[23] is interpolated from a grid of height differences and the interpolation requires the latitude and longitude components of the geographic 3D coordinate reference system as arguments.

If **h** is the ellipsoidal height (height of point above the ellipsoid, positive if up) in the geographic 3D CRS and **H** is the gravity-related height in a vertical CRS, then

$$H = h - \zeta$$

Note that unlike the general convention adopted for offsets described in Section §1.5, geoid-ellipsoid separation and height correction models conventionally use the true mathematical convention for sign and the value is applied as a subtraction.

In almost all cases, the offset ζ is interpolated from a gridded file. The CRS in which the file is constructed and interpolation is made must be the horizontal component of the geographic 3D CRS. The EPSG Dataset differentiates between the formats of the gridded height files and distinguishes separate coordinate operation methods for each file format. The coordinate operation method may also define the interpolation technique to be used. However, the density of grid nodes is usually sufficient for any reasonable interpolation technique to be used, with bilinear interpolation usually being applied.

Reverse
The reverse transformation, from gravity-related height in the vertical coordinate reference system to the ellipsoidal height component of the geographic3D coordinate reference system, requires that a horizontal position be associated with the gravity-related height. This is indeterminate unless a compound coordinate reference system is involved. Geographic3D to GravityRelatedHeight methods therefore are not reversible. But see the Geographic3D to Compound Geographic2D+GravityRelatedHeight method described in the following section for a reversible operation.

[23] Geodetic science recognises several types of gravity-related height, differentiated by the relationship between the height and the gravitational field. A discussion of these types is beyond the scope of this document. In this document the symbol ζ is used to indicate the correction to be applied to the ellipsoid height. In texts discussing correction to the geoid, the symbol N may be encountered (not to be confused with the symbol for northing).

4.11.2 Geographic3D to Geographic2D+GravityRelatedHeight

4.11.2.1 Using a geoid model file

This method transforms coordinates between a geographic 3D CRS and a compound CRS consisting of separate geographic 2D and vertical CRSs. The transformation is three-dimensional and reversible.

A transformation using this operation method consists of two parts:

a) The transformation of coordinates from the geographic 3D to or from the geographic 2D CRS
b) The transformation of ellipsoidal height to or from gravity-related height

However, complexities arise if the source 3D and the horizontal component of the target compound CRS are based on different geodetic datums. To account for any offset between and non-parallelism of the ellipsoid surfaces, the method is entirely dependent upon a suitable *three-dimensional* transformation existing between the geographic 3D and the horizontal component of the compound CRS, and this may not always be the case. For this reason *the general case is out of scope here and this method is restricted to circumstances where the source geographic 3D CRS and the target geographic 2D CRS are both based on the same geodetic datum*, i.e., the horizontal component of the compound CRS is the two-dimensional subset of the geographic 3D CRS. This geographic 2D CRS is also the CRS used for interpolation of the geoid model file grid.

Forward transformation from geographic 3D to compound

Because of the restriction above, latitude and longitude of the point in the source CRS do not change as a result of the transformation; therefore $\varphi_t = \varphi_s$ and $\lambda_t = \lambda_s$, where the subscripts 's' and 't' indicates the coordinates in the source and target CRSs, respectively.

The vertical component of the forward transformation is as described in the previous Section §4.11.1. The method requires that the 'Interpolation CRS' - the horizontal CRS to which the grid containing the geoid heights (ζ) are referenced - is specified. That will be the geographic 2D CRS which is both the horizontal subset of the geographic 3D CRS and the horizontal component of the compound CRS. The unit of measure in which the geoid heights are given is part of the file format definition.

The forward case is straightforward:

$$\varphi_t = \varphi_s$$
$$\lambda_t = \lambda_s$$

Then, using (φ, λ) for interpolation for the geoid height at the point,

$$H = h - \zeta$$

Reverse transformation from compound to geographic 3D

For the reverse case, the height offset ζ is first interpolated using the geographic 2D coordinates (φ, λ). Then:

$$\varphi_t = \varphi_s$$
$$\lambda_t = \lambda_s$$
$$h = H + \zeta$$

Example

For coordinate transformation: ETRS89 to ETRS89 + NAP height:
The Interpolation CRS is ETRS89 geographic 2D, CRS code 4258.

A point in the geographic 3D coordinate reference system ETRS89, code 4937, with:

latitude	φ = 51°59'10.80033"N	= +51.986333425°
longitude	λ = 4°37'48.72315"E	= +4.630200875°
ellipsoidal height	h = 36.7595m	

is converted to the geographic 2D coordinate reference system ETRS89, code 4258, as:

point	φ = +51.986333426°	λ = +4.630200875°

To find its NAP height, first obtain the geoid height (ζ) at each of the surrounding grid nodes:

NW node	φ = +51.9875°	λ = +4.6200°	ζ = +43.5376m
NE node	φ = +51.9875°	λ = +4.6400°	ζ = +43.5398m
SE node	φ = +51.9750°	λ = +4.6400°	ζ = +43.5479m
SW node	φ = +51.9750°	λ = +4.6200°	ζ = +43.5455m

Use bilinear interpolation in ETRS89 to obtain the geoid height (ζ):

point	φ = +51.986333426°	λ = +4.630200875°	ζ = +43.5395m

Then the NAP height $H = h - \zeta = 36.7595 - 43.5395 = -6.7800$m

The coordinates in compound CRS "ETRS89 + NAP height", code 9283, are:

latitude	φ = 51°59'10.80033"N
longitude	λ = 4°37'48.72315"E
NAP height	H = −6.7800m

For the reverse transformation to find the ellipsoidal ETRS89 height, first obtain the geoid height (ζ) at each of the surrounding grid nodes and use bilinear interpolation in ETRS89 to obtain the geoid height (ζ) at the point, as for the forward computation above:

point	φ = +51.986333426°	λ = +4.630200875°	ζ = +43.5395m

Then the ellipsoidal ETRS89 height $h = H + \zeta = -6.7800 + 43.5395 = 36.7595$m

The coordinates in geographic 3D coordinate reference system ETRS89, code 4937, are:

latitude	φ = 51°59'10.80033"N	= +51.986333425°
longitude	λ = 4°37'48.72315"E	= +4.630200875°
ellipsoidal height	h = 36.7595m	

4.11.2.2 Using a fixed geoid height

(EPSG Dataset coordinate operation method code 1131)

Over small distances, the geoid slope is sometimes considered to be insignificant and the geoid height given a fixed value. This is the case on the Netherlands Caribbean island of Saba. The forward and reverse formulas are as in the general case in Section §4.11.2.1 above.

Forward case from geographic 3D CRS to compound geographic 2D + vertical CRS where (as in the previous Section) the geographic 2D CRS is the horizontal subset of the geographic 3D CRS:

$$\varphi_t = \varphi_s$$
$$\lambda_t = \lambda_s$$
$$H = h - \zeta$$

Reverse case from compound CRS to geographic 3D CRS:

$$\varphi_t = \varphi_s$$
$$\lambda_t = \lambda_s$$
$$h = H + \zeta$$

Note that, in the forward transformation, the geoid height ζ is applied to the source CRS ellipsoidal height as a subtractive correction.

Example
For coordinate transformation: ETRS89 to hypothetical compound CRS "ETRS89 + fixed offset height".

A point in the geographic 3D coordinate reference system ETRS89, code 4937, with:
latitude φ = 51°59'10.80033"N = +51.986333425°
longitude λ = 4°37'48.72315"E = +4.630200875°
ellipsoidal height h = 36.7595m

Geoid height ζ fixed value = +43.5395m

Then, the "fixed offset" gravity-related height $H = h - \zeta = 36.7595 - 43.5395 = -6.7800$m

The coordinates in compound CRS "ETRS89 + fixed offset height", code 9283, are:
latitude φ = 51°59'10.80033"N
longitude λ = 4°37'48.72315"E
fixed offset height $H = -6.7800$m

For the reverse transformation from "fixed offset" gravity-related height H of -6.7800m., the ETRS89 ellipsoidal height $h = H + \zeta = -6.7800 + 43.5395 = 36.7595$m

The coordinates in geographic 3D coordinate reference system ETRS89, code 4937, are:
latitude φ = 51°59'10.80033"N = +51.986333425°
longitude λ = 4°37'48.72315"E = +4.630200875°
ellipsoidal height h = 36.7595m

4.11.3 <u>Geographic2D with Height Offsets</u>
(EPSG Dataset coordinate operation method code 9618)

This method used in Japan is a form of the general Geographic3D to Geographic2D+GravityRelatedHeight method mentioned in the previous section applied in the opposite direction, that is compound Geographic2D+GravityRelatedHeight to Geographic 3D, combined with the geographic 2D offset method described in Section §4.4.4 above. An ellipsoidal height to gravity-related height geoid-ellipsoid separation value A is applied as a vertical offset as a positive correction to the gravity-related height.

$$\varphi_{WGS84} = \varphi_{Tokyo} + d\varphi$$
$$\lambda_{WGS84} = \lambda_{Tokyo} + d\lambda$$
$$h_{WGS84} = H_{JSLD} + A$$

4.12 Hydroid Models

These methods are a form of the Geographic3D to GravityRelatedHeight methods described in Sections §4.11.1 and §4.11.2 above applied to hydrographic surveying and marine navigation. The hydroid models describe the offset of a tidal surface such as lowest astronomical tide (LAT) from a reference ellipsoid, positive when the tidal surface is above the ellipsoid. The same conventions and reversibility constraints as for geoid and height correction models apply.

4.12.1 Geographic3D to Depth

Because it is applied in the depth domain, the height correction model relationship to ellipsoidal height described in Section §4.11.1 has its sign reversed:

$$H = h - \zeta$$

becomes

$$D = \zeta - h$$

In a marine context, depth of the seabed below the tidal surface is derived from observed water depth and observed three-dimensional position including ellipsoidal height, both measurements reduced to a common vessel reference point, and the ellipsoid height of the tidal surface interpreted from the hydroid model.

$$D = (D_{obs} - h_{obs}) + \zeta$$

where D is the depth below the tidal surface (for example, depth below LAT)
D_{obs} is an observed depth below the vessel reference point
h_{obs} is an observed ellipsoidal height of the vessel reference point
ζ is the height of the tidal surface above the reference ellipsoid interpolated from the hydroid model

Example

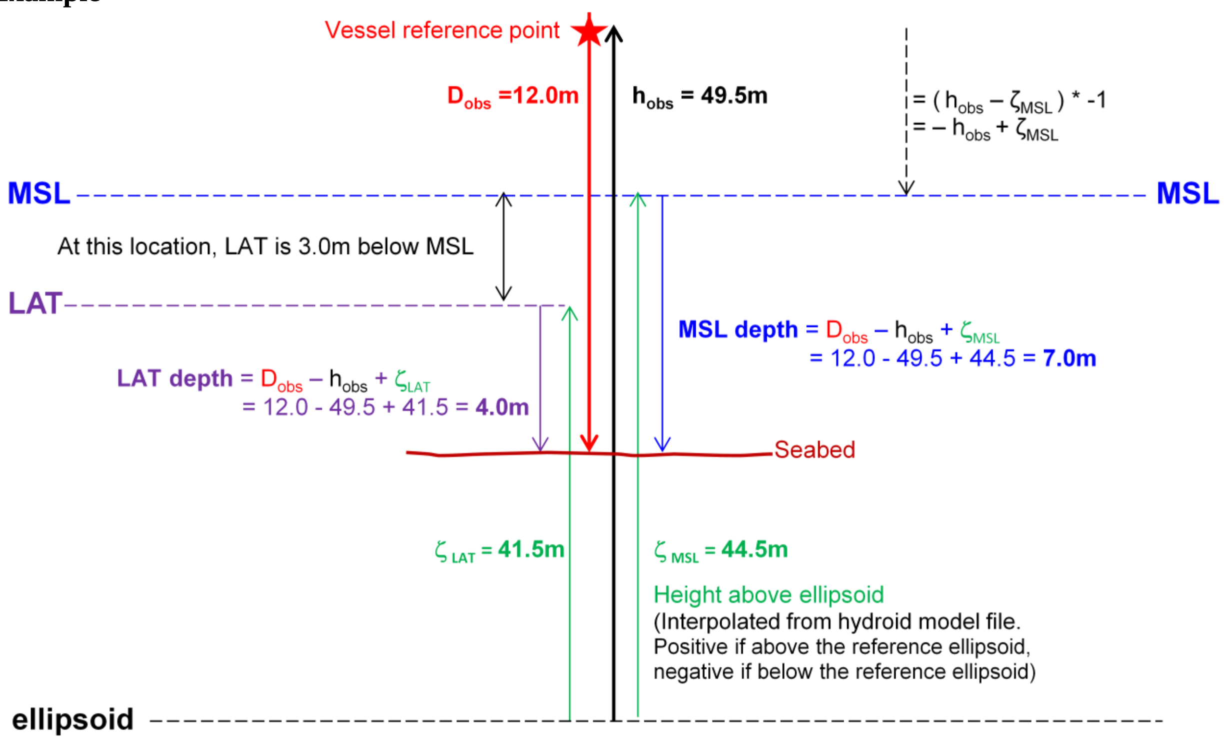


Figure 18 – Application of hydroid models.

Reverse
The reverse transformation, from depth in the vertical coordinate reference system to the ellipsoidal height component of the geographic3D coordinate reference system, requires that a horizontal position be associated with the depth. This is indeterminate unless a compound coordinate reference system is involved; Geographic3D to Depth methods therefore are not reversible. But see the Geographic3D to Compound Geographic2D+Depth method described below for a reversible operation.

4.12.2 Geographic3D to Geographic2D+Depth

This method transforms coordinates between a geographic 3D CRS and a compound CRS consisting of separate geographic 2D and vertical CRSs where the vertical axis is a depth. As with the Geographic3D to Compound Geographic2D+GravityRelatedHeight method described in Section §4.11.2 above, it extends the Geographic3D to Depth method in previous Section §4.12.1 to be reversible. As with the Geographic3D to Compound Geographic2D+GravityRelatedHeight method, the general case is considered to be out of scope here and this method is restricted to circumstances where the source geographic 3D CRS and the target geographic 2D CRS are both based on the same geodetic datum. The transformation is three-dimensional and reversible.

The forward case is straightforward:

$$\varphi_t = \varphi_s$$
$$\lambda_t = \lambda_s$$

Then, using (φλ) for interpolation for the hydroid height at the point and the formula from the previous section:

$$D = \zeta - h$$

where h is ellipsoid height, ζ is height offset from ellipsoid to tidal surface and D is depth below tidal surface.

In a marine context,

$$D = (D_{obs} - h_{obs}) + \zeta$$

Reverse transformation from compound to geographic 3D
For the reverse case the height offset ζ is first interpolated using the geographic 2D coordinates (φλ).
Then:

$$\varphi_t = \varphi_s$$
$$\lambda_t = \lambda_s$$
$$h = \zeta + (D_{obs} - D)$$

where h and ζ are heights and D is depth.

Example
For coordinate transformation: ETRS89 to ETRS89 + CD Norway depth:
The Interpolation CRS is ETRS89 geographic 2D, CRS code 4258.

A point in the geographic 3D coordinate reference system ETRS89, code 4937, with:

latitude	φ = 60°00'05.4"N	= +60.0015°
longitude	λ = 4°59'45.6"E	= +4.9960°
ellipsoidal height	h = 50.000m	

has an observed water depth D_{obs} of 12.00m

This is converted to the geographic 2D coordinate reference system ETRS89, code 4258, as:

point for interpolation: φ = +60.0015° λ = +4.9960°

To find its CD Norway depth, first obtain the hydroid model height (ζ) at each of the surrounding grid nodes:

NW node	φ = +60.005°	λ = +4.990°	ζ = +43.885m
NE node	φ = +60.005°	λ = +5.000°	ζ = +43.887m
SE node	φ = +60.000°	λ = +5.000°	ζ = +43.882m
SW node	φ = +60.000°	λ = +4.990°	ζ = +43.880m

Use bilinear interpolation in ETRS89, code 4258, to obtain the hydroid model height at the observation point φ = +60.0015°, λ = +4.9960°:

ζ = +43.8827m which rounded to three decimal places gives ζ = +43.883m

Then the depth D below CD Norway = $(D_{obs} – h_{obs}) + \zeta$ = (12.00 − 50.000) + 43.883 = +5.883m

The coordinates in compound CRS "ETRS89 + CD Norway depth", code 9883, are:

latitude	φ = 60°00'05.4"N
longitude	λ = 4°59'45.6"E
CD Norway depth	D = +5.883m

For the reverse transformation to find the ellipsoidal ETRS89 height, first obtain the hydroid height at each of the surrounding grid nodes and use bilinear interpolation in ETRS89 to obtain the hydroid height (ζ) at the point, as for the forward computation above:

point	φ = +60.0015°	λ = +4.9960°	ζ = +43.883m

Then the ETRS89 ellipsoidal height h = $\zeta + (D_{obs} – D)$ = 43.883 + 12.000 – 5.883 = 50.000m

The coordinates in geographic 3D coordinate reference system ETRS89, code 4937, are:

latitude	φ = +60.0015°N	= 60°00'05.400"N
longitude	λ = +4.9960°	= 4°59'45.600"E
ellipsoidal height	h = 50.000m	

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6 Revision history

Version	Date	Amendments
1	December 1993	First release – POSC Epicentre
10	May 1998	Additionally issued as an EPSG guidance note.
11	November 1998	Polynomial for Spain and Tunisia Mining Grid methods added.
12	February 1999	Abridged Molodensky formulas corrected.
13	July 1999	Lambert Conic Near Conformal and American Polyconic methods added.
14	December 1999	Stereographic and Tunisia Mining Grid formulas corrected. Krovak method added.
15	June 2000	General Polynomial and Affine methods added
16	December 2000	Lambert Conformal (Belgium) remarks revised; Oblique Mercator methods consolidated and formulas added. Similarity Transformation reversibility remarks amended.
17	June 2001	Lambert Conformal, Mercator and Helmert formulas corrected.
18	August 2002	Revised to include ISO 19111 terminology. Section numbering revised. Added Preface. Lambert Conformal (West Orientated), Lambert Azimuthal Equal Area, Albers, Equidistant Cylindrical (Plate Carrée), TM zoned, Bonne, Molodensky-Badedas methods added. Errors in Transverse Mercator (South Orientated) formula corrected.
19	December 2002	Polynomial formulas amended. Formula for spherical radius in Equidistant Cylindrical projection amended. Formula for Krovak projection amended. Degree representation conversions added. Editorial amendments made to subscripts and superscripts.
20	May 2003	Font for Greek symbols in Albers section amended.
21	October 2003	Typographic errors in example for Lambert Conic (Belgium) corrected. Polar Stereographic formulas extended for secant variants. General polynomial extended to degree 13. Added Abridged Molodensky and Lambert Azimuthal Equal Area examples and Reversible polynomial formulas.
22	December 2003	Errors in FE and FN values in example for Lambert Azimuthal Equal Area corrected.
23	January 2004	Database codes for Polar Stereographic variants corrected. Degree representation conversions withdrawn.
24	October 2004 From this revision, published as part 2 of a two-part set.	Corrected equation for u in Oblique Mercator. Added Guam projection, Geographic 3D to 2D conversion, vertical offset and gradient method, geoid models, bilinear interpolation methods. Added tables giving projection parameter definitions. Amended Molodensky-Badekas method name and added example. Added section on reversibility to Helmert 7-parameter transformations. Transformation section 2 reordered. Section §3 (concatenated operations) added.
25	May 2005	Amended reverse formulas for Lambert Conic Near-Conformal. Corrected Lambert Azimuthal Equal Area formulas. Symbol for latitude of pseudo standard parallel parameter made consistent. Corrected Affine Orthogonal Geometric transformation reverse example. Added Modified Azimuthal Equidistant projection.
26	July 2005	Further correction to Lambert Azimuthal Equal Area formulas. Correction to Moldenski-Badekas example.
27	September 2005	Miscellaneous linear coordinate operations paragraphs re-written to include reversibility and UKOOA P6. Improved formula for r’ in Lambert Conic Near-Conformal.
28	November 2005	Corrected error in formula for t and false grid coordinates of 2SP example in Mercator projection.
29	April 2006	Typographic errors corrected. (For oblique stereographic, corrected formula for w. For Lambert azimuthal equal area, changed example. For Albers equal area, corrected formulas for alpha. For modified azimuthal equidistant, corrected formula for c. For Krovak, corrected formula for theta’, clarified formulas for tO and lat. For Cassini, in example corrected radian value of longitude of natural origin). References to EPSG updated.

Version	Date	Amendments
30	June 2006	Added Hyperbolic Cassini-Soldner. Corrected FE and FN values in example for Modified Azimuthal Equidistant. Added note to Krovak. Amended Abridged Molodensky description, corrected example.
31	August 2006	Corrected sign of value for G in Modified Azimuthal Equidistant example.
32	February 2007	Descriptive text for Oblique Mercator amended; formula for Laborde projection for Madagascar added. Added polar aspect equations for Lambert Azimuthal Equal Area. Corrected example in polynomial transformation for Spain. For Lambert 1SP, corrected equation for r'.
33	March 2007	For Krovak example, corrected axis names.
34	July 2007	Note on longitude wrap-around added prior to preample to map projection formulas, Section §1.4. For Laborde, corrected formula for q'. For Albers Equal Area, corrected formulas for α and ß'.
35	April 2008	Longitude wrap-around note clarified. For Oblique Mercator, corrected symbol in formula for longitude. For Krovak, clarified defining parameters. Amended Vertical Offset description and formula. Added geographic/topocentric conversions, geocentric/topocentric conversions, Vertical Perspective, Orthographic, Lambert Cylindrical Equal Area, ellipsoidal development of Equidistant Cylindrical. Removed section on identification of map projection method.
36	July 2008	For Lambert Conic Near Conformal, corrected equations for φ'.
37	August 2008	Corrected general polynomial example.
38	January 2009	For Mercator (1SP), clarified use of φ_O. For Molodensky-Badekas, augmented example. Added Popular Visualisation Pseudo Mercator method, added formulas and examples for Mercator (Spherical) and formulas for American Polyconic.
39	April 2009	Preface revised to be consistent with other parts of GN7. For Lambert Azimuth Equal Area, in example corrected symbol for ß'. For Krovak, corrected formulas. For Equidistant Cylindrical (spherical) corrected fomula for R; comments on R added to all spherical methods. For Equidistant Cylindrical updated formula to hamonize parameters and symbols with similar methods.
40	November 2009	For geographic/geocentric conversions, corrected equation for φ. For Tranverse Mercator (South Oriented), added example. Corrected equation for computation of radius of authalic sphere (optionally referenced by equal area methods developed on a sphere). Augmented description of geocentric methods to clearly discriminate the coordinate domain to which they are applied.
41	October 2010	Augmented Transverse Mercator for wide area. Amended TM(SO) example. Amended Mercator and Oblique Mercator variant names, added Mercator variant C method. Corrected Oblique Mercator (Hotine variant B) formulas for azimuth = 90º case. Added Krovak variants. Corrected Polar Stereographic equations for calculation of longitude. Augmented reverse case for Similarity Transformation. References to IOGP publication 430 (GIGS test data) added.
42	November 2010	Corrected formula for t in Mercator variant C reverse case.
43	July 2011	Corrected formulas for ξ_O parameters in Transverse Mercator. Corrected coefficient B1 and longitude example in Polynomial transformation for Spain.
44	April 2012	Added missing equations for reverse case of American Polyconic. Amended Transverse Mercator JHS formula to allow for origin at pole. Corrected Modified Krovak formula and example. Corrected equation for u' in Hotine Oblique Mercator.
45	July 2012	Added left-handed bin grid method.
46	June 2013	Added Lambert Conic Conformal 2SP (Michigan), Colombia Urban and time-dependent Helmert 7-parameter transformation methods and Affine Parametric Transformation example. Amended Tunisia Mining Grid description.
47	June 2014	Added Molodensky-Badekas (PV).
48	July 2014	Re-write of P6 Seismic Bin Grid transformation text for consistency with P6/11 exchange format v1.1.
49	February 2015	Helmert and Molodensky-Badekas descriptions rewritten, corrected time-dependent Helmert example. Document sections reordered.
50	April 2015	Complex polynomial reverse example added. Conformal sphere and Albers unbalanced brackets corrected. Minor editorial corrections made.

Version	Date	Amendments
51	September 2016	Horizontal axis order reversal added. Pseudo Mercator section rewritten.
52	October 2017	Added time-specific transformations and point motion operations. Amended vertical transformation descriptions (splitting vertical offset into three methods). Note regarding coefficients with zero values added to Polynomial Transformations. Zoned Transverse Mercator formulas modified. Minor changes to Krovak examples.
53	April 2018	Added reverse example to Change of Vertical Unit method. Harmonised case for Helmert transformation names.
54	August 2018	Added implementation notes section to preface. Reviewed use of atan and atan2 functions throughout document. In point motion operation with velocity grid, removed reference to Interpolation CRS.
55	October 2018	Added Equal Earth projection.
56	February 2019	Added Wellbore and New Zealand vertical offset methods. Editorial amendments to Affine and Helmert transformation descriptions. Re-ordered sections.
57	May 2019	Corrected Equal Earth reverse formula and example and clarified Mercator reverse formulas.
58	March 2020	Amended Lambert Conic Conformal and Albers reverse formula for atan2 function in southwest hemisphere, added examples. Revised France geocentric interpolation description (Section §4.3.1.1) and added example. Minor revision to Vertical Offset by Interpolation of Gridded Data description.
59	October 2020	Corrected example for Geographic/Geocentric conversion (Section §4.1.1). Updated Geographic3D to Geographic2D+GravityRelatedHeight (Section §4.11.2) and added hydroid model transformations (Section §4.12).
60	February 2021	Added Lambert conic conformal 1SP variant B (Section §3.1.1). Updated P6 seismic bin grid text (Section §4.8)
61	November 2021	Clarified Transverse Mercator (JHS method) formula for pole (Section §3.4.3) and Mercator (variant C) (Section §3.4.1). Corrected corruption of text for Krovak (Section §3.3.2). Added example to hydroid model (Section §4.12.2).
62	December 2021	Inserted Section §3.3 Map Projections supporting projected 3D CRS (former sections 3.3 through 3.6 become 3.4 through 3.7). Corrected Transverse Mercator (JHS method) formula for pole (Section §3.5.3). Clarified formulas and corrected example for Oblique Stereographic (Section §3.6.1.1).
63	January 2022	Text on 3D projections augmented (Section §3.3). Corrected starting latitude value in example 2 for Lambert Conic Conformal (2SP) method (Section §3.4.1.1) and CRS name in Tranverse Mercator example (Section §3.5.3.1).
64	May 2022	Added description of velocity grid as transformation (Sections §4.3.5, §4.4.4 and §4.10.2.1).
65	March 2023	Inserted descriptions for Equidistant Conic projection (Section §3.4.5) and geocentric method translations using grid interpolation (Section §4.2.4.1). Treatment of reversiblity in point motion operations (Section §4.4) augmented. Minor edits to map projection examples to more clearly identify initial and intermediate values.
66	December 2023	Inserted description for rigorous Azimuthal Equidistant priojection (Section §3.6.3.3) and vertical change by grid interpolation (Section §4.10.5). Moved point motion operation (former Section §4.4) before former Sections §4.2 and §4.3. Minor update to velocity grid method descriptions ([new] Sections §4.2.1, §4.2.2, §4.2.3 and §4.4.5 and §4.10.2.1).
67	June 2024	Added Local Orthographic projection (Section §3.6.6). Added summary of affine and polyomial relationship with CS axes in Section §1.7; added note cross-referencing this in Sections §4.6 and §4.7. For geocentric offset by grid interpolation method, removed duplication in Sections §4.3.3.1 and §4.4.1.1, consolidating description in Section §4.4.1.1.
68	July 2024	Supplemented Helmert full matrix description (Section §4.3.3), inserted longitude rotation (Section §4.4.3), supplemented geoid model application (Section §4.11.2). Minor changes to text throughout document to align with IOGP style.

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