Network Dynamics

Event Rate

Hidden Layers

At time t, input $\mathbf{h}^0(t-1)$ is received by the first hidden layer and used to compute its somatic voltages $\mathbf{v}^1(t)$:

$$\mathbf{v}^1(t) = \mathbf{W}^1\mathbf{h}^0(t-1) + \mathbf{b}^1 \tag{1}$$

Then, the *event rates* of the first hidden layer units, $\mathbf{h}^1(t)$, are given by applying the softplus function σ:

$$\mathbf{h}^1(t) = \sigma(\mathbf{v}^1(t)) \tag{2}$$

Similarly, for all other hidden layers i of the network,

$$\mathbf{v}^i(t) = \mathbf{W}^i\mathbf{h}^{i-1}(t-1) + \mathbf{b}^i \tag{3}$$

$$\mathbf{h}^i(t) = \sigma(\mathbf{v}^i(t)) \tag{4}$$

Output Layer

The somatic voltages of output layer k, $\mathbf{v}^k(t)$, are calculated as in equation (3). Unlike hidden layers, output layer units have a rectified linear activation function ψ:

$$\mathbf{h}^k(t) = \psi(\mathbf{v}^k(t)) \tag{5}$$

During training, a teaching signal $\mathbf{t}$ is randomly presented at the output layer. When the teaching signal is present, the event rates of the final layer are equal to the teaching signal. That is, if a teaching signal arrives at time t^*, the event rates of the output layer k become:

$$\mathbf{h}^k(t^*) = \mathbf{t} \tag{6}$$

Burst Probability & Burst Rate

Output Layer

At the output layer k, we set the *burst probabilities* to be a constant value ω:

$$\mathbf{p}^k(t) = \omega \tag{7}$$

The *burst rates* of the output layer k at time t are given by:

$$\boldsymbol{\beta}^k(t) = \mathbf{p}^k(t)\mathbf{h}^k(t) \tag{8}$$

Hidden Layers

At time t, for the last hidden layer $k-1$, let the apical dendritic voltages $\mathbf{u}^{k-1}(t)$ be:

$$\mathbf{u}^{k-1}(t) = \mathbf{W}^{k^T} \cdot (\boldsymbol{\beta}^k(t-1) \odot \omega H(\mathbf{v}^k(t-1)) \tag{9}$$

where $\boldsymbol{\beta}^k(t)$ are the *burst rates* of output layer k at time t (see equation (8)), ω is the (constant) burst probability of the output layer units, and H is the Heaviside step function (ie. the derivative of the rectified linear activation function of the output layer).

For any *other* hidden layer i, we let the apical dendritic voltages $\mathbf{u}^i(t)$ be:

$$\mathbf{u}^i(t) = \mathbf{W}^{i+1^T} \cdot (\boldsymbol{p}^{i+1}(t-1) \odot \sigma'(\mathbf{v}^{i+1}(t-1))) \tag{10}$$

where σ' is the derivative of the softplus activation function.

The *burst probabilities* of any hidden layer i, $\mathbf{p}^i(t)$, are given by:

$$\mathbf{p}^i(t) = \boldsymbol{u}^i(t) \tag{11}$$

Finally, the *burst rates* of any hidden layer i are as in equation (8).

Feedforward Weight Updates

For a network with output layer k, the feedforward weights $\mathbf{W}^i$ are updated continuously in order to descend the global loss function $L(t)$:

$$L(t) = ||\boldsymbol{\beta}^k(t) - \boldsymbol{\beta}^k(t-1)||_2^2 \tag{12}$$

First, we will show the feedforward weight updates that would be prescribed by backpropagation of error. For the output layer,

$$\begin{aligned}
\Delta\mathbf{W}^k(t) &= -\eta \frac{\partial L(t)}{\partial \mathbf{W}^k} && (13)\\
&= -\eta \frac{\partial L(t)}{\partial \boldsymbol{\beta}^k(t-1)} \frac{\partial \boldsymbol{\beta}^k(t-1)}{\partial \mathbf{h}^k(t-1)} \frac{\partial \mathbf{h}^k(t-1)}{\partial \mathbf{v}^k(t-1)} \frac{\partial \mathbf{v}^k(t-1)}{\partial \mathbf{W}^k} && (14)\\
&= \eta\Big((\boldsymbol{\beta}^k(t) - \boldsymbol{\beta}^k(t-1)) \odot \omega H(\mathbf{v}^k(t-1))\Big) \cdot \mathbf{h}^{k-1}(t-2) && (15)\\
&= -\eta \boldsymbol{\delta}^k \cdot \mathbf{h}^{k-1}(t-2) && (16)
\end{aligned}$$

where H is the Heaviside step function, and we define $\boldsymbol{\delta}^k$ as:

$$\boldsymbol{\delta}^k := -(\boldsymbol{\beta}^k(t) - \boldsymbol{\beta}^k(t-1)) \odot \omega H(\mathbf{v}^k(t-1)) \tag{17}$$

For the last hidden layer $k-1$,

$$
\begin{aligned}
\Delta \mathbf{W}^{k-1}(t) &= -\eta \frac{\partial L(t)}{\partial \mathbf{W}^{k-1}} && (18)\\
&= -\eta \boldsymbol{\delta}^k \frac{\partial \mathbf{v}^k(t-1)}{\partial \mathbf{h}^{k-1}(t-2)} \frac{\partial \mathbf{h}^{k-1}(t-2)}{\partial \mathbf{v}^{k-1}(t-2)} \frac{\partial \mathbf{v}^{k-1}(t-2)}{\partial \mathbf{W}^{k-1}} && (19)\\
&= -\eta \Big((\mathbf{W}^{k^T} \cdot \boldsymbol{\delta}^k) \odot \sigma'(\mathbf{v}^{k-1}(t-2)) \Big) \cdot \mathbf{h}^{k-2}(t-3) && (20)\\
&= -\eta \boldsymbol{\delta}^{k-1} \cdot \mathbf{h}^{k-2}(t-3) && (21)
\end{aligned}
$$

where $\boldsymbol{\delta}^{k-1}$ is defined as:

$$
\boldsymbol{\delta}^{k-1} := (\mathbf{W}^{k^T} \cdot \boldsymbol{\delta}^k) \odot \sigma'(\mathbf{v}^{k-1}(t-2)) \tag{22}
$$

Then, For any other hidden layer i,

$$
\begin{aligned}
\Delta \mathbf{W}^{i}(t) &= -\eta \frac{\partial L(t)}{\partial \mathbf{W}^{i}} && (23)\\
&= -\eta \Big((\mathbf{W}^{i+1^T} \cdot \boldsymbol{\delta}^{i+1}) \odot \sigma'(\mathbf{v}^{i}(t-(k-i+1))) \Big) \cdot \mathbf{h}^{i-1}(t-3) && (24)\\
&= -\eta \boldsymbol{\delta}^{i} \cdot \mathbf{h}^{i-1}(t-3) && (25)
\end{aligned}
$$

where $\boldsymbol{\delta}^{i}$ is similarly defined as:

$$
\boldsymbol{\delta}^{i} := (\mathbf{W}^{i+1^T} \cdot \boldsymbol{\delta}^{i+1}) \odot \sigma'(\mathbf{v}^{i}(t-(k-i+1))) \tag{26}
$$

With the dynamics described in equations (9), (10) and (11), we can see that

$$
\boldsymbol{\delta}^{k-1} = -(\boldsymbol{p}^{k-1}(t) - \boldsymbol{p}^{k-1}(t-1)) \odot \sigma'(\mathbf{v}^{k-1}(t-2)) \tag{27}
$$

and therefore,

$$
\Delta \mathbf{W}^{k-1}(t) = \eta \Big((\boldsymbol{p}^{k-1}(t) - \boldsymbol{p}^{k-1}(t-1)) \odot \sigma'(\mathbf{v}^{k-1}(t-2)) \Big) \cdot \mathbf{h}^{k-2}(t-3) \tag{28}
$$

Hidden Layers

At time t, for hidden layer i, let the apical dendritic voltages $\mathbf{u}^i(t)$ be:

$$
\mathbf{u}^i(t) = \frac{\mathbf{Y}^i \boldsymbol{\beta}^{i+1}(t-1)}{\mathbf{Z}^i \boldsymbol{\beta}^{i}(t-1)} \tag{29}
$$

where $\boldsymbol{\beta}^i(t)$ are the *burst rates* of layer i at time t (see equation (8)). The *burst probabilities* of hidden layer i, $\mathbf{p}^i(t)$, are given by applying the sigmoid function σ:

$$
\mathbf{p}^i(t) = \sigma(\boldsymbol{u}^i(t)) \tag{30}
$$

Finally, the *burst rates* of hidden layer i are as in equation (8).

Recurrent Weight Updates

The recurrent weights $\mathbf{Z}^i$ are updated continuously in order to descend the loss function $L^{\mathbf{Z}^i}(t)$:

$$L^{\mathbf{Z}^i}(t) = \|\mathbf{u}^i(t)\|_2^2 + \|\alpha - \mathbf{Z}^i\|_2^2 \tag{31}$$

where α is a constant. The first term will update $\mathbf{Z}^i$ in order to push the apical voltages $\mathbf{u}^i$ toward 0, while the second term is a regularization term that will push $\mathbf{Z}$ toward small values as long as α is small. The parameter α allows us to control how small $\mathbf{Z}$ should be allowed to get. Thus, at every time step t, $\mathbf{Z}^i$ is updated according to:

$$\begin{aligned}
\Delta\mathbf{Z}^i(t) &= -\eta \frac{\partial L^{\mathbf{Z}^i}(t)}{\partial \mathbf{Z}^i} && (32)\\
&= \eta\left(\left(\frac{(\mathbf{u}^i(t))^2}{\mathbf{Z}^i\boldsymbol{\beta}^i(t-1)}\right)\otimes \boldsymbol{\beta}^i(t-1) + (\alpha - \mathbf{Z}^i)\right) && (33)
\end{aligned}$$

Feedback Weight Updates

The feedback weights $\mathbf{Y}^i$ are updated continuously in order to descend the loss function $L^{\mathbf{Y}^i}(t)$:

$$L^{\mathbf{Y}^i}(t) = \|\gamma - \mathbf{u}^i_{\max}\|_2^2 \tag{34}$$

where γ is a constant, and $\mathbf{u}^i_{\max}$ are the maximum possible values for the apical voltages that can be driven by feedback, if recurrent activity is kept fixed:

$$\mathbf{u}^i_{\max} := \frac{|\mathbf{Y}^i|\mathbf{1}}{\mathbf{Z}^i\boldsymbol{\beta}^i(t-1)} \tag{35}$$

where $\mathbf{1}$ is a vector of ones of the same size as layer $i+1$. Descending this loss function will update $\mathbf{Y}^i$ in order to produce an upper bound and lower bound of $\mathbf{u}^i_{\max}$ and $-\mathbf{u}^i_{\max}$, respectively, for the apical voltages $\mathbf{u}^i$. The parameter γ allows us to control what value the upper and lower bounds should have. Thus, at every time step t, $\mathbf{Y}^i$ is updated according to:

$$\begin{aligned}
\Delta\mathbf{Y}^i(t) &= -\eta \frac{\partial L^{\mathbf{Y}^i}(t)}{\partial \mathbf{Y}^i} && (36)\\
&= \eta\left(\frac{(\gamma - \mathbf{u}^i_{\max})}{\mathbf{Z}^i\boldsymbol{\beta}^i(t-1)}|\mathbf{Y}^i|\right) && (37)
\end{aligned}$$